

NONLINEAR DYNAMICS AND SYSTEMS THEORY

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PERSONAGE IN SCIENCE

Professor N.V. Azbelev

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Professor Nikolay Viktorovich Azbelev, a well-known Russian mathematician, has been a leading figure in the differential and integral equations profession for about five decades. To commemorate Professor Azbelev's valuable contribution to nonlinear dynamics, the Editorial Board of the Journal presents a biographical sketch of his life and academic activities. The main stages of Azbelev's life and activity are also presented in the special papers in "Differential'nye Uravneniya" 18, No.4, 1982; 33, No.4, 1997; 38, No.4, 2002; 43, No.5, 2007, "Mem. Differential Equations Math. Phys.", 26, 2002; 41, 2007 and "Functional Differential Equations", 9, No.3-4, 2002.

1 N.V. Azbelev's life

Nikolay Viktorovich Azbelev was born on April 15, 1922 in selo (small village) Bazlovo, Pskov Region, Russia, in a physician's family. His mother, Antonina Fedorovna Khlebnikova, was a scholar and collaborator of a famous botanist V.L. Komarov, later the President of the Academy of Sciences of the USSR. His father, Viktor Nikolaevich, graduated from Military Medical Academy in St.Petersburg in 1905, further attended lectures on microbiology at Robert Koch's Institute, Berlin, Germany, and was a physician in a field hospital during the World War I. Later he was the director of Polar Institute of Bacteriology in Arkhangelsk, Russia.

In 1941 Nikolay Viktorovich enrolled at Moscow State University (MSU). His studies at the Faculty of Mechanics and Mathematics of MSU were interrupted due to his military service in the Soviet Army during the World War II. In 1945 he entered Moscow Aviation Institute, from which he graduated with the degree in engineering in 1949. The same year he started to work at the Design Bureau headed by Prof. A.A. Mikulin, a member of the Academy of Sciences. At this place Nikolay Viktorovich gained an experience in several areas of applied mathematics and solved several important problems. For example, he

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proposed an original computational method in the study of strength properties of a kind of ball bearing. In 1947 he was among the first to make use of the so-called method of electrical analogy as applied to turbine dynamics. He also designed an analog computer to find the frequencies of the shift vibration in turbo-jet engines.

In 1951-54 he was a post-graduate student at the Department of Higher Mathematics of Moscow Machine and Instruments Institute under supervision of Prof. B.I. Segal. In 1954 Nikolay Viktorovich defended his Candidate of Sciences (the Soviet equivalent of Ph.D. degree) thesis "On the boundaries of feasibility of Chaplygin's theorem on differential inequalities" at Moscow State University (in his report, V.V. Nemytskyi, the official reader of the thesis, emphasized a very high mathematical level of the work). The same year he left Moscow for Izhevsk, a city situated in the vicinity of the Ural Mountains, to become the head of the Higher Mathematics Department at Izhevsk Mechanical Institute (IMI). One of the first things Nikolay Viktorovich did upon his arrival at IMI was to found the Izhevsk Mathematical Seminar. It soon became the central meeting point for mathematicians and engineers. Azbelev's warmth and sensitiveness were tremendously important for the creation of the mathematical community around IMI. The works of the participants of this seminar concerning the theory of integral, differential and difference inequalities allowed to solve a number of problems on existence, uniqueness, and asymptotic behavior of solutions to differential equations. Other works of the Izhevsk Mathematical Seminar are devoted to the search of effective conditions and criteria for unique solvability of boundary value problems for ordinary differential equations and investigation of the properties of Green's function for those problems. Since 1961 the major attention of N.V. Azbelev and his seminar was focused on the problems of the general theory of equations with discontinuous operators. In Izhevsk he wrote his Doctoral thesis "On the Chaplygin problem", which was defended in 1962 at Kazan State University. In 1964 he was granted the title of professor.

In 1966 Professor Azbelev was elected to be the Head of the Higher Mathematics Department in Tambov Institute for Chemical Engineering. Nikolay Viktorovich moved to Tambov together with his wife Lina Fazylovna Rakhmatullina, a brilliant mathematician and the closest collaborator. A large group of postgraduate students and colleagues of Prof. Azbelev from Izhevsk joined them in Tambov. Soon after that the Tambov Seminar under the leadership of N.V. Azbelev and L.F. Rakhmatullina started its work. It dealt with equations with deviating argument. The activity of the Tambov Seminar implied the creation of an effective theory of differential equations with deviating argument. This theory became a basis of the contemporary Theory of Functional Differential Equations.

In 1975 Professor Azbelev accepted invitation of the Rector of Perm Polytechnic Institute (PPI), Prof. M.N. Dedyukin, and moved to Perm, where he founded the Department of Mathematical Analysis. Azbelev's scientific expertise and leadership contributed immensely to the development of this department. As a result, it had become one of the well known mathematical centers and the core of the Perm Seminar on Functional Differential Equations. Since 1994 to his last day N. Azbelev has been the head of the Research Center on Functional Differential Equations at Perm State Technical University (former PPI).

Nikolay Viktorovich was a true representative of the Russian intelligentsia. He was a connoisseur of the Russian poetry and a great admirer of the classical music. As a passionate traveller he together with Lina Fazylovna travelled across the Caucasus, Middle Asia, and Central part of Russia first by his motorbike and later by his car.

2 Main Direction of His Research

Nikolay Azbelev's research covers integral, differential and functional differential equations and inequalities, numerical methods, stability theory, boundary value problems and calculus of variations. He is one of the founders of the Russian scientific school of differential and integral inequalities. In his first papers N. Azbelev gave a solution to the Chaplygin problem on the boundaries of feasibility of the differential inequality theorem. His works essentially expanded the area of applications of differential inequalities. The activity of Azbelev and his Tambov Seminar implied the creation of the theory of differential equations with deviating argument. This theory became a basis of the contemporary Theory of Functional Differential Equations which was worked out by the members of Perm Seminar under the leadership of Azbelev. In 1991 the Publishing House Nauka, Moscow, published N. Azbelev's book (with V. Maksimov and L. Rakhmatullina) "Introduction to the Theory of Functional Differential Equations". It can be said that up to now this monograph remains a reference book for specialists in the theory of FDE. The further development of the FDE theory was treated thoroughly in eight books, four of them in English. On his last day N. Azbelev dealt with the galley proof of a new book. Now this theory covers many classes of equations containing the ordinary derivatives of the solution function. Of special importance are the contributions of N. Azbelev to creation and development (jointly with L. Rakhmatullina) of the theory of Abstract FDE, further generalization of the equations with ordinary derivatives, covering wide classes of n -th order FDEs, systems with impulses, singular equations. It is worth noting that this theory has become a very useful tool for solving some variational problems, especially in the cases when the problem of minimization of a functional is unsolvable within the framework of the classical calculus of variations, as well as for the study of boundary value problems with arbitrary finite number of boundary conditions in the form of equalities and inequalities.

3 General Education and Science Activity

N. Azbelev's influence was not limited to the original and fundamental contributions to the theory of integral and functional differential equations. A characteristic feature of N. Azbelev's activity was his ability to unite around himself colleagues and all those who were enthusiastic about Science. He significantly contributed to the education of young mathematicians, supervised over 60 Candidates and 10 Doctors of Sciences. In the 60's he became a founder of mathematical schools for gifted children in Izhevsk, Russia. N. Azbelev was a member of editorial boards of "Differentsial'nye Uravneniya" (for more than 25 years), "Nonlinear Dynamics and System Theory", "Memoirs on Differential Equations and Mathematical Physics", "Functional Differential Equations" and the editor-in-chief of periodical interuniversity proceedings of scientific works "Functional-differential equations" and "Boundary Value Problems" (Perm, Russia).

Professor Azbelev received many honors and awards in the course of his career. He was awarded orders and medals, recognized as a Meritorious Science Worker of the Russian Federation, awarded the Grant of the Russian Federation President for Leading Scientist, selected as a Georg Soros Emeritus Professor, conferred the title of Honored member of the Academy of Nonlinear Sciences.

4 List of Monographs by N.V. Azbelev

- [1] Introduction to the Theory of Functional Differential Equations, Moscow: Nauka, 1991, 280 p. (in Russian, with V.P. Maksimov, L.F. Rakhmatullina)
- [2] Introduction to the Theory of Linear Functional Differential Equations, Atlanta: World Federat. Publ. Company, 1995, 213 p. (with V.P. Maksimov, L.F. Rakhmatullina).
- [3] Theory of Linear Abstract Functional Differential Equations and Applications// Memoirs on Differential Equations and Mathematical Physics, 8 (1996), pp. 1-102. Tbilisi: Publishing House GCI, 1996 (with L.F. Rakhmatullina).
- [4] Methods of the Contemporary Theory of Linear Functional Differential Equations, Moscow-Izhevsk: Regular and Chaotic Dynamics, 2000, 300 p. (in Russian, with V.P. Maksimov, L.F. Rakhmatullina)
- [5] Stability of Solutions to Equations with Ordinary Derivatives, Perm: Perm State University, 2001, 230 p. (in Russian, with P.M. Simonov)
- [6] Elements of the Contemporary Theory of Functional Differential Equations. Methods and Applications, Moscow: Institute of Computer-Assisted Study, 2002, 384 c. (in Russian, with V.P. Maksimov, L.F. Rakhmatullina)
- [7] Stability of Differential Equations with Aftereffect, London: Taylor and Francis Publishing Group, 2002, 222 p. (with P.M. Simonov)
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On the Minimum Free Energy for a Rigid Heat Conductor with Memory Effects

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Abstract: A general closed expression for the minimum free energy, related to a state of a rigid heat conductor with memory, is derived in terms of Fourier-transformed functions, by using the coincidence of this quantity with the maximum recoverable work obtainable from that state. The linearized constitutive equations both for the internal energy and for the heat flux consider the effects of the actual values of the temperature and of its gradient, together with the ones of the integrated histories of such quantities, which are chosen to characterize the states of the material. An equivalent formulation for the minimum free energy is given and also used to derive explicit formulae for a discrete spectrum model.

Keywords: *fading memory; heat conduction; thermodynamics.*

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1 Introduction

A general nonlinear theory of rigid heat conductors with memory effects was proposed by Gurtin and Pipkin in [21], by using Coleman's results for materials with memory [10]. The constitutive equation for the heat flux, derived in [21] for isotropic media when small variations of the temperature and of its gradient are studied, is well known. Such a relation is expressed by a linear functional of the history of the temperature gradient and gives a generalization of the Cattaneo–Maxwell equation [9], which, therefore, is a special case of the theory in [21]. In the framework proposed in [21], later on, a linear theory of rigid heat conductors has been considered in particular in [14].

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Nunziato in [25] has subsequently developed a slightly different memory theory, which had to include the effects of the present value of the temperature gradient in the constitutive equation of the heat flux, together with the ones due to the history of the temperature gradient [11, 22]. Such a theory yields a linearized constitutive equation, for the heat flux in isotropic materials, characterized by two terms, one of which is similar to the corresponding Gurtin–Pipkin’s relation, i.e. a linear functional of the past history of the temperature gradient with an integral kernel $k'(s)$, while the other term is expressed by Fourier’s law, that is a term proportional to the present value of the temperature gradient with a coefficient $k_0 = k(0)$. Thus, Nunziato’s linearized constitutive equation, in absence of memory effects, that is when $k'(s) = 0$, coincides with Fourier’s law; moreover, if $k_0 = 0$, it reduces to Gurtin–Pipkin’s linearized equation.

It is well known that the thermodynamic principles impose restrictions on any constitutive equation; the constraints related to Nunziato’s relation have been derived in [19] and there used to prove a theorem of existence, uniqueness and stability of solutions to the heat equation.

In [3] this constitutive equation has been considered to derive explicit formulae for the minimum free energy for a rigid heat conductor, thus, generalizing previous articles [2, 4] related to the use of Gurtin–Pipkin’s relation.

The problem of finding explicit forms for the minimum free energy associated with a given state of a material is particularly important, since it coincides with the maximum recoverable work, i.e. it allows us to determine the amount of energy available from that state. In many papers this subject has been considered, particularly for linear viscoelastic solids, see [6]–[7], [12]–[13], [15]–[17] and especially, [20], [18] and [26].

In this work we use again Nunziato’s general constitutive equation to solve the analogous problems of [3], but, instead of the histories of temperature and of its gradient assumed in [3], we now choose the integrated histories of these two quantities to characterize the state of the material. The integrated histories of the temperature gradient, already introduced in the pioneer work [21], has been preferred by some authors, see for example [23]; thus, it seems interesting to study the said problems with this point of view. Therefore, in the present article the material states of the rigid body are characterized by the actual values of the temperature, as in [3], and by the integrated histories of the temperature and of its gradient.

To study these problems, in this paper, we shall refer to [14], for the linearization of the Clausius–Duhem inequality, and to [19], for the thermodynamic constraints on the constitutive equations of the internal energy and of the heat flux; finally, to derive the expression for the minimum free energy we shall follow the procedure used in [20] and [18].

As we have already observed in [3], contrary to what occurs for viscoelastic solids, for which the method used to evaluate the minimum free energy yields a Wiener–Hopf integral equation of the first kind, the use of Nunziato’s relation for a rigid heat conductor yields two Wiener–Hopf integral equations but of the second kind. These two integral equations of second kind can be easily solved in the frequency domain by virtue of the thermodynamic properties of the kernels related to the expressions for the internal energy and for the heat flux, together with some theorems on factorization. Hence, an explicit expression for the minimum free energy is derived.

Another different but equivalent expression is also deduced for the minimum free energy and used to study the discrete spectrum model material response.

The layout of the paper is as follows. In Sect. 2, the linear theory, the linearized

form of the Second Law of Thermodynamics and the thermodynamic constraints on the constitutive equations of the internal energy and of the heat flux are examined. In Sect. 3, we introduce the notions of states and processes together with the prolongation of histories; two particular histories are also considered. Then, in Sect. 4, we define an equivalence relation between states. After giving the expression for the thermal work in Sect. 5, in Sect. 6 we consider another equivalence relation between states by means of work and we prove its equivalence with the previous one. In Sect. 7, we derive an explicit expression for the minimum free energy. Another equivalent expression for this minimum free energy is given in Sect. 8, and, in Sect. 9, it is used to obtain explicit results for a discrete spectrum model material response.

2 Fundamental Relationships

We denote by \mathcal{B} a homogeneous and isotropic rigid heat conductor, endowed of memory effects, which occupies a fixed bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If we are concerned only with small variations of the temperature ϑ , relative to a uniform absolute temperature Θ_0 , and of the temperature gradient $\mathbf{g} = \nabla\vartheta$, we can consider the linearization of the theory developed in [14]. Thus, we consider the following constitutive equations

$$e(\mathbf{x}, t) = e_0 + \alpha_0\vartheta(\mathbf{x}, t) + \int_0^{+\infty} \alpha'(s) {}_r\vartheta^t(\mathbf{x}, s) ds, \tag{2.1}$$

$$\mathbf{q}(\mathbf{x}, t) = -k_0\mathbf{g}(\mathbf{x}, t) - \int_0^{+\infty} k'(s) {}_r\mathbf{g}^t(\mathbf{x}, s) ds \tag{2.2}$$

for the internal energy e and the heat flux \mathbf{q} of \mathcal{B} [25].

Here, we have denoted by $\mathbf{x} \in \Omega$ the vector position and by ${}_r\vartheta^t(\mathbf{x}, s) = \vartheta(\mathbf{x}, t - s)$ and ${}_r\mathbf{g}^t(\mathbf{x}, s) = \mathbf{g}(\mathbf{x}, t - s) \forall s \in \mathbb{R}^{++} \equiv (0, +\infty)$ the past histories of ϑ and \mathbf{g} . The history up to time t of any function f can be expressed by means of the couple $(f(t), {}_r f^t)$, where $f(t)$ is the present value of f and ${}_r f^t$ its past history.

The kernels $\alpha' : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $k' : \mathbb{R}^+ \rightarrow \mathbb{R}$ are the relaxation functions such that $\alpha', \alpha'', \alpha''' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ and $k', k'' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. Moreover,

$$\alpha(t) = \alpha_0 + \int_0^t \alpha'(s) ds, \quad k(t) = k_0 + \int_0^t k'(s) ds \tag{2.3}$$

are the heat capacity and the thermal conductivity of \mathcal{B} , the asymptotic values of which

$$\alpha_\infty = \lim_{t \rightarrow +\infty} \alpha(t), \quad k_\infty = \lim_{t \rightarrow +\infty} k(t) \tag{2.4}$$

are said to be the equilibrium heat capacity and the thermal conductivity of \mathcal{B} .

The physical consideration that the internal energy increases, if the temperature of a body, constant up a time $t = 0$, instantaneously increases, justifies the assumption

$$\alpha_0 > 0. \tag{2.5}$$

If, for any function f , we introduce its integrated history,

$$\bar{f}^t(\mathbf{x}, s) = \int_0^s f^t(\mathbf{x}, \xi) d\xi = \int_{t-s}^t f(\mathbf{x}, \lambda) d\lambda, \tag{2.6}$$

the internal energy (2.1) and the heat flux (2.2), by integrating by parts, assume the following forms

$$e(\mathbf{x}, t) = e_0 + \alpha_0 \vartheta(\mathbf{x}, t) - \int_0^{+\infty} \alpha''(s) \bar{\vartheta}^t(\mathbf{x}, s) ds, \quad (2.7)$$

$$\mathbf{q}(\mathbf{x}, t) = -k_0 \mathbf{g}(\mathbf{x}, t) + \int_0^{+\infty} k''(s) \bar{\mathbf{g}}^t(\mathbf{x}, s) ds. \quad (2.8)$$

In order to give the restrictions imposed on the constitutive equation by the thermodynamic principles, derived in [19], we recall that the Fourier transform of any function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(s) e^{-i\omega s} ds = f_-(\omega) + f_+(\omega) \quad \forall \omega \in \mathbb{R}, \quad (2.9)$$

where

$$f_-(\omega) = \int_{-\infty}^0 f(s) e^{-i\omega s} ds, \quad f_+(\omega) = \int_0^{+\infty} f(s) e^{-i\omega s} ds. \quad (2.10)$$

Moreover, the half-range Fourier cosine and sine transforms are defined by

$$f_c(\omega) = \int_0^{+\infty} f(s) \cos \omega s ds, \quad f_s(\omega) = \int_0^{+\infty} f(s) \sin \omega s ds; \quad (2.11)$$

we observe that they hold even if f is defined only on \mathbb{R}^+ , as it occurs also for f_+ .

Any function f , which is defined only on \mathbb{R}^+ , can be extended on \mathbb{R} ; thus, we recall that the Fourier transform of the new function is given by

$$f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -2if_s(\omega), \quad f_F(\omega) = f_c(\omega) - if_s(\omega), \quad (2.12)$$

when, respectively, the extension is made with an even function ($f(\xi) = f(-\xi) \forall \xi < 0$), or an odd one ($f(\xi) = -f(-\xi) \forall \xi < 0$), or by using the causal extension ($f(\xi) = 0 \forall \xi < 0$). Moreover, we remember that, if f and f' belong to $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, we obtain

$$f'_s(\omega) = -\omega f_c(\omega) \quad (2.13)$$

and, if $f'' \in L^1(\mathbb{R}^+)$, we have

$$\omega f'_s(\omega) = f'(0) + f''_c(\omega). \quad (2.14)$$

The thermodynamic constraints on (2.1)-(2.2) are expressed by the following inequalities [19]

$$\omega \alpha'_s(\omega) > 0 \quad \forall \omega \neq 0, \quad k_0 + k'_c(\omega) > 0 \quad \forall \omega \in \mathbb{R}. \quad (2.15)$$

Hence, useful relations have been deduced, always in [19].

For α , by using (2.13), (2.14), (2.15)₁ and the inverse half-range Fourier transforms, we have

$$\alpha''_c(\omega) = \omega \alpha'_s(\omega) - \alpha'(0), \quad \alpha(t) - \alpha_0 = \frac{2}{\pi} \int_0^{+\infty} \frac{\alpha'_s(\omega)}{\omega} (1 - \cos \omega t) d\omega > 0, \quad (2.16)$$

under the further hypothesis that $\alpha'' \in L^1(\mathbb{R}^+)$. Thus, we also have

$$\alpha_\infty - \alpha_0 = \frac{2}{\pi} \int_0^{+\infty} \frac{\alpha'_s(\omega)}{\omega} d\omega > 0, \quad \lim_{\omega \rightarrow +\infty} \omega \alpha'_s(\omega) = \alpha'(0) \geq 0 \quad (2.17)$$

and hence

$$\alpha_0 < \alpha(t) < 2\alpha_\infty - \alpha_0. \tag{2.18}$$

For k , we have

$$k_0 \geq 0, \quad k_\infty = k_0 + k'_c(0) > 0. \tag{2.19}$$

We shall assume

$$k_0 > 0, \quad \alpha'(0) > 0. \tag{2.20}$$

We note that the functions $f_\pm(\omega)$, given by (2.10), can be extended to the complex z -plane \mathbb{C} . Thus, they become analytic functions in the subsets $\mathbb{C}^{(\mp)}$ so defined

$$\mathbb{C}^{(-)} = \{z \in \mathbb{C}; \text{Im } z \in \mathbb{R}^{--}\}, \quad \mathbb{C}^{(+)} = \{z \in \mathbb{C}; \text{Im } z \in \mathbb{R}^{++}\}, \tag{2.21}$$

where $\mathbb{R}^{--} = (-\infty, 0)$ and $\mathbb{R}^{++} = (0, +\infty)$. Following [20], we assume the analyticity of the Fourier transforms on \mathbb{R} ; therefore, $f_\pm(z)$ become analytic on an open region containing \mathbb{C}^\mp , given by

$$\mathbb{C}^- = \{z \in \mathbb{C}; \text{Im } z \in \mathbb{R}^-\}, \quad \mathbb{C}^+ = \{z \in \mathbb{C}; \text{Im } z \in \mathbb{R}^+\}. \tag{2.22}$$

Finally, will shall use the notation $f_{(\pm)}(z)$ to denote that the zeros and the singularities of f are in \mathbb{C}^\pm .

Since we are concerned with a linear theory of rigid heat conductors, the linearization of the Clausius-Duhem inequality is also required. Such a linearization, derived in [14], involves the first order approximation for e and \mathbf{q} and the second order one for the free energy and the entropy. By introducing the pseudo-free energy $\psi = \Theta_0(e - \Theta_0\eta)$, whose properties closely resemble those of the canonical free energy, the factor $1/\Theta_0$, involved in the linearization of the term $\mathbf{q} \cdot \mathbf{g}$, is eliminated by the presence of the factor Θ_0 in the definition of ψ ; thus, the authors have derived the following linearized local form of the Second Law of Thermodynamics

$$\dot{\psi}(\mathbf{x}, t) \leq \dot{e}(\mathbf{x}, t)\vartheta(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t). \tag{2.23}$$

Henceforth, since our attention shall be fixed on a specific point $\mathbf{x} \in \Omega$, the dependence on such a vector will be omitted in the equations.

3 States and Processes of the Rigid Heat Conductor

The behaviour of our rigid heat conductor \mathcal{B} is characterized by the constitutive equations (2.1)-(2.2) in the form given by (2.7)-(2.8), which allow us to consider \mathcal{B} as a simple material, that we can describe in terms of states and processes.

Thus, we assume that the material state of the conductor at time t is expressed by

$$\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t), \tag{3.1}$$

where $\vartheta(t)$ is the instantaneous value of ϑ , $\bar{\vartheta}^t$ and $\bar{\mathbf{g}}^t$ are the integrated histories of the temperature and of its gradient.

The thermodynamic process of the conductor is given by a piecewise continuous map $P : [0, d] \rightarrow \mathbb{R} \times \mathbb{R}^3$ defined by

$$P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \quad \forall \tau \in [0, d], \tag{3.2}$$

which can be applied to the body at any time $t \geq 0$ and generally has a finite duration d .

The sets of the states and of the processes, which are possible for the material, are denoted by Σ and Π . We can introduce the state transition function $\rho : \Sigma \times \Pi \rightarrow \Sigma$, which maps any initial state $\sigma^i \in \Sigma$ and any process $P \in \Pi$ into the final state $\sigma^f = \rho(\sigma^i, P) \in \Sigma$. We can consider any restriction of a given $P \in \Pi$ to a subset $[\tau_1, \tau_2) \subset [0, d)$, denoted by $P_{[\tau_1, \tau_2)}$, as well as the composition of two processes $P_j \in \Pi$ ($j = 1, 2$) with durations d_j ($j = 1, 2$) so defined

$$P_1 * P_2(\tau) = \begin{cases} P_1(\tau) & \forall \tau \in [0, d_1), \\ P_2(\tau - d_1) & \forall \tau \in [d_1, d_1 + d_2); \end{cases} \tag{3.3}$$

both also belong to Π . In particular, the restriction $P_{[0, \tau)}$ applied to $\sigma^i = \sigma(0)$ yields the final state $\sigma(\tau) = \rho(\sigma(0), P_{[0, \tau)})$. Finally, the pair (σ, P) is said to be a cycle if $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$.

Let $\sigma(0) = (\vartheta_*(0), \bar{\vartheta}_*^0, \bar{\mathbf{g}}_*^0)$ be the initial state at time $t = 0$, when a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ is applied for any $\tau \equiv t \in [0, d)$. In particular, we have

$$\vartheta(t) = \vartheta_*(0) + \int_0^t \dot{\vartheta}_P(s) ds, \quad r\vartheta^t(s) = \begin{cases} \vartheta(t-s) & \forall s \in (0, t], \\ \vartheta_*^0(s-t) & \forall s > t \end{cases} \tag{3.4}$$

and hence the subsequent states are expressed by

$$\bar{\vartheta}^t(s) = \begin{cases} \int_{t-s}^t \vartheta_P(\xi) d\xi & \forall s \in [0, t), \\ \int_0^t \vartheta_P(\lambda) d\lambda + \bar{\vartheta}_*^0(s-t) & \forall s \geq t, \end{cases} \tag{3.5}$$

$$\bar{\mathbf{g}}^t(s) = \begin{cases} \int_{t-s}^t \mathbf{g}_P(\xi) d\xi & \forall s \in [0, t), \\ \int_0^t \mathbf{g}_P(\lambda) d\lambda + \bar{\mathbf{g}}_*^0(s-t) & \forall s \geq t, \end{cases} \tag{3.6}$$

together with the values $\vartheta(t)$ of the temperature given by (3.4)₁.

When $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \forall \tau \in [0, d)$ is applied at time $t > 0$ to the initial state $\sigma^i(t) = (\vartheta_i(t), \bar{\vartheta}_i^t, \bar{\mathbf{g}}_i^t)$, we must consider the continuations of the integrated histories of ϑ and \mathbf{g} to express the ensuing states. Now, we have

$$\vartheta_P(\tau) \equiv \vartheta(t + \tau) = \vartheta_i(t) + \int_0^\tau \dot{\vartheta}_P(\eta) d\eta, \tag{3.7}$$

$$\vartheta(t + d - s) = (\vartheta_P * \vartheta_i)^{t+d}(s) = \begin{cases} \vartheta_P(d-s) & \forall s \in [0, d), \\ \vartheta_i(t+d-s) & \forall s \geq d, \end{cases} \tag{3.8}$$

and hence

$$\bar{\vartheta}(t + d - s) = (\vartheta_P * \bar{\vartheta}_i)^{t+d}(s) = \begin{cases} \int_{d-s}^d \vartheta_P(s) ds = \bar{\vartheta}_P^d(s) & \forall s \in [0, d), \\ \bar{\vartheta}_P^d(d) + \bar{\vartheta}_i^t(s-d) & \forall s \geq d, \end{cases} \tag{3.9}$$

$$\bar{\mathbf{g}}(t + d - s) = (\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+d}(s) = \begin{cases} \int_{d-s}^d \mathbf{g}_P(\xi) d\xi = \bar{\mathbf{g}}_P^d(s) & \forall s \in [0, d), \\ \bar{\mathbf{g}}_P^d(d) + \bar{\mathbf{g}}_i^t(s-d) & \forall s \geq d. \end{cases} \tag{3.10}$$

These prolongations $(\vartheta_P * \bar{\vartheta}_i)^{t+d}$ and $(\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+d}$ given in (3.9)-(3.10) allow us to evaluate the final values of the internal energy (2.7) and the heat flux (2.8) at the end of a process.

Let the restriction $P_{[0,\tau]}$ be applied at time $t > 0$ to the state $\sigma^i(t) = (\vartheta_i(t), \bar{\vartheta}_i^t, \bar{\mathbf{g}}_i^t)$. Then, the expressions (2.7) and (2.8) for e and \mathbf{q} , by replacing d with τ in (3.9)-(3.10), become

$$e(t + \tau) = e_0 + \alpha_0 \vartheta_P(\tau) - \int_0^\tau \alpha''(s) \bar{\vartheta}_P^\tau(s) ds - \int_\tau^{+\infty} \alpha''(s) [\bar{\vartheta}_P^\tau(\tau) + \bar{\vartheta}_i^t(s - \tau)] ds, \tag{3.11}$$

$$\mathbf{q}(t + \tau) = -k_0 \mathbf{g}_P(\tau) + \int_0^\tau k''(s) \bar{\mathbf{g}}_P^\tau(s) ds + \int_\tau^{+\infty} k''(s) [\bar{\mathbf{g}}_P^\tau(\tau) + \bar{\mathbf{g}}_i^t(s - \tau)] ds. \tag{3.12}$$

We observe that in these last two relations, in particular, we have

$$\bar{\vartheta}_P^\tau(\tau) = \int_0^\tau \vartheta_P(\xi) d\xi, \quad \bar{\mathbf{g}}_P^\tau(\tau) = \int_0^\tau \mathbf{g}_P(\xi) d\xi, \tag{3.13}$$

which immediately follow from (2.6).

These formulae are now applied to the particular cases of a static continuation of histories and of constant histories.

Firstly, we examine the static continuation of two assigned histories $(\vartheta(t), {}_r\vartheta^t)$ and $(\mathbf{g}(t), {}_r\mathbf{g}^t)$, with a finite duration $a \in \mathbb{R}^{++}$, defined by

$$\vartheta^{t(a)}(s) = \begin{cases} \vartheta(t) & \forall s \in [0, a], \\ \vartheta^t(s - a) & \forall s > a, \end{cases} \quad \mathbf{g}^{t(a)}(s) = \begin{cases} \mathbf{g}(t) & \forall s \in [0, a], \\ \mathbf{g}^t(s - a) & \forall s > a; \end{cases} \tag{3.14}$$

hence, the corresponding integrated histories of ϑ and \mathbf{g} are expressed by

$$\bar{\vartheta}^{t+a}(s) = \begin{cases} \int_{a-s}^a \vartheta(t) d\xi = s\vartheta(t) & \forall s \in [0, a], \\ \int_0^a \vartheta(t) d\xi + \int_{t-(s-a)}^t \vartheta(\xi) d\xi = a\vartheta(t) + \int_0^{s-a} \vartheta^t(\rho) d\rho & \forall s > a, \end{cases} \tag{3.15}$$

$$\bar{\mathbf{g}}^{t+a}(s) = \begin{cases} \int_{a-s}^a \mathbf{g}(t) d\xi = s\mathbf{g}(t) & \forall s \in [0, a], \\ \int_0^a \mathbf{g}(t) d\xi + \int_{t-(s-a)}^t \mathbf{g}(\xi) d\xi = a\mathbf{g}(t) + \int_0^{s-a} \mathbf{g}^t(\rho) d\rho & \forall s > a. \end{cases} \tag{3.16}$$

Thus, (3.11) and (3.12), by substituting (3.15) and (3.16), give

$$e(t + a) = e_0 + \alpha(a)\vartheta(t) - \int_0^{+\infty} \alpha''(\xi + a) \bar{\vartheta}^t(\xi) d\xi, \tag{3.17}$$

$$\mathbf{q}(t + a) = -k(a)\mathbf{g}(t) + \int_0^{+\infty} k''(\xi + a) \bar{\mathbf{g}}^t(\xi) d\xi. \tag{3.18}$$

Now, let $\vartheta(t - s) = \vartheta^\dagger(s) = \vartheta$ and $\mathbf{g}(t - s) = \mathbf{g}^\dagger(s) = \mathbf{g} \forall s \in \mathbb{R}^+$ be two given constant histories; the internal energy and the heat flux at time t can be evaluated directly from (2.7)-(2.8) and are expressed by

$$e(t) = e_0 + \alpha_\infty \vartheta, \quad \mathbf{q}(t) = -k_\infty \mathbf{g}, \tag{3.19}$$

where the asymptotic values (2.4) of α and of k are involved; in particular, we note that the heat flux has the opposite versus of the temperature gradient.

Taking account of the constitutive equations (2.7) and (2.8), we can introduce the functionals $\tilde{e} : \mathbb{R} \times \Gamma_\alpha \rightarrow \mathbb{R}$ and $\tilde{\mathbf{q}} : \Gamma_k \rightarrow \mathbb{R}^3$ defined by

$$e(\sigma(t)) = \tilde{e}(\vartheta(t), \bar{\vartheta}^t), \quad \mathbf{q}(\sigma(t), P_t) = \tilde{\mathbf{q}}(\mathbf{g}(t), \bar{\mathbf{g}}^t), \tag{3.20}$$

where Γ_α and Γ_k denote the function spaces of the integrated histories of ϑ and \mathbf{g} up to time t , which, by virtue of (3.17)-(3.18), are so defined

$$\Gamma_\alpha = \left\{ \bar{\vartheta}^t : [0, +\infty) \rightarrow \mathbb{R}; \left| \int_0^{+\infty} \alpha''(\eta + \tau) \bar{\vartheta}^t(\eta) d\eta \right| < +\infty \quad \forall \tau \in \mathbb{R}^+ \right\}, \quad (3.21)$$

$$\Gamma_k = \left\{ \bar{\mathbf{g}}^t : [0, +\infty) \rightarrow \mathbb{R}^3; \left| \int_0^{+\infty} k''(\eta + \tau) \bar{\mathbf{g}}^t(\eta) d\eta \right| < +\infty \quad \forall \tau \in \mathbb{R}^+ \right\}. \quad (3.22)$$

4 An Equivalence Relation Between States

An equivalence relation can be introduced in the state space Σ with this definition.

Definition 4.1 Two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t) \in \Sigma$ ($j = 1, 2$) of a rigid heat conductor, characterized by the constitutive equations (2.7) and (2.8), are said to be equivalent if, for every process $P_\tau \in \Pi$ and for every $\tau > 0$,

$$e(\rho(\sigma_1(t), P_\tau)) = e(\rho(\sigma_2(t), P_\tau)), \quad \mathbf{q}(\rho(\sigma_1(t), P_\tau), P_\tau) = \mathbf{q}(\rho(\sigma_2(t), P_\tau), P_\tau). \quad (4.1)$$

Such a definition of equivalence requires the coincidence of the response of the material, expressed by the values of e and \mathbf{q} ; this implies some consequences, which are shown in the following theorem.

Theorem 4.1 For a conductor, characterized by the constitutive equations (2.7) and (2.8), two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t) \in \Sigma$ ($j = 1, 2$) are equivalent if and only if

$$\vartheta_1(t) = \vartheta_2(t), \quad \int_0^{+\infty} \alpha''(\xi + \tau) [\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi)] d\xi = 0, \quad (4.2)$$

$$\int_0^{+\infty} k''(\xi + \tau) [\bar{\mathbf{g}}_1^t(\xi) - \bar{\mathbf{g}}_2^t(\xi)] d\xi = \mathbf{0} \quad (4.3)$$

for every $\tau > 0$.

Proof If $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ ($j = 1, 2$) are two equivalent states, then the equalities (4.1) are satisfied for every $P_\tau \in \Pi$ and every $\tau > 0$; thus, we have

$$\tilde{e}(\vartheta_{P_1}(\tau), (\vartheta_{P_1} * \bar{\vartheta}_1)^{t+\tau}) = \tilde{e}(\vartheta_{P_2}(\tau), (\vartheta_{P_2} * \bar{\vartheta}_2)^{t+\tau}), \quad (4.4)$$

$$\tilde{\mathbf{q}}(\mathbf{g}_{P_1}(\tau), (\mathbf{g}_{P_1} * \bar{\mathbf{g}}_1)^{t+\tau}) = \tilde{\mathbf{q}}(\mathbf{g}_{P_2}(\tau), (\mathbf{g}_{P_2} * \bar{\mathbf{g}}_2)^{t+\tau}). \quad (4.5)$$

These two equalities, by using (3.11), (3.12) and (3.7), (3.9), (3.10), yield

$$\alpha(\tau) [\vartheta_1(t) - \vartheta_2(t)] - \int_\tau^{+\infty} \alpha''(s) [\bar{\vartheta}_1^t(s - \tau) - \bar{\vartheta}_2^t(s - \tau)] ds = 0, \quad (4.6)$$

$$\int_\tau^{+\infty} k''(s) [\bar{\mathbf{g}}_1^t(s - \tau) - \bar{\mathbf{g}}_2^t(s - \tau)] ds = \mathbf{0}, \quad (4.7)$$

which must be satisfied for arbitrary values of τ .

Taking the limit $\tau \rightarrow +\infty$ in (4.6), we have

$$\alpha_\infty [\vartheta_1(t) - \vartheta_2(t)] = 0 \quad (4.8)$$

and hence (4.2)₁ follows. Thus, (4.6) reduces to its integral, which, by changing the variable of integration, coincides with (4.2)₂. An analogous change in (4.7) gives (4.3).

Obviously, the converse follows from these same relations. □

Consequently, using (3.11)-(3.12), it follows that a state $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ is equivalent to the zero state $\sigma_0(t) = (0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger)$, where $\bar{0}^\dagger(s) = \bar{\vartheta}^t(s) = 0 \ \forall s \in \mathbb{R}^+$ and $\bar{\mathbf{0}}^\dagger(s) = \bar{\mathbf{g}}^t(s) = \mathbf{0} \ \forall s \in \mathbb{R}^+$ denote the zero integrated histories of ϑ and \mathbf{g} , if

$$\vartheta(t) = 0, \quad \int_{\tau}^{+\infty} \alpha''(s) \bar{\vartheta}^t(s - \tau) ds \equiv \int_0^{+\infty} \alpha''(\xi + \tau) \bar{\vartheta}^t(\xi) d\xi = 0, \quad (4.9)$$

$$\int_{\tau}^{+\infty} k''(s) \bar{\mathbf{g}}^t(s - \tau) ds \equiv \int_0^{+\infty} k''(\xi + \tau) \bar{\mathbf{g}}^t(\xi) d\xi = \mathbf{0}. \quad (4.10)$$

Moreover, it follows that the difference of two given equivalent states $\sigma_j(t)$ ($j = 1, 2$), i.e. $\sigma_1(t) - \sigma_2(t) = (\vartheta_1(t) - \vartheta_2(t), \bar{\vartheta}_1^t - \bar{\vartheta}_2^t, \bar{\mathbf{g}}_1^t - \bar{\mathbf{g}}_2^t)$ is a state equivalent to the zero state $\sigma_0(t) = (0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger)$.

5 Thermal Work

The linearized form of the Clausius-Duhem inequality, expressed by (2.23), gives for the thermal power the following form [14]

$$w(t) = \dot{e}(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t). \quad (5.1)$$

Therefore, during the application of a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \ \forall \tau \in [0, d]$, starting at time $t > 0$ when $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ is the initial state, the thermal work done on the material is

$$W(\sigma, P) = \tilde{W}(\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) = \int_0^d [\dot{e}(t + \tau)\vartheta_P(\tau) - \mathbf{q}(t + \tau) \cdot \mathbf{g}_P(\tau)] d\tau. \quad (5.2)$$

To evaluate the derivative of the internal energy, which appears in (5.2), we observe that (3.11), with an integration, can be written as follows

$$\begin{aligned} e(t + \tau) &= e_0 + \alpha_0 \vartheta_P(\tau) + \alpha'(\tau) \bar{\vartheta}_P^\tau(\tau) - \int_0^\tau \alpha''(s) \bar{\vartheta}_P^\tau(s) ds \\ &\quad - \int_0^{+\infty} \alpha''(\xi + \tau) \bar{\vartheta}^t(\xi) d\xi. \end{aligned} \quad (5.3)$$

Thus, we can differentiate with respect to τ this expression (5.3); by using (3.13) and the relation $\frac{d}{d\tau} \bar{\vartheta}_P^\tau(s) \equiv \dot{\bar{\vartheta}}^\tau(s) = \vartheta_P(\tau) - \vartheta_P^\tau(s)$, we obtain

$$\begin{aligned} \dot{e}(t + \tau) &= \alpha_0 \dot{\vartheta}_P(\tau) + \alpha'(0) \vartheta_P(\tau) + \int_0^\tau \alpha''(s) \vartheta_P^\tau(s) ds \\ &\quad - \int_0^{+\infty} \alpha'''(\xi + \tau) \bar{\vartheta}^t(\xi) d\xi. \end{aligned} \quad (5.4)$$

Moreover, the expression (3.12) for $\mathbf{q}(t + \tau)$, by replacing $\bar{\mathbf{g}}_i^t$ with the integrated history $\bar{\mathbf{g}}^t$ of the initial state $\sigma(t)$, with two integrations, can be rewritten as

$$\mathbf{q}(t + \tau) = -k_0 \mathbf{g}_P(\tau) - \int_0^\tau k'(s) \mathbf{g}_P^\tau(s) ds + \int_0^{+\infty} k''(\xi + \tau) \bar{\mathbf{g}}^t(\xi) d\xi. \quad (5.5)$$

We firstly consider the particular case when the process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ of duration $d < +\infty$ is applied at time $t = 0$ to the initial state $\sigma_0(0) = (0, \bar{\mathbf{0}}^\dagger, \bar{\mathbf{0}}^\dagger)$, in order to derive the amount of work due only to P . Denoting the ensuing fields by $(\vartheta_0, \bar{\vartheta}_0^t, \bar{\mathbf{g}}_0^t)$ and using (3.5)-(3.6), we have

$$\vartheta_0(t) = \int_0^t \dot{\vartheta}_P(s) ds, \quad \bar{\vartheta}_0^t(s) = (\vartheta_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\vartheta}_0^t(s) & \forall s \in [0, t), \\ \bar{\vartheta}_0^t(t) & \forall s \geq t, \end{cases} \quad (5.6)$$

$$\bar{\mathbf{g}}_0^t(s) = (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\mathbf{g}}_0^t(s) & \forall s \in [0, t), \\ \bar{\mathbf{g}}_0^t(t) & \forall s \geq t. \end{cases} \quad (5.7)$$

Thus, (5.4) and (5.5) become

$$\dot{e}(t) = \alpha_0 \dot{\vartheta}_0(t) + \alpha'(0) \vartheta_0(t) + \int_0^t \alpha''(s) \vartheta_0^t(s) ds, \quad (5.8)$$

$$\mathbf{q}(t) = -k_0 \mathbf{g}_0(t) - \int_0^t k'(s) \mathbf{g}_0^t(s) ds. \quad (5.9)$$

By substituting (5.8)-(5.9) into (5.2), we have

$$\begin{aligned} W(\sigma_0(0), P) &= \bar{W}(0, \bar{\mathbf{0}}^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) = \frac{1}{2} \alpha_0 \vartheta_0^2(d) \\ &+ \alpha'(0) \int_0^d \vartheta_0^2(t) dt + \int_0^d \left[\int_0^t \alpha''(s) \vartheta_0^t(s) ds \right] \vartheta_0(t) dt \\ &+ k_0 \int_0^d \mathbf{g}_0^2(t) dt + \int_0^d \left[\int_0^t k'(s) \mathbf{g}_0^t(s) ds \right] \cdot \mathbf{g}_0(t) dt. \end{aligned} \quad (5.10)$$

Definition 5.1 A process $P = (\dot{\vartheta}_P, \mathbf{g}_P)$ with a duration d , applied at time $t = 0$ and related to (5.8)-(5.9), is said to be a finite work process if

$$\bar{W}(0, \bar{\mathbf{0}}^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) < +\infty. \quad (5.11)$$

Theorem 5.1 *The work done during the application of any finite work process is positive.*

Proof Let the process P be extended to \mathbb{R} by assuming that $P(t) = (0, \mathbf{0}) \forall t > d$ and $\vartheta_0(t) = 0 \forall t > d$; the expression of the work, by applying Plancherel's theorem and using $*$ to denote the complex conjugate, can be written as follows

$$\begin{aligned} W(\sigma_0(0), P) &= \frac{1}{2} \alpha_0 \vartheta_0^2(d) + \frac{\alpha'(0)}{2\pi} \int_{-\infty}^{+\infty} \vartheta_{0_F}(\omega) [\vartheta_{0_F}(\omega)]^* d\omega \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \alpha''_F(\omega) \vartheta_{0_F}(\omega) [\vartheta_{0_F}(\omega)]^* d\omega + \frac{k_0}{2\pi} \int_{-\infty}^{+\infty} \mathbf{g}_{0_F}(\omega) [\mathbf{g}_{0_F}(\omega)]^* d\omega \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} k'_F(\omega) \mathbf{g}_{0_F}(\omega) \cdot [\mathbf{g}_{0_F}(\omega)]^* d\omega. \end{aligned} \quad (5.12)$$

Using (2.12)₃ for the Fourier transforms of any function which vanishes on \mathbb{R}^- , it follows that $\alpha''_F(\omega) = \alpha''_c(\omega) - i\alpha''_s(\omega)$ and $k'_F(\omega) = k'_c(\omega) - ik'_s(\omega)$ are expressed in terms

of the cosine and the sine transforms, which are even and odd functions, respectively; therefore, (5.12) can be rewritten as

$$\begin{aligned} W(\sigma_0(0), P) &= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\alpha'(0) + \alpha''(\omega)] [\vartheta_{0_c}^2(\omega) + \vartheta_{0_s}^2(\omega)] d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} [k_0 + k'_c(\omega)] [\mathbf{g}_{0_c}^2(\omega) + \mathbf{g}_{0_s}^2(\omega)] d\omega \\ &= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \{ \omega\alpha'_s(\omega) [\vartheta_{0_c}^2(\omega) + \vartheta_{0_s}^2(\omega)] \\ &\quad + [k_0 + k'_c(\omega)] [\mathbf{g}_{0_c}^2(\omega) + \mathbf{g}_{0_s}^2(\omega)] \} d\omega > 0, \end{aligned} \tag{5.13}$$

by virtue of (2.16)₁ and (2.15). □

Thus, the work $W(\sigma_0(0), P)$ depends on the ensuing field of the temperature $\vartheta_0(t)$, which is related to P through ϑ_P by means of (3.4) or (3.5), and on the temperature gradient $\mathbf{g}_P(t) \equiv \mathbf{g}_0(t)$ assigned with P . Consequently, we can characterize the finite work processes by introducing the following function spaces [18]

$$\tilde{H}_\alpha(\mathbb{R}^+, \mathbb{R}) = \left\{ \vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}; \int_{-\infty}^{+\infty} \omega\alpha'_s(\omega)\vartheta_+(\omega) [\vartheta_+(\omega)]^* d\omega < +\infty \right\}, \tag{5.14}$$

$$\tilde{H}_k(\mathbb{R}^+, \mathbb{R}^3) = \left\{ \mathbf{g} : \mathbb{R}^+ \rightarrow \mathbb{R}^3; \int_{-\infty}^{+\infty} [k_0 + k'_c(\omega)] \mathbf{g}_+(\omega) \cdot [\mathbf{g}_+(\omega)]^* d\omega < +\infty \right\}, \tag{5.15}$$

which, with the completions with respect to the norms corresponding to the following two inner products $(\vartheta_1, \vartheta_2)_\alpha = \int_{-\infty}^{+\infty} \omega\alpha'_s(\omega)\vartheta_{1+}(\omega) [\vartheta_{2+}(\omega)]^* d\omega$ and $(\mathbf{g}_1, \mathbf{g}_2)_k = \int_{-\infty}^{+\infty} [k_0 + k'_c(\omega)] \mathbf{g}_{1+}(\omega) \cdot [\mathbf{g}_{2+}(\omega)]^* d\omega$, respectively, yield two Hilbert spaces $H_\alpha(\mathbb{R}^+, \mathbb{R})$ and $H_k(\mathbb{R}^+, \mathbb{R}^3)$.

Now, we consider the general case when the initial state of \mathcal{B} at time $t > 0$ is $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$, where $\bar{\vartheta}^t$ and $\bar{\mathbf{g}}^t$, belonging to the function spaces Γ_α and Γ_k , introduced in (3.21)-(3.22), are possible integrated histories, which yield a finite work during any process P , characterized by $\mathbf{g}_P \in H_k(\mathbb{R}^+, \mathbb{R}^3)$ and related to $\vartheta_P \in H_\alpha(\mathbb{R}^+, \mathbb{R})$. If, as above, we extend any of these processes $P = (\vartheta_P, \mathbf{g}_P)$ with a finite duration $d < +\infty$ to \mathbb{R}^+ , by assuming that $P(\tau) = (0, \mathbf{0}) \forall \tau \geq d$ and that $\vartheta_P(\tau) = 0 \forall \tau > d$, the work done during the application of any of these processes can be derived by means of (5.2), where $\dot{e}(t + \tau)$ has the form (5.4) and $\mathbf{q}(t + \tau)$ is given by (5.5). Thus, we obtain

$$\begin{aligned} W(\sigma(t), P) &= \tilde{W}(\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t; \vartheta_P, \mathbf{g}_P) \\ &= \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] + \alpha'(0) \int_0^{+\infty} \vartheta_P^2(\tau) d\tau + k_0 \int_0^{+\infty} \mathbf{g}_P^2(\tau) d\tau \\ &\quad + \int_0^{+\infty} \left[\int_0^\tau \alpha''(\tau - \eta)\vartheta_P(\eta) d\eta - \int_0^{+\infty} \alpha'''(\xi + \tau)\bar{\vartheta}^t(\xi) d\xi \right] \vartheta_P(\tau) d\tau \\ &\quad + \int_0^{+\infty} \left[\int_0^\tau k'(\tau - \eta)\mathbf{g}_P(\eta) d\eta - \int_0^{+\infty} k''(\xi + \tau)\bar{\mathbf{g}}^t(\xi) d\xi \right] \cdot \mathbf{g}_P(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] + \alpha'(0) \int_0^{+\infty} \vartheta_P^2(\tau)d\tau + k_0 \int_0^{+\infty} \mathbf{g}_P^2(\tau)d\tau \\
 &+ \int_0^{+\infty} \left[\frac{1}{2} \int_0^{+\infty} \alpha''(|\tau - \eta|)\vartheta_P(\eta)d\eta - I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) \right] \vartheta_P(\tau)d\tau \\
 &+ \int_0^{+\infty} \left[\frac{1}{2} \int_0^{+\infty} k'(|\tau - \eta|)\mathbf{g}_P(\eta)d\eta - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) \right] \cdot \mathbf{g}_P(\tau)d\tau, \tag{5.16}
 \end{aligned}$$

where

$$I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) = \int_0^{+\infty} \alpha'''(\xi + \tau)\bar{\vartheta}^t(\xi)d\xi, \quad \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) = \int_0^{+\infty} k''(\xi + \tau)\bar{\mathbf{g}}^t(\xi)d\xi \quad \forall \tau \geq 0. \tag{5.17}$$

Contrary to what occurs for $\mathbf{I}_{(k)}^t$ in (3.18), the quantity $I_{(\alpha)}^t$ is not present in (3.17), which gives the value of the internal energy after a static continuation with a fixed duration, $\tau = a$. The reason for such a result is due to the fact that in the expression (5.2) we have the presence of \dot{e} , instead of e , as already observed in [4]. We only observe that such quantities are related to the minimal state of the rigid heat conductor (see, for example, [8]).

6 The Equivalence Between States by Means of the Work

In Section 4 we have called equivalent two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ ($j = 1, 2$) if the application of the same process to each of them yields the same response of the material, that is the final values of the internal energy and of the heat flux, corresponding to the two cases, coincide.

A new but equivalent definition of this relation can be given in terms of the work.

Definition 6.1 Two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ ($j = 1, 2$) are said to be w-equivalent if

$$W(\sigma_1(t), P_\tau) = W(\sigma_2(t), P_\tau) \tag{6.1}$$

for every process $P : [0, \tau) \rightarrow \mathbb{R} \times \mathbb{R}^3$ and for every $\tau > 0$.

The two definitions, we have introduced, are equivalent by virtue of the following theorem.

Theorem 6.1 *Two states are equivalent in the sense of Definition 4.1 if and only if they are w-equivalent.*

Proof Let $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ ($j = 1, 2$) be two states equivalent in the sense of Definition 4.1, then (4.1) and hence (4.2)-(4.3) hold for every process P_τ and for every $\tau > 0$. Therefore, it follows that

$$\int_0^d [\dot{e}_1(t + \tau)\vartheta_{P_1}(\tau) - \mathbf{q}_1(t + \tau) \cdot \mathbf{g}_P(\tau)]d\tau = \int_0^d [\dot{e}_2(t + \tau)\vartheta_{P_2}(\tau) - \mathbf{q}_2(t + \tau) \cdot \mathbf{g}_P(\tau)]d\tau. \tag{6.2}$$

In fact, on account of (4.1)₁, the derivatives with respect to τ of e_1 and e_2 coincide; $\vartheta_{P_j}(\tau)$ ($j = 1, 2$) are expressed by (3.7), where we have $\vartheta_1(t) = \vartheta_2(t)$, by virtue of (4.2)₁, and the same $\dot{\vartheta}_P(\tau)$; finally, $\mathbf{q}_j(t + \tau)$ ($j = 1, 2$), given by (3.12) or equivalently by (5.5), where we have the same \mathbf{g}_P in $[0, \tau)$ and the last integrals related to $j = 1$ and

$j = 2$ coincide because (4.3) holds by hypothesis, assume the same value. Since such an equality expresses the equality of the two works done during the application of the same process to $\sigma_j(t)$ ($j = 1, 2$), (6.1) is satisfied.

Let us now suppose that two states $\sigma_j(t)$ ($j = 1, 2$) are w-equivalent; then, they satisfy (6.1) for any P with any arbitrary duration $d > 0$. From (6.1), taking account of the expression for the work (5.2) in the form (5.16)₂, it follows that

$$\begin{aligned} & \alpha_0 \int_0^d \dot{\vartheta}_P(\tau) [\vartheta_{P_1}(\tau) - \vartheta_{P_2}(\tau)] d\tau + \alpha'(0) \int_0^d [\vartheta_{P_1}^2(\tau) - \vartheta_{P_2}^2(\tau)] d\tau \\ & + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|) [\vartheta_{P_1}(\eta)\vartheta_{P_1}(\tau) - \vartheta_{P_2}(\eta)\vartheta_{P_2}(\tau)] d\eta d\tau \\ & - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t)\vartheta_{P_1}(\tau) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\vartheta_{P_2}(\tau) \right] d\tau \\ & - \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0, \end{aligned} \tag{6.3}$$

where the integrals with the factor k_0 and the one with the kernel k' have been eliminated since expressed by means of the same \mathbf{g}_P . From (3.7) we have $\vartheta_{P_j}(\tau) = \vartheta_j(t) + \int_0^\tau \dot{\vartheta}_P(\eta) d\eta$ ($j = 1, 2$), which allow us to rewrite (6.3) as

$$\begin{aligned} & \alpha_0 [\vartheta_1(t) - \vartheta_2(t)] \int_0^d \dot{\vartheta}_P(\tau) d\tau + \alpha'(0) \int_0^d \{ [\vartheta_1^2(t) - \vartheta_2^2(t)] \\ & + 2[\vartheta_1(t) - \vartheta_2(t)] \int_0^\tau \dot{\vartheta}_P(\xi) d\xi \} d\tau + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|) \{ [\vartheta_1^2(t) - \vartheta_2^2(t)] \\ & + [\vartheta_1(t) - \vartheta_2(t)] \left[\int_0^\eta \dot{\vartheta}_P(\nu) d\nu + \int_0^\tau \dot{\vartheta}_P(\rho) d\rho \right] \} d\eta d\tau - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t)\vartheta_1(t) \right. \\ & \left. - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\vartheta_2(t) \right] d\tau - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t) \right] \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi \right] d\tau \\ & - \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0. \end{aligned} \tag{6.4}$$

Since this relation must hold for any P and any $d > 0$, we can choose the arbitrary quantities $\dot{\vartheta}_P$ and \mathbf{g}_P equal to zero; thus, the sum of the remaining terms must vanish. Consequently, (6.4) reduces to

$$\begin{aligned} & [\vartheta_1(t) - \vartheta_2(t)] \left\{ \alpha_0 \int_0^d \dot{\vartheta}_P(\tau) d\tau + 2\alpha'(0) \int_0^d \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi \right] d\tau \right. \\ & \left. + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|) \left[\int_0^\eta \dot{\vartheta}_P(\nu) d\nu + \int_0^\tau \dot{\vartheta}_P(\rho) d\rho \right] d\eta d\tau \right\} \\ & - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t) \right] \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi \right] d\tau \\ & - \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0. \end{aligned} \tag{6.5}$$

We now observe that

$$\alpha''(|s_1 - s_2|) = -2\delta(s_1 - s_2)\alpha'(|s_1 - s_2|) - \alpha_{12}(|s_1 - s_2|), \tag{6.6}$$

where $\alpha_{12} = \partial^2 \alpha / \partial s_1 \partial s_2$; hence, substituting it into (6.5) and recalling that both $\dot{\vartheta}_P(\tau)$ and $\vartheta_P(\tau)$ are equal to zero for any $\tau > d$, we obtain

$$\begin{aligned} & [\vartheta_1(t) - \vartheta_2(t)] \left\{ \alpha_0 f(d) - \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha_{12}(|\tau - \eta|) [f(\eta) + f(\tau)] d\eta d\tau \right\} \\ & - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t) \right] f(\tau) d\tau \\ & - \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0, \end{aligned} \tag{6.7}$$

where we have put

$$f(\tau) \equiv \int_0^\tau \dot{\vartheta}_P(\xi) d\xi, \quad f(\tau) = f(d) \quad \forall \tau > d, \tag{6.8}$$

whence $f'(\tau) \equiv \dot{\vartheta}_P(\tau)$.

From the relation (6.7), with an integration of each term of the integral with the kernel α_{12} , we have

$$\begin{aligned} & -\frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha_{12}(|\tau - \eta|) [f(\eta) + f(\tau)] d\eta d\tau \\ & = \int_0^{+\infty} \alpha'(\tau) f(\tau) d\tau = - \int_0^{+\infty} \alpha(\tau) f'(\tau) d\tau + \alpha_\infty f(d). \end{aligned} \tag{6.9}$$

By substituting this result into (6.7), after putting $\mathbf{g}_P(\tau) = \mathbf{0}$, we obtain

$$\begin{aligned} & [\vartheta_1(t) - \vartheta_2(t)] \left[(\alpha_0 + \alpha_\infty) f(d) - \int_0^{+\infty} \alpha(\tau) f'(\tau) d\tau \right] \\ & = \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t) \right] f(\tau) d\tau, \end{aligned} \tag{6.10}$$

which also implies

$$\int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0. \tag{6.11}$$

In (6.10), with an integration by parts and taking account of (6.8)₂, we have

$$\begin{aligned} \mathcal{I} & \equiv \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t) \right] f(\tau) d\tau \\ & = - \int_0^{+\infty} \left\{ \int_0^\tau \left[I_{(\alpha)}^t(\beta, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\beta, \bar{\vartheta}_2^t) \right] d\beta \right\} f'(\tau) d\tau \\ & \quad + f(d) \int_0^{+\infty} \left[I_{(\alpha)}^t(\rho, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\rho, \bar{\vartheta}_2^t) \right] d\rho. \end{aligned} \tag{6.12}$$

After substituting the expression (5.17)₁ for $I_{(\alpha)}^t$, which has α'' as kernel, into this relation, we can integrate with respect to β in $(0, \tau)$ and to ρ in $(0, +\infty)$; thus, \mathcal{I} reduces to

$$\mathcal{I} = - \int_0^{+\infty} \int_0^{+\infty} \alpha''(\xi + \tau) [\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi)] f'(\tau) d\xi d\tau. \tag{6.13}$$

Therefore, (6.10) can be written as

$$\int_0^{+\infty} \{[\vartheta_1(t) - \vartheta_2(t)] [\alpha_0 + \alpha_\infty - \alpha(\tau)] + \int_0^{+\infty} \alpha''(\xi + \tau) [\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi)] d\xi\} f'(\tau) d\tau = 0. \tag{6.14}$$

Hence, the arbitrariness of $f'(\tau) = \dot{\vartheta}_P(\tau)$ yields

$$[\vartheta_1(t) - \vartheta_2(t)] [\alpha_0 + \alpha_\infty - \alpha(\tau)] = - \int_0^{+\infty} \alpha''(\xi + \tau) [\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi)] d\xi. \tag{6.15}$$

Hence the limit as $\tau \rightarrow +\infty$, by virtue of (2.5), yields $\vartheta_1(t) = \vartheta_2(t)$, that is (4.2)₁.

This result implies that also the right-hand side of (6.15) vanishes so that (4.2)₂ is satisfied.

Finally, from (6.11), which must hold for any non-zero $\mathbf{g}_P(\tau)$, it follows that

$$\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) = \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t), \tag{6.16}$$

which, by virtue of the definition (5.17)₂, yields (4.3).

Thus, all the equalities required by Theorem 4.1 hold; consequently, the two w-equivalent $\sigma_j(t)$ ($j = 1, 2$) are also equivalent in the sense of Definition 4.1. \square

7 A First Expression for the Minimum Free Energy

Let $\psi_m(t)$ denote the minimum free energy and Π be the set of finite work processes of \mathcal{B} , we have

$$\psi_m(t) \equiv W_R(\sigma) = \sup \{-W(\sigma, P) : P \in \Pi\}, \tag{7.1}$$

where $W_R(\sigma)$ is the maximum recoverable work from a given state σ of the body [15, 18, 20].

The work $W_R(\sigma)$ is a non-negative function of the state, since in Π there exists the null process, for which the work done on the body starting from σ vanishes; moreover, by virtue of thermodynamic considerations, it follows that $W_R(\sigma) < +\infty$.

Let $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ be the initial state at time $t > 0$, when a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ is applied to the body for any $\tau \in [0, d]$. By extending P on \mathbb{R}^+ by means of $P(\tau) = (0, \mathbf{0}) \forall \tau \in [0, +\infty)$, the work done on the body has the expression given by (5.16)₂, which, if we assume that $\vartheta_P(d) \equiv \vartheta(t + d) = 0$, can be written as

$$\begin{aligned} W(\sigma, P) = & -\frac{1}{2} \alpha_0 \vartheta^2(t) + \alpha'(0) \int_0^{+\infty} \vartheta_P^2(\tau) d\tau \\ & + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|) \vartheta_P(\eta) \vartheta_P(\tau) d\eta d\tau - \int_0^{+\infty} I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) \vartheta_P(\tau) d\tau \\ & + k_0 \int_0^{+\infty} \mathbf{g}_P^2(\tau) d\tau + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} k'(|\tau - \eta|) \mathbf{g}_P(\eta) \cdot \mathbf{g}_P(\tau) d\eta d\tau \\ & - \int_0^{+\infty} \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) \cdot \mathbf{g}_P(\tau) d\tau. \end{aligned} \tag{7.2}$$

In order to obtain the maximum recoverable work, we must evaluate the maximum of $-W(\sigma, P)$, which will correspond to an opportune process, denoted by $P^{(m)}(\tau) = (\dot{\vartheta}^{(m)}(\tau), \mathbf{g}^{(m)}(\tau))$, where $\dot{\vartheta}^{(m)}(\tau)$ will be related to the optimal temperature $\vartheta^{(m)}$, which characterizes the maximum together with $\mathbf{g}^{(m)}$. We can consider the ensuing field ϑ_P with \mathbf{g}_P expressed by means of the quantities $\vartheta^{(m)}$ and $\mathbf{g}^{(m)}$ by assuming

$$\vartheta_P(\tau) = \vartheta^{(m)}(\tau) + \gamma\varphi(\tau), \quad \mathbf{g}_P(\tau) = \mathbf{g}^{(m)}(\tau) + \delta\mathbf{e}(\tau) \quad \forall \tau \in \mathbb{R}^+, \quad (7.3)$$

with γ and δ being two real parameters, φ and \mathbf{e} being two arbitrary smooth functions such that $\varphi(0) = 0$ and $\mathbf{e}(0) = \mathbf{0}$.

Substitution of (7.3) into (7.2) gives

$$\begin{aligned} -W(\sigma, P) &= -\tilde{W}(\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t; \dot{\vartheta}^{(m)} + \gamma\dot{\varphi}, \mathbf{g}^{(m)} + \delta\mathbf{e}) \\ &= \frac{1}{2}\alpha_0\vartheta^2(t) - \alpha'(0) \int_0^{+\infty} \left\{ \left[\vartheta^{(m)}(\tau) \right]^2 + 2\vartheta^{(m)}(\tau)\varphi(\tau)\gamma + \varphi^2(\tau)\gamma^2 \right\} d\tau \\ &\quad - \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|) \left\{ \vartheta^{(m)}(\eta)\vartheta^{(m)}(\tau) + [\vartheta^{(m)}(\eta)\varphi(\tau) \right. \\ &\quad \left. + \varphi(\eta)\vartheta^{(m)}(\tau)]\gamma + \varphi(\eta)\varphi(\tau)\gamma^2 \right\} d\eta d\tau + \int_0^{+\infty} I_{(\alpha)}^t(\tau, \bar{\vartheta}^t)[\vartheta^{(m)}(\tau) \\ &\quad + \varphi(\tau)\gamma] d\tau - k_0 \int_0^{+\infty} \left\{ \left[\mathbf{g}^{(m)}(\tau) \right]^2 + 2\mathbf{g}^{(m)}(\tau) \cdot \mathbf{e}(\tau)\delta + \mathbf{e}^2(\tau)\delta^2 \right\} d\tau \\ &\quad - \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} k'(|\tau - \eta|) \left\{ \mathbf{g}^{(m)}(\eta) \cdot \mathbf{g}^{(m)}(\tau) + [\mathbf{g}^{(m)}(\eta) \cdot \mathbf{e}(\tau) \right. \\ &\quad \left. + \mathbf{e}(\eta) \cdot \mathbf{g}^{(m)}(\tau)]\delta + \mathbf{e}(\eta) \cdot \mathbf{e}(\tau)\delta^2 \right\} d\eta d\tau \\ &\quad + \int_0^{+\infty} \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) \cdot [\mathbf{g}^{(m)}(\tau) + \mathbf{e}(\tau)\delta] d\tau, \end{aligned} \quad (7.4)$$

whence, the derivatives with respect to γ and δ yield

$$\begin{cases} \frac{\partial}{\partial \gamma} [-W(\sigma, P)] |_{\gamma=0} = \int_0^{+\infty} \varphi(\tau) \left[-2\alpha'(0)\vartheta^{(m)}(\tau) \right. \\ \quad \left. - \int_0^{+\infty} \alpha''(|\tau - \eta|)\vartheta^{(m)}(\eta)d\eta + I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) \right] d\tau = 0 \\ \frac{\partial}{\partial \delta} [-W(\sigma, P)] |_{\delta=0} = \int_0^{+\infty} \mathbf{e}(\tau) \cdot \left[-2k_0\mathbf{g}^{(m)}(\tau) \right. \\ \quad \left. - \int_0^{+\infty} k'(|\tau - \eta|)\mathbf{g}^{(m)}(\eta)d\eta + \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) \right] d\tau = 0. \end{cases} \quad (7.5)$$

From the arbitrariness of φ and \mathbf{e} in (7.5) it follows that

$$\begin{cases} \int_0^{+\infty} \alpha''(|\tau - \eta|)\vartheta^{(m)}(\eta)d\eta + 2\alpha'(0)\vartheta^{(m)}(\tau) = I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) \\ \int_0^{+\infty} k'(|\tau - \eta|)\mathbf{g}^{(m)}(\eta)d\eta + 2k_0\mathbf{g}^{(m)}(\tau) = \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) \end{cases} \quad \forall \tau \in \mathbb{R}^+. \quad (7.6)$$

In this system we have two Wiener-Hopf integral equations of the second kind, which are solvable by virtue of the thermodynamic properties of the kernels and of some theorems on factorization; thus, we are able to derive the solutions $\vartheta^{(m)}$ and $\mathbf{g}^{(m)}$, which give the maximum recoverable work.

Such a work, by substituting the expressions of $I_{(\alpha)}^t$ and $\mathbf{I}_{(k)}^t$, given by (7.6), into

(7.2), assumes the form

$$\begin{aligned}
 W_R(\sigma) &= \frac{1}{2}\alpha_0\vartheta^2(t) + \alpha'(0) \int_0^{+\infty} [\vartheta^{(m)}(\tau)]^2 d\tau + k_0 \int_0^{+\infty} [\mathbf{g}^{(m)}(\tau)]^2 d\tau \\
 &\quad + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(|\tau - \eta|)\vartheta^{(m)}(\eta)\vartheta^{(m)}(\tau)d\eta d\tau \\
 &\quad + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} k'(|\tau - \eta|)\mathbf{g}^{(m)}(\eta) \cdot \mathbf{g}^{(m)}(\tau)d\eta d\tau. \tag{7.7}
 \end{aligned}$$

This relation can be expressed in terms of Fourier’s transform, by using Plancherel’s theorem and (2.16)₁, as follows

$$\begin{aligned}
 W_R(\sigma) &= \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega\alpha'_s(\omega)\vartheta_+^{(m)}(\omega) [\vartheta_+^{(m)}(\omega)]^* d\omega \\
 &\quad + \frac{1}{2\pi} \int_{-\infty}^{+\infty} [k_0 + k'_c(\omega)] \mathbf{g}_+^{(m)}(\omega) \cdot [\mathbf{g}_+^{(m)}(\omega)]^* d\omega. \tag{7.8}
 \end{aligned}$$

It remains to solve the Wiener-Hopf integral equations in (7.6). To do this we introduce

$$r^{(\alpha)}(\tau) = \begin{cases} \int_{-\infty}^{+\infty} \alpha''(|\tau - \eta|)\vartheta^{(m)}(\eta)d\eta & \forall \tau \in \mathbb{R}^-, \\ 0 & \forall \tau \in \mathbb{R}^{++}, \end{cases} \tag{7.9}$$

$$\mathbf{r}^{(k)}(\tau) = \begin{cases} \int_{-\infty}^{+\infty} k'(|\tau - \eta|)\mathbf{g}^{(m)}(\eta)d\eta & \forall \tau \in \mathbb{R}^-, \\ \mathbf{0} & \forall \tau \in \mathbb{R}^{++}, \end{cases} \tag{7.10}$$

which allow us to give to (7.6) the following form

$$\begin{cases} \int_{-\infty}^{+\infty} \alpha''(|\tau - \eta|)\vartheta^{(m)}(\eta)d\eta + 2\alpha'(0)\vartheta^{(m)}(\tau) = I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) + r^{(\alpha)}(\tau) \\ \int_{-\infty}^{+\infty} k'(|\tau - \eta|)\mathbf{g}^{(m)}(\eta)d\eta + 2k_0\mathbf{g}^{(m)}(\tau) = \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) + \mathbf{r}^{(k)}(\tau) \end{cases} \quad \forall \tau \in \mathbb{R}. \tag{7.11}$$

Hence, using the Fourier transform, we obtain

$$\begin{cases} 2[\alpha'_c(\omega) + \alpha'(0)]\vartheta_+^{(m)}(\omega) = I_{(\alpha)_+}^t(\omega, \bar{\vartheta}^t) + r_-^{(\alpha)}(\omega), \\ 2[k'_c(\omega) + k_0]\mathbf{g}_+^{(m)}(\omega) = \mathbf{I}_{(k)_+}^t(\omega, \bar{\mathbf{g}}^t) + \mathbf{r}_-^{(k)}(\omega). \end{cases} \tag{7.12}$$

Since, in particular, $\alpha''_c(\omega) + \alpha'_0 = \omega\alpha'_s(\omega)$ by virtue of (2.16)₁, we can put

$$H^{(\alpha)}(\omega) = \omega\alpha'_s(\omega) \geq 0, \quad H^{(k)}(\omega) = k_0 + k'_c(\omega) > 0, \tag{7.13}$$

because of the thermodynamic restrictions (2.15).

We note that $H^{(\alpha)}$ is an even function, that is

$$H^{(\alpha)}(\omega) = H^{(\alpha)}(-\omega), \tag{7.14}$$

and goes to zero at least quadratically at the origin; we assume for such a function a behaviour no stronger than the quadratic one. Moreover, using (2.16)₁ and (2.20)₂, it follows that

$$H^{(\alpha)}_\infty = \lim_{\omega \rightarrow +\infty} \omega\alpha'_s(\omega) = \alpha'(0) > 0. \tag{7.15}$$

We also observe that

$$H_\infty^{(k)} = \lim_{\omega \rightarrow +\infty} [k_0 + k'_c(\omega)] = k_0 > 0, \quad H^{(k)}(0) = \lim_{\omega \rightarrow 0} [k_0 + k'_c(\omega)] = k_\infty > 0, \quad (7.16)$$

by virtue of (2.20)₁ and (2.19)₂.

Hence, the functions $H^{(\alpha)}(\omega)$ and $H^{(k)}(\omega)$ can be factorized [20]

$$H^{(\alpha)}(\omega) = H_{(+)}^{(\alpha)}(\omega)H_{(-)}^{(\alpha)}(\omega), \quad H^{(k)}(\omega) = H_{(+)}^{(k)}(\omega)H_{(-)}^{(k)}(\omega), \quad (7.17)$$

where the extensions to the complex plane \mathbb{C} of $H_{(+)}^{(\alpha)}(\omega)$ and $H_{(+)}^{(k)}(\omega)$ have no singularities and zeros in $\mathbb{C}^{(-)}$ and, therefore, they are analytic in \mathbb{C}^- , while the extensions of $H_{(-)}^{(\alpha)}(\omega)$ and $H_{(-)}^{(k)}(\omega)$, without zeros and singularities in $\mathbb{C}^{(+)}$, are analytic in \mathbb{C}^+ .

Thus, from (7.12), by using (2.16)₁, (7.13) and (7.17), we obtain

$$H_{(+)}^{(\alpha)}(\omega)\vartheta_+^{(m)}(\omega) = \frac{I_{(\alpha)+}^t(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} + \frac{r_-^{(\alpha)}(\omega)}{2H_{(-)}^{(\alpha)}(\omega)}, \quad (7.18)$$

$$H_{(+)}^{(k)}(\omega)\mathbf{g}_+^{(m)}(\omega) = \frac{\mathbf{I}_{(k)+}^t(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} + \frac{\mathbf{r}_-^{(k)}(\omega)}{2H_{(-)}^{(k)}(\omega)}. \quad (7.19)$$

Using the Plemelj formulae [24], we have

$$\frac{I_{(\alpha)+}^t(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} = P_{(\alpha)(-)}^t(\omega) - P_{(\alpha)(+)}^t(\omega), \quad (7.20)$$

$$\frac{\mathbf{I}_{(k)+}^t(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} = \mathbf{P}_{(k)(-)}^t(\omega) - \mathbf{P}_{(k)(+)}^t(\omega), \quad (7.21)$$

where

$$P_{(\alpha)}^t(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{I_{(\alpha)+}^t(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} \frac{d\omega}{\omega - z}, \quad P_{(\alpha)(\pm)}^t(\omega) = \lim_{\beta \rightarrow 0^\mp} P_{(\alpha)}^t(\omega + i\beta), \quad (7.22)$$

$$\mathbf{P}_{(k)}^t(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\mathbf{I}_{(k)+}^t(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} \frac{d\omega}{\omega - z}, \quad \mathbf{P}_{(k)(\pm)}^t(\omega) = \lim_{\beta \rightarrow 0^\mp} \mathbf{P}_{(k)}^t(\omega + i\beta). \quad (7.23)$$

From (7.18)-(7.19), by virtue of (7.20)-(7.21), we obtain

$$H_{(+)}^{(\alpha)}(\omega)\vartheta_+^{(m)}(\omega) + P_{(\alpha)(+)}^t(\omega) = P_{(\alpha)(-)}^t(\omega) + \frac{r_-^{(\alpha)}(\omega)}{2H_{(-)}^{(\alpha)}(\omega)}, \quad (7.24)$$

$$H_{(+)}^{(k)}(\omega)\mathbf{g}_+^{(m)}(\omega) + \mathbf{P}_{(k)(+)}^t(\omega) = \mathbf{P}_{(k)(-)}^t(\omega) + \frac{\mathbf{r}_-^{(k)}(\omega)}{2H_{(-)}^{(k)}(\omega)}, \quad (7.25)$$

where $P_{(\alpha)(\pm)}^t(\omega)$ and $\mathbf{P}_{(k)(\pm)}^t(\omega)$, considered as functions of $z \in \mathbb{C}$, are analytic in $\mathbb{C}^{(\mp)}$ but also in \mathbb{R} , by virtue of the assumption on the Fourier transforms. Hence, the functions

at the left-hand sides are analytic in \mathbb{C}^- , while the others at the right-hand sides in \mathbb{C}^+ ; moreover, they vanish at infinity. Therefore, both sides must be equal to zero and, in particular, give

$$\vartheta_+^{(m)}(\omega) = -\frac{P_{(\alpha)(+)}^t(\omega)}{H_{(+)}^{(\alpha)}(\omega)}, \quad \mathbf{g}_+^{(m)}(\omega) = -\frac{\mathbf{P}_{(k)(+)}^t(\omega)}{H_{(+)}^{(k)}(\omega)}, \quad (7.26)$$

which, substituted into (7.8), yield

$$\psi_m(t) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |P_{(\alpha)(+)}^t(\omega)|^2 d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathbf{P}_{(k)(+)}^t(\omega)|^2 d\omega. \quad (7.27)$$

8 Another Equivalent Expression for the Minimum Free Energy

The expression (7.27), now derived, can be changed in an equivalent one by considering the relation between $P_{(\alpha)(+)}^t(\omega)$ and $\bar{\vartheta}^t(\omega)$ and the one between $\mathbf{P}_{(k)(+)}^t(\omega)$ and $\mathbf{g}_+^t(\omega)$.

In order to derive this new expression, firstly we consider the casual extensions of $\bar{\vartheta}^t$ and $\bar{\mathbf{g}}^t$, by assuming $\bar{\vartheta}^t(s) = 0$ and $\bar{\mathbf{g}}^t(s) = \mathbf{0} \forall s \in \mathbb{R}^-$; then, we consider the odd extension of $\alpha'''(s)$ and $k''(s)$ on \mathbb{R}^- , denoted by $\alpha'''^{(o)}(s)$ and $k''^{(o)}(s)$, that is

$$\alpha'''^{(o)}(s) = \begin{cases} \alpha'''(s) & \forall s \geq 0, \\ -\alpha'''(-s) & \forall s < 0, \end{cases} \quad k''^{(o)}(s) = \begin{cases} k''(s) & \forall s \geq 0, \\ -k''(-s) & \forall s < 0. \end{cases} \quad (8.1)$$

Hence, we can give to (5.17) the new forms

$$I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi + \tau)\bar{\vartheta}^t(\xi)d\xi, \quad \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi + \tau)\bar{\mathbf{g}}^t(\xi)d\xi \quad \forall \tau \geq 0 \quad (8.2)$$

and, by putting

$$I_{(\alpha)}^{t(n)}(\tau, \bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi + \tau)\bar{\vartheta}^t(\xi)d\xi, \quad \mathbf{I}_{(k)}^{t(n)}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi + \tau)\bar{\mathbf{g}}^t(\xi)d\xi \quad \forall \tau < 0, \quad (8.3)$$

extend the functions (8.2) on \mathbb{R} as follows

$$I_{(\alpha)}^{t(\mathbb{R})}(\tau, \bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi + \tau)\bar{\vartheta}^t(\xi)d\xi = \begin{cases} I_{(\alpha)}^t(\tau, \bar{\vartheta}^t) & \forall \tau \geq 0, \\ I_{(\alpha)}^{t(n)}(\tau, \bar{\vartheta}^t) & \forall \tau < 0, \end{cases} \quad (8.4)$$

$$\mathbf{I}_{(k)}^{t(\mathbb{R})}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi + \tau)\bar{\mathbf{g}}^t(\xi)d\xi = \begin{cases} \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}^t) & \forall \tau \geq 0, \\ \mathbf{I}_{(k)}^{t(n)}(\tau, \bar{\mathbf{g}}^t) & \forall \tau < 0. \end{cases} \quad (8.5)$$

Let us introduce the functions $\bar{\vartheta}_n^t(s) = \bar{\vartheta}^t(-s)$ and $\bar{\mathbf{g}}_n^t(s) = \bar{\mathbf{g}}^t(-s)$ for any $s \leq 0$. Then, we extend them on \mathbb{R} by assuming $\bar{\vartheta}_n^t(s) = 0$ and $\bar{\mathbf{g}}_n^t(s) = \mathbf{0} \forall s > 0$. We can rewrite (8.4)-(8.5) as follows

$$I_{(\alpha)}^{t(\mathbb{R})}(\tau, \bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\tau - s)\bar{\vartheta}_n^t(s)ds, \quad \mathbf{I}_{(k)}^{t(\mathbb{R})}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\tau - s)\bar{\mathbf{g}}_n^t(s)ds. \quad (8.6)$$

Using their Fourier's transforms, given by (2.10)₁ and expressed by

$$\bar{\vartheta}_{n_F}^t(\omega) = \bar{\vartheta}_{n_-}^t(\omega) = [\bar{\vartheta}_+^t(\omega)]^*, \quad \bar{\mathbf{g}}_{n_F}^t(\omega) = \bar{\mathbf{g}}_{n_-}^t(\omega) = [\bar{\mathbf{g}}_+^t(\omega)]^*, \quad (8.7)$$

we obtain

$$\begin{aligned} I_{(\alpha)_F}^{t(\mathbb{R})}(\omega, \bar{\vartheta}^t) &= -2i\alpha_s'''(\omega)\bar{\vartheta}_{n_F}^t(\omega) = 2i\omega\alpha_c''(\omega) [\bar{\vartheta}_+^t(\omega)]^* \\ &= 2i\omega[H^{(\alpha)}(\omega) - \alpha'(0)] [\bar{\vartheta}_+^t(\omega)]^*, \end{aligned} \quad (8.8)$$

$$\begin{aligned} \mathbf{I}_{(k)_F}^{t(\mathbb{R})}(\omega, \bar{\mathbf{g}}^t) &= -2ik_s''(\omega)\bar{\mathbf{g}}_{n_F}^t(\omega) = 2i\omega k_c'(\omega) [\bar{\mathbf{g}}_+^t(\omega)]^* \\ &= 2i\omega [H^{(k)}(\omega) - k_0] [\bar{\mathbf{g}}_+^t(\omega)]^*, \end{aligned} \quad (8.9)$$

where we have used (2.12)₂, (2.13) for $\alpha_s'''(\omega)$ and $k_s''(\omega)$, (2.16)₁ and (7.13).

These last two Fourier transforms can be evaluated also by means of (8.4)-(8.5); thus, we obtain

$$I_{(\alpha)_F}^{t(\mathbb{R})}(\omega, \bar{\vartheta}^t) = I_{(\alpha)_-}^{t(n)}(\omega, \bar{\vartheta}^t) + I_{(\alpha)_+}^t(\omega, \bar{\vartheta}^t), \quad (8.10)$$

$$\mathbf{I}_{(k)_F}^{t(\mathbb{R})}(\omega, \bar{\mathbf{g}}^t) = \mathbf{I}_{(k)_-}^{t(n)}(\omega, \bar{\mathbf{g}}^t) + \mathbf{I}_{(k)_+}^t(\omega, \bar{\mathbf{g}}^t), \quad (8.11)$$

which, by virtue of (7.20)-(7.21), give

$$\frac{I_{(\alpha)_F}^{t(\mathbb{R})}(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} = \frac{I_{(\alpha)_-}^{t(n)}(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} + P_{(\alpha)(-)}^t(\omega) - P_{(\alpha)(+)}^t(\omega), \quad (8.12)$$

$$\frac{\mathbf{I}_{(k)_F}^{t(\mathbb{R})}(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} = \frac{\mathbf{I}_{(k)_-}^{t(n)}(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} + \mathbf{P}_{(k)(-)}^t(\omega) - \mathbf{P}_{(k)(+)}^t(\omega). \quad (8.13)$$

Hence, using the Plemelj formulae, we also have

$$\frac{I_{(\alpha)_F}^{t(\mathbb{R})}(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} = P_{(\alpha)_1(-)}^t(\omega) - P_{(\alpha)_1(+)}^t(\omega), \quad (8.14)$$

$$\frac{\mathbf{I}_{(k)_F}^{t(\mathbb{R})}(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)} = \mathbf{P}_{(k)_1(-)}^t(\omega) - \mathbf{P}_{(k)_1(+)}^t(\omega), \quad (8.15)$$

where the new functions $P_{(\alpha)_1(\pm)}^t(\omega)$ and $\mathbf{P}_{(k)_1(\pm)}^t(\omega)$, as in (7.22)-(7.23), are defined by

$$P_{(\alpha)_1}^t(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{I_{(\alpha)_F}^{t(\mathbb{R})}(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)}}{\omega - z} d\omega, \quad (8.16)$$

$$\mathbf{P}_{(k)_1}^t(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{\mathbf{I}_{(k)_F}^{t(\mathbb{R})}(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)}}{\omega - z} d\omega. \quad (8.17)$$

Using (8.12)-(8.15), we can consider two functions

$$\begin{aligned} F_{(\alpha)}(\omega) &\equiv P_{(\alpha)(+)}^t(\omega) - P_{(\alpha)_1(+)}^t(\omega) \\ &= P_{(\alpha)(-)}^t(\omega) - P_{(\alpha)_1(-)}^t(\omega) + \frac{I_{(\alpha)_-}^{t(n)}(\omega, \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega)} \end{aligned} \quad (8.18)$$

and

$$\begin{aligned} \mathbf{F}_{(k)}(\omega) &\equiv \mathbf{P}_{(k)(+)}^t(\omega) - \mathbf{P}_{(k)1(+)}^t(\omega) \\ &= \mathbf{P}_{(k)(-)}^t(\omega) - \mathbf{P}_{(k)1(-)}^t(\omega) + \frac{\mathbf{I}_{(k)-}^{t(n)}(\omega, \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega)}, \end{aligned} \tag{8.19}$$

defined by means of two different quantities, which are analytic in \mathbb{C}^- and in \mathbb{C}^+ , respectively, and vanish at infinity. Consequently, these functions must vanish, i. e. $F_{(\alpha)}(\omega) = 0$ and $\mathbf{F}_{(k)}(\omega) = \mathbf{0}$; thus, taking account of (8.16)-(8.17), it follows that

$$P_{(\alpha)(+)}^t(\omega) \equiv P_{(\alpha)1(+)}^t(\omega) = \lim_{z \rightarrow \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{I_{(\alpha)F}^{t(\mathbb{R})}(\omega', \bar{\vartheta}^t)}{2H_{(-)}^{(\alpha)}(\omega') \omega' - z} d\omega', \tag{8.20}$$

$$\mathbf{P}_{(k)(+)}^t(\omega) \equiv \mathbf{P}_{(k)1(+)}^t(\omega) = \lim_{z \rightarrow \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\mathbf{I}_{(k)F}^{t(\mathbb{R})}(\omega', \bar{\mathbf{g}}^t)}{2H_{(-)}^{(k)}(\omega') \omega' - z} d\omega'. \tag{8.21}$$

Hence, using (8.8)₃, (8.9)₃ and (7.17), we have

$$\begin{aligned} P_{(\alpha)(+)}^t(\omega) &= \lim_{z \rightarrow \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(+)}^{(\alpha)}(\omega') [\bar{\vartheta}_+^t(\omega')]^*}{\omega' - z} d\omega' \\ &\quad - \lim_{z \rightarrow \omega^-} \frac{\alpha'(0)}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{[\bar{\vartheta}_+^t(\omega')]^*}{H_{(-)}^{(\alpha)}(\omega')}}{\omega' - z} d\omega', \end{aligned} \tag{8.22}$$

$$\begin{aligned} \mathbf{P}_{(k)(+)}^t(\omega) &= \lim_{z \rightarrow \omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(+)}^{(k)}(\omega') [\bar{\mathbf{g}}_+^t(\omega')]^*}{\omega' - z} d\omega' \\ &\quad - \lim_{z \rightarrow \omega^-} \frac{k_0}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{[\bar{\mathbf{g}}_+^t(\omega')]^*}{H_{(-)}^{(k)}(\omega')}}{\omega' - z} d\omega', \end{aligned} \tag{8.23}$$

which yield

$$\begin{aligned} [P_{(\alpha)(+)}^t(\omega)]^* &= \lim_{w \rightarrow \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(-)}^{(\alpha)}(\omega') \bar{\vartheta}_+^t(\omega')}{\omega' - w} d\omega' \\ &\quad - \lim_{w \rightarrow \omega^+} \frac{\alpha'(0)}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{\bar{\vartheta}_+^t(\omega')}{H_{(+)}^{(\alpha)}(\omega')}}{\omega' - w} d\omega' \\ &= \lim_{w \rightarrow \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(-)}^{(\alpha)}(\omega') \bar{\vartheta}_+^t(\omega')}{\omega' - w} d\omega', \end{aligned} \tag{8.24}$$

$$\begin{aligned}
\left[\mathbf{P}_{(k)(+)}^t(\omega) \right]^* &= \lim_{w \rightarrow \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(-)}^{(k)}(\omega') \bar{\mathbf{g}}_+^t(\omega')}{\omega' - w} d\omega' \\
&\quad - \lim_{w \rightarrow \omega^+} \frac{k_0}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{\bar{\mathbf{g}}_+^t(\omega')}{H_{(+)}^{(k)}(\omega')}}{\omega' - w} d\omega' \\
&= \lim_{w \rightarrow \omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H_{(-)}^{(k)}(\omega') \bar{\mathbf{g}}_+^t(\omega')}{\omega' - w} d\omega'. \tag{8.25}
\end{aligned}$$

We note that, in these last two relations (8.24)₁ and (8.25)₁, the two integrals with $\alpha'(0)$ and k_0 are equal to zero, since they can be evaluated by closing the contour in $\mathbb{C}^{(-)}$, where $\bar{\vartheta}_+^t(\omega)$ with $H_{(+)}^{(\alpha)}(\omega)$ and $\bar{\mathbf{g}}_+^t(\omega)$ with $H_{(+)}^{(k)}(\omega)$ have no singularities and hence they are analytic in \mathbb{C}^- , by virtue of the hypothesis assumed for the Fourier transforms after (2.21); moreover, for the integral with $\alpha'(0)$ in (8.24)₁, we observe that the zero at the origin of $H_{(+)}^{(\alpha)}(\omega)$ is eliminated by the presence of the factor ω .

The application of the Plemelj formulae to (8.24)₂ and (8.25)₂ yields

$$\omega H_{(-)}^{(\alpha)}(\omega) \bar{\vartheta}_+^t(\omega) = Q_{(\alpha)(-)}^t(\omega) - Q_{(\alpha)(+)}^t(\omega), \tag{8.26}$$

where

$$Q_{(\alpha)(\pm)}^t(\omega) = \lim_{z \rightarrow \omega^\mp} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\omega' H_{(-)}^{(\alpha)}(\omega') \bar{\vartheta}_+^t(\omega')}{\omega' - z} d\omega' \tag{8.27}$$

and

$$\omega H_{(-)}^{(k)}(\omega) \bar{\mathbf{g}}_+^t(\omega) = \mathbf{Q}_{(k)(-)}^t(\omega) - \mathbf{Q}_{(k)(+)}^t(\omega) \tag{8.28}$$

with

$$\mathbf{Q}_{(k)(\pm)}^t(\omega) = \lim_{z \rightarrow \omega^\mp} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\omega' H_{(-)}^{(k)}(\omega') \bar{\mathbf{g}}_+^t(\omega')}{\omega' - z} d\omega'. \tag{8.29}$$

Thus, comparison of (8.24)₂ and (8.25)₂ with (8.27) and (8.29), respectively, yields

$$\left[P_{(\alpha)(+)}^t(\omega) \right]^* = iQ_{(\alpha)(-)}^t(\omega), \quad \left[\mathbf{P}_{(k)(+)}^t(\omega) \right]^* = i\mathbf{Q}_{(k)(-)}^t(\omega), \tag{8.30}$$

which give the required new expression

$$\psi_m(t) = \frac{1}{2} \alpha_0 \vartheta^2(t) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Q_{(\alpha)(-)}^t(\omega)|^2 d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\mathbf{Q}_{(k)(-)}^t(\omega)|^2 d\omega. \tag{8.31}$$

9 The Discrete Spectrum Model

We now apply the results of the previous section to study the particular class of response functions, which characterize the discrete spectrum model.

For this purpose, we assume the following relaxation functions

$$\alpha(t) = \begin{cases} \alpha_\infty - \sum_{i=1}^n h_i e^{-\alpha_i t} & \forall t \in \mathbb{R}^+, \\ 0 & \forall t \in \mathbb{R}^-, \end{cases} \quad k(t) = \begin{cases} k_\infty - \sum_{i=1}^n g_i e^{-k_i t} & \forall t \in \mathbb{R}^+, \\ 0 & \forall t \in \mathbb{R}^-, \end{cases} \tag{9.1}$$

where n is a positive integer, the inverse decay times α_i , k_i and the coefficients h_i , g_i ($i = 1, 2, \dots, n$) are also assumed to be positive; moreover, α_i and k_i are such that $\alpha_j < \alpha_{j+1}$ and $k_j < k_{j+1}$ ($j = 1, 2, \dots, n - 1$).

From (9.1) we have

$$\alpha'(t) = \sum_{i=1}^n \alpha_i h_i e^{-\alpha_i t}, \quad \alpha'_F(\omega) = \sum_{i=1}^n \frac{\alpha_i h_i}{\alpha_i + i\omega} = \sum_{i=1}^n \alpha_i h_i \frac{\alpha_i - i\omega}{\alpha_i^2 + \omega^2}, \quad (9.2)$$

$$k'(t) = \sum_{i=1}^n k_i g_i e^{-k_i t}, \quad k'_F(\omega) = \sum_{i=1}^n \frac{k_i g_i}{k_i + i\omega} = \sum_{i=1}^n k_i g_i \frac{k_i - i\omega}{k_i^2 + \omega^2}. \quad (9.3)$$

Hence, by virtue of (2.12)₃, it follows that

$$\alpha'_s(\omega) = \omega \sum_{i=1}^n \frac{\alpha_i h_i}{\alpha_i^2 + \omega^2}, \quad k'_c(\omega) = \sum_{i=1}^n \frac{k_i^2 g_i}{k_i^2 + \omega^2}. \quad (9.4)$$

These Fourier's transforms allow us to derive the expressions for the two functions defined in (7.13); we have

$$H^{(\alpha)}(\omega) = \omega^2 \sum_{i=1}^n \frac{\alpha_i h_i}{\omega^2 + \alpha_i^2} \geq 0 \forall \omega \in \mathbb{R}, \quad H_\infty^{(\alpha)} = \sum_{i=1}^n \alpha_i h_i = \alpha'(0) > 0, \quad (9.5)$$

$$H^{(k)}(\omega) = k_0 + \sum_{i=1}^n \frac{k_i^2 g_i}{k_i^2 + \omega^2} > 0 \forall \omega \in \mathbb{R}, \quad H_\infty^{(k)} = k_0 > 0. \quad (9.6)$$

We observe that the relaxation functions α and k , we have assumed in (9.1), satisfy all the conditions of Sect. 2. In fact we have

$$\alpha_\infty - \alpha_0 = \sum_{i=1}^n h_i > 0, \quad k_\infty - k_0 = \sum_{i=1}^n g_i > 0, \quad (9.7)$$

which satisfy (2.17)₁, (2.19)₂ and (2.20), by virtue of (9.5)₂.

Firstly, we consider the kernel α , for which the expression (9.5)₁ is analogous to the one obtained in [20], where however a minus sign occurs at the right-hand side of (9.5)₁ and the numerators are negative; it coincides with the one derived in [4] and [3]. We recall the results deduced in these works.

Let $n \neq 1$.

The function $f_{(\alpha)}(z) = H^{(\alpha)}(\omega) |_{z=-\omega^2}$ has n simple poles at α_i^2 ($i = 1, 2, \dots, n$) and n simple zeros at $\gamma_1 = 0$ and γ_j^2 ($j = 2, 3, \dots, n$), which are so ordered

$$\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < \dots < \alpha_{n-1}^2 < \gamma_n^2 < \alpha_n^2. \quad (9.8)$$

Consequently, we can give a new form to (9.5)₁, since it easily yields the factorization (7.17)₁; we have

$$H^{(\alpha)}(\omega) = H_\infty^{(\alpha)} \prod_{i=1}^n \left\{ \frac{\gamma_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\} = H_\infty^{(\alpha)} \prod_{i=1}^n \left\{ \frac{(\omega - i\gamma_i)(\omega + i\gamma_i)}{(\omega - i\alpha_i)(\omega + i\alpha_i)} \right\}, \quad (9.9)$$

and, in particular,

$$H_{(-)}^{(\alpha)}(\omega) = h_\infty^{(\alpha)} \prod_{i=1}^n \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\} \equiv h_\infty^{(\alpha)} \left(1 + i \sum_{i=1}^n \frac{R_i}{\omega + i\alpha_i} \right), \quad (9.10)$$

with

$$R_i = (\gamma_i - \alpha_i) \prod_{j=1, j \neq i}^n \left\{ \frac{\gamma_j - \alpha_i}{\alpha_j - \alpha_i} \right\}, \quad h_\infty^{(\alpha)} = \sqrt{H_\infty^{(\alpha)}}. \tag{9.11}$$

We must consider (8.27) to derive the quantity $Q_{(\alpha)(-)}^t(\omega)$ in (8.27). For this purpose we note that, contrary to what occurs in the previous works [4] and [3], $H_{(-)}^{(\alpha)}(\omega)$ is now multiplied by ω in the integrand of (8.27).

Substitution of (9.10)₂ into (8.27) yields

$$Q_{(\alpha)(-)}^t(\omega) = \frac{h_\infty^{(\alpha)}}{2\pi i} \left[\int_{-\infty}^{+\infty} \frac{\omega' \bar{\vartheta}_+^t(\omega')}{\omega' - \omega^+} d\omega' + i \sum_{r=1}^n R_r \int_{-\infty}^{+\infty} \frac{\bar{\vartheta}_+^t(\omega') \frac{\omega'}{\omega' - \omega^+}}{\omega' - (-i\alpha_r)} d\omega' \right]. \tag{9.12}$$

The first integral of this relation vanishes since it can be extended to an infinite contour in $\mathbb{C}^{(-)}$, where the integrand function, considered as a function of $z \in \mathbb{C}$, is analytic. By closing again in $\mathbb{C}^{(-)}$ and taking account of the sense of the integrations, we can evaluate the other integrals; we have

$$Q_{(\alpha)(-)}^t(\omega) = h_\infty^{(\alpha)} \sum_{r=1}^n \frac{\alpha_r R_r}{\omega + i\alpha_r} \bar{\vartheta}_+^t(-i\alpha_r). \tag{9.13}$$

Hence, we obtain

$$\left[Q_{(\alpha)(-)}^t(\omega) \right]^* = h_\infty^{(\alpha)} \sum_{l=1}^n \frac{\alpha_l R_l}{\omega - i\alpha_l} \bar{\vartheta}_+^t(-i\alpha_l), \tag{9.14}$$

since, by virtue of (2.10)₂,

$$\bar{\vartheta}_+^t(-i\alpha_r) = \int_0^{+\infty} \bar{\vartheta}^t(s) e^{-\alpha_r s} ds = [\bar{\vartheta}_+^t(-i\alpha_r)]^*. \tag{9.15}$$

By closing now in $\mathbb{C}^{(+)}$, the first integral of (8.31) can be evaluated; we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Q_{(\alpha)(-)}^t(\omega)|^2 d\omega \\ &= \left[h_\infty^{(\alpha)} \right]^2 \sum_{r,l=1}^n \alpha_r \alpha_l R_r R_l \bar{\vartheta}_+^t(-i\alpha_r) \bar{\vartheta}_+^t(-i\alpha_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i}{\omega - i\alpha_r} d\omega \\ &= H_\infty^{(\alpha)} \sum_{r,l=1}^n \frac{B_r B_l}{\alpha_r + \alpha_l} \bar{\vartheta}_+^t(-i\alpha_r) \bar{\vartheta}_+^t(-i\alpha_l), \end{aligned} \tag{9.16}$$

where we have put

$$B_r = \alpha_r R_r \quad (r = 1, 2, \dots, n). \tag{9.17}$$

Let $n = 1$.

From (9.10)₁, since $\gamma_1 = 0$, we have

$$H_{(-)}^{(\alpha)}(\omega) = h_\infty^{(\alpha)} \frac{\omega}{\omega + i\alpha_1} = h_\infty^{(\alpha)} \left(1 + i \frac{R_1}{\omega + i\alpha_1} \right), \quad R_1 = -\alpha_1, \quad h_\infty^{(\alpha)} = \sqrt{\alpha_1 h_1}. \tag{9.18}$$

Now, we consider the kernel k .

Case $n \neq 1$.

Taking in mind the expression (9.6)₁, we observe that the function $f_{(k)}(z) = H^{(k)}(\omega) |_{z=-\omega^2}$, such that

$$f_{(k)}(0) = k_0 + \sum_{i=1}^n g_i, \quad \lim_{z \rightarrow \pm\infty} f_{(k)}(z) = k_0^\mp, \quad \lim_{z \rightarrow (k_i^2)^\mp} f_{(k)}(z) = \pm\infty, \quad (9.19)$$

has n simple poles at k_i^2 ($i = 1, 2, \dots, n$) and n simple zeros in ν_i^2 ($i = 1, 2, \dots, n$) so ordered

$$k_1^2 < \nu_1^2 < k_2^2 < \dots < \nu_{n-1}^2 < k_n^2 < \nu_n^2; \quad (9.20)$$

thus, (9.6)₁ can be written as

$$H^{(k)}(\omega) = H_\infty^{(k)} \prod_{i=1}^n \left\{ \frac{\nu_i^2 + \omega^2}{k_i^2 + \omega^2} \right\} = k_0 \prod_{i=1}^n \left\{ \frac{(\omega - i\nu_i)(\omega + i\nu_i)}{(\omega - ik_i)(\omega + ik_i)} \right\}. \quad (9.21)$$

Hence, it follows that

$$H_{(-)}^{(k)}(\omega) = k_0^{1/2} \prod_{i=1}^n \left\{ \frac{\omega + i\nu_i}{\omega + ik_i} \right\} \equiv k_0^{1/2} \left(1 + i \sum_{i=1}^n \frac{S_i}{\omega + ik_i} \right), \quad (9.22)$$

where

$$S_r = (\nu_r - k_r) \prod_{j=1, j \neq r}^n \left\{ \frac{\nu_j - k_r}{k_j - k_r} \right\} \quad (r = 1, 2, \dots, n). \quad (9.23)$$

As above for $Q_{(\alpha)(-)}^t(\omega)$, still now in the expression for $Q_{(k)(-)}^t(\omega)$ given by (8.29), the quantity $H_{(-)}^{(k)}(\omega)$ is multiplied by ω . Thus, by substituting (9.22)₂, with S_r given by (9.23), into (8.29), we obtain

$$\begin{aligned} Q_{(k)(-)}^t(\omega) &= \frac{k_0^{1/2}}{2\pi i} \left[\int_{-\infty}^{+\infty} \frac{\omega' \bar{\mathbf{g}}_+^t(\omega')}{\omega' - \omega^+} d\omega' + i \sum_{r=1}^n S_r \int_{-\infty}^{+\infty} \frac{\bar{\mathbf{g}}_+^t(\omega') \frac{\omega'}{\omega' - \omega^+}}{\omega' - (-ik_r)} d\omega' \right] \\ &= k_0^{1/2} \sum_{r=1}^n \frac{k_r S_r}{\omega + ik_r} \bar{\mathbf{g}}_+^t(-ik_r), \end{aligned} \quad (9.24)$$

since, as we have noted above for $Q_{(\alpha)(-)}^t(\omega)$ after (9.12), still now the first integral vanishes and the other integrals can be evaluated by closing again in $\mathbb{C}^{(-)}$ and taking account of the sense of integrations.

Using (2.10)₂, as for $\bar{\vartheta}_+^t$ in (9.15),

$$\bar{\mathbf{g}}_+^t(-ik_r) = [\bar{\mathbf{g}}_+^t(-ik_r)]^* \quad (9.25)$$

also holds; thus, (9.24)₂ gives

$$[Q_{(k)(-)}^t(\omega)]^* = k_0^{1/2} \sum_{r=1}^n \frac{k_r S_r}{\omega - ik_r} \bar{\mathbf{g}}_+^t(-ik_r). \quad (9.26)$$

Hence, the second integral of (8.31) can be evaluated by closing in $\mathbb{C}^{(+)}$; we obtain

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} | \mathbf{Q}_{(k)(-)}^t(\omega) |^2 d\omega \\ &= k_0 \sum_{r,l=1}^n k_r k_l S_r S_l \bar{\mathbf{g}}_+^t(-ik_r) \cdot \bar{\mathbf{g}}_+^t(-ik_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i}{\omega - ik_r} d\omega \\ &= k_0 \sum_{r,l=1}^n \frac{C_r C_l}{k_r + k_l} \bar{\mathbf{g}}_+^t(-ik_r) \cdot \bar{\mathbf{g}}_+^t(-ik_l), \end{aligned} \tag{9.27}$$

where we have put

$$C_r = k_r S_r \quad (r = 1, 2, \dots, n). \tag{9.28}$$

Let $n = 1$.

In this particular case from (9.6)₁ we can evaluate the zero $\nu_1 = k_1 \sqrt{1 + \frac{g_1}{k_0}}$; then, (9.22)₂ gives

$$H_{(-)}^{(k)}(\omega) = k_0^{\frac{1}{2}} \frac{\omega + i\nu_1}{\omega + ik_1} \equiv k_0^{\frac{1}{2}} \left(1 + i \frac{S_1}{\omega + ik_1} \right), \tag{9.29}$$

with

$$S_1 = \nu_1 - k_1 = k_1 \left(\sqrt{1 + \frac{g_1}{k_0}} - 1 \right). \tag{9.30}$$

Therefore, when $n \neq 1$, by using (9.15)₁ for $\bar{\vartheta}_+^t$ and the analogous relation (9.25) for $\bar{\mathbf{g}}_+^t$, in (9.16)₂ and in (9.27)₂, respectively, (8.31) yields the required expression

$$\begin{aligned} \psi_m(t) &= \frac{1}{2} \alpha_0 \vartheta^2(t) \\ &+ \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \left[2H_{\infty}^{(\alpha)} \sum_{r,l=1}^n \frac{B_r B_l}{\alpha_r + \alpha_l} e^{-(\alpha_r s_1 + \alpha_l s_2)} \bar{\vartheta}^t(s_1) \bar{\vartheta}^t(s_2) \right. \\ &\left. + 2k_0 \sum_{r,l=1}^n \frac{C_r C_l}{k_r + k_l} e^{-(k_r s_1 + k_l s_2)} \bar{\mathbf{g}}^t(s_1) \cdot \bar{\mathbf{g}}^t(s_2) \right] ds_1 ds_2. \end{aligned} \tag{9.31}$$

In the simple case when $n = 1$, (9.31), by using (9.18) and (9.29), becomes

$$\begin{aligned} \psi_m(t) &= \frac{1}{2} \alpha_0 \vartheta^2(t) + \frac{1}{2} h_1 \alpha_1^4 \left[\int_0^{+\infty} e^{-\alpha_1 s} \bar{\vartheta}^t(s) ds \right]^2 \\ &+ \frac{1}{2} k_0 k_1^3 \left(\sqrt{1 + \frac{g_1}{k_0}} - 1 \right)^2 \left[\int_0^{+\infty} e^{-k_1 s} \bar{\mathbf{g}}^t(s) ds \right]^2. \end{aligned} \tag{9.32}$$

Finally, we want to observe that, at least in this last case characterized by $n = 1$, from (9.32), by means of the following integrations by parts

$$\begin{aligned} \int_0^{+\infty} e^{-\alpha_1 s} \bar{\vartheta}^t(s) ds &= \frac{1}{\alpha_1} \int_0^{+\infty} e^{-\alpha_1 s} {}_r \vartheta^t(s) ds, \\ \int_0^{+\infty} e^{-k_1 s} \bar{\mathbf{g}}^t(s) ds &= \frac{1}{k_1} \int_0^{+\infty} e^{-k_1 s} {}_r \mathbf{g}^t(s) ds, \end{aligned}$$

it is easy to obtain the corresponding result derived in [3] for $\psi_m(t)$, that is

$$\begin{aligned} \psi_m(t) = & \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2} \left\{ \alpha_1^2 h_1 \left[\int_0^{+\infty} e^{-\alpha_1 s} {}_r\vartheta^t(s) ds \right]^2 \right. \\ & \left. + k_0 k_1 \left(\sqrt{1 + \frac{g_1}{k_0}} - 1 \right)^2 \left[\int_0^{+\infty} e^{-k_1 s} {}_r\mathbf{g}^t(s) ds \right]^2 \right\} \end{aligned} \quad (9.33)$$

expressed in terms of $(\vartheta(t), {}_r\vartheta^t(s), {}_r\mathbf{g}^t(s))$, there assumed as the material state of the body.

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Positive Solutions of Semipositone Singular Dirichlet Dynamic Boundary Value Problems

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Abstract: We obtain a sufficient condition for the existence of a positive solution for a second-order superlinear semipositone singular Dirichlet dynamic boundary value problem by constructing a special cone. As a special case when $\mathbb{T} = \mathbb{R}$, this result includes those of Zhang and Liu [9]. This result is new in a general time scale setting and can be applied to q -difference equations. Two examples are given at the end of this paper to demonstrate the result.

Keywords: *semipositone; cone; time scale; delta derivative; nabla derivative.*

Mathematics Subject Classification (2000): 39A10, 34B10.

1 Introduction

We consider the following Dirichlet boundary value problem (BVP)

$$Lx = f(t, x) + h(t), \quad t \in (\rho(a), \sigma(b))_{\mathbb{T}}, \quad (1.1)$$

$$x(\rho(a)) = 0, \quad (1.2)$$

$$x(\sigma(b)) = 0, \quad (1.3)$$

where the operator L is defined by $Lx := -(p(t)x^{\Delta})^{\nabla}$, and \mathbb{T} is a time scale containing a and b . We define the time scale interval $(a, b)_{\mathbb{T}}$ by $(a, b)_{\mathbb{T}} := (a, b) \cap \mathbb{T}$, and similarly for other types of intervals. If \mathbb{T} has a right-scattered minimum m , we define $\mathbb{T}_{\kappa} := \mathbb{T} \setminus \{m\}$; otherwise, we set $\mathbb{T}_{\kappa} = \mathbb{T}$. The backward graininess ν is defined by $\nu := t - \rho(t)$. Then

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the nabla derivative of x at t , denoted by $x^\nabla(t)$, is defined to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|x(\rho(t)) - x(s) - x^\nabla(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \forall s \in U.$$

For $\mathbb{T} = \mathbb{R}$, we have $x^\nabla = x'$, and for $\mathbb{T} = \mathbb{Z}$, we have $x^\nabla(t) = \nabla x(t) = x(t) - x(t-1)$, which is the *backward difference operator*. An introduction of Time Scales Calculus can be found in Chapter 1 of [4], and in [5]. The domain D of L is the set of functions $x : \mathbb{T} \rightarrow \mathbb{R}$ such that x is continuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$, x^Δ is continuous on $[\rho(a), b]_{\mathbb{T}}$, and $(p(t)x^\Delta)^\nabla$ is continuous on $[a, b]_{\mathbb{T}}$. Since f may have singularities with respect to t at one or both end points, we shall assume, either f is continuous on $(a, b)_{\mathbb{T}} \times \mathbb{R}$ if f is singular at both a and b , or f is continuous on $(a, b]_{\mathbb{T}} \times \mathbb{R}$ if f is not singular at b , or f is continuous on $[a, b)_{\mathbb{T}} \times \mathbb{R}$ if f is not singular at a . If either f or h has a singularity at a , we assume $\rho(a) = a = \sigma(a)$, and if f or h has a singularity at b , then we assume $\rho(b) = b = \sigma(b)$. Let a and b be such that $0 \leq \rho(a) \leq a < b < \infty$ with $(a, b)_{\mathbb{T}} \neq \emptyset$, and $h : (\rho(a), \sigma(b))_{\mathbb{T}} \rightarrow (-\infty, \infty)$ is Lebesgue ∇ -integrable. Also $p > 0$ is continuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$, and there are constants m, M such that

$$0 < m \leq p(t) \leq M.$$

The BVP (1.1) – (1.3) arises in chemical reactor theory [2] when we consider the domain to be the set of real numbers. Since the function $h(t)$ in the above BVP may change sign we say this type of problem is semipositone. Special cases are studied in [8], [1] and the references therein. In the applications one is interested in finding positive solutions.

We impose the following conditions:

(H₁) For any $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, $f(t, 1) > 0$, and there exist constants $\lambda_1 \geq \lambda_2 > 1$ such that for any $(t, u) \in (\rho(a), \sigma(b))_{\mathbb{T}} \times [0, \infty)$

$$c^{\lambda_1} f(t, u) \leq f(t, cu) \leq c^{\lambda_2} f(t, u), \quad c \in [0, 1]. \quad (1.4)$$

(H₂) Let $r := \frac{M^3(\sigma(b) - \rho(a))}{m^4} \int_{\rho(a)}^b h^-(t) \nabla t > 0$, where m , and M are such that $0 < m \leq p(t) \leq M$, and $h^\pm(t) := \max\{\pm h(t), 0\}$, and assume

$$\int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s)[f(s, 1) + h^+(s)] \nabla s < \frac{m^2 r (\sigma(b) - \rho(a))}{M[(r + 1)^{\lambda_1} + 1]}. \quad (1.5)$$

Remark 1.1 Note that it is easy to see for $c \geq 1$, from (1.4) that

$$c^{\lambda_2} f(t, u) \leq f(t, cu) \leq c^{\lambda_1} f(t, u) \quad (1.6)$$

for any $(t, u) \in (\rho(a), \sigma(b))_{\mathbb{T}} \times [0, \infty)$.

A solution u_0 of the BVP (1.1) – (1.3) with $u_0(t) > 0$, $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, is called positive solution of the BVP (1.1) – (1.3).

2 Preliminary Lemmas

We state the following lemmas which we will use later in this section.

Lemma 2.1 [7] *Let X be a real Banach space, Ω be a bounded open subset of X with $0 \in \Omega$, and $A : \bar{\Omega} \cap P \rightarrow P$ be a completely continuous operator, where P is a cone in X .*

- (i) *Suppose that $Au \neq \lambda u$, for all $u \in \partial\Omega \cap P$, $\lambda \geq 1$. Then $i(A, \Omega \cap P, P) = 1$.*
- (ii) *Suppose that $Au \not\leq u$, for all $u \in \partial\Omega \cap P$. Then $i(A, \Omega \cap P, P) = 0$.*

Lemma 2.2 *If $f(t, u)$ satisfies (H_1) , then for any $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, $f(t, u)$ is non-decreasing in $u \in [0, \infty)$, and for any nonempty $[\alpha, \beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$,*

$$\lim_{u \rightarrow \infty} \min_{t \in [\alpha, \beta]_{\mathbb{T}}} \frac{f(t, u)}{u} = \infty.$$

Proof Let $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, and $x, y \in [0, \infty)$ be arbitrary. Without loss of generality assume $0 \leq x \leq y$. Now, if $y = 0$, then $f(t, x) \leq f(t, y)$ is clear. If $y \neq 0$, let $c_0 = \frac{x}{y}$, then $0 \leq c_0 \leq 1$. Now by (1.4),

$$f(t, x) = f(t, c_0 y) \leq c_0^{\lambda_2} f(t, y) \leq f(t, y).$$

Thus $f(t, u)$ is non-decreasing in u on $[0, \infty)$.

Next choose $u > 1$. Then it follows from (1.6) that $f(t, u) \geq u^{\lambda_2} f(t, 1)$. So we get

$$\frac{f(t, u)}{u} \geq u^{\lambda_2 - 1} f(t, 1), \quad \forall t \in (\rho(a), \sigma(b))_{\mathbb{T}}.$$

So for any nonempty $[\alpha, \beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$\min_{t \in [\alpha, \beta]_{\mathbb{T}}} \frac{f(t, u)}{u} \geq u^{\lambda_2 - 1} \min_{t \in [\alpha, \beta]_{\mathbb{T}}} f(t, 1).$$

Since $f(t, 1) > 0$ (by (H_1)),

$$\lim_{u \rightarrow \infty} \min_{t \in [\alpha, \beta]_{\mathbb{T}}} \frac{f(t, u)}{u} = \infty. \quad \square$$

Let $X := \{x \in C([\rho(a), \sigma(b)]_{\mathbb{T}}, \mathbb{R})\}$ with $\|x\| = \sup_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} |x(t)|$, and define

$$P := \{x \in X : x(t) \geq 0, t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\},$$

$$Q := \{x \in P : x(t) \geq \|x\| \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2}, t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\},$$

where $0 < m \leq p(t) \leq M$.

Then one can easily verify that X is a real Banach space, and P, Q are cones in X , and clearly $Q \subset P$.

Note that the Green's function for the BVP

$$\begin{aligned} -(p(t)x^\Delta)^\nabla &= 0, & t \in (\rho(a), \sigma(b))_{\mathbb{T}} \\ x(\rho(a)) &= 0 \\ x(\sigma(b)) &= 0 \end{aligned}$$

can be shown to be given by (see [3] for more information)

$$G(t, s) = \frac{1}{\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau} \begin{cases} \int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau, & \text{for } t \leq s; \\ \int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau \int_t^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau, & \text{for } s \leq t. \end{cases} \quad (2.1)$$

Also note that

$$0 \leq G(t, s) \leq G(s, s) = \frac{\int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau}{\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau} \leq \frac{M(s - \rho(a))(\sigma(b) - s)}{m^2(\sigma(b) - \rho(a))}. \quad (2.2)$$

Now set $w(t) := \int_{\rho(a)}^b G(t, s)h^-(s)\nabla s$, where $G(t, s)$ is as defined above. Then $w(t)$ is the unique solution of the BVP

$$(p(t)x^\Delta)^\nabla + h^-(t) = 0, \quad t \in (\rho(a), \sigma(b))_{\mathbb{T}}, \quad x(\rho(a)) = 0 = x(\sigma(b)). \quad (2.3)$$

To see that $w(t)$ is well defined note that

$$\begin{aligned} w(t) &= \int_{\rho(a)}^b G(t, s)h^-(s)\nabla s \\ &\leq \int_{\rho(a)}^b G(s, s)h^-(s)\nabla s \\ &\leq \frac{M(\sigma(b) - \rho(a))}{m^2} \int_{\rho(a)}^b h^-(s)\nabla s \\ &< \infty, \quad \text{for all } t \in [\rho(a), \sigma(b)]_{\mathbb{T}}. \end{aligned}$$

Also,

$$\begin{aligned} w(\rho(a)) &= \int_{\rho(a)}^b G(\rho(a), s)h^-(s)\nabla s = 0. \\ w(\sigma(b)) &= \int_{\rho(a)}^b G(\sigma(b), s)h^-(s)\nabla s = 0. \end{aligned}$$

It remains to show that

$$-(p(t)w^\Delta)^\nabla = h^-(t). \quad (2.4)$$

To verify this last statement we will use the formulas [5][Theorem 5.37]

$$\begin{aligned} \left(\int_a^t f(t, s)\nabla s \right)^\Delta &= \int_a^t f^\Delta(t, s)\nabla s + f(\sigma(t), \sigma(t)); \\ \left(\int_a^t f(t, s)\nabla s \right)^\nabla &= \int_a^t f^\nabla(t, s)\nabla s + f(\rho(t), t). \end{aligned}$$

Note that

$$w(t) = \frac{1}{\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau} \left[\int_{\rho(a)}^t \left(\int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau \int_t^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau \right) h^-(s) \nabla s + \int_t^b \left(\int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau \right) h^-(s) \nabla s \right]$$

Then we get,

$$\begin{aligned} \left(\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau \right) w^\Delta(t) &= \left(\int_{\rho(a)}^t \int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau \int_t^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \right)^\Delta \\ &\quad + \left(\int_t^b \int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \right)^\Delta \\ &= - \int_{\rho(a)}^t \frac{1}{p(t)} \int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \\ &\quad + \int_{\rho(a)}^{\sigma(t)} \frac{1}{p(\tau)} \Delta\tau \int_{\sigma(t)}^{\sigma(b)} \frac{1}{p(\tau)} \nabla\tau h^-(\sigma(t)) \\ &\quad + \int_t^b \frac{1}{p(t)} \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \\ &\quad - \int_{\rho(a)}^{\sigma(t)} \frac{1}{p(\tau)} \Delta\tau \int_{\sigma(t)}^{\sigma(b)} \frac{1}{p(\tau)} \nabla\tau h^-(\sigma(t)) \\ &= - \int_{\rho(a)}^t \frac{1}{p(t)} \int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \\ &\quad + \int_t^b \frac{1}{p(t)} \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s. \end{aligned}$$

So,

$$\begin{aligned} -(p(t)w^\Delta)^\nabla(t) &= \frac{1}{\left(\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau \right)} \left[\left(\int_{\rho(a)}^t \int_{\rho(a)}^s \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \right)^\nabla \right. \\ &\quad \left. - \left(\int_t^b \int_s^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(s) \nabla s \right)^\nabla \right] \\ &= \frac{1}{\left(\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau \right)} \left(\int_{\rho(a)}^t \frac{1}{p(\tau)} \Delta\tau h^-(t) + \int_t^{\sigma(b)} \frac{1}{p(\tau)} \Delta\tau h^-(t) \right) \\ &= h^-(t). \end{aligned}$$

Now we define an operator T on P by

$$(Tu)(t) := \int_{\rho(a)}^b G(t, s) [f(s, [u - w]^+(s)) + h^+(s)] \nabla s, \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}.$$

Claim: $T : P \rightarrow P$.

Proof of claim: Let $u \in P$ be fixed but arbitrary. Choose $0 < c < 1$ such that $c\|u\| < 1$. Then

$$c[u - w]^+(s) \leq cu(s) \leq c\|u\| < 1$$

Then by (1.4), (1.6), and Lemma 2.2, we get, for all $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$f(t, [u - w]^+(t)) \leq \left(\frac{1}{c}\right)^{\lambda_1} f(t, c[u - w]^+(t)) \leq c^{\lambda_2 - \lambda_1} \|u\|^{\lambda_2} f(t, 1). \quad (2.5)$$

So for any $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get using (2.2), (2.5), and (1.5) that

$$\begin{aligned} (Tu)(t) &= \int_{\rho(a)}^b G(t, s) [f(s, [u - w]^+(s)) + h^+(s)] \nabla s \\ &\leq \int_{\rho(a)}^b G(s, s) \left[c^{\lambda_2 - \lambda_1} \|u\|^{\lambda_2} f(s, 1) + h^+(s) \right] \nabla s \\ &\leq \frac{M \left(c^{\lambda_2 - \lambda_1} \|u\|^{\lambda_2} + 1 \right)}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s \\ &< \infty. \end{aligned}$$

Note that $Tu \in C[\rho(a), \sigma(b)]_{\mathbb{T}}$, and $Tu(t) \geq 0$, $\forall t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ are clear.

Thus $T : P \rightarrow P$ is well defined.

So from the definition of the operator T , we can easily prove the following theorem:

Theorem 2.1 *Suppose that (H_1) , and (H_2) hold. Then the operator T has a fixed point in $C[\rho(a), \sigma(b)]_{\mathbb{T}}$ iff the BVP*

$$\begin{cases} (p(t)u^\Delta)^\nabla + f(t, [u - w]^+(t)) + h^+(t) = 0 & \rho(a) < t < \sigma(b) \\ u(\rho(a)) = 0 = u(\sigma(b)) \end{cases} \quad (2.6)$$

has a positive solution where w is given as in (2.3).

Proof The operator T has a fixed point u ,

$$\begin{aligned} \implies u(t) &= (Tu)(t) \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}} \\ \implies u(t) &= \int_{\rho(a)}^b G(t, s) [f(s, [u - w]^+(s)) + h^+(s)] \nabla s, \quad u(\rho(a)) = 0 = u(\sigma(b)) \end{aligned}$$

Now using properties of the Green's function (the same steps that are used above to verify (2.4)), we get

$$-(p(t)u^\Delta)^\nabla = f(t, [u - w]^+(t)) + h^+(t), \quad u(\rho(a)) = 0 = u(\sigma(b)).$$

The other direction is similar. \square

Now we have the following lemma:

Lemma 2.3 *If the singular BVP (2.6) has a positive solution $u(t) \geq w(t)$ for all $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, then the BVP (1.1) – (1.3) has a $C[a, b]_{\mathbb{T}} \cap C^2(a, b)_{\mathbb{T}}$ positive solution $y(t) = u(t) - w(t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$.*

Proof Let $u(t) = y(t) + w(t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. Then by the first equation of (2.6), it follows that

$$\begin{aligned} (p(t)y^\Delta)^\nabla + (p(t)w^\Delta)^\nabla + f(t, y(t)) + h^+(t) &= 0, \\ \text{i.e., } (p(t)y^\Delta)^\nabla - h^-(t) + f(t, y(t)) + h^+(t) &= 0, \\ \text{i.e., } (p(t)y^\Delta)^\nabla + f(t, y(t)) + h(t) &= 0. \end{aligned}$$

Also,

$$\begin{aligned} y(\rho(a)) &= u(\rho(a)) - w(\rho(a)) = 0, \\ y(\sigma(b)) &= u(\sigma(b)) - w(\sigma(b)) = 0. \end{aligned}$$

Thus $y(t) = u(t) - w(t)$ is a positive solution of the BVP (1.1) – (1.3). \square

Lemma 2.4 *Assume (H_1) and (H_2) hold. Then $T : Q \rightarrow Q$ is a completely continuous operator.*

Proof For any $u \in Q$, let $y(t) = Tu(t)$. Then $y(\rho(a)) = 0 = y(\sigma(b))$. So there exists $t_0 \in (\rho(a), \sigma(b))$ such that $\|y\| = y(t_0)$. Note that for any $t, s \in (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$\begin{aligned} \frac{G(t, s)}{G(t_0, s)} &\geq \begin{cases} \frac{m(t-\rho(a))}{M(t_0-\rho(a))}, & \text{for } t, t_0 \leq s; \\ \frac{m^2(t-\rho(a))(\sigma(b)-s)}{M^2(s-\rho(a))(\sigma(b)-t_0)}, & \text{for } t \leq s \leq t_0; \\ \frac{m(\sigma(b)-t)}{M(\sigma(b)-t_0)}, & \text{for } t, t_0 \geq s; \\ \frac{m^2(s-\rho(a))(\sigma(b)-t)}{M^2(t_0-\rho(a))(\sigma(b)-s)}, & \text{for } t \geq s \geq t_0; \end{cases} \\ &\geq \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2}. \end{aligned}$$

Then for all $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$\begin{aligned} y(t) = (Tu)(t) &= \int_{\rho(a)}^b G(t, s) [f(s, [u-w]^+(s)) + h^+(s)] \nabla s \\ &= \int_{\rho(a)}^b \frac{G(t, s)}{G(t_0, s)} G(t_0, s) [f(s, [u-w]^+(s)) + h^+(s)] \nabla s \\ &\geq \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} y(t_0) \\ &= \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} \|y\|. \end{aligned}$$

Thus, $Tu \in Q$, and hence $T : Q \rightarrow Q$.

Next we show that $T : Q \rightarrow Q$ is a completely continuous operator.

First we show $T : Q \rightarrow Q$ is continuous. Let $\{x_n\}_{n=0}^\infty \subset Q$ be such that $x_n \rightarrow x_0$ when $n \rightarrow \infty$. Then there is a constant $M_1 > 0$ such that $\|x_n\| \leq M_1$ for all $n = 0, 1, 2, \dots$. Since for any $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$[x_n - w]^+(s) \leq x_n(s) \leq \|x_n\| \leq M_1 < M_1 + 1,$$

by (1.6), and Lemma 2.2 (since (H_1) holds for f), we get

$$\begin{aligned} f(s, [x_n - w]^+(s)) + h^+(s) &\leq f(s, M_1 + 1) + h^+(s) \\ &\leq (M_1 + 1)^{\lambda_1} f(s, 1) + h^+(s) \\ &\leq [(M_1 + 1)^{\lambda_1} + 1] [f(s, 1) + h^+(s)]. \end{aligned}$$

Then using (2.2) and (1.5), we get

$$\begin{aligned} &\int_{\rho(a)}^b G(t, s) [f(s, [x_n - w]^+(s)) + h^+(s)] \nabla s \\ &\leq [(M_1 + 1)^{\lambda_1} + 1] \int_{\rho(a)}^b G(s, s) [f(s, 1) + h^+(s)] \nabla s \\ &\leq \frac{M [(M_1 + 1)^{\lambda_1} + 1]}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s \\ &< \infty. \end{aligned}$$

Note that by the continuity of f ,

$$\lim_{n \rightarrow \infty} f(s, [x_n - w]^+(s)) = f(s, [x_0 - w]^+(s)).$$

Then by the Lebesgue Dominated Convergence Theorem [5, page 159], we get

$$\begin{aligned} &\lim_{n \rightarrow \infty} \|Tx_n - Tx_0\| \\ &= \lim_{n \rightarrow \infty} \sup_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} |Tx_n - Tx_0| \\ &\leq \lim_{n \rightarrow \infty} \sup_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} \int_{\rho(a)}^b G(t, s) |f(s, [x_n - w]^+(s)) - f(s, [x_0 - w]^+(s))| \nabla s \\ &\leq \lim_{n \rightarrow \infty} \int_{\rho(a)}^b \frac{M(s - \rho(a))(\sigma(b) - s)}{m^2(\sigma(b) - \rho(a))} |f(s, [x_n - w]^+) - f(s, [x_0 - w]^+)| \nabla s \\ &\leq \frac{M}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) \lim_{n \rightarrow \infty} |f(s, [x_n - w]^+(s)) \\ &\quad - f(s, [x_0 - w]^+(s))| \nabla s \\ &= 0. \end{aligned}$$

Thus $T : Q \rightarrow Q$ is continuous.

Finally, we show that $T : Q \rightarrow Q$ is relatively compact.

To see this let $D \subset Q$ be any bounded set. Then there exists $M_2 > 0$ such that $\|x\| \leq M_2$ for all $x \in D$. So for any $x \in D$ and $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get

$$[x - w]^+(s) \leq x(s) \leq \|x\| \leq M_2 < M_2 + 1.$$

So for all $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$f(s, [x - w]^+(s)) + h^+(s) \leq f(s, M_2 + 1) + h^+(s) \leq [(M_2 + 1)^{\lambda_1} + 1] [f(s, 1) + h^+(s)].$$

Then for all $x \in D$, and $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get using (2.2) and (1.5),

$$\begin{aligned} |Tx(t)| &= \left| \int_{\rho(a)}^b G(t, s) [f(s, [x - w]^+(s)) + h^+(s)] \nabla s \right| \\ &\leq \frac{M [(M_2 + 1)^{\lambda_1} + 1]}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s \\ &< \infty. \end{aligned}$$

Thus $T(D)$ is uniformly bounded.

Again by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} |Tx(t_1) - Tx(t_2)| &\leq \int_{\rho(a)}^b |G(t_1, s) - G(t_2, s)| [f(s, [x - w]^+(s)) + h^+(s)] \nabla s \\ &\leq [(M_2 + 1)^{\lambda_1} + 1] \int_{\rho(a)}^b |G(t_1, s) - G(t_2, s)| [f(s, 1) + h^+(s)] \nabla s \\ &\rightarrow 0 \quad \text{as } t_1 \rightarrow t_2. \end{aligned}$$

Since this is true for any $t_1, t_2 \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and the RHS is independent of x , $T(D)$ is equicontinuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$. Then by the Arzela-Ascoli Theorem, $T : Q \rightarrow Q$ is relatively compact.

Thus, $T : Q \rightarrow Q$ is a completely continuous operator. \square

Lemma 2.5 *Assume (H_1) and (H_2) hold. Let $Q_r = \{x \in Q : \|x\| < r\}$, and $\partial Q_r = \{x \in Q : \|x\| = r\}$, where $r := \frac{M^3(\sigma(b) - \rho(a))}{m^4} \int_{\rho(a)}^b h^-(t) \nabla t$ as defined in (H_2) . Then $i(T, Q_r, Q) = 1$.*

Proof Assume that there exist $z_0 \in \partial Q_r$, $\mu \geq 1$ such that $\mu z_0 = Tz_0$. Then $z_0 = \frac{1}{\mu} Tz_0$, and $0 < \frac{1}{\mu} \leq 1$. Since $z_0 \in \partial Q_r$,

$$[z_0 - w]^+(s) \leq z_0(s) \leq \|z_0\| = r < r + 1,$$

then for $s \in (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$f(s, [z_0 - w]^+(s)) + h^+(s) \leq [(r + 1)^{\lambda_1} + 1] [f(s, 1) + h^+(s)].$$

Now

$$\begin{aligned} r = \|z_0\| &= \left\| \frac{1}{\mu} Tz_0 \right\| \\ &\leq \|Tz_0\| \\ &= \sup_{t \in [\rho(a), \sigma(b)]} \left| \int_{\rho(a)}^b G(t, s) [f(s, [z_0 - w]^+(s)) + h^+(s)] \nabla s \right| \\ &\leq \int_{\rho(a)}^b G(s, s) [f(s, [z_0 - w]^+(s)) + h^+(s)] \nabla s \\ &\leq \frac{M[(r + 1)^{\lambda_1} + 1]}{m^2[\sigma(b) - \rho(a)]} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s. \end{aligned}$$

This implies,

$$\int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s)[f(s, 1) + h^+(s)]\nabla s \geq \frac{m^2 r(\sigma(b) - \rho(a))}{M[(r+1)^{\lambda_1} + 1]}$$

which is a contradiction to (1.5). So $Tz_0 \neq \mu z_0$ for all $z_0 \in \partial Q_r$, $\mu \geq 1$.

Then by Lemma 2.1, $i(T, Q_r, Q) = 1$. \square

Lemma 2.6 *Assume (H_1) holds. Then there exists a constant $R > r$ such that $i(T, Q_R, Q) = 0$ where $Q_R := \{x \in Q : \|x\| < R\}$, and $\partial Q_R := \{x \in Q : \|x\| = R\}$.*

Proof Assume $x \not\geq Tx$ for all $x \in \partial Q_R$ is false. Then there exists $y_1 \in \partial Q_R$ such that $y_1 \geq Ty_1$.

Choose constants α, β so that $[\alpha, \beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$, and K such that

$$K > \frac{2M^2(\sigma(b) - \rho(a))^2}{m^2(\alpha - \rho(a))(\sigma(b) - \beta) \max_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} \int_{\alpha}^{\beta} G(t, s)\nabla s}. \quad (2.7)$$

From Lemma 2.2 there exists $R_1 > 2r$ such that when $t \in [\alpha, \beta]_{\mathbb{T}}$, and $x \geq R_1$, we get

$$\frac{f(t, x)}{x} \geq K$$

That is,

$$f(t, x) \geq Kx, \quad t \in [\alpha, \beta]_{\mathbb{T}}, \quad x \geq R_1.$$

Let

$$R \geq \frac{2R_1 M^2(\sigma(b) - \rho(a))^2}{m^2(\alpha - \rho(a))(\sigma(b) - \beta)}. \quad (2.8)$$

Then clearly $R > R_1 > 2r$, and so $\frac{r}{R} < \frac{1}{2}$.

Now for the above mentioned y_1 , we have for all $t \in [\alpha, \beta]_{\mathbb{T}}$,

$$\begin{aligned} y_1(t) - w(t) &= y_1(t) - \int_{\rho(a)}^b G(t, s)h^-(s)\nabla s \\ &\geq y_1(t) - \frac{M(t - \rho(a))(\sigma(b) - t)}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b h^-(s)\nabla s \\ &= y_1(t) - \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} r \\ &\geq y_1(t) - \frac{y_1(t)}{\|y_1\|} r = y_1(t) - \frac{r}{R} y_1(t) \\ &\geq y_1(t) - \frac{1}{2} y_1(t) = \frac{1}{2} y_1(t) \\ &\geq \frac{1}{2} \|y_1\| \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} \quad (\text{as } y_1 \in Q) \\ &\geq \frac{1}{2} R \frac{m^2(\alpha - \rho(a))(\sigma(b) - \beta)}{M^2(\sigma(b) - \rho(a))^2} \\ &\geq R_1 > 0. \quad (\text{using (2.8)}) \end{aligned} \quad (2.9)$$

So,

$$\begin{aligned}
 R &= \|y_1\| \geq y_1(t) \\
 &\geq Ty_1(t) = \int_{\rho(a)}^b G(t, s) [f(s, [y_1 - w]^+(s)) + h^+(s)] \nabla s \\
 &\geq \int_{\alpha}^{\beta} G(t, s) [f(s, (y_1(s) - w(s)) + h^+(s)] \nabla s \\
 &\geq \int_{\alpha}^{\beta} G(t, s) f(s, (y_1(s) - w(s))) \nabla s \\
 &\geq \int_{\alpha}^{\beta} G(t, s) K(y_1(s) - w(s)) \nabla s \\
 &\geq \int_{\alpha}^{\beta} G(t, s) K \frac{1}{2} R \frac{m^2(\alpha - \rho(a))(\sigma(b) - \beta)}{M^2(\sigma(b) - \rho(a))^2} \nabla s \quad (\text{using (2.9)}) \\
 &= \frac{1}{2} KR \frac{m^2(\alpha - \rho(a))(\sigma(b) - \beta)}{M^2(\sigma(b) - \rho(a))^2} \int_{\alpha}^{\beta} G(t, s) \nabla s, \quad \forall t \in [\rho(a), \sigma(b)]_{\mathbb{T}} \\
 &\geq \frac{1}{2} KR \frac{m^2(\alpha - \rho(a))(\sigma(b) - \beta)}{M^2(\sigma(b) - \rho(a))^2} \max_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} \int_{\alpha}^{\beta} G(t, s) \nabla s \\
 \Rightarrow K &\leq \frac{2M^2(\sigma(b) - \rho(a))^2}{m^2(\alpha - \rho(a))(\sigma(b) - \beta) \max_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} \int_{\alpha}^{\beta} G(t, s) \nabla s}
 \end{aligned}$$

which is a contradiction to our choice of K above.

Thus $x \not\leq Tx$ for all $x \in \partial Q_R$, so by Lemma 2.1, we get

$$i(T, Q_R, Q) = 0. \quad \square$$

3 Main Result

Now we state and prove our main result.

Theorem 3.1 *Suppose that (H_1) , and (H_2) hold. Then the BVP (1.1) – (1.3) has at least one $C[a, b]_{\mathbb{T}} \cap C^2(a, b)_{\mathbb{T}}$ positive solution $u_0(t)$, and there exists $k > 0$ such that $u_0(t) \geq k(t - \rho(a))(\sigma(b) - t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$.*

Proof By Lemmas 2.5, 2.6, and by a property of the fixed point index, we get

$$\begin{aligned}
 i(T, Q_R \setminus \bar{Q}_r, Q) &= i(T, Q_R, Q) - i(T, Q_r, Q) \\
 &= 0 - 1 \\
 &= -1 \quad (\neq 0).
 \end{aligned}$$

So T has a fixed point z_0 in $Q_R \setminus \bar{Q}_r$, with $r < \|z_0\| < R$.

Then for all $t \in [\rho(a), \sigma(b)]$,

$$\begin{aligned}
z_0(t) - w(t) &\geq \|z_0\| \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} - \int_{\rho(a)}^b G(t, s) h^-(s) \nabla s \\
&\geq \|z_0\| \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} - \frac{M(t - \rho(a))(\sigma(b) - t)}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^b h^-(s) \nabla s \\
&= \|z_0\| \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} - r \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} \\
&= \frac{m^2(t - \rho(a))(\sigma(b) - t)}{M^2(\sigma(b) - \rho(a))^2} [\|z_0\| - r] \\
&= k(t - \rho(a))(\sigma(b) - t) \quad \text{where } k := \frac{m^2[\|z_0\| - r]}{M^2(\sigma(b) - \rho(a))^2} > 0 \\
&\geq 0, \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}.
\end{aligned}$$

Now let $u_0(t) := z_0(t) - w(t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. Then from Lemma 2.3, it follows that $u_0(t)$ is a positive solution of the BVP (1.1) – (1.3), and there exists a $k > 0$ such that $u_0(t) \geq k(t - \rho(a))(\sigma(b) - t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. The proof is now completed. \square

4 Examples

In this section we give two examples as applications of Theorem 3.1.

Example 4.1 Let $\mathbb{T} = \left\{ \frac{1}{q^n} \right\}_{n=0}^{\infty} \cup \{0, 2\}$, $q > 1$. Then we claim the BVP

$$\begin{cases} u^{\Delta \nabla} + \frac{u^{3/2}}{5t} - \frac{1}{\sqrt{t}} = 0, & t \in (0, 2)_{\mathbb{T}}, \\ u(0) = 0 = u(2) \end{cases} \quad (4.1)$$

has a positive solution.

First note that the BVP (4.1) is of the form (1.1) – (1.3) with $a = 0$, $b = 1$ and

$$p(t) \equiv 1, \quad f(t, u) = \frac{u^{3/2}}{5t}, \quad h^-(t) = \frac{1}{\sqrt{t}}, \quad h^+(t) = 0.$$

Also note that f and h have a singularity at $t = 0$, and $m = M = 1$. Then since $q > 1$,

$$\begin{aligned}
r &= \frac{M^3(\sigma(b) - \rho(a))}{m^4} \int_{\rho(a)}^b h^-(t) \nabla t \\
&= 2 \int_0^1 \frac{1}{\sqrt{t}} \nabla t \\
&= 2 \left[1 \left(1 - \frac{1}{q} \right) + \sqrt{q} \left(\frac{1}{q} - \frac{1}{q^2} \right) + \sqrt{q^2} \left(\frac{1}{q^2} - \frac{1}{q^3} \right) + \dots \right] \\
&= 2 \left[1 + \frac{1}{\sqrt{q}} \right].
\end{aligned}$$

Take $\lambda_1 = \lambda_2 = 3/2$, then (H_1) is satisfied.

For (H_2) note that,

$$\begin{aligned} \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s \\ = \frac{1}{5} \int_0^1 (2 - s) \nabla s = \frac{2 + q}{5 + 5q}. \end{aligned}$$

Also note that,

$$\frac{m^2 r (\sigma(b) - \rho(a))}{M((r + 1)^{\lambda_1} + 1)} \geq \frac{2r}{(r + 1)^2 + 1} = \frac{2q + 2\sqrt{q}}{5q + 6\sqrt{q} + 2}.$$

Now, it is easy to see that $\frac{2+q}{5+5q} < \frac{2q+2\sqrt{q}}{5q+6\sqrt{q}+2}$ for $q > 1$.

Thus, (H_2) is also satisfied. Hence the existence of a positive solution is now guaranteed from Theorem 3.1.

Example 4.2 Let $\mathbb{T} =$ The Cantor Set. (See pages 18–19 of [4] for more information regarding this time scale.)

Consider the following BVP for $k > \frac{20}{7}$,

$$\begin{cases} u^{\Delta \nabla} + \frac{u^2}{k(1-t)} - \frac{1}{\sqrt{t} + \sqrt{\rho(t)}} = 0, & t \in (0, 1)_{\mathbb{T}} \\ u(0) = 0 = u(1). \end{cases} \tag{4.2}$$

Again we apply Theorem 3.1. First note that

$$\begin{aligned} r &= \frac{M^3(\sigma(b) - \rho(a))}{m^4} \int_{\rho(a)}^b h^-(t) \nabla t \\ &= \int_0^1 \frac{1}{\sqrt{t} + \sqrt{\rho(t)}} \nabla t \\ &= \int_0^1 (\sqrt{t})^\nabla \nabla t = 1. \end{aligned}$$

Take $\lambda_1 = \lambda_2 = 2$, then (H_1) is satisfied.

In [6] the authors show that

$$\int_0^1 t \Delta t = \frac{3}{7},$$

where $t \in \mathbb{T}$, and \mathbb{T} is the Cantor set. Using similar arguments we get that

$$\int_0^1 t \nabla t = \frac{4}{7}$$

which we use below.

To see that (H_2) holds, note that

$$\begin{aligned} & \int_{\rho(a)}^b (s - \rho(a))(\sigma(b) - s) [f(s, 1) + h^+(s)] \nabla s \\ &= \int_0^1 s(1-s) \frac{1}{k(1-s)} \nabla s \\ &= \frac{1}{k} \int_0^1 s \nabla s = \frac{4}{7k}. \end{aligned}$$

Now it is clear that

$$\frac{4}{7k} < \frac{m^2 r (\sigma(b) - \rho(a))}{M[(r+1)^{\lambda_1} + 1]} = \frac{r}{(r+1)^2 + 1} = \frac{1}{5} \text{ for } k > \frac{20}{7}.$$

Thus (H_2) is also satisfied. Hence the existence of a positive solution is now guaranteed from Theorem 3.1.

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Analytical Methods for Analysis of Transitions to Chaotic Vibrations in Mechanical Systems

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Abstract: Some analytical methods for the analysis of transition to a chaotic behavior in nonlinear mechanical systems are considered here. First, the subharmonic Melnikov–Morozov theory, which is used to study a sequence of the saddle-node bifurcations, is considered. It is shown, that such bifurcations sequence occurs before an appearance of chaotic vibrations. Then, the chaotic dynamics in modulation equations is considered to study chaos in mechanical systems under the action of almost-periodic force, and the heteroclinic Melnikov functions are used to study the chaotic dynamics region. Finally, new approach for the construction of homo- and heteroclinic trajectories in some 2-DOF non-linear dynamical systems is used. Use of the Pade and quasi-Pade approximants, as well these approximants convergence condition make possible to solve boundary-value problems formulated for these orbits and to determine initial amplitude values of the trajectories with admissible precision.

Keywords: *chaotic vibrations; subharmonic bifurcations; homo- and heteroclinic trajectories.*

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1 Introduction

Analysis of chaotic vibrations is a subject of intensive investigations during the last decades. One of the principal scenarios of the transient to chaotic behavior is a cascade of bifurcations of the period doubling. Two approaches exist to predict the chaotic behavior. Both these approaches hang one upon other. In the first approach, intersections of invariant manifolds, which lead to appearance of Smale horseshoe, are investigated to predict the chaotic dynamics. In the second approach, bifurcations of periodic and almost periodic vibrations are investigated to determine regions of the chaotic behavior. The subharmonic Melnikov–Morozov theory [1, 2 et al], which is considered in Section 2 of this paper, is related to the second approach. The methods, which are considered in Sections 3-6, are related to the first group of approaches. In particular, in Section 3 the Melnikov function is used to determine the region, where the heteroclinic structure exists in nonlinear mechanical systems under the action of almost-periodic excitation. Formation of homo- and heteroclinic trajectories (HT) in phase plane is a criterion of the chaotic behavior in dynamical systems [1, 2 et al]. Methods, based on investigation of the HT formation, are related to the first group of approaches. (Note that the small dissipation leads to the complicated behavior near a separatrix of the Hamiltonian systems. Solutions that cross the separatrix due to the dissipation, were analyzed, for example, in [3]). The closed HT formation is possible in dynamical systems with dissipation and external periodic excitation. To construct the HT in such dynamical systems it is necessary to determine some important parameters. Namely, in such single-DOF system, it is necessary to know corresponding initial conditions of HT, and the functional dependence of the system parameters. For example, it may be a dependence of the external excitation amplitude on the dissipation parameter. In most cases the authors of the last and recent publications on HT construction make use of the well-known Melnikov function for the analysis of homoclinic structure [4-7], which gives a single equation for determination of unknown parameters. As a result, in the Melnikov condition, separatrix trajectories of the unperturbed autonomous equations are used. A problem of effective analytic approximation of HT of non-autonomous system is difficult and it is not solved up to now. Here a new approach for the HT construction in the nonlinear dynamical systems with phase space of dimensions equal to two is proposed and used. Pade approximants (PA) [8] and quasi-Pade approximants (QPA) are used to construct the HT in the dynamical system phase space and for the corresponding time history solution. Note that QPA which contain both powers of some parameter, and exponential functions were considered in Ref. [9]. Convergence condition were used earlier in the theory of non-linear normal vibration modes [10-12] as well as the conditions at infinity. This made possible to solve the boundary-value problem formulated for the HT and evaluate initial amplitude values with admissible precision. We suppose that the HT construction criterion of the chaos beginning proposed in this paper is more exact than the generally accepted Melnikov criterion, because it is not necessary to use separatrix trajectories of the Hamiltonian equations.

This work is structured as follows. First, the subharmonic Melnikov–Morozov theory and its application to parametric dynamics of beams are considered in Section 2. The method for determination of domains of chaos in mechanical systems under the action of quasiperiodic forces is considered in Section 3. The Pade approximants convergence condition is discussed in Section 4, and the HT boundary values problem is formulated in this Section. The approach proposed here was realized for the homoclinic solution of

the non-autonomous Duffing equation. Corresponding results are presented in Section 5. Construction of the homo- and heteroclinic orbits for the different dynamical systems is discussed in Section 6.

2 Subharmonic Melnikov–Morozov Theory and its Applications to Dynamics of Beam

Subharmonic Melnikov function analysis is carried out on the example of beam vibrations analysis. This beam is presented in Figure 2.1. Mass M is attached to the end of the beam. Transverse beam motions $W(s, t)$ induce displacements $\eta(t)$ of the mass M . Therefore, linear viscous damping force $R_L = c_L \dot{\eta}$ acts on the mass M . The nonlinear curvature and nonlinear inertia are taken into account in the model, so the equation of the beam parametric oscillations has the following form [13]:

$$\begin{aligned}
 EJw'''' + \frac{EJ}{2} (w''w'')'' + \left\{ P_0 + P_t \cos(\bar{\Omega}t) - \frac{M}{2} \int_0^l (w'^2)''_{tt} ds - \frac{c_L}{2} \int_0^l (w'^2)'_t ds \right\} w'' \\
 + c\dot{w} + \mu\ddot{w} - (Nw')' = 0, \\
 N = \frac{\mu}{2} \int_s^l ds_1 \int_0^{s_1} (w'^2)''_{tt} ds_2,
 \end{aligned} \tag{2.1}$$

where $\dot{w} = w'_t; w' = w'_s; \mu$ is the mass per unit of length; \dot{w} is the material damping; the term $w'''' + \frac{1}{2}(w''w'')''$ describes the beam curvature. The nonlinear inertia is presented by the term $(Nw')'$ in equation (2.1). The following dimensionless parameters are used:

$$\begin{aligned}
 \varepsilon\delta = \frac{l^2}{\sqrt{EJ\mu}}; \varepsilon\delta_L = \frac{c_L w_*^2}{2l\sqrt{EJ\mu}}; \varepsilon\Gamma_t = \frac{l^2 P_t}{EJ}; \Gamma_0 = \frac{P_0 l^2}{EJ}; \varepsilon\gamma = \frac{w_*^2}{2l^2}; m = \frac{M}{\mu l}; \\
 u = \frac{w}{w_*}; \tau = \sqrt{\frac{EJ}{\mu l^4}} t; \xi = \frac{s}{l}; \Omega = \frac{\bar{\Omega} l^2 \sqrt{\mu}}{\sqrt{EJ}}; w_* = \frac{l 2\sqrt{2}}{\pi} \sqrt{\frac{P_0}{P_*} - 1},
 \end{aligned} \tag{2.2}$$

where $\varepsilon \ll 1; P_*$ is the buckling force; w_* is the static deflection at $s = \frac{l}{2}$. Equation (2.1) is rewritten in the dimensionless form:

$$\begin{aligned}
 u'''' + \Gamma_0 u'' + \ddot{u} + \alpha(u''u'^2)'' + \varepsilon \left[-m\gamma u'' \int_0^1 (u'^2)'' d\xi - \gamma \left(u' \int_\xi^1 d\eta \int_0^\eta (u'^2)'' dh \right)' + \right. \\
 \left. + \delta\dot{u} + \Gamma_t \cos(\Omega\tau) u'' - \delta_L u'' \int_0^1 (u'^2)'' d\xi \right] = 0,
 \end{aligned} \tag{2.3}$$

where $\alpha = 0.5 w_*^2 l^{-2}; u' = u'_\xi; \dot{u} = u'_\tau$. Dimensionless fundamental frequencies of the corresponding linearized system (2.1) are the following: $p_k = k^2 \pi^2$. The frequency Ω is varied in the next range: $0.5 < \Omega < 4$. Therefore, one mode approximation, $u = q(t) \sin(\pi\xi)$, accurately describes the beam dynamics. The following differential equation is derived by the Galerkin method:

$$\ddot{q} + \lambda(q^3 - q) + \varepsilon [\gamma\rho\pi^4(q\dot{q}^2 + q\ddot{q}) + \delta\dot{q} - \Gamma_t\pi^2 q \cos(\Omega\tau) + \delta_L\pi^4 \dot{q}q^2] + O(\varepsilon^2) = 0, \tag{2.4}$$

$$\lambda = \Gamma_0\pi^2 - \pi^4; \rho = m + \frac{1}{3} - \frac{3}{8\pi^2}. \tag{2.5}$$

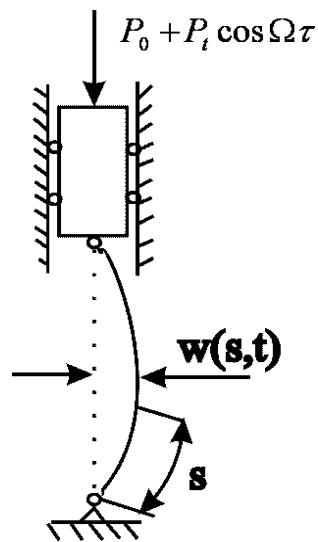


Figure 2.1: Transverse parametric oscillations of beam.

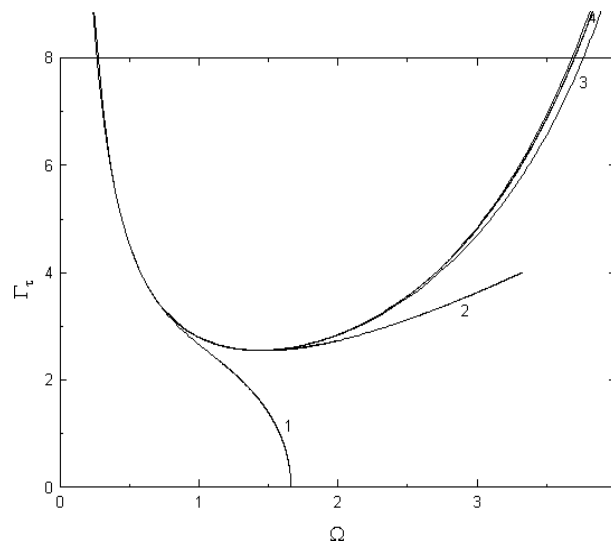


Figure 2.2: The saddle-node bifurcations curves of the subharmonic oscillations of orders 1,2,3,4. The curves are denoted by the same numbers. The calculations were performed with the following parameters: $\varepsilon = 0.01$; $\varepsilon \delta = \varepsilon \delta_L = 0.18$; $\varepsilon \gamma = 1.84 \cdot 10^{-3}$; $\rho = 3.4$.

The equation (2.4) can be presented as

$$\ddot{q} + \lambda(q^3 - q) + \varepsilon [-\gamma\rho\lambda\pi^4(q^5 - q^3) + \gamma\rho\pi^4qq\dot{q}^2 + \delta\dot{q} - \Gamma_t\pi^2q \cos(\Omega\tau) + \delta_L\pi^4\dot{q}q^2] = 0. \tag{2.6}$$

We stress that, for $\varepsilon=0$ the system (2.6) is a nonlinear conservative one.

In the future calculations the beam dynamics is considered with the following parameters [14]:

$$E = 2.013 \cdot 10^{11} \frac{N}{m^2}; \rho = 7.80 \cdot 10^3 \frac{Kg}{m^3}; l = 558mm; b = 11.95mm; h = 1mm;$$

$$M = 0.162Kg; \mu = 9.3 \cdot 10^{-2} \frac{Kg}{m}; P_* = 6.39N; P_0 = 6.42N; c = 7.8 \cdot 10^{-2} \frac{Kg}{s};$$

$$EJ = 0.201Nm^2; w_* = 3.4 \times 10^{-2}m.$$

Let us analyze the application of the Melnikov–Morozov method [1, 2, 4] for the saddle-node bifurcations analysis. It is known, that for $\varepsilon=0$ the system (2.6) allows the following periodic motions:

$$(q_0, \dot{q}_0) = \left\{ \sqrt{\frac{2}{2-k^2}} dn\tau; -\frac{k^2\sqrt{2\lambda}}{2-k^2} sn\tau cn\tau \right\}; \quad \tau = t\sqrt{\frac{\lambda}{2-k^2}}, \tag{2.7}$$

where k is the elliptic integral modulus; $dn; sn; cn$ are elliptic functions [15]. The equation: $H = \lambda(k^2 - 1)(2 - k^2)^{-2}$ connects the Hamiltonian H of system (2.6) for $\varepsilon=0$ with the modulus of the elliptic integral. Let us consider motions of the system (2.6) meeting the resonance conditions:

$$T(k) = mT; T = 2\pi/\Omega; T(k) = 2K\sqrt{2-k^2}/\lambda, \tag{2.8}$$

where K is the complete elliptic integral of the first kind; $T(k)$ is the period of the unperturbed system ($\varepsilon=0$) orbits. The subharmonic Melnikov–Morozov method permits to determine the subharmonic oscillations of a single DOF system with essential nonlinear unperturbed part. The simple roots of the subharmonic Melnikov function define these subharmonic oscillations. If the subharmonic Melnikov function roots meet the equation $|\sin(\Omega t_0)| = 1$, the saddle-node bifurcation set is taken place. The subharmonic Melnikov function of system (2.6) is derived as

$$\bar{M}_1^{m/1} = -\delta\sqrt{\lambda}J_1(k) + \Gamma_t\pi^2J_3(k) \sin(\Omega t_0) - \delta_L\pi^4\sqrt{\lambda}J_2(k), \tag{2.9}$$

where

$$J_1(k) = \frac{1}{\sqrt{\lambda}} \int_0^{mT} \dot{q}_0^2 dt = \frac{4}{3} [(2-k^2)E - 2k'^2K] (2-k^2)^{-3/2};$$

$$J_2(k) = \frac{1}{\sqrt{\lambda}} \int_0^{mT} \dot{q}_0^2 q_0^2 dt = \frac{8}{15} [2(k^4 + k'^2)E + (k^2 - 2)k'^2K] (2-k^2)^{-5/2}; \tag{2.10}$$

$$J_3 \sin(\Omega t_0) = \int_0^{mT} q_0 \dot{q}_0 \cos(\Omega\tau + \Omega t_0) d\tau = \frac{\Omega^2 \pi}{\lambda sh \left(\frac{m\pi K'}{K}\right)} \sin(\Omega t_0).$$

Note that E is the complete elliptic integral of the second kind.

From the equation (2.9) the parametric set of saddle-node bifurcations is derived as

$$\Gamma_t = \pm \frac{\sqrt{\lambda}}{\pi^2 J_3(k)} [\delta_L \pi^4 J_2(k) + \delta J_1(k)]. \quad (2.11)$$

Figure 2.2 shows the saddle-node bifurcations curves on the parametric plane (Ω, Γ_t) . The bifurcations curves of the subharmonic oscillations of orders 1,2,3,4 are denoted by the same numbers on this figure

$$(q_0, \dot{q}_0) = \left\{ \sqrt{\frac{2k^2}{2k^2-1}} cn\tau; -\frac{k\sqrt{2\lambda}}{2k^2-1} sn\tau dn\tau; \right\}, \quad \tau = \frac{\sqrt{\lambda}t}{\sqrt{2k^2-1}}. \quad (2.12)$$

In this case, the equation $H = k^2 k'^2 \lambda (2k^2 - 1)^{-2}$ connects the Hamiltonian H with the elliptic integral modulus k . The subharmonic Melnikov function of these motions has the following form:

$$\bar{M}_1^{m/1} = -\delta \sqrt{\lambda} \hat{J}_1(k) + \Gamma_t \pi^2 \hat{J}_3(k) \sin(\Omega t_0) - \delta_L \pi^4 \sqrt{\lambda} \hat{J}_2(k), \quad (2.13)$$

$$\hat{J}_1(k) = \frac{8}{13} \{k'^2 K - (1 - 2k^2)E\} (2k^2 - 1)^{-3/2};$$

$$\hat{J}_2(k) = \frac{16}{15} \{K k'^2 (k^2 - 2) + 2E(k'^2 + k^4)\} (2k^2 - 1)^{-5/2}; \quad \hat{J}_3(k) = \frac{2\Omega^2 \pi}{\lambda sh\left(\frac{l\pi K'}{K}\right)}.$$

where $m = 2l; l = 1, 2, \dots$. If $J_i (i = \overline{1, 3})$ is replaced with \hat{J}_i in formula (2.9), the equation of saddle-node bifurcations of the motions outside homoclinic orbit is obtained. Figure 2.3 shows the saddle-node bifurcations curves of subharmonic oscillations outside the homoclinic orbits of orders $m=2; m=4; m=6$ for the system parameters from Section 1. The periodic motions of system (2.6) for $\varepsilon=0$ outside the homoclinic orbit have the form:

Now the saddle-node bifurcations on the plane (δ_L, Γ_t) is considered. We study the limit cycles of the right homoclinic orbit on the system parametric plane. The equation (2.11) is rewritten as:

$$\Gamma_t = \pm \pi^2 \sqrt{\lambda} J_2(k) J_3^{-1}(k) [\delta_L - \delta_L^*(m)], \quad (2.14)$$

where $\delta_L^*(m) = -\delta J_1(k) \pi^{-4} J_2^{-1}(k)$. Following [16], the values $\delta_L^*(m)$ are called the resonance numbers. Figure 2.4 shows qualitatively the bifurcations curves (2.14). As the elliptic integral modulus k satisfies the resonance condition (2.8), the following inequalities are true: $k(2.1) < k(2.2) < \dots < k(\infty) = 1$. Note that the resonance numbers $\delta_L^*(m)$ satisfy the following relations: $\delta_L^*(\infty) = -1.25\delta\pi^{-4}; \lim_{k \rightarrow 0} \delta_L^*(m) = -\delta\pi^{-4}$. From the analysis of the resonance numbers the following inequality is obtained: $\frac{d}{dk} \delta_L^*(k) < 0; k \in [k_1; 1]$. Therefore, integer number m_* can be selected, that the following inequalities are meet: $-1.25\delta\pi^{-4} = \delta_L^*(\infty) < \dots < \delta_L^*(m_* + 1) < \delta_L^*(m_*)$.

The intersections of the invariant manifolds of the saddle point are considered now. It is well known, that these intersections are the necessary condition for the existence of chaos [1, 2]. The homoclinic Melnikov function of beam has the following form:

$$M(t_0) = -\frac{4\sqrt{\lambda}\delta}{3} + \frac{\Gamma_t \pi^3 \Omega^2}{\lambda sh\left(\frac{\pi\Omega}{2\sqrt{\lambda}}\right)} \sin(\Omega t_0) - \frac{16}{15} \delta_L \pi^4 \sqrt{\lambda}. \quad (2.15)$$

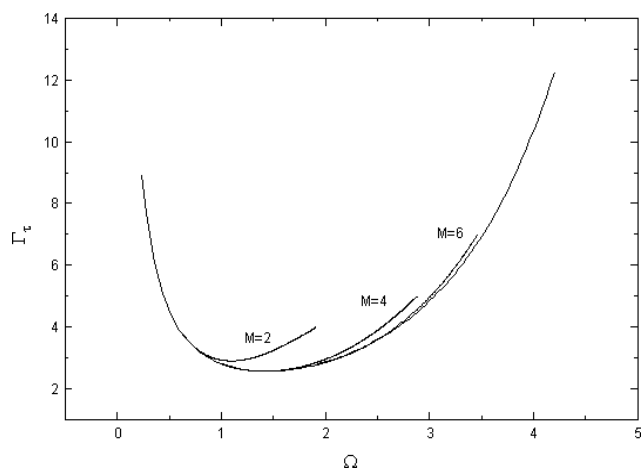


Figure 2.3: The saddle-node bifurcations curves of the subharmonic oscillations of orders $m=2$, $m=4$, $m=6$. The calculations are produced with the following parameters: $\varepsilon = 0.01$; $\varepsilon\delta = \varepsilon\delta_L = 0.18$; $\varepsilon\gamma = 1.84 \cdot 10^{-3}$; $\rho = 3.4$.

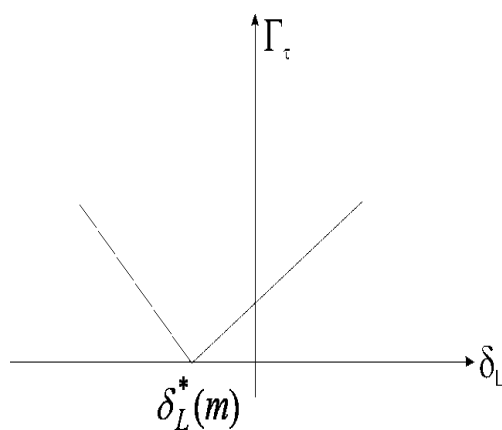


Figure 2.4: The qualitative behavior of the saddle-node bifurcation curve on the plane (δ_L, Γ_t) .

The function M and subharmonic Melnikov function $\bar{M}_1^{m/1}$ satisfy the following limits:

$$\lim_{m \rightarrow \infty} \bar{M}_1^{m/1} = M. \quad (2.16)$$

Thus, the saddle-node bifurcations are obtained. However, the periodic motions, which undergo these bifurcations, are not studied. It is clear that these cycles may undergo others bifurcations. Here the Melnikov–Morozov method, which is considered in [4, 17], is used to study other bifurcations of parametric oscillations of beams.

The system (2.6) with respect to the action-angle coordinates (I, θ) can be written in the next form [17]:

$$\dot{I} = \varepsilon F(I, \theta, t); \dot{\theta} = \Omega_\Sigma(I) + \varepsilon G(I, \theta, t), \quad (2.17)$$

where $\Omega_\Sigma(I)$ is the frequency of the system (2.6) for $\varepsilon=0$. Let us consider the following motions:

$$I = I^{m,1} + \sqrt{\varepsilon} h(t); \theta = \Omega_\Sigma(I^{m,1})t + \phi, \quad (2.18)$$

where the values $I^{m,1}$ are obtained from the resonance conditions (2.8). Following [17], the oscillations $I = I^{m,1} + \sqrt{\varepsilon} h(t)$ are called the motions close to the resonance energetic level. The aim of the present study is an analysis of the topology of the Poincaré sections close to the resonance energetic levels. Then the equations of the motions have the following form [1, 2]:

$$\dot{h} = \frac{\sqrt{\varepsilon}}{2\pi} \bar{M}_1^{m/1} \left(\frac{\bar{\phi}}{\Omega_\Sigma(I^{m,1})} \right) + \varepsilon \bar{F}'_I \bar{h}; \quad \dot{\bar{\phi}} = \sqrt{\varepsilon} \frac{\partial \Omega(I^{m,1})}{\partial I} \bar{h} + \varepsilon \left[\frac{\Omega''(I^{m,1})}{2} \bar{h}^2 + \bar{G}(\bar{\phi}) \right]. \quad (2.19)$$

The system (2.19) can be rewritten in the following form:

$$\begin{aligned} \dot{h} &= \frac{1}{2\pi} \left(-\Delta_1 \pi^4 \sqrt{\lambda} J_2 + \Gamma_t \pi^2 J_3 \sin m\phi \right) + \varepsilon h \left[\chi(\Delta_1) + \Gamma_t \pi^2 K_3 \sin m\phi \right]; \\ \dot{\phi} &= \Omega'_\Sigma h + \sqrt{\varepsilon} \left[\frac{\Omega''_\Sigma}{2} h^2 - \frac{\Gamma_t \pi^2 K_3}{m} \cos m\phi \right], \end{aligned} \quad (2.20)$$

where $\Delta_1 = \delta_L - \delta_L^*$; $\chi = \delta \sqrt{\lambda} \frac{\Omega(2-k^2)^2 \sigma(k)}{60\pi m \lambda k^3 J_2} - \Delta_1 \pi^4 \sqrt{\lambda} K_2$;

$\sigma(k) = 80(2-k^2)E^2(k) - 160k'^2 K(k)E(k) - 32(k^4 + k'^2)K(k)E(k) + 16k'^2(2-k^2)K^2(k)$;

$$K_3 = \frac{\Omega\pi}{\lambda sh \left(\frac{m\pi K'}{K} \right)} \left[\frac{(2-k^2)^3 \Omega^2 \pi}{8\lambda k^4 K^2 k'^2} \operatorname{cth} \left(\frac{m\pi K'}{K} \right) + \omega(k) \right];$$

$$K_2 = \frac{2E\Omega}{\lambda\pi m \sqrt{2-k^2}} + \frac{\omega(k)}{\Omega} J_2; \quad \Omega'_\Sigma = -\frac{\sqrt{\lambda}\pi^2(2-k^2)[(2-k^2)E - 2k'^2 K]}{2K^3 k'^2 k^4};$$

$$\Omega''_\Sigma = -\frac{\sqrt{\lambda}\pi^3(2-k^2)^{5/2}}{4k^8 k'^4} \left[\frac{2E(k'^6 + 3k'^2 + k^4)}{K^4} + \frac{k'^2}{K^3}(4k'^2 - k^4) - \frac{3E^2}{K^5}(2-k^2)^2 \right].$$

The fixed points of the system (20) are the following:

$$(\phi_\nu, h_\nu) = \left(\frac{(-1)^\nu}{m} \arcsin(a) + \frac{\pi\nu}{m}; 0 \right) + O(\varepsilon); \quad \nu \in Z, \quad a = \frac{\sqrt{\lambda}}{\Gamma_t \pi^2 J_3} (\delta J_1 + \delta_L \pi^4 J_2). \quad (2.21)$$

If $a > 0$ ($a < 0$), then ν is changed as: $\nu = 0, \dots, 2m - 1$ ($\nu = 1, \dots, 2m$).

Let us study these fixed points stability. System (20) is linearized and the eigenvalues λ of the constant matrix of the linear system are derived. The values λ of the saddle fixed points are the following:

$$\lambda_{1,2}^{(A)} = \pm \sqrt{\frac{\varepsilon}{2} |\Omega'_\Sigma| \Gamma_t \pi J_3 m \sqrt{1 - a^2} + O(\varepsilon)}. \tag{2.22}$$

The other group of the fixed points is denoted by B. The values λ of these fixed points are the following:

$$\lambda_{1,2}^{(B)} = \frac{1}{2} tr(\tilde{A}) \pm i \sqrt{\frac{\varepsilon}{2} |\Omega'_\Sigma| \Gamma_t \pi J_3 m \sqrt{1 - a^2}}, \tag{2.23}$$

where $tr(\tilde{A})$ is the trace of matrix $[\tilde{A}]$, which meets the following limit:

$$\lim_{k \rightarrow 1} tr(\tilde{A}) = \lim_{k \rightarrow 1} \varepsilon \frac{\sqrt{\lambda} (\delta_L \pi^4 J_2 + \delta J_1)}{2m T k'^2 K(k)}. \tag{2.24}$$

Motions close to the resonance energetic levels have values k near 1. Using (2.24) we conclude that if

$$\delta_L < \delta_L^*(m) \text{ (} \delta_L > \delta_L^*(m) \text{)}, \tag{2.25}$$

the fixed points B are stable (unstable), respectively. From the inequality (2.25) the following parameter is introduced:

$$\alpha(k) = \delta \sqrt{\lambda} \frac{\Omega(2 - k^2)^2 \sigma(k)}{60\pi m \lambda k^3 J_2} + \Delta_1 \pi^4 \sqrt{\lambda} \left(2K_3 \frac{J_2}{J_3} - K_2 \right).$$

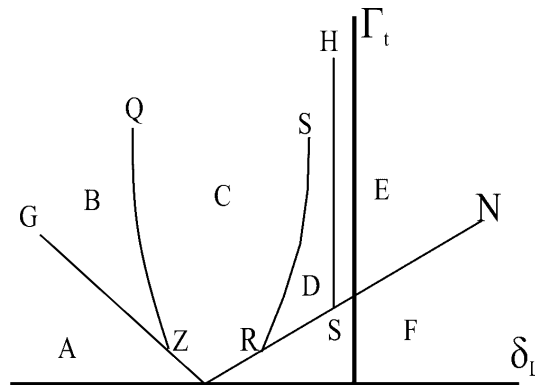


Figure 2.5: The curves of the saddle-node bifurcations and the heteroclinic bifurcations are shown. (QZ) and (RS) are the heteroclinic bifurcations curves. The letters denote the regions of the different dynamical behavior.

Then the inequality (2.25) can be rewritten as $\alpha < 0$ ($\alpha > 0$). Therefore, if $\alpha < 0$ ($\alpha > 0$), the fixed point B is stable (unstable), respectively. Therefore, the bifurcation set

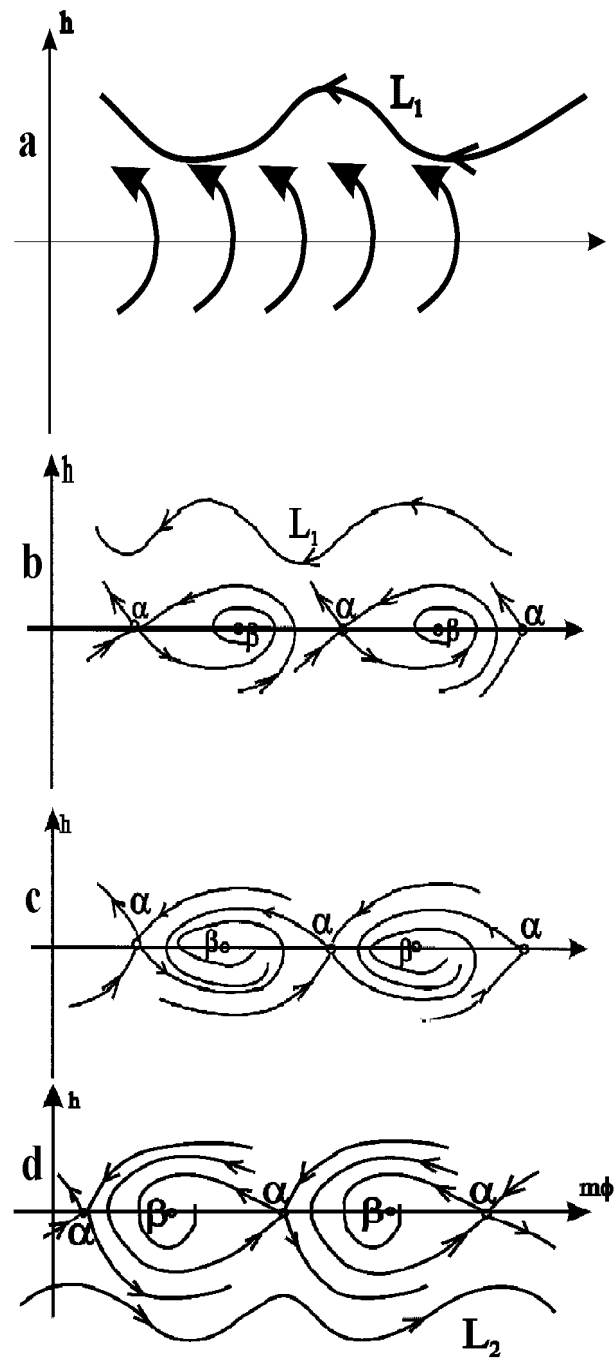


Figure 2.6: The phase portraits of dynamical system.

satisfies the equation $\alpha=0$. This equation describes the bifurcation curve H (Figure 2.5) on the parameter plane $(\delta_L, \Gamma_t) \in R^2$, which can be written as

$$\Delta_1 = \Delta_1^{(H)}(k); \lim_{k \rightarrow 1} \Delta_1^{(H)}(k) = \frac{15\delta}{8\pi^4\sqrt{\lambda}} \left[\frac{\Omega\pi}{\lambda} \operatorname{cth}\left(\frac{\Omega\pi}{2\sqrt{\lambda}}\right) + 1 \right]^{-1} \lim_{k \rightarrow 1} k'^2 K^2. \quad (2.26)$$

The bifurcation behavior of the system (20) is considered. The bifurcation structure on the parametric plane (δ_L, Γ_t) is qualitatively presented in Figure 2.5. In the regions A and B the motions are qualitatively different, as saddle-node bifurcation line (GZ) separates them. The phase trajectories of the region B are shown qualitatively in Figure 2.6b. Here, the saddle fixed point α , the stable fixed point β and stable periodic motions L_1 take place. As a result of the saddle-node bifurcation (GZ) these fixed points are coupled and disappeared. Therefore, there are no fixed points in the region A (Figure 2.5). In this case, only the stable periodic motions L_1 take place (Figure 2.6a). The same bifurcation behavior is observed in $E - F$ region transitions. The saddle-node bifurcation (RN) separates these regions.

Heteroclinic orbits of the system (20) are considered. The following values of δ_1 are chosen:

$$\delta_1 = \delta_*(m) + \sqrt{\varepsilon}\Delta; \quad \Delta = O(2.1). \quad (2.27)$$

The equations (2.27) are substituted into (20) and the Hamiltonian of the system (20) is the following:

$$H = \frac{\sqrt{\varepsilon}}{2} \frac{\partial \omega}{\partial I} \bar{h}^2 + \frac{\sqrt{\varepsilon}\Gamma\pi J_3}{2m} \cos m\bar{\phi}. \quad (2.28)$$

The dynamical system (2.28) has the following fixed points: centers $(\bar{\phi}_\nu, \bar{h}_\nu) = \left(\frac{2\nu}{m}\pi; 0\right); \nu=0; \pm 1 \dots$; and saddles $(\bar{\phi}_\nu, \bar{h}_\nu) = \left(\frac{2\nu+1}{m}\pi; 0\right)$. The heteroclinic orbits joint the saddles fixed points. Taking into account (2.28), the heteroclinic orbits in dissipative dynamical system (20) are calculated by means of the following Melnikov function:

$$\bar{M} = -\frac{\sqrt{\varepsilon}}{2} \frac{\partial \omega}{\partial I} \Delta \pi^3 \sqrt{\lambda} J_2 \int_{-\infty}^{\infty} \bar{h} dt + \sqrt{\varepsilon} \frac{\partial \omega}{\partial I} \chi|_{\Delta=0} \int_{-\infty}^{\infty} \bar{h}^2 dt. \quad (2.29)$$

Integration in the equation (2.29) is performed taking into account the Hamiltonian (2.28). Then the heteroclinic bifurcations take place, if the system parameters satisfy the following equation:

$$\Delta = \pm \frac{4 \chi|_{\Delta=0}}{J_2 \sqrt{\lambda}} \sqrt{\frac{2\Gamma J_3}{m\pi^7 \left|\frac{\partial \omega}{\partial I}\right|}}. \quad (2.30)$$

The heteroclinic bifurcations sets (ZQ) and (RS) are presented qualitatively in Figure 2.5. Let us consider the bifurcations, when the system passes from the region B to the regions C and D . The periodic motions L_1 are observed in the region B (Figure 2.6b). These periodic motions are connected with the heteroclinic orbit. Note, that this heteroclinic trajectory is observed on the bifurcation curve (ZQ) . There are no periodic motions in the region C (Figure 2.6c). The heteroclinic trajectory is observed on the bifurcation curve (RS) . Moreover, the periodic motion L_2 is born from this heteroclinic trajectory. These periodic motions take place in the region D (Figure 2.6d).

The system (20) having the homoclinic trajectory was obtained by the averaging method. This method is applied to nonautonomous systems, which is derived from the

system (2.17) using the change of the variables (2.18). At some system parameters, the homoclinic trajectory of the autonomous system (20) corresponds to the separatrix manifolds intersections of saddle periodic motions in nonautonomous equations. The Smale horseshoes arise due to these intersections in phase space. Such phenomenon for the Duffing–Van-der-Pol oscillator is considered in the paper [16]. The intersections of invariant manifolds in the system (2.6) are not considered in this paper.

3 Domains of Chaotic Frictional Vibrations Under the Action of Almost Periodic Excitation

One degree-of-freedom system (Figure 3.1) is considered in this section. The Duffing oscillator under the action of almost periodic force interacts with moving belt. The vibrations of the discrete mass is described by the general coordinate x . It is assumed, that the belt moves with constant velocity v_* , interacting with oscillator due to the dry friction $f(v_R)$, where v_R is relative velocity of rubbing surfaces. The nonlinear spring is described by the restoring force: $R = cx + c_3x^3$. The system vibrations are excited by the following almost periodic force:

$$p(t) = \Gamma_1 \cos \omega_1 t + \Gamma_2 \cos \omega_2 t.$$

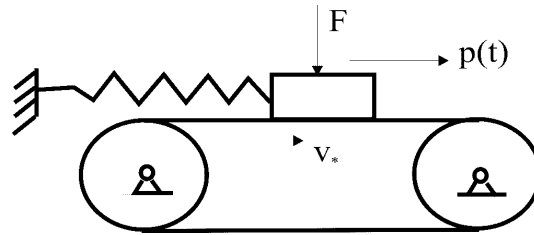


Figure 3.1: The Duffing oscillator interacting with moving belt.

The equation of the system motions has the following form:

$$m\ddot{x} + cx + c_3x^3 = \Gamma_1 \cos \omega_1 t + \Gamma_2 \cos \omega_2 t - f(\dot{x} - v_*); \tag{3.1}$$

$$f(\dot{x} - v_*) = \theta_0 \text{sign}(\dot{x} - v_*) - A(\dot{x} - v_*) + B(\dot{x} - v_*)^3. \tag{3.2}$$

We use the next dimensionless variables and parameters:

$$\begin{aligned} \varepsilon\mu\gamma_2 &= \frac{\Gamma_2}{cx_*}; \alpha = \frac{A\omega_0x_*}{\theta_0}; \beta = \frac{B\omega_0^3x_*^3}{\theta_0}; \\ x &= x_*\xi(t); \tau = \omega_0 t; \Omega_1 = \frac{\omega_1}{\omega_0}; \Omega_2 = \frac{\omega_2}{\omega_0}; \varepsilon\lambda = \frac{c_3x_*^2}{c}; \varepsilon\mu\tilde{\theta} = \frac{\theta_0}{cx_*}; \varepsilon\gamma_1 = \frac{\Gamma_1}{cx_*}, \end{aligned} \tag{3.3}$$

where μ, ε are two independent small parameters: $0 < \varepsilon < \mu < 1$. The mechanical system (3.1) with respect to dimensionless variables and parameters is written in the form:

$$\xi'' + \xi = \varepsilon \left\{ -\lambda\xi^3 + \gamma_1 \cos \Omega_1 \tau + \mu \left[\gamma_2 \cos \Omega_2 \tau - \tilde{\theta}P(\xi' - v_B) \right] \right\}; \tag{3.4}$$

$$P(\xi' - v_B) = \text{sign}(\xi' - v_B) - \alpha(\xi' - v_B) + \beta(\xi' - v_B)^3.$$

The second small parameter μ points, that the friction force is essentially smaller than the nonlinear part of the restoring force and the amplitude of the harmonic γ_1 is significantly larger than the amplitude $\mu \gamma_2$.

In the future analysis the vibrations are treated for the resonance case:

$$\Omega_1 = 1 + \varepsilon\sigma; \quad \Omega_2 = \Omega_1 + \varepsilon \Delta, \tag{3.5}$$

where σ, Δ are two independent detuning parameters. Note, that in the case of the resonance (3.5) the external force is almost periodic. Using the multiple scales method [18], the following system of modulation equations is derived:

$$\begin{aligned} \rho' = & \sqrt{\rho} \frac{\gamma_1}{\sqrt{2}} \sin \theta + \mu \left\{ \rho \tilde{\theta} (\alpha - 3\beta v_B^2) - \tilde{\theta} \alpha_1 \sqrt{2\rho} - \frac{3}{2} \tilde{\theta} \beta \rho^2 + \right. \\ & \left. + \sqrt{\rho} \frac{\gamma_2}{\sqrt{2}} \sin \theta \cos \Delta T_1 + \sqrt{\rho} \frac{\gamma_2}{\sqrt{2}} \cos \theta \sin \Delta T_1 \right\}; \end{aligned} \tag{3.6}$$

$$\theta' = \sigma - \frac{3\lambda}{4} \rho + \frac{\gamma_1}{2\sqrt{2\rho}} \cos \theta + \mu \frac{\gamma_2}{2\sqrt{2\rho}} (\cos \theta \cos \Delta T_1 - \sin \theta \sin \Delta T_1); \tag{3.7}$$

$$\alpha_1(\rho) = \begin{cases} 0; v_B > \sqrt{2\rho}; \\ \frac{2}{\pi} \sqrt{1 - \frac{v_B^2}{2\rho}}; v_B < \sqrt{2\rho}, \end{cases}$$

where $(\cdot)' = \frac{d(\cdot)}{dT_1}$; $T_1 = \varepsilon\tau$. Note, that the dynamical system (36, 37) has small parameter μ .

The general coordinate ξ of the system (3.4) and the modulation variables (ρ, θ) are connected as

$$\xi = \sqrt{2\rho} \cos(\Omega_1\tau - \theta) + O(\varepsilon). \tag{3.8}$$

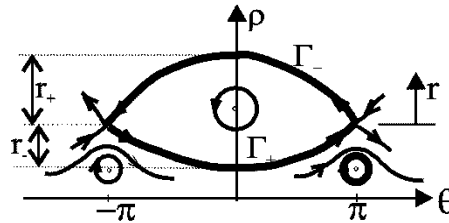


Figure 3.2: The phase portraits of the Hamiltonian system (3.9).

Unperturbed system (36, 37) ($\mu=0$) has the following Hamiltonian:

$$H = -\sqrt{2\rho} \frac{\gamma_1}{2} \cos \theta + \frac{3\lambda}{8} \rho^2 - \sigma\rho. \tag{3.9}$$

The system with Hamiltonian (3.9) has two groups of fixed points (θ_1, ρ_1) and (θ_2, ρ_2) , which satisfy the next cubic equation:

$$\sigma - \frac{3\lambda}{4} \rho_{1,2} \pm \frac{\gamma_1}{2\sqrt{2\rho_{1,2}}} = 0; \theta_1 = 0; \theta_2 = \pm\pi. \tag{3.10}$$

These fixed points are shown in Figure 3.2. They are two types of the points: centers and saddles. The saddle fixed points are connected by heteroclinic orbits $\Gamma_-(\rho_-(T_1); \theta_-(T_1))$ and $\Gamma_+(\rho_+(T_1); \theta_+(T_1))$ (Figure 3.2). The trajectories Γ_- and Γ_+ correspond to upper and lower heteroclinic orbit, respectively. From the equation (3.7) at $\mu=0$, it is derived:

$$\theta_{\pm}(\tau) = \arcsin \left(\frac{\sqrt{2}\rho'_{\pm}}{\gamma_1\sqrt{\rho_{\pm}}} \right).$$

This equation is substituted into (3.6) assuming that $\mu=0$. As a result one has:

$$\rho'^2 = \left(\frac{\gamma_1\sqrt{\rho}}{\sqrt{2}} + \sigma\rho - \frac{3\lambda}{8}\rho^2 + H_s \right) \left(\frac{\gamma_1\sqrt{\rho}}{\sqrt{2}} - \sigma\rho + \frac{3\lambda}{8}\rho^2 - H_s \right). \tag{3.11}$$

The equation (3.11) is solved using the change of the variables

$$\rho(T_1) = \rho_2^{(2.1)} + r(T_1);$$

and the initial conditions

$$\theta_{\pm}(0) = 0; \rho_{\pm}(0) = \rho_2^{(2.1)} + \tilde{r}_{\pm}, \tilde{r}_{\pm} = 2(k \pm \sqrt{2k\rho_2^{(2.1)}}); k = \frac{4\sigma}{3\lambda} - \rho_2^{(2.1)}.$$

As a result it is derived:

$$\rho_{\pm}(T_1) = \rho_2^{(2.1)} \pm \frac{2\tilde{r}_-\tilde{r}_+}{(\tilde{r}_+ - \tilde{r}_-)ch(\tilde{a}T_1) \pm (\tilde{r}_+ + \tilde{r}_-)}, \tag{3.12}$$

where $\tilde{a} = \frac{3\lambda}{8}\sqrt{-\tilde{r}_-\tilde{r}_+}$, $\rho_2^{(2.1)}$ is coordinate of the saddle fixed point.

The intersections of the invariant manifolds take place in the perturbed system (36, 37). The theory of such intersections is treated in books [1,2]. The Smale horseshoe, which is the simplest mathematical pattern of chaotic vibrations, appears due to such heteroclinic structure. The Melnikov function [1] is used to determine the region, where the heteroclinic structure exists. The method for these functions calculations is considered in [1]. Here this approach is used to determine the region of heteroclinic orbits existence in the system of modulation equations (36, 37). The heteroclinic Melnikov function of the system (36, 37) has the following form:

$$\begin{aligned} \tilde{M} = & \int_{-\infty}^{\infty} \left\{ -\frac{\gamma_1\gamma_2}{4} \sin \theta \cos(\Delta t - \Delta t_0 + \theta) + \frac{\gamma_2}{\sqrt{2}}\sqrt{\rho} \sin(\Delta t - \Delta t_0 + \theta) \times \right. \\ & \left. \times \left(\sigma - \frac{3}{4}\lambda\rho + \frac{\gamma_1}{2\sqrt{2\rho}} \cos \theta \right) \right\} dt + \int_{-\infty}^{\infty} P(\rho) \left(\sigma - \frac{3}{4}\lambda\rho + \frac{\gamma_1}{2\sqrt{2\rho}} \cos \theta \right) dt, \end{aligned} \tag{3.13}$$

where $P(\rho) = -\tilde{\theta}\alpha_1\sqrt{2\rho} + \rho\tilde{\theta}(\alpha - 3\beta v_B^2) - \frac{3}{2}\tilde{\theta}\beta\rho^2$. The integrals (3.13) are determined using the heteroclinic trajectories of the system (36, 37) at $\mu=0$. On performing the integration (3.13), the following equations are taken into account: $\rho(T_1) = \rho(-T_1); \theta(T_1) = -\theta(-T_1)$. As a result of the transformations, the Melnikov function is derived in the following form:

$$\begin{aligned} \tilde{M} = & \frac{\gamma_2}{2} \sin(\Delta t_0) \left\{ \int_{-\infty}^{\infty} \left(-\sigma\sqrt{2\rho} \cos \theta + \frac{3\lambda}{4}\rho\sqrt{2\rho} \cos \theta - \frac{\gamma_1}{2} \right) \cos(\Delta t) dt + \right. \\ & \left. + \sigma \int_{-\infty}^{\infty} \sqrt{2\rho} \sin \theta \sin(\Delta t) dt - \frac{3\lambda}{4} \int_{-\infty}^{\infty} \rho\sqrt{2\rho} \sin \theta \sin(\Delta t) dt \right\} + \\ & + \int_{-\infty}^{\infty} P(\rho) \left(\sigma - \frac{3}{4}\lambda\rho + \frac{\gamma_1}{2\sqrt{2\rho}} \cos \theta \right) dt. \end{aligned} \tag{3.14}$$

Conclusively, the Melnikov function is written in the form:

$$\tilde{M}_{\pm}(t_0) = \frac{\gamma_2}{2} A_{\pm} \sin(\Delta t_0) + D_{\pm}. \tag{3.15}$$

The parameter D_{\pm} is derived as:

$$D_{\pm} = \mp (3\beta v_B^2 - \alpha) \tilde{\theta} \frac{16}{3\lambda} \left(\sigma \theta_0^{\pm} \mp \frac{9\lambda}{16} \tilde{\rho} \right) + \tilde{\theta} \beta \left\{ \frac{14\tilde{\rho}\sigma}{\lambda} \mp \theta_0^{\pm} \left[\left(9\rho_2^{(1)} - \frac{4\sigma}{\lambda} \right) \times \right. \right. \\ \left. \left. \times \left(\frac{4\sigma}{\lambda} - 2\rho_2^{(1)} \right) + \frac{7\sigma(\tilde{r}_+ \tilde{r}_-)}{\lambda} \right] \right\} + \tilde{\theta} \frac{9\lambda}{16} J_2^{(\pm)} + \tilde{\theta} \left(\frac{9}{8} \lambda \rho_2^{(1)} - \frac{\sigma}{2} \right) J_1^{(\pm)},$$

where $\theta_0^+ = \theta_0; \theta_0^- = \pi - \theta_0; \theta_0 = \arccos \left(\frac{\tilde{r}_+ + \tilde{r}_-}{r_+ - r_-} \right); \tilde{\rho} = \sqrt{-\tilde{r}_+ \tilde{r}_-}$.

The parameters $J_2^{(\pm)}$ and $J_1^{(\pm)}$ are not presented here for brevity. The values A_{\pm} are determined as:

$$A_{\pm}(\Delta, \lambda, \gamma_1, \sigma) = \frac{9\lambda^2}{16\gamma_1} K_3^{\pm} - \frac{9\lambda}{8\gamma_1} \left(2\sigma - \frac{3}{2} \rho_2^{(1)} \lambda \right) K_2^{\pm} - \\ - \frac{1}{2\gamma_1} \left[\frac{9}{4} \gamma_1 \lambda \sqrt{2\rho_2^{(1)} - \sigma(4\sigma - 3\lambda\rho_2^{(1)})} \right] K_1^{\pm} - \frac{3\lambda}{2\gamma_1} L_1^{\pm} + \left(\frac{2\sigma}{\gamma_1} - \frac{3\lambda}{2\gamma_1} \rho_2^{(1)} \right) L_0^{\pm}, \tag{3.16}$$

where

$$K_n^{\pm} = \int_{-\infty}^{\infty} r_{\pm}^n(t) \cos(\Delta t) dt; L_n^{\pm} = \int_{-\infty}^{\infty} r_{\pm}^n(t) \dot{r}_{\pm}(t) \sin(\Delta t) dt; n = 1, 2, 3. \tag{3.17}$$

The integrals (3.17) satisfy the following equations:

$$L_0 = -\Delta K_1; K_2 = -\frac{2}{\Delta} L_1. \tag{3.18}$$

Values of the integrals are determined using the residuals. As a result, the following parameters are calculated:

$$K_1^{\pm} = \mp \frac{16\pi \operatorname{sh}(\Delta' \theta_0^{\pm})}{3\lambda \operatorname{sh}(\Delta' \pi)}; K_2^{\pm} = \frac{16\pi \tilde{\rho}}{3\lambda} \left[\frac{\Delta' \operatorname{ch}(\Delta' \theta_0^{\pm})}{\operatorname{sh}(\Delta' \pi)} \mp \operatorname{ctg} \theta_0 \frac{\operatorname{sh}(\Delta' \theta_0^{\pm})}{\operatorname{sh}(\Delta' \pi)} \right]; \\ K_3^{\pm} = \mp \frac{8\pi \tilde{\rho}^2}{3\lambda \operatorname{sh}(\Delta' \pi)} \left\{ \operatorname{sh}(\Delta' \theta_0^{\pm}) (1 + 3\operatorname{ctg}^2 \theta_0 + \Delta'^2) \mp 3\Delta' \operatorname{ctg} \theta_0 \operatorname{ch}(\Delta' \theta_0^{\pm}) \right\}; \\ L_0^{\pm} = \pm \frac{16\Delta\pi \operatorname{sh}(\Delta' \theta_0^{\pm})}{3\lambda \operatorname{sh}(\Delta' \pi)}; L_1^{\pm} = -\frac{8\Delta\pi \tilde{\rho}}{3\lambda} \left[\frac{\Delta' \operatorname{ch}(\Delta' \theta_0^{\pm})}{\operatorname{sh}(\Delta' \pi)} \mp \operatorname{ctg} \theta_0 \frac{\operatorname{sh}(\Delta' \theta_0^{\pm})}{\operatorname{sh}(\Delta' \pi)} \right],$$

where $\Delta' = \frac{8\Delta}{3\tilde{\rho}\lambda}$. Then, finally, the value of $A_{\pm}(\Delta, \lambda, \gamma_1, \sigma)$ has the following form:

$$A_{\pm}(\Delta, \lambda, \gamma_1, \sigma) = \operatorname{cosech} \left(\frac{8\Delta\pi}{3\tilde{\rho}\lambda} \right) \left\{ \mp \left[\frac{8\pi\Delta}{3\lambda\gamma_1} \left(\frac{3}{2} \lambda \tilde{\rho} \operatorname{ctg} \theta_0 - 2 \left[2\sigma - \frac{3}{2} \lambda \rho_2^{(1)} \right] \right) \right] + \right. \\ \left. \frac{2\pi}{3\lambda\gamma_1} \left\{ \frac{27}{4} \lambda^2 \tilde{\rho}^2 \operatorname{ctg}^2 \theta_0 - 9\lambda \tilde{\rho} \left(2\sigma - \frac{3}{2} \rho_2^{(1)} \lambda \right) \operatorname{ctg} \theta_0 - 9\gamma_1 \lambda \sqrt{2\rho_2^{(1)} +} \right. \right. \\ \left. \left. + 8\sigma \left(2\sigma - \frac{3}{2} \rho_2^{(1)} \lambda \right) + \frac{9}{4} \lambda^2 \tilde{\rho}^2 \right\} + \frac{\pi \Delta^2 32}{3\gamma_1 \lambda} \right] \operatorname{sh}(\Delta' \theta_0^{\pm}) + \\ \left. + \left[\frac{\Delta^2 \pi 32}{3\gamma_1 \lambda} + \frac{8\pi\Delta}{\lambda\gamma_1} \left\{ \frac{3}{2} \tilde{\rho} \lambda \operatorname{ctg} \theta_0 - 2 \left(2\sigma - \frac{3}{2} \rho_2^{(1)} \lambda \right) \right\} \right] \operatorname{ch}(\Delta' \theta_0^{\pm}) \right\}. \tag{3.19}$$

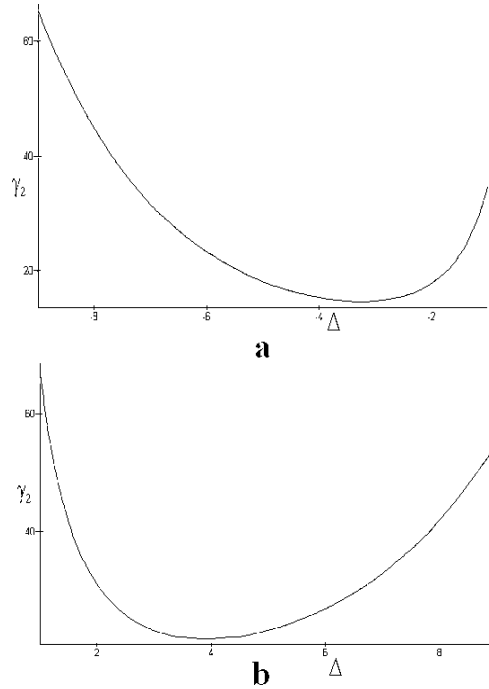


Figure 3.3: The boundaries of the regions of chaotic vibrations.

As shown in the book [1], the intersections of invariant manifolds are described by the simple roots of the equation: $\tilde{M}_{\pm}(t_0) = 0$. The homoclinic structure is observed in the region, where the following inequality is met:

$$|D_{\pm}A_{\pm}^{-1}| < 0.5\gamma_2^{(\pm)}. \quad (3.20)$$

The region of chaotic vibrations (3.20) is studied numerically. The following parameters of the mechanical system (3.1) are used in the future analysis [19]:

$$m = 0.981kg; c = 9.81 \cdot 10^3 \frac{N}{m}; c_3 = 1.67 \cdot 10^3 \frac{N}{m^3}; \Gamma_1 = 100N; \theta_0 = 4.9N;$$

$$A = 0.2 \frac{kg}{s}; B = 3 \cdot 10^{-6} \frac{kg \cdot s}{m^2}.$$

Then dimensionless parameters (3.3) have the following values:

$$\varepsilon = 0.01; \mu = 0.1; \lambda = 17; \tilde{\theta} = 0.5; \gamma_1 = 1.02; \alpha = 4.08; \beta = 0.61; \sigma = 10; \nu_B = 4.$$

Figure 3.3a, b shows boundaries of the chaotic vibrations regions, $\gamma_2^{(+)}(\Delta)$ and $\gamma_2^{(-)}(\Delta)$, for the above-presented system parameters. The heteroclinic structures of the modulation equations (36, 37) take place above these boundaries.

4 Boundary Values Problem for the HT Construction

4.1 Convergence condition

Let us assume that there are local expansions of solution obtained at small and large values of a parameter c (for example, the parameter is an amplitude value or initial energy of the system). For small values of c the local expansion can be determined as a power series in c , while for large values of c it can be determined as a power series in c^{-1} :

$$y^{(0)} = \alpha_0 + \alpha_1 c + \alpha_2 c^2 + \dots, \quad y^{(\infty)} = \beta_0 + \beta_1 c^{-1} + \beta_2 c^{-2} + \dots \quad (4.1)$$

In order to join local expansions (4.1), fractional rational diagonal two-point PA [8] can be used. Let us consider the PA of the form:

$$PA_s = \frac{\sum_{j=0}^s a_j c^j}{\sum_{j=0}^s b_j c^j} = \frac{\sum_{j=0}^s a_j c^{j-s}}{\sum_{j=0}^s b_j c^{j-s}}, \quad s = 1, 2, \dots \quad (4.2)$$

By comparison of expressions (4.1) and (4.2) and retaining only terms with the order of r ($-s \leq r \leq s$), one obtains a system of $2(s+1)$ linear algebraic equations for the determination of coefficients a_j, b_j ($j = 0, \dots, s$). Since generally the determinant of the system Δ_s , is not equal to zero, the system has a single trivial exact solution. But we need in PA corresponding to the retaining terms in Eq. (2.1) having non-zero coefficients a_j, b_j . Without loss of generality it can be assumed that $b_0 = 1$. Now, the system of algebraic equations for determination of a_j, b_j becomes overdetermined. All of the unknown coefficients can be determined from $(2s+1)$ equations while the “residual” of this approximate solution can be obtained by substitution of all the coefficients into the remaining equation. Obviously, the residual (or “error”) is determined by the value of Δ_s (it can be proved), since the non-zero solution for coefficients and consequently PA will be obtained in the given approximation by c only in the case when $\Delta_s = 0$. Hence the following is a necessary condition for convergence of the succession of PA_s in the form (2.2) at $s \rightarrow \infty$ to the fractional rational function which gives us a presentation of the solution for all values of the parameter c [10-12]. Namely,

$$\lim_{s \rightarrow \infty} \Delta_s^{(i)} = 0 \quad (i = 2, 3, \dots, n). \quad (4.3)$$

It is possible to generalize the necessary condition for convergence (4.3) to quasi- PA_s which contain both powers of some unknown parameter, and exponential functions [9]. Besides, it is possible to utilize the condition (4.3) for obtaining some unknown parameters which are contained in local expansions [20].

4.2 Potentiality condition and condition at infinity

It is assumed that along the closed HT the dynamical system energy is saved in average. For the single-DOF dynamical system of the form,

$$\ddot{x} + f(x, \dot{x}, t) = 0,$$

Multiplying the last equation by $\dot{x}(t)$ and integrating within limits from $t = 0$ to $t = \pm\infty$, (or from $t = -\infty$ to $t = +\infty$) along the HT one has the following:

$$\oint f(x, \dot{x}, t) \dot{x} dt = 0. \quad (4.4)$$

Note that such condition is used in many problems of the perturbation theory [18] to construct periodic solutions. In this case the integration is made by the solution period. This condition for periodic solution is called the periodicity one.

Additionally we are going to find the solution $x(t)$ as an analytic function along the HT satisfying the next condition:

$$(x(t), \dot{x}(t)) \xrightarrow{t \rightarrow \pm\infty} (b_0, 0). \quad (4.5)$$

That is we suppose that HT tends to the equilibrium point (saddle point) at infinity.

The convergence condition (4.3), and conditions presented in this subsection, permit to solve uniquely the boundary-value problem for the HT. It is possible to construct both this trajectory and the corresponding solution in time.

5 Non-autonomous Duffing Equation

5.1 Analytical construction of the homoclinic trajectory

One considers in details the construction of HT for the well-known non-autonomous Duffing equation. In general case this equation has a form

$$\ddot{y} + \delta \dot{y} - \beta y + \alpha y^3 = f \cos \omega t, \quad (5.1)$$

A lot of papers are devoted to the investigation of this equation and systems described by it [1, 4-7, 22-24]. Chaotic behavior of solutions can be observed at different choices of elastic characteristics, namely for soft elasticity ($\beta < 0$, $\alpha < 0$) [21], rigid one ($\beta < 0$, $\alpha > 0$) [22], with zero ($k = 0$, $\gamma > 0$) [23] or negative ($\beta > 0$, $\alpha > 0$) [5, 24] linear elasticity.

Here the last variant, namely, $\beta > 0$, $\alpha > 0$, $\delta > 0$, $\delta \ll 1$, $f \ll 1$, is considered. In this case the unperturbed system has three equilibrium positions, namely one unstable saddle point $(0, 0)$ and two stable nodes $(\pm \sqrt{\beta/\alpha}, 0)$. To simplify notations let us do the change of variables $y = \lambda x$, $t = \mu \tau$ to make coefficients of x and x^3 equal to -1 and 1, correspondingly. Then equation (2.1) can be rewritten as

$$y'' + \delta y' - y + y^3 = f \cos \omega t. \quad (5.2)$$

A problem of effective analytic approximation of HT in non-autonomous system is not solved up to now. Here PA and QRA [8, 9] are used for the HT and the corresponding time solution construction in the case of small dissipation.

To construct the HT in this system we should determine values of the system parameters δ , ω , f , corresponding to this trajectory and the coordinates of the shifted saddle point $(b_0, 0)$. The coordinates of the initial point for this trajectory (a_0, a_1) are also required. Thus we have to construct system of four equations to find unknowns. The condition (4.5) at infinity will be used.

Thus, we can consider the next expansion of the solution of equation (5.2) in Taylor series at zero:

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \dots, \quad (5.3)$$

where a_0, a_1 are arbitrary constants. After substitution of (5.3) to the equation (5.2) and equating the coefficients at equal powers of variable t , we get the following expressions

for series coefficients:

$$\begin{aligned} a_2 &= -(a_0^3 - a_0 - f + \delta a_1)/2, & a_3 &= (a_1 - 3a_0^2 a_1 - 2\delta a_2)/6, \\ a_4 &= (2a_2 - f\omega^2 - 6\delta a_3 - 6a_1^2 a_0 - 6a_2 a_0^2)/24, & \dots \end{aligned}$$

Multiplying the equation (5.2) by $y'(t)$ and integrating within limits from $t = 0$ to infinity along the homoclinic trajectory, we have the following:

$$-\frac{b_0^2}{2} + \frac{a_0^2}{2} + \frac{b_0^4}{4} - \frac{a_0^4}{4} - \frac{a_1^2}{2} + \int_0^{\pm\infty} (\delta y' - f \cos \omega t) y' dt = 0. \tag{5.4}$$

Let us consider the integral $\int_0^T (\delta y' - f \cos \omega t) y' dt$. After substitution instead of $y(t)$ its Taylor series and integration one obtains:

$$\int_0^T (\delta y' - f \cos \omega t) y' dt = A T + B T^2 + C T^3 + D T^4 + E T^5 + \dots, \tag{5.5}$$

where

$$\begin{aligned} A &= (\delta a_1 - f)a_1, & B &= (2(\delta a_1 - f)a_2 + 2\delta a_2 a_1)/2, \\ C &= (3(\delta a_1 - f)a_3 + 4\delta a_2^2 + (f\omega^2/2 + 3\delta a_3)a_1)/3, \\ D &= (4(\delta a_1 - f)a_4 + 4\delta a_4 a_1 + 6\delta a_2 a_3 + 2(f\omega^2/2 + 3\delta a_3)a_2)/4, \dots \end{aligned}$$

It is desirable to get presentation of the integral at infinity. For this the QPA is used as a form of analytical continuation of the expansion (5.5):

$$A T + B T^2 + C T^3 + \dots \rightarrow P A_3^p = \frac{\alpha_1 T + \alpha_2 T^2 + \alpha_3 T^3}{1 + \beta_1 T + \beta_2 T^2 + \beta_3 T^3}. \tag{5.6}$$

From here one has the following:

$$\begin{aligned} \alpha_1 &= A, \\ \alpha_2 &= -\frac{-ADC^2 - DA^2E + 2AD^2B + FA^2C - AFB^2 + B^3E - 2B^2DC + BC^3}{AEC - AD^2 - B^2E + 2BDC - C^3}, \\ \alpha_3 &= (-A^2E^2 + 2AEC^2 + 2AE BD - 2ACD^2 - 2ACBF + DFA^2 \\ &\quad - 2CB^2E + 3BDC^2 - D^2B^2 + FB^3 - C^4)/(AEC - AD^2 - B^2E + 2BDC - C^3), \\ \beta_2 &= -\frac{AE^2 - EC^2 - EBD + CD^2 + CBF - DFA}{AEC - AD^2 - B^2E + 2BDC - C^3}, \\ \beta_3 &= \frac{-BDF + BE^2 - 2DCE + C^2F + D^3}{AEC - AD^2 - B^2E + 2BDC - C^3}. \end{aligned}$$

Passing on to infinity in the fractional presentation (5.6), we can rewrite the equation (5.4) as:

$$-\frac{b_0^2}{2} + \frac{a_0^2}{2} + \frac{b_0^4}{4} - \frac{a_0^4}{4} - \frac{a_1^2}{2} + \frac{\alpha_3}{\beta_3} = 0. \tag{5.7}$$

Additional equation could be obtained from the convergence condition for the PA (5.6):

$$-2D^2BF + 2DBE^2 - 3D^2CE + 2DC^2F + D^4 - AE^3 + E^2C^2 - 2ECBF + 2EDFA - F^2AC + F^2B^2 + GAEC - GAD^2 - GB^2E + 2GBDC - GC^3 = 0. \tag{5.8}$$

In similar way we can construct the analytical continuation of the solution at infinity by means of quasi-rational approximation. This QPA is chosen in the form similar to the solution of autonomous Duffing equation (separatrix solution), namely:

$$y = a_0 + a_1t + a_2t^2 + a_3t^3 + \dots \rightarrow e^{-t} \frac{\alpha_0 + \alpha_1e^t + \alpha_2e^{2t} + \alpha_3e^{3t}}{1 + \beta_2e^{2t}}. \tag{5.9}$$

It follows from (5.9) that

$$b_0 = \frac{\alpha_3}{\beta_2}, \tag{5.10}$$

where coefficients α_i, β_j in (5.9) can be found as it is described below.

Final equation being the convergence condition for the approximation (5.9) is

$$24a_5a_3 + 2a_5a_1 + 12a_5a_2 - 24a_4^2 - 4a_1a_4 - \frac{4}{15}a_1a_2 - \frac{7}{10}a_1a_3 - \frac{1}{10}a_1^2 - 8a_2a_4 + \frac{5}{6}a_2^2 + 4a_2a_3 + 6a_3^2 - 12a_4a_3 = 0. \tag{5.11}$$

The system of algebraic equations (5.7), (5.8), (5.10) and (5.11) determines the unknown values a_0, a_1, b_0 and $f = f(\delta)$ while ω is fixed. They can be obtained from the essentially nonlinear system by means of the Newton method. Several examples of obtained phase trajectories are presented in Figure 5.1 and Figure 5.2. Figure 5.1 shows trajectories constructed by Runge–Kutta procedure with initial points obtained from the system. Here two sets of parameters are chosen, namely:

- a) $\delta = 0.001, a_0 = 1.21508, a_1 = 0.621819, b_0 = 0.00058, f = 0.00087; \omega = 1;$
- b) $\delta = 0.01, a_0 = 1.21609, a_1 = 0.621943, b_0 = 0.0058, f = 0.00878, \omega = 1.$

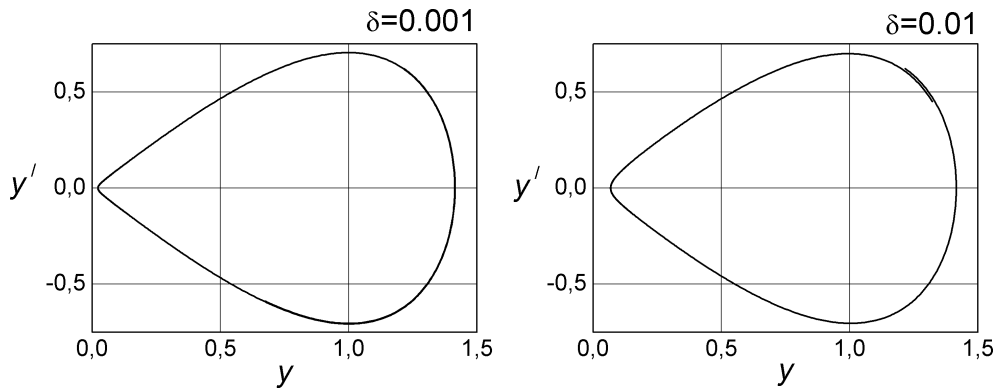


Figure 5.1: Trajectories constructed by Runge–Kutta procedure with initial points obtained from the system (5.7), (5.8), (5.10) and (5.11).

Figure 5.2 gives comparison of trajectories constructed with the same initial point obtained from the system but in different ways, namely by means of Runge–Kutta method

(solid line) and by the QPA (5.9) (dash line). Values of parameters are taken as for the previous figure.

Figure 5.3 and Figure 5.4 present parameter dependences corresponding to the HT creation, i.e. chaos onset. The solid lines show curves obtained by the proposed approach, but dash lines show the same curves obtained by the Melnikov method. Numerical investigation of chaos onset in the system under consideration shows that our curve is more exact.

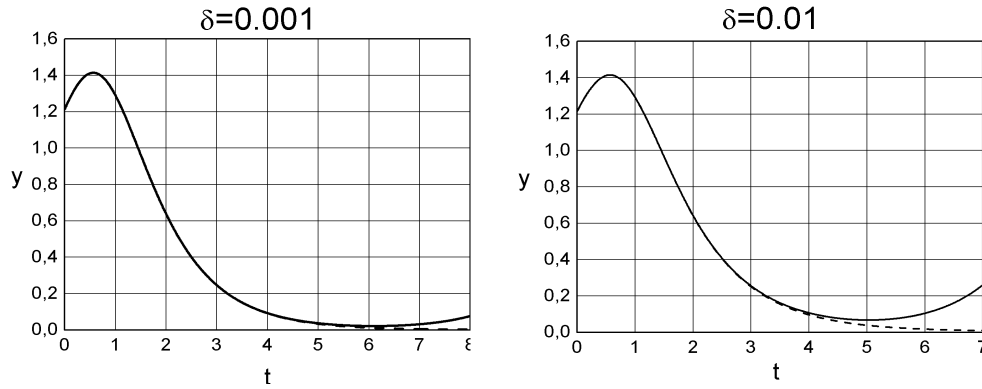


Figure 5.2: Comparison of the trajectories constructed with the obtained initial point by Runge–Kutta method (solid line) and by means of quasi-rational approximation (dash line).

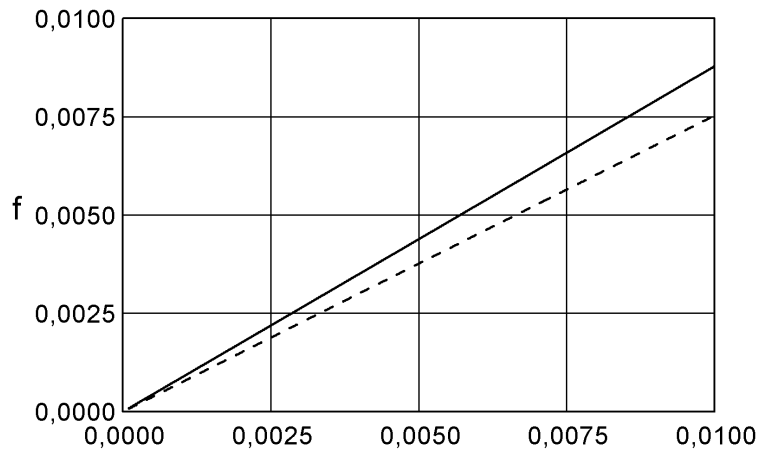


Figure 5.3: Dependence between the amplitude of external force and dissipation coefficient for $\omega = 1$.

Introduction of the phase φ permits to choose the point $(a_0, 0)$ as the HT initial point instead of such point (a_0, a_1) as was made earlier. The corresponding HT construction is not presented here, and it can be found in [25].

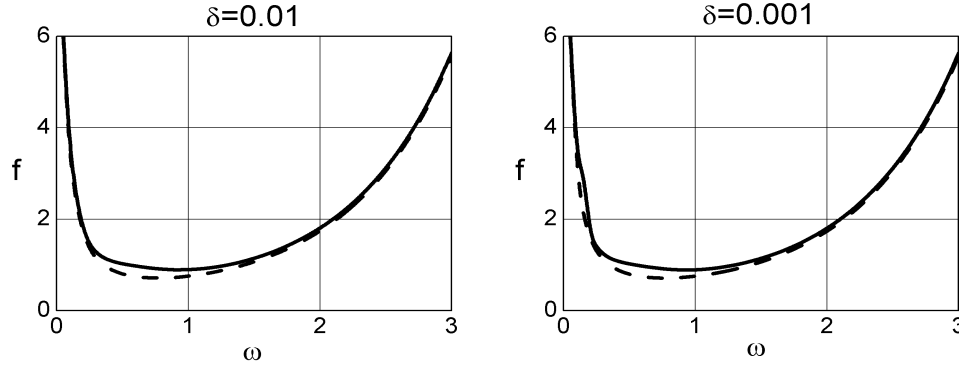


Figure 5.4: Dependence (ω, f) corresponding to HT appearance.

5.2 Comparison of analytical construction and numerical simulation

The obtained analytical results can be compared with numerical simulation. The numerical construction of the manifolds was fulfilled to find the moment of the separatrix branches touching, which corresponds to the chaos onset. The method of Latte [26] was used for this. The main idea of this approach is to consider quadratic approximation of manifolds:

$$y - y_0 = \alpha(x - x_0) = \alpha_1^\pm (x - x_0) + \frac{\alpha_2^\pm}{2} (x - x_0)^2 + O(|x - x_0|^3),$$

where “+” corresponds to unstable manifold but “-” corresponds to stable one. Here (x_0, y_0) is saddle point in phase space. Figure 5.5 and Figure 5.6 present the fulfilled investigation and demonstrate the accuracy of the obtained above analytic results (values of parameters corresponding to manifold touching are the same as obtained above analytically).

Value of the force amplitude corresponding to a point of the HT formation obtained by the analytical approach is equal to 0.004465 for some fixed values of ω and δ . The same result is observed in Figure 5.5.

Figure 5.6 presents phase portraits when $\omega = 2$, $\delta = 0.001$ and $\delta = 0.01$. Corresponding analytic results are $f = 0.0018$ and $f = 0.018$.

6 Construction of the HT in Different Dynamical Systems

6.1 The Van der Pol–Duffing equation

One considers the model which describes, in particular, the panel flutter in the supersonic air flow [25]:

$$\ddot{x} + \delta(\alpha - \beta x^2)\dot{x} - x + x^3 = 0, \quad (6.1)$$

where $\alpha, \beta > 0$, δ is the small parameter ($0 < \delta \ll 1$).

To construct the HT, the procedure presented in the previous Section, is used here. At first, local expansions near the unstable equilibrium point are selected. These expansions, corresponding to stable and unstable branches, can be obtained by using the small

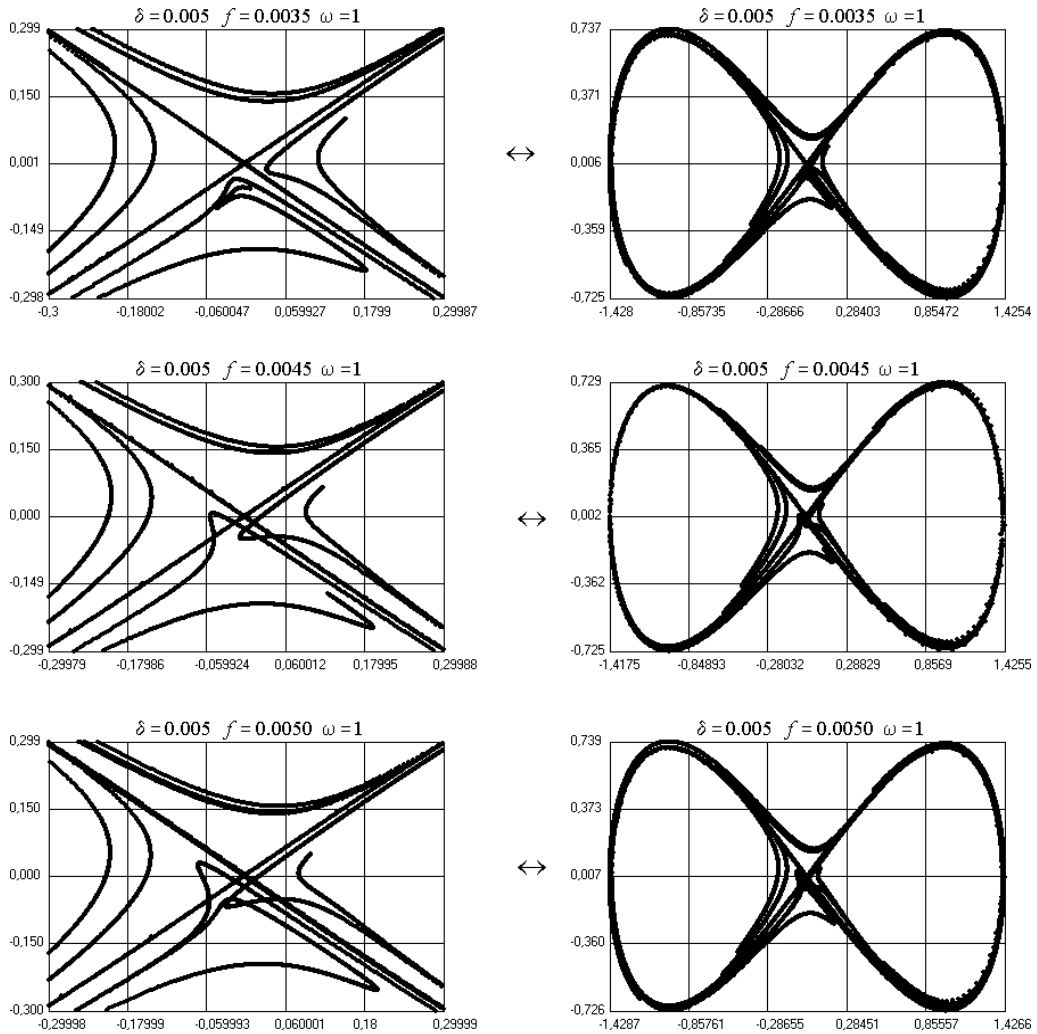


Figure 5.5: Phase portraits for Duffing equation when $\delta = 0.005$, $\omega = 1$.

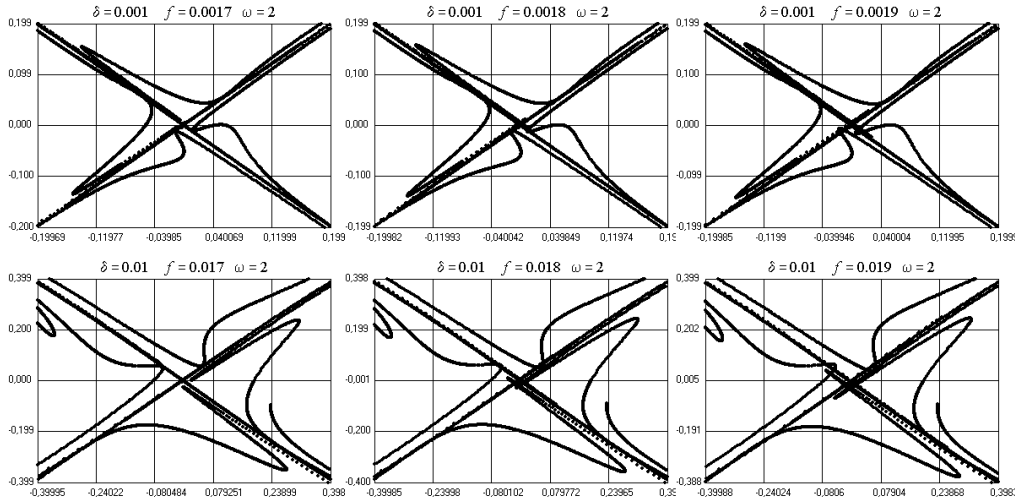


Figure 5.6: Phase portraits of Duffing system in the vicinity of the saddle point.

parameter method:

$$x = c_0 e^{k_1 t} - c_0^3 \frac{1 - k_1 \delta \beta}{9k_1^2 + 3\delta \alpha k_1 - 1} e^{3k_1 t} + \dots, \quad t \rightarrow +\infty; \tag{6.2}$$

$$x = c_1 e^{k_2 t} - c_1^3 \frac{1 - k_2 \delta \beta}{9k_2^2 + 3\delta \alpha k_2 - 1} e^{3k_2 t} + \dots, \quad t \rightarrow -\infty, \tag{6.3}$$

here $k_1 = \frac{-\delta \alpha - \sqrt{\delta^2 \alpha^2 + 4}}{2}$, $k_2 = \frac{-\delta \alpha + \sqrt{\delta^2 \alpha^2 + 4}}{2}$ are roots of the characteristic equation $k^2 + \delta \alpha k - 1 = 0$, and c_0, c_1 are arbitrary constants. One writes too the Taylor series for a solution $x(t)$ at point $t = 0$:

$$x = a_0 + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \dots, \tag{6.4}$$

where a_0 is an arbitrary constant, $a_2 = \frac{a_0 - a_0^3}{2}$, $a_3 = -\frac{\delta(-\alpha + \beta a_0^2) a_0 (a_0^2 - 1)}{6}, \dots$

Thus, to construct the HT it is necessary to find values of the three pointed out arbitrary constants, c_0, c_1, a_0 . Respectively, three algebraic equations to obtain these constants must be constructed.

Multiplying the equation (6.2) by $\dot{x}(t)$ and integrating within limits from $t = 0$ to infinity along the homoclinic trajectory, we have the following:

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \delta \int_0^{\pm\infty} (\alpha - \beta x^2) \dot{x}^2 dt = 0.$$

Using in this integral the local expansion (6.4), and rebuilding the obtained expression to the Pade approximation, we can write the following:

$$\int_0^t (\alpha - \beta x^2) \dot{x}^2 dt = At^3 + Bt^4 + Ct^5 + \dots = \frac{\alpha_3 t^3 + \alpha_4 t^4}{1 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4},$$

One has in the limit at $t \rightarrow \pm\infty$:

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \delta \frac{\alpha_4}{\beta_4} = 0. \tag{6.5}$$

Taking into account the local expansions at infinity (6.2)–(6.3), it is possible to match these expansions with the expansion (6.4) by using the QPA of the form

$$P_{+\infty} = e^{k_1 t} \frac{\alpha_0 + \alpha_2 e^{2k_1 t} + \alpha_4 e^{4k_1 t}}{1 + \beta_2 e^{2k_1 t} + \beta_4 e^{4k_1 t}}, \quad P_{-\infty} = e^{k_2 t} \frac{\alpha_0 + \alpha_2 e^{2k_2 t} + \alpha_4 e^{4k_2 t}}{1 + \beta_2 e^{2k_2 t} + \beta_4 e^{4k_2 t}}, \tag{6.6}$$

where coefficients of approximations $P_{+\infty}$, $P_{-\infty}$ are calculated by comparing them with the expansions (6.2), (6.4)–(6.3), (6.4), respectively.

So, there are two solution presentations for positive and for negative values of the variable t , and we can obtain two additional equations which are the convergence conditions (4.3) for approximations $P_{+\infty}$, $P_{-\infty}$. These equations together with the condition of potentiality (6.5) form a system of nonlinear algebraic equations to determine unknown constants presented in local expansions.

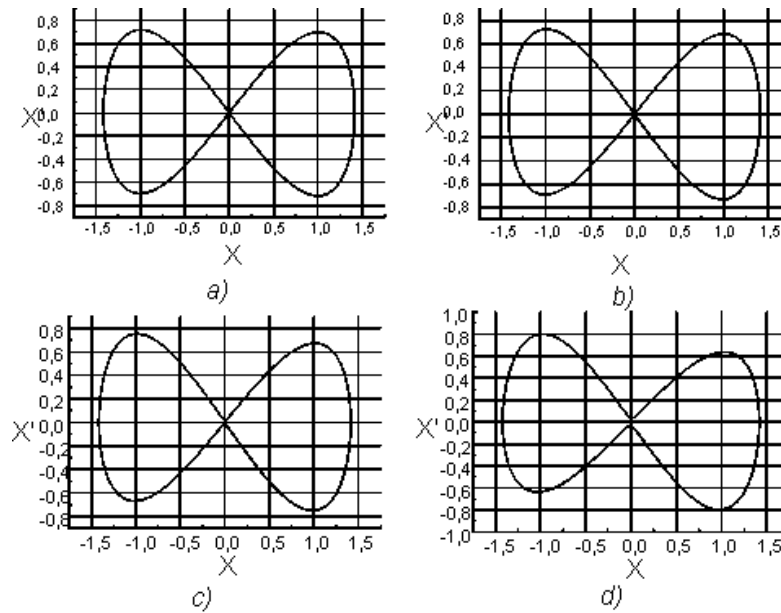


Figure 6.1:

In Figure 6.1 the examples of HT determined by the Runge–Kutta method are shown for different values of the parameter δ , namely: a) $\delta = 0.05$; b) $\delta = 0.1$; c) $\delta = 0.2$; d) $\delta = 0.4$, where the initial values determined from the algebraic equations are used. Figure 6.2 presents a comparison of the HT, obtained by the Runge–Kutta method (line a) and the QPA (6.6) (i.e. $P_{-\infty}$ (line b) and $P_{+\infty}$ (line c) for $\delta = 0.01$.

6.2 Equation of a parametrically excited damped pendulum

Let us use the same technique to investigate the behavior of pendulum with periodically excited point of pendulum suspension [27]. This system is governed by the following

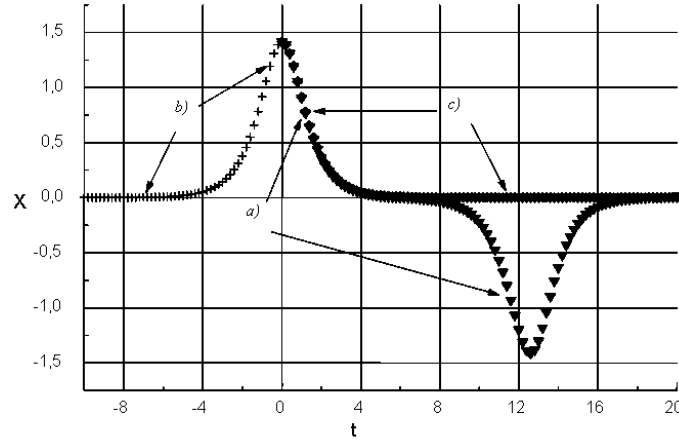


Figure 6.2:

equation:

$$x'' + \delta x' + (1 + f \cos \omega t) \sin x = 0, \quad (6.7)$$

where x is an angle of deviation from the vertical line. We rewrite this equation (6.7) in the form

$$y'' + \delta y' - (1 + f \cos \omega t) \sin y = 0, \quad (6.8)$$

after change $y = x + \pi$. This system possesses infinite number of saddle points $(2\pi n, 0)$ ($n \in \mathbb{Z}$), therefore we will consider heteroclinic trajectory construction as criterion of chaos onset. We make the same assumptions as for Duffing equation and obtain the following:

$$\begin{aligned} & \int_0^{\infty} (y'' + \delta y' - (1 + f \cos \omega t) \sin y) y' dt = \\ & = -\frac{a_1^2}{2} + \cos(b_0) - \cos(a_0) + \int_0^{\infty} (\delta y' - f \cos \omega t \sin y) y' dt = 0. \end{aligned}$$

$$\int_0^{\infty} (\delta y' - f \cos \omega t \sin y) y' dt = (A t + B t^2 + C t^3 + D t^4 + E t^5 + \dots) \Big|_0^{\infty},$$

where $A = a_1 (\delta a_1 - f \sin a_0)$, $B = a_1 (2\delta a_2 - f a_1 \cos a_0)/2 + a_2 (\delta a_1 - f \sin a_0), \dots$

For analytic continuation we use the quasi-rational approximation:

$$A t + B t^2 + C t^3 + D t^4 + E t^5 + \dots \rightarrow \frac{\alpha_1 t + \alpha_2 t^2}{1 + \beta_1 t + \beta_2 t^2}, \quad (6.9)$$

where $\alpha_1 = A$, $\alpha_2 = (-2ABC + DA_2 + B_3)/(B_2 - AC)$, $\beta_1 = (DA - BC)/(B_2 - AC)$, $\beta_2 = (BD - C_2)/(AC - B_2)$.

Thus we have:

$$-\frac{a_1^2}{2} + \cos(b_0) - \cos(a_0) + \frac{\alpha_2}{\beta_2} = 0. \quad (6.10)$$

To simplify the problem we accept $b_0 = 0$. Additionally we continue the Taylor series of solution in quasi-rational approximation:

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots \rightarrow e^{-t} \frac{\alpha_0 + \alpha_1 e^t + \alpha_2 e^{2t}}{1 + \beta_2 e^{2t}}. \tag{6.11}$$

The following two algebraic equations are obtained from existence condition for (76) and (78):

$$2CBD - C^3 - D^2A + EAC - EB^2 = 0, \tag{6.12}$$

$$6a_1a_2 - 72a_2a_3 + 42a_1^2 - 30a_1a_3 + 3a_0a_1 - 18a_0a_3 + 36a_4a_0 + 72a_4a_1 + 72a_4a_2 - 72a_3^2 - 3a_2a_0 - 6a_2^2 = 0. \tag{6.13}$$

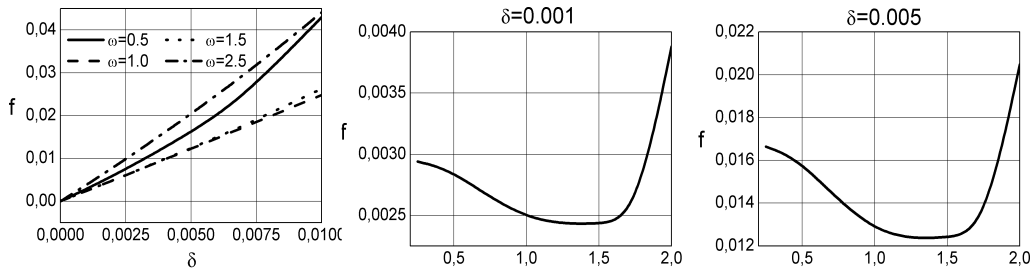


Figure 6.3: Dependences between parameters corresponding to HT creation.

Equations (6.10), (79), (80) form the system of nonlinear algebraic equations to determine parameters of (6.8) and a_0, a_1 while ω is fixed. Figure 6.3 demonstrates the dependences between parameters obtained from constructed system.

At values of force amplitude f less than 0.2, the instability domain is observed only in vicinity of heteroclinic trajectory but at rising of f the domain enlarges as well.

Figure 6.4 demonstrates the results of manifolds construction for the system (6.8) for $\delta = 0.001$ and $\omega = 1$. Figure 6.5 presents the same construction for $\delta = 0.001$ and $\omega = 2$.

Analytic results obtained above by proposed approach are $f = 0.00243$ ($\omega = 1$) and $f = 0.0038$ ($\omega = 2$). So, comparing the obtained analytical results with the phase portraits we can observe a good accuracy of the analytical results and efficiency of the proposed approach.

Similar equations with parametric periodic excitation can be obtained in a problem of the elastic oscillations absorption by using the snap-through truss as absorber. It was shown that the snap-through truss can be used for effective absorption of longitudinal oscillations of some elastic solid [28]. In this case a big part of the energy of elastic oscillations is transferred to the truss, which has a capacity to jump. But it was shown too [29] that the chaotic behavior, which is not appropriate for this absorption, can appear in this system.

7 The One-degree-of-freedom Weakly Forced Oscillator with Nonlinear Dissipation Forces

Mechanical system with a small periodic external excitation, nonlinear dissipation forces and the Duffing type stiffness is governed by the following second order differential equa-

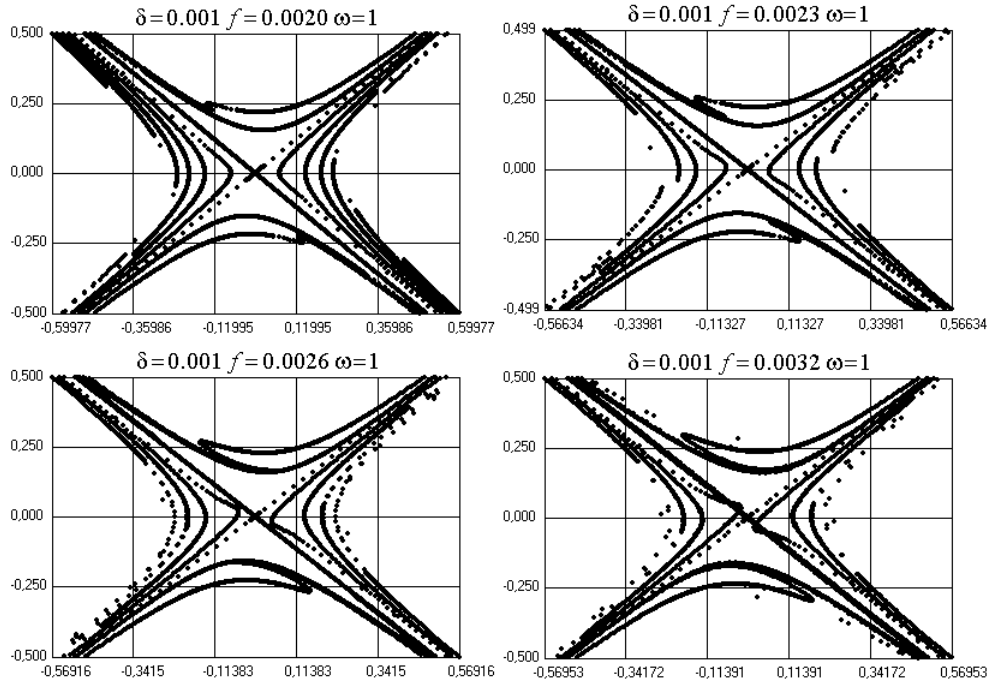


Figure 6.4: Phase portrait of (6.8) in the vicinity of saddle point $(0, 0)$ for $\omega = 1$.

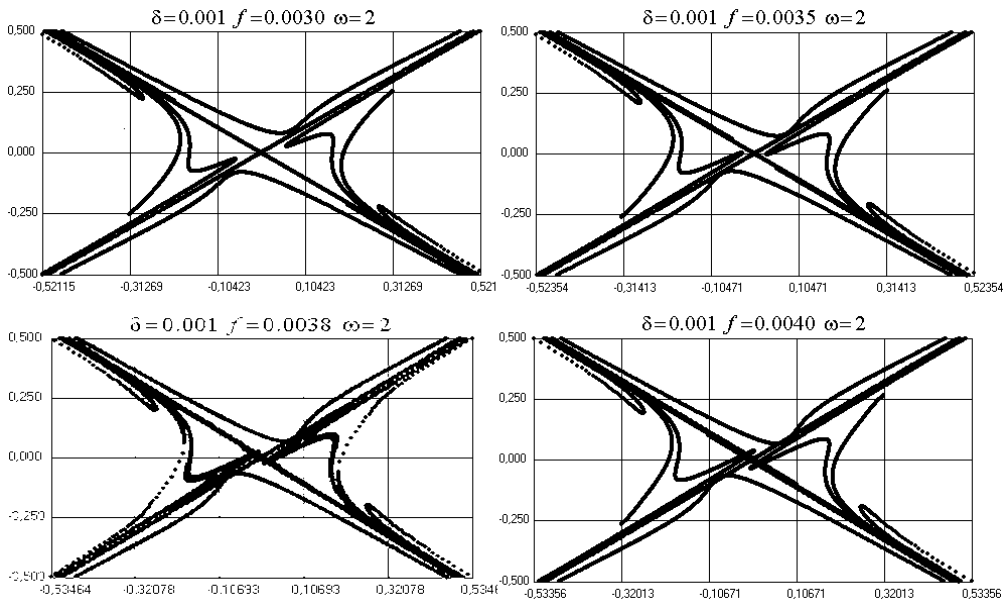


Figure 6.5: Phase portrait of (2.15) in the vicinity of saddle point $(0, 0)$ for $\omega = 2$.

tion:

$$y'' - y + y^3 = f \cos(\omega t + \varphi) - \theta(y' - \nu^*), \tag{7.1}$$

where $\theta(y' - \nu^*) = T_0 \text{sign}(y' - \nu^*) - \alpha(y' - \nu^*) + \beta(y' - \nu^*)^3$ is the nonlinear dissipation characteristic.

To construct a homoclinic trajectory we need to know the initial point $(a_0, 0)$, the phase ϕ corresponding to a moment $t = 0$, and the relation of the system parameters ω , f and θ corresponding to HT appearing. Thus we should construct the algebraic system to determine the unknown values.

Let us make some assumption like for the previous systems. One assumes that $(y, y') \xrightarrow{t \rightarrow \pm\infty} (0, 0)$. We will construct the analytical approximation for the sought solution. First, we can consider the Taylor expansion at zero of the solution $y(t)$:

$$y = a_0 + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \dots, \tag{7.2}$$

where a_0 is an arbitrary constant, and $a_j = a_j(a_0, \varphi, f, T_0, \alpha, \beta)$ ($j = \overline{2, \infty}$)

Then multiplying the equation (7.1) by $y'(t)$ and integrating within the limits from $t = 0$ to $t = +\infty$ and from $t = 0$ to $t = -\infty$, one has the following equations where several integrals are calculated along the separatrix zero approximation (for $\theta = 0, f = 0$), $y_0 = \sqrt{2}/ch(t)$:

$$\begin{aligned} \frac{a_0^2}{2} - \frac{a_0^4}{4} - (\alpha\nu^* - \beta\nu^{*3} - T_0) a_0 - \frac{2\alpha}{3} + \frac{8\beta}{35} + \frac{4\sqrt{2}\beta\nu^*}{5} + 2\beta\nu^{*2} + \\ + f \sin \varphi \int_0^{+\infty} \sin \omega t y'_0 dt - f \cos \varphi \int_0^{+\infty} \cos \omega t y'_0 dt = 0; \end{aligned} \tag{7.3}$$

$$\begin{aligned} \frac{a_0^2}{2} - \frac{a_0^4}{4} - (\alpha\nu^* - \beta\nu^{*3}) a_0 + \frac{2\alpha}{3} - \frac{8\beta}{35} + \frac{4\sqrt{2}\beta\nu^*}{5} - 2\beta\nu^{*2} - f \sin \varphi \int_0^{+\infty} \sin \omega t y'_0 dt - \\ - f \cos \varphi \int_0^{+\infty} \cos \omega t y'_0 dt - T_0 \int_{-\infty}^0 \text{sign}(y'_0 - \nu^*) y'_0 dt = 0. \end{aligned} \tag{7.4}$$

Here

$$\int_0^{+\infty} \sin \omega t y'_0 dt = \int_{-\infty}^0 \sin \omega t y'_0 dt = -\frac{\omega\sqrt{2}\pi}{2} \cdot \frac{1}{ch\frac{\omega\pi}{2}};$$

$$\int_0^{+\infty} \cos \omega t y'_0 dt = - \int_{-\infty}^0 \cos \omega t y'_0 dt = -\sqrt{2} + \omega\sqrt{2} \left(-\frac{\pi}{2} th \frac{\omega\pi}{2} + 4\omega \sum_{k=0}^{\infty} \frac{1}{\omega^2 + (1 + 4k)^2} \right).$$

The integral $\int_{-\infty}^0 \text{sign}(y'_0 - \nu^*) y'_0 dt$ is evaluated as a function of the parameter ν^* computationally.

For the continuation of the local expansion at infinitum we rebuild it to QPA:

$$y = a_0 + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \dots \rightarrow e^{-t} \frac{\alpha_0 + \alpha_1 e^t + \alpha_2 e^{2t}}{1 + \beta_1 e^t + \beta_2 e^{2t}}. \tag{7.5}$$

So, the additional equation may be obtained by means of the convergence equation (4.3) for the QPA (7.5).

$$\begin{aligned}
 &72a_5a_1a_3 + 72a_5a_2a_1 + 12a_5a_1a_0 - 144a_4a_2a_3 - 72a_4a_1a_3 - \\
 &- 60a_4a_2a_1 - 72a_5a_3a_0 - 72a_4a_2^2 + 72a_2a_3^2 + 72a_5a_2^2 + \\
 &+ 30a_5a_1^2 + 30a_3a_2^2 + 72a_3^3 + 3/5a_1a_1a_3 - 12a_0a_4a_2 + \\
 &+ 72a_4^2a_0 - \frac{1}{10}a_0a_1^2 + \frac{1}{2}a_0a_2^2 - 6a_4a_1^2 - \frac{11}{10}a_2a_1^2 - \\
 &- \frac{9}{10}a_1^2a_3 + \frac{9}{10}a_1a_2^2 + \frac{1}{3}a_1^3 + 6a_2^3 + 72a_4^2a_1 = 0.
 \end{aligned}
 \tag{7.6}$$

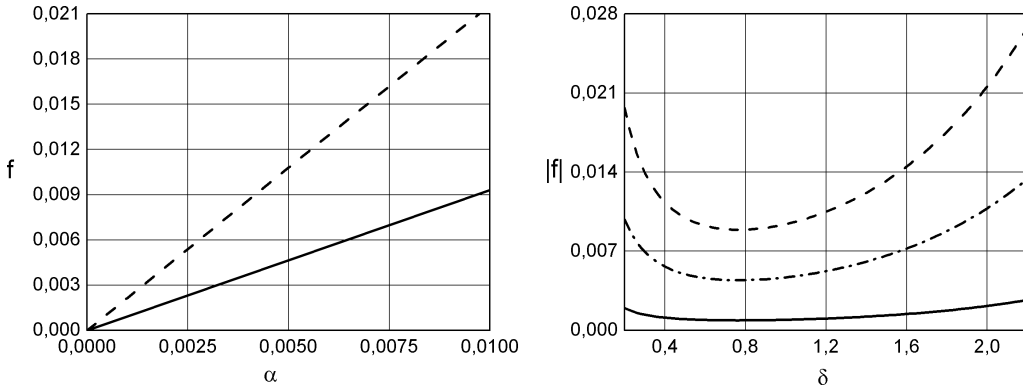


Figure 7.1: Boundaries of the chaotic behavior regions in planes ω, f and δ, f , for $\nu^*=0.5$, $\delta = 0.001$ (solid line), $\delta = 0.005$ (“point-dash” line), $\delta = 0.01$ (dash line).

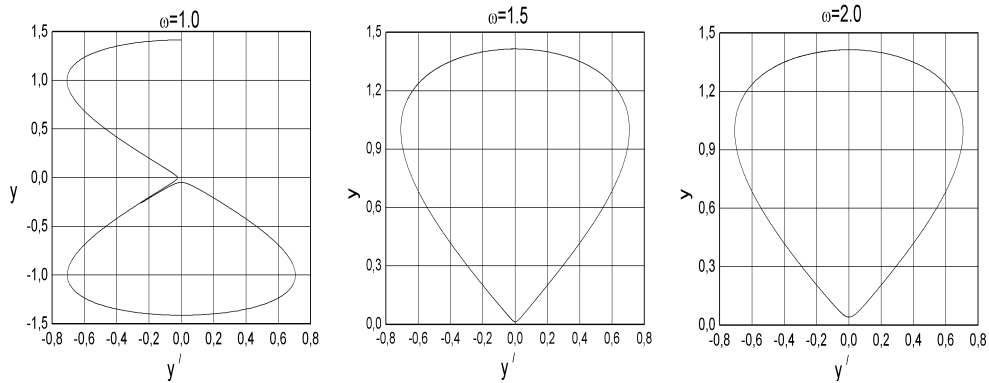


Figure 7.2: Haotic behavior boundaries in parameter spaces and the homoclinic trajectories in phasespace while $\nu^*=0.5$, $T_0 = \alpha = \beta=0.001$.

Nonlinear algebraic equations (7.3), (7.4) and (7.6) form the system which allows to determine unknown parameters a_0, φ and the relation $f = f(\omega)$ while the dissipation parameters T_0, α, β are fixed.

Figure 7.1 shows the dependences between the parameters of the system corresponding to HT and the one obtained by the method proposed here.

Also the example of homoclinic trajectory and comparison of the trajectory evaluated by Runge-Kutta method (when it is used the initial values, obtained from the obtained above algebraic equations) and by means of QPA (7.5) are presented in Figure 7.2.

8 Concluding Remarks

The methodologies presented in this work is sufficiently general to be applicable to other types of non-linear dynamical systems. The subharmonic Melnikov–Morozov theory is utilized to describe a sequence of the saddle-node bifurcations in the process of transition to the chaotic behavior in some mechanical systems. An appearance of heteroclinic structures in mechanical systems under the almost-periodic excitation, is described too by using the Melnikov functions. The multiple-scale method is used here successfully. Other approach of detection of the chaotic behavior is a construction of homo- or heteroclinic trajectories (HT) by using the Pade- and quasi-Pade approximants. It seems more exact that the generally used Melnikov function approach. The presented approach realizes the analytical continuation of the local expansions connected with these HT, to infinity. The necessary condition of convergence of the PA or QPA, as well additional conditions at infinity permit to solve corresponding boundary-value problem for the closed HT. Checking of numerical calculations of the HT with initial amplitudes values obtained by using the analytical approach, shows an acceptable precision of the proposed analytical procedure.

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The Fell Topology for Dynamic Equations on Time Scales

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Abstract: In order to study the changing dynamics of solutions of a dynamic equation on time scales as the time scales change, we must determine appropriate topologies on the set of time scales and the set of solutions of dynamic equations. As a first step, we prove a natural characterization of the Fell topology on the space of time scales.

Keywords: *time scales; dynamic equations; Fell topology.*

Mathematics Subject Classification (2000): 39A05, 54B20.

1 Introduction

Dynamic equations on times scales were introduced by S. Hilger in [10] in 1988. A thorough introduction is contained in [2]. A time scale is a nonempty closed subset of \mathbb{R} . Hilger's Δ -derivative is defined for a real-valued function f whose domain is a time scale \mathbb{T} and is denoted by $f^\Delta(t)$ at any $t \in \mathbb{T}$, where $t < \sup T$.

By design, $f^\Delta(t)$ mimics the standard right-hand derivative $f'(t)$ when there exists a strictly decreasing sequence convergent to t in \mathbb{T} and a scaled difference operator otherwise. In particular, $f^\Delta(t) = f'(t)$ on \mathbb{R} and $f^\Delta(t) = \Delta f(t)$ on \mathbb{Z} . While the Δ -derivative is a “forwards” operator, an analogous “backwards” operator exists called the ∇ -derivative.

Generalizing differential and difference equations are dynamic equations, which involve Δ -derivatives (or ∇ -derivatives, etc.). Given a dynamic equation, say the initial value problem

$$x^\Delta = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

the solution inherently depends on the time scale. Broadly, we would like to examine how the solution of (1.1) depends on the time scale that is its domain.

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1.1 An example

The following illustrative example has been considered in [4], [8], [12], and [14]. Consider the initial value problem:

$$x^\Delta = 4x \left(\frac{3}{4} - x \right), \quad x(0) = x_0.$$

Over the eulerian time scales $\mu\mathbb{Z}_+$ for $0 < \mu \leq 1$, the solution is found by iterating

$$L_\mu(x) = 4\mu x \left(\frac{3\mu+1}{4\mu} - x \right).$$

starting from $x(0) = x_0$. When $\mu = 1$, the difference equation is solved by iteration of

$$L_1(x) = 4x(1 - x)$$

over \mathbb{Z}_+ . On the other hand, as $\mu \rightarrow 0$, the solutions appear to tend towards the solution of the logistic differential equation over \mathbb{R}_+ .

The dynamics of the quadratic polynomial L_μ is easily understood: L_μ is topologically conjugate to

$$Q_c(x) = x^2 + c, \text{ where } c = \frac{1}{4}(1 - 9\mu^2).$$

Every value of $\mu \in (0, 1]$ corresponds exactly to one value of $c \in [-2, 1/4)$, with $\mu = 1$ corresponding to $c = -2$ and $c \rightarrow 1/4$ as $\mu \rightarrow 0$.

Note that the real interval $[-2, 1/4]$ is the real part of the Mandelbrot set for the family Q_c . Hence, passing through the time scales $\mu\mathbb{Z}_+$ —from a difference equation when $\mu = 1$ towards a differential equation as $\mu \rightarrow 0$ —all of the interesting dynamics of real quadratic polynomials, including all of their bifurcations, are displayed! (Of course, the issue of $\mu\mathbb{Z}_+$ converging to \mathbb{R}_+ must be dealt with also.)

1.2 The goal

In the example of subsection 1.1, we have realized the domain of the solutions on eulerian time scales as a parameter of a family of dynamical systems. This is a simple case. We do not know what happens when non-eulerian time scales are used in this example. Also, we have not dealt with an equation that has non-unique solutions.

As indicated in [14], we propose the following project. For any given initial value problem, treat the time scales as a parameter. Let A denote the set of all time scales and let B denote the set of all solutions of the initial value problem on all possible time scales. Consider the canonical projection:

$$\begin{array}{c} B \\ \downarrow \pi \\ A \end{array} \tag{1.2}$$

That is, an element of B , a solution $f : \mathbb{T} \rightarrow \mathbb{R}$, projects to its domain, \mathbb{T} . What can be said about this projection? Hopefully, this approach will help explain the changes in dynamics of solutions caused by changes in time scales and make for better modeling of applications.

In Section 2, we examine the Fell topology on the space of time scales. We prove a recent conjecture in [14] giving a natural characterization of convergence in the Fell topology.

Section 3 considers the compatible topology on the space of partial mappings, *i.e.*, continuous function on time scales.

The first natural example, that of equations with unique solutions, is treated in Section 3.3. Of course, the projection is a homeomorphism onto its image in this case.

2 Convergence of Sets in Terms of Convergence of Their Elements

A *hyperspace* is a set of closed subsets of a topological space X . The set of all closed subsets of X is denoted $CL(X)$. See [11] for an introduction. For example, $CL(\mathbb{R})$ is the set of all time scales.

Hausdorff (for metrizable X), Vietoris, and Fell defined topologies on hyperspaces in [9], [15], and [6], respectively. These are all equivalent on a compact metrizable space. However, the Vietoris and Fell topologies are not metrizable on $CL(\mathbb{R})$.

2.1 The Fell topology on $CL(X)$

We set the following notation that will assist in defining the Fell topology on $CL(X)$. For any $E \subset X$, let

$$E^- = \{\mathbb{A} \in CL(X) \mid \mathbb{A} \cap E \neq \emptyset\}$$

and

$$\begin{aligned} E^+ &= \{\mathbb{A} \in CL(X) \mid \mathbb{A} \subset E\} \\ &= \{\mathbb{A} \in CL(X) \mid \mathbb{A} \cap (X - E) = \emptyset\}. \end{aligned}$$

We say that every $\mathbb{A} \in E^-$ *hits* E and every $\mathbb{A} \in E^+$ *misses* $X - E$; E^- and E^+ are called *hit* and *miss* sets, respectively. Note that $E^+ \subset E^-$ for every E . Also, we call a subset of X *cocompact* if its complement is compact.

The Fell, as well as the Vietoris, topologies are defined by hit and miss sets; these topologies are called *hit-and-miss* topologies. (In fact, the Hausdorff metric topology is also a hit-and-miss topology. See [13].) The Fell topology, denoted by $\tau(F)$, is generated by the hit sets U^- for all open subsets U of X and the miss sets V^+ for all cocompact subsets V of X . The Vietoris topology, denoted by $\tau(V)$, is similarly generated except that the V 's need only be open. (If X is Hausdorff, then the Vietoris topology is finer than the Fell topology.)

Remark 2.1 By convergence in $CL(X)$, we will mean convergence with respect to the Fell topology on $CL(X)$ unless otherwise indicated.

2.2 Convergence through a sequence in $CL(X)$

In [14], we defined another kind of convergence. (This was also discussed in [12] and it inspired [3] and [4].)

Let $\{\mathbb{T}_n\}$ be a sequence in $CL(X)$ and let $t \in X$. t is called a *sequential limit point* of the sequence $\{\mathbb{T}_n\}$ if there exists a sequence $\{t_n\}$ such that $t_n \in \mathbb{T}_n$ for all $n \in \mathbb{N}$ and t_n converges to t in X . Analogously, t is called a *subsequential limit point* of the sequence $\{\mathbb{T}_n\}$ if t is a sequential limit point of a subsequence $\{\mathbb{T}_{n_i}\}$. We denote the set of all sequential limit points of $\{\mathbb{T}_n\}$ by \mathbb{T} and the set of all subsequential limit points of

$\{\mathbb{T}_n\}$ by \mathbb{T}' . We say that $\{t_n\}$ converges to a sequential limit point t through the \mathbb{T}_n 's. Similarly, $\{t_{n_i}\}$ converges to a subsequential limit point t through the \mathbb{T}_{n_i} 's.

It is always the case that $\mathbb{T} \subset \mathbb{T}'$. Two obvious questions are whether \mathbb{T} is in $\text{CL}(X)$ and whether \mathbb{T} is the limit of the sequence \mathbb{T}_n .

Lemma 2.1 *If X is metrizable, then \mathbb{T} is closed in X .*

Proof Choose a metric d on X . Suppose that a sequence $\{s_i\}$ in \mathbb{T} converges to t in X . We wish to show that $t \in \mathbb{T}$.

Since the sequence $\{s_i\}$ converges to t , for every $n \in \mathbb{N}$, there exists a natural number N_n such that

$$d(s_i, t) < \frac{1}{2n} \quad (2.1)$$

whenever $i \geq N_n$.

Since, for each $i \in \mathbb{N}$, $s_i \in \mathbb{T}$, there exist sequences $\{t_{i,j}\}$ converging to s_i through the \mathbb{T}_j 's. Set $M_0 = 1$. For all $i \in \mathbb{N}$, there exists a natural number M_i such that $M_i > M_{i-1}$ and

$$d(t_{i,j}, s_i) < \frac{1}{2i} \quad (2.2)$$

whenever $j \geq M_i$.

We wish to construct a sequence $\{t_j\}$ converging to t through the \mathbb{T}_j 's. For each i , for $M_{i-1} \leq j < M_i$, set $t_j = t_{i,j}$.

Take an arbitrary $\varepsilon > 0$. When $i > 1/\varepsilon$ and $i \geq N_n$, by (2.2) and (2.1),

$$\begin{aligned} d(t_j, t) &\leq d(t_j, s_i) + d(s_i, t) \\ &< \frac{1}{2i} + \frac{1}{2n} < \varepsilon. \end{aligned}$$

Therefore, the sequence $\{t_j\}$ converges to t through the \mathbb{T}_j 's and $t \in \mathbb{T}$. \square

Remark 2.2 Therefore, in the setting of a metric space X , the sequential limit set of a sequence in $\text{CL}(X)$ is either empty or in $\text{CL}(X)$. For example, the sequence of singleton sets $\{\{n\}\}$ in $\text{CL}(\mathbb{R})$ has empty sequential limit set and $\emptyset \notin \text{CL}(\mathbb{R})$ by definition.

2.3 A characterization of the Fell topology on $\text{CL}(X)$

In [14], it was conjectured that a sequence is convergent in $\text{CL}(\mathbb{R})$ if and only if the sequential and subsequential limit sets of the sequence are equal. We prove this in the more general setting of a metric space X .

Theorem 2.1 *Let X be metrizable. Let $\{\mathbb{T}_n\}$ be a sequence in $\text{CL}(X)$. $\{\mathbb{T}_n\}$ converges in $\text{CL}(X)$ if and only if $\mathbb{T} = \mathbb{T}' \neq \emptyset$. Moreover, in this situation, $\{\mathbb{T}_n\}$ converges to \mathbb{T} .*

Proof Choose a metric d on X . First, let us suppose that $\mathbb{T} = \mathbb{T}' \neq \emptyset$. We consider two cases of subbasic open sets containing \mathbb{T} in order to prove that $\{\mathbb{T}_n\}$ converges to \mathbb{T} .

Case 1: Let $U \subset X$ be open such that $\mathbb{T} \in U^-$. Choose $t \in \mathbb{T} \cap U$ and $\varepsilon > 0$ sufficiently small such that

$$B_\varepsilon(t) = \{b \in X \mid d(b, t) < \varepsilon\} \subset U.$$

Since $t \in \mathbb{T}$ is a sequential limit point, there exists a sequence $\{t_n\}$ that converges to t through the \mathbb{T}_n 's. Therefore, there exists N such that $t_n \in B_\varepsilon(t) \subset U$ for all $n \geq N$. Hence, $\mathbb{T}_n \cap U \neq \emptyset$ and $\mathbb{T}_n \in U^-$ for all $n \geq N$.

Case 2: Let $K \subset X$ be compact such that $V = X - K$ and $\mathbb{T} \in V^+$. Assume that there is no N such that $\mathbb{T}_n \in V^+$ for all $n \geq N$. So there exists a subsequence $\{\mathbb{T}_{n_i}\}$ such that $\mathbb{T}_{n_i} \notin V^+$. Therefore, for each i , there exists $t_i \in \mathbb{T}_{n_i} \cap K$. If the set $\{t_i\}$ is finite, then $\{t_i\}$ has a constant subsequence $\{t\}$ for some $t \in K$. Alternatively, the infinite set $\{t_i\}$ has a limit point t in the compact set K . In either case, $t \in \mathbb{T}'$, but $t \notin \mathbb{T}$. This contradicts the fact that $\mathbb{T} = \mathbb{T}'$.

Therefore, $\mathbb{T} = \mathbb{T}' \neq \emptyset$ implies that $\{\mathbb{T}_n\}$ converges to \mathbb{T} , which is in $\text{CL}(X)$ by Lemma 2.1.

Conversely, let us suppose that $\{\mathbb{T}_n\}$ converges to \mathbb{S} in $\text{CL}(X)$. We know that $S \neq \emptyset$ since $\mathbb{S} \in \text{CL}(X)$. We wish to show that $\mathbb{S} \subset \mathbb{T} \subset \mathbb{T}' \subset \mathbb{S}$. We know that $\mathbb{T} \subset \mathbb{T}'$. It remains to show that $\mathbb{S} \subset \mathbb{T}$ and $\mathbb{T}' \subset \mathbb{S}$.

First, we choose $s \in \mathbb{S}$. For every $m \in \mathbb{N}$, let

$$U_m = B_{1/m}(s) = \left\{ u \in X \mid d(u, s) < \frac{1}{m} \right\}.$$

For every m , $\mathbb{S} \in U_m^-$ since $s \in \mathbb{S} \cap U_m$. Since $\{\mathbb{T}_n\}$ converges to \mathbb{S} , for every m , there exists an integer N_m such that $\mathbb{T}_n \in U_m^-$ whenever $n \geq N_m$. If necessary, adjust the sequence $\{N_m\}$ to be increasing. For every m and every integer n such that $N_m \leq n < N_{m+1}$, choose $t_n \in \mathbb{T}_n \cap U_m$. This yields a sequence $\{t_n\}$ that converges to s through the \mathbb{T}_n 's. Therefore, $\mathbb{S} \subset \mathbb{T}$.

Next, we choose $t \in \mathbb{T}'$. Thus, there exists a sequence t_{n_i} that converges to t through the \mathbb{T}_{n_i} 's. Assume that $t \notin \mathbb{S}$. Choose a cocompact V such that $\mathbb{S} \subset V$ and choose $\varepsilon > 0$ such that

$$B_\varepsilon(t) \cap V = \{ u \in X \mid d(u, t) < \varepsilon \} \cap V = \emptyset.$$

Since $\{\mathbb{T}_n\}$ converges to \mathbb{S} , there exists N such that $\mathbb{T}_n \in V^+$ whenever $n \geq N$. For every $n \geq N$ and for every $t' \in \mathbb{T}_n$,

$$d(t', t) \geq \varepsilon > 0.$$

This contradicts that $\{t_{n_i}\}$ converges to t through the \mathbb{T}_{n_i} 's. Therefore, $\mathbb{T}' \subset \mathbb{S}$. \square

Remark 2.3 In particular, Theorem 2.1 characterizes convergence in $\text{CL}(\mathbb{R})$, the space of all time scales.

2.4 Examples

Example 2.1 The sequence of singleton sets $\{\{n\}\}$ does not converge since its sequential limit set is empty. While we could say the sequence converges to the empty set, we do not include the empty set in $\text{CL}(X)$. Similarly, the sequence of intervals $\{[n, n+1]\}$ fails to converge.

Example 2.2 The sequence of intervals $\{[-n, n]\}$ converges to its sequential limit set \mathbb{R} . This fails to converge in the Hausdorff topology since the distance between $\{[-n, n]\}$ and \mathbb{R} is bounded away from 0. (See [14].) How about in the Vietoris topology?

Let us see if the proof of Theorem 2.1 holds for $\{[-n, n]\}$ in the Vietoris topology rather than the Fell topology. That is, we allow V to just be open rather than cocompact.

The sequential limit set is \mathbb{R} . If $\mathbb{R} \in V^+$, then $V = \mathbb{R}$. But then $\{[-n, n]\} \in V^+$ for every n and the convergence holds in the Vietoris topology.

Example 2.3 The sequence $\{\mathbb{Z} + \frac{1}{n}\}$ converges to its sequential limit set \mathbb{Z} . It also converges in the Hausdorff topology, but not in the Vietoris topology. Here the proof would break down for

$$V = \bigcup_{k=1}^{\infty} \left(k - \frac{1}{k}, k + \frac{1}{k} \right),$$

which is not cocompact. (See [14].)

Example 2.4 The sequence $\frac{1}{n}\mathbb{Z}_+$ converges to its sequential limit set \mathbb{R}_+ .

2.5 Properties of the Fell topology on $\text{CL}(X)$

Many properties of the Fell topology on $\text{CL}(X)$ for a metrizable space X may be found in [1], wherein references to primary sources can be found.

The Fell topology on the one-point compactification of $\text{CL}(X)$ —extended to include the empty set—is compact Hausdorff; we denote this by $\overline{\text{CL}(X)}$. The Fell topology on $\text{CL}(X)$ is locally compact Hausdorff. For example, this implies that the Fell topology on $\text{CL}(X)$ is completely regular.

Since $\overline{\text{CL}(X)}$ is compact, every sequence $\{\mathbb{T}_n\}$ in $\text{CL}(X)$ or $\overline{\text{CL}(X)}$ must have a convergent subsequence. So the subsequential limit set in $\overline{\text{CL}(X)}$ is never empty, but may be $\{\emptyset\}$.

Giving a subset $\mathcal{S} \subset \text{CL}(X)$, the induced topology, the Hausdorff, Vietoris, and Fell topologies always agree if X is a compact metric space. So, when considering uniformly bounded time scales, we can revert to Hausdorff metric.

3 The Topology on The Solution Spaces

Recall that for Hausdorff spaces X and Y , a subbasis for the *compact-open topology* on the set, $C(X, Y)$, of continuous functions from X to Y is given by

$$S(K, U) = \{f \in C(X, Y) \mid K \subset X \text{ is compact, } U \text{ is open in } Y, \text{ and } f(K) \subset U\}.$$

If Y is a metric space, this is the *topology of compact convergence*, *i.e.*, sequences converge if and only if they converge uniformly on compact subsets. If X is compact and Y is a metric space, this is the *topology of uniform convergence*.

3.1 The space of continuous functions on time scales

Since we are interested in function spaces over variable domains, we must unite the standard function spaces.

For a closed subset K of X , a function $f : K \rightarrow Y$ can be thought of as a *partial function* from X to Y —the domain of definition is K rather than X . By a *partial mapping*, we will mean a continuous partial function. (See [7].) The set of all partial mappings from X to Y is

$$C_F(X, Y) = \cup \{ C(K, Y) \mid K \in \text{CL}(X) \}.$$

The subscript “F” is a reminder that we will be using the Fell topology to build a compatible topology on this set. *E.g.*, $C_F(\mathbb{R}, \mathbb{R})$ is the set of all continuous real-valued functions on time scales.

Suppose that X and Y are metric spaces. So $X \times Y$ is metrizable. We wish to give a topology on $C_F(X, Y)$ that is consistent with the compact-open topology on $C(X, Y)$.

Consider the function $\text{Gr} : C_F(X, Y) \rightarrow \text{CL}(X \times Y)$ that sends each partial mapping to its graph. Since Gr is injective, we can pull back the Fell topology on $\text{CL}(X \times Y)$ to give a topology on $C_F(X, Y)$: \mathcal{S} is open in $C_F(X, Y)$ if and only if $\text{Gr}(\mathcal{S})$ is open in $\text{Gr}(C_F(X, Y))$ as a subspace of $\text{CL}(X \times Y)$.

Following [7], Theorem 3.1 follows from the facts that projection from $X \times Y$ to X is continuous and induces a continuous mapping from $\text{CL}(X \times Y)$ to $\text{CL}(X)$.

Theorem 3.1 *The canonical projection $\pi : C_F(X, Y) \rightarrow \text{CL}(X)$ is continuous.*

3.2 The case of unique solutions

Recall the goal proposed in subsection 1.2. We examine the case of a dynamic equation whose solutions are always unique (for example, $x^\Delta = 0$).

Let \mathcal{S} denote the set of all solutions of a given initial value problem over all possible time scales. Consider the restriction of the projection π :

$$\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \text{CL}(\mathbb{R}).$$

That is, an element of \mathcal{S} , a solution $f : \mathbb{T} \rightarrow \mathbb{R}$ of the initial value problem, projects to its domain, \mathbb{T} . Since all solutions are unique on their domains, $\pi_{\mathcal{S}}$ is a bijection onto its range. The construction of the topology on $C_F(X, Y)$ now shows the following:

Corollary 3.1 *$\pi_{\mathcal{S}}$ is a homeomorphism onto its range.*

3.3 Open problem: the case of non-unique solutions

In the non-unique case, the projection $\pi_{\mathcal{S}}$ may be far more interesting. Hopefully, the topology will tell us something about the dynamics. A question to whet one’s appetite: can there be monodromy? Can we lift a loop with a base point in the space of time scales so that we start and end at different solutions?

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Independent of this paper, Esty and Hilger have concluded, in [5], that the Fell topology is best suited for the space of time scales. They give an interesting characterization of the Fell topology that extends the topologies induced by the Hausdorff metric on compact sets. The present paper seeks to extend the same topologies from the viewpoint of the Vietoris topology as a hit-and-miss topology. This seems to be somewhat more natural and dynamic. Probably that is because of the similarity of the hit-and-miss constructions of the Vietoris and Fell topologies; it is far less natural to think of the Hausdorff topology as a hit-and-miss topology.

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Application of Generalized Hamiltonian Systems to Chaotic Synchronization[◇]

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Abstract: In this paper, a method of chaotic synchronization is introduced which is obtained from the perspective of passivity-based state observer design in the context of Generalized Hamiltonian systems including dissipative and destabilizing vector fields. Two cases of chaotic synchronization, namely, the synchronization of some famous chaotic systems with and without time delay are analyzed. The numerical results are obtained by the nonlinear dynamical software, WinPP in this paper. The numerical results are in very good agreement with the theoretical analysis.

Keywords: *chaotic synchronization; generalized Hamiltonian canonical form; observer approach; time-delay systems.*

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1 Introduction

Because of the rapid development of chaos-based cryptography in secure communication, chaotic synchronization has become an active research area. Many results on all kinds of chaotic synchronization and its applications have been systematically summarized in [1]. Synchronization is ubiquitous in many natural and engineering systems. Synchronization literally means two identical, near-identical or even different chaotic systems tend to move at the same state, velocity, acceleration and phase, if one of them is coupled or both coupled with each other. The relevant research on synchronization can be dated back

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to Huygens who investigated frequency locking between two clocks, which is perhaps the first synchronization phenomenon observed. Chaotic synchronization has become an active research in nonlinear dynamics [2, 3] since the early 1990s when researchers realized that chaotic dynamical systems can be synchronized and recognized its potential applications to the secure communication. In addition to the classical complete synchronization [3, 4], there are some other new types of synchronization, such as Pecora–Carroll synchronization [5], phase synchronization [6], frequency synchronization [7], anticipating synchronization [8, 9], quasiperiodic synchronization [10], lag synchronization [11], inverse synchronization [12] and generalized synchronization [13].

Synchronization of chaotic dynamical systems has received a lot of attention in recent years. The potential application of chaotic synchronization to signal masking and private communication [14, 15, 16, 17, 18, 19] is very interesting. Chaos in nonlinear dynamical system is a well-established discipline in physics, chemistry, power, electronics, biology, ecology, economics, etc in the meantime. Chaotic behavior can be found in systems described by ordinary differential equations (ODE), discrete dynamical systems and delay differential equations (DDE), etc [20]. In other words, chaos is a multidisciplinary research field and ubiquitous phenomenon. The main property of chaotic dynamics is its critical sensitivity to initial conditions in the systems' evolution. For many years this property made chaos unpredictable, since the sensitivity to initial conditions reduces the long-term predictability of such chaotic dynamical systems. But the recent investigations have shown, in fact, this property of chaotic dynamical systems could practically be very beneficial [21].

Time delay does also widely exist in the natural world and the human society. Finite signal transmission, switching speeds and memory effects make it ubiquitous in nature, technology and society [8]. Therefore the study of the effect of time delay on the systems' dynamics is of considerable practical importance. Time-delayed dynamical systems are also interesting since they have infinite-dimensional state spaces and the number of their positive Lyapunov exponents can be made arbitrarily large because of the existence of the time delay. From this point of view, such systems are especially appealing for secure communication scheme [1].

The objective of this paper is to apply the Generalized Hamiltonian forms and observer approach developed in [22] to the synchronization of some chaotic dynamical systems. Besides the observer perspective on synchronization, some works, such as the concept of synchronization is revisited in the light of the classical notion of observers from (non)linear control theory, are obtained in [23, 24]. As described in [25] this method has several advantages over the existing synchronization methods: (1) it enables synchronization achieved in a systematic way; (2) it can be successfully applied to several well-known chaotic or hyperchaotic oscillators; (3) it does not require the computation of any Lyapunov exponent; (4) it does not require initial conditions belonging to the same basin of attraction. In Section 2, the Generalized Hamiltonian forms and observer approach [22, 25] are first introduced. Then the synchronization of some kinds of chaotic dynamical systems such as $L\ddot{u}$ system, Van der Pol–Duffing system, Genesio system and SMIB power system, which without time delay, employed by the Generalized Hamiltonian forms and observer approach is considered in Section 3. That of the delayed chaotic dynamical systems, i.e., SMIB power system and Van der Pol–Duffing system, is also investigated in Section 4. At last, the conclusion and discussion are presented.

2 The design in Generalized Hamiltonian system

Consider a smooth nonlinear system given in the following “Generalized Hamiltonian” canonical form,

$$\dot{x} = \mathcal{J}(x) \frac{\partial H}{\partial x} + \mathcal{S}(x) \frac{\partial H}{\partial x} + \mathcal{F}(x), \quad x \in \mathbf{R}^n, \tag{2.1}$$

where $H(x)$ denotes a smooth energy function which is globally positive definite in \mathbf{R}^n . The column gradient vector of $H(x)$, denoted by $\partial H/\partial x$, is assumed to exist everywhere. We usually use quadratic energy function $H(x) = \frac{1}{2}x^T \mathcal{M}x$ with \mathcal{M} being a, constant, symmetric positive definite matrix. So $\partial H/\partial x = \mathcal{M}x$. The square matrices $\mathcal{J}(x)$ and $\mathcal{S}(x)$ satisfy for all $x \in \mathbf{R}^n$, the properties: $\mathcal{J}(x) + \mathcal{J}^T(x) = 0$, and $\mathcal{S}(x) = \mathcal{S}^T(x)$. The vector field $\mathcal{J}(x)\partial H/\partial x$ exhibits the conservative part of the system and it is also referred to as the workless part, or workless forces of the system; and $\mathcal{S}(x)$ depicts the working or nonconservative part of the system. For certain systems, $\mathcal{S}(x)$ is negative definite or negative semi-definite. If, on the other hand, $\mathcal{S}(x)$ is positive definite, positive semi-definite, or indefinite, it clearly represents, respectively, the global, semi-global, and local destabilizing part of the system. And where $\mathcal{F}(x)$ is a locally destabilizing vector field. Consider now the following dynamical system

$$\dot{x} = f(x, t). \tag{2.2}$$

It can be rewritten as

$$\dot{x} = A \frac{\partial H}{\partial x} + \mathcal{F}(x, t). \tag{2.3}$$

Since $A = \frac{A-A^T}{2} + \frac{A+A^T}{2}$, we have

$$\dot{x} = \frac{A - A^T}{2} \frac{\partial H}{\partial x} + \frac{A + A^T}{2} \frac{\partial H}{\partial x} + \mathcal{F}(x, t), \tag{2.4}$$

Let $\mathcal{J}(x) = \frac{A-A^T}{2}$, $\mathcal{S}(x) = \frac{A+A^T}{2}$. The equation (2.2) can be written in the Generalized Hamiltonian canonical form (2.1). This form is not only used for autonomous systems, but also for non-autonomous systems and delay differential equations.

In the context of observer design, we consider a special class of Generalized Hamiltonian systems with destabilizing vector field and liner output map, $y(t)$, given by

$$\begin{cases} \dot{x} = \mathcal{J}(y) \frac{\partial H}{\partial x} + (\mathcal{I} + \mathcal{S}) \frac{\partial H}{\partial x} + \mathcal{F}(y), & x \in \mathbf{R}^n, \\ y = \mathcal{C} \frac{\partial H}{\partial x}, & y \in \mathbf{R}^m, \end{cases} \tag{2.5}$$

where S is a constant symmetric matrix, the matrix \mathcal{I} is a constant skew symmetric matrix. The vector variable $y(t)$ is referred to as the system output. The matrix \mathcal{C} is a constant matrix.

We denote the estimate of the state vector x by ξ , and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of H in terms of ξ . Similarly, we denote by η the estimated output, computed in terms of the estimated state ξ . The gradient vector $\partial H/\partial \xi$ is, naturally, of the form $\mathcal{M}\xi$ with \mathcal{M} being a constant symmetric positive definite matrix.

A dynamic nonlinear state observer for (2.5) is obtained as

$$\begin{cases} \dot{\xi} = \mathcal{J}(y) \frac{\partial H}{\partial \xi} + (\mathcal{I} + \mathcal{S}) \frac{\partial H}{\partial \xi} + \mathcal{F}(y) + K(y - \eta), & \xi \in \mathbf{R}^n, \\ \eta = \mathcal{C} \frac{\partial H}{\partial \xi}, & \eta \in \mathbf{R}^m, \end{cases} \quad (2.6)$$

where K is a constant matrix, known as the observer gain. The state estimation error, defined as $e = x - \xi$ and the output estimation error, defined as $e_y = y - \eta$, are governed by

$$\begin{cases} \dot{e} = \mathcal{J}(y) \frac{\partial H}{\partial e} + (\mathcal{I} + \mathcal{S} - KC) \frac{\partial H}{\partial e}, & e \in \mathbf{R}^n, \\ e_y = \mathcal{C} \frac{\partial H}{\partial e}, & e_y \in \mathbf{R}^m, \end{cases} \quad (2.7)$$

where the vector, $\partial H / \partial e$ actually stands, with some abuse of notation, for the gradient vector of energy function, $\partial H / \partial e = \partial H / \partial x - \partial H / \partial \xi = \mathcal{M}e$. When needed, set $\mathcal{I} + \mathcal{S} = \mathcal{W}$.

Definition 2.1 (Synchronization) [1] We say that the receiver dynamics (2.6) synchronizes with the transmitter dynamics (2.5), if

$$\lim_{t \rightarrow \infty} \|x(t) - \xi(t)\| = 0, \quad (2.8)$$

no matter which initial conditions $x(0)$ and $\xi(0)$ have.

Theorem 2.1 (Stability of the estimation/synchronization error [22]) *The state x of the nonlinear system (2.5) can be globally exponentially asymptotically estimated by the state ξ of the nonlinear observer (2.6) if and only if there exists a constant matrix K such that the symmetric matrix*

$$[\mathcal{W} - KC] + [\mathcal{W} - KC]^T = [\mathcal{S} - KC] + [\mathcal{S} - KC]^T = 2[\mathcal{S} - \frac{1}{2}(KC + C^T K^T)]$$

is negative definite.

In the latter synchronized programs, we mainly use Theorem 2.1. Most time we only consider the matrix \mathcal{S} , but not the matrix $\mathcal{I} + \mathcal{S}$.

And a sufficient but not necessary condition based on the observability condition for asymptotical stability of the synchronization was obtained.

Theorem 2.2 [22] *The state $x(t)$ of the nonlinear system (2.5) can be globally exponentially asymptotically estimated by the state ξ of the nonlinear observer (2.6), if the pair of matrices $(\mathcal{C}, \mathcal{W})$ or the pair $(\mathcal{C}, \mathcal{S})$, is either observable or, at least, detectable.*

3 Synchronization of some chaotic systems

3.1 Lü system

Consider Lü system [26]

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = -x_1 x_3 + c x_2, \\ \dot{x}_3 = x_1 x_2 - b x_3. \end{cases} \quad (3.1)$$

The system (3.1) can be easily written in the following Generalized Hamiltonian form,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix} \frac{\partial H}{\partial x}, \tag{3.2}$$

where $H(x)$ is the Hamiltonian energy scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.3}$$

we choose $y = [x_1, x_2]^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \frac{\partial H}{\partial x}$ as the output signal to be transmitted. The matrices \mathcal{C} , \mathcal{S} , \mathcal{I} , and $\mathcal{J}(y)$ are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{S} = \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \mathcal{I} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{J}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix}.$$

Using Theorem 2.2 [22], the pair $(\mathcal{C}, \mathcal{W})$ or $(\mathcal{C}, \mathcal{S})$ already constitutes a detectable, but not observable pair for the chaotic parameters $a = 36, b = 3, c = 20$. Because the output contains two states, namely, x_1 and x_2 , so the gain parameters should be chosen as K_1, \dots, K_6 , this results in the receiver

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \tag{3.4}$$

where $\eta = \mathcal{C} \frac{\partial H}{\partial \xi}$. One may now choose the gain vector $K = \begin{pmatrix} K_1 & K_3 & K_5 \\ K_2 & K_4 & K_6 \end{pmatrix}^T$. The synchronization error, corresponding to this receiver, is

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} - \frac{K_2}{2} + \frac{K_3}{2} & \frac{K_5}{2} \\ -\frac{a}{2} + \frac{K_2}{2} - \frac{K_3}{2} & 0 & -x_1 + \frac{K_6}{2} \\ -\frac{K_5}{2} & x_1 - \frac{K_6}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -a - K_1 & \frac{a}{2} - \frac{K_2}{2} - \frac{K_3}{2} & -\frac{K_5}{2} \\ \frac{a}{2} - \frac{K_2}{2} - \frac{K_3}{2} & c - K_4 & -\frac{K_6}{2} \\ -\frac{K_5}{2} & -\frac{K_6}{2} & -b \end{pmatrix} \frac{\partial H}{\partial e}. \tag{3.5}$$

From Theorem 2.1, the following expression is obtained

$$2[\mathcal{S} - \frac{1}{2}(\mathcal{K}\mathcal{C} + \mathcal{C}^T \mathcal{K}^T)] = \begin{pmatrix} -2a - 2K_1 & -K_2 - K_3 + a & -K_5 \\ -K_2 - K_3 + a & -2K_4 + 2c & -K_6 \\ -K_5 & -K_6 & -2b \end{pmatrix},$$

we may prescribe K_1, K_2, K_3, K_4, K_5 and K_6 in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester’s Criterion—which provides a test for definite negativity of a matrix—thus, this is achieved by setting

$$\begin{aligned}
&K_1 > -a, \\
&(K_2 + K_3 - a)^2 < 4(K_1 + a)(K_4 - c), \\
&(K_1 + a)[4b(K_4 - c) - K_6^2] - (K_2 + K_3 - a)[b(K_2 + K_3 - a) - K_5K_6] \\
&- K_5(K_4 - c) > 0.
\end{aligned}$$

Figure 3.1 shows the performance of the designed receiver with the following parameter values and for the constant gains $a = 36$, $b = 3$, $c = 20$, $K_1 = 0$, $K_2 = 3$, $K_3 = 3$, $K_4 = 30$, $K_5 = 0$, $K_6 = 2$. From Figure 3.1, it can be easily known that after a very short time, $L\ddot{u}$ system is synchronized.

3.2 Van der Pol–Duffing system

The initial mathematical model is Van der Pol–Duffing system with external excitation given by

$$\ddot{x} + \omega_0^2 x - (\alpha - \gamma x^2)\dot{x} + \beta x^3 = k \cos(\Omega t). \quad (3.6)$$

By setting $x_1 = x, x_2 = \dot{x}$, we can write the system (14) in the following form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = (\alpha - \gamma x_1^2)x_2 - \omega_0^2 x_1 - \beta x_1^3 + k \cos(\Omega t). \end{cases} \quad (3.7)$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2], \quad (3.8)$$

we write the system in Generalized Hamiltonian canonical form as

$$\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix} \frac{\partial H}{\partial x} \\
&+ \begin{pmatrix} 0 \\ -\gamma x_1^2 x_2 - \beta x_1^3 + k \cos \Omega t \end{pmatrix}. \quad (3.9)
\end{aligned}$$

The destabilizing vector requires two signals for complete cancellation at the receiver, namely, the variables, x_1 and x_2 . The output is then chosen as the vector $y = [x_1, x_2]^T$. The matrices \mathcal{C} , \mathcal{S} and \mathcal{I} are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix},$$

the pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. In order to achieve chaotic behavior, we should choose suitable parameters. The receiver would then be designed as follows

$$\begin{aligned}
\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix} \frac{\partial H}{\partial \xi} \\
&+ \begin{pmatrix} 0 \\ -\gamma x_1^2 x_2 - \beta x_1^3 + k \cos \Omega t \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} (y - \eta), \quad (3.10)
\end{aligned}$$

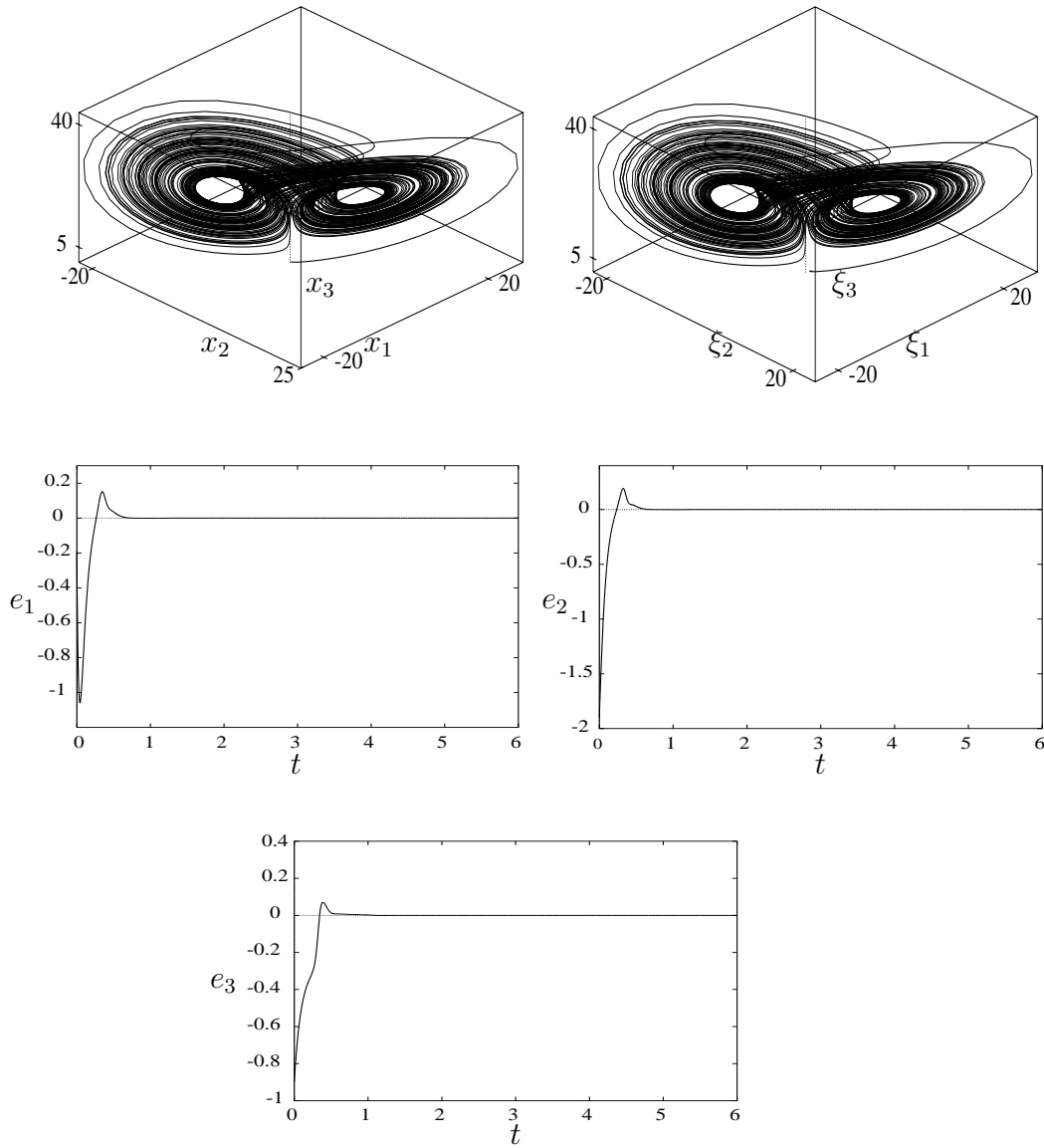


Figure 3.1: The synchronization of the Lü systems (3.2) and (3.4) with the following parameter values and for the constant gains $a = 36$, $b = 3$, $c = 20$, $K_1 = 0$, $K_2 = 3$, $K_3 = 3$, $K_4 = 30$, $K_5 = 0$, $K_6 = 2$ and the initial conditions $x(0) = (0.01, 0.1, 1)^T, \xi(0) = (1, 0.5, 2)^T$.

where $\eta = \mathcal{C} \frac{\partial H}{\partial \xi}$, the synchronization error dynamics is governed by

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} + \frac{1}{2}\omega_0^2 + \frac{K_3}{2} - \frac{K_2}{2} \\ -\frac{1}{2} - \frac{1}{2}\omega_0^2 + \frac{K_2}{2} - \frac{K_3}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -K_1 & \frac{1}{2} - \frac{1}{2}\omega_0^2 - \frac{K_2}{2} - \frac{K_3}{2} \\ \frac{1}{2} - \frac{1}{2}\omega_0^2 - \frac{K_2}{2} - \frac{K_3}{2} & -K_4 + \alpha \end{pmatrix} \frac{\partial H}{\partial e}, \quad (3.11)$$

we could prescribe K_1, K_2, K_3 , and K_4 , in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester's Criterion, this is achieved by setting

$$K_1 > 0, \\ K_4 > \alpha + \frac{1}{4K_1}(K_2 + K_3 + \omega_0^2 - 1)^2.$$

Figure 3.2 shows the synchronization of the systems (3.9) and (3.10), the chosen parameters were set as following^[27], $\alpha = 1, \gamma = 1, \omega_0^2 = 1, \beta = 0.01, k = 5, \Omega = 2.463$, with receiver parameter gains $K_1 = 1, K_2 = 0, K_3 = 0, K_4 = 9$.

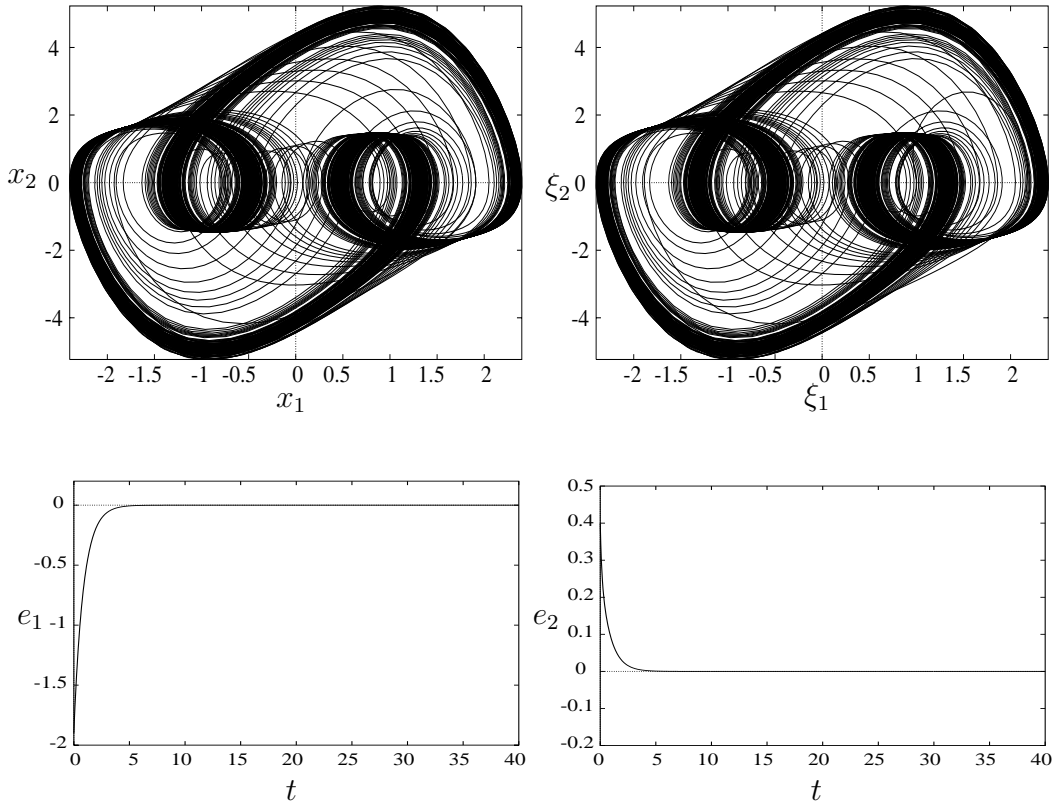


Figure 3.2: The synchronization of the van der Pol–Duffing systems (3.9) and (3.10) with the following parameter values and for the constant gains $\alpha = 1, \gamma = 1, \omega_0^2 = 1, \beta = 0.01, k = 5, \Omega = 2.463, K_1 = 1, K_2 = 0, K_3 = 0, K_4 = 9$ and the initial conditions $x(0) = (0.1, 0.5)^T, \xi(0) = (2, 0.1)^T$.

3.3 Genesio system

Genesio system, proposed by Genesio and Tesi [28], is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three negative parameters.

The dynamic equations of the system is given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = ax_1 + bx_2 + cx_3 + x_1^2, \end{cases} \tag{3.12}$$

where x_1, x_2, x_3 are state variables, indeed

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix}, \tag{3.13}$$

taking the Hamiltonian energy function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.14}$$

the destabilizing vector field and the lacking damping in x_2 variable, call for $y = [x_1, x_2]^T$ to be used as the output of the transmitter. The matrices \mathcal{C}, \mathcal{S} are found to be

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix}, \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. The receiver would then be designed as

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &+ \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{3.15}$$

where $\eta = \mathcal{C} \frac{\partial H}{\partial \xi}$.

The synchronization error evolves according to

$$\begin{aligned} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} - \frac{K_2}{2} + \frac{K_3}{2} & -\frac{a}{2} + \frac{K_5}{2} \\ -\frac{1}{2} + \frac{K_2}{2} - \frac{K_3}{2} & 0 & \frac{1}{2} - \frac{b}{2} + \frac{K_6}{2} \\ \frac{a}{2} - \frac{K_5}{2} & -\frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} \\ &+ \begin{pmatrix} -K_1 & \frac{1}{2} - \frac{K_2}{2} - \frac{K_3}{2} & \frac{a}{2} - \frac{K_5}{2} \\ \frac{1}{2} - \frac{K_2}{2} - \frac{K_3}{2} & -K_4 & \frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} \\ \frac{a}{2} - \frac{K_5}{2} & \frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} & c \end{pmatrix} \frac{\partial H}{\partial e}. \end{aligned} \tag{3.16}$$

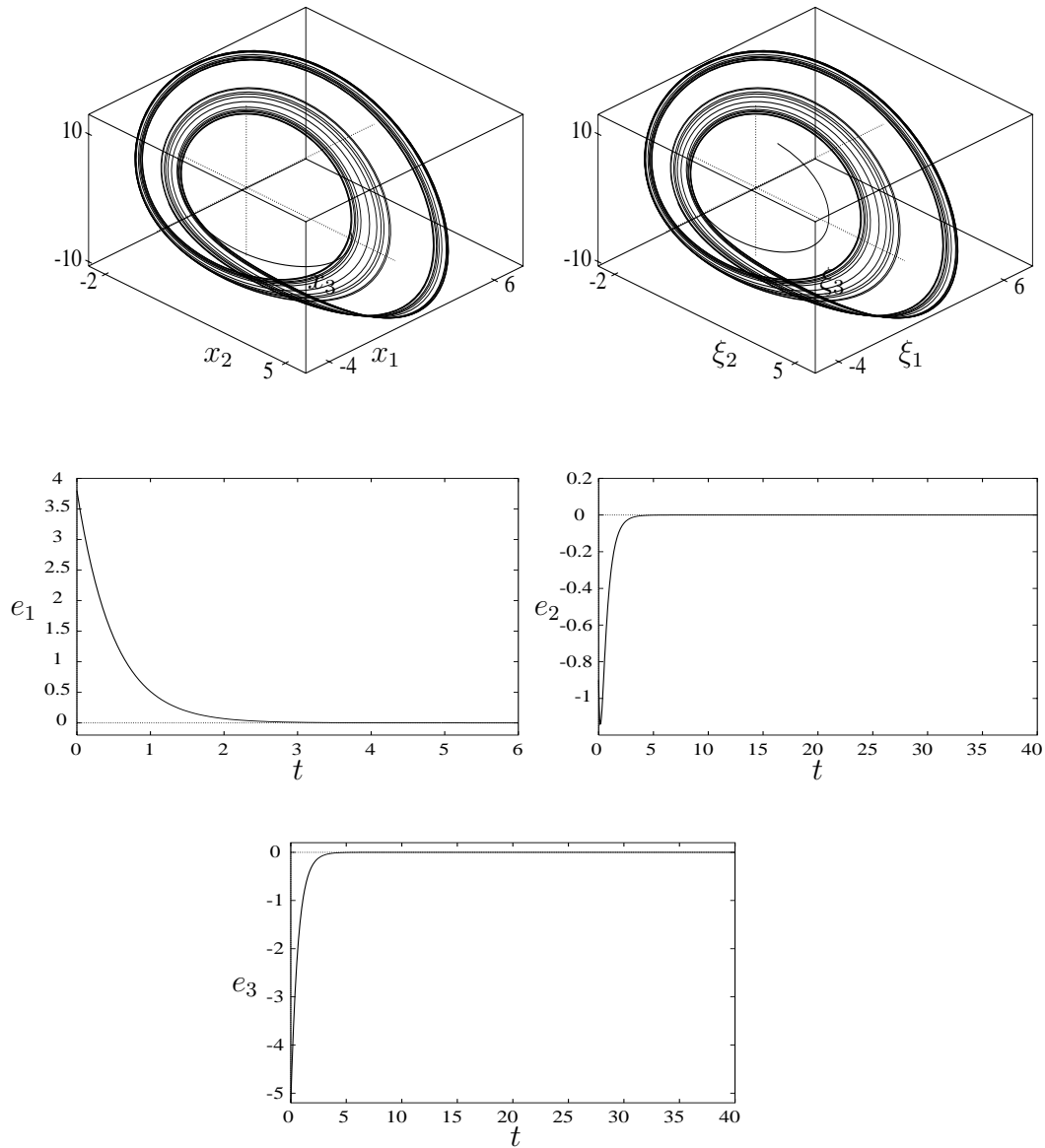


Figure 3.3: The synchronization of the Genesis systems (3.13) and (3.15) with the following parameter values and for the constant gains $a = -6, b = -2.92, c = -1.2, K_1 = 2, K_2 = 1, K_3 = 1, K_4 = 5, K_5 = -6, K_6 = -1.92$ and the initial conditions $x(0) = (4, 0.1, 0.8)^T, \xi(0) = (0.2, 1, 6)^T$.

To guarantee asymptotic stability of zero of the error dynamics, we should choose suitable $K_1, K_2, K_3, K_4, K_5, K_6$. By applying the Sylvester’s Criterion, this is achieved by setting

$$\begin{aligned} &K_1 > 0, \\ &(K_2 + K_3 - 1)^2 < 4K_1K_4, \\ &2K_1[-4cK_4 - (K_6 - b - 1)^2] + 2c(K_2 + K_3 - 1)^2 \\ &+ 2(K_5 - a)(K_2 + K_3 - 1)(K_6 - b - 1) - 2K_4(K_5 - a)^2 > 0. \end{aligned}$$

Figure 3.3 shows the synchronization of the systems (3.13) and (3.15). The chosen parameters were set, following [21], as $a = -6, b = -2.92, c = -1.2$, with receiver parameter gains $K_1 = 2, K_2 = 1, K_3 = 1, K_4 = 5, K_5 = -6, K_6 = -1.92$.

3.4 SMIB power system

Consider SMIB power system [29], called swing equation

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -cx_2 - \beta \sin x_1 + f \sin x_3, \\ \dot{x}_3 = \omega. \end{cases} \tag{3.17}$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.18}$$

we write the system in Generalized Hamiltonian canonical form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 \\ \omega \end{pmatrix} \tag{3.19}$$

The destabilizing vector field requires two signals for complete cancellation at the receiver. Namely, the variables, x_1 and x_2 . The output is then chosen as the vector $y = [y_1, y_2]^T = [x_1, x_3]^T$, the matrices \mathcal{C}, \mathcal{S} and \mathcal{I} are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable, \mathcal{S} is therefore of indefinite sign. The receiver would then be designed as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &+ \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 \\ \omega \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{3.20}$$

where $\eta = \mathcal{C} \frac{\partial H}{\partial \xi}$, the synchronization error, corresponding to this receiver, is found to be

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{K_3+1}{2} & \frac{K_5-K_2}{2} \\ -\frac{K_3+1}{2} & 0 & -\frac{K_4}{2} \\ \frac{K_2-K_5}{2} & \frac{K_4}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -K_1 & \frac{1-K_3}{2} & -\frac{K_2+K_5}{2} \\ \frac{1-K_2}{2} & -c & -\frac{K_4}{2} \\ -\frac{K_2+K_5}{2} & -\frac{K_4}{2} & -K_6 \end{pmatrix} \frac{\partial H}{\partial e}, \tag{3.21}$$

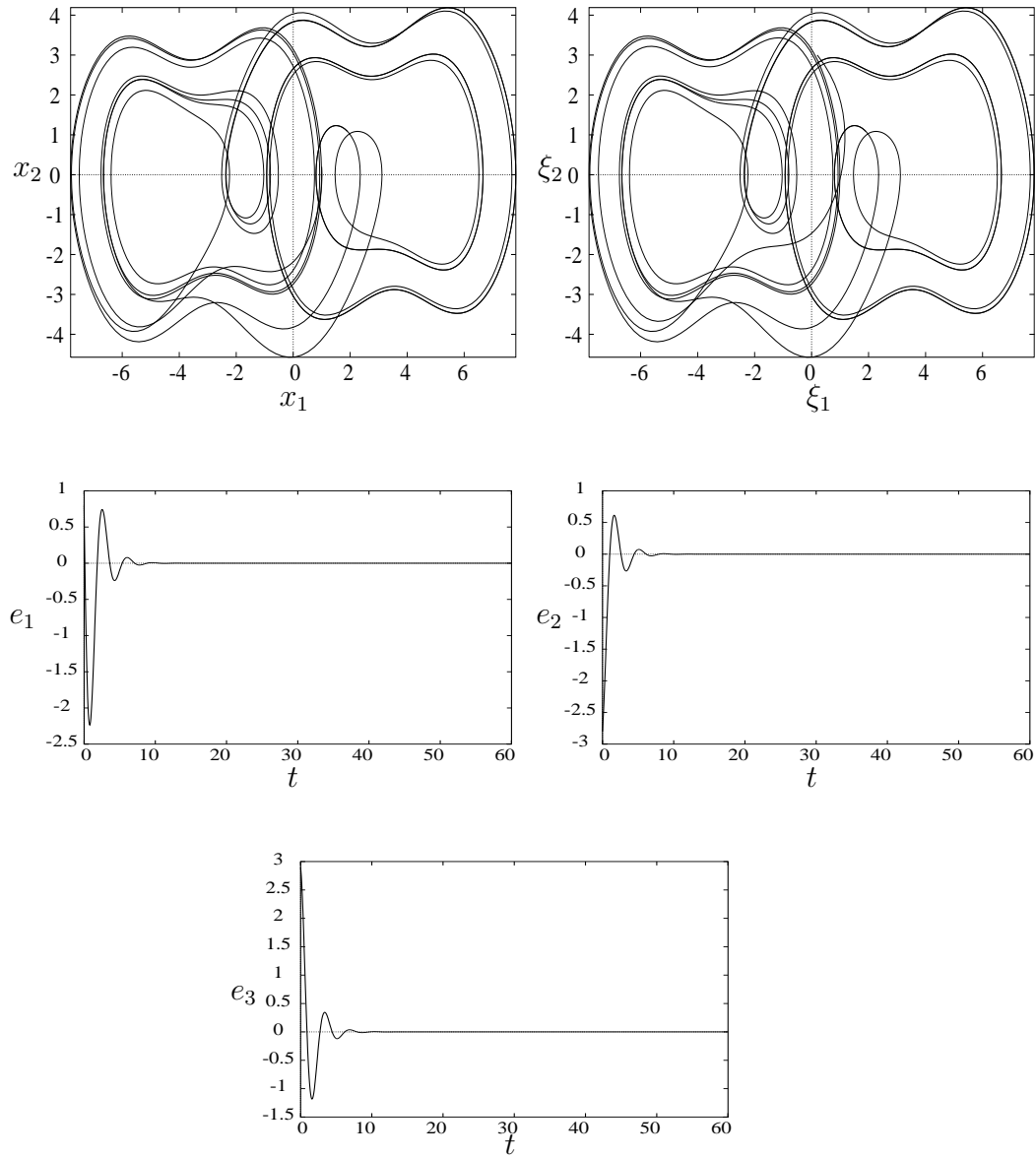


Figure 3.4: The synchronization of the SMIB power systems (3.19) and (3.20) with the following parameter values and for the constant gains $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$ and the initial conditions $x(0) = (1, 0.2, 3)^T, \xi(0) = (0.2, 3, 0.1)^T$.

we can easily know that x_3 and ξ_3 are synchronized with each other, so we only concern the synchronization of other variables, this was achieved by setting

$$K_1 > 0, \quad (K_3 - 1)^2 < 4cK_1, \\ K_1(4cK_6 - K_4^2) + K_4(K_3 - 1)(K_2 + K_5) - K_6(K_3 - 1)^2 - c(K_2 + K_5)^2 > 0$$

Figure 3.4 shows the performance of the designed receiver with the following parameter values for the system and for the constant gains: $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$.

4 Synchronization of Time–Delay Chaotic Systems

Following [22], that the time–delay system, a mathematic description by a delay differential equation (DDE), which in its simplest form of a single fixed time–delay τ is given by

$$\dot{x} = f(x, x(t - \tau)), \quad x \in \mathbf{R}^n, \tag{4.1}$$

can be written in the Generalized Hamiltonian canonical form,

$$\dot{x} = \mathcal{J} \frac{\partial H}{\partial x} + \mathcal{S}(x) \frac{\partial H}{\partial x} + \mathcal{F}(x, x(t - \tau)), \quad x \in \mathbf{R}^n. \tag{4.2}$$

The properties of \mathcal{J}, \mathcal{S} and $\mathcal{F}(x, x(t - \tau))$ as before in Section 2. The observer design is also similar to (2.6). Now we use two examples to describe it.

4.1 Delayed Duffing–Van der Pol system

The system under consideration is a nonlinear oscillator governed by equation

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\alpha(1 - x_1^2)x_2 + F \cos(\omega t) - \beta x_1^3 + \gamma x_1(t - \tau), \end{cases} \tag{4.3}$$

taking $H(x) = \frac{1}{2}[x_1^2 + x_2^2]$ as the Hamiltonian energy function, we write the system in Generalized Hamiltonian form as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 \\ \alpha x_1^2 x_2 + F \cos(\omega t) - \beta x_1^3 + \gamma x_1(t - \tau) \end{pmatrix}. \tag{4.4}$$

The destabilizing vector field requires for complete cancelation at the receiver, namely, the variables x_1 and x_2 . Then the output is chosen as $y = [y_1, y_2]^T = [x_1, x_2]^T$, the matrices \mathcal{C}, \mathcal{S} and \mathcal{I} are given by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. But we can observe that the pair of matrices $(\mathcal{C}_1, \mathcal{S})$ is also a detectable pair. An injection of the synchronization

error $e_1 = x_1 - \xi_1$ suffices to have an asymptotically stable trajectory convergence. The receiver would then be designed, exploiting this last observation, as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &+ \begin{pmatrix} 0 \\ \alpha x_1^2 x_2 + F \cos(\omega t) - \beta x_1^3 + \gamma x_1(t - \tau) \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (y - \eta), \end{aligned} \quad (4.5)$$

where $\eta = C_1 \frac{\partial H}{\partial \xi}$ corresponding to this receiver, we can obtain the synchronization error

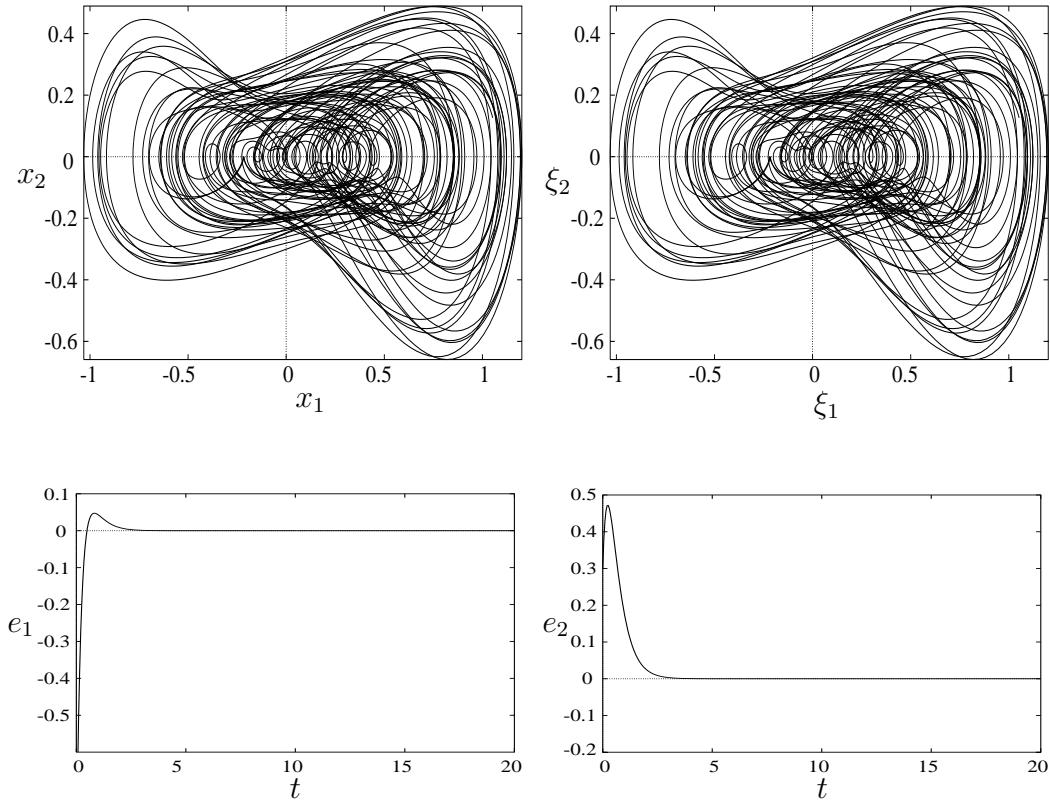


Figure 4.1: Delayed Duffing–Van der Pol system (4.4) and (4.5) with the following parameter values and for the constant gains $\alpha = 1.2$, $\beta = 0.75$, $F = 0.2$, $\gamma = 0.4$, $\omega = 0.5$, $\tau = 25$, $K_1 = 5$, $K_2 = 2.5$. and the initial conditions $x(0) = (0.1, 0.5)^T$, $\xi(0) = (1, 0.2)^T$.

as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} + \frac{K_2}{2} \\ -\frac{1}{2} - \frac{K_2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -K_1 & \frac{1}{2} - \frac{K_2}{2} \\ \frac{1}{2} - \frac{K_2}{2} & -\alpha \end{pmatrix} \frac{\partial H}{\partial e}. \quad (4.6)$$

We could prescribe K_1 , and K_2 , in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester’s Criterion, this is achieved by setting $K_1 > 0$, $2K_1\alpha > (K_2 - 1)^2$.

Figure 4.1 shows the synchronization of the systems (4.4) and (4.5), the chosen parameters are set as the following $\alpha = 1.2$, $\beta = 0.75$, $F = 0.2$, $\gamma = 0.4$, $\omega = 0.5$, $\tau = 25$, with receiver parameter gains $K_1 = 5$, $K_2 = 2.5$.

4.2 Delayed SMIB power system

Let's also consider the classical SMIB power system (3.17) with a time-delay τ ,

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -cx_2 - \beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t - \tau)), \\ \dot{x}_3 = \omega, \end{cases} \tag{4.7}$$

taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{4.8}$$

we write the system in the Generalized Hamiltonian canonical form as

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} \\ &+ \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t - \tau)) \\ \omega \end{pmatrix}. \end{aligned} \tag{4.9}$$

Following the analysis in Section 3.4, taking the variable $y = [y_1, y_2]^T = [x_1, x_3]^T$ as the output, then the matrices \mathcal{C} , \mathcal{S} and \mathcal{I} are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the receiver would then be designed as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \\ &\begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t - \tau)) \\ \omega \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{4.10}$$

Figure 4.2 shows the performance of the designed receiver with the following parameter values of the system and the constant gains: $c = 2$, $\beta = 6$, $f = 9$, $R = 50$, $\omega = 1$, $\epsilon = 10$, $\tau = 0.6$, $K_1 = 0.5$, $K_2 = 1.5$, $K_3 = 1$, $K_4 = 0.1$, $K_5 = -1.5$, $K_6 = 0.75$.

5 Conclusion

In this paper, we have considered the problem of synchronization of several famous chaotic dynamical systems, including two types of synchronization which are respectively the dynamical systems with time-delay and that without any time-delay, from the perspective

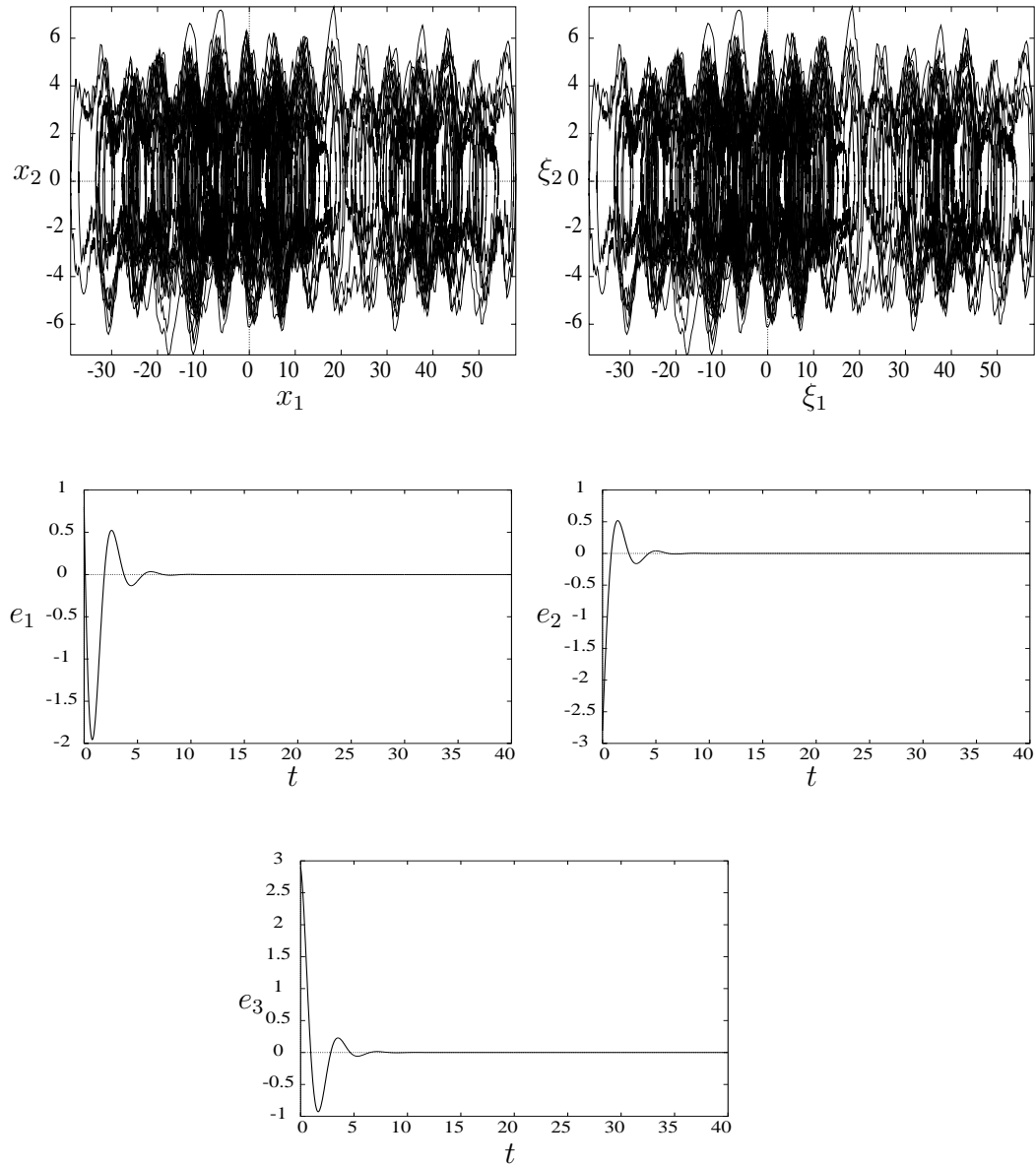


Figure 4.2: The synchronization of the SMIB power systems (4.9) and (4.10) with the following parameter values and for the constant gains $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$ and the initial conditions $x(0) = (1, 0.2, 3)^T, \xi(0) = (0.2, 3, 0.1)^T$.

of Generalized Hamiltonian systems (developed by Sira–Ramírez and Cruz^[22]). Several chaotic dynamical systems, consisting of ones which are without any time–delay and that with time–delay, are analyzed from this perspective and their synchronization were both confirmed. The six figures show that the chaotic systems under consideration achieved synchronization with their receivers immediately, respectively.

In the course of applying this method to the synchronization of some dynamical systems, we confront one problem: for some systems as we choose one variable as the output signal, we can't find suitable values of K_i , $i = 1, 2, \dots$, for the receiver to synchronize with the master. In order to overcome such problem we add one or more output signals, to increase the number of the constants K_i , $i = 1, 2, \dots$, and extend the flexibility of the constants to be chosen, easily we obtain the synchronization. But a small problem is that the computation may be a little more complex.

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Residual Generator Based Measurement of Current Input Into a Cell

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Abstract: We address the problem of real-time estimation of the excitation current into a cell. The membrane voltages can be measured experimentally, even in vivo. On the other hand, a direct measurement of the current into a cell interferes with the voltage activity. We propose a method to estimate the current input into a cell using the measured voltage and an observer based residual generator scheme. Our approach can be applied to all cell models of the Hodgkin-Huxley type.

Keywords: *cell models; observer; residual generator; nonlinear control.*

Mathematics Subject Classification (2000): 92C37, 93B30, 93C10, 93D25.

1 Introduction

The paper addresses a measurement problem from cell biology. In particular, the potential on the membrane of a cell such as a neuron can be recorded directly via voltage measurement. The electrophysiological behaviour of the cell, especially the membrane voltage dynamics, is influenced by ionic channels which allow ions to move through the cell's membrane.

Voltage clamp is a standard method to measure ionic currents across a membrane [1, 2]. The technique developed by Cole [3] uses two electrodes. One electrode is used to measure the intracellular voltage relative to ground. This voltage is amplified and compared with a given reference voltage. The difference of these signals is amplified again and feed back via the second electrode. With this feedback the potential on the membrane is hold at a specific level. The current injected into the cell by this feedback

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structure compensates the ionic current, i.e., the output current of the second amplifier has equal value but opposite sign of the ionic current.

During the last decades, several more advanced measurement techniques such as patch clamp [4] have been developed, see [5,6] and references cited there. Unfortunately, most measurement methods have at least one the following disadvantages: measurement inferences with the cell's activities, or requirement of very specific laboratory conditions (limiting the applicability *in vivo*), or expensive measurement devices, or slow dynamics.

The author suggests a new technique for a real-time estimation of the current input into a cell. The paper extends preliminary results published in [7,8]. Our approach requires a dynamical model describing the interaction between the membrane voltage and the different kinds of current. To estimate the input current we use an observer based residual generator. The usage of an observer to estimate quantities which are not directly available for measurement is quite common in system and control theory [9]. Originally, residual generators are used to detect the occurrence of faults in a given system [10]. Here, we apply such a generator to reconstruct the input current quantitatively.

In Section 2 we present a large class of cell models. An observer based method to estimate the current input into a cell is derived in Section 3. Our estimation scheme is used in Section 4 to reconstruct the input current for the Hodgkin–Huxley model of a neuron.

2 Cell Models

2.1 Conductance-based models

Important aspects of the biophysical behaviour of an excitable cell such as a neuron can be represented by an equivalent circuit model. The dynamics of the membrane voltage V is governed by a differential equation

$$C\dot{V} = I - \sum_j \tilde{g}_j (V - V_j) \quad (2.1)$$

with a capacitance $C > 0$. The current I is injected into the cell, either from a coupling with other cells, or by an electrode. The sum in (2.1) represents the ionic currents, that is the leak current and the currents flowing through ionic channels. The reversal potential of the i th channel is denoted by V_i . The ionic channels, which describe the concentrations of certain ions, have two states: open and closed. The probability of a channel to be open is represented by a so-called gating variable. In particular, the conductances \tilde{g}_j may depend on some gating variables. The dynamics of the gating variables w_1, \dots, w_p is governed by differential equations of the form

$$\dot{w}_i = \alpha_i(V)(1 - w_i) - \beta_i(V)w_i \quad \text{for } i = 1, \dots, p \quad (2.2)$$

with functions α_i and β_i . These functions result from the Markov model of the associated ionic channel (see [11]). More precisely, the functions α_i and β_i are the transition rates for opening and closing an channel. Furthermore, we have $\alpha_i(V), \beta_i(V) > 0$ for all V .

2.2 Hodgkin–Huxley model

The most well-known simulation model for excitable cells such as neurons and cardiac myocytes was developed by Hodgkin and Huxley [12]. Originally, the model describes

the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. The model takes the concentrations of sodium ions (Na^+) and potassium ions (K^+) into account. In this case, Eq. (2.1) becomes

$$C\dot{V} = I - I_{Na} - I_K - I_L \quad (2.3)$$

with $C = 1\mu\text{F}/\text{cm}^2$. The ionic currents I_{Na} and I_K , and the leak current I_L are given by

$$\begin{aligned} I_{Na} &= g_{Na}m^3h(V - V_{Na}), \\ I_K &= g_Kn^4(V - V_K), \\ I_L &= g_L(V - V_L), \end{aligned} \quad (2.4)$$

with constant conductances $g_{Na} = 120\text{mS}/\text{cm}^2$, $g_K = 36\text{mS}/\text{cm}^2$, $g_L = 0.3\text{mS}/\text{cm}^2$, the potentials $V_{Na} = 50\text{mV}$, $V_K = -77\text{mV}$, $V_L = -54.4\text{mV}$ and the gating variables m, h, n . The gating variables are governed by differential equations of the form (2.2), namely

$$\begin{aligned} \dot{m} &= \alpha_m(V)(1 - m) - \beta_m(V)m, \\ \dot{h} &= \alpha_h(V)(1 - h) - \beta_h(V)h, \\ \dot{n} &= \alpha_n(V)(1 - n) - \beta_n(V)n, \end{aligned} \quad (2.5)$$

with the normalized functions

$$\begin{aligned} \alpha_m(V) &= 0.1(V + 40)/(1 - \exp(-(V + 40)/10)), \\ \beta_m(V) &= 4\exp(-(V + 65)/18), \\ \alpha_h(V) &= 0.07\exp(-V + 65)/20, \\ \beta_h(V) &= 1/(1 + \exp(-(V + 35)/10)), \\ \alpha_n(V) &= 0.01(V + 55)/(1 - \exp(-(V + 55)/10)), \\ \beta_n(V) &= 0.125\exp(-(V + 65)/80). \end{aligned} \quad (2.6)$$

The voltage V in Eq. (2.6) is in mV. The gating variables as well as the function values of the transition rates (2.6) are dimensionless.

2.3 Similar models

There are many other models of cells and fibers that are based on the Hodgkin-Huxley formalism. One of the most well-know models of this type, namely the Morris-Lecar model [13], arose from studies of the excitability of the barnacle muscle fiber. The model takes ionic currents resulting from potassium ions (K^+) and calcium ions (Ca^+) into account. The channels' behaviour is modelled by $p = 2$ gating variables. Overall, the model has the dimension $n = 3$. Under some circumstances, the model can be reduced further to the dimension $n = 2$ (see [13]).

Several system-theoretical approaches have been used to reduce the dimension of the Hodgkin-Huxley model. For example, FitzHugh [14] observed that the spike-like oscillations of the Hodgkin-Huxley model are similar to oscillations generated by the Bonhoeffer-van der Pol equation [15]. An equivalent circuit model using a tunnel diode was derived in [16]. As a whole, the FitzHugh-Nagumo model has the dimension $n = 2$.

In the past, the low dimensional models have been simulated on analog computers. Today fast digital computers allow the simulation of significantly more complicated models. During the last decades, several advanced models have been developed. The Connor-Stevens model [17] takes $p = 5$ gating variables into account and is therefore 6-dimensional. Another widely used model was derived by Traub [18, 19] and is 5-dimensional. Further informations on the modelling of excitable cells etc. can be found in [11, 20, 21].

3 Observer Based Residuum Generation

3.1 Observer structure

The 4-dimensional Hodgkin–Huxley model (2.3)-(2.5) can be written as

$$\dot{V} = f(V, w) + \frac{1}{C}I, \quad (3.1a)$$

$$\dot{w} = g(V, w) \quad (3.1b)$$

with smooth nonlinear maps $f : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, and $w = (m, h, n)^T$. To estimate the unknown input current I via the measured voltage V we consider the following dynamic system:

$$\dot{\hat{V}} = f(\hat{V}, \hat{w}) + k(V - \hat{V}), \quad (3.2a)$$

$$\dot{\hat{w}} = g(V, \hat{w}), \quad (3.2b)$$

$$\tilde{V} = V - \hat{V}. \quad (3.2c)$$

The first part (3.2a),(3.2b) is a high-gain observer for (3.1), where the constant observer gain $k > 0$ acts only on the first subsystem (3.2a). A difference to standard high-gain observers is the direct injection of the measured voltage V into the second subsystem (3.2b). The output of (3.2) is the observation output error \tilde{V} given in Eq. (3.2c). As a whole, system (3.2) has the structure of an observer based residual generator used for fault detection (see [9] and references cited there). The observation error is governed by the error dynamics

$$\dot{\tilde{V}} = f(V, w) - f(\hat{V}, \hat{w}) - k\tilde{V} + \frac{1}{C}I, \quad (3.3a)$$

$$\dot{\tilde{w}} = g(V, w) - g(V, \hat{w}), \quad (3.3b)$$

$$\tilde{V} = V - \hat{V}, \quad (3.3c)$$

where $\tilde{w} = w - \hat{w}$.

3.2 Passivity

For a given initial value of (3.1) and a bounded input I , the trajectories of the original system stay in a compact subset $\mathbb{X} \subset \mathbb{R}^n$. Since the map f is continuously differentiable (see Eqs. (2.3) and (2.4)), it is also Lipschitz continuous on \mathbb{X} . We assume that there exist constants $L_1, L_2 > 0$ such that

$$\left| f(V, w) - f(\hat{V}, \hat{w}) \right| \leq L_1 \left| V - \hat{V} \right| + L_2 \|w - \hat{w}\|$$

holds on $(V, w), (\hat{V}, \hat{w}) \in \mathbb{X}$, where $|\cdot|$ is the absolute value and $\|\cdot\|$ is the euclidean norm.

In contrast to classical observer design we are not directly interested in the stability of the error dynamics (3.3), but in its input-output behaviour. Instead, we will describe the input-output behaviour qualitatively using the concept of passivity [22]. The candidate storage function

$$S(\tilde{V}, \tilde{w}) = \frac{C}{2} \tilde{V}^2 + \frac{1}{2} \sum_{i=1}^p \tilde{w}_i^2$$

is positive definite and radially unbounded. The derivative along the error dynamics (3.3) reads as

$$\dot{S}(\tilde{V}, \tilde{w}) \Big|_{(3.3)} = C\tilde{V}\dot{\tilde{V}} + \sum_{i=1}^p \tilde{w}_i \dot{\tilde{w}}_i. \tag{3.4}$$

For the first summand of (3.4) we obtain

$$\begin{aligned} C\tilde{V}\dot{\tilde{V}} &= C\tilde{V} \left(f(V, w) - f(\hat{V}, \hat{w}) - k\tilde{V} + \frac{1}{C}I \right) \\ &\leq C\tilde{V} \left| f(V, w) - f(\hat{V}, \hat{w}) \right| - Ck\tilde{V}^2 + \tilde{V}I \\ &\leq CL_1\tilde{V} \left| \tilde{V} \right| + CL_2\tilde{V} \|\tilde{w}\| - Ck\tilde{V}^2 + \tilde{V}I \\ &\leq CL_1\tilde{V}^2 + \theta CL_2\tilde{V}^2 + CL_2\theta^{-1} \|\tilde{w}\|^2 - Ck\tilde{V}^2 + \tilde{V}I \\ &= C(L_1 + \theta L_2 - k)\tilde{V}^2 + CL_2\theta^{-1} \|\tilde{w}\|^2 + \tilde{V}I \end{aligned}$$

for any $\theta > 0$ because $ab \leq \theta a^2 + \theta^{-1}b^2$ for all $a, b \in \mathbb{R}$. Taking the special form (2.2) of subsystem (3.3b) into account, the second summand of (3.4) is bounded by

$$\sum_{i=1}^p \tilde{w}_i \dot{\tilde{w}}_i = - \sum_{i=1}^p (\alpha_i(V) + \beta_i(V)) \tilde{w}_i^2 \leq -\mu \|\tilde{w}\|^2 \tag{3.5}$$

with

$$\mu := \inf_{i,V} (\alpha_i(V) + \beta_i(V)) > 0$$

since the functions α_i and β_i are positive and the measured voltage in bounded. Altogether we obtain

$$\begin{aligned} \dot{S}(\tilde{V}, \tilde{w}) \Big|_{(3.3)} &\leq -C(k - L_1 - \theta L_2)\tilde{V}^2 - (\mu - CL_2\theta^{-1}) \|\tilde{w}\|^2 + I\tilde{V} \\ &\leq -\rho\tilde{V}^2 - \nu \|\tilde{w}\|^2 + I\tilde{V}, \end{aligned}$$

where $\rho := C(k - L_1 - \theta L_2)$ and $\nu := \mu - CL_2\theta^{-1}$. Choosing $\theta > CL_2/\mu$ and $k > L_1 + \theta L_2$ yields $\rho, \nu > 0$. For $I = 0$, the scalar field S is a Lyapunov function, i.e., the point $\tilde{V} = 0, \tilde{w} = 0$ is a globally asymptotically stable equilibrium. However, we also have

$$\dot{S}(\tilde{V}) \Big|_{(3.3)} \leq -\rho\tilde{V}^2 + I\tilde{V},$$

which implies that the error system (3.3) is not only passive, but also output feedback passive [22] with respect to the input I and the output \tilde{V} . Physically, the supply rate $I\tilde{V}$ is the difference of electric power of systems (3.1) and (3.2) provided by the input current source, i.e., the rate of increase of energy is not bigger than the input power.

3.3 Input reconstruction

The residual \tilde{V} generated by (3.2) describes the degree of consistency between the model (3.1) and the observer scheme (3.2). Since the input current is missing in (3.2a), one would expect that the residual \tilde{V} is somehow related to the unknown input I . In the classical application of residual generators, namely in fault detection, the residual is

only used qualitatively to indicate that a fault occurred (and which one). Here, we want to use the residual \tilde{V} quantitatively to obtain an estimate of the input current I .

From Eq. (3.5) we conclude that the equilibrium $\tilde{w} = 0$ of subsystem (3.3b) is globally asymptotically stable (uniform in V). After the transient oscillations of (3.3) we can expect $\tilde{w} \approx 0$, i.e., $w \approx \hat{w}$. To ensure the passivity of (3.3), we have to choose $k > 0$ sufficiently large. This implies

$$\left| f(V, w) - f(\hat{V}, w) \right| \ll k \left| \tilde{V} \right| \quad \text{for } t \gg 0, \quad (3.6)$$

which means that the observer correction term is much stronger than the difference between the two systems (3.1) and (3.2).

Next, we consider Eq. (3.3a) near an equilibrium point, i.e., $\dot{\tilde{V}} \approx 0$. From (3.6) we conclude that

$$0 \approx -k \tilde{V} + \frac{1}{C} I.$$

Hence, an estimate of the input current I can be obtained from \tilde{V} by

$$I \approx k C \tilde{V}. \quad (3.7)$$

If the current input I exceeds a certain level, the original system (3.1) oscillates. These oscillations can also be seen at the output \tilde{V} of (3.2), although the oscillations are better suppressed using a large observer gain k . Therefore, we will smooth the current estimate from (3.7) by a m th order low-pass with the continuous time transfer function

$$F(s) = \frac{1}{(1 + sT)^m}$$

with a time constant $T > 0$. For simplicity, the transfer function used here has a multiple real pole at $-1/T$. However, one could also choose from several other filter design techniques (e.g. Bessel, Butterworth or Cauer filter). Combining time and frequency domain as well as taking the scaling (3.7) into account, the final estimate \hat{I} of I results from

$$\hat{I}(t) = \frac{k C}{(1 + sT)^m} \circ \tilde{V}(t). \quad (3.8)$$

The whole estimation scheme is given in Fig. 3.1.

4 Simulation Results

For the simulation of the Hodgkin–Huxley model (2.3)-(2.6) we used the initial values $V(0) = 65\text{mV}$, $m(0) = 0.1$, $h(0) = 0.6$ and $n(0) = 0.3$. The input current was chosen as follows:

$$I(t) = \begin{cases} 10\mu\text{A}/\text{cm}^2 & \text{for } 0\text{ms} \leq t < 80\text{ms}, \\ 25\mu\text{A}/\text{cm}^2 & \text{for } 80\text{ms} \leq t < 140\text{ms}, \\ 15\mu\text{A}/\text{cm}^2 & \text{for } t \geq 140\text{ms}. \end{cases} \quad (4.1)$$

The simulation was carried out by the scientific software package Scilab [23]. The generated output trajectory is shown in Fig. 4.1. The membrane voltage shows spike-like oscillations, whose amplitude and frequency vary according to the current in the specific time interval.

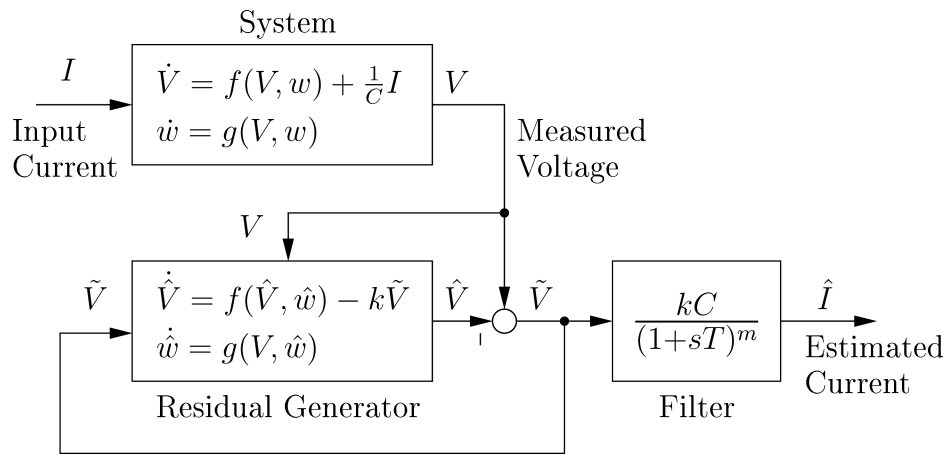


Figure 3.1: Current Estimation Scheme.

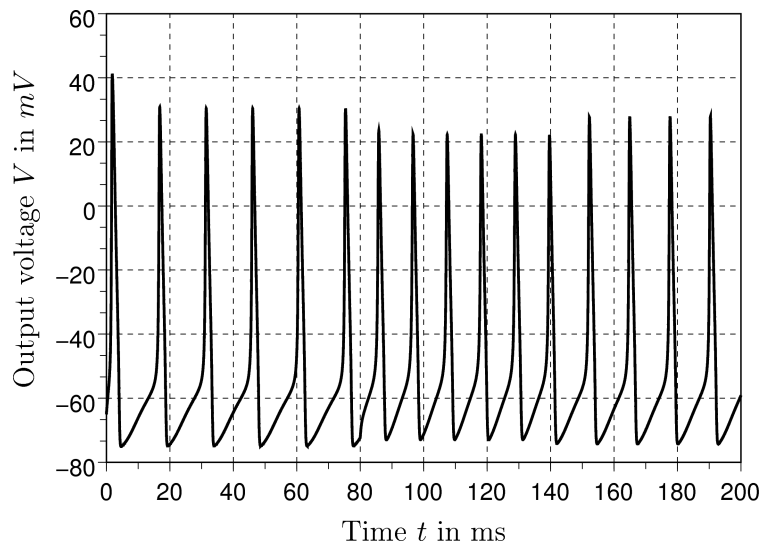


Figure 4.1: Output voltage of the Hodgkin–Huxley model (2.3)-(2.6).

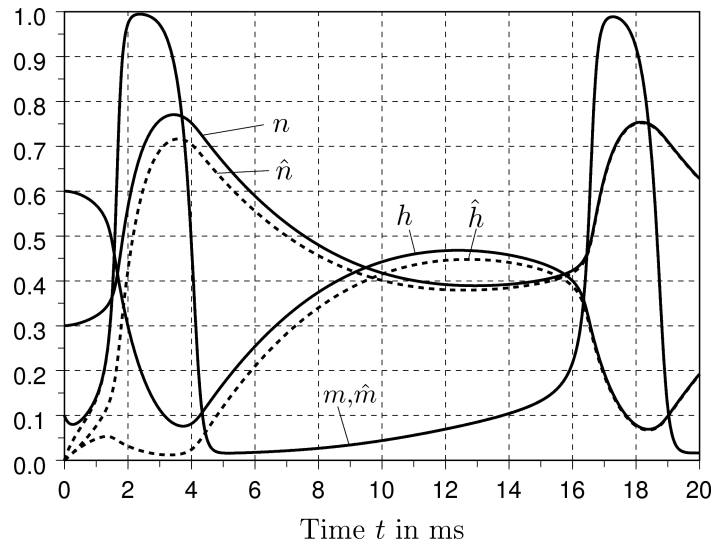


Figure 4.2: Trajectories of the gating variables and its estimates.

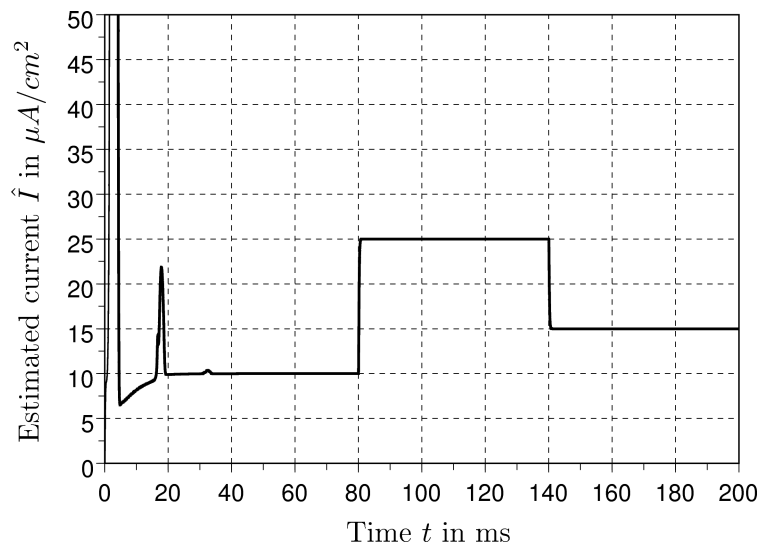


Figure 4.3: Estimated current \hat{I} according to Eq. (3.8).

For the observer (3.2a) we have chosen the initial value $\hat{V}(0) = V(0) = 65\text{mV}$ to be consistent with the measurement (see [24, Section 3.3]). Since we have no further information on the state of the gating variables we used the 3-dimensional zero vector as initial value of (3.2b), i.e., $\hat{m}(0) = \hat{h}(0) = \hat{n}(0) = 0$. The transient behaviour of the observer (3.2b) for the gating variables is shown in Fig. 4.2. We used solid lines for the original gating variables m, n, h and dashed lines for its estimates $\hat{m}, \hat{h}, \hat{n}$ generated by the observer (3.2b). Note that Fig. 4.2 has a different time domain as Fig. 4.1, i.e., we only show the first 20ms in Fig. 4.2. After that, the gating variables and its estimates basically coincide, i.e., they cannot be separated visually.

The observer scheme (3.2) with the gain $k = 1000$ yields the residual \tilde{V} . To obtain an estimate \hat{I} for the current I we have to scale and filter this voltage difference according to Eq. (3.8), where we used the normalized filter time constant $T = 0.1$. The result is shown in Fig. 4.3. After some transient oscillations the estimated current matches the input current (4.1) almost perfectly.

5 Conclusions

We suggested a new approach to estimate the current input into a (possibly living) cell. This method requires a reasonable precise model of the cell under consideration. In contrast to voltage and patch clamp techniques, our approach cannot be used to analyze new cell types or cells with significant anomalies. However, our measurement technique can be used to verify given models and to study the interaction of cells (such as neurons interconnected by synapses).

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