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Dominant and Recessive Solutions of Self-Adjoint Matrix Systems on Time Scales

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Abstract: In this study, linear second-order self-adjoint delta-nabla matrix systems on time scales are considered with the motivation of extending the analysis of dominant and recessive solutions from the differential and discrete cases to any arbitrary dynamic equations on time scales. These results emphasize the case when the system is non-oscillatory.

Keywords: *time scales; self-adjoint; matrix equations; second-order; non-oscillation; linear.*

Mathematics Subject Classification (2000): 39A11, 34C10.

1 Introduction

To motivate this study of dominant and recessive solutions, consider the self-adjoint second-order scalar differential equation

$$(px')'(t) + q(t)x(t) = 0.$$

According to the classical formulation by Kelley and Peterson [1, Section 5.6], a solution u is recessive at ω and a second, linearly-independent solution v is dominant at ω if the conditions

$$\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0, \qquad \int_{t_0}^{\omega} \frac{1}{p(t)u^2(t)}dt = \infty, \qquad \int_{t_0}^{\omega} \frac{1}{p(t)v^2(t)}dt < \infty$$

all hold; see also a related discussion for three-term difference equations in Ahlbrandt [2], Ahlbrandt and Peterson [3, Section 5.10], Ma [4], and scalar dynamic equations in Bohner

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and Peterson [5, Section 4.3], Messer [6], and [7, Section 4.5]. It is the purpose of this work to introduce a robust treatment of these types of solutions for the corresponding self-adjoint second-order matrix dynamic equation on time scales. Dynamic equations on time scales have been introduced by Hilger and Aulbach [8, 9] to unify, extend, and generalize the theory of ordinary differential equations, difference equations, quantum equations, and all other differential systems defined over nonempty closed subsets of the real line. We use this overarching theory to extend from the discrete case [3, 4] the matrix difference system

$$\Delta\left(P(t)\Delta X(t-1)\right)+Q(t)X(t)=0, \tag{1.1}$$

for $q>1$ the quantum system [10]

$$D^q\left(PD_qX\right)(t)+Q(t)X(t)=0, \tag{1.2}$$

and the continuous case developed by Reid [11–15]

$$\left(PX'\right)'(t)+Q(t)X(t)=0, \tag{1.3}$$

to the general time scale setting, which admits the self-adjoint delta-nabla matrix system

$$\left(PX^{\Delta}\right)^{\nabla}(t)+Q(t)X(t)=0. \tag{1.4}$$

Only recently has (formal) self-adjointness been investigated for arbitrary time scales, even in the scalar case, by Messer [6], Anderson, Guseinov and Hoffacker [16], and Atici and Guseinov [17]; self-adjoint matrix systems on time scales are relatively unexplored at this time [18]. More commonly authors Bohner and Peterson [5, Chapter 5] and Erbe and Peterson [19] focus on

$$\left(PX^{\Delta}\right)^{\Delta}(t)+Q(t)X^{\sigma}(t)=0, \tag{1.5}$$

which they term "self-adjoint" since it admits a Lagrange identity. Thus, these results connected to the self-adjoint system (1.4) extend and generalize the results related to (1.1), (1.2) and (1.3), and are different from those worked out for (1.5).

2 Technical Results on Time Scales

Any arbitrary nonempty closed subset of the reals $\mathbb{R}$ can serve as a time scale $\mathbb{T}$; see the books by Bohner and Peterson [5, 7] and the papers by Hilger and Aulbach [8, 9]. Here and in the sequel we assume a working knowledge of basic time-scale notation and the time-scale calculus. In addition, the following results will prove to be useful.

Theorem 2.1 *If f is delta differentiable at $t\in\mathbb{T}^{\kappa}$, then $f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t)$. If f is nabla differentiable at $t\in\mathbb{T}_{\kappa}$, then $f^{\rho}(t)=f(t)-\nu(t)f^{\nabla}(t)$.*

Theorem 2.2 *Let $f:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ be a continuous function of two variables $(t,s)\in\mathbb{T}\times\mathbb{T}$, and $a\in\mathbb{T}$. Assume that f has continuous derivatives f^{Δ} and f^{∇} with respect to t. Then the following formulas hold:*

(i) $\left(\int_a^t f(t,s)\Delta s\right)^{\Delta}=f(\sigma(t),t)+\int_a^t f^{\Delta}(t,s)\Delta s,$

(ii) $\left(\int_a^t f(t,s)\Delta s\right)^{\nabla}=f(\rho(t),\rho(t))+\int_a^t f^{\nabla}(t,s)\Delta s,$

(iii) $\left(\int_a^t f(t,s)\nabla s\right)^\Delta = f(\sigma(t),\sigma(t)) + \int_a^t f^\Delta(t,s)\nabla s,$

(iv) $\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f^\nabla(t,s)\nabla s.$

The following sets and statement [6, Theorem 2.6] (see also [17]) will play an important role in many of our calculations.

Definition 2.1 Let the time-scale sets A and B be given by

$$A := \{t \in \mathbb{T} : t \text{ is a left-dense and right-scattered point}\}, \tag{2.1}$$

and

$$B := \{t \in \mathbb{T} : t \text{ is a right-dense and left-scattered point}\}. \tag{2.2}$$

It follows that for $t \in A$,

$$\lim_{s\to t^-} \sigma(s) = t,$$

and for $t \in \mathbb{T}\backslash A$, $\sigma(\rho(t)) = t$. Likewise for $t \in B$,

$$\lim_{s\to t^+} \rho(s) = t,$$

and for $t \in \mathbb{T}\backslash B$, $\rho(\sigma(t)) = t$.

Theorem 2.3 *Let the sets A and B be given as in* (2.1) *and* (2.2)*, respectively.*

(i) If $f : \mathbb{T} \to \mathbb{R}$ is Δ differentiable on $\mathbb{T}^\kappa$ and f^Δ is right-dense continuous on $\mathbb{T}^\kappa$, then f is ∇ differentiable on $\mathbb{T}_\kappa$, and

$$f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) & : t \in \mathbb{T}\backslash A, \\ \lim_{s\to t^-} f^\Delta(s) & : t \in A. \end{cases}$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is ∇ differentiable on $\mathbb{T}_\kappa$ and f^∇ is left-dense continuous on $\mathbb{T}_\kappa$, then f is Δ differentiable on $\mathbb{T}^\kappa$, and

$$f^\Delta(t) = \begin{cases} f^\nabla(\sigma(t)) & : t \in \mathbb{T}\backslash B, \\ \lim_{s\to t^+} f^\nabla(s) & : t \in B. \end{cases}$$

The statements of the previous theorem can be formulated as $\left(f^\Delta\right)^\rho = f^\nabla$ and $\left(f^\nabla\right)^\sigma = f^\Delta$ provided that f^Δ and f^∇ are continuous, respectively.

3 Self-Adjoint Matrix Equations

All of the results in this section are from Anderson and Buchholz [18]. Let P and Q be Hermitian $n \times n$-matrix-valued functions on a time scale $\mathbb{T}$ such that $P > 0$ (positive definite) and Q are continuous for all $t \in \mathbb{T}$. (A matrix M is *Hermitian* iff $M^* = M$, where * indicates conjugate transpose.) In this section we are concerned with the second-order (formally) self-adjoint matrix dynamic equation

$$LX = 0, \quad \text{where} \quad LX(t) := \left(PX^\Delta\right)^\nabla(t) + Q(t)X(t), \quad t \in \mathbb{T}_\kappa^\kappa. \tag{3.1}$$

Definition 3.1 Let $\mathbb{D}$ denote the set of all $n \times n$ matrix-valued functions X defined on $\mathbb{T}$ such that X^{Δ} is continuous on $\mathbb{T}^{\kappa}$ and $(PX^{\Delta})^{\nabla}$ is left-dense continuous on $\mathbb{T}^{\kappa}_{\kappa}$. Then X is a solution of (3.1) on $\mathbb{T}$ provided $X \in \mathbb{D}$ and $LX(t) = 0$ for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Definition 3.2 (Regressivity) An $n \times n$ matrix-valued function M on a time scale $\mathbb{T}$ is *regressive* with respect to $\mathbb{T}$ provided

$$I + \mu(t)M(t) \text{ is invertible for all } t \in \mathbb{T}^{\kappa}, \tag{3.2}$$

and the class of all such regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}).$$

Theorem 3.1 *Let $a \in \mathbb{T}^{\kappa}$ be fixed and X_a, X_a^{Δ} be given constant $n \times n$ matrices. Then the initial boundary value problem*

$$\left(PX^{\Delta}\right)^{\nabla}(t) + Q(t)X(t) = 0, \quad X(a) = X_a, \quad X^{\Delta}(a) = X_a^{\Delta}$$

has a unique solution.

Definition 3.3 If $X, Y \in \mathbb{D}$, then the (generalized) *Wronskian matrix* of X and Y is given by

$$W(X,Y)(t) = X^*(t)P(t)Y^{\Delta}(t) - [P(t)X^{\Delta}(t)]^*Y(t)$$

for $t \in \mathbb{T}^{\kappa}$.

Theorem 3.2 (Lagrange identity) *If $X, Y \in \mathbb{D}$, then*

$$W(X,Y)^{\nabla}(t) = X^*(t)(LY)(t) - (LX(t))^*\,Y(t), \quad t \in \mathbb{T}^{\kappa}_{\kappa}.$$

Definition 3.4 Define the inner product of $n \times n$ matrices M and N on $[a,b]_{\mathbb{T}}$ for $a < b$ to be

$$\langle M, N \rangle = \int_a^b M^*(t)N(t)\nabla t, \quad M, N \in C_{ld}(\mathbb{T}), \quad a, b \in \mathbb{T}^{\kappa}. \tag{3.3}$$

Corollary 3.1 (Self-adjoint operator) *The operator L in* (3.1) *is formally self adjoint with respect to the inner product* (3.3)*; that is, the identity*

$$\langle LX, Y \rangle = \langle X, LY \rangle$$

holds provided $X, Y \in \mathbb{D}$ and X, Y satisfy $W(X,Y)(t)\big|_a^b = 0$, called the self-adjoint boundary conditions.

Corollary 3.2 (Abel's formula) *If X, Y are solutions of* (3.1) *on $\mathbb{T}$, then*

$$W(X,Y)(t) \equiv C, \quad t \in \mathbb{T}^{\kappa}_{\kappa},$$

where C is a constant matrix.

From Abel's formula we get that if $X \in \mathbb{D}$ is a solution of (3.1) on $\mathbb{T}$, then

$$W(X,X)(t) \equiv C, \quad t \in \mathbb{T}^{\kappa}_{\kappa},$$

where C is a constant matrix. With this in mind we make the following definition.

Definition 3.5 Let $X, Y \in \mathbb{D}$ and W be given as in (3.3).

(i) $X \in \mathbb{D}$ is a prepared (conjoined, isotropic) solution of (3.1) iff X is a solution of (3.1) and
$$W(X,X)(t) \equiv 0, \quad t \in \mathbb{T}^\kappa.$$

(ii) $X, Y \in \mathbb{D}$ are normalized prepared bases of (3.1) iff X, Y are two prepared solutions of (3.1) with
$$W(X,Y)(t) \equiv I, \quad t \in \mathbb{T}^\kappa.$$

Theorem 3.3 *Assume that $X \in \mathbb{D}$ is a solution of* (3.1) *on $\mathbb{T}$. Then the following are equivalent:*

(i) X is a prepared solution;

(ii) $X^*(t)P(t)X^\Delta(t)$ is Hermitian for all $t \in \mathbb{T}^\kappa$;

(iii) $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}^\kappa$.

Note that one can easily get prepared solutions of (3.1) by taking initial conditions at $t_0 \in \mathbb{T}$ so that $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian.

In the Sturmian theory for (3.1) the matrix function X^*PX^σ is important. We note the following result.

Lemma 3.1 *Let X be a solution of* (3.1). *If X is prepared, then*
$$X^*(t)P(t)X^\sigma(t) \quad \text{is Hermitian for all} \quad t \in \mathbb{T}^\kappa.$$
Conversely, if there is $t_0 \in \mathbb{T}^\kappa$ such that $\mu(t_0) > 0$ and $X^*(t_0)P(t_0)X^\sigma(t_0)$ is Hermitian, then X is a prepared solution of (3.1). *Moreover, if X is an invertible prepared solution, then*
$$P(t)X^\sigma(t)X^{-1}(t),\ P(t)X(t)(X^\sigma)^{-1}(t),\ \text{and } Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$
are Hermitian for all $t \in \mathbb{T}^\kappa$.

Lemma 3.2 *Assume that X is a prepared solution of* (3.1) *on $\mathbb{T}$. Then the following are equivalent:*

(i) $(X^*)^\sigma PX = X^*PX^\sigma > 0$ on $\mathbb{T}^\kappa$;

(ii) X is invertible and $PX^\sigma X^{-1} > 0$ on $\mathbb{T}^\kappa$;

(iii) X is invertible and $PX(X^\sigma)^{-1} > 0$ on $\mathbb{T}^\kappa$.

Theorem 3.4 (Reduction of order I) *Let $t_0 \in \mathbb{T}^\kappa$, and assume X is a prepared solution of* (3.1) *with X invertible on $\mathbb{T}$. Then a second prepared solution Y of* (3.1) *is given by*
$$Y(t) := X(t)\int_{t_0}^{t} \left(X^*PX^\sigma\right)^{-1}(s)\Delta s, \quad t \in \mathbb{T}^\kappa$$
such that X, Y are normalized prepared bases of (3.1).

Lemma 3.3 *Assume $X, Y \in \mathbb{D}$ are normalized prepared bases of (3.1). Then $U := XE + YF$ is a prepared solution of (3.1) for constant $n \times n$ matrices E, F if and only if F^*E is Hermitian. If $F = I$, then X, U are normalized prepared bases of (3.1) if and only if E is a constant Hermitian matrix.*

Theorem 3.5 (Reduction of order II) *Let $t_0 \in \mathbb{T}^\kappa$, and assume X is a prepared solution of (3.1) with X invertible on $\mathbb{T}$. Then U is a second $n \times n$ matrix solution of (3.1) iff U satisfies the first-order matrix equation*

$$(X^{-1}U)^\Delta(t) = (X^*PX^\sigma)^{-1}(t)F, \quad t \in \mathbb{T}^\kappa, \quad t \geq t_0, \tag{3.4}$$

for some constant $n \times n$ matrix F iff U is of the form

$$U(t) = X(t)E + X(t)\left(\int_{t_0}^t (X^*PX^\sigma)^{-1}(s)\Delta s\right)F, \quad t \in \mathbb{T}, \quad t \geq t_0, \tag{3.5}$$

where E and F are constant $n \times n$ matrices. In the latter case,

$$E = X^{-1}(t_0)U(t_0), \qquad F = W(X,U)(t_0), \tag{3.6}$$

such that U is a prepared solution of (3.1) iff $F^*E = E^*F$.

4 Factorization of the Self-Adjoint Operator

In this section we introduce the Pólya factorization for the self-adjoint matrix-differential operator L defined in (3.1).

Theorem 4.1 (Pólya factorization) *If (3.1) has a prepared solution $U > 0$ (positive definite) on an interval $\mathcal{I} \subset \mathbb{T}$ such that $U^*PU^\sigma > 0$ on $\mathcal{I}$, then for any $X \in \mathbb{D}$ we have on $\mathcal{I}$ a Pólya factorization*

$$LX = M_1^*\left\{M_2(M_1X)^\Delta\right\}^\nabla, \quad M_1 := U^{-1} > 0, \quad M_2 := U^*PU^\sigma > 0.$$

Proof Assume $U > 0$ is a prepared solution of (3.1) on $\mathcal{I} \subset \mathbb{T}$ such that $U^*PU^\sigma > 0$ on $\mathcal{I}$, and let $X \in \mathbb{D}$. Then U is invertible and

$$\begin{aligned}
LX &\overset{\text{Thm 3.2}}{=} (U^*)^{-1}W(U,X)^\nabla \\
&\overset{\text{Def 3.3}}{=} (U^*)^{-1}\left\{U^*PX^\Delta - U^{\Delta*}PX\right\}^\nabla \\
&= M_1^*\left\{U^*[PX^\Delta - (U^*)^{-1}U^{\Delta*}PX]\right\}^\nabla \\
&\overset{\text{Thm 3.1}}{=} M_1^*\left\{U^*[PX^\Delta - PU^\Delta U^{-1}X]\right\}^\nabla \\
&= M_1^*\left\{M_2[(U^\sigma)^{-1}X^\Delta - (U^\sigma)^{-1}U^\Delta U^{-1}X]\right\}^\nabla \\
&= M_1^*\left\{M_2[(U^\sigma)^{-1}X^\Delta + (U^{-1})^\Delta X]\right\}^\nabla \\
&= M_1^*\left\{M_2(U^{-1}X)^\Delta\right\}^\nabla \\
&= M_1^*\left\{M_2(M_1X)^\Delta\right\}^\nabla,
\end{aligned}$$

for M_1 and M_2 as defined in the statement of the theorem. □

5 Dominant and Recessive Solutions

Throughout the rest of the paper assume $a \in \mathbb{T}$, and set $\omega := \sup \mathbb{T}$. If $\omega < \infty$, assume $\rho(\omega) = \omega$. We focus on extending the analysis of dominant and recessive solutions developed in the case of difference system (1.1), quantum system (1.2), and differential system (1.3) to the general time-scale setting in (3.1).

Definition 5.1 A solution X of (3.1) is a basis iff $\operatorname{rank}\begin{pmatrix} X(t_0) \\ (PX^\Delta)(t_0) \end{pmatrix} = n$ for some $t_0 \geq a$. A solution V of (3.1) is dominant at ω iff V is a prepared basis and there exists a $t_0 \in [a, \omega)_\mathbb{T}$ such that V is invertible on $[t_0, \omega)_\mathbb{T}$ and

$$\int_{t_0}^{\omega} \left(V^* P V^\sigma\right)^{-1}(t)\Delta t$$

converges to a Hermitian matrix with finite entries.

Lemma 5.1 *Assume the self-adjoint equation $LX = 0$ has a dominant solution V at ω. If X is any other $n \times n$ solution of* (3.1)*, then*

$$\lim_{t \to \omega} V^{-1}(t)X(t) = K$$

for some $n \times n$ constant matrix K.

Proof Since V is a dominant solution at ω of (3.1), there exists a $t_0 \in [a, \omega)_\mathbb{T}$ such that V is invertible on $[t_0, \omega)_\mathbb{T}$. By the second reduction of order theorem, Theorem 3.5,

$$X(t) = V(t)V^{-1}(t_0)X(t_0) + V(t)\left(\int_{t_0}^{t} \left(V^* P V^\sigma\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

Multiplying on the left by V^{-1} we have

$$V^{-1}(t)X(t) = V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{t} \left(V^* P V^\sigma\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

Since V is dominant at ω, the following limit exists:

$$\lim_{t \to \omega} V^{-1}(t)X(t) = K := V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{\omega} \left(V^* P V^\sigma\right)^{-1}(s)\Delta s\right) W(V, X)(t_0).$$

□

Definition 5.2 A solution U of (3.1) is recessive at ω iff U is a prepared basis and whenever X is any other $n \times n$ solution of (3.1) such that $W(X, U)$ is invertible, X is eventually invertible and

$$\lim_{t \to \omega} X^{-1}(t)U(t) = 0.$$

Lemma 5.2 *If U is a solution of* (3.1) *which is recessive at ω, then for any invertible constant matrix K, the solution UK of* (3.1) *is recessive at ω as well.*

Proof The proof follows from the definition. $\square$

Lemma 5.3 *If U is a solution of* (3.1) *which is recessive at ω, and V is a prepared solution of* (3.1) *such that $W(V,U)$ is invertible, then V is dominant at ω.*

Proof By the definition of recessive, $W(V,U)$ invertible implies that V is invertible on $[t_0,\omega)_{\mathbb{T}}$ for some $t_0 \in [a,\omega)_{\mathbb{T}}$, and

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0. \tag{5.1}$$

Let $K := W(V,U)$; by assumption K is invertible, and by Definition 3.3

$$K = (V^*PV^\sigma)(V^\sigma)^{-1}U^\Delta - (V^{\Delta *}PV)V^{-1}U$$

for all $t \in [t_0,\omega)_{\mathbb{T}}$. Since V is prepared,

$$(V^*PV^\sigma)^{-1}K = (V^\sigma)^{-1}U^\Delta - (V^\sigma)^{-1}V^\Delta V^{-1}U = \left(V^{-1}U\right)^\Delta.$$

Delta integrating from t_0 to ω and using (5.1) yields that

$$\int_{t_0}^{\omega}(V^*PV^\sigma)^{-1}(t)\Delta t = -V^{-1}(t_0)U(t_0)K^{-1}$$

converges. Thus V is dominant at ω. $\square$

Theorem 5.1 *Assume* (3.1) *has a solution V which is dominant at ω. Then*

$$U(t) := V(t)\int_t^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s$$

is a solution of (3.1) *which is recessive at ω and $W(V,U) = -I$.*

Proof Since V is dominant at ω, U is a well-defined function and can be written as

$$U(t) = V(t)\left[\int_{t_0}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s - \left(\int_{t_0}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s\right)I\right];$$

by the second reduction of order theorem, Theorem 3.5, U is a solution of (3.1) of the form (3.5) with

$$E = \int_{t_0}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s, \qquad F = -I.$$

From (3.6), $W(V,U) = F = -I$. Since

$$E^*F = -\int_{t_0}^{\omega}(V^*PV^\sigma)^{-1}(s)\Delta s$$

is Hermitian, U is a prepared solution of (3.1), and $W(-V,U) = I$ implies that U and $-V$ are normalized prepared bases. Let X be an $n\times n$ matrix solution of $LX = 0$ such that $W(X,U)$ is invertible. By the second reduction of order theorem,

$$\begin{aligned} X(t) &= V(t)\left[V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{t}(V^*PV^\sigma)^{-1}(s)\Delta s\right)W(V,X)\right] \\ &= V(t)C_1 + U(t)C_2, \end{aligned} \tag{5.2}$$

where

$$C_1 := V^{-1}(t_0)X(t_0) + \left(\int_{t_0}^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s\right) W(V,X)$$

and

$$C_2 := -W(V,X).$$

Note that

$$W(X,U) = C_1^* W(V,U) + C_2^* W(U,U) = -C_1^*.$$

As $W(X,U)$ is invertible by assumption, C_1 is invertible. From (5.2),

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \lim_{t\to\omega} \left(C_1 + V^{-1}(t)U(t)C_2\right)$$
$$= \lim_{t\to\omega} \left(C_1 + \int_t^{\omega} (V^*PV^\sigma)^{-1}(s)\Delta s C_2\right) = C_1$$

is likewise invertible. Consequently for large t, $X(t)$ is invertible. Lastly,

$$\begin{aligned}\lim_{t\to\omega} X^{-1}(t)U(t) &= \lim_{t\to\omega} \left[V(t)C_1 + U(t)C_2\right]^{-1} U(t)\\ &= \lim_{t\to\omega} \left[C_1 + V^{-1}(t)U(t)C_2\right]^{-1} V^{-1}(t)U(t) = \left[C_1 + 0\right]^{-1} 0 = 0.\end{aligned}$$

Therefore U is a recessive solution at ω. □

Theorem 5.2 *Assume* (3.1) *has a solution U which is recessive at ω, and $U(t_0)$ is invertible for some $t_0 \in [a,\omega)_{\mathbb{T}}$. Then U is uniquely determined by $U(t_0)$, and* (3.1) *has a solution V which is dominant at ω.*

Proof Assume $U(t_0)$ is invertible; let V be the unique solution of the initial value problem

$$LV = 0, \quad V(t_0) = 0, \quad V^\Delta(t_0) = I.$$

Then V is a prepared basis and

$$W(V,U) = W(V,U)(t_0) = (V^*PU^\Delta)(t_0) - (PV^\Delta)^*(t_0)U(t_0) = -P(t_0)U(t_0)$$

is invertible. It follows from Lemma 5.3 that V is dominant at ω. Let Γ be an arbitrary but fixed $n \times n$ constant matrix. Let X solve the initial value problem

$$LX = 0, \quad X(t_0) = I, \quad X^\Delta(t_0) = \Gamma.$$

By Theorem 5.1,

$$\lim_{t\to\omega} V^{-1}(t)X(t) = K,$$

where K is an $n\times n$ constant matrix; note that K is independent of the recessive solution U. By using the initial conditions at t_0, by uniqueness of solutions it is easy to see that there exist constant $n \times n$ matrices C_1 and C_2 such that

$$U(t) = X(t)C_1 + V(t)C_2,$$

where $C_1 = U(t_0)$ is invertible. Consequently, using the recessive nature of U, we have

$$0 = \lim_{t\to\omega} V^{-1}(t)U(t) = \lim_{t\to\omega} \left(V^{-1}(t)X(t)U(t_0) + C_2\right) = KU(t_0) + C_2,$$

so that $C_2 = -KU(t_0)$. Thus the initial condition for U^Δ is

$$U^\Delta(t_0) = (\Gamma - K)U(t_0),$$

and the recessive solution U is uniquely determined by its initial value $U(t_0)$. □

Theorem 5.3 *Assume* (3.1) *has a solution U which is recessive at ω and a solution V which is dominant at ω. If U and $\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s$ are both invertible for large $t \in \mathbb{T}$, then there exists an invertible constant matrix K such that*

$$U(t) = V(t) \left(\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s \right) K$$

for large t. In addition, $W(U,V)$ is invertible and

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0.$$

Proof For sufficiently large $t \in \mathbb{T}$ define

$$Y(t) = V(t) \int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s.$$

By Theorem 5.1 Y is also a recessive solution of (3.1) at ω and $W(V,Y) = -I$. Because U and $\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s$ are both invertible for large $t \in \mathbb{T}$, Y is likewise invertible for large t, and

$$\lim_{t\to\omega} V^{-1}(t)Y(t) = 0$$

by the recessive nature of Y. Choose $t_0 \in [a,\omega)_\mathbb{T}$ large enough to ensure that U and Y are invertible in $[t_0,\omega)_\mathbb{T}$. By Lemma 5.2 the solution given by

$$X(t) := Y(t)Y^{-1}(t_0)U(t_0), \quad t \in [t_0,\omega)_\mathbb{T}$$

is yet another recessive solution at ω. Since U and X are recessive solutions at ω and $U(t_0) = X(t_0)$, we conclude from the uniqueness established in Theorem 5.2 that $X \equiv U$. Thus for $t \in [t_0,\omega)_\mathbb{T}$ we have

$$U(t) = Y(t)Y^{-1}(t_0)U(t_0) = V(t) \left(\int_t^\omega (V^*PV^\sigma)^{-1}(s)\Delta s \right) K,$$

where $K := Y^{-1}(t_0)U(t_0)$ is an invertible constant matrix. □

The next result, when $\mathbb{T} = \mathbb{Z}$, relates the convergence of infinite series, the convergence of certain continued fractions, and the existence of recessive solutions; for more see [3] and the references therein.

Theorem 5.4 (Connection theorem) *Let X and V be solutions of* (3.1) *determined by the initial conditions*

$$X(t_0) = I, \quad X^\Delta(t_0) = P^{-1}(t_0)K, \quad \textit{and} \quad V(t_0) = 0, \quad V^\Delta(t_0) = P^{-1}(t_0),$$

respectively, where $t_0 \in [a,\omega)_\mathbb{T}$ and K is a constant Hermitian matrix. Then X, V are normalized prepared bases of (3.1)*, and the following are equivalent:*

(i) V is dominant at ω;

(ii) V is invertible for large $t \in \mathbb{T}$ and $\lim_{t\to\omega} V^{-1}(t)X(t)$ exists as a Hermitian matrix $\Omega(K)$ with finite entries;

(iii) there exists a solution U of (3.1) *which is recessive at ω, with $U(t_0)$ invertible.*

If (i), (ii), and (iii) hold then

$$U^{\Delta}(t_0)U^{-1}(t_0) = X^{\Delta}(t_0) - V^{\Delta}(t_0)\Omega(K) = -P^{-1}(t_0)\Omega(0).$$

Proof Since $V(t_0) = 0$, V is a prepared solution of (3.1). Also,

$$W(X,X) = W(X,X)(t_0) = (X^*PX^{\Delta} - X^{\Delta *}PX)(t_0) = IK - K^*I = 0$$

as K is Hermitian, making X a prepared solution of (3.1) as well. Checking

$$W(X,V) = W(X,V)(t_0) = (X^*PV^{\Delta} - X^{\Delta *}PV)(t_0) = I - 0 = I,$$

we see that X, V are normalized prepared bases of (3.1). Now we show that (i) implies (ii). If V is a dominant solution of (3.1) at ω, then there exists a $t_1 \in [a,\omega)_{\mathbb{T}}$ such that $V(t)$ is invertible for $t \in [t_1,\omega)_{\mathbb{T}}$, and the delta integral

$$\int_{t_1}^{\omega} (V^*PV^{\sigma})^{-1}(s)\Delta s$$

converges to a Hermitian matrix with finite entries. By the second reduction of order theorem,

$$X(t) = V(t)E + V(t)\left(\int_{t_1}^{t} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s\right)F, \tag{5.3}$$

where

$$E = V^{-1}(t_1)X(t_1), \qquad F = W(V,X)(t_1) = -W(X,V)^* = -I.$$

Since X is prepared, $E^*F = -E^*$ is Hermitian, whence E is Hermitian. As a result, by (5.3)

$$\lim_{t\to\omega} V^{-1}(t)X(t) = E - \int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s$$

converges to a Hermitian matrix with finite entries, and (ii) holds. Next we show that (ii) implies (iii). If V is invertible on $[t_1,\omega)_{\mathbb{T}}$ and

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \Omega \tag{5.4}$$

exists as a Hermitian matrix, then from (5.3) and (5.4),

$$\Omega = \lim_{t\to\omega} V^{-1}(t)X(t) = E - \int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s;$$

in other words,

$$\int_{t_1}^{\omega} \left(V^*PV^{\sigma}\right)^{-1}(s)\Delta s = E - \Omega.$$

Define

$$U(t) := X(t) - V(t)\Omega. \tag{5.5}$$

Then

$$\begin{aligned} W(U,U) &= W(X - V\Omega, X - V\Omega) \\ &= W(X,X) - W(X,V)\Omega - \Omega^*W(V,X) + \Omega^*W(V,V)\Omega \\ &= -\Omega + \Omega^* = 0, \end{aligned}$$

and $U(t_0) = X(t_0) = I$, making U a prepared basis for (3.1). If X_1 is an $n \times n$ matrix solution of $LX = 0$ such that $W(X_1, U)$ is invertible, then

$$X_1(t) = V(t)C_1 + U(t)C_2 \tag{5.6}$$

for some constant matrices C_1 and C_2 determined by the initial conditions at t_0. It follows that

$$\begin{aligned} W(X_1, U) &= W(VC_1 + UC_2, U) = C_1^* W(V, U) + C_2^* W(U, U) \\ &= C_1^* W(V, U) = C_1^* W(V, U)(t_0) = -C_1^* \end{aligned}$$

by (5.5), so that C_1 is invertible. From (5.4) and (5.5) we have that

$$\lim_{t\to\omega} V^{-1}(t)U(t) = \lim_{t\to\omega} \left[V^{-1}(t)X(t) - \Omega\right] = 0,$$

resulting in

$$\lim_{t\to\omega} V^{-1}(t)X_1(t) = \lim_{t\to\omega} \left[C_1 + V^{-1}(t)U(t)C_2\right] = C_1,$$

which is invertible. Thus $X_1(t)$ is invertible for large $t \in \mathbb{T}$, and

$$\begin{aligned} \lim_{t\to\omega} X_1^{-1}(t)U(t) &= \lim_{t\to\omega} [V(t)C_1 + U(t)C_2]^{-1} U(t) \\ &= \lim_{t\to\omega} [C_1 + V^{-1}(t)U(t)C_2]^{-1} V^{-1}(t)U(t) \\ &= C_1^{-1}(0) = 0. \end{aligned}$$

Hence U is a recessive solution of (3.1) at ω and (iii) holds. Finally we show that (iii) implies (i). If U is a recessive solution of (3.1) at ω with $U(t_0)$ invertible, then

$$W(V, U) = W(V, U)(t_0) = -U(t_0)$$

is also invertible. Hence by Lemma 5.3, V is a dominant solution of (3.1) at ω.

To complete the proof, assume (i), (ii), and (iii) hold. It can be shown via initial conditions at t_0 that

$$U(t) = X(t)U(t_0) + V(t)C$$

for some suitable constant matrix C. By (ii),

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \Omega(K),$$

and thus

$$V^{-1}(t)U(t) = V^{-1}(t)X(t)U(t_0) + C.$$

As U is a recessive solution at ω by (iii),

$$0 = \lim_{t\to\omega} \left(V^{-1}(t)X(t)U(t_0) + C\right) = \Omega(K)U(t_0) + C,$$

yielding $U(t) = \left[X(t) - V(t)\Omega(K)\right] U(t_0)$. Delta differentiation at t_0 gives

$$U^{\Delta}(t_0)U^{-1}(t_0) = X^{\Delta}(t_0) - V^{\Delta}(t_0)\Omega(K).$$

Now let Y be the unique solution of the initial value problem

$$LY = 0, \quad Y(t_0) = I, \quad Y^{\Delta}(t_0) = 0.$$

Using the initial conditions at t_0 we see that $X(t) = Y(t) + V(t)K$. Consequently,

$$\lim_{t\to\omega} V^{-1}(t)X(t) = \lim_{t\to\omega} V^{-1}(t)Y(t) + K$$

implies, by (ii) and the fact that $X = Y$ when $K = 0$, that $\Omega(K) = \Omega(0) + K$. Therefore

$$X^{\Delta}(t_0) - V^{\Delta}(t_0)\Omega(K) = -V^{\Delta}(t_0)\Omega(0) = -P^{-1}(t_0)\Omega(0).$$

Thus the proof is complete. □

Theorem 5.5 (Variation of parameters) *Let H be an $n\times n$ matrix function that is left-dense continuous on $[t_0,\omega)_{\mathbb{T}}$. If the homogeneous matrix equation (3.1) has a prepared solution X with $X(t)$ invertible for $t\in[t_0,\omega)_{\mathbb{T}}$, then the nonhomogeneous equation $LY = H$ has a solution $Y\in\mathbb{D}$ given by*

$$\begin{aligned} Y(t) \quad &= \quad X(t)X^{-1}(t_0)Y(t_0) + X(t)\int_{t_0}^{t}\left(X^*PX^{\sigma}\right)^{-1}(\tau)\Delta\tau W(X,Y)(t_0) \\ &\quad + X(t)\int_{t_0}^{t}\left(\left(X^*PX^{\sigma}\right)^{-1}(\tau)\int_{t_0}^{\tau}X^*(s)H(s)\nabla s\right)\Delta\tau. \end{aligned}$$

Proof Let $Y\in\mathbb{D}$ and assume X is a prepared solution of (3.1) invertible on $[t_0,\omega)_{\mathbb{T}}$. As in Theorem 4.1, we factor LY to get

$$H = LY = X^{*-1}\left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)^{\nabla}.$$

Multiplying by X^* and nabla integrating from t_0 to t we arrive at

$$\left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)(t) - W(X,Y)(t_0) = \int_{t_0}^{t}X^*(s)H(s)\nabla s,$$

where $W(X,Y)(t_0) = \left(X^*PX^{\sigma}(X^{-1}Y)^{\Delta}\right)(t_0)$ since X is prepared. This leads to

$$(X^{-1}Y)^{\Delta}(t) = (X^*PX^{\sigma})^{-1}(t)\left(W(X,Y)(t_0) + \int_{t_0}^{t}X^*(s)H(s)\nabla s\right),$$

which is then delta integrated from t_0 to t to obtain the form for Y given in the statement of the theorem. Clearly the right-hand side of the form of Y above reduces to $Y(t_0)$ at t_0, and since X is an invertible prepared solution, by Theorem 3.1 the delta derivative reduces to $Y^{\Delta}(t_0)$ at t_0. □

Corollary 5.1 *Let H be an $n\times n$ matrix function that is left-dense continuous on $[t_0,\omega)_{\mathbb{T}}$. If the homogeneous matrix equation (3.1) has a prepared solution X with $X(t)$ invertible for $t\in[t_0,\omega)_{\mathbb{T}}$, then the nonhomogeneous initial value problem*

$$LY = (PY^{\Delta})^{\nabla} + QY = H, \quad Y(t_0) = Y_0, \quad Y^{\Delta}(t_0) = Y_0^{\Delta} \tag{5.7}$$

has a unique solution.

Proof By Theorem 5.5, the nonhomogeneous initial value problem (5.7) has a solution. Suppose Y_1 and Y_2 both solve (5.7). Then $X = Y_1 - Y_2$ solves the homogeneous initial value problem

$$LX = 0, \quad X(t_0) = 0, \quad X^{\Delta}(t_0) = 0;$$

by Theorem 3.1, this has only the trivial solution $X = 0$. $\square$

We will also be interested in analyzing the self-adjoint vector dynamic equation

$$Lx = 0, \quad \text{where} \quad Lx(t) := \left(Px^\Delta\right)^\nabla(t) + Q(t)x(t), \quad t \in [a, \omega)_\mathbb{T}, \tag{5.8}$$

where x is an $n \times 1$ vector-valued function defined on $\mathbb{T}$ such that x^Δ is continuous and $(Px^\Delta)^\nabla$ is left-dense continuous on $[a, \omega)_\mathbb{T}$. We will see interesting relationships between the so-called unique two-point property (defined below) of the nonhomogeneous vector equation $Lx = h$, disconjugacy of $Lx = 0$, and the construction of recessive solutions to the matrix equation $LX = 0$. The following theorem can be proven by modifying the proof of Theorem 5.5 and its corollary.

Theorem 5.6 *Let h be an $n \times 1$ vector function that is left-dense continuous on $[t_0, \omega)_\mathbb{T}$. If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_\mathbb{T}$, then the nonhomogeneous vector initial value problem*

$$Ly = (Py^\Delta)^\nabla + Qy = h, \quad y(t_0) = y_0, \quad y^\Delta(t_0) = y_0^\Delta \tag{5.9}$$

has a unique solution.

Definition 5.3 Assume h is an $n \times 1$ left-dense continuous vector function on $[t_0, \omega)_\mathbb{T}$. Then the vector dynamic equation $Lx = h$ has the unique two-point property on $[t_0, \omega)_\mathbb{T}$ provided given any $t_0 \le t_1 < t_2$ in $\mathbb{T}$, if u and v are solutions of $Lx = h$ with $u(t_1) = v(t_1)$ and $u(t_2) = v(t_2)$, then $u \equiv v$ on $[t_0, \omega)_\mathbb{T}$.

Theorem 5.7 *If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_\mathbb{T}$, and if the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_\mathbb{T}$, then the boundary value problem*

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta,$$

where $t_0 \le t_1 < t_2$ in $\mathbb{T}$ and $\alpha, \beta \in \mathbb{C}^n$, has a unique solution on $[t_0, \omega)_\mathbb{T}$.

Proof If t_1 is a right-scattered point and $t_2 = \sigma(t_1)$, then the boundary value problem is an initial value problem and the result holds by Theorem 5.6. Assume $t_2 > \sigma(t_1)$. Let $X(t, t_1)$ and $Y(t, t_1)$ be the unique $n \times n$ matrix solutions of (3.1) determined by the initial conditions

$$X(t_1, t_1) = 0, \quad X^\Delta(t_1, t_1) = I, \quad \text{and} \quad Y(t_1, t_1) = I, \quad Y^\Delta(t_1, t_1) = 0;$$

by variation of constants, Theorem 5.5,

$$X(t, t_1) = X(t) \int_{t_1}^{t} (X^* P X^\sigma)^{-1}(\tau) \Delta\tau X^*(t_1) P(t_1)$$

and

$$Y(t, t_1) = X(t) X^{-1}(t_1) - X(t) \int_{t_1}^{t} (X^* P X^\sigma)^{-1}(\tau) \Delta\tau X^{\Delta *}(t_1) P(t_1).$$

Then a general solution of (5.8) is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta, \tag{5.10}$$

for $\gamma, \delta \in \mathbb{C}^n$, as $x(t_1) = \delta$ and $x^{\Delta}(t_1) = \gamma$. By the unique two-point property the homogeneous boundary value problem

$$Lx = 0, \quad x(t_1) = 0, \quad x(t_2) = 0$$

has only the trivial solution. For x given by (5.10), the boundary condition at t_1 implies that $\delta = 0$, and the boundary condition at t_2 yields

$$X(t_2, t_1)\gamma = 0;$$

by uniqueness and the fact that x is trivial, $\gamma = 0$ is the unique solution, meaning $X(t_2, t_1)$ is invertible. Next let v be the solution of the initial value problem

$$Lv = h, \quad v(t_1) = 0, \quad v^{\Delta}(t_1) = 0.$$

Then the general solution of $Lx = h$ is given by

$$x(t) = X(t, t_1)\gamma + Y(t, t_1)\delta + v(t).$$

We now show that the boundary value problem

$$Lx = h, \quad x(t_1) = \alpha, \quad x(t_2) = \beta$$

has a unique solution. The boundary condition at t_1 implies that $\delta = \alpha$. The condition at t_2 leads to the equation

$$\beta = X(t_2, t_1)\gamma + Y(t_2, t_1)\alpha + v(t_2);$$

since $X(t_2, t_1)$ is invertible, this can be solved uniquely for γ. □

Corollary 5.2 *If the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_{\mathbb{T}}$, and if the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_{\mathbb{T}}$, then the matrix boundary value problem*

$$LX = 0, \quad X(t_1) = M, \quad X(t_2) = N$$

has a unique solution, where M and N are given constant $n \times n$ matrices.

Proof Modify the proof of Theorem 5.7 to get existence and uniqueness. □

Theorem 5.8 *Assume the homogeneous matrix equation* (3.1) *has a prepared solution X with $X(t)$ invertible for $t \in [t_0, \omega)_{\mathbb{T}}$, and the homogeneous vector equation* (5.8) *has the unique two-point property on $[t_0, \omega)_{\mathbb{T}}$. Further assume U is a solution of* (3.1) *which is recessive at ω with $U(t_0)$ invertible. For each fixed $s \in (t_0, \omega)_{\mathbb{T}}$, let $Y(t, s)$ be the solution of the boundary value problem*

$$LY(t, s) = 0, \quad Y(t_0, s) = I, \quad Y(s, s) = 0.$$

Then the recessive solution $U(t)U^{-1}(t_0)$ is uniquely determined by

$$U(t)U^{-1}(t_0) = \lim_{s \to \omega} Y(t, s). \tag{5.11}$$

Proof Assume U is a solution of (3.1) which is recessive at ω with $U(t_0)$ invertible. Let V be the unique solution of the initial value problem

$$LV = 0, \quad V(t_0) = 0, \quad V^{\Delta}(t_0) = P^{-1}(t_0).$$

By the connection theorem, Theorem 5.4, V is invertible for large t. By checking boundary conditions at t_0 and s for s large, we get that

$$Y(t,s) = -V(t)V^{-1}(s)U(s)U^{-1}(t_0) + U(t)U^{-1}(t_0).$$

Then

$$W(V,U) = W(V,U)(t_0) = (V^*PU^{\Delta} - V^{\Delta *}PU)(t_0) = -U(t_0)$$

is invertible, and by the recessive nature of U,

$$\lim_{t\to\omega} V^{-1}(t)U(t) = 0.$$

As a result,

$$\lim_{s\to\omega} Y(t,s) = 0 + U(t)U^{-1}(t_0),$$

and the proof is complete. $\square$

Definition 5.4 A prepared vector solution x of (5.8) has a generalized zero at a iff $x(a) = 0$, and x has a generalized zero at $t_0 > a$ iff $x(t_0) = 0$, or if t_0 is a left-scattered point and $x^{*\rho}(t_0)P^{\rho}(t_0)x(t_0) < 0$. Equation (5.8) is disconjugate on $[a,\omega)_{\mathbb{T}}$ iff no nontrivial prepared vector solution of (5.8) has two generalized zeros in $[a,\omega)_{\mathbb{T}}$.

Definition 5.5 A prepared basis X of (3.1) has a generalized zero at a iff $X(a)$ is noninvertible, and X has a generalized zero at $t_0 \in (a,\omega)_{\mathbb{T}}$ iff $X(t_0)$ is noninvertible, or $X^{*\rho}(t_0)P^{\rho}(t_0)X(t_0)$ is invertible but $X^{*\rho}(t_0)P^{\rho}(t_0)X(t_0) \le 0$.

Lemma 5.4 *If a prepared basis X of* (3.1) *has a generalized zero at $t_0 \in [a,\omega)_{\mathbb{T}}$, then there exists a vector $\gamma \in \mathbb{C}^n$ such that $x = X\gamma$ is a nontrivial prepared solution of* (5.8) *with a generalized zero at t_0.*

Proof The proof follows from Definitions 5.4 and 5.5. $\square$

Lemma 5.5 *If f and g are continuous on $[t_0,\omega)_{\mathbb{T}}$, then*

$$\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s = \int_{t_0}^{t} f(s)g^{\sigma}(s)\Delta s, \qquad t \in [t_0,\omega)_{\mathbb{T}}.$$

Proof Set

$$F(t) := \int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s - \int_{t_0}^{t} f(s)g^{\sigma}(s)\Delta s;$$

clearly $F(t_0) = 0$, and

$$F^{\Delta}(t) = \left[\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s\right]^{\Delta} - f(t)g^{\sigma}(t).$$

Using Theorem 2.2 (iii) and the set B in (2.2),

$$\left[\int_{t_0}^{t} f^{\rho}(s)g(s)\nabla s\right]^{\Delta} = \begin{cases}(f^{\rho}g)(\sigma(t)) & : t\in\mathbb{T}\backslash B,\\ \lim_{s\to t^+}(f^{\rho}g)(s) & : t\in B.\end{cases}$$

For $t \in \mathbb{T}\backslash B$, $\rho(\sigma(t)) = t$, so that $(f^{\rho}g)(\sigma(t)) = (fg^{\sigma})(t)$. For $t \in B$, $t = \sigma(t)$ and $\lim_{s\to t^+} \rho(s) = t$, yielding

$$\lim_{s\to t^+}(f^{\rho}g)(s) = (fg)(t) = (fg^{\sigma})(t).$$

Thus in either case $F^{\Delta}(t) = 0$. By the uniqueness property, $F \equiv 0$, and the result follows. □

Theorem 5.9 *If the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*, then the matrix equation* (3.1) *has a solution* V *which is dominant at* ω *and a solution* U *which is recessive at* ω*, with* V *and* U *both invertible such that* $PV^{\Delta}V^{-1} > PU^{\Delta}U^{-1}$ *on* $(\sigma(t_0), \omega)_{\mathbb{T}}$*.*

Proof Let X be the solution of the initial value problem

$$LX = 0, \quad X^{\rho}(t_0) = 0, \quad X^{\Delta\rho}(t_0) = I.$$

If X is not invertible on $(t_0, \omega)_{\mathbb{T}}$, then there exists a $t_1 > t_0$ such that $X(t_1)$ is singular. But then there exists a nontrivial vector $\delta \in \mathbb{C}^n$ such that $X(t_1)\delta = 0$. If $x(t) := X(t)\delta$, then x is a nontrivial prepared solution of (5.8) with

$$x^{\rho}(t_0) = 0, \quad x(t_1) = 0,$$

a contradiction of disconjugacy. Hence X is invertible in $(t_0, \omega)_{\mathbb{T}}$. We next claim that

$$(X^{*\rho}P^{\rho}X)(t) > 0, \quad t \in (\sigma(t_0), \omega)_{\mathbb{T}}; \tag{5.12}$$

if not, there exists $t_2 \in (\sigma(t_0), \omega)_{\mathbb{T}}$ such that

$$(X^{*\rho}P^{\rho}X)(t_2) \not> 0.$$

It follows that there exists a nontrivial vector γ such that $x(t) := X(t)\gamma$ is a nontrivial prepared vector solution of $Lx = 0$ with a generalized zero at t_2. Using the initial condition for X, however, we have $x^{\rho}(t_0) = 0$, another generalized zero, a contradiction of the assumption that the vector equation (5.8) is disconjugate on $[\rho(t_0), \omega)_{\mathbb{T}}$. Thus (5.12) holds, in particular for any $t_2 \in (\sigma(t_0), \omega)_{\mathbb{T}}$. Define for $t \in [t_2, \omega)_{\mathbb{T}}$

$$V(t) := X(t)\left[I + \int_{t_2}^{t}(X^{*}PX^{\sigma})^{-1}(s)\Delta s\right] = X(t)\left[I + \int_{t_2}^{t}(X^{*\rho}P^{\rho}X)^{-1}(s)\nabla s\right],$$

where the second equality follows from Lemma 5.5. By Theorem 3.5, V is a prepared solution of $LV = 0$ with $W(X, V) = I$. Note that V is also invertible on $[t_2, \omega)_{\mathbb{T}}$, so that by the reduction of order theorem again,

$$X(t) = V(t)\left[I - \int_{t_2}^{t}(V^{*}PV^{\sigma})^{-1}(s)\Delta s\right], \quad t \in [t_2, \omega)_{\mathbb{T}}.$$

Consequently,

$$I = [V^{-1}(t)X(t)][X^{-1}(t)V(t)] = \left[I - \int_{t_2}^{t} (V^* P V^\sigma)^{-1}(s)\Delta s\right] \left[I + \int_{t_2}^{t} (X^* P X^\sigma)^{-1}(s)\Delta s\right].$$

Since the second factor is strictly increasing and bounded below by I, the first factor is positive definite and strictly decreasing, ensuring the existence of a limit, in other words, we have

$$0 \le I - \int_{t_2}^{\omega} (V^* P V^\sigma)^{-1}(s)\Delta s < I - \int_{t_2}^{t} (V^* P V^\sigma)^{-1}(s)\Delta s \le I.$$

It follows that

$$0 \le \int_{t_2}^{t} (V^* P V^\sigma)^{-1}(s)\Delta s < \int_{t_2}^{\omega} (V^* P V^\sigma)^{-1}(s)\Delta s \le I, \quad t \in [t_2, \omega)_{\mathbb{T}}, \tag{5.13}$$

and V is a dominant solution of (3.1) at ω. Set

$$U(t) := V(t) \int_{t}^{\omega} (V^* P V^\sigma)^{-1}(s)\Delta s.$$

By Theorem 5.1, U is a recessive solution of (3.1) at ω, and $W(U, V) = I$. Since

$$U(t) = V(t) \left[\int_{t_2}^{\omega} (V^* P V^\sigma)^{-1}(s)\Delta s - \int_{t_2}^{t} (V^* P V^\sigma)^{-1}(s)\Delta s\right],$$

V is invertible on $[t_2, \omega)_{\mathbb{T}}$, and the difference in brackets is positive definite on $[t_2, \omega)_{\mathbb{T}}$, we get that U is invertible on $[t_2, \omega)_{\mathbb{T}}$ as well. Then on $[t_2, \omega)_{\mathbb{T}}$, we have

$$\begin{aligned}
PV^\Delta V^{-1} - PU^\Delta U^{-1} &= U^{*-1}U^* PV^\Delta V^{-1} - X^{*-1}X^{\Delta *}PVV^{-1} \\
&= U^{*-1}\left[U^* PV^\Delta - U^{\Delta *}PV\right] V^{-1} \\
&= U^{*-1}\left[W(U, V)\right] V^{-1}UU^{-1} \\
&= U^{*-1}\left[V^{-1}U\right] U^{-1} \\
&= U^{*-1}\left[\int_{t}^{\omega} (V^* P V^\sigma)^{-1}(s)\Delta s\right] U^{-1} > 0
\end{aligned}$$

by (5.13). Since t_2 in $(\sigma(t_0), \omega)_{\mathbb{T}}$ arbitrary, the conclusions of the theorem follow. □

Corollary 5.3 *Assume the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*, and* K *is a constant Hermitian matrix. Let* U, V *be the matrix solutions of* $LX = 0$ *satisfying the initial conditions*

$$U(t_2) = I, \quad U^\Delta(t_2) = P^{-1}(t_2)K, \quad \textit{and} \quad V(t_2) = 0, \quad V^\Delta(t_2) = P^{-1}(t_2)$$

for any $t_2 \in (\sigma(t_0), \omega)_{\mathbb{T}}$*. Then* V *is invertible in* $(\sigma(t_2), \omega)_{\mathbb{T}}$*,* V *is a dominant solution of* (3.1) *at* ω*, and*

$$\lim_{t \to \omega} V^{-1}(t)U(t)$$

exists as a Hermitian matrix.

Proof By Theorem 5.9, the matrix equation (3.1) has a solution U which is recessive at ω with $U(t)$ invertible for $t \in [t_2, \omega)_{\mathbb{T}}$. Thus (iii) of the connection theorem, Theorem 5.4 holds; by (i), then, V is a dominant solution of (3.1) at ω, and by (ii),

$$\lim_{t \to \omega} V^{-1}(t)U(t)$$

exists as a Hermitian matrix. Since $V(t_2) = 0$ and the vector equation (5.8) is disconjugate on $[\rho(t_0), \omega)_{\mathbb{T}}$,

$$\left(V^{*\rho} P^{\rho} V\right)(t) > 0, \quad t \in (\sigma(t_2), \omega)_{\mathbb{T}}.$$

In particular, V is invertible in $(\sigma(t_2), \omega)_{\mathbb{T}}$. □

Theorem 5.10 *If the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*, then* $Lx(t) = h(t)$ *has the unique two-point property in* $[t_0, \omega)_{\mathbb{T}}$*. In particular, every boundary value problem of the form*

$$Lx(t) = h(t), \qquad x(\tau_1) = \alpha, \qquad x(\tau_2) = \beta,$$

where $\tau_1, \tau_2 \in [t_2, \omega)_{\mathbb{T}}$ *for* $t_2 \in (\sigma(t_0), \omega)_{\mathbb{T}}$ *with* $\tau_1 < \tau_2$*, and where* α, β *are given n-vectors, has a unique solution.*

Proof By Theorem 5.9, disconjugacy of (5.8) implies the existence of a prepared, invertible matrix solution of (3.1). Thus by Theorem 5.7, it suffices to show that (5.8) has the unique two-point property in $[t_2, \omega)_{\mathbb{T}}$. To this end, assume u, v are solutions of $Lx = 0$, and there exist points $s_1, s_2 \in \mathbb{T}$ such that $t_2 \le s_1 < s_2$ and

$$u(s_1) = v(s_1), \qquad u(s_2) = v(s_2).$$

If s_1 is a right-scattered point and $s_2 = \sigma(s_1)$, then u and v satisfy the same initial conditions and $u \equiv v$ by uniqueness; hence we assume $s_2 > \sigma(s_1)$. Setting $x = u - v$, we see that x solves the initial value problem

$$Lx = 0, \qquad x(\tau_1) = 0, \qquad x(\tau_2) = 0.$$

Since $Lx = 0$ is disconjugate and x is a prepared solution with two generalized zeros, it must be that $x \equiv 0$ in $[t_2, \omega)_{\mathbb{T}}$. Consequently, $u = v$ and the two-point property holds. □

Corollary 5.4 (Construction of the recessive solution) *Assume the vector equation* (5.8) *is disconjugate on* $[\rho(t_0), \omega)_{\mathbb{T}}$*. For each* $s \in (t_0, \omega)_{\mathbb{T}}$*, let* $U(t, s)$ *be the solution of the boundary value problem*

$$LU(\cdot, s) = 0, \quad U(t_0, s) = I, \quad U(s, s) = 0.$$

Then the solution U *with* $U(t_0) = I$ *which is recessive at* ω *is given by*

$$U(t) = \lim_{s \to \omega} U(t, s),$$

satisfying

$$(U^{*\rho} P^{\rho} U)(t) > 0, \quad t \in [t_0, \omega)_{\mathbb{T}}. \tag{5.14}$$

Proof By Theorem 5.9 and Theorem 5.10, $LX = 0$ has a recessive solution and $Lx = h$ has the unique two-point property. The conclusion then follows from Theorem 5.8, except for (5.14). From the boundary condition $U(s,s) = 0$ and the fact that $Lx = 0$ is disconjugate, it follows that $U^*(\rho(t), s)P^\rho(t)U(t,s) > 0$ holds in $[t_0, s)_{\mathbb{T}}$. Again from Theorem 5.8,

$$\lim_{s\to\omega} U(t,s) = U(t)U^{-1}(t_0) = U(t),$$

so that U invertible on $[t_0, \omega)_{\mathbb{T}}$ and (5.14) holds. □

References

[1] Kelley, W. and Peterson, A. *The Theory of Differential Equations: Classical and Qualitative.* Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

[2] Ahlbrandt, C. D. Dominant and recessive solutions of symmetric three term recurrences. *J. Differ. Equ.* **107**(2) (1994) 238–258.

[3] Ahlbrandt, C. D. and Peterson, A. C. *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations.* Kluwer Academic Publishers, Dordrecht, 1996.

[4] Ma, M. Dominant and recessive solutions for second-order self-adjoint linear difference systems. *Appl. Math. Lett.* **18** (2005) 179–185.

[5] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales, An Introduction with Applications.* Birkhäuser, Boston, 2001.

[6] Messer, K. A second-order self-adjoint equation on a time scale. *Dyn. Sys. Appl.* **12** (2003) 201–215.

[7] Bohner, M. and Peterson, A., editors. *Advances in Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2003.

[8] Aulbach, B. and Hilger, S. Linear dynamic processes with inhomogeneous time scale. *Nonlinear Dynamics and Quantum Dynamical Systems* (Gaussig, 1990) **59** *Math. Res.*, 9–20. Akademie Verlag, Berlin, 1990.

[9] Hilger, S. Analysis on measure chains - a unified approach to continuous and discrete calculus. *Results Math.* **18** (1990) 18–56.

[10] Anderson, D. R. and Moats, L. M. q-Dominant and q-recessive matrix solutions for linear quantum systems. *Electronic J. Qualitative Theory Diff. Eq.* **2007**(11) (2007) 1–29.

[11] Reid, W. T. Oscillation criteria for linear differential systems with complex coefficients. *Pacific J. Math.* **6** (1956) 733–751.

[12] Reid, W. T. Principal solutions of non-oscillatory self-adjoint linear differential systems. *Pacific J. Math.* **8** (1958) 147–169.

[13] Reid, W. T. *Ordinary Differential Equations.* Wiley, New York, 1971.

[14] Reid, W. T. *Riccati Differential Equations.* Academic Press, New York, 1972.

[15] Reid, W. T. *Sturmian Theory for Ordinary Differential Equations.* Springer-Verlag, New York, 1980.

[16] Anderson, D. R., Guseinov, G. Sh. and Hoffacker, J. Higher-order self adjoint boundary value problems on time scales. *J. Comput. Appl. Math.* **194**(2) (2006) 309–342.

[17] Atici, F. M. and Guseinov, G. Sh. On Green's functions and positive solutions for boundary value problems on time scales. *J. Comput. Appl. Math.* **141** (2002) 75–99.

[18] Anderson, D. R. and Buchholz, B. Self-adjoint matrix equations on time scales. *PanAmerican Math. J.* **17**(2) (2007) 81–104.

[19] Erbe, L. H. and Peterson, A. C. Oscillation criteria for second-order matrix dynamic equations on a time scale. *J. Comput. Appl. Math.* **141** (2002) 169–185.

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Dynamic Inequalities, Bounds, and Stability of Systems with Linear and Nonlinear Perturbations

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Abstract: Generalized dynamic inequalities are introduced to the time scales scene, mainly as generalizations of Gronwall's inequality. Linear systems with linear and nonlinear perturbations and their stability characteristics versus the unperturbed system are investigated. Bounds for solutions to linear dynamic systems are stated using the system matrix.

Keywords: *stability; perturbed linear system; dynamic inequality; system bounds; time scales.*

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1 Introduction

It is useful to consider state equations that are close (in an appropriate sense) to another linear state equation that is uniformly stable or uniformly exponentially stable. Prompted by Lyapunov [6], DaCunha [4] showed that if the stability of the uniformly regressive time varying linear dynamic system

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \tag{1.1}$$

has already been determined by an appropriate *generalized Lyapunov function*, then certain conditions on the perturbation matrix $F(t)$ guarantee specific stability characteristics of the perturbed linear system

$$z^{\Delta}(t) = [A(t) + F(t)]z(t), \qquad z(t_0) = z_0. \tag{1.2}$$

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In Brogan [2], Chen [3], and Rugh [8], the stability of linear systems and perturbed linear systems is investigated on the lackluster time scales of $\mathbb{R}$ and $\mathbb{Z}$. As is known in the time scales community, analysis on either of these two domains rarely offers the complexity and challenge of the same study on an arbitrary closed set of the reals. One of the main reasons for this is that the uniform graininess of each makes for a run of the mill investigation. Despite these shortcomings of $\mathbb{R}$ and $\mathbb{Z}$, this paper is motivated by these works to unify and extend to the more general area of time scales, as were Gard and Hoffacker [5] in the scalar dynamic equation case and Pötzsche, Siegmund, and Wirth [7] in the constant and Jordan reducible linear systems case. Our aim in this exposition is to prove analogous results for the universal time scales setting.

This paper is organized as follows. Section 2 introduces two dynamic inequalities which are generalizations of Gronwall's inequality. In addition to bounds for solutions to linear dynamic systems using the system matrix coefficients, linear systems with perturbations and their stability characteristics versus the unperturbed system are investigated in Section 3. Section 4 gives slightly more general stability results for linear systems with nonlinear perturbations. The author's conclusions end the paper.

2 Generalizations of Gronwall's Inequality

To begin with, we state two theorems from the introductory time scales text [1]. One important result that is supplied from the following is a way to show uniqueness of solutions for initial value problems of linear dynamic systems.

Theorem 2.1 *[1, Thm. 6.1] Let $f, x \in \mathrm{C_{rd}}$ and $p \in \mathcal{R}^+$. Then*

$$x^\Delta(t) \le p(t)x(t) + f(t), \quad \textit{for all}\, t \in \mathbb{T}$$

implies

$$x(t) \le e_p(t,t_0)x_0 + \int_{t_0}^t e_p(t,\sigma(s))f(t)\Delta s, \quad \textit{for all}\, t \in \mathbb{T}.$$

Theorem 2.2 (Gronwall's inequality) *[1, Thm. 6.4] Let $f, x \in \mathrm{C_{rd}}$, $p \in \mathcal{R}^+$, and $p \ge 0$ for all $t \ge t_0$. Then*

$$x(t) \le f(t) + \int_{t_0}^t p(s)x(s)\Delta s, \quad \textit{for all}\, t \in \mathbb{T}$$

implies

$$x(t) \le f(t) + \int_{t_0}^t e_p(t,\sigma(s))f(s)p(s)\Delta s, \quad \textit{for all}\, t \in \mathbb{T}. \tag{2.1}$$

By employing these previous two theorems, in particular, the generalized Gronwall inequality, we have the following two new generalized dynamic inequalities.

Theorem 2.3 *Let $x \in \mathrm{C_{rd}}$, $f \in \mathrm{C^1_{rd}}$, $p \in \mathcal{R}^+$, and $p \ge 0$ for all $t \ge t_0$. Then*

$$x(t) \le f(t) + \int_{t_0}^t p(s)x(s)\Delta s, \quad \textit{for all}\, t \in \mathbb{T} \tag{2.2}$$

implies

$$x(t) \le e_p(t,t_0)f(t_0) + \int_{t_0}^t e_p(t,\sigma(s))f^\Delta(s)\Delta s, \quad \textit{for all}\, t \in \mathbb{T}. \tag{2.3}$$

Proof Applying Gronwall's inequality from Theorem 2.2 to the inequality (2.2), we obtain the inequality (2.1).

Defining the function $r(t)$ as the right hand side of the inequality (2.1), using the fact that $p \geq 0$, and then delta differentiating $r(t)$ we obtain

$$r^{\Delta}(t) = f^{\Delta}(t) + f(t)p(t) + \int_{t_0}^{t} p(t)e_p(t,\sigma(s))f(s)p(s)\Delta s = f^{\Delta}(t) + p(t)r(t).$$

Multiplying both sides by the positive function $e_{\ominus p}(\sigma(t), t_0)$ we have

$$e_{\ominus p}(\sigma(t), t_0)(r^{\Delta}(t) - p(t)r(t)) = e_{\ominus p}(\sigma(t), t_0)f^{\Delta}(t)$$

which is equivalent to

$$[e_{\ominus p}(t, t_0)r(t)]^{\Delta} = e_{\ominus p}(\sigma(t), t_0)f^{\Delta}(t).$$

On both sides, integrate from t_0 to t, then multiply by $e_p(t, t_0)$ and obtain

$$r(t) = e_p(t, t_0)r(t_0) + \int_{t_0}^{t} e_{\ominus p}(\sigma(s), t)f^{\Delta}(s)\Delta s.$$

Thus, the desired inequality (2.3) is obtained. □

Theorem 2.4 *Let $f, w, x \in \mathrm{C_{rd}}$, where f is a constant, $p \in \mathcal{R}^+$, and $p \geq 0$ for all $t \geq t_0$. Then*

$$x(t) \leq f + \int_{t_0}^{t} w(s) + p(s)x(s)\Delta s, \qquad \text{for all } t \in \mathbb{T} \tag{2.4}$$

implies

$$x(t) \leq e_p(t, t_0)f + \int_{t_0}^{t} e_p(t, \sigma(s))w(s)\Delta s, \qquad \text{for all } t \in \mathbb{T}. \tag{2.5}$$

Proof We define the function $r(t)$ by writing the right hand side of the inequality (2.4). Observe that with (2.4) and the fact that $p \geq 0$,

$$r^{\Delta}(t) = w(t) + p(t)x(t) \leq w(t) + p(t)r(t).$$

Multiplying both sides by the positive function $e_{\ominus p}(\sigma(t), t_0)$ we have

$$e_{\ominus p}(\sigma(t), t_0)(r^{\Delta}(t) - p(t)r(t)) = e_{\ominus p}(\sigma(t), t_0)w(t)$$

which is equivalent to

$$[e_{\ominus p}(t, t_0)r(t)]^{\Delta} = e_{\ominus p}(\sigma(t), t_0)w(t).$$

On both sides, integrate from t_0 to t, then multiply by $e_p(t, t_0)$ and obtain

$$r(t) = e_p(t, t_0)r(t_0) + \int_{t_0}^{t} e_{\ominus p}(\sigma(s), t)w(s)\Delta s.$$

Thus, we obtain the desired inequality (2.5). □

Example 2.1 Given the time varying system (1.1), we can use Theorem 2.1 (with $f(t) \equiv 0$) or Theorem 2.4 (with $w \equiv 0$) to derive a bound on the solution using the system matrix. Observe that

$$||x(t)|| \leq ||x_0|| + \int_{t_0}^{t} ||A(s)||\,||x(s)||\Delta s \implies ||x(t)|| \leq e_{||A||}(t, t_0)||x_0||, \quad \text{for all } t \in \mathbb{T}.$$

3 Linear Perturbations

We begin this section with a few useful definitions.

Definition 3.1 [7, Lem. 4.5] A regressive mapping $\lambda \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{C})$ is *uniformly regressive* on the time scale $\mathbb{T}$ if there exists a constant $\delta > 0$ such that

$$0 < \delta^{-1} \leq |1 + \mu(t)\lambda(t)|, \tag{3.1}$$

for all $t \in \mathbb{T}$.

Further, the $n \times n$ linear dynamic system (1.1) is *uniformly regressive* if all eigenvalues $\{\lambda_i\}_{i=1}^k$, $k \leq n$, of A satisfy (3.1) for all $t \in \mathbb{T}$.

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the uniformly regressive time varying linear dynamic equation (1.1).

Definition 3.2 The time varying linear dynamic equation (1.1) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq \gamma ||x(t_0)||, \qquad t \geq t_0.$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (1.1), but also the asymptotic characteristics of the solutions as well. If the solutions to (1.1) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 3.3 The time varying linear dynamic equation (1.1) is called *uniformly exponentially stable* if there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq ||x(t_0)|| \gamma e_{-\lambda}(t, t_0), \qquad t \geq t_0.$$

It is obvious by inspection of the previous definitions that we must have $\gamma \geq 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0.

Definition 3.4 [7] The *regressive stability region* for the scalar IVP is defined to be the set

$$\mathcal{S}(\mathbb{T}) = \left\{ \gamma(t) \in \mathbb{C} : \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(\tau)} \frac{\log|1 + s\gamma(\tau)|}{s} \Delta\tau < 0 \right\}.$$

It is easy to see that the regressive stability region is always contained in $\{\gamma \in \mathbb{C} : \mathrm{Re}(\gamma) < 0\}$. The reader is referred to [7] for more explanation.

Theorem 3.1 *Suppose the linear system* (1.1) *is uniformly stable. Then there exists some* $\beta > 0$ *such that if*

$$\int_{\tau}^{\infty} ||F(s)|| \Delta s \leq \beta$$

for all $\tau \in \mathbb{T}$*, the perturbed system* (1.2) *is uniformly stable.*

Proof See [4] for proof. □

Theorem 3.2 *Suppose the linear system* (1.1) *is uniformly exponentially stable. Then there exists some* $\beta > 0$ *such that if*

$$\int_{\tau}^{\infty} ||F(s)||\Delta s \leq \beta$$

for all $\tau \in \mathbb{T}$, *the perturbed system* (1.2) *is uniformly exponentially stable.*

Proof For any initial conditions, the solution of (1.2) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t, t_0)$ is the transition matrix for the system (1.1). By the uniform exponential stability of (1.1), there exist constants λ, $\gamma > 0$ with $-\lambda \in \mathcal{R}^+$ uniformly such that $||\Phi_A(t, \tau)|| \leq \gamma e_{-\lambda}(t, \tau)$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. Taking the norms of both sides and utilizing the uniform regressivity, we see

$$||z(t)|| \leq \gamma e_{-\lambda}(t, t_0)||z_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t, s)\delta||F(s)||\, ||z(s)||\, \Delta s, \quad t \geq t_0.$$

Defining $\psi(t) := e_{-\lambda}(t_0, t)||z(t)||$, this implies

$$\psi(t) \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\delta||F(s)||\, \psi(s)\Delta s.$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}
||z(t)|| &\leq \gamma||z_0||e_{-\lambda\oplus\gamma\delta||F||}(t, t_0) \\
&= \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\int_{t_0}^{t} \frac{\mathrm{Log}(1 + \mu(s)\gamma\delta||F(s)||)}{\mu(s)}\Delta s\right) \\
&\leq \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\int_{t_0}^{\infty} \frac{\mathrm{Log}(1 + \mu(s)\gamma\delta||F(s)||)}{\mu(s)}\Delta s\right) \\
&\leq \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\gamma\delta \int_{t_0}^{\infty} ||F(s)||\Delta s\right) \\
&\leq \gamma||z_0||e^{\gamma\delta\beta}e_{-\lambda}(t, t_0), \quad t \geq t_0.
\end{aligned}$$

Since γ and $-\lambda$ can be used for any initial conditions, the system (1.2) is uniformly exponentially stable. □

Theorem 3.3 *Suppose the linear system* (1.1) *is uniformly exponentially stable. Then there exists some* $\beta > 0$ *such that if*

$$||F(t)|| \leq \beta \tag{3.2}$$

for all $t \geq t_0$ *with* $t, t_0 \in \mathbb{T}$, *the perturbed system* (1.2) *is uniformly exponentially stable.*

Proof For any initial conditions, the solution of (1.2) satisfies

$$z(t) = \Phi_A(t,t_0)z_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t,t_0)$ is the transition matrix for the system (1.1). By the uniform exponential stability of (1.1), there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t,\tau)|| \leq \gamma e_{-\lambda}(t,\tau)$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides, we have

$$||z(t)|| \leq \gamma e_{-\lambda}(t,t_0)||z_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t,\sigma(s))||F(s)||\,||z(s)||\,\Delta s, \quad t \geq t_0.$$

Rearranging and applying the uniform regressivity bound and the inequality (3.2), we obtain

$$e_{-\lambda}(t_0,t)||z(t)|| \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta e_{-\lambda}(t_0,s)||z(s)||\,\Delta s, \quad t \geq t_0.$$

Defining $\psi(t) := e_{-\lambda}(t_0,t)||z(t)||$, we now have

$$\psi(t) \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta\psi(s)\,\Delta s, \quad t \geq t_0.$$

By Gronwall's inequality, we obtain

$$\psi(t) \leq \gamma||z_0||e_{\gamma\beta\delta}(t,t_0), \quad t \geq t_0.$$

Thus, substituting back in for $\psi(t)$, we conclude

$$||z(t)|| \leq \gamma||z_0||e_{-\lambda\oplus\gamma\beta\delta}(t,t_0), \quad t \geq t_0.$$

We need $-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$ and negative for all $t \in \mathbb{T}$. Observe, since $\gamma\beta\delta > 0$, it is positively regressive, and so $\gamma\beta\delta \in \mathcal{R}^+$. Since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we see that $-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$. So we must have

$$-\lambda \oplus \gamma\beta\delta < 0 \implies \beta < \frac{\lambda}{\gamma\delta(1-\mu(t)\lambda)},$$

for all $t \in \mathbb{T}$. Thus, by choosing β accordingly and since γ is independent of the initial conditions, the system (1.2) is uniformly exponentially stable. $\square$

Theorem 3.4 *Consider the uniformly regressive linear dynamic system* (1.2)*, with the matrices $A(t)$ and $F(t)$ constant. Let the uniformly regressive constants $\lambda \in \mathcal{R}^+$ and $\gamma > 0$ such that*

$$||e_A(t,t_0)|| \leq \gamma e_\lambda(t,t_0), \qquad t \geq t_0.$$

Then the bound

$$||e_{A+F}(t,t_0)|| \leq \gamma e_{(\lambda\oplus\gamma\delta||F||)}(t,t_0), \qquad t \geq t_0,$$

is valid.

Proof We begin by noting that the solution X to (1.2) with constant system matrices is given by

$$e_{A+F}(t,t_0) = X(t) = e_A(t,t_0) + \int_{t_0}^{t} e_A(t,\sigma(s))FX(s)\Delta s. \tag{3.3}$$

The solution (3.3) can be bounded by the following

$$||X(t)|| \le \gamma e_\lambda(t,t_0) + \int_{t_0}^{t} \gamma e_\lambda(t,\sigma(s))||F||\;||X(s)||\Delta s. \tag{3.4}$$

We now employ Gronwall's inequality on (3.4) by defining $\psi(t) := e_\lambda(t_0,t)||X(t)||$. Thus,

$$\psi(t) \le \gamma + \int_{t_0}^{t} \gamma e_\lambda(s,\sigma(s))||F||\;\psi(s)\Delta s \le \gamma + \int_{t_0}^{t} \gamma\delta||F||\;\psi(s)\Delta s$$

which implies

$$||e_{A+F}(t,t_0)|| \le \gamma e_{(\lambda\oplus\delta\gamma||F||)}(t,t_0). \;\square$$

Theorem 3.5 *Given the uniformly regressive system* (1.2) *with* $A(t) \equiv A$ *a constant matrix, suppose all eigenvalues of* A *belong to* $\mathcal{S}(\mathbb{T})$*, the matrix* $F(t) \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ *satisfies*

$$\lim_{t\to\infty} ||F(t)|| = 0, \tag{3.5}$$

and the solution $x(t) \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ *is defined for all* $t \ge t_0$*. Then given any initial conditions* $x(t_0) = x_0$*, the solution to* (1.2) *satisfies*

$$\lim_{t\to\infty} x(t) = 0. \tag{3.6}$$

Proof Since $\mathrm{spec}(A) \in \mathcal{S}(\mathbb{T})$ for all $t \in \mathbb{T}$ and the system is uniformly regressive, we have

$$||e_A(t,t_0)|| \le \gamma e_{-\lambda}(t,t_0), \tag{3.7}$$

for some $\gamma, \lambda > 0$ with $-\lambda \in \mathcal{R}^+$, and all $t \ge t_0$. Using (3.7), we can bound the solution by

$$||x(t)|| \le \gamma e_{-\lambda}(t,t_0) + \int_{t_0}^{t} \gamma e_{-\lambda}(t,\sigma(s))||F(s)||\;||x(s)||\Delta s.$$

Choose an $\varepsilon > 0$ such that $-\lambda \oplus \varepsilon < 0$ and $-\lambda \oplus \varepsilon \in \mathcal{R}^+$ for all $t \in \mathbb{T}$. By Gronwall's inequality, we have

$$||x(t)||e_{-\lambda}(t_0,t) \le \gamma||x_0||\exp\left[\int_{t_0}^{t} \lim_{s\searrow\mu(\tau)} \frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(\tau)||]\Delta\tau\right]. \tag{3.8}$$

Denoting the upper bound of the graininess of $\mathbb{T}$ by μ^* and employing the generalized version of L'Hôpital's rule [1] and (3.5), we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_{t_0}^{t} \lim_{s\searrow\mu(\tau)} \frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(\tau)||]\Delta\tau}{\int_{t_0}^{t} \lim_{s\searrow\mu(\tau)} \frac{1}{s}\mathrm{Log}[1+s\varepsilon]\Delta\tau} &= \lim_{t\to\infty} \frac{\lim_{s\searrow\mu(t)} \frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(t)||]}{\lim_{s\searrow\mu(t)} \frac{1}{s}\mathrm{Log}[1+s\varepsilon]} \\
&\le \frac{\gamma\delta\lim_{t\to\infty}||F(t)||}{\frac{1}{\mu^*}\mathrm{Log}[1+\mu^*\varepsilon]} \\
&= 0,
\end{aligned}$$

thus implying that there exists a $T \in \mathbb{T}$ such that for $t \geq T$ we have

$$\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \mathrm{Log}[1 + s\gamma\delta||F(\tau)||]\Delta\tau \leq \int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \mathrm{Log}[1 + s\varepsilon]\Delta\tau.$$

From (3.8), for $t \geq T$ we obtain

$$||x(t)||e_{-\lambda}(t_0, t) \leq \gamma||x_0||e_{\varepsilon}(t, t_0).$$

With a correct choice of ε above, it easily follows that

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda\oplus\varepsilon}(t, t_0)$$

which implies the claim (3.6). □

4 Nonlinear Perturbations

In the following theorem, we show that under certain conditions on the linear and nonlinear perturbations, the resulting perturbed nonlinear initial value problem will still yield uniformly exponentially stable solutions.

Theorem 4.1 *Given the nonlinear uniformly regressive initial value problem*

$$x^{\Delta}(t) = [A(t) + F(t)]\, x(t) + g(t, x(t)), \qquad x(t_0) = x_0, \tag{4.1}$$

and an arbitrary time scale $\mathbb{T}$*, suppose* (1.1) *is uniformly exponentially stable, the matrix* $F(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *satisfies* $||F(t)|| \leq \beta$ *for all* $t \in \mathbb{T}$*, the vector-valued function* $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *satisfies* $||g(t, x(t))|| \leq \epsilon||x(t)||$ *for all* $t \in \mathbb{T}$ *and* $x(t)$*, and the solution* $x(t) \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *is defined for all* $t \geq t_0$*. Then if* β *and* ϵ *are sufficiently small, there exist constants* γ*,* $\lambda^* > 0$ *with* $-\lambda^* \in \mathcal{R}^+$ *such that*

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$*.*

Proof Observe that the solution to (4.1) is given by

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))[F(s)x(s) + g(s, x(s))]\Delta s, \tag{4.2}$$

for all $t \geq t_0$. Since (1.1) is uniformly exponentially stable, there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$. Recall $||F(t)|| \leq \beta$, $||g(t, x(t))|| \leq \epsilon||x(t)||$ for all $t \in \mathbb{T}$, and since the decay factor $-\lambda$ is uniformly regressive on $\mathbb{T}$, there exists a $\delta > 0$ such that $0 < \delta^{-1} \leq (1 - \mu(t)\lambda)$ for all $t \in \mathbb{T}$ which implies that $0 < (1 - \mu(t)\lambda)^{-1} \leq \delta$. Taking the norms of both sides of (4.2), we obtain

$$\begin{aligned} ||x(t)|| &\leq ||\Phi_A(t, t_0)||\; ||x_0|| + \int_{t_0}^{t} ||\Phi_A(t, \sigma(s))||(||F(s)||\; ||x(s)|| + ||g(s, x(s))||)\Delta s \\ &= e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma\delta(\beta + \epsilon)e_{-\lambda}(t_0, s)||x(s)||\Delta s\right], \end{aligned}$$

for all $t \geq t_0$.

By Gronwall's inequality,

$$||x(t)|| \leq \gamma ||x_0|| e_{-\lambda \oplus \gamma\delta(\beta+\epsilon)}(t, t_0).$$

To conclude, we need $-\lambda \oplus \gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$ as well as $-\lambda \oplus \gamma\delta(\beta + \epsilon) < 0$. Observe that $\gamma\delta(\beta + \epsilon) > 0$ implies $\gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$ and since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we have $-\lambda \oplus \gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$. So we need

$$-\lambda \oplus \gamma\delta(\beta + \epsilon) < 0 \implies \beta < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta} - \epsilon.$$

From this result, we must have $\frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta} - \epsilon > 0$ for all $t \in \mathbb{T}$, i.e. $\epsilon < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta}$ for all $t \in \mathbb{T}$.

Thus, to fulfill the requirements of the theorem, we must satisfy the following:

$$0 < \epsilon < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta}, \quad 0 < \beta < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta} - \epsilon, \quad \text{and} \quad -\lambda^* := -\lambda \oplus \gamma\delta(\beta + \epsilon)$$

for all $t \in \mathbb{T}$. $\Box$

Corollary 4.1 *Given the nonlinear uniformly regressive initial value problem* (4.1) *with $A(t) \equiv A$ a constant matrix, suppose* $\text{spec}(A) \in \mathcal{S}(\mathbb{T})$ *for all $t \in \mathbb{T}$, the matrix $F(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ satisfies $||F(t)|| \leq \beta$ for all $t \in \mathbb{T}$, the vector-valued function $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfies $||g(t, x(t))|| \leq \epsilon ||x(t)||$ for all $t \in \mathbb{T}$ and $x(t)$, and the solution $x(t) \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}^n)$ is defined for all $t \geq t_0$. Then if β and ϵ are sufficiently small, there exist constants γ, $\lambda^* > 0$ with $-\lambda^* \in \mathcal{R}^+$ such that*

$$||x(t)|| \leq \gamma ||x_0|| e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$.

Proof The proof follows exactly as in Theorem 4.1, with the observation that $\Phi_A(t, t_0) \equiv e_A(t, t_0)$. Since $\text{spec}(A) \in \mathcal{S}(\mathbb{T})$, there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||e_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$, we now have the bound $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$, for some constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$. $\Box$

Conclusions

The intent of this paper was to add to the completeness of bounds on solutions to linear systems on time scales. In particular, in Section 2 this was done via introduction of two generalizations of Gronwall's inequality, thereby creating addition possibilities for bounding solutions to systems of the form (1.1) and (1.2).

In Section 3 and Section 4, certain bounds were given on the linear and nonlinear perturbations which maintained stability of the system (1.2) were investigated. This included integral bounds and asymptotic bounds on the perturbation matrix F.

References

[1] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2001.

[2] Brogan, W.L. *Modern Control Theory*. Prentice-Hall, Upper Saddle River, 1991.

[3] Chen, C.T. *Linear System Theory and Design*. Oxford University Press, New York, 1999.

[4] DaCunha, J.J. Stability for time varying linear dynamic systems on time scales. *J. Comput. Appl. Math.* **176**(2) (2005) 381–410.

[5] Gard, T and Hoffacker, J. Asymptotic behavior of natural growth on time scales. *Dynam. Systems Appl.* **12** (2003) 131–147.

[6] Lyapunov, A.M. The general problem of the stability of motion. *Internat. J. Control* **55** (1992) 521–790.

[7] Pötzsche, C., Siegmund, S., and Wirth, F. A spectral characterization of exponential stability for linear time-invariant systems on time scales. *Discrete Contin. Dyn. Syst.* **9** (2003) 1223–1241.

[8] Rugh, W.J. *Linear System Theory*. Prentice-Hall, Englewood Cliffs, 1996.

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Stability Properties for Some Non-autonomous Dissipative Phenomena Proved by Families of Liapunov Functionals

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Abstract: We prove some new results regarding the boundedness, stability and attractivity of the solutions of a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may contain time-dependent coefficients. The class includes equations arising in superconductor theory, and in the theory of viscoelastic materials. In the proof we use a family of Liapunov functionals W depending on two parameters, which we adapt to the 'error', i.e. to the size σ of the chosen neighbourhood of the null solution.

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1 Introduction

In this paper we study the boundedness and stability properties of a large class of initial-boundary-value problems of the form

$$\begin{cases} -\varepsilon(t)u_{xxt} + u_{tt} - C(t)u_{xx} + a'u_t = F(u) - au_t, & x \in]0,\pi[, \quad t > t_0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \end{cases} \tag{1.1}$$

$$u(x,t_0) = u_0(x), \qquad u_t(x,t_0) = u_1(x). \tag{1.2}$$

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Here $t_0 \geq 0$, $\varepsilon \in C^2(I,I)$, $C \in C^1(I,\mathbb{R}^+)$ (with $I := [0,\infty[$) are functions of t, with $C(t) \geq \overline{C} = \text{const} > 0$, the conservative force fulfills $F(0) = 0$, so that the equation admits the trivial solution $u(x,t) \equiv 0$; $a' = \text{const} \geq 0$, $a = a(x,t,u,u_x,u_t,u_{xx}) \geq 0$, $\varepsilon(t) \geq 0$, so that the corresponding terms are dissipative [1].

Solutions u of such problems describe a number of physically remarkable continuous phenomena occurring on a finite space interval.

For instance, when $F(u) = b\sin u$, $a = 0$ we deal with a perturbed Sine–Gordon equation which is used to describe the classical Josephson effect [8] in the theory of superconductors, which is at the base (see e.g. [12, 1] and references therein) of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3–6 in [2]): $u(x,t)$ is the phase difference of the macroscopic quantum wave functions describing the Bose–Einstein condensates of Cooper pairs in two superconductors separated by a very thin and narrow dielectric strip (a socalled "Josephson junction"), the dissipative term $(a'+a)u_t$ is due to Joule effect of the residual current across the junction due to single electrons, whereas the third order dissipative term is due to the surface impedence of the two superconductors of the strip. Usually the model is considered with constant (dimensionless) coefficients $\varepsilon, C, (a'+a)$, but in fact the latter depend on other physical parameters like the temperature or the voltage difference applied to the junction (see e.g. [12]), which can be controlled and varied with time; in a more accurate description of the model one should take a non-constant $a = \beta\cos u$, where β also depends on temperature and voltage difference applied and therefore can be varied with time.

Other applications of problem (1.1)–(1.2) include heat conduction at low temperature [13, 7], sound propagation in viscous gases [10], propagation of plane waves in perfect incompressible and electrically conducting fluids [15], motions of viscoelastic fluids or solids [9, 14, 16]. For instance, problem (1.1)–(1.2) with $a = 0 = a'$ describes [14] the evolution of the displacement $u(x,t)$ of the section of a rod from its rest position x in a Voigt material when an external force F is applied; in this case $c^2 = E/\rho$, $\varepsilon = 1/(\rho\mu)$, where ρ is the (constant) linear density of the rod at rest, and E, μ are respectively the elastic and viscous constants of the rod, which enter the stress-strain relation $\sigma = E\nu + \partial_t\nu/\mu$, where σ is the stress, ν is the strain. Again, some of these parameters, like the viscous constant of the rod, may depend on the temperature of the rod, which can be controlled and varied with time.

The problem (1.1)–(1.2) considered here generalizes those considered in [3, 4, 5, 6], in that the square velocity C and the dissipative coefficient ε can depend on t. The physical phenomena just described provide the motivations for such a generalization. While we require C to have a positive lower bound, in order not to completely destroy the wave propagation effects due to the operator $\partial_t^2 - C\partial_x^2$, we wish to include the cases that ε goes to zero as $t \to \infty$, vanishes at some point t, or even vanishes identically. To that

[1] This follows from the non-positivity of the corresponding terms in the time derivative of the Hamiltonian:

$$H = \int_0^\pi dx \left[\frac{u_t^2 + Cu_x^2}{2} - \int_0^{u(x)} F(z)dz\right] \quad \Rightarrow \quad \dot{H} = -\int_0^\pi dx \left[(a+a')u_t^2 + \varepsilon u_{xt}^2\right] + \int_0^\pi dx\, \dot{C}\frac{u_x^2}{2}.$$

We also see that the last term is respectively dissipative, forcing if $\dot{C}$ is negative, positive. H can play the role of Liapunov functional w.r.t. the reduced norm $d_{\varepsilon=0}(u,u_t)$.

end, we consider the t-dependent norm

$$d^2(\varphi,\psi) \equiv d^2_\varepsilon(\varphi,\psi) = \int_0^\pi dx\,[\varepsilon^2(t)\varphi^2_{xx} + \varphi^2_x + \varphi^2 + \psi^2]. \tag{1.3}$$

ε^2 plays the role of a weight for the second order derivative term φ^2_{xx} so that for $\varepsilon = 0$ this automatically reduces to the proper norm needed for treating the corresponding second order problem. Imposing the condition that φ, ψ vanish in $0, \pi$ one easily derives that $|\varphi(x)|, \varepsilon|\varphi_x(x)| \le d(\varphi,\psi)$ for any x; therefore a convergence in the norm d implies also a uniform (in x) pointwise convergence of φ and a uniform (in x) pointwise convergence of φ_x for $\varepsilon(t) \neq 0$. To evaluate the distance of u from the trivial solution we shall use the t-dependent norm $d(t) \equiv d_{\varepsilon(t)}\big[u(x,t), u_t(x,t)\big]$; we use the abbreviation $d(t)$ whenever this is not ambiguous.

In Section 2 we state the hypotheses necessary to prove our results, give the relevant definitions of boundedness and (asymptotic) stability, introduce a 2-parameter family of Liapunov functionals W and tune these parameters in order to prove bounds for $W, \dot W$. In Sections 3, 4 we prove the main results: a theorem of stability and (exponential) asymptotic stability of the null solution (Section 3), under stronger assumptions theorem of eventual and/or uniform boundedness of the solutions and eventual and/or exponential asymptotic stability in the large of the null solution (Section 4). In Section 5, we mention some examples to which these results can be applied.

We note that for constant ε the existence and uniqueness of the solution of the problem (1.1)–(1.2) follows from the theorem in section 2 of [6], as we can replace at the left-hand side $C(t)$ by $\inf_t C$ and include in the right-hand side the difference $[\inf_t C - C(t)]u_{xx}$.

2 Main Assumptions, Definitions and Preliminary Estimates

For any function $f(t)$, we denote $\overline{f} = \inf_{t>0} f(t)$, $\overline{\overline{f}} = \sup_{t>0} f(t)$. We assume that there exist constants $A \ge 0$, $\tau > 0$, $k \ge 0$, $\rho > 0$, $\mu > 0$ such that

$$F(0) = 0 \quad \& \quad F_z(z) \le k \qquad \text{if } |z| < \rho. \tag{2.1}$$

$$\overline{C} \ge k, \quad C - \dot\varepsilon \ge \mu(1+\varepsilon), \quad \mu + \frac{\overline{C}}{2} - 2k > 0, \quad \overline{\overline{\varepsilon}} > -\infty. \tag{2.2}$$

$$0 \le a \le A d^\tau(u, u_t), \qquad a' + \frac{\overline{\varepsilon}}{2} > 0 \tag{2.3}$$

We are not excluding the following cases: $\varepsilon(t) = 0$ for some t, $\varepsilon \xrightarrow{t\to\infty} 0$, $\varepsilon(t) \equiv 0$, $\varepsilon \xrightarrow{t\to\infty} \infty$ [in view of $(2.2)_2$ the latter condition requires also $C \xrightarrow{t\to\infty} \infty$]; but by condition $(2.3)_2$ at least one of the dissipative terms must be nonzero. Eq. (2.1) implies

$$\int_0^\varphi F(z)dz \le k\frac{\varphi^2}{2}, \qquad \varphi F(\varphi) \le k\varphi^2 \quad \text{if } |\varphi| < \rho. \tag{2.4}$$

We shall consider also the cases that, in addition to (2.1), either one of the following inequalities [which are stronger than (2.4)] holds:

$$\int_0^\varphi F(z)dz \le 0, \qquad \varphi F(\varphi) \le 0 \quad \text{if } |\varphi| < \rho. \tag{2.4'}$$

To formulate our results, we need the following definitions. Fix once and for all $\kappa \in \mathbb{R}$, $\xi > 0$ and let $I_\kappa := [\kappa, \infty[$, $d(t) := d_{\varepsilon(t)}\big[u(x,t), u_t(x,t)\big]$.

Definition 2.1 The solution $u(x,t) \equiv 0$ of (1.1) is stable if for any $\sigma \in]0, \xi]$ and $t_0 \in I_\kappa$ there exists a $\delta(\sigma, t_0) > 0$ such that

$$d(t_0) < \delta(\sigma, t_0) \quad \Rightarrow \quad d(t) < \sigma \quad \forall t \geq t_0.$$

If δ can be chosen independent of t_0, $\delta = \delta(\sigma)$, $u(x,t) \equiv 0$ is uniformly stable.

Definition 2.2 The solution $u(x,t) \equiv 0$ of (1.1) is asymptotically stable if it is stable and moreover for any $t_0 \in I_\kappa$ there exists a $\delta(t_0) > 0$ such that $d(t_0) < \delta(t_0)$ implies $d(t) \to 0$ as $t \to \infty$, namely for any $\nu > 0$ there exists a $T(\nu, t_0, u_0, u_1) > 0$ such that

$$d(t_0) < \delta(t_0) \quad \Rightarrow \quad d(t) < \nu \quad \forall t \geq t_0 + T.$$

The solution $u(x,t) \equiv 0$ is uniformly asymptotically stable if it is uniformly stable and moreover δ, T can be chosen independent of t_0, u_0, u_1, i.e. $d(t) \to 0$ as $t \to \infty$ uniformly in t_0, u_0, u_1.

Definition 2.3 The solutions of (1.1) are eventually uniformly bounded if for any $\delta > 0$ there exist a $s(\delta) \geq 0$ and a $\beta(\delta) > 0$ such that if $t_0 \geq s(\delta)$, $d(t_0) \leq \delta$, then $d(t) < \beta(\delta)$ for all $t \geq t_0$. If $s(\delta) = 0$ the solutions of (1.1) are uniformly bounded.

Definition 2.4 The solutions of (1.1) are bounded if for any $\delta > 0$ there exist a $\tilde{\beta}(\delta, t_0) > 0$ such that if $d(t_0) \leq \delta$, then $d(t) < \tilde{\beta}(\delta, t_0)$ for all $t \geq t_0$.

Definition 2.5 The solution $u(x,t) \equiv 0$ of (1.1) is eventually exponential-asymptotically stable in the large if for any $\delta > 0$ there are a nonnegative constant $s(\delta)$ and positive constants $D(\delta), E(\delta)$ such that if $t_0 \geq s(\delta)$, $d(t_0) \leq \delta$, then

$$d(t) \leq D(\delta) \exp\left[-E(\delta)(t - t_0)\right] d(t_0), \qquad \forall t \geq t_0. \tag{2.5}$$

If $s(\delta) = 0$ then $u(x,t) \equiv 0$ is exponential-asymptotically stable in the large.

Definition 2.6 The solution $u(x,t) \equiv 0$ of (1.1) is (uniformly) exponential-asymptotically stable if there exist positive constants δ, D, E such that

$$d(t_0) < \delta \quad \Rightarrow \quad d(t) \leq D \exp\left[-E(t - t_0)\right] d(t_0), \quad \forall t \geq t_0. \tag{2.6}$$

Definition 2.7 The solution $u(x,t) \equiv 0$ of (1.1) is asymptotically stable in the large if it is stable and moreover for any $t_0 \in I_\kappa$, $\nu, \alpha > 0$ there exists $T(\alpha, \nu, t_0, u_0, u_1) > 0$ such that

$$d(t_0) < \alpha \quad \Rightarrow \quad d(t) < \nu \quad \forall t \geq t_0 + T.$$

We recall Poincaré inequality, which easily follows from Fourier analysis:

$$\phi \in C^1(]0,\pi[), \quad \phi(0) = 0, \quad \phi(\pi) = 0 \quad \Rightarrow \quad \int_0^\pi dx\, \phi_x^2(x) \geq \int_0^\pi dx\, \phi^2(x). \tag{2.7}$$

We introduce the non-autonomous family of Liapunov functionals

$$\begin{aligned} W \equiv W(\varphi, \psi, t; \gamma, \theta) \quad := \quad & \int_0^\pi \frac{1}{2}\Big\{\gamma\psi^2 + (\varepsilon\varphi_{xx} - \psi)^2 + \left[C(1+\gamma) - \dot{\varepsilon} + \varepsilon(a' + \theta)\right]\varphi_x^2 \\ & + a'\theta\varphi^2 + 2\theta\varphi\psi - 2(1+\gamma)\int_0^{\varphi(x)} F(z)dz\Big\} dx \end{aligned} \tag{2.8}$$

where θ, γ are for the moment unspecified positive parameters. W coincides with the Liapunov functional of [3] for constant ε, C and $\gamma = 3$, $\theta = a'$. Let $W(t;\gamma,\theta) := W(u,u_t,t;\gamma,\theta)$. Using (1.1), from (2.8) one finds

$$\begin{aligned}
\dot W(t;\gamma,\theta) &= \int_0^\pi \Big\{(\varepsilon u_{xx}-u_t)(\varepsilon u_{xxt}-u_{tt}+\dot\varepsilon u_{xx})+[\dot C(1+\gamma)-\ddot\varepsilon+\dot\varepsilon(a'+\theta)]\frac{u_x^2}{2}\\
&\quad+[C(1+\gamma)-\dot\varepsilon+\varepsilon(a'+\theta)]u_x u_{xt}+a'\theta u u_t+\theta u_t^2+(\gamma u_t+\theta u)u_{tt}-(1+\gamma)F(u)u_t\Big\}\,dx\\
&= \int_0^\pi \Big\{(\varepsilon u_{xx}-u_t)[(a+a')u_t-Cu_{xx}-F(u)+\dot\varepsilon u_{xx}]+[\dot C(1+\gamma)-\ddot\varepsilon+\dot\varepsilon(a'+\theta)]\frac{u_x^2}{2}\\
&\quad-[C(1+\gamma)-\dot\varepsilon+\varepsilon(a'+\theta)]u_{xx}u_t+a'\theta u u_t+\theta u_t^2\\
&\quad+(\gamma u_t+\theta u)[Cu_{xx}+\varepsilon u_{xxt}+F(u)-(a+a')u_t]-(1+\gamma)F(u)u_t\}\,dx\\
&= \int_0^\pi \Big\{\varepsilon u_{xx}[(\dot\varepsilon-C)-F(u)]u_{xx}+[\varepsilon u_{xx}(a+a')-(a+a')u_t+Cu_{xx}+F(u)-\dot\varepsilon u_{xx}-C(1+\gamma)u_{xx}\\
&\quad+\dot\varepsilon u_{xx}-\varepsilon(a'+\theta)u_{xx}+a'\theta u+\theta u_t+\gamma Cu_{xx}+\gamma\varepsilon u_{xxt}+\gamma F(u)-(a+a')\gamma u_t-\theta(a+a')u\\
&\quad-(1+\gamma)F(u)]u_t+\theta u[Cu_{xx}+\varepsilon u_{xxt}+F(u)]+[\dot C(1+\gamma)-\ddot\varepsilon+\dot\varepsilon(a'+\theta)]\frac{u_x^2}{2}\Big\}\,dx\\
&= \int_0^\pi \Big\{\varepsilon[(\dot\varepsilon-C)u_{xx}-F(u)]u_{xx}+u_t[\varepsilon a u_{xx}-(a+a')(1+\gamma)u_t-\varepsilon\theta u_{xx}\\
&\quad+\theta u_t+\gamma\varepsilon u_{xxt}-a\theta u]+\theta u[Cu_{xx}+\varepsilon u_{xxt}+F(u)]+[\dot C(1+\gamma)-\ddot\varepsilon+\dot\varepsilon(a'+\theta)]\frac{u_x^2}{2}\Big\}\,dx\\
&= -\int_0^\pi \Big\{\varepsilon(C-\dot\varepsilon)u_{xx}^2+[(a+a')(1+\gamma)-\theta]\,u_t^2+\Big[2\theta C+\ddot\varepsilon-\dot\varepsilon(a'+\theta)-(1+\gamma)\dot C\Big]\,\frac{u_x^2}{2}+\varepsilon\gamma u_{xt}^2\\
&\quad+\theta a u u_t-\theta u F(u)+\varepsilon[-a u_t+F(u)]u_{xx}\Big\}dx. \qquad (2.9)
\end{aligned}$$

2.1 Upper bound for $\dot W$

After some rearrangement of terms and integration by parts of the last term, we obtain

$$\begin{aligned}
\dot W = -\int_0^\pi \Big\{&\varepsilon\gamma u_{xt}^2+\Big[(a+a')(1+\gamma)-\theta-\varepsilon\frac{a^2}{C-\dot\varepsilon}-\theta\frac{a^2}{C}\Big]\,u_t^2+\varepsilon(C-\dot\varepsilon)\,\Big[\frac{a}{C-\dot\varepsilon}u_t-\frac{u_{xx}}{2}\Big]^2\\
&+\frac{3}{4}\varepsilon(C-\dot\varepsilon)u_{xx}^2+\Big[C\left(\frac{\theta}{2}-a'\right)+\ddot\varepsilon+(C-\dot\varepsilon)(a'+\theta)-(1+\gamma)\dot C-2\varepsilon F_u\Big]\,\frac{u_x^2}{2}\\
&+\frac{\theta C}{4}(u_x^2-u^2)+\frac{\theta C}{4}\,\Big[u+\frac{2a}{C}u_t\Big]^2-\theta u F(u)\Big\}\,dx.
\end{aligned}$$

Using (2.7) with $\phi(x)=u_t(x,t), u(x,t)$ we thus find, provided $|u|<\rho$, $\theta>2a'$, $\mu(a'+\theta)>2k$

$$
\begin{aligned}
\dot{W}\leq &-\int_0^{\pi}\!\Bigg\{\!\left[\varepsilon\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{C}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\
&\qquad\left[C\left(\frac{\theta}{2}-a'\right)+\ddot{\varepsilon}+\mu(1+\varepsilon)(a'+\theta)-(1+\gamma)\dot{C}-2\varepsilon k\right]\frac{u_x^2}{2}-\theta k u^2\Bigg\}dx\\
\leq &-\int_0^{\pi}\!\Bigg\{\!\left[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\
&\left[\overline{C}\left(\frac{\theta}{2}-a'\right)+\overline{\ddot{\varepsilon}}+\mu(a'+\theta)+[\mu(a'+\theta)-2k]\varepsilon-(1+\gamma)\dot{C}-2k\theta\right]\frac{u_x^2}{2}\Bigg\}dx.
\end{aligned}
\tag{2.10}
$$

We now assume that there exists $\bar{t}(\gamma)\in[0,\infty[$ such that

$$
\dot{C}(1+\gamma)\leq 1\quad\text{for } t\geq\bar{t},\qquad \dot{C}(1+\gamma)>1\quad\text{for } 0\leq t<\bar{t}. \tag{2.11}
$$

This is clearly satisfied with $\bar{t}(\gamma)\equiv 0$ if $\dot{C}\leq 0$, whereas it is satisfied with some $\bar{t}(\gamma)\geq 0$ if $\dot{C}\overset{t\to\infty}{\longrightarrow}0$. We fix θ by choosing

$$
\theta>\theta_1:=\max\left\{2a',\,\frac{2k}{\mu}-a',\,\frac{5-\overline{\ddot{\varepsilon}}-a'(\mu-\overline{C})}{\mu+\overline{C}/2-2k}\right\}. \tag{2.12}
$$

Then for all $t>\bar{t}$

$$
\theta\left(\mu+\frac{\overline{C}}{2}-2k\right)+[\mu(a'+\theta)-2k]\overline{\varepsilon}+\overline{\ddot{\varepsilon}}-(1+\gamma)\dot{C}+a'(\mu-\overline{C})>4. \tag{2.13}
$$

Next, provided $d(u,u_t)\leq\sigma<\rho$, we choose

$$
\gamma>\gamma_1(\sigma):=\frac{1+\theta}{a'+\overline{\varepsilon}}+\gamma_{32}\sigma^{2\tau},\qquad \gamma_{32}:=\frac{A^2}{(a'+\overline{\varepsilon})}\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right), \tag{2.14}
$$

what implies, for $d\leq\sigma$,

$$
\begin{aligned}
&\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)=a+a'+(a+a'+\overline{\varepsilon})\gamma-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\\
&\geq a'+\frac{a+a'+\overline{\varepsilon}}{a'+\overline{\varepsilon}}\left[(1+\theta)+A^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\sigma^{2\tau}\right]-\theta-A^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)d^{2\tau}\geq 1+a'.
\end{aligned}
\tag{2.15}
$$

Equations (2.10), (2.13) and (2.15) imply for all $t\geq\bar{t}$

$$
\begin{aligned}
\dot{W}(u,u_t,t;\gamma,\theta)\leq&-\int_0^{\pi}\!\Bigg\{\!\left[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\
&\left[\theta\left(\mu+\frac{\overline{C}}{2}-2k\right)+[\mu(a'+\theta)-2k]\overline{\varepsilon}+\overline{\ddot{\varepsilon}}-(1+\gamma)\dot{C}+a'(\mu-\overline{C})\right]\frac{u_x^2+u^2}{4}\Bigg\}dx\\
<&-\eta\, d^2(t),\qquad \eta:=\min\{1,3\mu/4\}
\end{aligned}
\tag{2.16}
$$

provided $0<d(t)<\sigma$. If, in addition to (2.3) with $k>0$, the inequality (2.4') [which is stronger than (2.4)] holds, then it is easy to check that we can avoid assuming $(2.2)_3$ and obtain again the previous inequality, provided we replace k by 0 in the definition (2.12) of θ_1.

Remark 2.1 One can check that if we had adopted the same Liapunov functional as in [5, 6] formulae (4.2), i.e. W of (2.8) with $\theta=0=a'$, we would have not been able to obtain (2.16) (which is essential to prove the asymptotic stability of the null solution) in a number of situations, e.g. if $\varepsilon\to 0$ sufficiently fast as $t\to\infty$.

2.2 Lower bound for W

From the definition (2.8) it immediately follows

$$W(\varphi,\psi,t;\gamma,\theta)=\int_0^\pi \frac{1}{2}\Bigg\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\frac{(\varepsilon\varphi_{xx}-2\psi)^2}{4}+\frac{(\varepsilon\varphi_{xx}-\psi)^2}{2}+\varepsilon^2\frac{\varphi_{xx}^2}{4}$$

$$+[C(1+\gamma)-\dot\varepsilon+\varepsilon(a'+\theta)]\varphi_x^2+(a'\theta-1)\varphi^2+[\theta\psi+\varphi]^2-2(1+\gamma)\int_0^{\varphi(x)}F(z)dz\Bigg\}dx. \quad (2.17)$$

Using $(2.2)_2$, (2.4) and (2.7) with $\phi(x)=\varphi(x)$ we find for $|\varphi|<\rho$

$$W\geq\int_0^\pi\frac{1}{2}\left\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\varepsilon^2\frac{\varphi_{xx}^2}{4}+[(C-k)\gamma+\mu+(\mu+a'+\theta)\varepsilon]\,\varphi_x^2+[a'\theta-1-k]\,\varphi^2\right\}dx$$

$$\geq\int_0^\pi\frac{1}{2}\Bigg\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\varepsilon^2\frac{\varphi_{xx}^2}{4}+\left[(\overline{C}-k)\gamma+\mu+\left(\mu+a'+\frac{\theta}{2}\right)\overline{\varepsilon}\right]\varphi_x^2$$

$$+\left[\left(a'+\frac{\overline{\varepsilon}}{2}\right)\theta-1-k\right]\varphi^2\Bigg\}\,dx. \quad (2.18)$$

Choosing

$$\theta>\theta_2:=\max\left\{\theta_1,\frac{k+5/4}{a'+\overline{\varepsilon}/2}\right\},\qquad \gamma\geq\gamma_2(\sigma):=\gamma_1(\sigma)+\theta^2+1, \quad (2.19)$$

we find that for $d\leq\sigma$

$$W(\varphi,\psi,t;\gamma,\theta)\geq\chi d^2(\varphi,\psi),\qquad \chi:=\frac{1}{2}\min\left\{\frac{1}{4},(\overline{C}-k)\gamma+\mu+\left(\mu+a'+\frac{\theta}{2}\right)\overline{\varepsilon}\right\}. \quad (2.20)$$

(Note that $0<\chi\leq 1/8$). If, in addition to (2.1) (with some $k>0$), the inequality $(2.4')_1$ holds, then it is easy to check that we obtain (2.20) [with the replacement $k\to 0$ in the definition of χ] by choosing θ,γ as in (2.19), but replacing $k\to 0$ there.

Finally, we note that if $\tau=0$ in (2.3), i.e. $a\leq A=$const, then $\gamma,\bar t(\gamma)$ are independent of σ.

2.3 Upper bound for W

As argued in [3],

$$\left|\int_0^\varphi F(z)dz\right|=\left|\int_0^\varphi dz\int_0^\zeta F_\zeta(\zeta)d\zeta\right|=\left|\int_0^\varphi F_\zeta(\zeta)(\varphi-\zeta)d\zeta\right|.$$

Consequently, introducing the non-decreasing funtion $m(r):=\max\{|F_\zeta(\zeta)|\ :\ |\zeta|\leq r\}$ and in view of the inequality $|\varphi|\leq d(\varphi,\psi)$ we obtain

$$\left|\int_0^\varphi F(z)dz\right|\leq m(|\varphi|)\frac{\varphi^2}{2}\leq m(d)\frac{d^2}{2}. \quad (2.21)$$

Thus, from definition (2.8) and the inequalities $-2\epsilon\varphi_{xx}\psi \leq \epsilon^2\varphi_{xx}^2+\psi^2$, $2\theta\varphi\psi \leq \theta(\varphi^2+\psi^2)$, $(2.2)_3$ we easily find

$$W(\varphi,\psi,t;\gamma,\theta) \leq \int\limits_0^\pi \frac{1}{2}\{(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2+[C(1+\gamma)-\dot{\varepsilon}$$

$$+\varepsilon(a'+\theta)]\,\varphi_x^2+(a'+1)\theta\varphi^2\}\,dx+(1+\gamma)m(d)\frac{d^2}{2} \leq \int\limits_0^\pi \frac{1}{2}\Big\{(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2$$

$$+\left[C\gamma+(C-\dot{\varepsilon})\left(1+\frac{a'+\theta}{\mu}\right)\right]\varphi_x^2+(a'+1)\theta\varphi^2\Big\}\,dx+(1+\gamma)m(d)\frac{d^2}{2}.$$

Choosing

$$\gamma \geq \gamma_3(\sigma) := \gamma_2(\sigma)+1+\tfrac{a'+\theta}{\mu}+(a'+1)\theta = \gamma_{31}+\gamma_{32}\sigma^{2\tau}, \tag{2.22}$$

where $\gamma_{31} := \frac{1+\theta}{a'+\bar{\varepsilon}}+\theta^2+2+\frac{a'+\theta}{\mu}+(a'+1)\theta$ and setting

$$g(t) := C(t)-\dot{\varepsilon}(t)/2+1 > 1, \qquad B^2(d) := [1+m(d)]\,d^2, \tag{2.23}$$

we find that for $d \leq \sigma$

$$\begin{aligned} W(\varphi,\psi,t;\gamma,\theta) &\leq \int\limits_0^\pi \frac{1}{2}\left[(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2+\gamma\,(2C-\dot{\varepsilon})\,\varphi_x^2+\gamma\varphi^2\right]dx+(1+\gamma)m(d)\frac{d^2}{2} \\ &\leq [2\gamma g(t)+(1+\gamma)m(d)]\,\frac{d^2}{2} \leq (1+\gamma)\,[g(t)+m(d)]\,d^2 \\ &\leq [1+\gamma(\sigma)]\,g(t)B^2(d). \end{aligned} \tag{2.24}$$

The map $d \in [0,\infty[\to B(d) \in [0,\infty[$ is continuous and increasing, therefore also invertible. Moreover, $B(d) \geq d$.

3 Asymptotic Stability of the Null Solution

Theorem 3.1 *Assume that conditions (2.1)-(2.3) are fulfilled. Then the null solution $u(x,t)$ of (1.1) is stable if one of the following conditions is fulfilled:*

$$\dot{C} \leq 0, \quad \forall t \in I, \tag{3.1}$$

$$\dot{C} \xrightarrow{t\to\infty} 0; \tag{3.2}$$

the stability is uniform if the function $g(t)$ defined by (2.23) fulfills $\overline{\overline{g}} < \infty$. The ξ appearing in Definition 2.1 is a suitable positive constant, more precisely $\xi \in]0,\rho]$ if $\rho < \infty$. The null solution is asymptotically stable if, in addition,

$$\int_0^\infty \frac{dt}{g(t)} = \infty, \tag{3.3}$$

and uniformly exponential-asymptotically stable if $\overline{\overline{g}} < \infty$.

Proof As a first step, we analyze the behaviour of

$$\frac{\sigma^2}{1+\gamma_3(\sigma)} = \frac{\sigma^2}{1+\gamma_{31}+\gamma_{32}\sigma^{2\tau}} =: r^2(\sigma).$$

The positive constants γ_{31}, γ_{32}, defined in (2.22), are independent of σ, t_0. The function $r(\sigma)$ is an increasing and therefore invertible map $r: [0, \sigma_M[\to [0, r_M[$, where:

$$\begin{aligned}
&\sigma_M = \infty, && r_M = \infty, && \text{if } \tau \in [0,1[,\\
&\sigma_M = \infty && r_M = 1/\sqrt{\gamma_{32}}, && \text{if } \tau = 1,\\
&\sigma_M^{2\tau} := \tfrac{1+\gamma_{31}}{\gamma_{32}(\tau-1)}, && r_M = [\tfrac{\tau-1}{1+\gamma_{31}}]^{\frac{\tau-1}{2\tau}}/\sqrt{\tau}\gamma_{32}^{\frac{1}{2\tau}}, && \text{if } \tau > 1,
\end{aligned} \tag{3.4}$$

(in the latter case $r(\sigma)$ is decreasing beyond σ_M).

Next, let $\xi := \min\{\sigma_M, \rho\}$ if the rhs is finite, otherwise choose $\xi \in \mathbb{R}^+$; we shall consider an "error" $\sigma \in]0, \xi[$. We define

$$\delta(\sigma, t_0) := B^{-1}\left[r(\sigma)\frac{\sqrt{\chi}}{\sqrt{g(t_0)}}\right], \qquad \kappa := \bar{t}[\gamma_3(\xi)]. \tag{3.5}$$

$\delta(\sigma, t_0)$ belongs to $]0, \sigma[$, because $B(d) \geq d$ implies $B^{-1}\left[r(\sigma)\sqrt{\chi}/\sqrt{g(t_0)}\right] \leq \sqrt{\chi}\sigma \leq \sigma/2$ and is an increasing function of σ. The function $\bar{t}(\gamma)$ was defined in (2.11); $\bar{t}[\gamma_3(\sigma)] \leq \kappa$ as the function $\bar{t}[\gamma_3(\sigma)]$ is non-decreasing. Mimicking an argument of [6], we can show that for any $t_0 \geq \kappa$

$$d(t_0) < \delta(\sigma, t_0) \quad \Rightarrow \quad d(t) < \sigma \quad \forall t \geq t_0. \tag{3.6}$$

Ad absurdum, assume that there exists a finite $t_1 > t_0$ such that (3.6) is fulfilled for all $t \in [t_0, t_1[$, whereas

$$d(t_1) = \sigma. \tag{3.7}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W[u, u_t, t; \gamma_3(\sigma), \theta]$ is a decreasing function of t in $[t_0, t_1]$. Using (2.20), (2.24) we find the following contradiction with (3.7):

$$\begin{aligned}
&\chi d^2(t_1) \leq W(t_1) < W(t_0) \leq [1+\gamma_3(\sigma)]\, g(t_0) B^2\, [d(t_0)] < [1+\gamma_3(\sigma)]\, g(t_0) B^2(\delta)\\
&= [1+\gamma_3(\sigma)]\, g(t_0) \left\{B\left[B^{-1}\left(\sigma\frac{\sqrt{\chi}}{\sqrt{[1+\gamma_3(\sigma)]g(t_0)}}\right)\right]\right\}^2 = \chi\sigma^2.
\end{aligned}$$

Eq. (3.6) amounts to the stability of the null solution; if $\overline{\overline{g}} < \infty$ we obtain the uniform stability replacing $(3.5)_1$ by $\delta(\sigma) := B^{-1}\left[r(\sigma)\sqrt{\chi}/\sqrt{\overline{\overline{g}}}\right]$.

Let now $\delta(t_0) := \delta(\xi, t_0)$. By (3.6) and the monotonicity of $\delta(\cdot, t_0)$ we find that for any $t_0 \geq \kappa$

$$d(t_0) < \delta(t_0) \quad \Rightarrow \quad d(t) < \xi \qquad \forall t \geq t_0. \tag{3.8}$$

Choosing $W(t) \equiv W[u, u_t, t; \gamma_3(\xi), \theta]$, (2.24) becomes

$$W(t) \leq h(\xi) g(t) d^2(t), \qquad h(\xi) := [1+\gamma_3(\xi)]\,[1+m(\xi)]\,, \tag{3.9}$$

which together with (2.16), implies $\dot{W}(t) \le -\eta W(t)/[hg(t)]$ and (by means of the comparison principle [17]) $W(t) < W(t_0)\exp\left[-\eta\int_{t_0}^{t} dz/[hg(z)]\right]$, whence

$$d^2(t) \le \frac{W(t)}{\chi} < \frac{W(t_0)}{\chi}\exp\left[-\frac{\eta}{h}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right]$$
$$\le \frac{hg(t_0)}{\chi}d^2(t_0)\exp\left[-\frac{\eta}{h}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right] < \frac{h(\xi)g(t_0)}{\chi}\xi^2\exp\left[-\frac{\eta}{h(\xi)}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right]$$

Condition (3.3) implies that the exponential goes to zero as $t\to\infty$, proving the asymptotic stability of the null solution; if $\overline{\overline{g}}<\infty$ we can replace $g(t_0), g(z)$ by $\overline{\overline{g}}$ in the last but one inequality and obtain

$$d^2(t) < \frac{h(\xi)\overline{\overline{g}}}{\chi}\exp\left[-\frac{\eta}{h(\xi)\overline{\overline{g}}}(t-t_0)\right]d^2(t_0),$$

which proves the uniform exponential-asymptotic stability of the null solution (just set $\delta = B^{-1}\left[r(\xi)\sqrt{\chi}/\sqrt{\overline{\overline{g}}}\right]$, $D=\sqrt{h(\xi)\overline{\overline{g}}/\chi}$, $E=\eta/\left[2h(\xi)\overline{\overline{g}}\right]$ in Def. 2.6). $\square$

Remark 3.1 We stress that the theorem holds also if $\rho=\infty$. In the latter case ξ is σ_M, if the latter is finite, an arbitrary positive constant, if also $\sigma_M=\infty$.

Next, we are going to extend some of the previous results *in the large*.

4 Boundedness of the Solutions and Asymptotic Stability in the Large

Theorem 4.1 *Assume that: conditions (2.1)-(2.3), and possibly either one of (2.4'), are fulfilled with $\rho=\infty$ and $\tau<1$; the function $g(t)$ defined by (2.23) fulfills $\overline{\overline{g}}<\infty$; (3.1) is fulfilled. Then:*
1. the solutions of (1.1) are uniformly bounded;
2. the null solution of (1.1) is exponential-asymptotically stable in the large.
If only (3.2), instead of (3.1), is satisfied, then:
3. the solutions of (1.1) are eventually uniformly bounded;
4. the null solution of (1.1) is eventually exponential-asymptotically stable in the large.

Proof As noted, $r(\sigma)$ can be inverted to an increasing map $r^{-1}:[0,r_M[\mapsto[0,\sigma_M[$, whence also

$$\beta(\delta) := r^{-1}\left[\frac{\sqrt{\overline{\overline{g}}}B(\delta)}{\sqrt{\chi}}\right] \tag{4.1}$$

defines an increasing map $\beta:[0,\delta_M[\mapsto[0,\sigma_M[$, where $\delta_M := B^{-1}(r_M\sqrt{\chi}/\sqrt{\overline{\overline{g}}})$. Note that $\beta(\delta)>\delta$. An immediate consequence of (4.1) is

$$\frac{\overline{\overline{g}}B^2(\delta)}{\chi} = r^2[\beta(\delta)] = \frac{\beta^2(\delta)}{1+\gamma_3[\beta(\delta)]}. \tag{4.2}$$

From (2.11) it immediately follows that

$$s(\delta) := \bar{t}\{\gamma_3[\beta(\delta)]\}\begin{cases} =0, & \text{if (3.1) is fulfilled,}\\ <\infty, & \text{if (3.2) is fulfilled.}\end{cases} \tag{4.3}$$

We can now show that for any $\delta \in]0, \delta_M[$, $t_0 \geq s(\delta)$

$$d(t_0) < \delta \quad \Rightarrow \quad d(t) < \beta(\delta), \quad \forall t \geq t_0. \tag{4.4}$$

Ad absurdum, assume that there exists a finite $t_2 > t_0$ such that (4.4) is fulfilled for all $t \in [t_0, t_2[$, whereas

$$d(t_2) = \beta(\delta). \tag{4.5}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W\{u, u_t, t; \gamma_3[\beta(\delta)], \theta\}$ is a decreasing function of t in $[t_0, t_2]$. Using (2.20), (2.24) and the (4.2) we find the following contradiction with (4.5):

$$\chi d^2(t_2) \leq W(t_2) < W(t_0) \leq \{1+\gamma_3[\beta(\delta)]\} g(t_0) B^2\,[d(t_0)] < \{1+\gamma_3[\beta(\delta)]\}\overline{\overline{g}} B^2(\delta) = \chi \beta^2(\delta).$$

Formula (4.4) together with (4.3) proves statements 1., 3. under the assumption $\tau \in [0, 1[$, because then by (3.4) $\delta_M = \infty$, so that we can choose any $\delta > 0$ in Definition 2.3.

With the above choice of θ, by (4.4), (3.9) we find that for $t \geq t_0 \geq s(\delta)$ the Liapunov functional $W_\delta(t) \equiv W\big\{u, u_t, t; \gamma_3\big[\beta(\delta)\big], \theta(\delta)\big\}$ fulfills

$$W_\delta(t) \leq h(\delta)\overline{\overline{g}} d^2(t); \tag{4.6}$$

this, together with (2.16) implies $\dot{W}_\delta(t) \leq -\eta W_\delta(t)/[h(\delta)\overline{\overline{g}}]$ and (by means of the comparison principle [17]) $W_\delta(t) < W_\delta(t_0) \exp\big[-\eta(t-t_0)/[h(\delta)\overline{\overline{g}}]\big]$. From the latter inequality, (2.20) and (4.6) with $t = t_0$ it follows

$$d^2(t) \leq \frac{W_\delta(t)}{\chi} < \frac{W_\delta(t_0)}{\chi} \exp\left[-\frac{\eta}{h(\delta)\overline{\overline{g}}}(t-t_0)\right] \leq \frac{h(\delta)\overline{\overline{g}}}{\chi} \exp\left[-\frac{\eta}{h(\delta)\overline{\overline{g}}}(t-t_0)\right] d^2(t_0)$$

for all $t \geq t_0 \geq s(\delta)$. Recalling again (4.3), we see that the latter formula proves statements 2., 4. □

In the case $\tau \geq 1$ we find, by (3.4),

$$\delta_M = B^{-1}\left(r_M \frac{\sqrt{\chi}}{\sqrt{\overline{\overline{g}}}}\right) = B^{-1}\left\{\left[\frac{\tau-1}{1+\gamma_{31}}\right]^{\frac{\tau-1}{2\tau}} \frac{\sqrt{\chi}}{\sqrt{\overline{\overline{g}}\tau\gamma_{32}^{1/\tau}}}\right\}.$$

The finiteness of δ_M prevents us from extending the results in the large of the previous theorem to the case $\tau \geq 1$. One might think to exploit the freedom in the choice of θ to make δ_M as large as we wish. From the θ-dependence of γ_{31}, γ_{32} [formulae (2.22), (2.14)] we see that δ_M decreases with θ, so this is impossible. However, we can prove boundedness and asymptotic stability in the large even for some unbounded $g(t)$, provided $\tau = 0$.

Theorem 4.2 *Assume that: conditions (2.3–2.1), and possibly either one of (2.4'), are fulfilled with $\rho = \infty$ and $\tau = 0$; the function $g(t)$ defined by (2.23) fulfills (3.3); either (3.1) or (3.2) is fulfilled. Then:*
1. the solutions of (1.1) are bounded;
2. the null solution of (1.1) is asymptotically stable in the large.

Proof The condition $\tau = 0$ means that γ does not depend on σ; then $r^{-1}(\beta) = \beta\sqrt{1+\gamma}$, which is an increasing map $r^{-1} : I \to I$. For any fixed t_0 setting

$$\tilde{\beta}(\alpha; t_0) := r^{-1}\left[\frac{\sqrt{g(t_0)}B(\alpha)}{\sqrt{\chi}}\right] = B(\alpha)\frac{\sqrt{g(t_0)(1+\gamma)}}{\sqrt{\chi}} \tag{4.7}$$

also defines an increasing map $\tilde{\beta}: I \to I$, with $\tilde{\beta}(\alpha;t_0) > \alpha$. We now prove statement 1, i.e. for any $\alpha > 0$, $t_0 \geq \kappa := \bar{t}(\gamma)$,

$$d(t_0) < \alpha \quad \Rightarrow \quad d(t) < \tilde{\beta}(\alpha;t_0) \quad \forall t \geq t_0. \tag{4.8}$$

Ad absurdum, assume that there exist a finite $t_2 \in [t_0, t]$ such that (4.8) is fulfilled for all $t \in [t_0, t_2[$, whereas

$$d(t_2) = \tilde{\beta}(\alpha;t_0). \tag{4.9}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W\{u(t), u_t(t), t; \gamma, \theta\}$ is a decreasing function of t in $[t_0, t_2]$. Using (2.20), (2.24) and (4.7) we find the following contradiction with (4.9):

$$\chi d^2(t_2) \leq W(t_2) < W(t_0) \leq (1+\gamma) g(t_0) B^2\left[d(t_0)\right] < (1+\gamma) g(t_0) B^2(\alpha) = \chi \tilde{\beta}^2(\alpha;t_0), \text{ Q.E.D.}$$

By Theorem 3.1 the null solution of (1.1) is stable. Moreover, by (4.8) relation (2.24) becomes

$$W(t) \leq \tilde{h}(\alpha, t_0) g(t) d^2(t), \qquad \tilde{h}(\alpha, t_0) := (1+\gamma)\left\{1+m\left[\tilde{\beta}(\alpha;t_0)\right]\right\},$$

which, together with (2.16), implies $\dot{W}(t) \leq -\eta W(t)/[\tilde{h} g(t)]$ and employing usual arguments, $W(t) < W(t_0) \exp\left[-\eta \int_{t_0}^t dz/[\tilde{h} g(z)]\right]$, whence, for all $t > t_0 \geq \kappa$,

$$\begin{aligned} d^2(t) &\leq \frac{W(t)}{\chi} < \frac{W(t_0)}{\chi} \exp\left[-\frac{\eta}{\tilde{h}} \int_{t_0}^{t} \frac{dz}{g(z)}\right] \leq \frac{\tilde{h} g(t_0)}{\chi} d^2(t_0) \exp\left[-\frac{\eta}{\tilde{h}} \int_{t_0}^{t} \frac{dz}{g(z)}\right] \\ &< \frac{\tilde{h}(\alpha, t_0) g(t_0)}{\chi} \alpha^2 \exp\left[-\frac{\eta}{h(\alpha, t_0)} \int_{t_0}^{t} \frac{dz}{g(z)}\right]. \end{aligned}$$

The function $G_{t_0}(t) := \int_{t_0}^t dz/g(z)$ is increasing and by (3.3) diverges with t, what makes the rhs go to zero as $t \to \infty$; more precisely, we can fulfill Definition 2.7 defining the corresponding function $T(\alpha, \nu, t_0, u_0, u_1)$ by the condition that the rhs of the previous equation equals $\nu_0^2 := \min\{\nu^2, \alpha^2\}$ at $t = t_0 + T$, or equivalently

$$T = G_{t_0}^{-1}\left\{-\frac{\tilde{h}(\alpha, t_0)}{\eta} \log\left[\frac{\chi\, \nu_0^2}{\tilde{h}(\alpha, t_0)\, g(t_0)\, \alpha^2}\right]\right\} - t_0$$

(the rhs is positive as the argument of the logarithm is less than 1, by the definitions of $\chi, \tilde{h}$ and by the inequality $\nu_0/\alpha \leq 1$); this proves statement 2. □

5 Examples

Out of the many examples of forcing terms fulfilling (2.1) we just mention $F(z) = b\sin(\omega z)$ (this has $F_z(z) \leq b\omega =: k$), which makes (1.1) into a modification of the sine-Gordon equation, and the possibly non-analytic ones $F(z) = -b|z|^q z$ with $b > 0$, $q \geq 0$ (this has $F_z(z) \leq 0 =: k$), or $F(z) = b|z|^q z$ (this has $F_z(z) = b(q+1)|z|^q < b(q+1)|\rho|^q =: k$ if $|z| < \rho$). Out of the many examples of t-dependent coefficients that fulfill (2.2-2.3) and either (3.1) or (3.2), but not the hypotheses of the theorems of [4, 5, 6], we just mention the following ones:

Example 5.1 $\varepsilon(t) = \varepsilon_0(1+t)^{-p}$ with constant $\varepsilon_0, p \geq 0$ and $C \equiv C_0 \equiv$constant, with $C_0 > \frac{4(1+\varepsilon_0)k}{3+\varepsilon_0}$. As a consequence $\overline{\varepsilon} = 0 \leq \varepsilon \leq \varepsilon_0 = \overline{\overline{\varepsilon}}$, $\overline{\dot{\varepsilon}} = -p\varepsilon_0 \leq \dot{\varepsilon} = -p\varepsilon_0[1+t]^{-p-1} \leq 0 = \overline{\overline{\dot{\varepsilon}}}$, $\ddot{\varepsilon} = p(p+1)\varepsilon_0[1+t]^{-p-2} \geq 0 = \overline{\ddot{\varepsilon}}$ [condition $(2.2)_4$ is fulfilled], ($\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon} \to 0$ as $t \to \infty$). Conditions $(2.2)_1$-$(2.2)_3$ are fulfilled with $\mu = C/(1+\varepsilon_0)$. We find $g(t) = C_0 + p\varepsilon_0[1+t]^{-p-1} + 1$, whence $\overline{\overline{g}} = C_0 + p\varepsilon_0 + 1$. Finally we assume that $a' > 0$ and a fulfills $(2.3)_1$. Then Theorems 3.1, 4.1, apply: the null solution of (1.1) is uniformly stable and uniformly exponential-asymptotically stable; it is also uniformly bounded and exponential-asymptotically stable in the large if in addition $\rho = \infty$, $\tau < 1$.

One can check that if we had adopted the same Liapunov functional as in [5, 6] formulae (4.2), i.e. W of (2.8) with $\theta = 0 = a'$, for $p > 1$ (namely $\varepsilon \to 0$ sufficiently fast as $t \to \infty$) we would have not been able to prove the asymptotic stability.

Example 5.2 $\varepsilon(t) = \varepsilon_0(1+t)^p$, $C(t) = C_0(1+t)^q$, with $1 > q \geq p \geq 0$, $\varepsilon_0 \geq 0$ and C_0 fulfilling

$$C_0 > p\varepsilon_0, \qquad C_0 > \frac{4(1+\varepsilon_0)k + 2p\,\varepsilon_0}{3+\varepsilon_0}.$$

If $q, p > 0$ then $C(t), \varepsilon(t)$ diverge as $t \to \infty$. We immediately find $\varepsilon(t) \geq \varepsilon_0 = \overline{\varepsilon}$, $\dot{\varepsilon} = p\varepsilon_0(1+t)^{p-1} \geq 0$, $\ddot{\varepsilon} = p(p-1)\varepsilon_0(1+t)^{p-2} \leq 0$, $\overline{\ddot{\varepsilon}} = p(p-1)\varepsilon_0$ [condition $(2.2)_4$ is fulfilled], $C(t) \geq C_0$,

$$\frac{C - \dot{\varepsilon}}{1+\varepsilon} = \frac{C_0(1+t)^q - p\varepsilon_0(1+t)^{p-1}}{1+\varepsilon_0(1+t)^p} = \frac{C_0(1+t)^{q-p} - p\varepsilon_0(1+t)^{-1}}{(1+t)^{-p}+\varepsilon_0} \geq \frac{C_0 - p\varepsilon_0}{1+\varepsilon_0},$$

and conditions $(2.2)_1$-$(2.2)_3$ are fulfilled with $\mu = (C_0 - p\varepsilon_0)/(1+\varepsilon_0)$. Moreover, $\dot{C} = qC_0(1+t)^{q-1} \to 0$ as $t \to \infty$ [condition (3.2) is fulfilled]; $g(t)$ grows as t^q, implying that (3.3) is fulfilled. Finally we assume that a fulfills $(2.3)_1$ [condition $(2.3)_2$ is already satisfied] . Then Theorem 3.1 applies: the null solution of (1.1) is asymptotically stable. If in addition $\rho = \infty$, $\tau = 0$ then Theorem 4.2 applies, and the null solution is also bounded and asymptotically stable in the large.

Example 5.3 $\varepsilon(t)$ fulfilling $\overline{\overline{\varepsilon}} < \infty$, $\overline{\overline{\dot{\varepsilon}}} < \infty$, $\overline{\dot{\varepsilon}} > -\infty$, $\overline{\ddot{\varepsilon}} > -\infty$ [condition (3.2)]; we note that this includes regular, periodic $\varepsilon(t)$. $C(t) = C_0 + C_1(1+t)^{-q}$ with constant C_0, C_1, q fulfilling $C_1 > 0$, $q \geq 0$ and

$$C_0 > \max\left\{0, \overline{\overline{\dot{\varepsilon}}}, \frac{4(1+\overline{\overline{\varepsilon}})k + 2\overline{\overline{\dot{\varepsilon}}}}{3+\overline{\overline{\varepsilon}}}\right\}, \qquad C_0 \geq k.$$

Then conditions $(2.2)_1$–$(2.2)_3$ are fulfilled with $\mu = (C_0 - \overline{\overline{\dot{\varepsilon}}})/(1+\overline{\overline{\varepsilon}})$. Moreover, $\dot{C} \leq 0$ (condition (3.1) is fulfilled). We find $g(t) \leq C_0 + C_1 - \overline{\dot{\varepsilon}} + 1 =: \overline{\overline{g}} < \infty$. Finally we assume that $a' > 0$ and a fulfills $(2.3)_1$. Then Theorems 3.1, 4.1, apply: the null solution of (1.1) is uniformly stable and uniformly exponential-asymptotically stable. It is also uniformly bounded and exponential-asymptotically stable in the large if in addition $\rho = \infty$, $\tau < 1$.

References

[1] Barone, A. and釘ernó, G. *Physics and Applications of the Josephson Effect*. Wiley-Interscience, New-York, 1982.

[2] Christiansen, P. I., Scott, A. C. and Sorensen, M. P. *Nonlinear Science at the Dawn of the 21st Century*. Lecture Notes in Physics 542, Springer, 2000.

[3] D'Acunto, B. and D'Anna, A. *Stabilità per un'equazione tipo Sine-Gordon perturbata*. Atti del XII Congresso dell'Associazione Italian di Meccanica Teorica ed Applicata (AIMETA), Napoli, 3-6.10.95, p. 65 (1995).

[4] D'Acunto, B. and D'Anna, A. Stability for a third order Sine–Gordon equation. *Rend. Mat. Serie* **VII**, Vol. 18 (1998) 347–365.

[5] D'Anna, A. and Fiore, G. Stability and attractivity for a class of dissipative phenomena. *Rend. Mat. Serie* **VII**, Vol. 21 (2000) 191–206.

[6] D'Anna, A. and Fiore, G. Global Stability properties for a class of dissipative phenomena via one or several Liapunov functionals. *Nonlinear Dyn. Syst. Theory* **5** (2005) 9–38.

[7] Flavin, J. N. and Rionero, S. *Qualitative estimates for partial differential equations. An introduction*. CRC Press, Boca Raton, FL, 1996, 368 pp.

[8] Josephson, B. D. Possible new effects in superconductive tunneling. *Phys. Lett.* **1** (1962) 251–253; The discovery of tunneling supercurrents. *Rev. Mod. Phys. B* **46** (1974) 251–254; and references therein.

[9] Joseph, D. D., Renardy, M. and Saut, J. C. Hyperbolicity and change of type in the flow of viscoelastic fluids. *Arch Rational Mech. Anal.* **87** (1985) 213–251.

[10] Lamb, H. Hydrodynamics. Cambridge University Press, Cambridge, 1959.

[11] Lomdhal, P. S., Soerensen, O. H. and Christiansen, P. L. Soliton Excitations in Josephson Tunnel Junctions. *Phys. Rev. B* **25** (1982) 5737–5748.

[12] Parmentier, R. D. Fluxons in Long Josephson Junctions. In: *Solitons in Action* Proceedings of a workshop sponsored by the Mathematics Division, Army Research Office held at Redstone Arsenal, October 26-27, 1977 (Eds Karl Lonngren and Alwyn Scott). Academic Press, New York, 1978.

[13] Morro, A., Payne, L. E. and Straughan, B. Decay, growth, continuous dependence and uniqueness results in generalized heat conduction theories. *Appl. Anal.* **38** (1990) 231–243.

[14] Morrison, J. A. Wave propagations in rods of Voigt material and visco-elastic materials with three-parameters models. *Quart. Appl. Math.* **14** (1956) 153–169.

[15] Nardini, R. Soluzione di un problema al contorno della magneto-idrodinamica. *Ann. Mat. Pura Appl.* **35** (1953) 269–290. (Italian)

[16] Renno, P. On some viscoelastic models. *Atti Acc. Lincei Rend. Fis.* **75** (1983) 1–10.

[17] Yoshizawa, T. *Stability Theory by Liapunov's second method*. The Mathematical Society of Japan, 1966.

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Complete Analysis of an Ideal Rotating Uniformly Stratified System of ODEs

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Abstract: In this paper we discuss a system of six coupled ODEs which arise in ODE reduction of the PDEs governing the motion of uniformly stratified fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat. We prove that this autonomous system of ODEs is completely integrable if Rayleigh number $Ra = 0$ and determine the stable, unstable and center manifold passing through the rest point and discuss the qualitative feature of the solutions of this system of ODEs.

Keywords: *rotating stratified Boussinesq equation; completely integrable systems.*

Mathematics Subject Classification (2000): 34A34, 37K10.

1 Introduction

In fluid dynamics, the flow of fluid in the atmosphere and in the ocean is governed by the Navier-Stokes equations. In the scale of Boussinesq approximation (i.e., flow velocities are to slow to account for compressible effect), the flow of fluid is given by rotating stratified Boussinesq equations. In the theory of basin scale dynamics Maas [1], has considered the flow of fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat. The container is assumed to be steady, uniform rotation on an f-plane. With this assumptions Maas [1] reduces the rotating stratified Boussinesq equations to an interesting six coupled system of ODEs. Our analysis is quite different from the one employed by Maas [1] in as much as we have obtained rather precise information concerning the global phase portrait of the system as well as analytical representation of the solution in terms of elliptic functions.

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The system of six coupled ODEs is completely integrable if Rayleigh number $Ra = 0$. We provide in this paper the complete analysis of this integrable system. Four functionally independent first integrals and zero divergence of vector field implying the existence of fifth first integral, thereby prove the complete integrability of the system. The four first integrals reduce the $\mathbb{R}^6$ into a family of two dimensional invariant surfaces (when rotation frequency f is less than the twice of horizontal Rayleigh damping coefficient otherwise either degenerate into a rest point or an empty surface). We observe that gluing these surfaces along a circle of transit points we get a torus of genus one. If there is a rest point which lies on the invariant surface then it is seen to be singular and one of the generating circles gets pinched to the rest point. We obtain the stable and unstable manifolds passing through the rest point. We also find the center manifold through the rest point which shows that rest point is unstable with two dimensional stable, unstable and center manifolds passing through it. In additional we carry out the complete integration of the system in terms of elliptic functions which degenerate in special case. In the last section we obtained a fifth first integral which is guaranteed by Jacobi's last integral theorem, it is quite non trivial and expressible in terms of elliptic functions.

2 An Ideal Rotating Uniformly Stratified System of ODEs

In the scale of Boussinesq approximation, the flow of fluid in the atmosphere and in the ocean is described by rotating stratified Boussinesq equations

$$\begin{aligned}
&\frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e_3}} \times \mathbf{v}) = -\nabla p + \nu(\Delta \mathbf{v}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}},\\
&\operatorname{div} \mathbf{v} = 0, \\
&\frac{D\tilde{\rho}}{Dt} = \kappa \Delta \tilde{\rho}.
\end{aligned} \tag{2.1}$$

Here $\mathbf{v}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturb density $\tilde{\rho}$, p the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may consult Majda [2]. In their study of onset of instability in stratified fluids at large Richardson number, Majda and Shefter [3] obtained the ODE reduction of (2.1) by neglecting the effects of rotation and viscosity, and complete analysis of that system and qualitative features of the solution are discussed by Srinivasan et al [4] in their paper. Whereas Maas [1] consider the effects of rotation to equation (2.1) in the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times B \times H$. In this context, Maas [1] reduces the system of equations (2.1) to six coupled system of ODEs (2.3) given below, which form a completely integrable Hamiltonian system if Rayleigh number Ra vanishes. In his study he considers a rectangular basin of size $L \times L \times H$, which is temperature-stratified with fixed zeroth order moments of mass and heat (so that there is no net evaporation or precipitation, nor any net river input or output, and neither a net heating nor cooling). The container is assumed to be in steady, uniform rotation on an f-plane (f-plane refers to the effective background rotation axis determined by the projection of the earth's rotation vector along the vertical.) Maas [1] appeals to the idea that the dynamics of the position vector of its center of mass may, to some extent, be representative of the basin scale dynamics of a mid-latitude lake or sea; in this context one may refer to Morgan [5], and Maas [6].

Maas [1] reduces the system of equations (2.1) into the following system of six coupled ODEs:

$$\begin{aligned} &Pr^{-1}\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e_3}} \times \mathbf{w} = \hat{\mathbf{e_3}} \times \mathbf{b} - (w_1,\, w_2,\, rw_3) + \hat{T}\mathbf{T}, \\ &\frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} = -(b_1,\, b_2,\, \mu b_3) + Ra\mathbf{F}. \end{aligned} \tag{2.2}$$

In these equations, $\mathbf{b} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\mathbf{T}$ is the differential momentum, $\mathbf{F}$ are buoyancy fluxes, $f' = f/2r_h$ is the earth's rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [1] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction $\mathbf{F} = (0,\, 1,\, 0)$; the wind effect is neglected i.e. $\mathbf{T} = 0$. For Prandtl number, Pr, equal to one the system of equations (2.2) reduces to the following an ideal rotating, uniformly stratified system of six coupled ODEs.

$$\begin{aligned} &\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e_3}} \times \mathbf{w} = \hat{\mathbf{e_3}} \times \mathbf{b}, \\ &\frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} = Ra\mathbf{F}. \end{aligned} \tag{2.3}$$

We see the system of equations (2.3) is divergence free and, when $Ra = 0$, admits the following four functionally independent first integrals

$$|\mathbf{b}|^2 = c_1, \quad \hat{\mathbf{e_3}} \cdot \mathbf{w} = c_2, \quad |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = c_3, \quad \mathbf{b} \cdot \mathbf{w} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = c_4. \tag{2.4}$$

Hence, by using Liouville theorem on integral invariants and theorem of Jacobi [7] there exists an additional first integral. Also we see from (2.4) that $|\mathbf{b}|$ and $|\mathbf{w}|$ remain bounded so that the invariant surface (2.4) is compact and the flow of the vector field $(\mathbf{w},\, \mathbf{b})$ is complete. Therefore, the system of equations (2.3) is completely integrable for $Ra = 0$. Maas [1] took $f' = 1$ and equations (2.3) show that the horizontal circulation (w_3) is constant hence without loss of generality he took $w_3 = 0$ which is one of the first integral of the system (2.3). Using the first integral $\frac{|\mathbf{w}|^2}{2} + b_3 = B\,(constant)$, he obtained the Hamiltonian

$$H = \frac{1}{2}\Big(r^2 + s^2 + \{B - (w_1^2 + w_2^2)/2\}^2\Big) + Ra\, w_1\,, \tag{2.5}$$

where $r = \dot{w_1}$ and $s = \dot{w_2}$. With this Hamiltonian H, Maas [1] has shown that the system of equations (2.3) is completely integrable if $Ra = 0$.

Here we see that if $Ra = 0$, the system of equations (2.3) is completely integrable and we can rewrite it as follows

$$\begin{aligned} &\dot{\mathbf{w}} = -f'\hat{\mathbf{e_3}} \times \mathbf{w} + \hat{\mathbf{e_3}} \times \mathbf{b}, \\ &\dot{\mathbf{b}} = \mathbf{w} \times \mathbf{b}. \end{aligned} \tag{2.6}$$

It is easy to see that the critical points (rest points) of the system (2.6) are $(\lambda_1\hat{\mathbf{e_3}},\, \lambda_2\hat{\mathbf{e_3}})$, $(\lambda_1\hat{\mathbf{e_3}},\, 0)$,

$(0,\, \lambda_2\hat{\mathbf{e_3}})$, $(0,\, 0)$, $(\mathbf{w},\, f'\mathbf{w})$ and $(\frac{1}{f'}\mathbf{b},\, \mathbf{b})$ where $\lambda_1,\ \lambda_2$ are arbitrary scalars. Of these critical points, $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is the only one lying on the invariant surface

$$|\mathbf{b}|^2 = 1, \quad \hat{\mathbf{e_3}} \cdot \mathbf{w} = 1, \quad |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = 3, \quad \mathbf{w} \cdot \mathbf{b} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1 + f'. \tag{2.7}$$

We give the details of the analysis of the system (2.6) in the following section.

3 Analytical Details

We have six coupled autonomous system of nonlinear ODEs (2.6) with four first integrals (2.4). We now proceed to analyzing the system (2.6). With nonzero values of c_1, c_2, c_3 and c_4 the possible critical points of the system (2.6) are $(\lambda_1\hat{\mathbf{e_3}},\, \lambda_2\hat{\mathbf{e_3}})$. With $c_1 = 1$, and $\mathbf{w} = \pm\hat{\mathbf{e_3}}$, c_3 may assume the value -1 or 3 (not both). Now take $c_3 = 3$ so that the possible critical points are $(\hat{\mathbf{e_3}},\, \pm\hat{\mathbf{e_3}})$ and at these critical points the value of c_2 is ± 1. Note that the case $c_2 = -1$ will be a surface disjoint from $\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1$ so with the specific values of $c_1 = 1$, $c_2 = 1$, and $c_3 = 3$ we have only one critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$. At this critical point the fourth first integral assumes the value $c_4 = 1 + f'$.

We find the eigenvalues of the matrix of linearized part of the system (2.6) at this critical point and these are given below

$$0,\, 0,\, \pm\frac{\sqrt{1 - f'^2 \pm (-1 + f')^{3/2}\sqrt{3 + f'}}}{\sqrt{2}}, \tag{3.1}$$

the double eigenvalue zero implying the critical point is degenerate. With all four possible distributions of sign and for $0 < f' < 1$, we see that among these six eigenvalues, two of them have positive real parts and two of them have negative real parts and the remaining of two eigenvalues are zero. This linear analysis suggests that when $0 < f' < 1$, the rest point is degenerate and unstable. In fact the critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is unstable with two dimensional stable, unstable and center manifolds. For $f' = 1$ the system degenerates with all the six eigenvalues being zero possessing four linearly independent eigen vectors $(0,\, \hat{\mathbf{e_3}})$, $(\hat{\mathbf{e_2}},\, \hat{\mathbf{e_2}})$, $(\hat{\mathbf{e_1}},\, \hat{\mathbf{e_1}})$, $(\hat{\mathbf{e_3}},\, 0)$. We shall now bifurcate the analysis in two parts. (i) When a critical point lies on the invariant surface determine by equations (2.7). (ii) When no critical point lies on the invariant surface (2.7).

3.1 Critical point lying on the invariant surface

Now we set up the local coordinates on the two dimensional invariant surface (2.7), we get $w_3 = 1$. The general solution of the inhomogeneous equation $\mathbf{w} \cdot \mathbf{b} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1 + f'$ is given below.

$$w_1 = \frac{-b_2 k}{1 - b_3} + \frac{(1 + f')b_1}{1 + b_3}, \quad w_2 = \frac{b_1 k}{1 - b_3} + \frac{(1 + f')b_2}{1 + b_3}, \quad w_3 = 1, \tag{3.2}$$

where k is arbitrary. To determine the k, substitute (3.2) in $|\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = 3$ to get

$$k^2 = \left(\frac{1 - b_3}{1 + b_3}\right)^2 \left[1 + 2b_3 - 2f' - (f')^2\right] = k(b_3). \tag{3.3}$$

From above equation and for $|\mathbf{b}|^2 = 1$, we see that k is real if and only if

$$0 \le f' \le 1\,. \tag{3.4}$$

Note that when $f' = 0$, the system of equations (2.6) disregards rotation. For $f' = 1$ the invariant set (2.7) degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ whereas for $f' > 1$ the invariant set (2.7) is empty. By use of the first integral $|\mathbf{b}|^2 = 1$ we can introduce the spherical polar coordinates in our system

$$b_1 = \cos\theta \sin\phi, \quad b_2 = \sin\theta\sin\phi, \quad b_3 = \cos\phi, \tag{3.5}$$

with this help of spherical polar coordinates we get k as a function of ϕ as given below

$$k^2 = \tan^4\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]$$

or

$$k = \pm\tan^2\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]^{1/2} \tag{3.6}$$

and

$$\begin{aligned} w_1 &= \tan\left(\frac{\phi}{2}\right)\left((1+f')\cos\theta \mp \sin\theta\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\right), \\ w_2 &= \tan\left(\frac{\phi}{2}\right)\left((1+f')\sin\theta \pm \cos\theta\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\right). \end{aligned} \tag{3.7}$$

To obtain an ODE for ϕ we observe that

$$\frac{d}{dt}(b_1^2 + b_2^2) = b_3(w_2 b_1 - w_1 b_2).$$

Substituting (3.5) and (3.7) into this we get

$$\dot{\phi} = \pm\tan\left(\frac{\phi}{2}\right)\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\,. \tag{3.8}$$

Finally using this in the equations for $\dot{b_1}$ and $\dot{b_2}$ in (2.6) we get the equation for θ namely,

$$\dot{\theta} = \frac{(1 - f'\cos\phi)}{2\cos^2\frac{\phi}{2}}. \tag{3.9}$$

Equations (3.8)-(3.9) admit solutions in terms of elementary functions implying the complete integrability of the system (2.6). The solutions of the more general equations (3.22)-(3.26) below involve elliptic integrals. We record these results below for this special case. Corresponding to the plus sign in (3.8) we get for an arbitrary constants of integration $C_1 > 0$ and C_2,

$$\begin{aligned} \phi(t) &= 2\sin^{-1}\left[\frac{C_1\sqrt{4-(1+f')^2}\, e^{-\frac{t}{2}\sqrt{4-(1+f')^2}}}{1 + C_1^2\, e^{-t\sqrt{4-(1+f')^2}}}\right], \\ \theta(t) &= C_2 + \frac{(1-f')}{2}\left[t + \frac{2(3+4f'+f'^2)\tan^{-1}\left(\frac{2e^{t\sqrt{3-2f'-f'^2}} - (1-2f'-f'^2)C_1^2}{\sqrt{(1+f')^2(3-2f'-f'^2)C_1^4}}\right)}{(1+f')(3-2f'-f'^2)}\right]. \end{aligned} \tag{3.10}$$

Corresponding to the negative sign in (3.8) we get

$$\phi(t) = 2\sin^{-1}\left[\frac{C_1\sqrt{4-(1+f')^2}\, e^{\frac{t}{2}\sqrt{4-(1+f')^2}}}{1+C_1^2\, e^{t\sqrt{4-(1+f')^2}}}\right],$$

$$\theta(t) = C_2 + \frac{(1-f')}{2}\left[t + \frac{2(3+4f'+f'^2)\tan^{-1}\left(\frac{2C_1^2\, e^{t\sqrt{3-2f'-f'^2}}-(1-2f'-f'^2)}{\sqrt{(1+f')^2(3-2f'-f'^2)}}\right)}{(1+f')(3-2f'-f'^2)}\right]. \tag{3.11}$$

To settle the ambiguity in sign in (3.8) note that the first integrals (2.4) except $\mathbf{w}\cdot\mathbf{b}+f'\hat{\mathbf{e_3}}\cdot\mathbf{b}$ are invariant under reflection

$$(b_1,\, b_2,\, b_3) \mapsto (-b_1,\, -b_2,\, b_3), \tag{3.12}$$

whereas the integral $\mathbf{w}\cdot\mathbf{b}+f'\hat{\mathbf{e_3}}\cdot\mathbf{b}$ remains invariant when (3.12) is simultaneously applied with the transformation $k \mapsto -k$.

From (3.6) we see that ϕ is constrained by the relation

$$0 \le \phi \le 2\cos^{-1}\left(\frac{1+f'}{2}\right), \tag{3.13}$$

and k vanishes at both extreme values. The critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is correspond to $\phi = 0$ and at other end of extreme value of $\phi = 2\cos^{-1}\left(\frac{1+f'}{2}\right)$ the system of ODEs, (3.8) has a periodic trajectory given by

$$\phi = 2\cos^{-1}\left(\frac{1+f'}{2}\right), \quad \dot{\theta} = \frac{2-f'(1+f')}{(1+f')}\,. \tag{3.14}$$

However, this does not correspond to a periodic solution of the original system (2.6) since the parametrization (3.5)-(3.7) fails to be Lipschitz along the locus given by (3.14). The locus (3.14) consists of *transit points*, which separate the stable and unstable manifolds. The locus given by (3.14) is a periodic orbit of the system (2.6) in a special case that we identify in section 3.2.

3.1.1 Stable and unstable manifolds

Let us denote by S the portion of sphere $|\mathbf{b}|^2 = 1$ defined by

$$\left\{(b_1,\, b_2,\, b_3)|b_1^2+b_2^2+b_3^2 = 1;\, 0 \le \phi \le 2\cos^{-1}\left(\frac{1+f'}{2}\right)\right\} \tag{3.15}$$

which is a closed spherical cap as shown in Figure 3.1 For each choice of the sign for $k(b_3)$ we denote the graph of function $\mathbf{w} = (w_1,\, w_2,\, w_3)$, as a function of $\mathbf{b}$ on S), by $\Gamma_\pm$ namely,

$$\Gamma_\pm = \left\{(\mathbf{w}(\mathbf{b}),\, \mathbf{b})|k = \pm\tan^2\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]^{1/2}\right\}. \tag{3.16}$$

Note that $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is defined in (3.2). Define functions $f_\pm : S \mapsto \Gamma_+$ as

$$\begin{array}{rcl} f_+(\mathbf{b}) & = & (\mathbf{w}(\mathbf{b}),\, \mathbf{b}), \quad k \ge 0, \\ f_-(\mathbf{b}) & = & (\mathbf{w}(\mathbf{b}),\, \mathbf{b}), \quad k \le 0. \end{array} \tag{3.17}$$

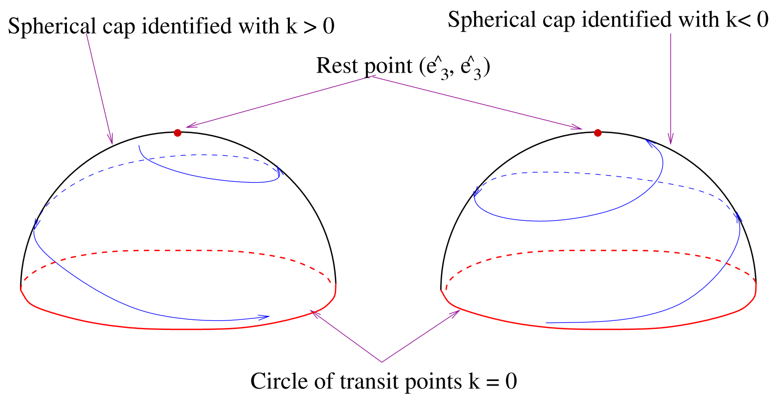

Figure 3.1: Stable and unstable manifolds.

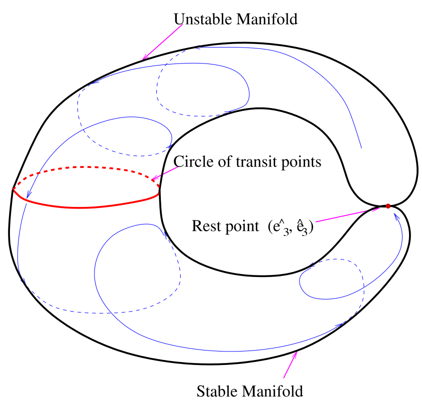

Figure 3.2: Torus pinched at critical point.

Both f_+ and f_- are homeomorphisms and they agree along the circle $k = 0$ as well as at the point $\mathbf{b} = \hat{\mathbf{e_3}}$. Thus the invariant surface is made up of the pieces $\Gamma_\pm$, each of which is homeomorphic to the closed spherical cap as shown in Figure 3.1 and given by (3.15). The invariant surface is obtained by gluing these pieces together at the critical point and the circle $k = 0$, as shown in Figure 3.2 This proves the invariant surface is a torus one of whose generating circle is pinched to a point.

Assume that for a solution starting near the critical point, $k(b_3) > 0$. Taking the plus sign in (3.8) we see that trajectories starting on Γ_+ recede away from the critical point since $\phi(t)$ monotonically increases, reaching the circle $k = 0$ in a finite time T given by

$$T = \int_\alpha^\beta \frac{\cot(\phi/2)d\phi}{\sqrt{4\cos^2(\phi/2) - (1 + f')^2}}\,. \tag{3.18}$$

Here α is the initial value of ϕ and β is the value of ϕ given by (3.8). The sign of $k(b_3)$ changes when $t > T$ whereby $\phi(t)$ decreases monotonically to zero and the trajectory, which now lies in Γ_-, approaches the critical point as $t \longrightarrow +\infty$.

On the other hand a trajectory starting on Γ_- stays in Γ_- and ultimately approaches the critical point as $t \longrightarrow +\infty$. We see that the part Γ_+ is the unstable manifold and Γ_-

the stable manifold of the system of ODEs (2.6). A trajectory starting on the unstable manifold reaches a point on (3.14) in a finite time and then enters the stable manifold.

A trajectory starting on the unstable manifold must reach a point on (3.14) in a finite time and subsequently must enter the stable manifold. This justifies the terminology "*transit points*".

3.2 When there are no critical points on the invariant surface

We perturb the initial conditions by assigning the values

$$c_1 = c_2 = 1, \quad c_4 = 1 + f', \quad c_3 = 3 + \epsilon, \tag{3.19}$$

to the first integrals (2.4). The compact invariant surface (2.4) no longer contains a rest point and so the Poincaré-Hopf index theorem shows that it is a torus. It is readily checked that the singularity $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ in the invariant surface that was initially present has smoothened out. Equations (3.2) continue to be valid except that $k(b_3)$ is now given by

$$(k(b_3))^2 = \Big(\frac{1-b_3}{1+b_3}\Big)^2 \Big[2(1+b_3) - (1+f')^2\Big] + \epsilon\Big(\frac{1-b_3}{1+b_3}\Big). \tag{3.20}$$

Parameterizing the sphere as in (3.5) we get in place of (3.6) the expression

$$k^2 = \tan^2\left(\frac{\phi}{2}\right)\Big[\tan^2\left(\frac{\phi}{2}\right)\Big(4\cos^2(\phi/2) - (1+f')^2\Big) + \epsilon\Big]. \tag{3.21}$$

Now using (2.6), $\frac{d}{dt}(b_1^2 + b_2^2) = 2kb_3(1+b_3)$, which is in polar coordinates assume the form

$$\dot{\phi} = k\cot\left(\frac{\phi}{2}\right) = \pm\Big[\tan^2\left(\frac{\phi}{2}\right)\Big(4\cos^2(\phi/2) - (1+f')^2\Big) + \epsilon\Big]^{1/2}. \tag{3.22}$$

The change of variable $v = \cos^2(\phi/2)$ transforms (3.22) into an ODE for elliptic integral:

$$\left(\frac{dv}{dt}\right)^2 = (v-1)\Big[4v^2 - \Big(4 + (1+f')^2 + \epsilon\Big)v + (1+f')^2\Big] = C(v)\,. \tag{3.23}$$

Note that for $\epsilon \leq -[2+(1+f')]^2$ or $\epsilon \geq -[2-(1+f')]^2$, the cubic polynomial $C(v)$ has three distinct real roots namely

$$\begin{aligned} \zeta_1 &= \tfrac{1}{8}\Big[(4+(1+f')^2+\epsilon) - \sqrt{(4+\epsilon)^2 + (1+f')^2[(1+f')^2+4+2\epsilon]}\Big], \\ \zeta_2 &= \tfrac{1}{8}\Big[(4+(1+f')^2+\epsilon) + \sqrt{(4+\epsilon)^2 + (1+f')^2[(1+f')^2+4+2\epsilon]}\Big], \\ v &= 1, \end{aligned} \tag{3.24}$$

two of which coalesce when $\epsilon \longrightarrow 0$.

For $\epsilon > 0$, $C(v)$ has real roots ζ_1, 1 and ζ_2 where $0 < \zeta_1 < 1 < \zeta_2$ and since $0 \leq v \leq 1$, we see that $C(v)$ is positive only on the interval $[\zeta_1, 1]$. The point $v(t)$ attains the value ζ_1 in time T_1 given by

$$T_1 = \int_\alpha^\beta \frac{d\phi}{\sqrt{\tan^2(\phi/2)\,[4\cos^2(\phi/2) - (1+f')^2] + \epsilon}},$$

where α is initial value of ϕ and β is the value of ϕ given by (3.22). After which k becomes negative, hence by equation (3.22), ϕ is decreasing and it decreases to zero in time T_2 given by

$$T_2 = -\int_{\beta}^{0} \frac{d\phi}{\sqrt{\tan^2(\phi/2)\,[4\cos^2(\phi/2) - (1+f')^2] + \epsilon}}.$$

Here we note that the value $v = 1$ corresponding to $\mathbf{b} = \hat{\mathbf{e_3}}$. However, $k \sim \tan(\frac{\phi}{2})\sqrt{\epsilon}$ and (3.2) gives

$$w_1 = -\sqrt{\epsilon}\sin\theta, \quad w_2 = \sqrt{\epsilon}\cos\theta, \quad \omega_3 = 1, \quad \text{as} \quad t \to T_2, \tag{3.25}$$

after which the value of k again becomes positive and ϕ increases from zero to its maximum value $2\cos^{-1}(\sqrt{\zeta_1})$ and this cycle repeats itself ad infinitum. Thus the points $v = 1$ and $v = \zeta_1$ represent a pair of circles of transit points and the solution of the system of ODEs (2.6) lying on the invariant surface (3.19) continuously oscillate between these circles of transit points in **b**-space.

On the other hand, for $\epsilon < 0$, equation (3.21) does not permit ϕ to approach zero. In fact the roots of the cubic polynomial $C(v)$ are real and satisfy $0 < \zeta_1 < \zeta_2 < 1$, forcing v to be in the interval $[\zeta_1, \zeta_2]$. Note that k vanishes along the pair of circles given by $2\cos^{-1}(\sqrt{\zeta_1})$ and $2\cos^{-1}(\sqrt{\zeta_2})$. These circles consist of *transit points* determining a frustum in which **b** is constrained to lie.

The equation governing θ is again (3.9) which in conjunction with (3.22) can be written as

$$\frac{d\theta}{d\phi} = \pm\frac{(1+f')\sec^2(\frac{\phi}{2}) - 2f'}{2\sqrt{\tan^2(\frac{\phi}{2})\Big(4\cos^2(\frac{\phi}{2}) - (1+f')^2\Big) + \epsilon}}\,. \tag{3.26}$$

Hence $\theta(t)$ may be expressed as an elliptic function of $\tan(\frac{\phi}{2})$.

In the special case when $\epsilon = -[2-(1+f')]^2$ the cubic polynomial $C(v)$ has two equal roots $\frac{(1+f')}{2}$, the frustum $\zeta_1 \le v \le \zeta_2$ is squeezed to a circle and the locus $k = 0$ does provide a periodic solution to the system (2.6) given by

$$\phi = 2\cos^{-1}\left(\sqrt{\frac{1+f'}{2}}\right), \quad \dot{\theta} = 1 - f'. \tag{3.27}$$

We summarize these results in the form of following theorem.

Theorem 3.1 *The solutions of the system of ODEs (2.6) lying on the two dimensional invariant surface (3.19) oscillate between circles of transit points and are expressible in terms of elliptic functions.*

3.2.1 The center manifold

We have noticed in previous section that if we perturb the initial conditions so that the first integrals assumes the values as indicated in equations (3.19), then the system admits a periodic solution lying on the invariant surface (3.19) when $\epsilon = -[2-(1+f')]^2$. This suggest the possibility of a more general perturbation that is, involving several parameters, resulting in a one parameter family of periodic solutions spanning a two dimensional invariant set that defines the center manifold.

We now proceed to obtain the center manifold through the rest point $(R_2\hat{\mathbf{e_3}},\, R_1\hat{\mathbf{e_3}})$ as the locus of a one parameter family of periodic solutions. At the place of equation (3.19) we assign to the constants the values given by

$$c_1 = R_1^2, \quad c_2 = R_2, \quad c_3 = R_2^2 + 2R_1 + \epsilon, \quad c_4 = R_1(R_2 + f'). \tag{3.28}$$

Instead of (3.2) we get

$$w_1 = \frac{-kR_2b_2}{R_1 - b_3} + \frac{(R_2 + f')b_1}{R_1 + b_3}, \quad w_2 = \frac{kR_2b_1}{R_1 - b_3} + \frac{(R_2 + f')b_2}{R_1 + b_3}, \quad w_3 = R_2. \tag{3.29}$$

Substituting in $|\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = R_2^2 + 2R_1 + \epsilon$ and using spherical polar coordinates, we find the value of k to be

$$k^2 = R_2^{-2} \tan^2\left(\frac{\phi}{2}\right)\left[4R_1 \sin^2\left(\frac{\phi}{2}\right) - (R_2 + f')^2 \tan^2\left(\frac{\phi}{2}\right) + \epsilon\right], \tag{3.30}$$

consequently we obtain the ODE for ϕ as given below

$$\left(\frac{d\phi}{dt}\right)^2 = \left[4R_1 \sin^2\left(\frac{\phi}{2}\right) - (R_2 + f')^2 \tan^2\left(\frac{\phi}{2}\right) + \epsilon\right].$$

Using the change of variable $v = \cos^2(\phi/2)$ the above equation transforms into the following ODE for elliptic function

$$\left(\frac{dv}{dt}\right)^2 = (v-1)\left[4R_1v^2 - \left(4R_1 + (R_2 + f')^2 + \epsilon\right)v + (R_2 + f')^2\right]. \tag{3.31}$$

The two roots of the cubic polynomial on the right hand side of (3.31) coincide (keeping v real) if and only if $\epsilon = -\left(R_2 + f' - 2\sqrt{R_1}\right)^2$, and corresponding repeated root is

$$\cos^2\left(\frac{\phi_0}{2}\right) = \frac{R_2 + f'}{2\sqrt{R_1}}. \tag{3.32}$$

The condition that the system of ODEs (2.6) admits a periodic solution $\cos^2(\frac{\phi_0}{2}) =$ constant is similar to the coalescence condition. Equation for $\dot{\theta}$ is

$$\dot{\theta} = \frac{R_2 + f' - 2f' \cos^2\left(\frac{\phi}{2}\right)}{2\cos^2\left(\frac{\phi}{2}\right)},$$

hence for the periodic trajectory we get $\dot{\theta} = \frac{R_2\sqrt{R_1} - f'(R_2 + f' - \sqrt{R_1})}{R_2 + f'}$. In particular, taking $R_1 = (\omega + f')^2$ we get the family of periodic trajectories parameterized by ω:

$$\begin{aligned} &w_1 = (R_2 + f')\tan\left(\frac{\phi_0}{2}\right)\cos(\omega t),\ w_2 = (R_2 + f')\tan\left(\frac{\phi_0}{2}\right)\sin(\omega t),\ w_3 = R_2, \\ &b_1 = R_1 \sin(\phi_0)\cos(\omega t),\ b_2 = R_1\sin(\phi_0)\sin(\omega t),\ b_3 = R_1\cos(\phi_0). \end{aligned} \tag{3.33}$$

We see that when $\omega = \left(\frac{R_2 - f'}{2}\right)$, the value of ϕ_0 vanishes and the periodic trajectory collapses to the rest point $(R_2\hat{\mathbf{e_3}},\, R_1\hat{\mathbf{e_3}})$ and the family (3.33) is the center manifold through the rest point.

We summarize our observations in the form of the following theorem.

Theorem 3.2 *The ODE reductions (2.3) of the Boussinesq equations with stratification and rotation form a completely integrable system if Rayleigh number Ra vanishes. Further, when $0 < f' < 1$, the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ is degenerate with two dimensional stable, unstable and center manifolds, and when $f' = 1$, the invariant surface (2.7), which is an intersection of four first integrals, degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$, whereas for $f' > 1$, the invariant surface is empty.*

4 Missing First Integral

Here we present some details on the computation of the evasive missing first integral whose existence is guaranteed by Jacobi's theorem.

$$\begin{aligned} z_j &= w_j, \qquad j = 1, 2, 3, \\ z_4 &= |\mathbf{b}|^2, \\ z_5 &= \mathbf{w} \cdot \mathbf{b} + f' \hat{\mathbf{e_3}} \cdot \mathbf{b}, \\ z_6 &= |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = z_1^2 + z_2^2 + z_3^2 + 2b_3\,. \end{aligned} \tag{4.1}$$

Now we determine the ODEs for z_j, $1 \le j \le 6$. From equations (2.6) and (2.4) we get

$$\dot{z_1} = f' z_2 - b_2, \quad \dot{z_2} = -f' z_1 + b_1, \quad \dot{z_j} = 0, \quad 3 \le j \le 6, \tag{4.2}$$

so that for $3 \le j \le 6$, z_j are constant and

$$z_5 = w_1 b_1 + w_2 b_2 + w_3 b_3 + f' b_3 = z_1 b_1 + z_2 b_2 + (z_3 + f') b_3,$$

$$\begin{aligned} z_1 b_1 + z_2 b_2 &= z_5 - \frac{(z_3 + f') z_6}{2} + \frac{(z_3 + f') z_3^2}{2} + \frac{(z_3 + f')}{2}(z_1^2 + z_2^2) \\ &= A + B(z_1^2 + z_2^2), \end{aligned} \tag{4.3}$$

where

$$A = z_5 - \frac{(z_3 + f')}{2}(z_6 - z_3^2), \quad B = \frac{z_3 + f'}{2}. \tag{4.4}$$

The general solution of equation (4.3) is given by

$$b_1 = \frac{-z_2 k}{z_1^2 + z_2^2} + \frac{A z_1}{z_1^2 + z_2^2} + B z_1, \quad b_2 = \frac{z_1 k}{z_1^2 + z_2^2} + \frac{A z_2}{z_1^2 + z_2^2} + B z_2\,, \tag{4.5}$$

where k is an arbitrary parameter. On substituting this in equation (4.1) we get

$$\begin{aligned} z_4 \;=\; & \left(\frac{-z_2 k}{z_1^2 + z_2^2} + \frac{A z_1}{z_1^2 + z_2^2} + B z_1 \right)^2 + \left(\frac{z_1 k}{z_1^2 + z_2^2} + \frac{A z_2}{z_1^2 + z_2^2} + B z_2 \right)^2 \\ & + \left(\frac{(z_6 - z_3^2) - (z_1^2 + z_2^2)}{2} \right)^2, \end{aligned}$$

which after simplification gives the value of k^2 as

$$k^2 = -A^2 + C(z_1^2 + z_2^2) + D(z_1^2 + z_2^2)^2 - \frac{1}{4}(z_1^2 + z_2^2)^3 := \psi(z_1^2 + z_2^2).$$

Here C and D are given by

$$C = z_4 - 2AB - \frac{1}{4}(z_6 - z_3^2)^2, \quad D = -B^2 + \frac{1}{2}(z_6 - z_3^2).$$

Rewriting the ODE (4.2) as

$$\frac{\dot{z_1}}{\dot{z_2}} = \frac{f' z_2 - b_2}{-f' z_1 + b_1}$$

and substituting for b_1 and b_2 from equation (4.5) we get

$$\frac{f'}{2}\frac{d(z_1^2+z_2^2)}{dt} - \left\{ \left(\frac{-z_2 k}{z_1^2+z_2^2} + \frac{Az_1}{z_1^2+z_2^2} + Bz_1 \right)\dot{z_1} + \left(\frac{z_1 k}{z_1^2+z_2^2} + \frac{Az_2}{z_1^2+z_2^2} + Bz_2 \right)\dot{z_2} \right\} = 0.$$

After simplification this can be written as

$$\left(\frac{f'-B}{4}\right)\frac{d}{dt}(z_1^2+z_2^2)^2 - \frac{A}{2}\frac{d}{dt}(z_1^2+z_2^2) - k(z_1\dot{z_2} - z_2\dot{z_1}) = 0\,,$$

which on integrating gives the first integral

$$\tan^{-1}(z_2/z_1) + \frac{1}{2}\int \left\{ (z_1^2+z_2^2)\sqrt{\psi(z_1^2+z_2^2)} \right\}^{-1} [A - (f'-B)(z_1^2+z_2^2)]d(z_1^2+z_2^2). \quad (4.6)$$

The integral term in equation (4.6) is an elliptic function and the term $\tan^{-1}(z_2/z_1)$ explains the spiraling of the solution curves on the surface of intersection of first integrals in equation (2.4). If $f' = 0$, then the equation (4.6) agrees with the missing first integral obtained by Srinivasan et al [4] in their study of integrable system of stratified Boussinesq equations without effects of rotation.

Note that the above first integral is singular in a neighborhood of the rest point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$. The values of A, B, C, D are given by

$$A = 0, \quad B = \frac{1+f'}{2}, \quad C = 0, \quad D = \frac{4-(1+f')^2}{4}$$

and a function ψ is given by

$$\psi(z_1^2+z_2^2) = (z_1^2+z_2^2)^2\left[\frac{4-(1+f')^2-(z_1^2+z_2^2)}{4}\right]$$

so (4.6) simplifies to

$$\tan^{-1}\left(\frac{z_2}{z_1}\right) + \frac{(1-f')}{2}\int \frac{d(z_1^2+z_2^2)}{(z_1^2+z_2^2)\sqrt{H-(z_1^2+z_2^2)}},$$

where $H = 4-(1+f')^2$. It implies that the first integral (4.6) is singular at $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$.

5 Conclusion

In this paper we have incorporated the effects of rotation in a stratified Boussinesq equations in the context of dynamics of an uniformly stratified fluid contained in a rectangular basin of dimension $L \times L \times H$. The ODE reductions provide a system of six coupled equations, which is completely integrable if a Rayleigh number $Ra = 0$. For

$0 < f' = \frac{f}{2r_h} < 1$, the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ of the system (2.6) is degenerate with two dimensional unstable, stable and center manifolds. For $f' = 1$ the invariant surface (2.7) degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ whereas for $f' > 1$ the invariant surface (2.7) is empty. The two dimensional compact invariant surface on which the solution curves develop is a torus, one of whose generating circle pinched to a critical point. We have obtained the analytical solutions of the system (2.6) lying on the invariant surface. Moreover these solutions are elementary functions, if a critical point lies on this invariant surface; whereas if there are no critical points lying on the invariant surface, the solutions are expressible in terms of elliptic functions.

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References

[1] Maas, L.R.M. Theory of Basin Scale Dynamics of a Stratified Rotating Fluid. *Surveys in Geophysics* **25** (2004) 249–279.

[2] Majda, A.J. *Introduction to PDEs and Waves for the Atmosphere and Ocean.* Courant Lecture Notes in Mathematics 9. American Mathematical Society, Providence, Rhode Island, 2003.

[3] Majda, A.J. and Shefter, M.G. Elementary stratified flows with instability at large Richardson number. *J. Fluid Mechanics* **376** (1998) 319–350.

[4] Srinivasan, G.K., Sharma, V.D. and Desale, B.S. An integrable system of ODE reductions of the stratified Boussinesq equations. *Computers and Mathematics with Applications* **53** (2007) 296–304.

[5] Morgan, G.W. On the Wind-driven Ocean Circulation. *Tellus* **8** (1956) 95–114.

[6] Maas, L.R.M. A Simple Model for the Three-dimensional, Thermally and Wind-driven Ocean Circulation. *Tellus* **46A** (1994) 671–680.

[7] Jacobi, C.G.J. *Vorlesungen über Dynamik*, Gesammelte werke, Druck and Verlag von Reimer. Berlin, 1842.

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Antagonistic Games with an Initial Phase †

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Abstract: We formalize and investigate an antagonistic game of two players (A and B), modeled by two independent marked Poisson processes forming casualties to the players. The game is observed by a third party point process. Unlike previous work on this topic, the initial observation moment is chosen not arbitrarily, but at some random moment of time following initial actions of the players. This caused an analytic complexity unresolved until recently. This, more realistic assumption, forms a new phase ("initial phase") of the game and it turns out to be a short game on its own. Following the initial phase, the main phase of the game lasts until one of the players' cumulative casualties exceed some specified threshold. We investigate the paths of the game in which player A loses the game.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time.*

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1 Introduction

We model an antagonistic stochastic game by two marked Poisson processes $\mathcal{A}$ and $\mathcal{B}$, each representing casualties incurred to players A and B. The mutual attacks are rendered in accordance with associated Poisson point processes and their marks are distributed arbitrary and position independent. The game is observed by a third party process $\mathcal{T}$. Consequently, the information on the game is available upon $\mathcal{T}$, thereby forming the embedding $\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}$. (The latter is a more general bivariate marked point process with marks being mutually and position dependent.) The game lasts until one of the players

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gets "exhausted" or "ruined". This happens whenever the total casualties to the players exceed some specified thresholds. The real exit from the game takes place with a delay in accordance with observations $\mathcal{T}$. This is one of the quite common scenarios of games, in which the co-authors [9] (and most recently, the first author [5–8, 12]) have been involved.

A realistic approach to the modeling was rendered through the embedded delayed process $\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}$ distorting the real time information. However, in the previous models the position of the first observation epoch was placed arbitrarily on the positive time axis with no regard to the start of the conflict. As the result, the initial observation point could have been placed before the game began. In a recent article by Dshalalow and Huang, this deficiency was overcome by placing the first observation at some random time after the conflict has emerged. This alone formed a separate initial phase of the conflict with a joint functional, which included the time of the beginning of the conflict and the amount of casualties to the players, all the way to the first observation. To merge this initial phase with the rest of the game, required some past information (non-Markovian), all resulting in two separate phases, which we thereby have come to identify. From the modeling point of view, the present game is simpler than that of [7], which in contrast, also included a second phase following the initial and first phases.

The first phase of this game ends with player A losing to player B (while in [7] it was not specified who of the two exactly loses, as their casualties were then limited).

Even though our model is not entirely characterized as a sequential game, it comes close enough to this literature [1, 3, 5–7, 11, 12, 14, 15, 18, 21, 24]. The tools we are using in this paper are mainly self-contained and developed methods of fluctuation theory that originated from applications to random walk processes. We hold on classic random walk fluctuation analysis, only in a generalized forms. We mention just a few pieces of literature where applications of the fluctuation theory takes place in the areas such as economics [17] and physics [20]. More on this can be found in [5–9]. Topically, the paper falls into the category of antagonistic stochastic games widely applied to economics [2, 16, 19, 24] and warfare [9, 12, 22, 23]. As in all previous work by the authors and the first author, the results are directly applicable to economics and warfare, in particular, in light of a high volatility of the global economy in the recent months. The latter can be interpreted as an "antagonism" between the economic actions (such as bailout of credit institutions) against the panic of the market.

Another area of applied mathematics that relates to our work includes *hybrid systems* [4, 13], in particular hybrid stochastic games [5]. For more references on this topic see [5].

The layoff of the paper is as follows. Section 2 deals with the formalism of the game. Section 3 takes on the initial phase. Section 4 continues with the game beyond the initial phase until player A is ruined. The merge between the two phases is the main contribution to this section.

2 A Formal Description of the Model

The results of Sections 2 and 3 are based on Dshalalow and Huang [7]. To make it self-contained we follow the initial phase of [7].

Let $(\Omega, \mathcal{F}(\Omega), \mathfrak{F}_t, P)$ be a filtered probability space and let $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}, \mathcal{F}_S \subseteq \mathcal{F}(\Omega)$ be independent sub-σ-algebras. We suppose that

$$\mathcal{A} := \sum_{j \geq 1} d_j \varepsilon_{r_j} \text{ and } \mathcal{B} := \sum_{k \geq 1} z_k \varepsilon_{w_k} \tag{2.1}$$

are $\mathcal{F}_{\mathcal{A}}$-measurable and $\mathcal{F}_{\mathcal{B}}$-measurable marked Poisson random measures (ε_a is a point mass at a) with respective intensities λ_A and λ_B and position independent marking. The random measures are specified by the transforms

$$Ee^{-u\mathcal{A}(\bullet)} = e^{\lambda_A|\bullet|[h_A(u)-1]},\ h_A(u) = Ee^{-ud_1},\ Re(u) \geq 0, \tag{2.2}$$

$$Ee^{-v\mathcal{B}(\bullet)} = e^{\lambda_B|\bullet|[h_B(v)-1]},\ h_B(v) = Ee^{-vz_1},\ Re(v) \geq 0, \tag{2.3}$$

where $|\bullet|$ is the Borel–Lebesgue measure and d_j and z_k are nonnegative r.v.'s representing the successive strikes of players B and A against each other, respectively, while r_j and w_k are the times of the strikes.

The game starts with hostile actions initiated by one of the players A or B at r_1 or w_1. The players can exchange with several more strikes before the first information is noticed by an observer at time t_0. We therefore assume that

$$t_0 \geq \max\{r_1, w_1\}. \tag{2.4}$$

The initial observation time t_0 will be formalized below. All forthcoming observations will be rendered in accordance with a point process

$$\begin{gathered} T_0 = \sum_{i\geq 0} \varepsilon_{t_i} = \varepsilon_{t_0} + S, \text{ with } S = \sum_{i\geq 1} \varepsilon_{t_i}, \\ 0 < t_0 < t_1 < \ldots < t_n < \ldots (t_n \to \infty, \text{ with } n \to \infty). \end{gathered} \tag{2.5}$$

We introduce the extension of $\mathcal{T}$:

$$\mathcal{T} := \varepsilon_{t_{-1}} + T_0, \text{ with } t_{-1} := \min\{r_1, w_1\}, \tag{2.6}$$

such that the tail $S = \sum_{i\geq 1} \varepsilon_{t_i}$ of T_0 is $\mathcal{F}_S$-measurable. The increments $\Delta_1 := t_1 - t_0,\ \Delta_2 := t_2 - t_1,\ \Delta_3 := t_3 - t_2,\ \ldots$ are all independent and identically distributed, and all belong to the equivalence class $[\Delta]$ of r.v.'s with the common Laplace-Stieltjes transform

$$\delta(\theta) := Ee^{-\theta\Delta}. \tag{2.7}$$

Now we define the initial observation as

$$t_0 = \max\{r_1, w_1\} + \Delta_0, \tag{2.8}$$

where $\Delta_0 \in [\Delta]$ and Δ_0 is independent from the rest of the Δ's. t_0 is included in T_0 of equation (2.5) and because it contains some of the $\mathcal{A}$ and $\mathcal{B}$, T_0 is not $\mathcal{F}_S$-measurable. However, T_0 is a delayed renewal process, while $\mathcal{T}$ is not.

We assign to t_{-1} the genuine start of the game at time $\min\{r_1, w_1\}$ of (2.6). That is,

$$t_{-1} = \min\{r_1, w_1\}. \tag{2.9}$$

Now, since t_{-1} and $t_0 - t_{-1}$ are dependent (through r_1 and w_1), the extended process $\mathcal{T}$ of (2.6) is not a renewal process, and not even a delayed renewal, as it was in [5, 6, 8, 9, 12].

It should be clear that t_0 depends upon r_1 and w_1 and thus on $\mathcal{A}$ and $\mathcal{B}$, which makes T_0 $\mathcal{A} \otimes \mathcal{B}$-measurable. Define the continuous time parameter process

$$(\alpha(t), \beta(t)) := \mathcal{A} \otimes \mathcal{B}([0, t]),\ t \geq 0, \tag{2.10}$$

to be adapted to the filtration $(\mathfrak{F}_t)_{t\geq 0}$. Also introduce its embedding over T_0 :

$$(\alpha_j, \beta_j) := (\alpha(t_j), \beta(t_j)) = \mathcal{A} \otimes \mathcal{B}([0, t_j]),\ j = 0, 1, \ldots, \tag{2.11}$$

which forms observations of $\mathcal{A} \otimes \mathcal{B}$ over T_0, with respective increments

$$(\xi_j, \eta_j) := \mathcal{A} \otimes \mathcal{B}((t_{j-1}, t_j]),\ j = 1, \ldots. \tag{2.12}$$

In addition, let

$$(\xi_0, \eta_0) := \mathcal{A} \otimes \mathcal{B}((\max\{r_1, w_1\}, t_0]) \tag{2.13}$$

to be used later on.

Introduce the embedded bivariate marked random measures

$$\mathcal{A}_{T_0} \otimes \mathcal{B}_{T_0} := (\alpha_0, \beta_0)\varepsilon_{t_0} + \sum_{j\geq 1} (\xi_j, \eta_j)\varepsilon_{t_j}, \tag{2.14}$$

where the marginal marked point processes

$$\mathcal{A}_{T_0} = \alpha_0 \varepsilon_{t_0} + \sum_{i\geq 1} \xi_i \varepsilon_{t_i} \text{ and } \mathcal{B}_{T_0} = \beta_0 \varepsilon_{t_0} + \sum_{i\geq 1} \eta_i \varepsilon_{t_i} \tag{2.15}$$

are with position dependent marking and with ξ_j and η_j being dependent. For the forthcoming sections we introduce the Laplace-Stieltjes transform

$$g(u, v, \theta) := Ee^{-u\xi_j - v\eta_j - \theta\Delta_j},\ Re(u) \geq 0,\ Re(v) \geq 0,\ Re(\theta) \geq 0,\ j \geq 1, \tag{2.16}$$

which will be evaluated as the follows:

$$\begin{aligned} E\left[e^{-u\xi_j - v\eta_j - \theta\Delta_j}\right] &= E\left[e^{-\theta\Delta_j} E\left[e^{-u\xi_j - v\eta_j} \middle| \Delta_j\right]\right] \\ &= E\left[e^{-\theta\Delta_j} E\left[e^{-u\mathcal{A}((t_{j-1}, t_j])} \middle| \Delta_j\right] E\left[e^{-v\mathcal{B}((t_{j-1}, t_j])} \middle| \Delta_j\right]\right] \\ &= E\left[e^{-\theta\Delta_j} \cdot e^{\lambda_A \Delta_j (h_A(u) - 1)} \cdot e^{\lambda_B \Delta_j (h_B(v) - 1)}\right] \\ &= E\left[e^{-\{\theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v))\}\Delta_j}\right] \\ &= \delta(\theta^*),\ j = 1, 2, \ldots, \end{aligned} \tag{2.17}$$

with

$$\theta^* := \theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v)), \tag{2.18}$$

and δ defined in (2.7).

3 The Initial Phase of the Game

The entire *game* will include the recording of the conflict between players A and B known to an observer upon process $\mathcal{T}$ (informally, $\{t_{-1}, t_0, t_1, \ldots\}$) from its inception upon t_{-1} followed by the initial observation at time t_0. $\mathcal{T}$ is defined below. The actual start of the game at t_{-1} is unknown to the observer, as this moment takes place prior to t_0. From the construction of the extended game, the point process $\mathcal{T}$ is obviously "doubly delayed" (in light of its attachment t_{-1}). The information on t_{-1} will be used in section 4 during the merging process.

The initial phase of the game is specified as follows. Define the respective damages to the players at t_{-1} as

$$(\xi_{-1},\eta_{-1}) := (\alpha_{-1},\beta_{-1}) := (\alpha(t_{-1}),\beta(t_{-1})) = (d_1\mathbf{1}_{\{r_1\le w_1\}}, z_1\mathbf{1}_{\{r_1\ge w_1\}}). \tag{3.1}$$

Therefore, the embedded process $\sum_{k\ge -1}\varepsilon_{t_k}(\alpha_k,\beta_k)$ satisfies the extended initial conditions

$$\mathcal{A}_{t_{-1}}\otimes\mathcal{B}_{t_{-1}} = (\alpha_{-1},\beta_{-1}) = (d_1,0),\ \text{on trace } \sigma\text{-algebra } \mathcal{F}(\Omega)\cap\{r_1<w_1\}, \tag{3.2}$$

$$\mathcal{A}_{t_{-1}}\otimes\mathcal{B}_{t_{-1}} = (\alpha_{-1},\beta_{-1}) = (0,z_1),\ \text{on } \mathcal{F}(\Omega)\cap\{r_1>w_1\}, \tag{3.3}$$

$$\mathcal{A}_{t_{-1}}\otimes\mathcal{B}_{t_{-1}} = (\alpha_{-1},\beta_{-1}) = (d_1,z_1),\ \text{on } \mathcal{F}(\Omega)\cap\{r_1=w_1\}. \tag{3.4}$$

The extended version of the game is defined as the bivariate marked point process

$$\mathcal{A}_{\mathcal{T}}\otimes\mathcal{B}_{\mathcal{T}} := (\xi_{-1},\eta_{-1})\varepsilon_{t_{-1}} + (\alpha_0-\xi_{-1},\beta_0-\eta_{-1})\varepsilon_{t_0} + \sum_{j\ge 1}(\xi_j,\eta_j)\varepsilon_{t_j} \tag{3.5}$$

(embedded over $\mathcal{T}$).

As we will see it in the next section, the game will require knowledge of $\mathcal{A}_{\mathcal{T}}\otimes\mathcal{B}_{\mathcal{T}}$ at t_{-1} and t_0. Consequently, we begin to work on the functional

$$\phi_0 := \phi_0(a_0,b_0,\vartheta_0,u_0,v_0,\theta_0) = E[e^{-a_0\alpha_{-1}-u_0\alpha_0-b_0\beta_{-1}-v_0\beta_0-\vartheta_0 t_{-1}-\theta_0 t_0}] \tag{3.6}$$

that describes what we call, the *initial phase* of the game. The following theorem is due to Dshalalow and Huang [7].

Theorem 3.1 *The functional ϕ_0 of the initial phase of the game satisfies the following formula:*

$$\phi_0 = \frac{\lambda_A\lambda_B\delta(\theta_0^*)}{\vartheta_0+\theta_0+\lambda_A+\lambda_B}\left(\frac{1}{\theta_A+\lambda_B}h_A(a_0+u_0)h_B(v_0) + \frac{1}{\theta_B+\lambda_A}h_A(u_0)h_B(b_0+v_0)\right), \tag{3.7}$$

where

$$\theta_0^* := \theta_0 + \lambda_A(1-h_A(u_0)) + \lambda_B(1-h_B(v_0)), \tag{3.8}$$

$$\theta_A := \theta_0 - \lambda_A(h_A(u_0)-1), \tag{3.9}$$

$$\theta_B := \theta_0 - \lambda_B(h_B(v_0)-1), \tag{3.10}$$

$$\delta(\theta) := E\big[e^{-\theta\Delta_0}\big],\ \Delta_0\in[\Delta]. \tag{3.11}$$

4 The Main Phase of the Game

After passing the initial phase, the game continues with its status registered at epochs $\mathcal{T}$ and it ends when at least one of the players sustains damages in excess of thresholds M or N. To further formalize the game past t_0 we introduce the following random *exit indices*

$$\mu := \inf\{j\ge 0 : \alpha_j = \alpha_0+\xi_1+\ldots+\xi_j > M\}, \tag{4.1}$$

$$\nu := \inf\{k\ge 0 : \beta_k = \beta_0+\eta_1+\ldots+\eta_k > N\}. \tag{4.2}$$

Related on μ and ν are the following r.v.'s:

t_μ is the nearest observation epoch when player A's damages exceed threshold M,

t_ν is the first observation of $\mathcal{T}$ when player B's damages exceed threshold N.

Apparently, α_μ and β_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. Accordingly, we will target the following functional

$$\phi_\mu := \phi_\mu(a, b, \vartheta, u, v, \theta) = E[e^{-a\alpha_{\mu-1} - u\alpha_\mu - b\beta_{\mu-1} - v\beta_\mu - \vartheta t_{\mu-1} - \theta t_\mu} \mathbf{1}_{\{\mu < \nu\}}]. \tag{4.3}$$

To calculate a tractable form of ϕ_μ we will use the bivariate Laplace-Carson transform

$$\mathcal{LC}_{pq}(\cdot)(x, y) := xy \int_{p=0}^{\infty} \int_{q=0}^{\infty} e^{-xp-yq}(\cdot) d(p, q),\ Re(x) > 0,\ Re(y) > 0, \tag{4.4}$$

with the inverse

$$\mathcal{LC}_{xy}^{-1}(\cdot)(p, q) = \mathcal{L}_{xy}^{-1}(\cdot\, \frac{1}{xy}), \tag{4.5}$$

where $\mathcal{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 4.1 *The functional ϕ_μ of the game on trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$ satisfies the following formula:*

$$\phi_\mu = \mathcal{LC}_{xy}^{-1} \left((\Phi_0^1 - \Phi_0) + \frac{\Phi_0^*}{1-g} (G^1 - G) \right) (M, N), \tag{4.6}$$

where

$$G := g(u + x, v + y, \theta), \tag{4.7}$$

$$G^1 := g(u, v + y, \theta), \tag{4.8}$$

$$\Phi_0^* := \phi_0(0, 0, 0, a + u + x, b + v + y, \vartheta + \theta), \tag{4.9}$$

$$\Phi_0 := \phi_0(a, b, \vartheta, u + x, v + y, \theta), \tag{4.10}$$

$$\Phi_0^1 := \phi_0(a + x, b, \vartheta, u, v + y, \theta), \tag{4.11}$$

with g and ϕ_0 of (2.16) and (3.7), respectively.

Proof: First we modify (4.1) and (4.2) for the random exit indices μ and ν which depend on parameters M and N, now to depend on p and q (being arbitrary nonnegative real numbers), respectively, and working with them as parametric families of r.v.'s:

$$\mu(p) := \inf\{j \geq 0 : \alpha_j = \alpha_0 + \xi_1 + \ldots + \xi_j > p\},\ p \geq 0, \tag{4.12}$$

$$\nu(q) := \inf\{k \geq 0 : \beta_k = \beta_0 + \eta_1 + \ldots + \eta_k > q\},\ q \geq 0. \tag{4.13}$$

The functional ϕ_μ will now change to

$$\Phi_{pq} = E[e^{-a\alpha_{\mu(p)-1} - u\alpha_{\mu(p)} - b\beta_{\mu(p)-1} - v\beta_{\mu(p)} - \vartheta t_{\mu(p)-1} - \theta t_{\mu(p)}} \mathbf{1}_{\{\mu(p) < \nu(q)\}}]. \tag{4.14}$$

This will follow the paths of the game on the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu(p) < \nu(q)\}$ and yield:

$$\Phi_{pq} = \sum_{j \geq 0} \sum_{k > j} E[e^{-a\alpha_{j-1} - u\alpha_j - b\beta_{j-1} - v\beta_j - \vartheta t_{j-1} - \theta t_j} \mathbf{1}_{\{\mu(p) = j, \nu(q) = k\}}]. \tag{4.15}$$

By Fubini's theorem, and that

$$\mathcal{LC}_{pq}\left(\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right)(x,y)=(e^{-x\alpha_{j-1}}-e^{-x\alpha_j})(e^{-y\beta_{k-1}}-e^{-y\beta_k}),$$

(which can be readily shown) we have

$$\begin{aligned}\mathcal{LC}_{pq}(\Phi_{pq})(x,y)=&\sum_{j\geq 0}\sum_{k>j}E[e^{-a\alpha_{j-1}-u\alpha_j-b\beta_{j-1}-v\beta_j-\vartheta t_{j-1}-\theta t_j}\\&\times(e^{-x\alpha_{j-1}}-e^{-x\alpha_j})(e^{-y\beta_{k-1}}-e^{-y\beta_k})].\end{aligned}\tag{4.16}$$

We distinguish two cases.

***(i)* Case $j=0$**. This case will include the entire information on the initial phase observed at t_0 and prior to t_0, including t_{-1}. In a few lines below, we are going to implement the result of Theorem 3.1 and utilize all necessary versions of the functional ϕ_0 :

$$\begin{aligned}&\sum_{k>0}E[e^{-a\alpha_{-1}-u\alpha_0-b\beta_{-1}-v\beta_0-\vartheta t_{-1}-\theta t_0}(e^{-x\alpha_{-1}}-e^{-x\alpha_0})(e^{-y\beta_{k-1}}-e^{-y\beta_k})]\\&=\sum_{k>0}E[e^{-a\alpha_{-1}-u\alpha_0-b\beta_{-1}-v\beta_0-\vartheta t_{-1}-\theta t_0}(e^{-x\alpha_{-1}}-e^{-x\alpha_0})\\&\quad\times e^{-y\beta_0}e^{-y(\eta_1+ldots+\eta_{k-1})}(1-e^{-y\eta_k})]\\&=\Big\{E[e^{-(a+x)\alpha_{-1}-u\alpha_0-b\beta_{-1}-(v+y)\beta_0-\vartheta t_{-1}-\theta t_0}]\\&\quad-E[e^{-a\alpha_{-1}-(u+x)\alpha_0-b\beta_{-1}-(v+y)\beta_0-\vartheta t_{-1}-\theta t_0}]\Big\}\sum_{k>0}E[e^{-y(\eta_1+\ldots+\eta_{k-1})}(1-e^{-y\eta_k})]\\&=\Big\{\phi_0(a+x,b,\vartheta,u,v+y,\theta)-\phi_0(a,b,\vartheta,u+x,v+y,\theta)\Big\}\\&\quad\times\sum_{k>0}[g(0,y,0)]^{k-1}\big(1-g(0,y,0)\big)\\&=\Phi_0^1-\Phi_0,\end{aligned}\tag{4.17}$$

where the summation over $k>0$ converges to 1 as per Lemma 1 of Dshalalow and Huang [5]: the associated convergence of $\sum_{k>0}[g(0,y,0)]^{k-1}$ is guaranteed provided that $Re(y)>0$. The last line in (4.17) is due to notation (4.9-4.11).

***(ii)* Case $j>0$**. This case also contains parts of functional ϕ_0 in the information related to the reference point t_0.
Transformation (4.16) for this case is

$$\begin{aligned}&\sum_{j>0}\sum_{k>j}E[e^{-a\alpha_{j-1}-u\alpha_j-b\beta_{j-1}-v\beta_j-\vartheta t_{j-1}-\theta t_j}(e^{-x\alpha_{j-1}}-e^{-x\alpha_j})(e^{-y\beta_{k-1}}-e^{-y\beta_k})]\\&\quad=\sum_{j>0}\sum_{k>j}\Big\{E[e^{-(a+u+x)\alpha_{j-1}-(b+v+y)\beta_{j-1}-(\vartheta+\theta)t_{j-1}}]\\&\quad\quad\times E[e^{-u\xi_j}(1-e^{-x\xi_j})e^{-(v+y)\eta_j-\theta\Delta_j}]E[e^{-y(\eta_{j+1}+\ldots+\eta_{k-1})}(1-e^{-y\eta_k})]\Big\}\\&\quad=\sum_{j>0}\Big\{E[e^{-(a+u+x)\alpha_0-(b+v+y)\beta_0-(\vartheta+\theta)t_0}]\\&\quad\quad\times E[e^{-(a+u+x)(\xi_1+\ldots+\xi_{j-1})-(b+v+y)(\eta_1+\ldots+\eta_{j-1})-(\vartheta+\theta)\left(\Delta_1+\ldots+\Delta_{j-1}\right)}]\\&\quad\quad\times E[e^{-u\xi_j}(1-e^{-x\xi_j})e^{-(v+y)\eta_j-\theta\Delta_j}]\sum_{k>j}E[e^{-y(\eta_{j+1}+\ldots+\eta_{k-1})}(1-e^{-y\eta_k})]\Big\},\end{aligned}\tag{4.18}$$

where the third factor can be written as

$$E[e^{-u\xi_j-(v+y)\eta_j-\theta\Delta_j}]-E[e^{-(u+x)\xi_j-(v+y)\eta_j-\theta\Delta_j}]=G^1-G$$

(as per notation (4.7-4.8)) and the summation over $k>j$ converges to 1, for $Re(y)>0$, as per Lemma 1 of [5]. Then, after some algebra in (4.18) and the use of notation (4.7-4.8) and (4.18), we arrive at

$$\begin{aligned}&\phi_0(0,0,0,a+u+x,b+v+y,\vartheta+\theta)\cdot\sum_{j>0}g^{j-1}\cdot(G^1-G)\\&=\varPhi_0^*\cdot\sum_{j>0}g^{j-1}\cdot(G^1-G)=\frac{\varPhi_0^*}{1-g}(G^1-G),\end{aligned}\tag{4.19}$$

with the convergence of $\sum_{j>0}g^{j-1}$ under the condition that the parameters of g satisfy

$$Re(a+u+x)>0,\ Re(b+v+y)>0,\ Re(\vartheta+\theta)>0,\tag{4.20}$$

with any two of the three strict inequalities relaxed with $\geq$.

With the cases $j=0$ and $j>0$ combined together, we will arrive at

$$\mathcal{LC}_{pq}(\varPhi_{pq})(x,y)=(\varPhi_0^1-\varPhi_0)+\frac{\varPhi_0^*}{1-g}(G^1-G).\tag{4.21}$$

□

Remark 4.1 For the particular case

$$\varphi_\mu=\varphi_\mu(u,v,\vartheta)=E[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}]\tag{4.22}$$

of the functional ϕ_μ we get from (4.21)

$$\mathcal{LC}_{pq}(\varphi_{pq})(x,y)=\varPhi_0^1-\varPhi_0\frac{1-G^1}{1-G},\tag{4.23}$$

where φ_{pq} is the corresponding marginal reduction of $\varPhi_{pq}$ while the rest of the marginal functionals $G,G^1,\varPhi_0$, and $\varPhi_0^1$ will shrink but for convenience carry the same characters:

$$G=g(u+x,v+y,\theta),\tag{4.24}$$

$$G^1=g(u,v+y,\theta),\tag{4.25}$$

$$\varPhi_0^*=\varPhi_0=\phi_0(0,0,0,u+x,v+y,\theta),\tag{4.26}$$

$$\varPhi_0^1=\phi_0(x,0,0,u,v+y,\theta).\tag{4.27}$$

Explicitly,

$$\begin{aligned}\mathcal{LC}_{pq}(\varphi_{pq})(x,y)&=\phi_0(x,0,0,u,v+y,\theta)\\&\quad-\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)},\end{aligned}\tag{4.28}$$

where from (3.7-3.10) and (2.18), the marginal versions of ϕ_0 needed for (4.28) are

$$\begin{aligned}&\phi_0(x,0,0,u,v,\theta)=E[e^{-x\alpha_{-1}-u\alpha_0-v\beta_0-\theta t_0}]\\&=\frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\left(\frac{1}{\theta_A+\lambda_B}h_A(x+u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\right),\end{aligned}\tag{4.29}$$

$$\begin{aligned}&\phi_0(0,0,0,u,v,\theta)=E[e^{-u\alpha_0-v\beta_0-\theta t_0}]\\&=\frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\left(\frac{1}{\theta_A+\lambda_B}h_A(u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\right),\end{aligned}\tag{4.30}$$

and

$$\theta_0^* := \theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v)), \tag{4.31}$$

$$\theta_A := \theta - \lambda_A(h_A(u) - 1), \tag{4.32}$$

$$\theta_B := \theta - \lambda_B(h_B(v) - 1). \tag{4.33}$$

□

Concluding Remarks. In this paper, we study fully antagonistic stochastic games of two players (A and B) (initiated in [5-7]), modeled by two independent marked Poisson processes recording times and quantities of casualties to the players. The game is observed by a third party renewal point process upon which the information is gathered (and a decision about upcoming steps can be made or modified). Unlike previous work in [5, 6, 8, 9], the initial observation moment is not arbitrarily chosen, but it is placed at random following some initial actions of the players. This caused an analytic complexity which was unresolved until recently. Due to this more realistic assumption a new phase in the game emerged, which we name the "initial phase". This initial phase turned out to be a short game on its own. Following the initial phase, the main phase of the game lasts until one of the players is ruined. This takes place when the cumulative casualties of a losing player exceed some specified threshold. We investigate the paths of the game in which player A loses the game. The general formulas are obtained in closed forms. In [10] we will render calculation for a variety of special cases.

References

[1] Altman, E. and Gaitsgory, V. A hybrid (differential-stochastic) zero-sum game with fast stochastic part. In: *New Trends in Dynamic Games* (ed. by Olsder, G.J.), Birkhäuser, 1995, 46–59.

[2] Bagwell, K. Commitment and Observability in Games. *Games and Economic Behavior* **8** (2) (1995) 271–280.

[3] Brandts, J. and Solàc, C. Reference Points and Negative Reciprocity in Simple Sequential Games. *Games and Economic Behavior* **36** (2) (2001) 138–157.

[4] Collins, P. Chaotic dynamics in hybrid systems. *Nonlinear Dynamics and Systems Theory* **8** (2) (2008) 169–194.

[5] Dshalalow, J.H. and Huang, W. On noncooperative hybrid stochastic games. *Nonliear Analysis*: Special Issue Section: Analysis and Design of Hybrid Systems **2** (3) (2008) 803–811.

[6] Dshalalow, J.H. and Huang, W. A stochastic game with a two-phase conflict. Jubilee Volume: *Legacy of the Legend, Professor V. Lakshmikantham.* Cambridge Scientific Publishers, Chapter 18, (2009) 201–209.

[7] Dshalalow, J.H. and Huang, W. Sequential antagonistic games with initial phase (jointly with Weijun Huang). To appear in *Functional Equations And Difference Inequalities and Ulam Stability Notions*, Dedicated to Stanislaw Marcin ULAM, on the occasion of his 100-th birthday anniversary. In Press.

[8] Dshalalow, J.H. and Ke, H-J. Layers of noncooperative games. *Nonliear Analysis, Series A.* In press.

[9] Dshalalow, J.H. and Treerattrakoon, A. Set-theoretic inequalities in stochastic noncooperative games with coalition. *Journal of Inequalities and Applications*. Art. ID 713642, 14 pp. (2008).

[10] Dshalalow, J.H. and Treerattrakoon, A. Operational calculus in noncooperative stochastic games, *Nonlinear Dynamics and Systems Theory* (accepted for publication).

[11] Exman, I. Solving sequential games with Boltzmann-learned tactics. In: *Lecture Notes In: Computer Science*, **496**, 216–220. Proceedings of the 1st Workshop on Parallel Problem Solving from Nature, Springer-Verlag London, UK, 1990.

[12] Huang W. and Dshalalow, J.H. Tandem Antagonistic Games, *Nonliear Analysis, Series A*, in press.

[13] Khusainov, D., Langerak, R., Kuzmych, O. Estimations of solutions convergence of hybrid systems consisting of linear equations with delay. *Nonlinear Dynamics and Systems Theory* **7**(2) (2007) 169–186.

[14] Kobayashi, N. Equivalence between quantum simultaneous games and quantum sequential games. Submitted to *Quantum Physics*.

[15] Kohler, D.A. and Chandrasekaran, R. A Class of Sequential Games. *Operations Research*, INFORMS **19**(2) (1971) 270–277.

[16] Konstantinov, R.V. and Polovinkin, E.S. Mathematical simulation of a dynamic game in the enterprise competition problem. *Cybernetics and Systems Analysis* **40** (5) (2004) 720–725.

[17] Kyprianou, A.E. and Pistorius, M.R. Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Prob.* **13** (3) (2003) 1077–1098.

[18] Radzik, T. and Szajowski, K. Sequential Games with Random Priority. *Sequential Analysis* **9**(4) (1990) 361–377.

[19] Ragupathy, R. and Das, T. A stochastic game approach for modeling wholesale energy bidding in deregulated power markets. *IEEE Tras. on Power Syst.* **19** (2) (2004) 849–856.

[20] Redner, S. *A Guide to First-Passage Processes*. Cambridge University Press, Cambridge, 2001.

[21] Siegrist, K. and Steele, J. Sequential Games. *J. Appl. Probab.* **38**(4) (2001) 1006–1017.

[22] Shashikin, V.N. Antagonistic game with interval payoff functions. *Cybernetics and Systems Analysis* **40**(4) (2004) 556–564.

[23] Shima, T. Capture Conditions in a Pursuit-Evasion Game between Players with Biproper Dynamics. *Journal of Optimization Theory and Applications* **126**(3) (2005) 503–528.

[24] Wen, Q. A Folk Theorem for Repeated Sequential Games. *The Review of Economic Studies* **69**(2) (2002) 493–512.

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Robust Controller Design for Active Flutter Suppression of a Two-dimensional Airfoil

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Abstract: This paper investigates the problem of active flutter suppression for a two-dimensional three degrees of freedom (3DOF) airfoil. With the influence of unsteady aerodynamic forces and parametric uncertainties, the output suboptimal control law design for a 3DOF airfoil control system is transformed into a constrained optimization problem. Then, the flutter robust suppression control law could be expediently obtained by linear matrix inequalities (LMIs), which realizes active flutter suppression by increasing the flutter critical speed. Simulation results show that the flutter phenomenon could be well suppressed in spite of the uncertainty of damping coefficients.

Keywords: *active flutter suppression; suboptimal control; linear matrix inequalities.*

Mathematics Subject Classification (2000): 93C95, 93B12, 93D21.

1 Introduction

Recently, techniques of active aeroelastic wing [8], thrust vector control [1, 4] and flying-wing layout [2, 4] have became the hottest issues in aeronautic area. At the same time, high-altitude long-endurance aircrafts are taken into account by more and more countries [7]. The general features of high-altitude long-endurance aircraft are high aspect ratio, light structural weight, and well flexibility. Therefore, the future aircrafts are in the nature of more flexibility. With the increase of flexibility, the flutter phenomenon is more and more prominent. Flutter is a vibration caused by airstream energy being absorbed by the lifting surface, which is more likely to occur in the wings, ailerons and other flexible parts. Furthermore, this aeroelastic phenomenon increasing with the flight velocities can cause the wing fatigue to be increased. If the flight velocity is above the critical flutter

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speed and the flutter phenomenon is not suppressed, the structure of aircrafts may be destroyed. To reduce or suppress this phenomenon is very important in the aeronautic industry.

Over the past several decades, this severe problem has been studied using many different techniques. Traditional technique is the passive flutter suppression method, which adds structural weight to change the aircraft stiffness, and some components have to be moved to keep balance. So this technique deteriorates some flight performances, and is not always feasible. Later the active flutter suppression method appears to suppresses flutter phenomenon without adding structural weight and redesign. The idea of this method is to introduce a certain deformation based on the structure flexibility, which can suppress the flutter actively. Therefore, there are above two main techniques that we can use.

With the development of active control technology in the aeronautic area, flexibility at the support of active control technology exhibits more potential. Nowadays, more and more active control techniques are used to suppress the flutter phenomenon. Shana D. Olds uses Linear Quadratic Regulation theory to design a state feedback controller for an aeroelastic system [6]. Good performances are illustrated, but the results are not feasible in practice because all states are assumed to be measurable. Samuel da Silva and Vicente L. Júnior used the LMI technique to solve the active flutter suppression problem with robustness to polytopic parametric uncertainties [9]. In their paper, they designed a state feedback control law based on full-order state observer. The dimension of state observer is equal to that of controlled plant. Therefore, there are twenty-order states in their closed-loop aeroelastic system. Though the state feed back control law and observer can be designed respectively according to separate principle, the full-order observer is difficult to carry out in actual engineering application because of high order. In the view of engineering practice, convenient and effective design process play an important role in actual aeroelastic system, which motivates us to carry out the present study.

In this paper, for the sake of analysis, the model is simplified on the assumption that the stiffness of control surface is very large, which is different from the aeroelastic model of aforementioned papers [6, 9]. We adopt the output as the feedback information to design a robust controller for active flutter suppression of a two-dimensional 3DOF airfoil aeroelastic system. Considering the system with polytopic parametric uncertainties and the influence of unsteady aerodynamic forces, we transform the output suboptimal control law design for a 3DOF airfoil control system into a constrained optimization problem, then obtain the output feedback control law by LMI technique and the minimum norm method. Despite the uncertainties of two-dimensional 3DOF airfoil aeroelastic system, this proposed approach makes it design easier for engineering application. In addition, it considers both response performance and control performance. This approach can conveniently and effectively realize robust active flutter suppression. The simulation results show that the flutter phenomenon could be well suppressed in spite of the uncertainty of damping coefficients.

2 Aeroelastic System Formulation

The schematic diagram of a 3DOF airfoil aeroelastic system with control surface is shown in Figure 2.1. Here, in order to develop the motion equations a coordinate system is introduced, which originates at the midpoint of airfoil chord. The x axis lies along the chord in the horizontal direction. The z axis shown in Figure 2.1 is perpendicular with x

direction. The quantity b is half chord. And two springs, one of which is line spring, the other is torsional spring, are put on the point E of airfoil elastic axis which is located at a distance of ab from the mid-chord. The flap hinge is located at a distance of cb from the mid-chord. Then, the three degrees of freedom are respectively the plunge h which is measured at the elastic axis E and positive in the downward direction, the pitching angle α which rotates on the elastic axis E and positive nose-up, the deflective angle of control surface β which represents the angular deflection of the flap about the flap hinge and positive for the flap trailing edge down.

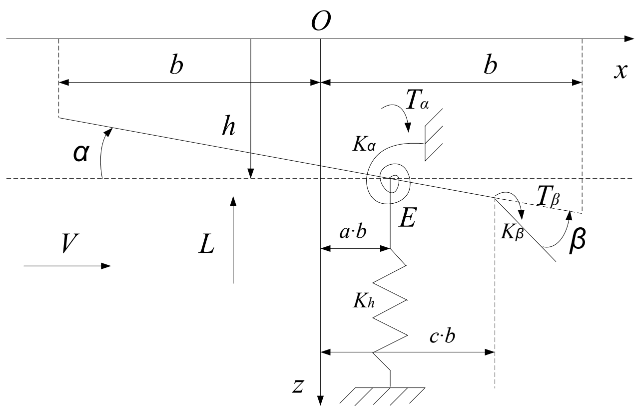

Figure 2.1: Configuration of a two-dimensional 3DOF airfoil.

2.1 Unsteady aerodynamic force calculation

The precise calculation of unsteady aerodynamic forces is an important step in two-dimensional airfoil flutter analysis. According to the Theodorsen theory, the aerodynamic lift L, pitching moment T_α, and control surface moment T_β of a unit wingspan length are respectively:

$$
\begin{aligned}
L &= \pi\rho b^2\left(\ddot{h} + V\dot{\alpha} - ba\ddot{\alpha} - \frac{V}{\pi}T_1\dot{\beta} - \frac{b}{\pi}T_4\ddot{\beta}\right) + 2\pi\rho V b T_0 C(k), \\
T_\alpha &= \pi\rho b^2\left[ba\ddot{h} - Vb\left(\frac{1}{2} - a\right)\dot{\alpha} - b^2\left(\frac{1}{8} + a^2\right)\ddot{\alpha} - \frac{V^2}{\pi}(T_4 + T_{10})\beta + \right. \\
&\quad \left. \frac{Vb}{\pi}\left(-T_1 + T_8 + (c-a)T_4 - \frac{1}{2}T_{11}\right)\dot{\beta} + \frac{b^2}{\pi}\left(T_7 + (c-a)T_1\ddot{\beta}\right)\right] \\
&\quad + 2\pi\rho V b^2\left(\bar{a} + \frac{1}{2}\right)T_0 C(k), \\
T_\beta &= \pi\rho b^2\left[\frac{b}{\pi}T_1\ddot{h} - \frac{Vb}{\pi}\left(2T_9 + T_1 - \left(a - \frac{1}{2}\right)T_4\right)\dot{\alpha} - \frac{2b^2}{\pi}T_{13}\ddot{\alpha}\right. \\
&\quad \left. - \left(\frac{V}{\pi}\right)^2(T_5 - T_4T_{10})\beta + \frac{Vb}{2\pi^2}T_4T_{11}\dot{\beta} + \left(\frac{b}{\pi}\right)^2 T_3\ddot{\beta}\right] - \rho V b^2 T_{12} T_0 C(k).
\end{aligned}
$$

where k is the air reduced frequency which is dimensionless, ρ is the air density, and V is the flow velocity. Definitions of other coefficients could be found in [10].

2.2 Aeroelastic System Modeling

In the dynamic schematic diagram Figure 2.1, any point displacement of the airfoil can be expressed as

$$z = h + (x - ab)\,\alpha + (x - cb)\,\beta U_{\text{step}}\,(x - cb)\,,$$

where $U_{\text{step}}\,(x - cb)$ is an unit step function.

Then, the system kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\int_{-b}^{b} \dot{z}^2 \bar{m} dx \\ &= \frac{1}{2} m\dot{h}^2 + \frac{1}{2} I_\alpha \dot{\alpha}^2 + \frac{1}{2} I_\beta \dot{\beta}^2 + S_\alpha \dot{h}\dot{\alpha} + S_\beta \dot{h}\dot{\beta} + \left[(c-a)\,bS_\beta + I_\beta\right]\dot{\alpha}\dot{\beta}, \end{aligned}$$

and the potential energy is

$$U = \frac{1}{2} k_h h^2 + \frac{1}{2} k_\alpha \alpha^2 + \frac{1}{2} k_\beta \beta^2,$$

where

$$\begin{aligned} m &= \int_{-b}^{b} \bar{m} dx, \\ S_\alpha &= \int_{-b}^{b} (x - ab)\,\bar{m} dx = m x_a, \\ I_\alpha &= \int_{-b}^{b} (x - ab)^2\,\bar{m} dx = m r_a^2, \\ S_\beta &= \int_{cb}^{b} (x - cb)\,\bar{m} dx = m x_\beta, \\ I_\beta &= \int_{cb}^{b} (x - cb)^2\,\bar{m} dx = m r_\beta^2, \end{aligned}$$

k_h, k_α, k_β are stiffness coefficients, $\bar{m}$ is airfoil mass of unit area. Definitions of other coefficients could be found in [11].

According to Lagrange's equation and principle of virtual work, the equation of motion for this two-dimensional 3DOF airfoil aeroelastic system is

$$\begin{bmatrix} m & m x_\alpha & m x_\beta \\ m x_\alpha & m r_\alpha^2 & m r_\beta^2 + m x_\beta\,(cb - ab) \\ m x_\beta & m r_\beta^2 + m x_\beta\,(cb - ab) & m r_\beta^2 \end{bmatrix} \begin{bmatrix} \ddot{h} \\ \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix}$$

$$+ \begin{bmatrix} d_h & 0 & 0 \\ 0 & d_\alpha & 0 \\ 0 & 0 & d_\beta \end{bmatrix} \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} k_h & 0 & 0 \\ 0 & k_\alpha & 0 \\ 0 & 0 & k_\beta \end{bmatrix} \begin{bmatrix} h \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -L \\ T_\alpha \\ T_\beta \end{bmatrix}$$

On the assumption of perfect rigidity, i.e. the stiffness of control surface is very large, after introducing some damping coefficients, and the unsteady aerodynamic forces, the open-loop motion model of a 3DOF airfoil can be represented as [11]

$$\left(s^2 \begin{bmatrix} M_s & M_c \end{bmatrix} + s \begin{bmatrix} D_s & 0 \end{bmatrix} + \begin{bmatrix} K_s & 0 \end{bmatrix}\right) \begin{bmatrix} q_s(s) \\ \beta(s) \end{bmatrix} + q_d \begin{bmatrix} \tilde{A}_s(s) & \tilde{A}_c(s) \end{bmatrix} \begin{bmatrix} q_s(s) \\ \beta(s) \end{bmatrix} = 0 \tag{2.1}$$

where $q_s = \begin{bmatrix} h & \alpha \end{bmatrix}^T$, M_s, D_s, K_s are respectively the mass matrix, structural damping matrix, and structural stiffness matrix of plunge and pitching modes, M_c is the coupled mass matrix among the control surface and structural modes, $\tilde{A}_s(s)$ and $\tilde{A}_c(s)$ are the matrices of aerodynamic forces, $q_d = \frac{1}{2}\rho V^2$ is the dynamic pressure of a gas flow.

For the sake of convenience, Eq. (2.1) could be rearranged into the following form:

$$\left(M_s s^2 + D_s s + K_s\right) q_s(s) + M_c s^2 \beta(s) + q_d \tilde{A}_s(s) q_s(s) + q_d \tilde{A}_c(s) \beta(s) = 0.$$

In order to obtain a state space representation, a rational function approximation, that is, the minimum states method, is adopted to fix the unsteady aerodynamic matrices in frequency domain to the matrices in Laplace domain. Therefore we have

$$\tilde{A}_s(s) = A_{s0} + \frac{b}{V} A_{s1} s + \frac{b^2}{V^2} A_{s2} s^2 + E\left(Is - \frac{V}{b} R\right)^{-1} F_s s, \tag{2.2}$$

$$\tilde{A}_c(s) = A_{c0} + \frac{b}{V} A_{c1} s + \frac{b^2}{V^2} A_{c2} s^2 + E\left(Is - \frac{V}{b} R\right)^{-1} F_c s. \tag{2.3}$$

And aerodynamic augmented states

$$x_a(s) = \left(Is - \frac{V}{b} R\right)^{-1} \left(F_s q_s(s) + F_c \beta(s)\right) s \tag{2.4}$$

are introduced.

According to formula (2.2), (2.3) and (2.4), Eq. (2.1) can be rewritten into the state space form:

$$\dot{X}_h = A_h X_h + B_h u_h,$$

where

$$X_h = \begin{bmatrix} q_s \\ \dot{q}_s \\ x_a \end{bmatrix}, \ u_h = \begin{bmatrix} \beta \\ \dot{\beta} \\ \ddot{\beta} \end{bmatrix},$$

$$\begin{aligned}
A_h &= \begin{bmatrix} 0 & I & 0 \\ -M^{-1}\left(K_s + q_d A_{s0}\right) & -M^{-1}\left(D_s + q_d \frac{b}{V} A_{s1}\right) & -q_d M^{-1} E \\ 0 & F_s & \frac{V}{b} R \end{bmatrix}, \\
B_h &= \begin{bmatrix} 0 & 0 & 0 \\ -q_d M^{-1} A_{c0} & -q_d \frac{b}{V} M^{-1} A_{c1} & -M^{-1}\left(M_c + q_d \frac{b^2}{V^2} A_{c2}\right) \\ 0 & F_c & 0 \end{bmatrix}, \\
M &= M_s + q_d \frac{b^2}{V^2} A_{s2}.
\end{aligned}$$

In practice, information of displacement, velocity, and acceleration can be obtained by sensors, such as accelerometers and angular rate gyros. It is assumed that in this two dimensional 3DOF aeroelastic system the acceleration information can be measured by gyros which takes the following form [11]

$$\begin{aligned} Y_h &= \Phi \begin{bmatrix} -M^{-1}(K_s + q_d A_{s0}) & -M^{-1}\left(D_s + q_d \frac{b}{V} A_{s1}\right) & -q_d M^{-1} E \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \\ x_a \end{bmatrix} \\ &\quad + \Phi \begin{bmatrix} -q_d M^{-1} A_{c0} & -q_d \frac{b}{V} M^{-1} A_{c1} & -M^{-1}\left(M_c + q_d \frac{b^2}{V^2} A_{c2}\right) \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\beta} \\ \ddot{\beta} \end{bmatrix}, \end{aligned}$$

where Φ is the coefficient matrix. Then the output state function of the two-dimensional 3DOF aeroelastic system could be denoted as

$$Y_h = C_h X_h + D_h u_h.$$

Furthermore, we adopt the following transfer function to describe the relation between the deflective angle of control surface and the command of actuator

$$\frac{\beta}{\delta_c} = \frac{a_3}{s^3 + a_2 s^2 + a_1 s + a_0},$$

which has the following representation in time domain

$$\begin{bmatrix} \dot{\beta} \\ \ddot{\beta} \\ \dddot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\beta} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} \delta_c.$$

Then, the final open-loop aeroelastic state and output functions are

$$\begin{aligned} \dot{X} &= AX + Bu, \\ Y &= CX, \end{aligned}$$

where

$$X = \begin{bmatrix} X_h & X_e \end{bmatrix}^T, \ u = \delta_c, \ X_e = \begin{bmatrix} \beta & \dot{\beta} & \ddot{\beta} \end{bmatrix}^T,$$

$$A = \begin{bmatrix} A_h & B_h \\ 0 & A_e \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ B_e \end{bmatrix}, \ C = \begin{bmatrix} C_h & D_h \end{bmatrix},$$

$$A_e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \ B_e = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}.$$

Since matrix A depends on the flow velocity V explicitly, in the following matrix A is substituted by $A(V)$. It is clear that eigenvalues of $A(V)$ change their positions on complex plan with V. According to the linear control theories, the system is stable if and only if the eigenvalues of state matrix are located in the open left-half complex plane. Therefore, when the locus of a eigenvalue crosses the imaginary axis from the left-half complex plane, the aeroelastic system is critically stable. And the corresponding flow velocity is called a critical flutter speed.

3 Robust Control Law Design for Active Flutter Suppression

3.1 Problem Formulation

In the aeroelastic control systems, the most common technique for active flutter suppression is the theory of Linear Quadratic Regulation by state feedback. Since the aerodynamic augmented states are immeasurable, this technique has difficulties to be applied in practice. Therefore, output feedback is adopted in this paper.

According to the two-dimensional 3DOF aeroelastic system model

$$\begin{aligned} \dot{X} &= A(V)X + Bu, \\ Y &= CX, \end{aligned} \tag{3.1}$$

and supposing that the matrix C is of full row rank, we design the following output feedback control law

$$u = -KY \tag{3.2}$$

to minimize the cost function

$$J = \frac{1}{2}\int_0^\infty \left(X^T QX + u^T Ru\right) dt. \tag{3.3}$$

Generally the weighting matrices Q and R are selected via engineering experiences. In this paper, the two weighting matrices are both assumed to be positive definite. Q is limited to 10^{-3} level, and R is limited to an identity matrix.

Usually there are three approaches, i.e. the Levine-Athans method, the least error excitation method, and the minimum norm method [13], to solve the output suboptimal problem and obtain the output feedback control law K indirectly. But the actual two-dimensional 3DOF system works in a changing environment, which differs from the model that we discuss and design, especially when the damping coefficients are difficult to be obtained precisely. Therefore the model we analysis possesses uncertainties. In this paper, we assume that the dynamic matrix has a parametric uncertainty which can be described by a polytope, i.e.

$$A \in \Omega = Co\{A_1, A_2, \cdots, A_n\} = \left\{\sum_{i=1}^{n} \lambda_i A_i; \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\right\},$$

where n is the number of vertexes of the polytopic system. In addition, the formula $q_d = \frac{1}{2}\rho V^2$ is included in every matrix A_i. Therefore, the matrix A_i also depends on the flow velocity V.

3.2 Robust Control Law Design

The problem to be investigated in this paper is how to design the output feedback control law (3.2). With the control law, the two-dimensional 3DOF aeroelastic system (3.1) can be represented as:

$$\dot{X} = (A - BKC)X, \ A \in \Omega.$$

Then, the cost function could be rewritten into the following form:

$$J = \frac{1}{2}\int_0^\infty X^T \left(Q + C^T K^T RKC\right) X dt.$$

The system described by (3.1) is quadratically stable if and only if there exists a symmetric matrix $P = P^T > 0$ such that

$$(A - BKC)^T P + P(A - BKC) + Q + C^T K^T RKC \le 0. \tag{3.4}$$

Along any trajectory of the closed-loop system, the derivative of $X^T(t) PX(t)$ is

$$\begin{aligned} \frac{d}{dt}\left[X^T(t) PX(t)\right] &= X^T(t)\left[(A - BKC)^T P + P(A - BKC)\right] X(t) \\ &\le -X^T(t)\left(Q + C^T K^T RKC\right) X(t). \end{aligned} \tag{3.5}$$

After integrating both sides of the inequality (3.5) from $t = 0$ to $t = \infty$, we have

$$J = \frac{1}{2}\int_0^\infty X^T\left(Q + C^T K^T RKC\right) X dt \le X^T(0) PX(0).$$

Therefore the suboptimal control problem could be transformed into a constrained optimization problem

$$\begin{gathered} \min \frac{1}{2} X^T(0) PX(0) \\ \text{s.t.} \begin{cases} (A - BKC)^T P + P(A - BKC) + Q + C^T K^T RKC \le 0, \\ P > 0,\ Q > 0,\ R > 0. \end{cases} \end{gathered} \tag{3.6}$$

It is noted that since our purpose is to determine the matrix K, inequality (3.4) is actually a nonlinear matrix inequality. This drawback can be overcome by defining $P_1 = P^{-1}$, $P_2 = -KCP_1$, and inequality (3.4) is equivalent to the following LMI

$$\begin{bmatrix} P_1 A^T + AP_1 + P_2^T B^T + BP_2 & P_1 & P_2^T \\ P_1 & -Q^{-1} & 0 \\ P_2 & 0 & -R^{-1} \end{bmatrix} \le 0.$$

Obviously, when the dynamic matrix A has a polytopic parametric variation, we only need analyze this problem on the vertexes [3, 5, 12]. Thus, the optimization problem (3.6) could be transformed further into the following form:

$$\begin{gathered} \min\ \gamma \\ \text{s.t.} \begin{cases} \begin{bmatrix} P_1 A_i^T + A_i P_1 + P_2^T B^T + BP_2 & P_1 & P_2^T \\ P_1 & -Q^{-1} & 0 \\ P_2 & 0 & -R^{-1} \end{bmatrix} \le 0, \\ \begin{bmatrix} \gamma & X^T(0) \\ X(0) & P_1 \end{bmatrix} \ge 0, \\ P_1 > 0, \end{cases} \end{gathered} \tag{3.7}$$

where $P_1 = P^{-1}$, $P_2 = -KCP_1$, $i = 1, 2, \cdots n$.

Because the output matrix C is not always square, we could not directly inverse CP_1 to derive K from equation $P_2 = -KCP_1$. In this paper, we apply the minimum norm method to determine the matrix K indirectly. Define $F^* \triangleq -P_2 P_1^{-1}$, $F \triangleq KC$. Supposing that the matrices P_1 and P_2 have been derived from the optimization problem (3.7), minimizing the following objective function

$$J = \| F - F^* \| = \sqrt{\mathrm{Trace}\,(F - F^*)^T (F - F^*)},$$

we can get the approximate solution

$$K = F^* C^T \left(CC^T \right)^{-1}.$$

4 Numerical Simulation

4.1 Open-loop Simulation

In order to validate the effectiveness of the proposed method, numerical simulation are set up in this section with the following parameters. Here parameter variations are not

Parameter	Value	Parameter	Value
m	1.285kg	S_α	0.0209kgm
S_β	0.0006608kgm	I_α	0.005142kgm^2
a	-0.5	b	0.1m
c	0.5	ρ	1.025kg/m^3
k_h	2742N/m	k_α	2.912Nm/rad
k_β	90042Nm/rad	d_h	30.43Ns/m
d_α	0.04Ns/m	d_β	418.8977Ns/m

Table 4.1: List parameters.

considered. Under the influence of the unsteady aerodynamic forces, the root locus of the open loop aeroelastic system are showed in Figure 4.1. And the real parts of the eigenvalues of $A(V)$ with respect to the flow velocities are showed in Figure 4.2. If the real parts of all of the eigenvalues of $A(V)$ are negative, that is, the eigenvalues are in the open left half plane, the two-dimensional 3DOF aeroelastic system is asymptotically stable. From Figure 4.1 and Figure 4.2 we can see that the pitching mode will be in the right half plane when the flow velocity exceeds 47.5m/s, and then flutter occurs. The flutter speed, $V_f = 47.5$m/s, is the speed at which the open loop system becomes marginally stable.

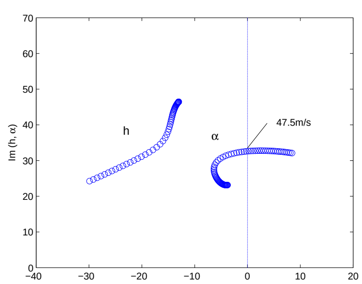

Figure 4.1: The root locus of the open loop aeroelastic system.

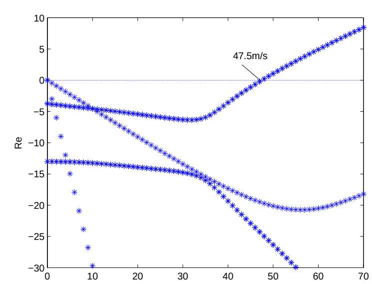

Figure 4.2: The relation between real parts of eigenvalues and flow velocity.

Here we select three velocity values to see the time response of each modes without considering uncertainties in any parameter. From Figures 4.3, 4.4 and 4.5 we could see the plunge, pitching and control surface states are asymptotically stable at $V = 46$m/s,

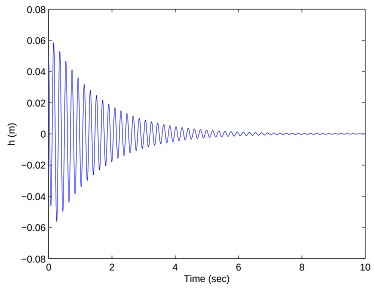

Figure 4.3: The time response curve of plunge mode at V=46m/s.

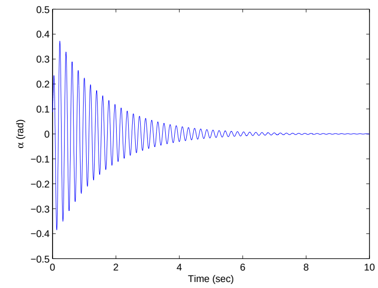

Figure 4.4: The time response curve of pitching mode at V=46m/s.

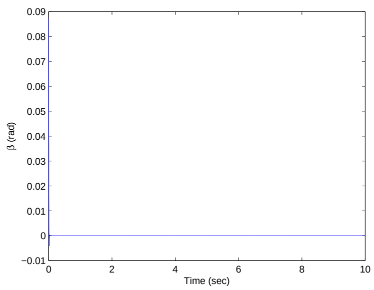

Figure 4.5: The time response curve of control surface mode at V=46m/s.

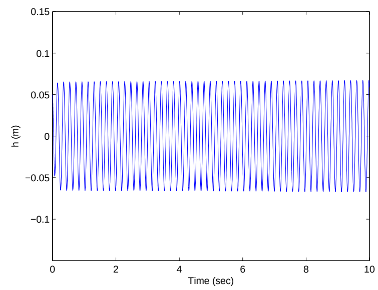

Figure 4.6: The time response curve of plunge mode at V=47.5m/s.

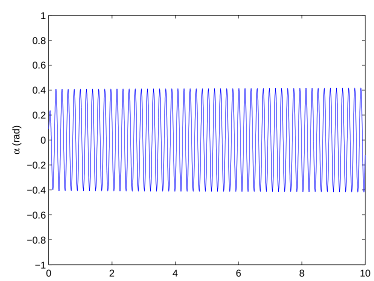

Figure 4.7: The time response curve of pitching mode at V=47.5m/s.

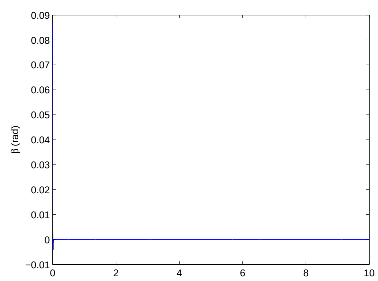

Figure 4.8: The time response curve of control surface mode at V=47.5m/s.

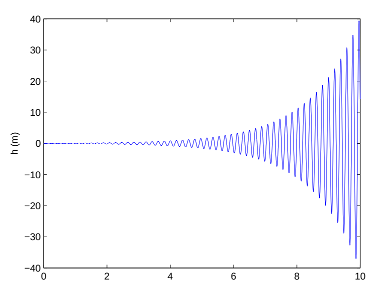

Figure 4.9: The time response curve of plunge mode at V=49m/s.

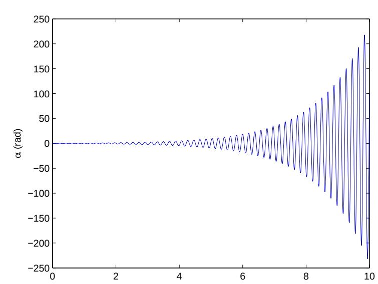

Figure 4.10: The time response curve of pitching mode at V=49m/s.

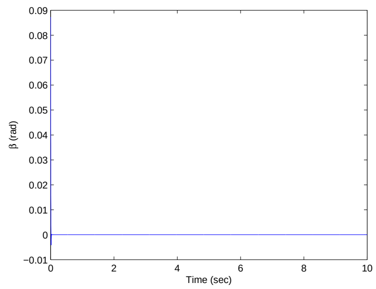

Figure 4.11: The time response curve of control surface mode at V=49m/s.

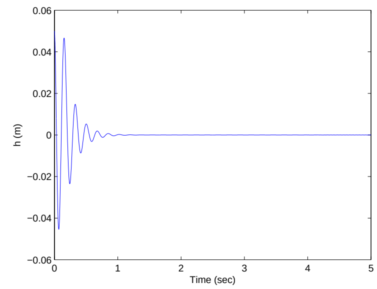

Figure 4.12: The time response curve of plunge mode at V=49m/s after robust flutter suppression.

and almost all oscillations disappear at $t = 7$ seconds. So, the flutter phenomenon could be suppressed by the aeroelastic system itself. At $V = V_f = 47.5$m/s, the sates are all settled into harmonic oscillations as shown in Figures 4.6, 4.7, 4.8. But in Figures 4.9, 4.10, and 4.11, with flow velocity $V = 49$m/s, the plunge, pitching and control surface states continue to increase without bound, and after about 6 seconds, the oscillations are so severe that the airfoil would become unstable. Furthermore, from Figure 4.11 we could see that the state of control surface β is always stable even though the flow velocity exceeds the critical flutter speed, which coincides with the assumption of the perfect rigid control surface.

In brief, for $V < V_f$ the system is asymptotically stable. And for $V > V_f$ the system is unable, in this case wing separation will occur which is dangerous for a real aircraft.

4.2 Closed-loop Simulation

In this section a robust controller is designed for the two-dimensional 3DOF airfoil aeroelastic system using the proposed method. Because the damping coefficients are difficult to be obtained precisely, the damping coefficients are assumed to be uncertain which have possible variations of $\pm 10\%$ around the nominal values. The robust output feedback gain matrix is obtained by $K = F^* C^T \left(CC^T\right)^{-1}$, where F^* is the solution to the optimization problem (3.7).

Figures 4.12, 4.13, and 4.14 illustrate the time response curves at $V = 49$m/s, from which we can see the flutter phenomenon is well suppressed after about 1 second and the output feedback is robust to the considered parametric variations.

Furthermore, we are interested in the performance when the flow velocity exceeds the critical flutter speed and the control is delayed by a few seconds. We investigate the system response with parametric uncertainties when the control is initiated at a time greater than $t = 0$ seconds. Consequently, with flow velocity 49m/s, and the control initiated at 2 seconds. the time responses are shown in Figures 4.15, 4.16, 4.17. The oscillation disappear at $t = 3$ seconds and the output feedback is robust to the considered parametric variations as well.

The relation between the real parts of $A(V)$ eigenvalues and the flow velocity with flutter robust suppression is shown in Figure 4.18, from which we could see the critical flutter speed is 57.8m/s, that is, the critical speed increase from the original speed 47.5m/s to 57.8m/s. The critical flutter speed increases 21.68%. From the simulations

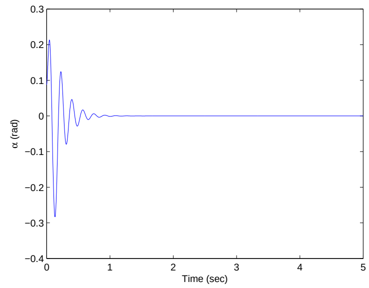

Figure 4.13: The time response curve of pitching mode at V=49m/s after robust flutter suppression.

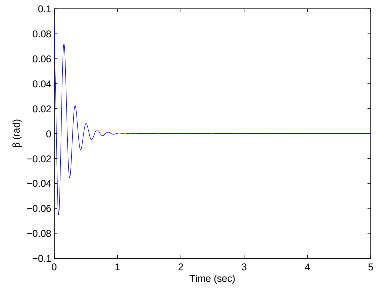

Figure 4.14: The time response curve of control surface mode at V=49m/s after robust flutter suppression.

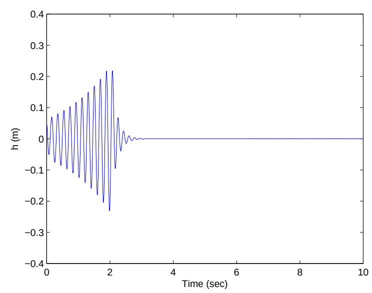

Figure 4.15: The time response curve of plunge mode at V=49m/s after robust flutter suppression: t=2 seconds.

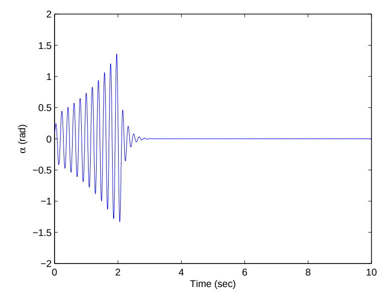

Figure 4.16: The time response curve of pitching mode at V=49m/s after robust flutter suppression: t=2 seconds.

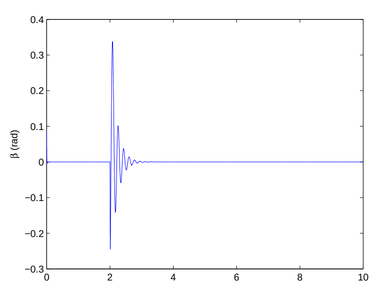

Figure 4.17: The time response curve of control surface mode at V=49m/s after robust flutter suppression: t=2 seconds.

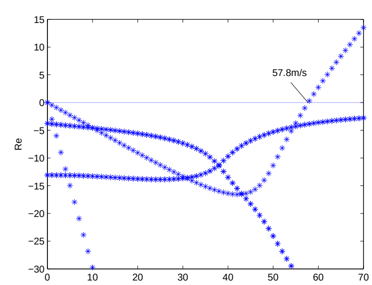

Figure 4.18: The relation between real parts of eigenvalues and flow velocity after robust flutter suppression.

we can conclude that the proposed method not only well suppresses flutter phenomenon, but also increases the critical flutter speed.

5 Conclusion

In the traditional aircraft design, a passive method is usually adopted, which increases the structure weight of the aircraft in order to increase the critical flutter speed. In this paper we present an active control approach, which transforms the suboptimal control law design problem into a constrained optimization problem, to design the robust control law of a two-dimensional 3DOF aeroelastic system. The introduced deformation can suppress the flutter phenomenon by the flexibility of structure. The simulation results show that the minimum norm method and the LMI technique adopted is valid with the uncertainties of damping coefficients. When the flow velocity exceeds the critical flutter speed, the two-dimensional 3DOF airfoil is still stable with the proposed robust controller.

References

[1] Atesoglu, Ö. and Özgören, M. High-Flight Maneuverability Enhancement of a Fighter Aircraft Using Thrust-Vectoring Control. *Journal of Guidance, Control and Dynamics* **30**(5) (2007) 1480–1493.

[2] Bolsunovsky, A., Buzoverya, N., Gurevich, B., Denisov, V., Dunaevsky, A., Shkadov, L., Sonin, O., Udzhuhu, A. and Zhurihin, J. Flying wing-problems and decisions. *Aircraft Design* **4**(4) (2001) 193–219.

[3] Boyd, S. and Vandenberghe, L. *Convex Optimization*. Cambridge University Press, Cambridge, 2004.

[4] Jadbabaie, A. and Hauser, J. Control of a thrust-vectored flying wing: a receding horizon-LPV approach. *International Journal of Robust and Nonlinear Control* **12**(9) (2002) 869–896.

[5] Karimi, H., Lohmann, B. and Buskens, C. An LMI Approach to H_∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs. *Nonlinear Dynamics and Systems Theory* **7**(4) (2006) 351–368.

[6] Olds, S. *Modeling and LQR Control of a Two-Dimensional Airfoil*. PhD thesis, Virginia Polytechnic Institute and State University, 1997.

[7] Patil, M., Hodges, D. and Cesnik, C. Nonlinear aeroelasticity and flight dynamics of high-altitude long-endurance aircraft. *Journal of Aircraft* **38**(1) (2001) 88–94.

[8] Pendleton, E., Bessette, D., Field, P., Miller, G. and Griffin, K. Active Aeroelastic Wing Flight Research Program: Technical Program and Model Analytical Development. *Journal of Aircraft* **37**(4) (2000) 554–561.

[9] Silva, S. and V. Lopes Júnior. Active flutter suppression in a 2-D airfoil using linear matrix inequalities techniques. *Journal of the Brazilian Society of Mechanical Sciences and Engineering* **28** (2006) 84–93.

[10] Theodorsen, T. General theory of aerodynamic instability and the mechanism of flutter. *NACA Report No.496* (1979) 291–311.

[11] Zhao, Y. *Aeroelastics and Control*. Science Press, Beijing, 2007. [Chinease]

[12] Zhai1, G., Yoshida, M., Imae1, J. and Kobayash, T. Decentralized H2 Controller Design for Descriptor Systems: An LMI Approach. *Nonlinear Dynamics and Systems Theory* **6**(1) (2006) 99–109.

[13] Gu, Z., Ma, K. and Chen W. *Vibration Active Control*. National Defence Industry Press, Beijing, 1997. [Chinease]

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H_∞ Filter Design for a Class of Nonlinear Neutral Systems with Time-Varying Delays

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Abstract: In this note, the problem of H_∞ filtering for a class of nonlinear neutral systems with delayed states and outputs is investigated. By introducing a descriptor technique, using Lyapunov-Krasovskii functional and a suitable change of variables, new required sufficient conditions are established in terms of delay-dependent linear matrix inequalities (LMIs) for the existence of the desired H_∞ filters. The explicit expression of the filters is derived to satisfy both asymptotic stability and a prescribed level of disturbance attenuation for all admissible known nonlinear functions. A numerical example is provided to show the proposed design approach.

Keywords: *neutral systems; H_∞ filtering; nonlinearity; LMI; time-delay.*

Mathematics Subject Classification (2000): 34K40, 93C10, 93E11.

1 Introduction

Delay (or memory) systems represent a class of infinite-dimensional systems [1, 2] largely used to describe propagation and transport phenomena or population dynamics [3, 4]. Delay differential systems are assuming an increasingly important role in many disciplines like economic, mathematics, science, and engineering. For instance, in economic systems, delays appear in a natural way since decisions and effects are separated by some time interval. The presence of a delay in a system may be the result of some essential simplification of the corresponding process model. The delay effects problem on the (closed-loop) stability of (linear) systems including delays in the state and/or input is a problem of recurring interest since the delay presence may induce complex behaviors (oscillation, instability, bad performances) for the (closed-loop) schemes [2, 5–9].

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Neutral delay systems constitute a more general class than those of the retarded type. It is important to point out that the highest order derivative of a retarded differential equation does not contain any delayed variables. When such a term does appear, then we have a differential equation of neutral type. Stability of these systems proves to be a more complex issue because the system involves the derivative of the delayed state. Especially, in the past few decades increased attention has been devoted to the problem of robust delay-independent stability or delay-dependent stability and stabilization via different approaches for linear neutral systems with delayed state and/or input and parameter uncertainties (see for instance [2, 10, 11]). Among the past results on neutral delay systems, the LMI approach is an efficient method to solve many control problems such as stability analysis and stabilization [12–17], H_∞ control problems [18–24] and guaranteed-cost (observer-based) control design [25–29].

On the other hand, the state estimation problem has been one of the fundamental issues in the control area and there have been many works following those of Kalman filter or H_2 optimal estimators (in the stochastic framework) and Luenberger filter (in the deterministic framework) [30]. Nevertheless there has been an increasing interest in the robust H_∞ filtering, which is concerned with the design of an estimator ensuring that the L_2-induced gain from the noise signal to the estimation error is less than a prescribed level, in the past years [31-35]. Compared with the conventional Kalman filtering, the H_∞ filter technique has several advantages. First, the noise sources in the H_∞ filtering setting are arbitrary signals with bounded energy or average power, and no exact statistics are required to be known [36]. Second, the H_∞ filter has been shown to be much more robust to parameter uncertainty in a control system. These advantages render the H_∞ filtering approach very appropriate to some practical applications. When parameter uncertainty arises in a system model, the robust H_∞ filtering problem has been studied, and a great number of results on this topic have been reported (see the references [37–39]). In the case when parameter uncertainty and time delays appear simultaneously in a system model, the robust H_∞ filtering problem was dealt with in [40] via LMI approach, respectively. The corresponding results for uncertain discrete delay systems can be found in [41]. However, it is noted that the H_∞ filtering of nonlinear neutral systems has not been been fully investigated in the past and remains to be important and challenging. This motivates the present study.

In this paper, we are concerned to develop a new delay-dependent stability criterion for H_∞ filtering problem of nonlinear neutral systems with known nonlinear functions which satisfy the Lipschitz conditions. The main merit of the proposed method is the fact that it provides a convex problem with additional degree of freedom which lead to less conservative results. Our analysis is based on the Hamiltonian–Jacoby–Isaac (HJI) method. By introducing a descriptor technique, using Lyapunov–Krasovskii functional and a suitable change of variables, we establish new required sufficient conditions in terms of delay-dependent LMIs under which the desired H_∞ filters exist, and derive the explicit expression of these filters to satisfy both asymptotic stability and H_∞ performance. A desired filter can be constructed through a convex optimization problem, which can be solved by using standard numerical algorithms. Finally, a numerical example is given to illustrate the proposed design method.

Notations. The superscript $'T'$ stands for matrix transposition; $\Re^n$ denotes the n-dimensional Euclidean space; $\Re^{n\times m}$ is the set of all real n by m matrices. $\|.\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. $col\{\cdots\}$ and $sym(A)$ represent, respectively, a column vector and the matrix $A + A^T$. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote, respectively, the smallest and largest eigenvalue of the square matrix A. The notation $P > 0$ means that P is real symmetric and positive definite; the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. In addition, $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2 Problem Description

We consider a class of nonlinear neutral systems with delayed states and outputs represented by

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-h(t)) + A_2\dot{x}(t-d(t)) + E_1f(x(t)) + E_2f(x(t-h(t))) + B_1w(t), \\ x(t) = \varphi(t), \quad t \in [-max\{h_1, d_1\},\, 0], \\ z(t) = C_1x(t), \\ y(t) = C_2x(t) + g(t, x(t)), \end{cases} \tag{1}$$

where $x(t) \in \Re^n$, $w(t) \in L_2^s[0,\infty)$, $z(t) \in \Re^z$ and $y(t) \in \Re^p$ are corresponded to state vector, disturbance input, estimated output and measured output. The time-varying function $\varphi(t)$ is continuous vector valued initial function and the parameters $h(t)$ and $d(t)$ are time-varying delays satisfying

$$0 \le h(t) \le h_1, \quad \dot{h}(t) \le h_2,$$

$$0 \le d(t) \le d_1, \quad \dot{d}(t) \le d_2 < 1.$$

Assumption 2.1 1) The nonlinear function $f : \Re^n \to \Re^n$ is continuous and satisfies $f(0) = 0$ and the Lipschitz condition, i.e., $\|f(x_0) - f(y_0)\| \le \|U_1(x_0 - y_0)\|$ for all $x_0, y_0 \in \Re^n$ and U_1 is a known matrix.

2) The nonlinear function $g : \Re \times \Re^n \to \Re^p$ is continuous and satisfies the Lipschitz condition, i.e., $\|g(t, x_0) - g(t, y_0)\| \le \|U_2(x_0 - y_0)\|$ for all $x_0, y_0 \in \Re^n$ and U_2 is a known matrix.

In this paper, the author's attention will be focused on the design of an $n-$th order delay-dependent H_∞ filter with the following state-space equations

$$\begin{cases} \dot{\hat{x}}(t) = F\hat{x}(t) + F_1\hat{x}(t-h(t)) + F_2\dot{\hat{x}}(t-d(t)) + F_3f(\hat{x}(t)) + F_4f(\hat{x}(t-h(t))) \\ \qquad +G(y(t) - C_2\hat{x}(t) - g(t, \hat{x}(t))), \\ \hat{x}(t) = 0, \quad t \in [-max\{h_1, d_1\},\, 0], \\ \hat{z}(t) = G_1\hat{x}(t), \end{cases} \tag{2}$$

where the state-space matrices F, F_1, F_2, F_3, F_4, G and G_1 of the appropriate dimensions are the filter design objectives to be determined. In the absence of $w(t)$, it is required that

$$\|x(t) - \hat{x}(t)\|_2 \to 0 \;\; as \;\; t \to \infty,$$

where $\hat{x}(t)$ and $\hat{z}(t)$ are the estimation of $x(t)$ and of $z(t)$, respectively, and $e(t) = x(t) - \hat{x}(t)$ is the estimation error. Then, the error dynamics between (1) and (2) can be expressed by

$$\dot{e}(t) = (A - F)\hat{x}(t) + (A_1 - F_1)\hat{x}(t-h(t)) + (A_2 - F_2)\dot{\hat{x}}(t-d(t))$$

$$+(F-GC_2)e(t)+F_1e(t-h(t))+F_2\dot{e}(t)-G\psi(t,e(t))+(E_1-F_3)f(x(t))$$
$$+(E_2-F_4)f(x(t-h(t)))+F_3\phi(e(t))+F_4\phi(e(t-h(t)))+B_1w(t), \tag{3}$$

where $\phi(e(t)) := f(x(t)) - f(x(t)-e(t))$ and $\psi(t,e(t)) := g(t,x(t)) - g(t,x(t)-e(t))$. Now, we obtain the following state-space model, namely filtering error system:

$$\begin{cases} \dot{X}(t) = \hat{A}X(t) + \hat{A}_1X(t-h(t)) + \hat{A}_2\dot{X}(t-d(t)) + \hat{G}\psi(t,e(t)) + \hat{E}_1f(x(t)), \\ \quad +\hat{E}_2f(x(t-h(t))) + \hat{E}_3\phi(e(t)) + \hat{E}_4\phi(e(t-h(t))) + \hat{B}w(t), \\ X(t) = \begin{bmatrix}\varphi(t)^T & \varphi(t)^T\end{bmatrix}^T, \quad t\in[-max\{h_1,d_1\},\,0], \\ z(t)-\hat{z}(t) = \hat{C}_1X(t), \end{cases} \tag{4}$$

where $X(t) = col\{x(t),e(t)\}$, $\hat{A} = \begin{bmatrix} A & 0 \\ A-F & F-GC_2 \end{bmatrix}$, $\hat{A}_1 = \begin{bmatrix} A_1 & 0 \\ A_1-F_1 & F_1 \end{bmatrix}$, $\hat{A}_2 = \begin{bmatrix} A_2 & 0 \\ A_2-F_2 & F_2 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B_1 \\ B_1 \end{bmatrix}$, $\hat{G} = \begin{bmatrix} 0 \\ -G \end{bmatrix}$, $\hat{E}_1 = \begin{bmatrix} E_1 \\ E_1-F_3 \end{bmatrix}$, $\hat{E}_2 = \begin{bmatrix} E_2 \\ E_2-F_4 \end{bmatrix}$, $\hat{E}_3 = \begin{bmatrix} 0 \\ F_3 \end{bmatrix}$, $\hat{E}_4 = \begin{bmatrix} 0 \\ F_4 \end{bmatrix}$ and $\hat{C}_1 = \begin{bmatrix} C_1-G_1 & G_1 \end{bmatrix}$.

Let $\alpha,\beta\in\Re$ and

$$s(\alpha,\beta) = \begin{cases} \frac{f(\alpha)-f(\beta)}{\alpha-\beta}, & \alpha\neq\beta, \\ \delta, & \alpha=\beta. \end{cases} \tag{5}$$

By Assumption 2.1, it is easy to see

$$\phi(e(t)) - \phi(e(t-h(t))) = s(t)(e(t)-e(t-h(t))) = s(t)\int_{t-h(t)}^{t} \dot{e}(s)\,ds. \tag{6}$$

Therefore, from the Leibniz-Newton formula, i.e., $x(t)-x(t-h) = \int_{t-h}^{t}\dot{x}(s)\,ds$, the filtering error system (4) can be represented in a descriptor model form as

$$\begin{cases} \dot{X}(t) = \eta(t), \\ \eta(t) = (\hat{A}+\hat{A}_1)X(t) + \hat{A}_2\eta(t-d(t)) + \hat{G}\psi(t,e(t)) + \hat{E}_1f(x(t)) + \hat{E}_2f(x(t-h(t))) \\ \quad +\hat{E}_3\phi(e(t)) - (\hat{A}_1+\hat{E}_4Js(t))\int_{t-h(t)}^{t}\eta(s)\,ds + \hat{B}w(t). \end{cases} \tag{7}$$

Definition 2.1 1. The delay-dependent H_∞ filter of the type (2) is said to achieve asymptotic stability in the Lyapunov sense for $w(t)=0$ if the augmented system (4) is asymptotically stable for all admissible nonlinear functions $f(x(t))$ and $g(t,x(t))$.

2. The delay-dependent H_∞ filter of the type (2) is said to guarantee robust disturbance attenuation if under zero initial condition

$$Sup_{\|w\|_2\neq0} \quad \frac{\|z(t)-\hat{z}(t)\|_2}{\|w(t)\|_2} \quad \leq\gamma \tag{8}$$

holds for all bounded energy disturbances and a prescribed positive value γ.

The filtering problem we address here is as follows: *Given a prescribed level of disturbance attenuation $\gamma>0$, find the delay-dependent H_∞ filter (2) in the sense of Definition 2.1.*

Before ending this section, we recall a well-known lemma, which will be used in the proof our main results.

Lemma 2.1 *([11]) For any arbitrary column vectors $a(t), b(t)$, matrices $\Phi(t), H, U$ and W the following inequality holds:*

$$-2\int_{t-r}^{t} a(s)^T\Phi(s)b(s)ds \leq \int_{t-r}^{t} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} H & U-\Phi(s) \\ * & W \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds,$$

where $\begin{bmatrix} H & U \\ * & W \end{bmatrix} \geq 0.$

3 H_∞ Filter Design

In this section, both the asymptotic stability and H_∞ performance of the filtering error system is investigated such a sufficient stability condition is derived for the existence of the filter (2). The approach employed here is to develop a criterion for the existence of such filter based on the LMI approach combined with the Lyapunov method. In the literature, extensions of the quadratic Lyapunov functions to the quadratic Lyapunov-Krasovskii functionals have been proposed for time-delayed systems (see for instance the references [2, 10, 11, 27, 29] and the references therein).

We choose a Lyapunov–Krasovskii functional candidate for the nonlinear neutral system (1) as

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{9}$$

where

$$V_1(t) = X(t)^T P_1 X(t) = \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix}^T TP \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix},$$

$$V_2(t) = \int_{t-h(t)}^{t} X(s)^T Q_1 X(s)\ ds + \int_{t-d(t)}^{t} \eta(s)^T Q_2 \eta(s)\ ds,$$

$$V_3(t) = \int_{t-h_1}^{t}\int_{s}^{t} \eta(\theta)^T (Q_3+Q_4)\eta(\theta)\ d\theta\, ds$$

with

$$P := \begin{bmatrix} P_1 & 0 \\ P_3 & P_2 \end{bmatrix}, \quad P_1 = P_1^T > 0, \quad T := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

In the following, we state our main results in terms of LMIs on the delay-dependent H_∞ filter design for the nonlinear neutral system (1) based on Lyapunov stability theory.

Theorem 3.1 *Consider system (1) and let the matrices U_1, U_2 and the scalars $h_1, d_1 > 0, d_2 < 1, h_2$ and $\gamma > 0$ be given scalars. If there exist the matrices $P_{11}, P_{12}, P_{22}, G_1, H, U, \{W_i\}_{i=1}^6, \{M_i\}_{i=1}^9$, the positive definite matrices $P_1, \{Q_i\}_{i=1}^4$ and the scalar ϵ, satisfying the following LMIs*

$$\begin{bmatrix} [1,1] & [1,2] & [1,3] & [1,4] \\ * & [2,2] & [2,3] & [2,4] \\ * & * & [3,3] & 0 \\ * & * & * & [4,4] \end{bmatrix} < 0, \tag{11a}$$

$$\begin{bmatrix} H & U \\ * & Q_3 \end{bmatrix} \geq 0, \tag{11b}$$

where

$$[1,1] := sym\left\{\begin{bmatrix} \epsilon(\Sigma_1+\Sigma_2) & P_1 - \epsilon\begin{bmatrix} P_{11}^T & P_{22}^T \\ P_{12}^T & P_{22}^T \end{bmatrix} \\ \Sigma_1+\Sigma_2 & -\begin{bmatrix} P_{11}^T & P_{22}^T \\ P_{12}^T & P_{22}^T \end{bmatrix} \end{bmatrix}\right\} - sym\left\{\begin{bmatrix} \epsilon\Sigma_2 \\ \Sigma_2 \end{bmatrix} J - (U+M_1)J\right\}$$

$$+h_1 H + \begin{bmatrix} Q_1 + J^T U_1^T U_1 J & 0 \\ * & Q_2 + h_1(Q_3+Q_4) + \hat{J}^T(U_1^T U_1 + U_2^T U_2)\hat{J} \end{bmatrix},$$

$$[1,2] := -U - M_1 + \begin{bmatrix} \epsilon\Sigma_2 \\ \Sigma_2 \end{bmatrix} + J^T M_2^T,$$

$$[2,2] := -(1-h_2)Q_1 - sym\{M_2\} + J^T U_1^T U_1 J + \hat{J}^T U_1^T U_1 \hat{J},$$

$$[1,3] := \left[\begin{bmatrix} \epsilon\Sigma_3 \\ \Sigma_3 \end{bmatrix} + J^T M_3^T \quad \begin{bmatrix} \epsilon\Sigma_4 \\ \Sigma_4 \end{bmatrix} + J^T M_4^T \quad \begin{bmatrix} \epsilon\Sigma_5 \\ \Sigma_5 \end{bmatrix} + J^T M_5^T \quad \begin{bmatrix} \epsilon(\Sigma_6-\Sigma_7) \\ \Sigma_6-\Sigma_7 \end{bmatrix} + J^T M_6^T\right],$$

$$[2,3] := \begin{bmatrix} -M_3^T & -M_4^T & -M_5^T & -M_6^T \end{bmatrix},$$

$$[3,3] := diag\{-(1-d_2)Q_2, -I, -I, -I\},$$

$$[1,4] := \left[\begin{bmatrix} \epsilon\Sigma_7 \\ \Sigma_7 \end{bmatrix} + J^T M_7^T \quad \begin{bmatrix} \epsilon\Sigma_8 \\ \Sigma_8 \end{bmatrix} + J^T M_8^T \quad \begin{bmatrix} \epsilon\Sigma_9 \\ \Sigma_9 \end{bmatrix} + J^T M_9^T \quad J^T \hat{C}_1^T\right],$$

$$[2,4] := \begin{bmatrix} -M_7^T & -M_8^T & -M_9^T & 0 \end{bmatrix},$$

$$[4,4] := diag\{-I, -I, -\gamma^2 I, -I\}$$

with

$$\Sigma_1 := \begin{bmatrix} (P_{11}^T + P_{22}^T)A - W_1 & W_1 - W_6 C_2 \\ (P_{11}^T + P_{22}^T)A - W_1 & W_1 - W_6 C_2 \end{bmatrix},\ \Sigma_2 := \begin{bmatrix} (P_{11}^T + P_{22}^T)A_1 - W_2 & W_2 \\ (P_{11}^T + P_{22}^T)A_1 - W_2 & W_2 \end{bmatrix},$$

$$\Sigma_3 := \begin{bmatrix} (P_{11}^T + P_{22}^T)A_2 - W_3 & W_3 \\ (P_{11}^T + P_{22}^T)A_2 - W_3 & W_3 \end{bmatrix},\ \Sigma_4 := \begin{bmatrix} (P_{11}^T + P_{22}^T)E_1 \\ (P_{12}^T + P_{22}^T)E_1 \end{bmatrix} - \Sigma_6,$$

$$\Sigma_5 := \begin{bmatrix} (P_{11}^T + P_{22}^T)E_2 \\ (P_{12}^T + P_{22}^T)E_2 \end{bmatrix} - \Sigma_7,\ \Sigma_6 := \begin{bmatrix} W_4 \\ W_4 \end{bmatrix},\ \Sigma_7 := \begin{bmatrix} W_5 \\ W_5 \end{bmatrix},$$

$$\Sigma_8 := -\begin{bmatrix} W_6 \\ W_6 \end{bmatrix},\ \Sigma_9 := \begin{bmatrix} (P_{11}^T + P_{22}^T)B_1 \\ (P_{12}^T + P_{22}^T)B_1 \end{bmatrix},$$

where $J := [I, 0]$ *and* $\hat{J} := [0, I]$, *then there exists a delay-dependent* H_∞ *filter of the type (2) which achieve the asymptotic stability and* H_∞ *performance, simultaneously, in the sense of Definition 2.1. Moreover, the state-space matrices of the filter are given by*

$$\begin{bmatrix} F & F_1 & F_2 & F_3 & F_4 & G \end{bmatrix} := (P_{22}^T)^{-1}\begin{bmatrix} W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \end{bmatrix},$$

$$and\ \ G_1\ from\ LMIs\ (11). \tag{12}$$

Proof Differentiating $V_1(t)$ in t along the trajectory of the filtering error system (4) we obtain

$$\dot{V}_1(t) = 2X(t)^T P_1 \dot{X}(t) = 2 \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix}^T P^T \begin{bmatrix} \dot{X}(t) \\ 0 \end{bmatrix} = 2 \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix}^T P^T \begin{bmatrix} \eta(t) \\ (.) \end{bmatrix}$$

$$= 2 \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix}^T P^T (\bar{A} \begin{bmatrix} X(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{A}_2 \end{bmatrix} \eta(t-d(t)) + \begin{bmatrix} 0 \\ \hat{G} \end{bmatrix} \psi(t,e(t)) + \begin{bmatrix} 0 \\ \hat{E}_1 \end{bmatrix} f(x(t))$$

$$+ \begin{bmatrix} 0 \\ \hat{E}_2 \end{bmatrix} f(x(t-h(t))) + \begin{bmatrix} 0 \\ \hat{E}_3 \end{bmatrix} \phi(e(t)) - \begin{bmatrix} 0 \\ \hat{A}_1 + \hat{E}_4 J s(t) \end{bmatrix} \int_{t-h(t)}^{t} \eta(s)\, ds + \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix} w(t)), \tag{13}$$

where

$$(.) := -\eta(t) + (\hat{A} + \hat{A}_1)X(t) + \hat{A}_2 \eta(t-d(t)) + \hat{G}\psi(t,e(t)) + \hat{E}_1 f(x(t))$$

$$+\hat{E}_2 f(x(t-h(t))) + \hat{E}_3 \phi(e(t)) - (\hat{A}_1 + \hat{E}_4 J s(t)) \int_{t-h(t)}^{t} \eta(s)\, ds + \hat{B} w(t)$$

and time derivative of the second and third terms of $V(t)$ are, respectively, as

$$\begin{aligned} \dot{V}_2(t) &= X(t)^T Q_1 X(t) - (1-\dot{h}(t)) X(t-h(t))^T Q_1 X(t-h(t)) \\ &\quad +\eta(t)^T Q_2 \eta(t) - (1-\dot{d}(t)) \eta(t-d(t))^T Q_2 \eta(t-d(t)) \\ &\leq X(t)^T Q_1 X(t) - (1-h_2) X(t-h(t))^T Q_1 X(t-h(t)) \\ &\quad +\eta(t)^T Q_2 \eta(t) - (1-d_2) \eta(t-d(t))^T Q_2 \eta(t-d(t)) \end{aligned} \tag{14}$$

and

$$\begin{aligned} \dot{V}_3(t) &= h_1 \eta(t)^T (Q_3 + Q_4) \eta(t) - \int_{t-h_1}^{t} \eta(s)^T (Q_3+Q_4) \eta(s)\ ds \\ &= h_1 \eta(t)^T (Q_3 + Q_4) \eta(t) - \int_{t-h_1}^{t} \eta(s)^T Q_3 \eta(s)\ ds \\ &\quad - \int_{t-h(t)}^{t} \eta(s)^T Q_4 \eta(s)\ ds - \int_{t-h_1}^{t-h(t)} \eta(s)^T Q_4 \eta(s)\ ds. \end{aligned} \tag{15}$$

Construct a HJI function in the form of

$$J[X(t), w(t)] = \frac{d}{dt} V(t) + (z(t) - \hat{z}(t))^T (z(t) - \hat{z}(t)) - \gamma^2 w(t)^T w(t), \tag{16}$$

where derivative of $V(t)$ is evaluated along the trajectory of the filtering error system (4). It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality $J[X(t), w(t)] < 0$ for every $w(t) \in L_2^s[0, \infty)$ results in a function $V(t)$, which is strictly radially unbounded (see for instance the reference [42]).

From (13)–(16) we obtain

$$J[X(t),w(t)] = 2\bar{\eta}(t)^T P^T(\bar{A}\bar{\eta}(t) + \begin{bmatrix} 0 \\ \hat{A}_2 \end{bmatrix} \eta(t-d(t)) + \begin{bmatrix} 0 \\ \hat{G} \end{bmatrix} \psi(t,e(t)) + \begin{bmatrix} 0 \\ \hat{E}_1 \end{bmatrix} f(x(t))$$

$$+ \begin{bmatrix} 0 \\ \hat{E}_2 \end{bmatrix} f(x(t-h(t))) + \begin{bmatrix} 0 \\ \hat{E}_3 \end{bmatrix} \phi(e(t)) - \begin{bmatrix} 0 \\ \hat{A}_1 + \hat{E}_4 Js(t) \end{bmatrix} \int_{t-h(t)}^{t} \eta(s)\ ds + \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix} w(t))$$

$$+X(t)^T(Q_1 + \hat{C}_1^T \hat{C}_1)X(t) - (1-h_2)X(t-h(t))^T Q_1 X(t-h(t))$$

$$+\eta(t)^T(Q_2 + h_1(Q_3+Q_4))\eta(t) - (1-d_2)\eta(t-d(t))^T Q_2 \eta(t-d(t)) - \int_{t-h_1}^{t} \eta(s)^T Q_3 \eta(s)\ ds$$

$$- \int_{t-h(t)}^{t} \eta(s)^T Q_4 \eta(s)\ ds - \int_{t-h_1}^{t-h(t)} \eta(s)^T Q_4 \eta(s)\ ds - \gamma^2 w(t)^T w(t), \tag{17}$$

where $\bar{\eta}(t) := col\{X(t), \eta(t)\}$ and $\bar{A} := \begin{bmatrix} 0 & I \\ \hat{A} + \hat{A}_1 & -I \end{bmatrix}$. By Lemma 2.1 and (11b), it is clear that

$$-2\bar{\eta}(t)^T P^T \begin{bmatrix} 0 \\ \hat{A}_1 + \hat{E}_4 Js(t) \end{bmatrix} \int_{t-h(t)}^{t} \eta(s)\, ds$$

$$\leq \int_{t-h(t)}^{t} \begin{bmatrix} \bar{\eta}(t) \\ \eta(s) \end{bmatrix}^T \begin{bmatrix} H & U - P^T \begin{bmatrix} 0 \\ \hat{A}_1 + \hat{E}_4 Js(t) \end{bmatrix} \\ * & Q_3 \end{bmatrix} \begin{bmatrix} \bar{\eta}(t) \\ \eta(s) \end{bmatrix}\ ds$$

$$\leq \int_{t-h_1}^{t} \eta(s)^T Q_3 \eta(s)\, ds + h_1 \bar{\eta}(t)^T H \bar{\eta}(t) + 2\bar{\eta}(t)^T (U - P^T \begin{bmatrix} 0 \\ \hat{A}_1 \end{bmatrix})(X(t) - X(t-h(t)))$$

$$-2\bar{\eta}(t)^T P^T \begin{bmatrix} 0 \\ \hat{E}_4 \end{bmatrix} (\phi(e(t)) - \phi(e(t-h(t)))). \tag{18}$$

Using Assumption 2.1, we have

$$0 \leq -f(x(t))^T f(x(t)) + x(t)^T U_1^T U_1 x(t), \tag{19a}$$

$$0 \leq -f(x(t-h(t)))^T f(x(t-h(t))) + x(t-h(t))^T U_1^T U_1 x(t-h(t)), \tag{19b}$$

$$0 \leq -\phi(e(t))^T \phi(e(t)) + e(t)^T U_1^T U_1 e(t), \tag{19c}$$

$$0 \leq -\phi(e(t-h(t)))^T \phi(e(t-h(t))) + e(t-h(t))^T U_1^T U_1 e(t-h(t)) \tag{19d}$$

and

$$0 \leq -\psi(t,e(t))^T \psi(t,e(t)) + e(t)^T U_2^T U_2 e(t). \tag{19e}$$

Moreover, from the Leibniz–Newton formula, the following equation holds for any matrix M with an appropriate dimension

$$2\upsilon(t)^T M(X(t) - X(t-h(t)) - \int_{t-h(t)}^{t} \eta(s)\ ds) = 0, \tag{20}$$

where $M := col\{M_1, M_2, \cdots, M_9\}$ and $\upsilon(t) := col\{\bar{\eta}(t), X(t-h(t)), \eta(t-d(t)), f(x(t)), f(x(t-h(t))), \phi(x(t)), \phi(x(t-h(t))), \psi(t,e(t)), w(t)\}$.

By adding the right- and the left-hand sides of (19) and (20), respectively, to (17) and using the inequality (18), it follows that

$$J[X(t),w(t)] \leq \upsilon(t)^T(\Pi + h_1 M Q_4^{-1} M^T)\upsilon(t) - \int_{t-h_1}^{t-h(t)} \eta(s)^T Q_4 \eta(s)\ ds$$

$$- \int_{t-h(t)}^{t} (\upsilon(t)^T M + \eta(s)^T Q_4) Q_4^{-1} (\upsilon(t)^T M + \eta(s)^T Q_4)^T\ ds, \tag{21}$$

where the matrix Π is given by

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & 0 \\ * & * & * & \Pi_{44} \end{bmatrix}$$

with

$$\Pi_{11} = sym\{P^T \bar{A}\} - sym\{P^T \begin{bmatrix} 0 \\ \hat{A}_1 \end{bmatrix} J - (U + M_1)J\} + h_1 H$$

$$+ \begin{bmatrix} Q_1 + \hat{C}_1^T \hat{C}_1 + J^T U_1^T U_1 J & 0 \\ * & Q_2 + h_1(Q_3 + Q_4) + \hat{J}^T(U_1^T U_1 + U_2^T U_2)\hat{J} \end{bmatrix},$$

$$\Pi_{12} = -U - M_1 + P^T \begin{bmatrix} 0 \\ \hat{A}_1 \end{bmatrix} + J^T M_2^T,$$

$$\Pi_{22} = -(1-h_2)Q_1 - sym\{M_2\} + J^T U_1^T U_1 J + \hat{J}^T U_1^T U_1 \hat{J}$$

$$\Pi_{13} = \begin{bmatrix} P^T \begin{bmatrix} 0 \\ \hat{A}_2 \end{bmatrix} + J^T M_3^T & P^T \begin{bmatrix} 0 \\ \hat{E}_1 \end{bmatrix} + J^T M_4^T & P^T \begin{bmatrix} 0 \\ \hat{E}_2 \end{bmatrix} + J^T M_5^T \end{bmatrix},$$

$$\Pi_{23} = \begin{bmatrix} -M_3^T & -M_4^T & -M_5^T \end{bmatrix}, \quad \Pi_{14} =$$

$$\begin{bmatrix} P^T \begin{bmatrix} 0 \\ \hat{E}_3 - \hat{E}_4 \end{bmatrix} + J^T M_6^T & P^T \begin{bmatrix} 0 \\ \hat{E}_4 \end{bmatrix} + J^T M_7^T & P^T \begin{bmatrix} 0 \\ \hat{G} \end{bmatrix} + J^T M_8^T & P^T \begin{bmatrix} 0 \\ \hat{B} \end{bmatrix} + J^T M_9^T \end{bmatrix},$$

$$\Pi_{24} = \begin{bmatrix} -M_6^T & -M_7^T & -M_8^T & -M_9^T \end{bmatrix},$$

$$\Pi_{33} = diag\{-(1-d_2)Q_2, -I, -I\},\ \Pi_{44} = diag\{-I, -I, -I, -\gamma^2 I\}.$$

Thus, if the inequality

$$\Pi + h_1 M Q_4^{-1} M^T < 0 \tag{22}$$

holds, it follows from $J[X(t),w(t)]|_{w(t)\equiv 0} \leq 0$ that $\frac{d}{dt}V(t) \leq 0$ or $V(t) \leq V(0)$. Then, from (9), it can be deduced

$$V(0) = X(0)^T P_1 X(0) + \int_{-h(0)}^{0} X(s)^T Q_1 X(s)\ ds + \int_{-d(0)}^{0} \eta(s)^T Q_2 \eta(s)\ ds$$

$$+ \int_{-h_1}^{0} \int_{s}^{0} \eta(\theta)^T (Q_3 + Q_4) \eta(\theta)\ d\theta\, ds$$

$$\leq \lambda_{max}(P_1)\|\varphi\|_2^2 + \lambda_{max}(Q_1) \int_{-h(0)}^{0} X(s)^T X(s)\, ds + \lambda_{max}(Q_2) \int_{-d(0)}^{0} \eta(s)^T \eta(s)\, ds$$

$$+\lambda_{max}(Q_3+Q_4)\int_{-h_1}^{0}\int_{s}^{0}\eta(\theta)^T\eta(\theta)\,d\theta\,ds \le \sigma_1\|\varphi\|_2^2+\sigma_2\|\eta\|_2^2,$$

where $\sigma_1 := \lambda_{max}(P_1) + h_1\lambda_{max}(Q_1)$ and $\sigma_2 := d_1\lambda_{max}(Q_2) + 0.5h_1^2\lambda_{max}(Q_3 + Q_4)$. Then, we have:

$$\lambda_{min}(P_1)\|\varphi\|_2^2 \le V(t) \le \sigma_1\|\varphi\|_2^2+\sigma_2\|\eta\|_2^2.$$

Therefore, we conclude that the filtering error system (4) is asymptotically stable. Notice that the matrix inequality (22) includes multiplication of filter matrices and Lyapunov matrices which are unknown and occur in nonlinear fashion. Hence, the inequality (22) cannot be considered an LMI problem. In the literature, more attention has been paid to the problems having this nature, which called bilinear matrix inequality (BMI) problems [43]. In the following, it is shown that, by considering $P_3 = \epsilon P_2$, where

$$P_2 = \begin{bmatrix} P_{11} & P_{12} \\ P_{22} & P_{22} \end{bmatrix} \tag{23}$$

and introducing change of variables

$$\begin{bmatrix} W_1 & W_2 & W_3 & W_4 & W_5 & W_6 \end{bmatrix} := P_{22}^T \begin{bmatrix} F & F_1 & F_2 & F_3 & F_4 & G \end{bmatrix} \tag{24}$$

the matrix inequality (22) is converted into LMI (11a) and can be solved via convex optimization algorithms. It is also easy to see that the inequality (22) implies $\Pi_{11} < 0$. Hence by Proposition 4.2 in the reference [19], the matrix P is nonsingular. Then, according to the structure of the matrix P in (10), the matrix P_2 (or P_{22}) is also nonsingular. This completes the proof.

Remark 3.1 It is worth noting that in the case when $x(t) \in \Re^n$, $w(t) \in \Re^s$, $z(t) \in \Re^z$ and $y(t) \in \Re^p$, the number of the variables to be determined in the LMIs (11) is $0.5n(17n + 2p + 2z + 5) + 5$. It is also observed that the LMIs (11) are linear in the set of matrices $P_{11}, P_{12}, P_{22}, G_1, H, U, \{W_i\}_{i=1}^6, \{M_i\}_{i=1}^9, P_1, \{Q_i\}_{i=1}^4$, and the scalars ϵ, γ^2. This implies that the scalar γ^2 can be included as one of the optimization variables in LMIs (11) to obtain the minimum disturbance attenuation level. Then, the optimal solution to the delay-dependent H_∞ filtering can be found by solving the following convex optimization problem

$$\min \lambda$$
$$\textit{subject to } (11) \textit{ with } \lambda := \gamma^2.$$

4 Simulation Results

In this section, we will verify the proposed methodology by giving an illustrative example. We solved LMIs (11) by using Matlab LMI Control Toolbox [44], which implements state-of-the-art interior-point algorithms and is significantly faster than classical convex optimization algorithms [45]. The example is given below.

Consider the system (1) with the following matrices

$$A = \begin{bmatrix} -1 & 0.5 \\ 0.3 & -2 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.5 & 0.1 \\ 0.1 & -0.6 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$E_1 = E_2 = I_2; C_1 = 10C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix},\ f(x(t)) = g(t, x(t)) = 0.5(|x(t)+1| - |x(t)-1|).$$

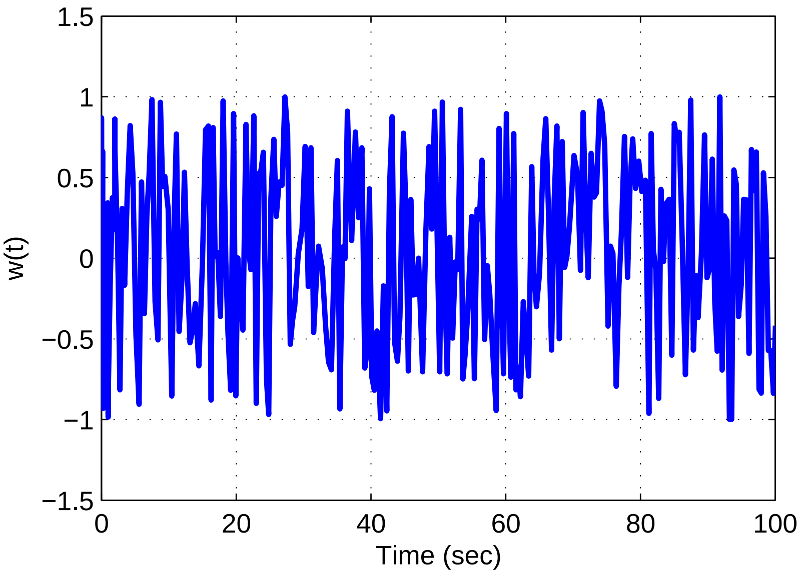

Figure 4.1: The disturbance signal.

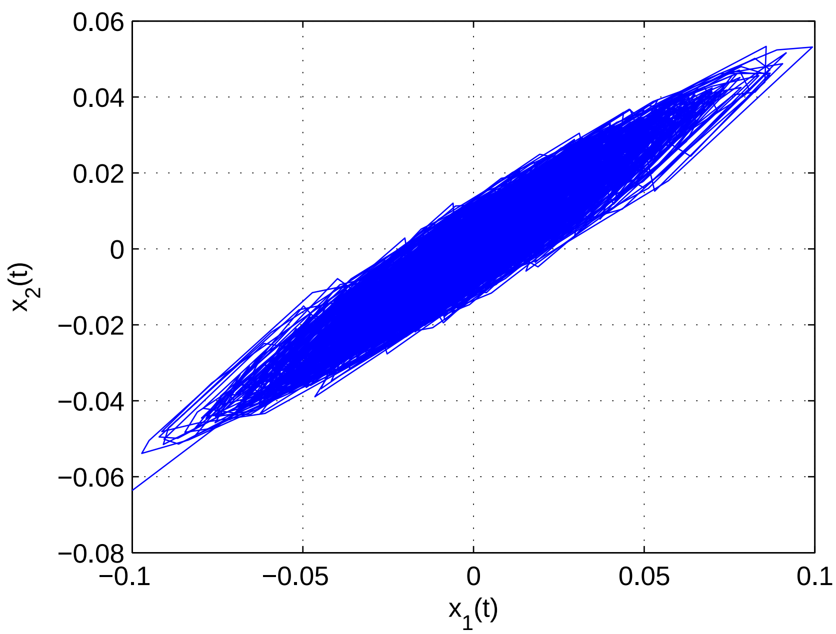

Figure 4.2: The phase trajectories.

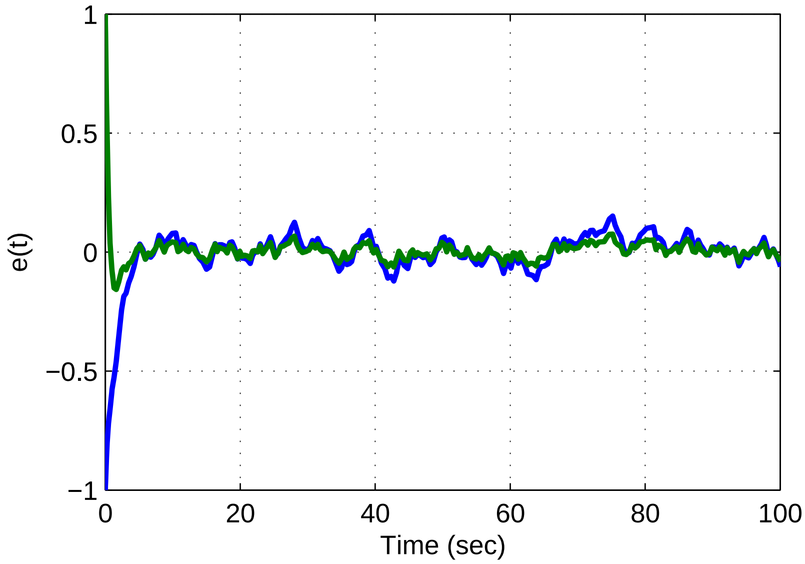

Figure 4.3: Curves of estimation error signal.

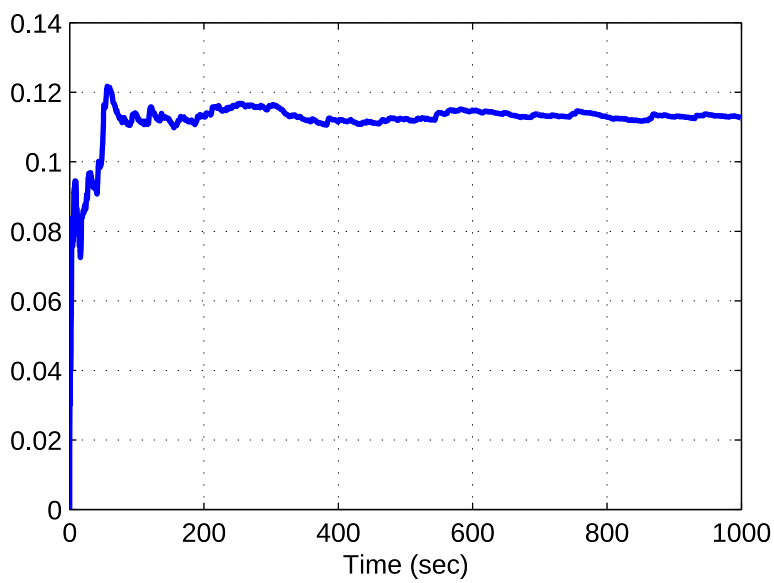

Figure 4.4: Curve of function $\|z(t) - \hat{z}(t)\|_2 / \|w(t)\|_2$.

The delays $h(t) = d(t) = (1 - e^{-t})/(1 + e^{-t})$ are time varying and satisfy $0 \le h(t) = d(t) \le 1$ and $\dot{h}(t) = \dot{d}(t) \le 0.5$. For simulation purposes, a uniformly distributed random signal, shown in Figure 4.1, with minimum and maximum -1 and 1, respectively, as the disturbance is imposed on the system. With the above parameters, the filtering error system (4) exhibits the chaotic behaviours such the state trajectories of the system with initial condition $x(0) = [0, 0]$ is depicted in Figure 4.2.

By solving the LMIs (11) in Theorem 3.1 with the disturbance attenuation $\gamma = 0.2$ we get the following state-space matrices of the delay-dependent H_∞ filter (2):

$$F = \begin{bmatrix} -2.8807 & 1.1770 \\ 1.0575 & -4.9106 \end{bmatrix},\ F_1 = \begin{bmatrix} -0.3991 & 0.2557 \\ 0.2297 & -0.7907 \end{bmatrix},\ F_2 = \begin{bmatrix} -0.0835 & -0.1410 \\ 0.0209 & -0.1002 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1.5747 & -0.4885 \\ -0.3693 & 2.7097 \end{bmatrix},\ F_4 = \begin{bmatrix} 1.1810 & -0.3664 \\ -0.2770 & 2.0323 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.0226 \\ -0.0662 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.5414 & 0.4628 \end{bmatrix}.$$

For initial conditions $x(0) = [-1, 1]$, the simulation results are shown in Figures 4.3 and 4.4. The trajectories of the estimation error are plotted in Figure 4.3. Finally, to observe the H_∞ performance, curve of the function $\|z(t) - \hat{z}(t)\|_2/\|w(t)\|_2$ is depicted in Figure 4.4 which shows that the H_∞ constraint in (8) is satisfied as well.

5 Conclusion

The problem of delay-dependent H_∞ filtering was proposed for a class of nonlinear neutral systems with delayed states and outputs. New required sufficient conditions were established in terms of delay-dependent LMIs for the existence of the desired robust H_∞ filters. The explicit expression of the robust H_∞ filters was derived to satisfy both asymptotic stability and a prescribed level of disturbance attenuation for all admissible known nonlinear functions. A numerical example was presented to illustrate the effectiveness of the designed filter.

References

[1] Malek–Zavarei, M. and Jamshidi, M. *Time-delay systems: Analysis, optimisation and application*. Amsterdam, The Netherlands, North–Holland, 1987.

[2] Niculescu, S.I. *Delay Effects on Stability: A Robust Control Approach*. Berlin, Springer, 2001.

[3] MacDonald, N. *Biological delay systems: linear stability theory*. Cambridge University Press, Cambridge, 1989.

[4] Kuang, Y. *Delay differential equations with applications in population dynamics*. Acadelic Press, Boston, 1993.

[5] Krasovskii, N.N. *Stability of motion*. Stanford, CA, Stanford University Press, 1963.

[6] Lou, X.Y. and Cui, B.T. Output feedback passive control of neutral systems with time-varying delays in state and control input. *Nonlinear Dynamics and Systems Theory* **8** (2) (2008) 195–204.

[7] Bahuguna, D. and Dabas, J. Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition. *Nonlinear Dynamics and Systems Theory* **8** (1) (2008) 7–19.

[8] Khusainov, D., Langerak, R. and Kuzmych O. Estimations of solutions convergence of hybrid systems consisting of linear equations with delay. *Nonlinear Dynamics and Systems Theory* **7** (2) (2007) 169–186.

[9] Cao, Y. and Cui, B. Existence and exponential stability of almost periodic solutions for a class of neural networks with variable delays. *Nonlinear Dynamics and Systems Theory* **8** (3) (2008) 287–298.

[10] Han, C.L. and Yu, L. Robust stability of linear neutral systems with nonlinear parameter perturbations. I*EE Proc. Control Theory Appl.* **151** (5) (2004) 539–546.

[11] Park, P. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Automatic Control* **44** (1999) 876–877.

[12] Fridman, E. New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters* **43** (2001) 309–319.

[13] Hu, G.D. and Hu, G.D. Some simple stability criteria of neutral delay-differential systems. *Applied Mathematics and Computation* **80** (1996) 257–271.

[14] Chen, W.H. and Zheng W.X. Delay-dependent robust stabilization for uncertain neutral systems with distributed delays. *Automatica* **43** (2007) 95–104.

[15] Chen, J.D., Lien, C.H., Fan, K.K. and Chou, J.H. Criteria for asymptotic stability of a class of neutral systems via a LMI approach. *IEE Proc. Control Theory and Applications* **148** (2001) 442–447.

[16] Yue, D., Won, S. and Kwon, O. Delay dependent stability of neutral systems with time delay: an LMI approach. *IEE Proc. Control Theory Appl.* **150** (1) (2003) 23–27.

[17] Xu, S., Lam, J. and Yang, C. Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay. *Systems & Control Letters* **43** (2001) 77–84.

[18] Chen, J.D. LMI-based robust H_∞ control of uncertain neutral systems with state and input Delays. *J. Optimization Theory and Applications* **126** (2005) 553–570.

[19] Fridman, E. and Shaked, U. Delay-dependent stability and H_∞ control: constant and time-varying delays. *Int. J. Control* **76** (2003) 48–60.

[20] Gao, H. and Wang, C. Comments and further results on 'A descriptor system approach to H_∞ control of linear time-delay systems'. *IEEE Transactions on Automatic control* **48** (2003) 520–525.

[21] Xu, S., Lam, J. and Yang, C. H_∞ and positive-real control for linear neutral delay systems. *IEEE Transactions on Automatic Control* **46** (2001) 1321–1326.

[22] Chen, J.D. LMI approach to robust delay-dependent mixed H_2/H_∞ controller of uncertain neutral systems with discrete and distributed time-varying delays. *J. Optimization Theory and Applications* **131** (3) (2006) 383–403.

[23] Xu, S., Lam, J. and Yang, C. Robust H_∞ control for uncertain linear neutral delay systems. *Optimal Control Applications and Methods* **23** (2002) 113–123.

[24] Xu, S., Chu, Y., Lu, J. and Zou, Y. Exponential dynamic output feedback controller design for stochastic neutral systems with distributed delays. *IEEE Transactions on Systems, Man, and Cybernetics A: Systems and Humans* **36** (2006) 540–548.

[25] Chen, B., Lam, J. and Xu, S. Memory state feedback guaranteed cost control for neutral systems. *Int. J. Innovative Computing, Information and Control* **2** (2) (2006) 293–303.

[26] Karimi, H.R., Zapateiro, M. and Lou, N. Robust mixed H_2/H_∞ delayed state-feedback control of neutral delay systems with time-varying delays. *Asian Journal of Control* **10** (5)(2008) 569–580.

[27] Lien, C.H. Guaranteed cost observer-based controls for a class of uncertain neutral time-delay systems. *J. Optimization Theory and Applications* **126** (1) (2005) 137–156.

[28] Lien, C.H. H_∞ observer-based control for a class of uncertain neutral time-delay systems via LMI optimization approach. *J. Optimization Theory and Applications* **127** (1) (2005) 129–144.

[29] Karimi, H.R. Observer-based mixed H_2/H_∞ control design for linear systems with time-varying delays: An LMI approach. *Int. J. of Control, Automation and Systems* **6** (1) (2008) 1–14.

[30] Anderson, B.D.O. and Moore, J.B. *Optimal filtering*. Englewood Cliffs, NJ, Prentice-Hall, 1979.

[31] Li, H. and Yang, C. Robust H_∞ filtering for uncertain linear neutral delay systems. *Proc. ACC* (2006) 2251–2255.

[32] Gao, H. and Wang, C. Delay-dependent robust H_∞ and $L_2 - L_\infty$ filtering for a class of uncertain nonlinear time-delay systems. *IEEE Trans. Automatic Control* **48** (9) (2003) 1661–1666.

[33] Gao, H. and Wang, C. A delay-dependent approach to robust H_∞ filtering for uncertain discrete-time state-delayed systems. *IEEE Trans. Signal Processing* **52** (6) (2004) 1631–1640.

[34] Germoel, J.C. and de Oliveira, M.C. H_2 and H_∞ robust filtering for convex bounded uncertain systems. *IEEE Trans. Automatic Control* **46** (1) (2001) 100–107.

[35] Guo, L., Yang F. and Fang, J. Multiobjective filtering for nonlinear time-delay systems with nonzero initial conditions based on convex optimization. *Circuits Systems Signal Processing* **25** (5) (2006) 591–607.

[36] Nagpal, K.M. and Khargonekar, P.P. Filtering and smoothing in an H_∞ setting. *IEEE Transactions on Automatic Control* **36** (1991) 152–166.

[37] de Souza, C.E., Xie, L. and Wang, Y. H_∞ filtering for a class of uncertain nonlinear systems. *Systems & Control Letter* **20** (1993) 419–426.

[38] Jin, S.H. and Park, J.B. Robust H_∞ filtering for polytopic uncertain systems via convex optimisation. *IEE Proceedings-Control Theory Applications* **148** (2001) 55–59.

[39] Karimi, H.R., Lohmann, B. and Buskens C. An LMI approach to H_∞ filtering for linear parameter-varying systems with delayed states and outputs. *Nonlinear Dynamics and Systems Theory* **7** (4) (2007) 351–368.

[40] de Souza, C.E., Palhares, R.M. and Peres, P.L.D. Robust H_∞ filter design for uncertain linear systems with multiple time-varying state delays. *IEEE Transactions on Signal Processing* **49** (2001) 569–576.

[41] Xu, S. Robust H_∞ filtering for a class of discrete-time uncertain nonlinear systems with state delay. *IEEE Transactions on Circuits Systems* I **49** (2002) 1853–1859.

[42] Zhou, K. and Khargonekar, P.P. Robust stabilization of linear systems with norm-bounded time-varying uncertainty. *System Control Letters* **10** (1988) 17–20.

[43] Safonov, M.G., Goh, K.C. and Ly, J.H. Control system synthesis via bilinear matrix inequalities. *Proc. ACC* (1994) 45–49.

[44] Gahinet, P., Nemirovsky, A., Laub, A.J. and Chilali, M. *LMI control Toolbox: For use with Matlab*. Natik, MA, The MATH Works, Inc., 1995.

[45] Boyd, S., Ghaoui, E.L., Feron, E. and Balakrishnan, V. *Linear matrix inequalities in systems and control theory*. Studies in Applied Mathematics. SIAM, Philadelphia, Pennsylvania, 15, 1994.

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Oscillation of Solutions and Behavior of the Nonoscillatory Solutions of Second-order Nonlinear Functional Equations

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Abstract: The aim of this study is to present new oscillation theorems for certain classes of second-order nonlinear functional differential equations of the type

$$x''(t) + p(t)f(x(t),\, x(\tau(t))) = 0, \qquad (*)$$
$$x''(t) + p_1(t)f_1(t,\, x(t),\, x'(t))x'(t) + q(t)g_1(x(\tau(t)) = 0,\ t \in [t_0,\, \infty),\, t_0 > 0.$$

In the study of Eq. $(*)$, no sign condition on $p(t)$ is explicitly assumed. Also, we study the behavior of the nonoscillatory solution of Eq. $(*)$.

Keywords: *nonlinear; functional differential equations; oscillatory solution; nonoscillatory solution.*

Mathematics Subject Classification (2000): 34K11, 34K12, 34C10.

1 Introduction

Over the last three decades, many studies have dealt with the oscillation theory for functional differential equations. For an excellent bibliography and later developments of this theory, we refer to the books by Agarwal, Bohner and Wan–Tong Li [1], Erbe, Kong and Zhang [3], Gopalsamy [4], Györi and Ladas [6], Ladde, Lakshmikantham and Zhang [10]. In this note, we consider the second-order nonlinear functional differential equations of the form

$$x''(t) + p(t)f(x(t),\, x(\tau(t))) = 0, \tag{1.1}$$
$$x''(t) + p_1(t)f_1(t,\, x(t),\, x'(t))x'(t) + q(t)g_1(x(\tau(t)) = 0,\ t \in [t_0,\, \infty), \tag{1.2}$$

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where $p \in C([t_0, \infty), \mathbb{R}), p_1, q \in C([t_0, \infty), \mathbb{R}^+)$, $f \in C(\mathbb{R}^2, \mathbb{R})$, $f_1 \in C([t_0, \infty) \times \mathbb{R}^2, \mathbb{R}^+)$, $g_1 \in C(\mathbb{R}, \mathbb{R})$, $yg_1(y) > 0$, $\forall\, 0 \neq y \in \mathbb{R}, \tau \in C^1([t_0, \infty), \mathbb{R}^+)$, $\tau'(t) > 0$ for all large t and $\tau(t) \to \infty$ as $t \to \infty$. In case $p(t)$ is positive, the oscillation criteria for Eq. (1.1) and its special case

$$x''(t) + p(t)F_1(x(\tau(t))) = 0,\ t \in [t_0, \infty)$$

is extensively studied by many investigators in this area (see, [7, 8, 13–15] and the references cited therein). All of them restrict the sign condition on $p(t)$; i.e., $p(t) \geq 0$, $\forall\, t \in [t_0, \infty)$. For the oscillation of Eq. (1.1), our study is free from such restriction. Also, as far as the author knows there is no oscillation result in literature for Eq. (1.2). The ideas of [2] are used to extend the oscillation results for Eq. (1.2). Let $\psi : [\tau(t_0), t_0] \to \mathbb{R}$ is a continuous function. By a solution of Eq. (1.1) (resp. Eq. (1.2)), we mean a continuously differentiable function $x : [\tau(t_0), \infty] \to \mathbb{R}$ such that $x(t) = \psi(t)$ for $\tau(t_0) < t_0$ and x satisfies Eq. (1.1) (resp. Eq. (1.2)) $\forall\, t \geq t_0$. We restrict our discussion to the nontrivial solutions of Eq. (1.1) (resp. Eq. (1.2)). A nontrivial solution of Eq. (1.1) (resp. Eq. (1.2)) is said to be oscillatory if it has arbitrarily large zeros, i.e., for any $T_1 > t_0$, $\exists\, t \geq T_1$ such that $x(t) = 0$, otherwise the solution is said to be nonoscillatory.

The paper is organized as follows. Section 2 deals with the oscillation theorems for Eqs. (1.1) and (1.2). The behavior of nonoscillatory solution of Eq. (1.1) is discussed in Section 3. In Section 4, we construct some examples for the illustration of these results.

2 Oscillation Theorems

We begin this section with the list of hypotheses:

(H1) $p(t) > 0$ for t sufficiently large.

(H2) $f(y_1, y_2) > 0$ if $y_i > 0$; $f(y_1, y_2) < 0$ if $y_i < 0$, $\forall\, i = 1, 2$.

(H3) $f(y_1, y_2)$ is a continuously differentiable function w. r. t. y_1 and y_2 and

suppose there exists $\alpha > 0$ such that $\frac{\partial}{\partial y_i} f(y_1, y_2) \geq \alpha$ for $y_i \neq 0, \forall\, i = 1, 2$.

(H4) There exist a C^1 function u defined on $[t_0, \infty)$, a C^1 function F on $\mathbb{R}$ and

a continuous function J on $\mathbb{R}$ such that $F'(u) = \sqrt{\alpha} J(u)$, $F(u) \geq \frac{(J(u))^2}{4}$.

(H5) $\liminf\limits_{t\to\infty} \frac{1}{t} \int\limits_{t_0}^{t} [(u'(s))^2 - p(s)F(u(s))]ds < 0.$

(H6) Let $U = \{(t, s) \in [t_0, \infty) \times [t_0, \infty)$ such that $t > s \geq 0\}$.

There exists a function $G \in C(U, \mathbb{R})$ such that $G(t, s) > 0$,

$\frac{\partial}{\partial s} G(t, s) \leq 0$ on U and $G(t, t) = 0$, $\forall\, t \geq t_0$.

(H7) Let there exist $h \in C^1(([t_0, \infty), (0, \infty))$ such that $h'(t) \leq 0, \forall t \in [t_0, \infty)$ and

(i) $\int\limits_{t_0}^{\infty} q(s)h(s)ds = \infty$.

(ii) $\limsup\limits_{t\to\infty} \frac{1}{G(t,\, t^*)} \int\limits_{t_0}^{t} G(t,\, s)q(s)h(s)ds = \infty, \;\; \forall\, t^* \geq t_0$.

(H8) Let there exist $h \in C^1([t_0,\, \infty),\, (0,\, \infty))$ such that $-\infty < \int\limits_{t_0}^{\infty} \frac{h'(t)}{h(t)} dt < \infty$ and $\int\limits_{t_0}^{\infty} q(t)h(t) \exp^{-\int\limits_{t^*}^{t} \frac{h'(s)}{h(s)}ds} dt = \infty$ for some $t^* > t_0$.

(H9) $g_1 \in C^1(B,\, \mathbb{R})$ such that $yg_1(y) > 0, \;\; \forall\, 0 \neq y \in \mathbb{R}$ and $\exists\, \beta > 0$ such that $g_1'(y) \geq \beta > 0, \;\; \forall\, 0 \neq y \in B,$ where $B = (-\infty,\, -N) \cup (N,\, \infty), \;\; N > 0$.

(H10) $\int\limits_{t_0}^{\infty} \left(\int\limits_{u}^{\infty} q(s)ds \right) du = \infty$.

Remark 2.1 Hypotheses (H4), (H5) are the extension of the conditions introduced by V. Komkov [9] and (H9), (H10) are given by Baculíková [2].

Lemma 2.1 *Let x be a nonoscillatory solution of (1.1) on $[T,\, \infty)$ and let* (H1)–(H3) *hold. Then for all large t, we have $x(t)x'(t) > 0$.*

Proof Without any loss of generality, this solution can be supposed to be such that $x(t) > 0$ for $t \geq T_1 \geq T$. Further, we observe that the substitution $u = -x$ transforms (1.1) into the Eq.

$$u''(t) + p(t)\bar{f}(u(t),\, u(\tau(t))) = 0, \tag{2.1}$$

where $\bar{f}(u_1,\, u_2) = -f(-u_1,\, -u_2)$. The function $\bar{f}$ is subject to the same conditions as f. So, there is no loss of generality to restrict our discussion to the case when the solution x is positive on $[T_1,\, \infty)$. If this lemma is not true, then either $x'(t) < 0$ for all large t or $x'(t)$ oscillates. By (H1), we choose T_1 sufficiently large so that $p(t) > 0, \;\; x'(t) < 0, \;\; \forall\, t \geq T_1$. This implies that

$$\int_{T_1}^{t} p(s)ds \geq 0, \;\; \text{and } x'(\tau(t)) < 0, \;\; \forall\, t \geq T_1.$$

Hence, we have

$$\int_{T_1}^{t} p(s)f(x(s),\, x(\tau(s)))ds = f(x(t),\, x(\tau(t)))\int_{T_1}^{t} p(s)ds - \int_{T_1}^{t} \big(\frac{\partial}{\partial x(s)} f(x(s),\, x(\tau(s))))x'(s)$$

$$+ \frac{\partial}{\partial x(\tau(s))} f(x(s),\, x(\tau(s))))x'(\tau(s))\tau'(s)\big) \left(\int_{T_1}^{s} p(\sigma)d\sigma \right) ds \geq 0, \forall\, t \geq T_1.$$

Now integrating (1.1), we get

$$x'(t) \leq x'(T_1) < 0, \;\; \forall\, t \geq T_1,$$

which contradicts the fact that $x(t)$ is nonoscillatory.

If $x'(t)$ is oscillatory. Then $\exists\, \{t_n\} \subset [t_0,\, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $x'(t_n) = 0, \forall\, n \in \mathbb{N}$. Let $\hat{t} > T_1$ be the zero of x'. This implies that $x'(\hat{t}) = 0, \;\; x''(\hat{t}) < 0$, from which one can prove that x' can not have another zero after it vanishes for large t, which is a contradiction. This completes the proof of the lemma.

Remark 2.2 For a lemma, similar to Lemma 2.1 under a similar hypothesis, we refer the reader to [11].

Theorem 2.1 *Under the hypotheses* (H1)–(H5)*, Eq. (1.1) is oscillatory.*

Proof Suppose on the contrary, (1.1) has a nonoscillatory solution $x(t)$. Then there exists some $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0,\ \forall t \geq t_1$.

Case 1. $x(t) > 0, \forall t \geq t_1$. By Lemma 2.1, we have $x(t)x'(t) > 0$, for all large t. So, we choose a T sufficiently large such that $x(t)x'(t) > 0,\ \forall t \geq T$. This implies that $x'(\tau(t)) > 0,\ \forall t \geq T$. Now we note that the following identity is valid on $[T, \infty)$:

$$\begin{aligned}
&(u'(t))^2 - p(t)F(u(t)) + \frac{F(u(t))}{f(x(t),\, x(\tau(t)))}[x''(t) + p(t)f(x(t),\, x(\tau(t)))] \\
&= \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \frac{(\frac{\partial}{\partial x(\tau(t))} f(x(t),\, x(\tau(t))))\ x'(t)x'(\tau(t))\tau'(t)F(u(t))}{(f(x(t),\, x(\tau(t)))^2} \\
&+ \frac{(\frac{\partial}{\partial x(t)} f(x(t),\, x(\tau(t))))x'(t)x'(t)F(u(t))}{(f(x(t),\, x(\tau(t))))^2} - \left(\frac{x'(t)F'(u(t))u'(t)}{f(x(t),\, x(\tau(t)))}\right) + (u'(t))^2.
\end{aligned}$$

$$\begin{aligned}
&(u'(t))^2 - p(t)F(u(t)) + \frac{F(u(t))}{f(x(t),\, x(\tau(t)))}\,[x''(t) + p(t)f(x(t),\, x(\tau(t)))] \\
&\geq \left(\frac{x'(t)F(u(t))}{f(x(t), x(\tau(t)))}\right)' - \left(\frac{x'(t)\sqrt{\alpha}\, J(u(t))u'(t)}{f(x(t),\, x(\tau(t)))}\right) + \frac{\alpha(x'(t))^2(J(u(t)))^2}{4(f(x(t),\, x(\tau(t))))^2} + (u'(t))^2 \\
&\geq \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \left[u'(t) - \frac{x'(t)\sqrt{\alpha}\, J(u(t))}{2f(x(t),\, x(\tau(t)))}\right]^2.
\end{aligned}$$

Since x being a solution of (1.1), so, we get

$$(u'(t))^2 - p(t)F(u(t)) \geq \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \left[u'(t) - \frac{x'(t)\sqrt{\alpha}\ J(u(t))}{2f(x(t),\, x(\tau(t)))}\right]^2.$$

An integration over $[T, \infty)$ yields

$$\begin{aligned}
&\int_T^t [(u'(s))^2 - p(s)F(u(s))]ds \\
&\geq \int_T^t \left(\frac{x'(s)F(u(s))}{f(x(s),\, x(\tau(s)))}\right)' ds \\
&\geq \frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))} - \frac{x'(T)F(u(T))}{f(x(T),\, x(\tau(T)))}.
\end{aligned}$$

So,

$$\frac{1}{t}\int_T^t [(u'(s))^2 - p(s)F(u(s))]ds \geq -\frac{1}{t}\frac{x'(T)F(u(T))}{f(x(T),\, x(\tau(T)))} \to 0 \text{ as } t \to \infty,$$

which contradicts to (H5).

Case 2. $x(t) < 0, \forall t \geq t_1$. For large t, we have, $x(t) < 0,\ x(\tau(t)) < 0,\ \forall t \geq T$, where T is sufficiently large. By Lemma 2.1, we have $x'(t) < 0,\ \forall t \geq T$. Now the rest of the proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

The next lemma is used in the proof of the next theorems.

Lemma 2.2 *Let $p_1(t) \geq 0$ and $q(t)$ be continuous non-negative and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geq t_0$ and assume that*

(i) $f_1(t, x, y) \leq \mid y \mid^\lambda$, $-\infty < x, y < \infty$, $t \geq t_0$ *and some constant* $\lambda \geq 0$.

(ii) $\left(1 + \int_{t_0}^t p_1(s)ds\right)^{-\frac{1}{\lambda}} \notin L(t_0, \infty)$, *if* $\lambda > 0$,

$\int_{t_0}^\infty \exp\left(\int_{t_0}^s -p_1(\sigma)d\sigma\right) ds = \infty$, *if* $\lambda = 0$.

If $x(t)$ is a non-oscillatory solution of Eq. (1.2), then $x(t)x'(t) > 0$ for all large t.

For the proof of this lemma, we refer the reader to [5].

Theorem 2.2 *Let $p_1(t) \geq 0$ and $q(t)$ be continuous non-negative and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geq t_0$. Let $\tau(t) < t$, for large t. Let the conditions* (i), (ii) *hold. Then under the hypotheses* (H8)–(H10), *Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$. Then there exists some $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$, $\forall\, t \geq t_1$.

Case 1. $x(t) > 0$, $\forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0$, $\forall\, t \geq T$, where $T > t_0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))}, \ \forall\, t \geq T, \tag{2.2}$$

where h is appearing in (H8). Differentiating $w(t)$ and by Eq. (1.2), we get

$$\begin{aligned} w'(t) &= \frac{-h(t)p_1(t)x'(t)f_1(t, x(t), x'(t))}{g_1(x(\tau(t)))} - q(t)h(t) + \frac{x'(t)h'(t)}{g_1(x(\tau(t)))} \\ &\quad - \frac{x'(t)g_1'(x(\tau(t)))x'(\tau(t))\tau'(t)h(t)}{(g_1(x(\tau(t))))^2} \\ &\leq -q(t)h(t) - \frac{w(t)g_1'(x(\tau(t)))x'(\tau(t))\tau'(t)}{g_1(x(\tau(t)))} + \frac{h'(t)w(t)}{h(t)}. \end{aligned}$$

Since x' is a decreasing function for $t \geq T$ and $\tau(t) < t$. So,

$$w'(t) \leq -q(t)h(t) - \frac{(w(t))^2 g_1'(x(\tau(t)))\tau'(t)}{h(t)} + \frac{h'(t)w(t)}{h(t)}. \tag{2.3}$$

Now we claim that $x(t) \to \infty$ as $t \to \infty$. Suppose not, then $0 < x(t) \leq M < \infty$, as $t \to \infty$. We may also assume that $0 < x(\tau(t)) \leq M < \infty$, as $t \to \infty$. Since $x'(t)$ is positive and decreasing, so $\lim_{t\to\infty} x'(t)$ exists and is finite. An integration of Eq. (1.2) from t to ∞, yields

$$\int_t^\infty x''(s)ds = -\int_t^\infty p(s)f_1(s, x(s), x'(s))x'(s)ds - \int_t^\infty q(s)g_1(x(\tau(s)))ds, \ t \geq T.$$

This implies that $x'(\infty) - x'(t) \leq -\int\limits_t^\infty q(s)g_1(x(\tau(s)))ds$ or

$$x'(t) \geq \int_t^\infty q(s)g_1(x(\tau(s)))ds, \ t \geq T. \tag{2.4}$$

Let

$$\delta = \min_{u \in [L,\, M]} g_1(u)$$

for some $L > 0$. Then $0 < \delta \leq g_1(x(\tau(s)))$. From inequality (2.4), we get

$$x'(t) \geq \delta \int_t^\infty q(s)ds.$$

An integration over $(t_0,\, t)$ of the above inequality yields

$$x(t) \geq x(0) + \delta \int_{t_0}^t \left(\int_u^\infty q(s)ds \right) du.$$

Letting $t \to \infty$ in above inequality, we get a contradiction from (H10). So, our claim is true and hence $x(\tau(t)) \in B$ for all large t. Now from (2.3) and (H9), we get

$$w'(t) \leq -q(t)h(t) - \frac{(w(t))^2 \beta \tau'(t)}{h(t)} + \frac{h'(t)w(t)}{h(t)} \leq -q(t)h(t) + \frac{h'(t)w(t)}{h(t)}. \tag{2.5}$$

From inequality (2.5), we get

$$w(t) \leq w(T_1) \exp^{-\int_T^{T_1} \frac{h'(s)}{h(s)} ds} \exp^{\int_T^t \frac{h'(s)}{h(s)} ds} - \exp^{\int_T^t \frac{h'(s)}{h(s)} ds} \int_{T_1}^t q(s)h(s) \exp^{-\int_T^s \frac{h'(u)}{h(u)} du} ds, \tag{2.6}$$

where $t \geq T_1 > T$. Letting $t \to \infty$, from (H8), we get $w(t) \to -\infty$, which is a contradiction as $w(t) > 0$.

Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

Theorem 2.3 *Let* (H8) *be replaced by* (H7(i)) *in Theorem 2.2. Then Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$. As in the foregoing text, there exists some $t_1 \geq 0$ such that either $x(t) > 0$ or $x(t) < 0,\ \forall\, t \geq t_1$.
Case 1. $x(t) > 0,\ \forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0,\ \forall\, t \geq T$, where $T > 0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))},\ \forall\, t \geq T, \tag{2.7}$$

where h is appearing in (H7). As in the proof of Theorem 2.2, we have Inequality (2.5)

$$w'(t) \leq -q(t)h(t) + \frac{h'(t)w(t)}{h(t)}.$$

In view of (H7), we get

$$w'(t) \leq -q(t)h(t). \tag{2.8}$$

An integration over $(T,\, \infty)$ yields

$$w(t) \leq w(T) - \int_T^t q(s)h(s)ds.$$

Letting $t \to \infty$ in above inequality, we get a contradiction from (H7(i)).
Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and we omit the proof. This completes the proof of the theorem.

Theorem 2.4 *Let* (H6) *hold and suppose* (H8) *be replaced by* (H7(ii)) *in Theorem 2.2. Then Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$.
Case 1. $x(t) > 0,\ \forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0,\ \ \forall\, t \geq T$, where $T > 0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))},\ \forall\, t \geq T,$$

where h is appearing in (H7). From (2.8), we have

$$\begin{aligned}\int_T^t G(t,\, s)q(s)h(s)ds \qquad &\leq -G(t,\, t)w(t) + G(t,\, T)w(T) + \int_T^t \frac{\partial G(t,\, s)}{\partial s} w(s)ds \\ &\leq\ G(t,\, T)w(T),\end{aligned}$$

which implies that

$$\frac{1}{G(t,\, T)} \int_T^t G(t,\, s)q(s)h(s)ds \leq w(T).$$

Letting $t \to \infty$, we get a contradiction from (H7(ii)).
Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and hence is omitted.

Remark 2.3 Theorems 2.2, 2.3 and 2.4 can be applied to sublinear and superlinear equations as the boundedness of $g_1'(y)$ is not required near zero.

3 Behavior of Nonoscillatory Solutions

In this section, we study the behavior of nonoscillatory solutions of Eq. $(*)$. In fact, we study the behavior of nonoscillatory solutions of

$$x''(t) + P(t)f(x(t),\, x(\tau(t)))g(x'(t)) = 0, t \in [t_0,\, \infty), \tag{3.1}$$

where $P \in C([t_0,\, \infty),\, \mathbb{R}^+),\ f \in C(\mathbb{R}^2,\, \mathbb{R}),\ g \in C(\mathbb{R},\, \mathbb{R})$. Let there exist $k > 0, l > 0$ such that $\frac{f(x,\, y)}{x} \geq k > 0, \forall\, 0 \neq x \in \mathbb{R}, y \in \mathbb{R}$ and $g(y) \geq l > 0,\ y \in \mathbb{R}$. Let $\tau \in C([t_0,\, \infty),\, \mathbb{R})$. Let there exists $\mu > 0$. Consider the second-order linear differential equation

$$x''(t) + \lambda P(t)x(t) = 0,\ \lambda > 0. \tag{3.2}$$

We establish that all nonoscillatory solutions $x(t)$ of Eq. (3.1) are such that $y(t) = O(x(t))$ as $t \to \infty$, where y is any oscillatory solution of Eq. (3.2), $\forall\, \lambda \in (0,\, \mu]$. The technique of Philos et al. [12] is employed to establish the following result. This result gives a new direction in the study of nonoscillatory behavior of functional differential equations.

Theorem 3.1 *Let x be any nonoscillatory solution of Eq. (3.1) and y be an oscillatory solution of Eq.* (3.2)*. Then $y(t) = O(x(t))$ as $t \to \infty$.*

Proof Since x is any nonoscillatory solution of Eq. (3.1), so there exists some $T_0 \geq t_0$ such that $x(t) \neq 0,\ \forall\, t \geq T_0$. There are two cases.

Case 1. $x(t) > 0,\ \forall\, t \geq T_0$. We define

$$v(t) = \frac{y(t)}{x(t)}, \quad \forall\, t \geq T_0.$$

We obtain

$$v'(t) = \frac{y'(t) - v(t)x'(t)}{x(t)},\ \forall\, t \geq T_0$$

and

$$v''(t) = \frac{y''(t) - v(t)x''(t) - 2v'(t)x'(t)}{x(t)},\ \forall\, t \geq T_0. \tag{3.3}$$

From Eqs. (3.1), (3.2) and (3.3), we get

$$v''(t) = -\frac{2v'(t)x'(t)}{x(t)} + \frac{-\lambda P(t)y(t)}{x(t)} + \frac{v(t)P(t)f(x(t),\, x(\tau(t)))g(x'(t))}{x(t)}. \tag{3.4}$$

Now we will show that v is bounded on the interval $[T_0,\, \infty)$. Assume on the contrary that v is unbounded on $[T_0,\, \infty)$. As $-y$ is also an oscillatory solution of Eq. (3.2) and $-v = \frac{-y}{x}$ on $[T_0,\, \infty)$. We may suppose that v is unbounded from above. Clearly, v is oscillatory. Thus, we can choose a sufficiently large $T \geq T_0$ so that

$$v'(T) = 0,\, v(T) >\mid v(t) \mid \ \text{ for } T_0 \leq t < T \tag{3.5}$$

and $v''(T) \leq 0$, (see, [Thm. 2, 12]). In view of Eq. (3.5), from Eq. (3.4), we get

$$v(T)P(T)[f(x(t),\, x(\tau(t)))g(x'(T)) - \lambda x(T)] \leq 0.$$

That is,

$$f(x(t),\, x(\tau(t)))g(x'(T)) - \lambda x(T) \leq 0. \tag{3.6}$$

From the hypotheses, we get

$$\frac{f(x(T),\, x(\tau(T)))}{x(T)} \geq k > 0, \quad \text{and } g(x'(T)) \geq l > 0. \tag{3.7}$$

That is,

$$\frac{f(x(T),\, x(\tau(T)))g(x'(T)) - klx(T)}{x(T)} \geq 0.$$

We choose $\mu = kl$, since $\lambda \in (0,\, \mu]$, we obtain

$$\frac{f(x(T),\, x(\tau(T)))g(x'(T)) - \lambda x(T)}{x(T)} \geq 0. \tag{3.8}$$

Eqs. (3.6) and (3.8) implies that $x(T) \leq 0$, which is a contradiction.

Case 2. $x(t) < 0,\ \forall\, t \geq T_0$. The proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

Remark 3.1 As a hypothesis, $''Eq.$ (3.2) is oscillatory $\forall\, \lambda > 0''$ is used by Lynn Erbe [11].

4 Examples

Finally, we give some examples to illustrate our results.

Example 4.1 Consider the differential equation

$$x''(t)+\left(1-\frac{\sin t}{t^2}\right)\left[x(t)+(x(t))^{2m+1}+x\left(\frac{t}{2}\right)+\left(x\left(\frac{t}{2}\right)\right)^{2n+1}\right]=0,\quad m,n\in\mathbb{N},\ t>0. \tag{4.1}$$

Eq. (4.1) can be viewed as Eq. (1.1) with $p(t)=1-\frac{\sin t}{t^2}$, $f(y_1,y_2)=y_1+y_1^{2m+1}+y_2+y_2^{2n+1}$, $\tau(t)=\frac{t}{2}$. With the choice of $\alpha=1$, $F(u)=u^2$, $u(t)=t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied. An application of Theorem 2.1 implies that (4.1) is oscillatory.

Remark 4.1 Here $p(t)\ngeq 0$, $\forall\, t\in[t_0,\infty)$, so none of the known criteria [8, 13, 14] can obtain this result to Eq. (4.1).

Example 4.2 Consider the differential equation

$$x''(t)+\left(e^{-t}+\frac{2}{t^2}+\frac{1}{t^4}\right)\left(x(t)+x\left(\frac{t}{3}\right)+x\left(\frac{t}{3}\right)^5\right)=0,\quad t>0. \tag{4.2}$$

Eq. (4.2) can be viewed as Eq. (1.1) with $p(t)=e^{-t}+\frac{2}{t^2}+\frac{1}{t^4}$, $f(y_1,y_2)=y_1+y_2+y_2^5$, $\tau(t)=\frac{t}{3}$. With the choice of $\alpha=1$, $F(u)=u^2$, $u(t)=t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied. An application of Theorem 2.1 implies that Eq. (4.2) is oscillatory, whereas none of the known criteria [8, 13, 14] can obtain this result to Eq. (4.2).

Example 4.3 Consider the differential equation

$$x''(t)+\frac{1}{t+1}x'(t)+\frac{1}{t^2}\left(\frac{\left(x\left(\frac{t}{3}\right)\right)^3}{\mid x\left(\frac{t}{3}\right)\mid+1}\right)=0,\quad t>0. \tag{4.3}$$

Eq. (4.3) can be viewed as Eq. (1.2) with $p_1(t)=\frac{1}{t+1}$, $f_1(t,x,y)=1$, $q(t)=\frac{1}{t^2}$, $g_1(y)=\frac{y^3}{|y|+1}$, $\tau(t)=\frac{t}{3}$. With the choice of $h(t)=1$, it is easy to see that the hypotheses of Theorem 2.2 are satisfied. So, by Theorem 2.2, Eq. (4.3) is oscillatory.

Example 4.4 Consider the differential equation

$$x''(t)+(x'(t))^2+e^t\left(x\left(\frac{t}{2}\right)\right)^3=0. \tag{4.4}$$

Eq. (4.4) can be viewed as Eq. (1.2) with $p_1(t)=1$, $f_1(t,x,y)=y$, $q(t)=e^t$, $g_1(y)=y^3$, $\tau(t)=\frac{t}{2}$. Since $f_1(t,x,y)=y$, so in view of Lemma 2.2(i), $\lambda=1$. With the choice of $h(t)=e^{-t}$, it is easy to see that the hypotheses of Theorem 2.3 are satisfied and by Theorem 2.3, Eq. (4.4) is oscillatory in view of Lemma 2.2(i).

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References

[1] Agarwal, R.P., Bohner, M. and Wan–Tong Li. Nonoscillation and oscillation: theory for functional differential equations. Marcel Dekker, New York, 2004.

[2] Blanka Baculíková. Oscillation criteria for second-order nonlinear differential equations, Arch. Math. (Brno) **42** (2006) 141–149.

[3] Erbe, L.H., Kong, Q. and Zhang, B.G. Oscillation theory for functional differential equations. Dekker, New York, 1995.

[4] Gopalsamy, K. Stability and oscillations in delay differential equations of population dynamics. Kluwer Academic, Dordrecht, 1992.

[5] Grace, S.R., Lalli, B.S. and Yeh, C.C. Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term. *SIAM J. Math. Anal.* **15** (1984) 1082–1093.

[6] Györi, I. and Ladas, G. Oscillation theory of delay differential equations with applications. Clarendon, Oxford, 1991.

[7] Hamedani, G.G. Oscillation theorems for second order functional differential equations *J. Math. Anal. Appl.* **135** (1988) 237–243.

[8] Bardley, J.S. Oscillation theorems for a second-order delay equation *J. Diff. Eqs.* **8** (1970) 397–403.

[9] Komkov, V. A generalization of Leighton's variational theorem *Applicable analysis* **1** (1972) 377–383.

[10] Ladde, G.S., Lakshmikantham, V. and Zhang, B.G. Oscillation theory of differential equations with deviating arguments. Dekker, New York, 1987.

[11] Lynn Erbe. Oscillation theorems for second-order nonlinear differential equations *Pro. Amer. Math. Soc.* **24** (1970) 811–814.

[12] Philos, Ch.G., Purnaras, I.K. and Sficas, Y.G. On the behavior of the oscillatory solutions of first or second order delay differential equations *J. Math. Anal. Appl.* **291** (2004) 764–774.

[13] Travis, C.C. Oscilllation theorems for second-order differential equations with functional arguments *Pro. Amer. Math. Soc.* **31** (1972) 199–202.

[14] Yeh, C.C. An oscilllation criterion for second-order nonlinear differential equations with functional arguments *J. Math. Anal. Appl.* **76** (1980) 72–76.

[15] Rogovchenko, Yu.V. Oscillation criteria for certain nonlinear differential equations *J. Math. Anal. Appl.* **229** (1999) 399–416.