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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

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Special Issue
on
Dynamic Equations on Time Scales: Qualitative Analysis and Applications

Research in the area of dynamic equations on time scales has followed a rapidly successful path since its beginning 20 years ago with Stefan Hilger's seminal work. A large part of this success is due to how topics involving the unification and extension of discrete and continuous dynamical systems have attracted the attention of researchers in a wide variety of scientific disciplines including pure and applied mathematics as well as biology, economics, engineering, and physics.

The purpose of this special issue is to appeal to that large cross-section of researchers by focusing on the interplay between dynamical systems theory and applications of time scales analysis. This collection of papers includes authors from all over the world each with a fundamentally new contribution to the field. While much progress has been made, there is an ever increasing supply of nontrivial questions that push the boundaries of the theory to new and exciting places. We hope that the readers of this special issue will find one or more papers here that resonate with them and in turn help drive the theory in another new direction.

This special issue on dynamic equations on time scales contains altogether fifteen carefully selected research papers, authored by thirty-one world-leading experts from eight different countries. Contributors are D. Anderson (USA), F. Atici (USA), E. Bairamov (Turkey), M. Benchohra (Algeria), F. Berhoun (Algeria), H. Berger (USA), J. DaCunha (USA), R. Dahal (USA), T. Doan (Germany), L. Erbe (USA), R. Ferreira (Portugal), S. Hamani (Algeria), T. Hassan (Egypt), J. Henderson (USA), Adil Huseynov (Turkey), Aydin Huseynov (Turkey), A. Kalauch (Germany), E. Kaufmann (USA), N. Kosmatov (USA), C. McMahan (USA), S. Ntouyas (Greece), R. Oberste-Vorth (USA), A. Ouahab (Algeria), A. Peterson (USA), I. Purnaras (Greece), S. Saker (Saudi Arabia), S. Siegmund (Germany), F. Topal (Turkey), D. Torres (Portugal), J. Weiss (USA), and A. Yantir (Turkey). We appreciate the contribution of each one above as well as the numerous referees whose behind the scenes work ensured the quality of this issue.

We are also grateful to Professor A.A. Martynyuk, Editor-in-Chief, for inviting us to guest edit this special issue.

May the next 20 years of research in dynamic equations on time scales be as successful as the first 20!

Martin Bohner[1] and John M. Davis[2],
Guest Editors

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List of papers

to be published in issues 2-4, 2009 as a continuation of the Special Issue on ***Dynamic Equations on Time Scales: Qualitative Analysis and Applications***

Dominant and Recessive Solutions of Self-Adjoint Matrix Systems on Time Scales (Submitted by J. Davis)
Douglas R. Anderson

Dynamic Inequalities, Bounds, and Stability of Systems with Linear and Nonlinear Perturbations (Submitted by J. Davis)
Jeffrey J. DaCunha

Some Linear and Nonlinear Integral Inequalities on Time Scales in Two Independent Variables (Submitted by M. Bohner)
R.A.C. Ferreira and D.F.M. Torres

Positive Solutions of Semipositone Singular Dirichlet Dynamic Boundary Value Problems (Submitted by M. Bohner)
R. Dahal

The Fell Topology for Dynamic Equations on Time Scales (Submitted by M. Bohner)
R.W. Oberste-Vorth

Positive Solutions of a Second Order m-Point BVP on Time Scales (Submitted by M. Bohner)
S. Gulsan Topal and Ahmet Yantir

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A Comparison in the Theory of Calculus of Variations on Time Scales with an Application to the Ramsey Model

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Abstract: The purpose of this paper is to provide a comparison between the existing results on the calculus of variations with the Δ and ∇ operators on time scales. We will also prove the theorems pertaining to free boundary conditions, for dynamic models missing one or both end points conditions. To illustrate our results we shall use a well known Ramsey model and an adjustment model in economics.

Keywords: *time scales; calculus of variations; nabla derivative; delta derivative; dynamic model.*

Mathematics Subject Classification (2000): 39J05, 49J15, 91B50, 91B62.

1 Introduction

The theory of calculus of variations on time scales has been developed in two directions, one with the Δ operator and one with the ∇ operator. It is possible to write one derivative in terms of the other derivative operator on time scales under certain continuity assumptions [4, Theorem 8.49]. On the other hand, it is not always possible to optimize the dynamic model on time scales by using the deterministic optimization method, namely calculus of variations, since the theory has been developed, to the authors' knowledge, for functionals of the form $\int_{[a,b]\cap\mathbb{T}} L(t, y(\sigma(t)), y^{\Delta}(t))\Delta t$ and $\int_{[a,b]\cap\mathbb{T}} N\left(t, y(\rho(t)), y^{\nabla}(t)\right)\nabla t$. As a result of this, here are some questions which need to be answered.

• Which is more advantageous using the Δ or ∇ derivative in dynamic modelling ?

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• Does there exist a dynamic model which can be formed with the Δ and ∇ operators, solved and compared ?
The aim of this paper is to answer these questions by illustrating some familiar discrete or continuous models from economics. The reader may find the papers [1, 2, 5, 6] interesting to see the development of the theory and some nice applications. We will start with listing two main theorems, one is obtained by Bohner in [5] for the Δ operator and the other one is obtained by Atıcı, Biles and Lebedinsky in [1] for the ∇ operator.

Let $\mathbb{T}$ be a time scale which is nonempty closed subset of reals $\mathbb{R}$. We refer to the books by Bohner and Peterson for further reading on time scales [3, 4].

Assume that $L(t,u,v)$ for each $t \in [\sigma(a), \sigma^2(b)] \subseteq \mathbb{T}$ is a class C^2_Δ function of (u,v). Let $y \in C^1_\Delta[a, \sigma^2(b)]$ with $y(\sigma(a)) = A$, $y(\sigma^2(b)) = B$, where

$$C^1_\Delta[a,\sigma^2(b)] = \{y : [a,\sigma^2(b)] \to \mathbb{R} \ \mid \ y^\Delta \text{ is continuous on } [a,\sigma^2(b)]^\kappa\}.$$

Theorem 1.1 *If a function $y(t)$ provides a local extremum to the functional*

$$J[y] = \int_{\sigma(a)}^{\sigma^2(b)} L(t, y(\sigma(t)), y^\Delta(t))\Delta t$$

where $y \in C^2_\Delta[a,\sigma^2(b)]$ and $y(\sigma(a)) = A$ and $y(\sigma^2(b)) = B$, then y must satisfy the Euler-Lagrange equation

$$L_{y^\sigma}(t, y^\sigma, y^\Delta) - L^\Delta_{y^\Delta}(t, y^\sigma, y^\Delta) = 0 \tag{1}$$

for $t \in [a, \sigma(b)]^\kappa_\kappa$.

Assume that $N(t,u,v)$ is a class C^2_∇ function of (u,v) for each $t \in [\rho^2(a), \rho(b)] \subseteq \mathbb{T}$. Let $y \in C^1_\nabla[\rho^2(a), \rho(b)]$ with $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, where

$$C^1_\nabla[\rho^2(a), \rho(b)] = \{y : [\rho^2(a), \rho(b)] \to \mathbb{R} \ \mid \ y^\nabla \text{ is continuous on } [\rho^2(a), \rho(b)]_\kappa\}.$$

Theorem 1.2 *If a function $y(t)$ provides a local extremum to the functional*

$$J[y] = \int_{\rho^2(a)}^{\rho(b)} N\left(t, y(\rho(t)), y^\nabla(t)\right) \nabla t$$

where $y \in C^2_\nabla\left[\rho^2(a), \rho(b)\right]$ and $y(\rho^2(a)) = A$, $y(\rho(b)) = B$, then y must satisfy the Euler-Lagrange equation

$$N_{y^\rho}\left(t, y^\rho, y^\nabla\right) - N^\nabla_{y^\nabla}\left(t, y^\rho, y^\nabla\right) = 0 \tag{2}$$

for $t \in [\rho(a), b]^\kappa_\kappa$.

The plan of this paper is as follows. In Section 2, we will introduce the Ramsey model [7] on time scales and write the model with the ∇ and Δ derivatives, respectively. We will then solve each model using the Euler-Lagrange equations (1) and (2). We shall then compare the solutions of each model on $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$. In Section 3, we shall state the free boundary conditions for the dynamic model which is missing one or two end-point conditions. We shall illustrate our results with an adjustment model [8].

2 The Ramsey Model

In this section, we study the Ramsey model which determines the behavior of saving/consumption as the result of optimal inter temporal choices by individual households. Before writing the model on time scales we will present its discrete and continuous versions so that one can see how the time scale model unifies its discrete and continuous counterparts.

Discrete Model:

We maximize the Ramsey model which is

$$\sum_{t=0}^{T-1}(1+p)^{-t}U[C_t]$$

subject to initial wealth W_0 that can always be invested for an exogeneously-given certain rate of yield r; or subject to the constraint

$$C_t = W_t - \frac{W_{t+1}}{1+r}, \tag{3}$$

or

$$\max_{[W_t]} \sum_{t=0}^{T-1}(1+p)^{-t}U\left[W_t - \frac{W_{t+1}}{1+r}\right],$$

where the quantities are defined as

C_t — consumption,
p — discount rate,
U_t — instantaneous utility function,
W_t — production function.

The Euler-Lagrange equation for the discrete model is as follows

$$\frac{r-p}{1+r}U'[C(t)] + \Delta[U'[C(t)]] = 0.$$

Continuous Model:

We maximize the Ramsey model

$$\int_0^T e^{-pt}U[C(t)]dt$$

subject to

$$C(t) = rW(t) - W'(t) \tag{4}$$

or

$$\max_{[W(t)]} \int_0^T e^{-pt}U[rW(t) - W'(t)]dt.$$

The Euler-Lagrange equation is as follows

$$(r-p)U'[C(t)]+[U'[C(t)]]'=0.$$

Let's now develop two formulations of the time scale Ramsey model, in order to employ both the nabla and delta calculus of variations.

The Ramsey Model with the Nabla Derivative:

Consider the constraint (4) for the continuous case which can be rewritten as

$$C(t)=-e^{rt}[e^{-rt}W(t)]'. \tag{5}$$

Now consider the discrete constraint (3) which can be rewritten as

$$C_{t-1}=-(1+r)^{t-1}\nabla\left[\frac{W(t)}{(1+r)^t}\right]. \tag{6}$$

Using the new formulations (5) and (6), of the continuous and discrete constraints, a generalization can be made in order to develop the time scale constraint

$$C(\rho(t))=-\left[\hat{e}_{-r}(\rho(t),0)\right]^{-1}\left[\hat{e}_{-r}(t,0)W(t)\right]^{\nabla}.$$

Then by taking the nabla derivative of $[\hat{e}_{-r}(t,0)W(t)]$ the following is obtained

$$\begin{aligned}C(\rho(t))&=-\left[\hat{e}_{-r}(\rho(t),0)\right]^{-1}\left[\hat{e}^{\nabla}_{-r}(t,0)W(\rho(t))+\hat{e}_{-r}(t,0)W^{\nabla}(t)\right]\\&=-\left[(1+\nu(t)r)\hat{e}_{-r}(t,0)\right]^{-1}\left[-r\hat{e}_{-r}(t,0)W(\rho(t))+\hat{e}_{-r}(t,0)W^{\nabla}(t)\right].\end{aligned}$$

Then by distributing through by $-\left[(1+\nu(t)r)\hat{e}_{-r}(t,0)\right]^{-1}$ the constraint for the nabla version of the Ramsey model is obtained

$$C(\rho(t))=\frac{rW(\rho(t))}{1+\nu(t)r}-\frac{W^{\nabla}(t)}{1+\nu(t)r}.$$

The Ramsey model with the nabla derivative is

$$\max_{[W(t)]}\int_{\rho^2(0)}^{\rho^2(T)}\hat{e}_{-p}(\rho(t),0)U\left[\frac{rW(\rho(t))}{1+\nu(t)r}-\frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\nabla t. \tag{7}$$

Note that this model includes the discrete case and the continuous case as special cases. First we derive the Euler-Lagrange equation using Theorem 1.2. In this model,

$$N(t,W^{\rho},W^{\nabla})=\hat{e}_{-p}(\rho(t),0)U\left[\frac{rW^{\rho}}{1+\nu(t)r}-\frac{W^{\nabla}}{1+\nu(t)r}\right]$$

so we obtain the following dynamic equation

$$\begin{aligned}&\hat{e}_{-p}(\rho(t),0)U'\left[\frac{rW(\rho(t))}{1+\nu(t)r}-\frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\left(\frac{r}{1+\nu(t)r}\right)\\&+\left[\hat{e}_{-p}(\rho(t),0)U'\left[\frac{rW(\rho(t))}{1+\nu(t)r}-\frac{W^{\nabla}(t)}{1+\nu(t)r}\right]\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla}=0.\end{aligned}$$

Then by substituting $C(\rho(t))$ in for $\dfrac{r}{1+\nu(t)r}W(\rho(t))-\dfrac{1}{1+\nu(t)r}W^{\nabla}(t)$ the following is obtained

$$\hat{e}_{-p}(\rho(t),0)U'\left(C(\rho(t))\right)\left(\frac{r}{1+\nu(t)r}\right)+\left[\hat{e}_{-p}(\rho(t),0)U'\left(C(\rho(t))\right)\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla}=0.$$

Then by using the product rule and by taking the nabla derivative of the nabla exponential function we have

$$\left[U'\left(C(\rho(t))\right)\left(\frac{1}{1+\nu(t)r}\right)\right]^{\nabla}=\frac{p(1-\nu^{\nabla}(t))-r}{(1+\nu(t)r)(1+\nu(\rho(t))p)}U'\left(C(\rho(t))\right),$$

where we assume that ν is a nabla differentiable function, note that ν is not necessarily nabla differentiable in general.
We let $\alpha(t) := \dfrac{1}{1+\nu(t)r}$. Then again using the product rule the following is obtained

$$\alpha(\rho(t))\left[U'\left(C(\rho(t))\right)\right]^{\nabla}+\alpha^{\nabla}(t)\left[U'\left(C(\rho(t))\right)\right]=\frac{p(1-\nu^{\nabla}(t))-r}{(1+\nu(t)r)(1+\nu(\rho(t))p)}U'\left(C(\rho(t))\right)$$

which is the same as

$$\left[U'\left(C(\rho(t))\right)\right]^{\nabla}=\left(\frac{p(1-\nu^{\nabla}(t))-r-\alpha^{\nabla}(t)(1+\nu(t)r)(1+\nu(\rho(t))p)}{(1+\nu(t)r)(1+\nu(\rho(t))p)\alpha(\rho(t))}\right)U'\left(C(\rho(t))\right),$$

then by substituting $\dfrac{1}{1+\nu(\rho(t))r}$ in for $\alpha(\rho(t))$ and rearranging the following is obtained

$$\frac{\left[U'\left(C(\rho(t))\right)\right]^{\nabla}}{U'\left(C(\rho(t))\right)}=\left(\frac{\left(p(1-\nu^{\nabla}(t))-r\right)(1+\nu(\rho(t))r)+\nu^{\nabla}(t)r(1+\nu(\rho(t))p)}{(1+\nu(t)r)(1+\nu(\rho(t))p)}\right) \tag{8}$$

for $t \in [\rho^2(0), \rho^2(T)]^{\kappa}_{\kappa}$.

The Ramsey Model with the Delta Derivative:

Consider the constraint for the continuous case (4) which can be rewritten as

$$C(t) = -e^{rt}[e^{-rt}W(t)]'. \tag{9}$$

Now consider the discrete constraint (3) which can be rewritten as

$$C_t = -(1+r)^{t-1}\Delta\left[\frac{W(t)}{(1+r)^{t-1}}\right]. \tag{10}$$

Using the new formulations (9) and (10), of the continuous and discrete constraints, a generalization can be made in order to develop the time scale constraint

$$C(t) = -[\hat{e}_{-r}(\rho(t),0)]^{-1}[\hat{e}_{-r}(\rho(t),0)W(t)]^{\Delta}.$$

Then by taking the delta derivative of $[\hat{e}_{-r}(\rho(t),0)W(t)]$ the following is obtained,

$$C(t)=-[\hat{e}_{-r}(\rho(t),0)]^{-1}\left[\frac{-r(1+\nu(t)r)+r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\hat{e}_{-r}(\rho(t),0)W(\sigma(t))+\hat{e}_{-r}(\rho(t),0)W^{\Delta}(t)\right],$$

where ν is assumed to be a delta differentiable function. Then by distributing through by $-[\hat{e}_{-r}(\rho(t),0)]^{-1}$ the constraint for the delta version of the Ramsey model is obtained which is

$$C(t)=\left[\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}W(\sigma(t))-W^{\Delta}(t)\right].$$

The Ramsey model with the delta derivative is

$$\max_{[W(t)]}\int_0^T \hat{e}_{-p}(t,0)U\left[\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}W(\sigma(t))-W^{\Delta}(t)\right]\Delta t. \tag{11}$$

Note that this model includes the discrete and continuous model as special cases. First we derive the Euler-Lagrange equation using Theorem 1.1. In this model,

$$L(t,W^{\sigma},W^{\Delta})=\hat{e}_{-p}(t,0)U\left[\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}W^{\sigma}-W^{\Delta}\right]$$

so we obtain the following dynamic equation

$$\hat{e}_{-p}(t,0)U'\left[\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\,W(\sigma(t))-W^{\Delta}(t)\right]\left(\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\right)$$
$$+\left[\hat{e}_{-p}(t,0)U'\left(\frac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}\,W(\sigma(t))-W^{\Delta}(t)\right)\right]^{\Delta}=0.$$

Then by substituting $C(t)$ in for $\dfrac{r(1+\nu(t)r)-r\nu^{\Delta}(t)}{(1+\mu(t)r)(1+\nu(t)r)}W(\sigma(t))-W^{\Delta}(t)$, using the product rule and taking the delta derivative of the nabla exponential we have

$$\frac{[U'(C(t))]^{\Delta}}{U'(C(t))}=\frac{(r\nu^{\Delta}(t)-r(1+\nu(t)r))(1+\mu(t)p)+p(1+\mu(t)r)(1+\nu(t)r)}{(1+\mu(t)r)(1+\nu(t)r)} \tag{12}$$

for $t\in[0,T]^{\kappa}_{\kappa}$.

We will end this section by comparing the solutions of the two models (7) and (11). The first comparison of the two solutions, (8) and (12), will be made where $\mathbb{T}=\mathbb{R}$. The solution (8) obtained from the Ramsey model with the nabla derivative becomes

$$\frac{[U'(C(t))]'}{U'(C(t))}=p-r$$

for $t\in[0,T]$. The solution (12) obtained from the Ramsey model with the delta derivative becomes

$$\frac{[U'(C(t))]'}{U'(C(t))}=p-r.$$

for $t\in[0,T]$. So when $\mathbb{T}=\mathbb{R}$ the two solutions are the same.
The next comparison will be made where $\mathbb{T}=h\mathbb{Z}$. The solution to the Ramsey model with the nabla derivative and $\mathbb{T}=h\mathbb{Z}$ is as follows

$$\frac{\nabla[U'(C(\rho(t)))]}{U'(C(\rho(t)))}=\frac{p-r}{1+hp}.$$

Then by taking the indicated backward difference we have

$$U'(C(\rho(t)))=\frac{1+hp}{1+hr}U'(C(\rho(\rho(t))))$$

for $t \in [-\frac{1}{h}, T - \frac{3}{h}]$. The solution to the Ramsey model with the delta derivative and $\mathbb{T} = h\mathbb{Z}$ is as follows

$$\frac{\Delta[U'(C(t))]}{U'(C(t))} = \frac{p-r}{1+hr}.$$

Then by taking the indicated forward difference we have

$$U'(C(\sigma(t))) = \frac{1+hp}{1+hr}U'(C(t))$$

for $t \in [\frac{1}{h}, T - \frac{1}{h}]$.

3 Free Boundary Conditions

In this section, we will form an adjustment model with Δ derivative on time scales. The Euler-Lagrange equation turns out to be a second order dynamic equation which currently has no closed solution. So to circumvent this issue we will consider a time scale $\mathbb{T} = \{[0,6) \cap h_1\mathbb{Z}\} \cup \{[6,14) \cap h_2\mathbb{Z}\} \cup \{[14,30] \cap h_3\mathbb{Z}\}$ where $h_1 = 1$, $h_2 = 0.5$, and $h_3 = 0.001$ in order to solve and compare the obtained solution and the desired target solution.

Discrete Model:

We want to minimize the dynamic model of adjustment

$$J[y] = \sum_{t=1}^{T} r^t[\alpha(y(t) - \overline{y}(t))^2 + (y(t) - y(t-1))^2],$$

where $y(t)$ is the output state variable, $r > 1$ is the exogenous rate of discount, $\overline{y}$ is the desired target level (which for the purposes of this paper we will consider two cases which are that $\overline{y}$ is either linear or exponential), and T is the horizon. The first component of the loss function above is the disequilibrium cost due to deviations from the desired target and the second component characterizes the agent's aversion to output fluctuations. The Euler-Lagrange equation for the discrete model is as follows

$$ry(t+1) - (r+\alpha+1)y(t) + y(t-1) + \alpha\overline{y}(t) = 0.$$

Continuous Model:

We want to minimize the dynamic model of adjustment

$$J[y] = \int_0^T e^{(r-1)t}[\alpha(y(t) - \overline{y}(t))^2 + (y'(t))^2]dt.$$

The Euler-Lagrange equation becomes

$$y''(t) + (r-1)y'(t) - \alpha y(t) + \alpha\overline{y}(t) = 0.$$

Time Scales Model:

The time scale model which we wish to minimize is

$$J[y]=\int_{\sigma(0)}^{\rho(T)} e_{r-1}(\sigma(t),0)[\alpha(y(\sigma(t))-\overline{y}(\sigma(t)))^2+(y^{\Delta}(t))^2]\Delta t.$$

Note that this model includes the discrete case and the continuous case as special cases. First we derive the Euler-Lagrange equation using Theorem 1.1. In this model,

$$L(t,y(\sigma(t)),y^{\Delta}(t))=e_{r-1}(\sigma(t),0)[\alpha(y(\sigma(t))-\overline{y}(\sigma(t)))^2+(y^{\Delta}(t))^2],$$

so we obtain the following dynamic equation

$$e_{r-1}(\sigma(t),0)[2\alpha(y(\sigma(t))-\overline{y}(\sigma(t))]-2[e_{r-1}(\sigma(t),0)]^{\Delta}y^{\Delta}(\sigma(t))-2e_{r-1}(\sigma(t),0)y^{\Delta\Delta}(t)=0.$$

Then using the identity $e^{\Delta}_{r-1}(\sigma(t),0)=(r-1)(\mu^{\Delta}(t)+1)e_{r-1}(\sigma(t),0)$, where μ is assumed to be a delta differentiable function, we have

$$\begin{aligned}e_{r-1}(\sigma(t),0)[2\alpha(y(\sigma(t))-\overline{y}(\sigma(t))]-2(r-1)(\mu^{\Delta}(t)+1)[e_{r-1}(\sigma(t),0)y^{\Delta}(\sigma(t))\\ -2e_{r-1}(\sigma(t),0)y^{\Delta\Delta}(t)=0,\end{aligned}$$

then dividing through by $-2e_{r-1}(\sigma(t),0)$ the equation simplifies to

$$y^{\Delta\Delta}(t)+(r-1)(\mu^{\Delta}(t)+1)y^{\Delta}(\sigma(t)))-\alpha y(\sigma(t))+\alpha\overline{y}(\sigma(t))=0.$$

This model differs from others that have been studied in the literature since there is no constraint or boundary condition. Next we derive the free boundary conditions and then apply the results to this adjustment model.

Theorem 3.1 *If*

$$J[y]=\int_{\sigma(a)}^{\sigma^2(b)} L(t,y(\sigma(t)),y^{\triangle}(t))\triangle t,$$

where $y\in C^2[a,\sigma^2(b)]$ and $y(\sigma(a))=A$, has a local extremum at $y(t)$, then $y(t)$ satisfies the Euler-Lagrange equation for $t\in[a,\sigma(b)]^{\kappa}_{\kappa}$, $y(\sigma(a))=A$ and $y(t)$ satisfies the condition

$$(\sigma^2(b)-\sigma(b))L_{y^{\sigma}}(\sigma(b),y(\sigma^2(b)),y^{\triangle}(\sigma(b)))+L_{y^{\triangle}}(\sigma(b),y(\sigma^2(b)),y^{\triangle}(\sigma(b)))=0 \quad (13)$$

Proof As in the proof of Theorem 1.1, $J_1[h]=0$ for all $h\in C^1[\sigma(a),\sigma^2(b)]$ with $h(\sigma(a))=0$. While getting Euler-Lagrange equation, if we use $h(\sigma(a))=0$, we get

$$\begin{aligned}&\int_{\sigma(a)}^{\sigma^2(b)}\{L_{y^{\sigma}}(t,y^{\sigma},y^{\triangle})-L^{\triangle}_{y^{\triangle}}(t,y^{\sigma},y^{\triangle})\}h^{\sigma}(t)\triangle t\\ &+\{(\sigma^2(b)-\sigma(b))L_{y^{\sigma}}(\sigma(b),y(\sigma^2(b)),y^{\triangle}(\sigma(b)))\\ &+L_{y^{\triangle}}(\sigma(b),y(\sigma^2(b)),y^{\triangle}(\sigma(b)))\}h(\sigma^2(b))=0\end{aligned}$$

for all $h\in C^1[\sigma(a),\sigma^2(b)]$. The conclusion of the theorem follows. □

Theorem 3.2 *If*

$$J[y]=\int_{\sigma(a)}^{\sigma^2(b)} L(t,y(\sigma(t)),y^{\triangle}(t))\triangle t,$$

where $y \in C^2[a, \sigma^2(b)]$ *and* $y(\sigma^2(b)) = B$*, has a local extremum at* $y(t)$*, then* $y(t)$ *satisfies the Euler-Lagrange equation for* $t \in [a, \sigma(b)]_\kappa^\kappa$*,* $y(\sigma^2(b)) = B$ *and* $y(t)$ *satisfies the condition*

$$L_{y^\triangle}(\sigma(a), y(\sigma^2(a)), y^\triangle(\sigma(a))) = 0. \tag{14}$$

In a similar way, we have the following theorem.

Theorem 3.3 *If* $y(t)$ *is a local extremum for* $J[y]$ *where* $y \in C^2[a, \sigma^2(b)]$*, then* $y(t)$ *satisfies the Euler-Lagrange equation for* $t \in [a, \sigma(b)]_\kappa^\kappa$ *and the conditions (*13*) and (*14*).*

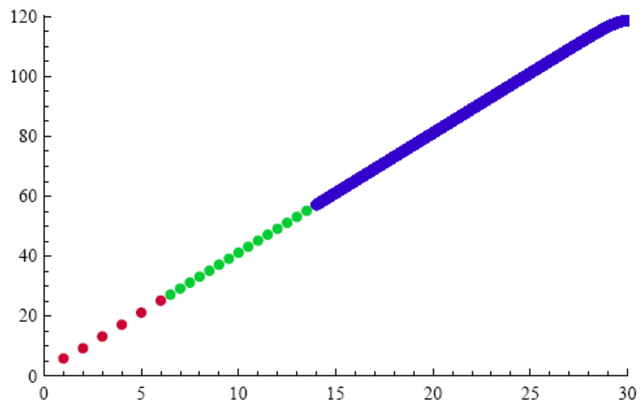

Figure 3.1: Linear case.

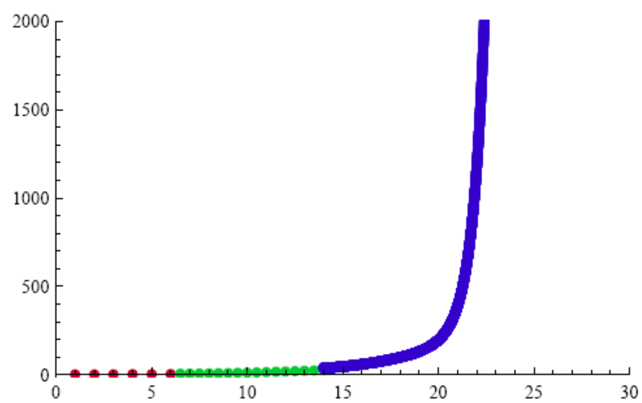

Figure 3.2: Exponential case.

So we have that the free boundary conditions for this model are $y^\Delta(\sigma(a)) = 0$ and $y^\Delta(\sigma(b)) = h\alpha[\overline{y}(\sigma^2(b)) - y(\sigma^2(b))]$. We will now illustrate the optimized solution of this problem for a time scale where $\mathbb{T} = \{[0, 6) \cap h_1\mathbb{Z}\} \cup \{[6, 14) \cap h_2\mathbb{Z}\} \cup \{[14, 30] \cap h_3\mathbb{Z}\}$ where

$h_1 = 1$, $h_2 = 0.5$, and $h_3 = 0.001$. This is accomplished by considering the optimized solution to be the following piecewise defined function

$$y(t) = \begin{cases} y_1(t) & \text{if } t \in [1, 6), \\ y_2(t) & \text{if } t \in [6, 14), \\ y_3(t) & \text{if } t \in [14, 30 - 0.002], \end{cases}$$

being optimized on the three separate intervals. Figures 3.1 and 3.2 are the graphs of the linear and exponential case with $r = 1.9$, $\alpha = 4$, $b = 0.25$, and $v = 4$ whose target functions are $\overline{y}(t) = vt + b$ and $\overline{y}(t) = e_b(t, 0)$.

4 Concluding Remarks

The techniques of modelling with dynamic equations on time scales are not widely used in economics. This may be due to the view that ordinary differential equations and difference equations are sufficient for modelling most interesting events in economy. However, economists encounter situations in which discrete and continuous models do not capture all the essential features of the events. In this sense, modelling with dynamic equations on time scales provides a more "complete" model for events at all level of time domains.

In Section 2, a well-known Ramsey model of economics has been used to illustrate that it is possible to write a model in economics with Δ operator as well as with ∇ operator. The existing theory, theory of calculus of variations on time scales, allows us to solve both models and compare the obtained solutions on time scales $\mathbb{R}$ and $h\mathbb{Z}$. Our calculations show that the solutions are exactly the same on certain time scales. In Section 3, we studied a model of economics where we cannot use both derivative operators. This is due to the fact that the theory of calculus of variations on time scales is very much a work in progress. At this time, the adjustment model can be solved if it is modelled with Δ operator only.

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Eigenvalues for Iterative Systems of Nonlinear Boundary Value Problems on Time Scales

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Abstract: Values of $\lambda_1, \ldots, \lambda_n$ are determined for which there exist positive solutions of the iterative system of dynamic equations, $u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0$, $1 \le i \le n$, $u_{n+1}(t) = u_1(t)$, for $t \in [0,1]_{\mathbb{T}}$, and satisfying the boundary conditions, $u_i(0) = 0 = u_i(\sigma^2(1))$, $1 \le i \le n$, where $\mathbb{T}$ is a time scale. A Guo-Krasnosel'skii fixed point theorem is applied.

Keywords: *time scales; boundary value problem; iterative system of dynamic equations; nonlinear; eigenvalue.*

Mathematics Subject Classification (2000): 39A10, 34B18.

1 Introduction

Let $\mathbb{T}$ be a time scale with $0,\ \sigma^2(1) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation,

$$J_{\mathbb{T}} := J \cap \mathbb{T}.$$

We are concerned with determining values of λ_i, $1 \le i \le n$, for which there exist positive solutions for the iterative system of dynamic equations,

$$\begin{gathered} u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0,\ 1 \le i \le n,\ t \in [0,1]_{\mathbb{T}}, \\ u_{n+1}(t) = u_1(t),\ t \in [0,1]_{\mathbb{T}}, \end{gathered} \tag{1}$$

satisfying the boundary conditions,

$$u_i(0) = 0 = u_i(\sigma^2(1)),\ 1 \le i \le n, \tag{2}$$

where

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(A) $f_i \in C([0,\infty),[0,\infty)),\ 1 \le i \le n$;

(B) $a_i \in C([0,\sigma(1)]_{\mathbb{T}},[0,\infty)),\ 1 \le i \le n$, and a_i does not vanish identically on any closed subinterval of $[0,\sigma(1)]_{\mathbb{T}}$;

(C) Each of $f_{i0} := \lim\limits_{x\to 0^+} \frac{f_i(x)}{x}$ and $f_{i\infty} := \lim\limits_{x\to\infty} \frac{f_i(x)}{x}$, $1 \le i \le n$, exists as a positive real number.

There is a great deal of research activity devoted to positive solutions of dynamic equations on time scales; see, for example [1, 3, 4, 5, 8, 10, 14]. This work entails an extension of the paper by Chyan and Henderson [9] to eigenvalue problems for systems of nonlinear boundary value problems on time scales, and also, in a very real sense, an extension of the recent paper by Benchohra, Henderson and Ntouyas [7]. Also, in that light, this paper is closely related to the works by Li and Sun [27, 29].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [11, 13, 21, 24, 31] and as applications for which only positive solutions are meaningful [2, 12, 25, 26]. These considerations are formulated primarily for scalar problems, but good attention also has been given to boundary value problems for systems of differential equations [6, 15, 16, 17, 18, 19, 20, 22, 23, 28, 30, 32].

The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [13]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

2 Some preliminaries

In this section, we state the well-known Guo-Krasnosel'skii fixed point theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta\Delta} = 0,$$
$$y(0) = 0 = y(\sigma^2(1)).$$

Erbe and Peterson [10] have found,

$$G(t,s) = \frac{1}{\sigma^2(1)} \begin{cases} t(\sigma^2(1)-\sigma(s)), & \text{if } t \le s, \\ \sigma(s)(\sigma^2(1)-t), & \text{if } \sigma(s) \le t, \end{cases}$$

from which

$$G(t,s) > 0, (t,s) \in (0,\sigma^2(1))_{\mathbb{T}} \times (0,\sigma(1))_{\mathbb{T}}, \tag{3}$$

$$G(t,s) \le G(\sigma(s),s) = \frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}, t \in [0,\sigma^2(1)]_{\mathbb{T}}, s \in [0,\sigma(1)]_{\mathbb{T}}, \tag{4}$$

and it is also shown in [10] that

$$G(t,s) \ge kG(\sigma(s),s) = k\frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)},\ t \in \left[\frac{\sigma^2(1)}{4}, \frac{3\sigma^2(1)}{4}\right]_{\mathbb{T}}, s \in [0,\sigma(1)]_{\mathbb{T}}, \tag{5}$$

where

$$k = \min\left\{\frac{1}{4}, \frac{\sigma^2(1)}{4(\sigma^2(1)-\sigma(0))}\right\}.$$

We note that an n-tuple $(u_1(t),\dots,u_n(t))$ is a solution of the eigenvalue problem (1), (2) if, and only if

$$u_i(t)=\lambda_i\int_0^{\sigma(1)} G(t,s)a_i(s)f_i(u_{i+1}(\sigma(s)))\Delta s,\ 0\le t\le\sigma^2(1),\ 1\le i\le n$$

and

$$u_{n+1}(t)=u_1(t),\ 0\le t\le \sigma^2(1),$$

so that, in particular,

$$\begin{aligned}u_1(t) &= \lambda_1\int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1\Big(\lambda_2\int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2)\times\\ &\quad\times f_2\Big(\lambda_3\int_0^{\sigma(1)} G(\sigma(s_2),s_3)a_3(s_3)\cdots\times\\ &\quad\times f_{n-1}\Big(\lambda_n\int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u_1(\sigma(s_n)))\Delta s_n\Big)\cdots\Delta s_3\Big)\Delta s_2\Big)\Delta s_1.\end{aligned}$$

Values of $\lambda_1,\dots,\lambda_n$, for which there are positive solutions (positive with respect to a cone) of (1), (2), will be determined via applications of the following fixed point theorem [13].

Theorem 2.1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P}\subset\mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, and let*

$$T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\|\le\|u\|, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\ge\|u\|, u\in\mathcal{P}\cap\partial\Omega_2$, *or*

(ii) $\|Tu\|\ge\|u\|, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\le\|u\|, u\in\mathcal{P}\cap\partial\Omega_2$.

Then T has a fixed point in $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3 Positive solutions in a cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (that is, positive solutions) of (1), (2). Assume throughout that $[0,\sigma^2(1)]_{\mathbb{T}}$ is such that

$$\xi=\min\left\{t\in\mathbb{T}\,|\,t\ge\frac{\sigma^2(1)}{4}\right\}$$

and

$$\omega=\max\left\{t\in\mathbb{T}\,|\,t\le\frac{3\sigma^2(1)}{4}\right\}$$

both exist and satisfy

$$\frac{\sigma^2(1)}{4}\le\xi<\omega\le\frac{3\sigma^2(1)}{4}.$$

Next, let $\tau_i\in[\xi,\omega]_{\mathbb{T}}$ be defined by

$$\int_\xi^\omega G(\tau_i,s)a(s)\Delta s=\min_{t\in[\xi,\omega]_{\mathbb{T}}}\int_\xi^\omega G(t,s)a_i(s)\Delta s.$$

Finally, we define

$$l = \min_{s \in [0,\sigma^2(1)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}$$

and let

$$m = \min\{k, l\}. \tag{6}$$

For our construction, let $\mathcal{B} = \{x \mid x : [0, \sigma^2(1)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm, $\|x\| = \sup\{|x(t)| : t \in [0, \sigma^2(1)]_{\mathbb{T}}\}$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0, \sigma^2(1)]_{\mathbb{T}} \text{ and } \min_{t \in [\xi, \sigma(\omega)]_{\mathbb{T}}} x(t) \geq m\|x\| \right\}.$$

We next define an integral operator $T : \mathcal{P} \to \mathcal{B}$, for $u \in \mathcal{P}$, by

$$\begin{aligned} Tu(t) \quad = \quad & \lambda_1 \int_0^{\sigma(1)} G(t, s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n))) \Delta s_n\Big) \cdots \Delta s_3\Big) \Delta s_2\Big) \Delta s_1. \end{aligned} \tag{7}$$

Notice from (A), (B) and (3) that, for $u \in \mathcal{P}$, $Tu(t) \geq 0$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. Also, for $u \in \mathcal{P}$, we have from (4) that

$$\begin{aligned} Tu(t) \quad \leq \quad & \lambda_1 \int_0^{\sigma(1)} G(\sigma(s_1), s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n))) \Delta s_n\Big) \cdots \Delta s_3\Big) \Delta s_2\Big) \Delta s_1. \end{aligned}$$

so that

$$\begin{aligned} \|Tu\| \quad \leq \quad & \lambda_1 \int_0^{\sigma(1)} G(\sigma(s_1), s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n))) \Delta s_n\Big) \cdots \Delta s_3\Big) \Delta s_2\Big) \Delta s_1. \end{aligned} \tag{8}$$

Next, if $u \in \mathcal{P}$, we have from (5), (6) and (8),

$$\begin{aligned}
&\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} Tu(t) \\
= &\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2)\times \\
&\times f_2\Big(\cdots f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\cdots\Big)\Delta s_2\Big)\Delta s_1 \\
\geq &\ \lambda_1 m \int_0^{\sigma(1)} G(\sigma(s_1),s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2)\times \\
&\times f_2\Big(\cdots f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\cdots\Big)\Delta s_2\Big)\Delta s_1 \\
\geq &\ m\|Tu\|.
\end{aligned}$$

Consequently, $T : \mathcal{P} \to \mathcal{P}$. In addition, the standard arguments can be used to verify that T is completely continuous.

By the remarks in Section 2, we seek suitable fixed points of T belonging to the cone $\mathcal{P}$.

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max_{1\leq i\leq n}\left\{\left[m\int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s f_{i\infty}\right]^{-1}\right\},$$

and

$$L_2 := \min_{1\leq i\leq n}\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a_i(s)\Delta s f_{i0}\right]^{-1}\right\},$$

where we recall that $G(\sigma(s),s) = \frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}$.

Theorem 3.1 *Assume conditions* (A), (B) *and* (C) *are satisfied. Then, for* $\lambda_1,\dots,\lambda_n$ *satisfying*

$$L_1 < \lambda_i < L_2,\ 1\leq i\leq n, \tag{9}$$

there exists an n*-tuple* $(u_1,\dots,u_n)$ *satisfying* (1), (2) *such that* $u_i(t) > 0$ *on* $(0,\sigma^2(1))_{\mathbb{T}}$, $1 \leq i \leq n$.

Proof. Let λ_j, $1 \leq j \leq n$, be as in (9). And let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n}\left\{\left[m\int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s(f_{i\infty}-\epsilon)\right]^{-1}\right\} \leq \min_{1\leq j\leq n}\lambda_j$$

and

$$\max_{1\leq j\leq n}\lambda_j \leq \min_{1\leq i\leq n}\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a_i(s)\Delta s(f_{i0}+\epsilon)\right]^{-1}\right\}.$$

We seek fixed points of the completely continuous operator $T : \mathcal{P} \to \mathcal{P}$ defined by (7).

Now, from the definitions of f_{i0}, $1 \le i \le n$, there exists an $H_1 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \le (f_{i0} + \epsilon)x, \ 0 < x \le H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (4) and the choice of ϵ, for $0 \le s_{n-1} \le \sigma(1)$,

$$\begin{aligned}
& \lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
\le \ & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
\le \ & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) (f_{n0} + \epsilon)(u(\sigma(s_n)))\Delta s_n \\
\le \ & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) \Delta s_n (f_{n0} + \epsilon)\|u\| \\
\le \ & \|u\| \\
= \ & H_1.
\end{aligned}$$

It follows in a similar manner from (4) and the choice of ϵ that, for $0 \le s_{n-2} \le \sigma(1)$,

$$\begin{aligned}
& \lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-2}), s_{n-1}) a_{n-1}(s_{n-1}) \times \\
& \qquad \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
\le \ & \lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_{n-1}) a_{n-1}(s_{n-1}) \Delta s_{n-1} (f_{n-1,0} + \epsilon)\|u\| \\
\le \ & \|u\| \\
= \ & H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we reach, for $0 \le t \le \sigma^2(1)$,

$$\lambda_1 \int_0^{\sigma(1)} G(t, s_1) a_1(s_1) f_1(\cdots f_n(u(\sigma(s_n)))\Delta s_n \cdots)\Delta s_1 \le H_1,$$

so that, for $0 \le t \le \sigma^2(1)$,

$$Tu(t) \le H_1,$$

or

$$\|Tu\| \ \le H_1 = \|u\|.$$

If we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \le \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{10}$$

Next, from the definition of $f_{i\infty}$, $1 \le i \le n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge (f_{i\infty} - \epsilon)x, \ x \ge \overline{H}_2.$$

Let

$$H_2 := \max\left\{2H_1, \frac{\overline{H}_2}{m}\right\}.$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then

$$\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} u(t) \geq m\|u\| \geq \overline{H}_2.$$

Consequently, from (5) and the choice of ϵ, for $0 \leq s_{n-1} \leq \sigma(1)$,

$$\begin{aligned}
&\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
\geq\ & \lambda_n \int_\xi^{\omega} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
\geq\ & \lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)(f_{n\infty} - \epsilon)(u(\sigma(s_n)))\Delta s_n \\
\geq\ & m\lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)\Delta s_n(f_{n\infty} - \epsilon)\|u\| \\
\geq\ & \|u\| \\
=\ & H_2.
\end{aligned}$$

It follows similarly from (5) and the choice of ϵ that, for $0 \leq s_{n-2} \leq \sigma(1)$,

$$\begin{aligned}
&\lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-2}), s_{n-1})a_{n-1}(s_{n-1}) \times \\
&\qquad \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
\geq\ & m\lambda_{n-1} \int_\xi^{\omega} G(\tau_{n-1}, s_{n-1})a_{n-1}(s_{n-1})\Delta s_{n-1}(f_{n-1,\infty} - \epsilon)\|u\| \\
\geq\ & \|u\| \\
=\ & H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we reach

$$Tu(\tau_1) = \lambda_1 \int_0^{\sigma(1)} G(\tau_1, s_1)a_1(s_1)f_1(\cdots f_n(u(\sigma(s_n)))\Delta s_n \cdots)\Delta s_1 \geq \|u\| = H_2,$$

so that $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \;|\|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{11}$$

Applying Theorem 2.1 to (10) and (11), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, setting $u_1 = u_{n+1} = u$, we obtain a positive solution $(u_1, \ldots, u_n)$ of (1), (2) given iteratively by

$$u_j(t) = \lambda_j \int_0^{\sigma(1)} G(t, s)a_j(s)f_j(u_{j+1}(\sigma(s)))\Delta s, \quad j = n, n-1, \ldots, 1.$$

The proof is complete. □

Prior to our next result, let ξ_i, $1 \leq i \leq n$, be defined by

$$\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s = \max_{t\in[1,\sigma^2(1)]_{\mathbb{T}}} \int_0^{\sigma(1)} G(t, s)a_i(s)\Delta s.$$

Then, we define positive numbers L_3 and L_4 by

$$L_3 := \max_{1\leq i\leq n} \left\{ \left[m \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s f_{i0} \right]^{-1} \right\}$$

and

$$L_4 := \min_{1\leq i\leq n} \left\{ \left[\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s f_{i\infty} \right]^{-1} \right\}.$$

Theorem 3.2 *Assume conditions* (A)–(C) *are satisfied. Then, for each* $\lambda_1, \ldots, \lambda_n$ *satisfying*

$$L_3 < \lambda_i < L_4, \quad 1 \leq i \leq n, \tag{12}$$

there exists an n*-tuple* $(u_1, \ldots, u_n)$ *satisfying* (1), (2) *such that* $u_i(t) > 0$ *on* $(0, \sigma^2(1))_{\mathbb{T}}$, $1 \leq i \leq n$.

Proof Let $\lambda_j, 1 \leq j \leq n$, be as in (12). And let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n} \left\{ \left[m \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s (f_{i0} - \epsilon) \right]^{-1} \right\} \leq \min_{1\leq j\leq n} \lambda_j$$

and

$$\max_{1\leq j\leq n} \lambda_j \leq \min_{1\leq i\leq n} \left\{ \left[\int_0^{\sigma(1)} G(\sigma(s), s)a_i(s)\Delta s (f_{i\infty} + \epsilon) \right]^{-1} \right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (7). From the definition of f_{i0}, $1 \leq i \leq n$, there exists $\overline{H_3} > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \geq (f_{i0} - \epsilon)x, \ 0 < x \leq \overline{H_3}.$$

Also, from the definition of f_{i0}, it follows that $f_{i0}(0) = 0$, $1 \leq i \leq n$, and so there exist $0 < K_n < K_{n-1} < \cdots < K_2 < \overline{H_3}$ such that

$$\lambda_i f_i(t) \leq \frac{K_{i-1}}{\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s}, \ t \in [0, K_i]_{\mathbb{T}}, \ 3 \leq i \leq n,$$

and

$$\lambda_2 f_2(t) \leq \frac{\overline{H_3}}{\int_0^{\sigma(1)} G(\xi_2, s)a_2(s)\Delta s}, \ t \in [0, K_2]_{\mathbb{T}}.$$

Choose $u \in \mathcal{P}$ with $\|u\| = K_n$. Then, we have

$$\begin{aligned} & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\ \leq \quad & \lambda_n \int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\ \leq \quad & \frac{\int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) K_{n-1} \Delta s_n}{\int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) \Delta s_n} \\ \leq \quad & K_{n-1}. \end{aligned}$$

Bootstrapping yields the standard iterative pattern, and it follows that

$$\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) f_2(\cdots)\Delta s_2 \leq \overline{H_3}.$$

Then

$$\begin{aligned} Tu(\tau_1) &= \lambda_1 \int_0^{\sigma(1)} G(\tau_1, s_1) a_1(s_1) f_1 \left(\lambda_2 \ \cdots \right) \Delta s_1 \\ &\geq \lambda_1 m \int_\xi^\omega G(\tau_1, s_1) a_1(s_1)(f_{1,0} - \epsilon)\|u\| \Delta s_1 \\ &\geq \|u\|. \end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < K_n\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1.$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leq i \leq n$, is unbounded at ∞.

For each $1 \leq i \leq n$, set

$$f_i^*(x) = \sup_{0 \leq s \leq x} f_i(s).$$

Then, it is straightforward that, for each $1 \leq i \leq n$, f_i^* is a nondecreasing real-valued function, $f_i \leq f_i^*$, and

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leq i \leq n$, there exists $\overline{H_4}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq (f_{i\infty} + \epsilon)x, \ x \geq \overline{H_4}.$$

It follows that there exists $H_4 > \max\{2\overline{H_3}, \overline{H_4}\}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq f_i^*(H_4), \quad 0 < x \leq H_4.$$

Choose $u \in \mathcal{P}$ with $||u|| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Tu(t) &= \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1(\lambda_2 \cdots)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1^*(\lambda_2 \cdots)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(\xi_1,s_1)a_1(s_1)f_1^*(H_4)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(\xi_1,s_1)a_1(s_1)\Delta s_1(f_{1\infty}+\epsilon)H_4 \\ &\leq H_4 \\ &= \|u\|, \end{aligned}$$

and so $\|Tu\| \leq \|u\|$. So, if we let

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_4\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2.$$

Application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which in turn, with $u_1 = u_{n+1} = u$, yields an n-tuple $(u_1, \ldots, u_n)$ satisfying (1), (2) for the chosen values of λ_i, $1 \leq i \leq n$. The proof is complete. □

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Nontrivial Solutions of Boundary Value Problems of Second-Order Dynamic Equations on an Isolated Time Scale

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Abstract: We will use Clark's theorem to show the existence of multiple solutions to the self–adjoint dynamic boundary value problem

$$
\begin{aligned}
&\big(p(t)u^{\Delta}(t)\big)^{\nabla} + q(t)u(t) + \lambda h(t, u(t)) = 0, \quad t \in [a, b]_{\mathbb{T}},\\
&u(\rho(a)) = u(\sigma(b)) = 0,
\end{aligned}
$$

where λ is a sufficiently large positive parameter and $\mathbb{T}$ is an isolated time scale. Examples of our results will be given.

Keywords: *Clark's theorem; isolated time scales; critical point theory.*

Mathematics Subject Classification (2000): 39A10, 34B10.

1 Introduction

A great deal of work has been done concerning the existence of solutions to discrete boundary value problems. Recently, techniques from critical point theory have been employed to show the existence of nontrivial solutions to discrete boundary value problems [4], [11], [13],[7]. These techniques are complementary to the fixed point theory that has also been utilized to study this area.

Throughout this paper, we assume the time scale $\mathbb{T}$ is isolated. Let $m = \min \mathbb{T}$ and $M = \max \mathbb{T}$. Then $\mathbb{T}$ is isolated if $\rho(t) < t < \sigma(t)$ $\forall t \in \mathbb{T}, t \neq m, M$ and

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$\rho(M) < M = \sigma(M)$, $\rho(m) = m < \sigma(m)$. Consider $[a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$, $\quad a < b$. By the interval $[a,b]_{\mathbb{T}}$ we mean the set $[a,b] \cap \mathbb{T}$. To avoid a trivialized problem, we assume throughout that there is at least one point in the time scale between the endpoints a and b. We will be concerned with the boundary value problem:

$$\left(p(t)u^{\Delta}(t)\right)^{\nabla} + q(t)u(t) + \lambda h(t, u(t)) = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{1}$$

$$u(\rho(a)) = u(\sigma(b)) = 0, \tag{2}$$

where λ is a positive parameter. The $\left(pu^{\Delta}\right)^{\nabla}$ term generalizes the central difference. The second–order mixed derivative problem was originally introduced in [1]. By examining this boundary value problem, we are extending the work done in [4]. Anderson considered the existence of solutions to a related second–order mixed derivative problem in [2]. We define the linear operator $\mathcal{L}$ on $\{u : [\rho(a), \sigma(b)]_{\mathbb{T}} \to \mathbb{R}\}$ by

$$\mathcal{L}u(t) = \left(p(t)u^{\Delta}(t)\right)^{\nabla} + q(t)u(t), \ t \in [a,b]_{\mathbb{T}}.$$

Then the formally self-adjoint nonlinear equation (1) can be written as

$$\mathcal{L}u = -\lambda h(t, u).$$

We assume:

$$p, \ q : [a,b]_{\mathbb{T}} \to \mathbb{R} \quad \text{and} \quad p > 0, \quad q < 0 \quad \text{on} \quad [a,b]_{\mathbb{T}}, \tag{3}$$

$$h : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R} \quad \text{is continuous with respect to the second variable}, \tag{4}$$

$$\exists \ \alpha > 0 \text{ such that } h(t, \alpha) = 0 \text{ and } h(t, u) > 0 \text{ for } u \in (0, \alpha), \ t \in [a,b]_{\mathbb{T}}, \tag{5}$$

$$h(t, u) \quad \text{is odd in } u. \tag{6}$$

This boundary value problem generalizes the important Sturm-Liouville problem. The time scale calculus was developed by Stefan Hilger [10] in 1988. The references [5], [6] provide excellent introductions to the theory of time scales. The following theorem provides a useful relationship between nabla and delta derivatives.

Theorem 1.1 [6] *If $\mathbb{T}$ is isolated and $f : \mathbb{T} \to \mathbb{R}$, then*

$$f^{\nabla}(t) = f^{\Delta}(\rho(t)),$$
$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)), \ \forall t \in \mathbb{T}.$$

Before proceeding, we need a few useful definitions and theorems pertaining to critical point theory.

Definition 1.1 Let E be a real Banach space and let $\varphi : E \to \mathbb{R}$ be a mapping. We say φ is *Fréchet differentiable* at $u \in E$ if there exists a continuous linear map $L = L(u) : E \to \mathbb{R}$ satisfying

$$\lim_{x \to u} \frac{\varphi(x) - \varphi(u) - L(x-u)}{\|x - u\|_E} = 0.$$

The mapping L will be denoted by $\varphi'(u)$. A *critical point* u of φ is a point at which $\varphi'(u) = 0$, i.e., $\varphi'(u)v = 0 \ \ \forall \, v \in E$. We write $\varphi \in C^1(E, \mathbb{R})$ provided $\varphi'(u)$ is continuous $\forall \, u \in E$.

The following remark will be useful.

Remark 1.1 If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined near $u^0 = (u_1^0, \cdots, u_n^0) \in \mathbb{R}^n$, and differentiable at u^0, then each of the partial derivatives $\frac{\partial \varphi}{\partial u_k}$ exists at u^0 and the Fréchet derivative of φ at u^0 is represented by the gradient:

$$\varphi'(u^0) = \nabla_u \varphi(u^0),$$

where

$$\nabla_u \varphi = \left(\frac{\partial \varphi}{\partial u_1}, \cdots, \frac{\partial \varphi}{\partial u_n} \right)$$

is the gradient of φ with respect to u.

Definition 1.2 [Palais–Smale condition] Let E be a real Banach space. A function $\varphi \in C^1(E, \mathbb{R})$ satisfies the *Palais–Smale condition* if every sequence $\{u_j\}$ in E such that $\{\varphi(u_j)\}$ is bounded and $\varphi'(u_j) \to 0$ as $j \to \infty$ contains a convergent subsequence.

We state Clark's theorem, which is crucial to proving the main results of the paper. Clark's Theorem was originally stated in [8]. The version we cite here comes from Rabinowitz [12] and Bai [4]. Let E be a real Banach space, with zero vector denoted by 0. Let Σ denote the family of sets $A \subset E \backslash \{0\}$ such that A is closed in E and symmetric to 0, i.e., $u \in A$ implies $-u \in A$. Suppose $u \in E$ satisfies Definition 1.1. In the case when $I : E \to \mathbb{R}$ is an even mapping, we say that $(u, -u)$ is a pair of critical points for I.

Theorem 1.2 (Clark's theorem) *Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ with I even, bounded from below, and satisfying the Palais–Smale condition. Suppose $I(0) = 0$, there is a set $K \in \Sigma$ such that K is homeomorphic to S^{j-1} (j-1 dimensional unit sphere in $\mathbb{R}^j$) by an odd map, and $\sup_K I < 0$. Then I has at least j distinct pairs of critical points.*

2 Preliminary Results

Definition 2.1 Real-valued functions α, β on $[\rho(a), \sigma(b)]_{\mathbb{T}}$ are called *lower and upper solutions,* respectively, for the BVP (1), (2) if

$$\begin{cases} (p\alpha^{\Delta})^{\nabla}(t) + q(t)\alpha(t) \geq -\lambda h(t, \alpha(t)), & \forall t \in [a, b]_{\mathbb{T}}, \\ \alpha(\rho(a)) \leq 0, \ \alpha(\sigma(b)) \leq 0 \end{cases}$$

and

$$\begin{cases} (p\beta^{\Delta})^{\nabla}(t) + q(t)\beta(t) \leq -\lambda h(t, \beta(t)), & \forall t \in [a, b]_{\mathbb{T}}, \\ \beta(\rho(a)) \geq 0, \ \beta(\sigma(b)) \geq 0. \end{cases}$$

Define $h_1 : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by

$$h_1(t, s) = \begin{cases} 0, & s > \alpha, \\ h(t, s), & |s| \leq \alpha, \\ 0, & s < -\alpha, \end{cases}$$

where α is as assumed in (5).

Lemma 2.1 *Let α be as in* (5). *If u satisfies the BVP*

$$\mathcal{L}u = -\lambda h_1(t,u), \quad t \in [a,b]_{\mathbb{T}}, \tag{7}$$

$$u(\rho(a)) = u(\sigma(b)) = 0, \tag{8}$$

then

$$\|u\|_1 := \sum_{i=1}^{T} u_i \leq \alpha$$

and consequently u is a solution of the BVP (1), (2).

Proof We first observe that by definition, $-\alpha,\ \alpha$ are lower and upper solutions, respectively, of the BVP (1), (2). We claim that $u(t) \leq \alpha$ on $[\rho(a),\sigma(b)]_{\mathbb{T}}$. Suppose not, then $w(t) := u(t) - \alpha > 0$ for at least one point in $[a,b]_{\mathbb{T}}$. Since $w(\rho(a)) \leq 0$ and $w(\sigma(b)) \leq 0$, we get that w has a positive maximum at some point $t_0 \in [a,b]_{\mathbb{T}}$. Furthermore, we may assume that t_0 is the last such maximum in $[a,b]_{\mathbb{T}}$, i.e., $w(t) < w(t_0)$ for $t \in (t_0,b]_{\mathbb{T}}$. Hence, by [6, Lemma 6.17],

$$w(t_0) > 0,\ w^{\Delta}(t_0) \leq 0,\ (pw^{\Delta})^{\nabla}(t_0) \leq 0.$$

This implies that

$$u(t_0) > \alpha,\ u^{\Delta}(t_0) \leq 0,\ (pu^{\Delta})^{\nabla}(t_0) \leq 0.$$

So

$$(pu^{\Delta})^{\nabla}(t_0) + q(t_0)u(t_0) - \alpha q(t_0) < 0.$$

But

$$(pu^{\Delta})^{\nabla}(t_0) + q(t_0)u(t_0) - \alpha q(t_0) \geq -\lambda h_1(t_0,u(t_0)) = 0.$$

This is a contradiction. Hence $u(t) \leq \alpha$ for $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$. A similar argument shows that $-\alpha \leq u(t)\ \forall t \in [a,b]_{\mathbb{T}}$. It follows that $u(t)$ is a solution of the BVP (1), (2). Thus, the lemma is proved. □

Let

$$E = \{u : [\rho(a),\sigma(b)]_{\mathbb{T}} \to \mathbb{R} : u(\rho(a)) = u(\sigma(b)) = 0\}.$$

Let $|S|$ denote the cardinality of the set S. Note that E can be identified with $\mathbb{R}^T$, where $T := |[a,b]_{\mathbb{T}}|$, by the correspondence

$$(0, u(a), u(\sigma(a)), \cdots, u(b), 0) \leftrightarrow (x_1, \cdots, x_T),$$

where $x_i = u^{\sigma^{i-1}}(a), 1 \leq i \leq T$.
Define an inner product on E by

$$< u,v >_E = \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[p^{\rho}(t)u^{\nabla}(t)v^{\nabla}(t) - q(t)u(t)v(t)\right]$$

with corresponding norm

$$\|u\|_E^2 =< u,u >_E = \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[p^{\rho}(t)\left(u^{\nabla}(t)\right)^2 - q(t)u^2(t)\right].$$

We note that since E is finite dimensional, E equipped with this inner product is a Hilbert space. In this definition of E, it is important that $\mathbb{T}$ is isolated to guarantee that E equipped with this inner product is indeed a Hilbert space. Work similar to that done in [3] would be invaluable to extend this work to more general time scales.

Definition 2.2 We define the nonlinear functional $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\|u\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t)), \quad \forall\, u \in E,$$

where $H(t,z) := \int_0^z h_1(t,s)ds$.

For our application, we will be interested in computing the Fréchet derivative of I. Here is a remark to aid in this calculation:

Remark 2.1 Let $\mathcal{H}$ be a real Hilbert space, let $f : \mathcal{H} \to \mathbb{R}$ be the function defined by $f(x) = \|x\|^2$ and let $u \in \mathcal{H}$. Then the Fréchet derivative of f at u is the linear functional on $\mathcal{H}$ given by $f'(u)x := 2 < x, u >$.

One could use Remark 1.1 to prove Remark 2.1. It is also an easy exercise to prove Remark 2.1 using the definition of the Fréchet derivative.

With the aid of Remark 2.1, we calculate the Fréchet derivative of our functional I:

Theorem 2.1 *For $u, v \in E$,*

$$\begin{aligned} I'(u)v &= < u,v >_E -\lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t) \\ &= -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\mathcal{L}u(t)v(t) - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t). \end{aligned}$$

Proof Let

$$I_1(u) = \frac{1}{2}\|u\|_E^2 \text{ and } I_2(u) = \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t)).$$

Then $I = I_1 - I_2$. By Remark 2.1,

$$I_1'(u)v =< u,v >_E = \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[p^\rho(t)u^\nabla(t)v^\nabla(t) - q(t)u(t)v(t)\right].$$

Using integration by parts, properties of the integral discussed in [5] and Theorem 1.1, we see

$$\begin{aligned} < u,v >_E &= p(t)u^\Delta(t)v(t)\Big|_{\rho(a)}^{\sigma(b)} \\ &\quad - \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^\Delta(t)\right]^\nabla + q(t)u(t)\right]v(t) \\ &= -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^\Delta(t)\right]^\nabla + q(t)u(t)\right]v(t) \end{aligned}$$

by the boundary conditions on u.

By Remark 1.1,

$$I_2'(u)v = \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t).$$

Thus,

$$I'(u)v = -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t)\right]v(t) - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t),$$

as desired. □

Corollary 2.1 *Let $u \in E$. The following are equivalent:*
1. u is a critical point of I,
2. u is a solution of (1), (2).
Furthermore, $I \in C^1(E,\mathbb{R})$.

Proof Let $u \in E$. Then

$$u \text{ is a critical point of } I$$

if and only if

$$I'(u)v = 0 \quad \forall\, v \in E,$$

if and only if

$$\sum_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t)\right]v(t) \\ +\lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t) = 0 \quad \forall\, v \in E,$$

if and only if

$$u \text{ is a solution of (1), (2).}$$

To see that the last statement holds, for any $m \in [a,b]_{\mathbb{T}}$, let

$$v_m(t) = \begin{cases} 1, & \text{if } t = m, \\ 0, & \text{if } t \neq m. \end{cases}$$

Then $v_m \in E$ and $I'(u)v_m = 0 \;\; \forall\, m \in [a,b]_{\mathbb{T}}$. But this implies: $\nu(t)\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t) - \lambda\nu(t)h_1(t,u(t)) = 0, \quad \forall\, t \in [a,b]_{\mathbb{T}}$. As $\nu(t) > 0$ on $\mathbb{T}$, these critical points correspond to solutions of (7), (8). By Lemma 2.1, we equivalently have solutions to (1), (2).

As E and $\mathbb{R}$ are Euclidean spaces, the continuity of h guarantees that $I \in C^1(E,\mathbb{R})$. □

3 Main Result and Proof

We note that if u is a solution of (1), (2) then $-u$ also solves (1), (2) and we say that $(u,-u)$ is a pair of solutions to (1), (2). The main result of this paper is:

Theorem 3.1 *Let* (3)–(6) *be satisfied. Then there exists a* $\lambda^* > 0$ *such that if* $\lambda > \lambda^*$, (1), (2) *has at least* $T := |[a,b]_{\mathbb{T}}|$ *distinct pairs of nontrivial solutions. Furthermore, each nontrivial solution* u *satisfies* $|u(t)| \le \alpha$, $t \in [a,b]_{\mathbb{T}}$ *and* α *as in* (5).

Proof We will use Theorem 1.2 and Lemma 2.1 to prove this result. As h_1 is odd in its second variable, we know that I is an even functional. Indeed,

$$\begin{aligned}
I(-u) &= \frac{1}{2}\|-u\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,-u(t)) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_0^{-u(t)} h_1(t,s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \left(- \int_{-u(t)}^{0} h_1(t,s)\, ds \right) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_{-u(t)}^{0} h_1(t,-s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_{u(t)}^{0} -h_1(t,\tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_0^{u(t)} h_1(t,\tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} H(t,u(t)) \\
&= I(u).
\end{aligned}$$

By construction, $I(0) = 0$. As $h_1(t,s) = 0$ for $|s| \ge \alpha$,

$$\begin{aligned}
\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)H(t,u(t)) &= \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \int_0^{u(t)} h_1(t,s)\, ds \\
&\le \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \int_{-\alpha}^{\alpha} |h_1(t,s)|\, ds =: C \quad \forall\, u \in E.
\end{aligned}$$

This implies:

$$I(u) \ge \frac{1}{2}\|u\|_E^2 - \lambda C \ge -\lambda C, \;\; \forall u \in E.$$

Hence, I is bounded from below.

Now we verify the Palais-Smale condition. Let $\{u_m\} \subset E$ be any sequence such that $\{I(u_m)\}$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$. Then there exist c_1, c_2 such that $c_1 \le I(u_m) \le c_2$, $\;m \in \mathbb{N}$. Then

$$\begin{aligned}
I(u_m) &= \frac{1}{2}\|u_m\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u_m(t)) \\
&\ge \frac{1}{2}\|u_m\|_E^2 - \lambda C
\end{aligned}$$

which implies that

$$\begin{aligned} \|u_m\|_E^2 &\leq 2I(u_m) + 2\lambda C \\ &\leq 2c_2 + 2\lambda C, \quad \forall\, m \in \mathbb{N}. \end{aligned}$$

Therefore, $\{u_m\}$ is a bounded sequence in a finite-dimensional space E and so has a convergent subsequence in E. Thus, the Palais-Smale condition is satisfied.

Recall that $T = |[a,b]_{\mathbb{T}}|$. We take $\{y_i\}_{i=1}^T$ as an orthonormal basis of E. Define

$$K(r) = \left\{ \sum_{i=1}^T \beta_i y_i : \sum_{i=1}^T \beta_i^2 = r^2 \right\}, \quad r > 0.$$

Then $0 \notin K(r)$ and $K(r)$ is symmetric with respect to 0.

It is immediate to see that $K(r)$ is compact. Indeed, fix $r > 0$. Define a map $f : K(r) \to S^{T-1}$ by

$$f(u) = f(\beta_1 y_1 + \cdots + \beta_T y_T) = \frac{< \beta_1, \cdots, \beta_T >}{r}.$$

Then f is an isomorphism that preserves inner products. As the inner product determines the topology of our spaces, it follows that f is a homeomorphism. Moreover, f is an odd map, so we have verified that $K(r)$ is homeomorphic to S^{T-1} by an odd map for any $r > 0$.

Now, let $u \in K(r)$. Then with the aid of Hölder's inequality,

$$\begin{aligned} \|u\|_E^2 &= \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^\rho(t) \left(\sum_{i=1}^T \beta_i y_i^\nabla(t) \right)^2 - q(t) \left(\sum_{i=1}^T \beta_i y_i(t) \right)^2 \right] \\ &\leq \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^\rho(t) \sum_{i=1}^T \beta_i^2 \sum_{i=1}^T (y_i^\nabla(t))^2 - q(t) \sum_{i=1}^T \beta_i^2 \sum_{i=1}^T y_i^2(t) \right] \\ &= r^2 \sum_{i=1}^T \sum_{t\in[a,\sigma(b)]} \nu(t) \left[p^\rho(t) (y_i^\nabla(t))^2 - q(t) y_i^2(t) \right] \\ &= r^2 \|y_i\|_E^2 T \\ &= r^2 T, \quad \text{since } \; \{y_i\}_{i=1}^T \; \text{ is an orthonormal basis of E.} \end{aligned} \tag{9}$$

As dim $E < \infty$, there exists a $c_0 > 0$ such that $\|u\|_1 \leq c_0 \|u\|_E$ for all $u \in E$. Fix r such that $0 < r \leq \dfrac{\alpha}{c_0\sqrt{T}}$. Using (9), we see that for $u \in K(r)$,

$$\|u\|_1 \leq c_0 \|u\|_E \leq c_0 r \sqrt{T} \leq \alpha.$$

Hence, $h(t,u(t)) = h_1(t,u(t))$ for all $u \in K(r)$. From assumption (5), we see that for $u \in K(r)$,

$$H(t,u(t)) = \int_0^{u(t)} h(t,s)ds > 0$$

if $u(t) \neq 0,\ t \in [a,b]_{\mathbb{T}}$. Since we know that $0 \notin K(r)$, we have

$$\sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) H(t,u(t)) = \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) \int_0^{u(t)} h(t,s)ds > 0.$$

Let $\tau = \inf_{u \in K(r)} \sum_{t \in [a,b]_{\mathbb{T}}} \nu(t) H(t, u(t))$. If $\tau = 0$, then by the compactness of $K(r)$, $0 \in K(r)$, which is a contradiction. Hence $\tau > 0$. Define $\lambda^* := \dfrac{\alpha^2}{2\tau c_0^2}$. For $u \in K(r)$, if $\lambda > \lambda^*$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_E^2 - \lambda \sum_{t \in [a,b]_{\mathbb{T}}} \nu(t) H(t, u(t)) \\ &\leq \frac{1}{2}\|u\|_E^2 - \lambda^* \tau \\ &\leq \frac{r^2}{2} T - \lambda^* \tau \\ &< \frac{\alpha^2}{2c_0^2} - \lambda^* \tau \\ &= 0. \end{aligned}$$

Thus, all the conditions of Clark's theorem (Theorem 1.2) are satisfied. Hence I has at least T distinct pairs of nonzero critical points. By construction and Lemma 2.1, the BVP (1), (2) has at least T distinct pairs of nontrivial solutions. □

We now examine two basic examples in which we find approximate upper bounds for λ^* explicitly, as predicted by Theorem 3.1. Since the 1970s, the theory of nonlinear difference equations has been widely studied due to its numerous applications in areas such as computer science, economics, and ecology, to name a few [9]. Analysis using time scales calculus could be used to extend and generalize these applications. These examples show a generalization of the important Sturm-Liouville problems to time scales, where $u^{\Delta\nabla}$ generalizes the central difference. Here the time scales are chosen to show the effect of the graininess ν on the value for λ^*.

Example 3.1 Consider the following difference equation boundary value problem:

$$\nabla \Delta u(t) - u(t) + \lambda \sin(\pi u(t)) = 0,\ t \in \{1, 2\}, \tag{10}$$

$$u(0) = 0 = u(3). \tag{11}$$

Then conditions (3)–(6) are satisfied, where $p(t) \equiv 1,\ q(t) \equiv -1$ for $t \in \{1, 2\}$, $h(t, s) = \sin(\pi s)$, and $\alpha = 1$. Using the Gram–Schmidt procedure, we can find an orthonormal basis for E. One such orthonormal basis is

$$y_1 = \left\langle \frac{1}{\sqrt{3}}, 0 \right\rangle \quad \text{and} \quad y_2 = \left\langle \frac{1}{6}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} \right\rangle.$$

We also need to find a constant $c_0 > 0$ such that $\|u\|_1 \leq c_0 \|u\|_E$. Then we know:

$$\begin{aligned} \|u\|_E^2 &= (\nabla u(1))^2 + u^2(1) + (\nabla u(2))^2 + u^2(2) + (\nabla u(3))^2 + u^2(3) \\ &= 3u^2(1) + 3u^2(2) - 2u(2)u(1). \end{aligned}$$

Due to symmetry, we may without loss of generality assume $u(1) \leq u(2)$. Hence:

$$\begin{aligned} \|u\|_E^2 &\geq 3u^2(1) + 3u^2(2) - 2u^2(2) \\ &\geq u^2(1) + u^2(2). \end{aligned}$$

So

$$\|u\|_E \geq \|u\|_2 \geq \frac{1}{\sqrt{2}}\|u\|_1.$$

So we can take $c_0 = \sqrt{2}$.

According to the proof of Theorem 3.1, we fix $0 < r \leq \frac{\alpha}{c_0\sqrt{T}}$. We take r as large as possible, so here $r = \frac{1}{2}$. Then $u \in K(\frac{1}{2})$ if and only if

$$u = \beta_1 \left\langle \frac{1}{\sqrt{3}}, 0 \right\rangle + \beta_2 \left\langle \frac{1}{6}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} \right\rangle,$$

where $\beta_1^2 + \beta_2^2 = \frac{1}{4}$.

Finally, we compute

$$\tau = \inf_{u \in K(\frac{1}{2})} \sum_{t \in \{1,2\}} \int_0^{u(t)} h(t,s)ds.$$

Note:

$$\begin{aligned}\sum_{t\in\{1,2\}} \int_0^{u(t)} h(t,s)\,ds &= \int_0^{\frac{\beta_1}{\sqrt{3}} + \frac{\beta_2}{6}\sqrt{\frac{3}{2}}} \sin \pi s\, ds + \int_0^{\frac{\beta_2}{2}\sqrt{\frac{3}{2}}} \sin \pi s\, ds \\ &= \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\frac{\beta_1}{\sqrt{3}} + \frac{\beta_2}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{\beta_2}{2}\sqrt{\frac{3}{2}}\right)\right].\end{aligned}$$

Thus, to find τ, we minimize

$$f(x,y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\frac{x}{\sqrt{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right]$$

subject to the constraint $x^2 + y^2 = \frac{1}{4}$. Solving for x:

$$x = \pm\sqrt{\frac{1}{4} - y^2}, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}.$$

So we minimize

$$f_-(y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(-\sqrt{\frac{\frac{1}{4} - y^2}{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right]$$

and

$$f_+(y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\sqrt{\frac{\frac{1}{4} - y^2}{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right],$$

$-\frac{1}{2} \leq y \leq \frac{1}{2}$. Running a script in Matlab, we find that we can take $\tau = 0.0957$. A graph of f_- and f_+ is shown above. Thus, by Theorem 3.1, if $\lambda > 2.61 \geq \lambda^*$, the boundary value problem (10), (11) has two distinct pairs of nontrivial solutions. Furthermore, each solution u satisfies $|u(t)| \leq 1,\ t \in \{1,2\}$.

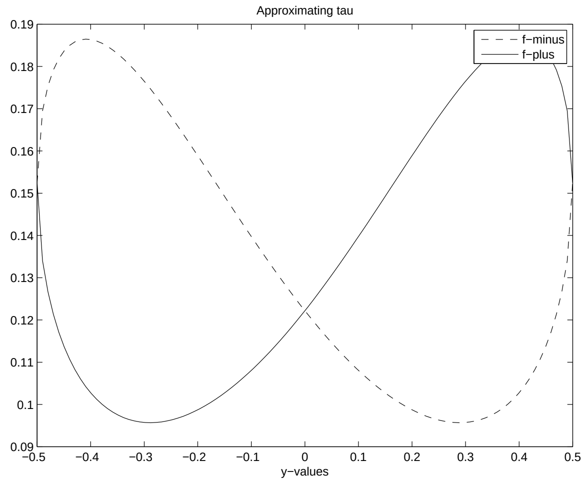

Figure 3.1: Approximating tau.

Example 3.2 Consider the following dynamic equation boundary value problem:

$$u^{\Delta\nabla}(t) - u(t) + \lambda \sin(\pi u(t)) = 0,\ t \in \left\{\frac{1}{4}, 2\right\}, \tag{12}$$

$$u(0) = 0 = u(3). \tag{13}$$

Then conditions (3)–(6) are satisfied, where $p(t) \equiv 1,\ q(t) \equiv -1$ for $t \in \left\{\frac{1}{4}, 2\right\}$, $h(t,s) = \sin(\pi s)$, and $\alpha = 1$. Using the Gram–Schmidt procedure, we can find an orthonormal basis for E. One such orthonormal basis is

$$y_1 = \left\langle \frac{2}{3}\sqrt{\frac{7}{15}}, 0 \right\rangle \quad \text{and} \quad y_2 = \left\langle \frac{32}{45}\sqrt{\frac{15}{1757}}, 6\sqrt{\frac{15}{1757}} \right\rangle.$$

We also need to find a constant $c_0 > 0$ such that $\|u\|_1 \le c_0 \|u\|_E$. There are two cases to consider:
Case 1: $|u\left(\frac{1}{4}\right)| \le |u(2)|$. Then we know:

$$\begin{aligned}
\|u\|_E^2 &= \frac{1}{4}\left[\left(u^{\nabla}\left(\frac{1}{4}\right)\right)^2 + u^2\left(\frac{1}{4}\right)\right] + \frac{7}{4}\left[(u^{\nabla}(2))^2 + u^2(2)\right] + (u^{\nabla}(3))^2 + u^2(3) \\
&\ge \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) - \frac{8}{7}|u(2)|\left|u\left(\frac{1}{4}\right)\right| \\
&\ge \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{61}{28}u^2(2) \\
&\ge \frac{61}{28}\left(u^2\left(\frac{1}{4}\right) + u^2(2)\right).
\end{aligned}$$

Case 2: $|u\left(\frac{1}{4}\right)| \geq |u(2)|$. Similarly,

$$\begin{aligned}
\|u\|_E^2 &\geq \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) - \frac{8}{7}|u(2)|\left|u\left(\frac{1}{4}\right)\right| \\
&\geq \frac{103}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) \\
&\geq \frac{103}{28}\left(u^2\left(\frac{1}{4}\right) + u^2(2)\right).
\end{aligned}$$

Hence, for all $u \in E$,

$$\|u\|_E \geq \frac{1}{2}\sqrt{\frac{61}{28}}\ \|u\|_2 \geq \frac{1}{2}\sqrt{\frac{61}{14}}\ \|u\|_1.$$

So we can take $c_0 = 2\sqrt{\frac{14}{61}}$.

According to the proof of Theorem 3.1, we fix $0 < r \leq \frac{\alpha}{c_0\sqrt{T}}$. We take r as large as possible, so here $r = \frac{1}{4}\sqrt{\frac{61}{7}}$. Then $u \in K\left(\frac{1}{4}\sqrt{\frac{61}{7}}\right)$ if and only if

$$u = \beta_1\left\langle \frac{2}{3}\sqrt{\frac{7}{15}}, 0\right\rangle + \beta_2\left\langle \frac{32}{45}\sqrt{\frac{15}{1757}}, 6\sqrt{\frac{15}{1757}}\right\rangle$$

where $\beta_1^2 + \beta_2^2 = \frac{61}{112}$.

Finally, we compute

$$\tau = \inf_{u\in K\left(\frac{1}{4}\sqrt{\frac{61}{7}}\right)} \sum_{t\in\{\frac{1}{4},2\}} \int_0^{u(t)} h(t,s)ds.$$

Note: $\sum_{t\in\{\frac{1}{4},2\}} \int_0^{u(t)} h(t,s)\ ds$

$$\begin{aligned}
&= \frac{1}{4}\int_0^{\frac{2\beta_1}{3}\sqrt{\frac{15}{1757}}+\frac{32\beta_2}{45}\sqrt{\frac{15}{1757}}} \sin\pi s\ ds + \frac{7}{4}\int_0^{6\beta_2\sqrt{\frac{15}{1757}}} \sin\pi s\ ds \\
&= \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2\beta_1}{3}\sqrt{\frac{7}{15}} + \frac{32\beta_2}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi\beta_2\sqrt{\frac{15}{1757}}\right)\right].
\end{aligned}$$

Thus, to find τ, we minimize

$$g(x,y) = \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2x}{3}\sqrt{\frac{7}{15}} + \frac{32y}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\frac{15}{1757}}\right)\right]$$

subject to the constraint $x^2 + y^2 = \frac{61}{112}$. Solving for x:

$$x = \pm\sqrt{\frac{61}{112} - y^2}, \quad -\frac{1}{4}\sqrt{\frac{61}{7}} \leq y \leq \frac{1}{4}\sqrt{\frac{61}{7}}.$$

So we minimize

$$g_-(y) = \tfrac{1}{\pi}\left[2 - \tfrac{1}{4}\cos\left[\pi\left(-\tfrac{2}{3}\sqrt{\tfrac{7}{15}}\sqrt{\tfrac{61}{112} - y^2} + \tfrac{32y}{45}\sqrt{\tfrac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\tfrac{15}{1757}}\right)\right]$$

and

$$g_+(y) = \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2}{3}\sqrt{\frac{7}{15}}\sqrt{\frac{61}{112} - y^2} + \frac{32y}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\frac{15}{1757}}\right)\right],$$

$-\frac{1}{4}\sqrt{\frac{61}{7}} \le y \le \frac{1}{4}\sqrt{\frac{61}{7}}$. Running a script in Matlab, we find that we can take $\tau = 0.0403$. A graph of g_- and g_+ is shown below.

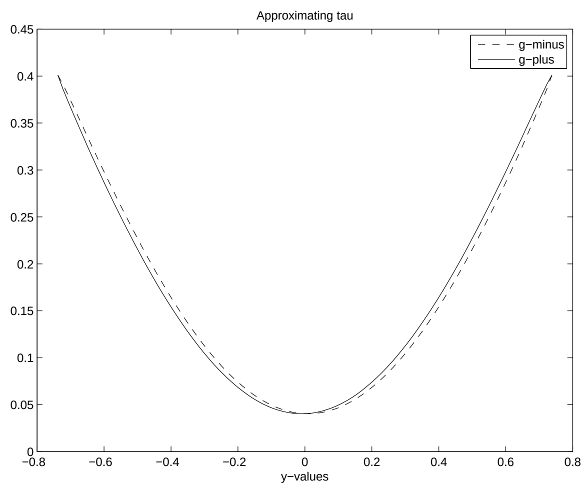

Figure 3.2: Approximating tau.

Thus, by Theorem 3.1, if $\lambda > 13.52 \ge \lambda^*$, the boundary value problem (12), (13) has two distinct pairs of nontrivial solutions. Furthermore, each solution u satisfies $|u(t)| \le 1,\ t \in \{\frac{1}{4}, 2\}$.

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Exponential Stability of Linear Time-Invariant Systems on Time Scales

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Abstract: Several notions of exponential stability of linear time-invariant systems on arbitrary time scales are discussed. We establish a necessary and sufficient condition for the existence of uniform exponential stability. Moreover, we characterize the uniform exponential stability of a system by the spectrum of its matrix. In general, exponential stability of a system can not be characterized by the spectrum of its matrix.

Keywords: *time scale; linear dynamic equation; exponential stability; uniform exponential stability.*

Mathematics Subject Classification (2000): 34C11, 39A10, 37B55, 34D99.

1 Introduction

It is well-known that exponential decay of the solution of a linear autonomous ordinary differential equation $\dot{x}(t) = Ax(t),\ t \in \mathbb{R}$, or of an autonomous difference equation $x_{t+1} = Ax_t,\ t \in \mathbb{Z}$, can be characterized by spectral properties of A. Namely, the solutions tend to 0 exponentially as $t \to \infty$, if and only if all the eigenvalues of $A \in \mathbb{C}^{d\times d}$ have negative real parts or a modulus smaller than 1, respectively. The question, which notion of stability of a linear time-invariant dynamic equation on a time scale inherits such a property, is answered partly in Pötzsche et al [16].

The history of asymptotic stability of an equation on a general time scale goes back to the work of Aulbach and Hilger [2]. Although it unifies the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z},\ h > 0$, its assumptions are often too pessimistic since the maximal graininess is involved. For a real scalar dynamic equation, stability and instability results are obtained

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by Gard and Hoffacker [7]. Another way to approach to the asymptotic stability of linear dynamic equations using Lyapunov functions can be found in Hilger and Kloeden [10]. Pötzsche [15, Abschnitt 2.1] provides sufficient conditions for the uniform exponential stability in Banach spaces, as well as spectral stability conditions for time-varying systems on time scales. Properties of exponential stability of a time-varying dynamic equation on a time scale have been also investigated recently by Bohner and Martynyuk [3], DaCunha [5], Du and Tien [6], Hoffacker and Tisdell [11], Martynyuk [13] and Peterson and Raffoul [14].

As a thorough introduction into dynamic equations on time scales we refer to the paper by Hilger [9] or the monograph by Bohner and Peterson [4]. The paper [2] presents the theory with a focus on linear systems.

A *time scale* $\mathbb{T}$ is a non-empty, closed subset of the reals $\mathbb{R}$. For the purpose of this paper we assume from now on that $\mathbb{T}$ is unbounded from above, i.e. $\sup \mathbb{T} = \infty$. On $\mathbb{T}$ the *graininess* is defined as

$$\mu^*(t) := \inf \{s \in \mathbb{T} : t < s\} - t.$$

This paper is organized as follows. In Section 2 we introduce the class of systems we wish to study and define the concepts of exponential, uniform exponential, robust exponential and weak-uniform exponential stability. In Section 3 we first provide a necessary and sufficient condition for the existence of a uniformly exponentially stable linear time-invariant system. We show that uniform exponential stability implies robust exponential stability. An example illustrates that robust exponential stability, in general, does not imply weak-uniform exponential stability. The uniform exponential stability and the robust exponential stability of a system are characterized by the spectrum of its matrix, respectively. In Section 4 we provide an example which indicates that, in general, exponential stability of a system is not determined by the spectrum. We intend to relate the stability of a scalar system to the stability of the according Jordan system. We arrive at the statement that weak-uniform exponential stability of a system is characterized by the spectrum of its matrix.

2 Preliminaries

In the following $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. As usual, $\mathbb{K}^{d\times d}$ is the space of square matrices with d rows, I_d is the identity mapping on the d-dimensional space $\mathbb{K}^d$ over $\mathbb{K}$ and $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{d\times d}$.

Let $A \in \mathbb{K}^{d\times d}$ and consider the d-dimensional linear system of dynamic equations

$$x^\Delta = Ax. \tag{1}$$

Let $e_A : \{(t, \tau) \in \mathbb{T} \times \mathbb{T} : t \geq \tau\} \to \mathbb{K}^{d\times d}$ denote the *transition matrix* corresponding to (1), that is, $\varphi(t, \tau, \xi) = e_A(t, \tau)\xi$ solves the initial value problem (1) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^d$ and $t, \tau \in \mathbb{T}$ with $t \geq \tau$. The classical examples for this setup are the following.

Example 2.1 If $\mathbb{T} = \mathbb{R}$ we consider linear time-invariant systems of the form $\dot{x}(t) = Ax(t)$. If $\mathbb{T} = h\mathbb{Z}$, then (1) reduces to $(x(t + h) - x(t))/h = Ax(t)$ or equivalently $x(t + h) = [I_d + hA]x(t)$.

The subsequent notions of exponential stability (i), (ii), (iii) of system (1) are introduced here as in Pötzsche et al [16].

Definition 2.1 (Exponential stability) Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (1)

(i) *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \geq 1$ with

$$\|e_A(t,s)\| \leq K(s)\exp(-\alpha(t-s)) \quad \text{for } t \geq s.$$

(ii) *uniformly exponentially stable* if K can be chosen independently of s in the definition of exponential stability.

(iii) *robustly exponentially stable* if there is an $\varepsilon > 0$ such that the exponential stability of (1) implies the exponential stability of $x^{\Delta} = Bx$ for any $B \in \mathbb{K}^{d\times d}$ with $\|B - A\| \leq \varepsilon$.

(iv)*weak-uniformly exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \geq 1$ with

$$\|e_A(t,\tau)\| \leq K(s)e^{-\alpha(t-\tau)} \qquad \text{for all } t \geq \tau \geq s.$$

Remark 2.1 (i) The different notions of stability (i), (ii) and (iii) are partly investigated in the paper by Pötzsche et al [16], where examples are provided which show that exponential stability, in general, does neither imply uniform exponential stability nor robust exponential stability.

(ii) The notion of weak-uniform exponential stability serves as an intermediate notion between exponential stability and uniform exponential stability. Note that weak-uniform exponential stability coincides with uniform exponential stability if we can choose a bounded function $K : \mathbb{T} \to \mathbb{R}^+$ in (iv).

One of the observations in this paper is the following diagram about the relations between the stability notions:

$$\begin{tikzcd}
 & \text{uniform e. s.} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{weak-uniform e. s.} \arrow[dr, Rightarrow] & & \text{robust e.s.} \arrow[dl, Rightarrow] \\
 & \text{exponential stability} &
\end{tikzcd}$$

3 Uniform Exponential Stability

In this section, we deal with some fundamental properties of uniform exponential stability. More precisely, the existence and robustness of uniform exponential stability are investigated. As a consequence, we obtain a characterization of uniform exponential stability for a linear time-invariant system based on the spectrum of its matrix.

Theorem 3.1 (Existence of a uniformly exponentially stable system) *Let $\mathbb{T}$ be a time scale which is unbounded above. Then there exists a uniformly exponentially stable system on $\mathbb{R}^d$*

$$x^{\Delta} = Ax, \qquad A \in \mathbb{R}^{d\times d}, x \in \mathbb{R}^d,$$

if and only if the graininess of $\mathbb{T}$ is bounded above, i.e. there exists $h > 0$ such that $\mu^*(t) \leq h$ for all $t \in \mathbb{T}$.

Proof ($\Rightarrow$) Assume that there exists $A \in \mathbb{R}^{d\times d}$ such that the system
$$x^\Delta = Ax, \tag{2}$$
is uniformly exponentially stable, i.e. there exist $K > 0$, $\alpha > 0$ such that
$$\|e_A(t,s)\| \le K \exp(-\alpha(t-s)) \quad \text{for all } t \ge s. \tag{3}$$
We first show that $A \neq 0$. Indeed, suppose that $A = 0$, then $e_A(t,s) = I_d$. Hence, we have $\|e_A(t,s)\| = 1$ for all $t \ge s$ and the inequality (3) thus does not hold.

Let $t_0 \in \mathbb{T}$ be an arbitrary right scattered point, i.e. $\mu^*(t_0) > 0$. Then at the point t_0 the equation (2) becomes
$$\frac{x(t_0 + \mu^*(t_0)) - x(t_0)}{\mu^*(t_0)} = Ax(t_0).$$
This implies that $e_A(t_0 + \mu^*(t_0), t_0) = I_d + \mu^*(t_0)A$ and then by using (3) we have
$$\|I_d + \mu^*(t_0)A\| \le K\exp(-\alpha\mu^*(t_0)), \quad \text{i.e.} \ -1 + \mu^*(t_0)\|A\| \le K.$$
Therefore,
$$\mu^*(t_0) \le \frac{K+1}{\|A\|}$$
for every right scattered point $t_0 \in \mathbb{T}$, i.e. $\mathbb{T}$ has bounded graininess.

($\Leftarrow$) Assume that there exists $h > 0$ so that $\mu^*(t) \le h$ for all $t \in \mathbb{T}$. Define $A = \frac{-1}{2h} I_d$. Clearly, $I_d + \mu^*(t)A$ is invertible for all $t \in \mathbb{T}$, i.e. A is a regressive matrix. Now we will show that the system
$$x^\Delta = Ax, \tag{4}$$
is uniformly exponentially stable. Since A is a regressive diagonal matrix, Hilger [9, Theorem 7.4(iii)] implies the following explicit representation of the norm of the transition matrix of (4)
$$\begin{aligned} \|e_A(t,s)\| &= \exp\int_s^t \lim_{u\searrow\mu^*(\tau)} \frac{\log|1-\frac{u}{2h}|}{u}\,\Delta\tau \\ &\le \exp\int_s^t \frac{-1}{2h}\,\Delta\tau = \exp\left(\frac{-1}{2h}(t-s)\right). \end{aligned}$$
This completes the proof.

From now on we only deal with a time scale with bounded graininess. In order to show the roughness of uniform exponential stability, we provide the following preparatory lemma.

Lemma 3.1 *Let $\alpha > 0$ be a positive number. Then for the corresponding scalar system $x^\Delta = \alpha x$ the following inequality holds*
$$e_\alpha(t,s) \le \exp(\alpha(t-s)) \quad \textit{for all } t \ge s.$$

Proof Since $\alpha > 0$ we have $1 + \mu^*(t)\alpha > 0$ for all $t \in \mathbb{T}$. Hence, by Hilger [9, Theorem 7.4(iii)] we have
$$\begin{aligned} \|e_\alpha(t,s)\| &= \exp\int_s^t \lim_{u\searrow\mu^*(\tau)} \frac{\log|1+\alpha u|}{u}\,\Delta\tau \\ &\le \exp\int_s^t \alpha\,\Delta\tau = \exp\left(\alpha(t-s)\right). \end{aligned}$$
This concludes the proof.

Proposition 3.1 (Robustness of uniform exponential stability) *Let $\mathbb{T}$ be a time scale which is unbounded above and with bounded graininess. Assume that the system*

$$x^{\Delta} = Ax, \tag{5}$$

where $A \in \mathbb{C}^{d\times d}$, is uniformly exponentially stable. Then there exists $\varepsilon > 0$ such that the system

$$x^{\Delta} = Bx \tag{6}$$

is also uniformly exponentially stable for all $B \in \mathbb{C}^{d\times d}$ with $\|A - B\| \le \varepsilon$.

Proof Let $K > 0$ and $\alpha > 0$ such that

$$\|e_A(t,s)\| \le K \exp(-\alpha(t-s)) \qquad \text{for all } t \ge s. \tag{7}$$

The equation (6) can be rewritten as follows

$$x^{\Delta} = Ax + (B - A)x.$$

Using the variation of constants formula (see Bohner and Peterson [4, pp 195]) with the inhomogeneous part $g(t) := (B - A)e_B(t,s)$ for a fixed $s \in \mathbb{T}$, the transition matrix of (6) is determined by

$$e_B(t,s) = e_A(t,s) + \int_s^t e_A(t, u + \mu^*(u))(B - A)e_B(u,s)\,\Delta u, \quad \text{for all } t \ge s.$$

Fix $s \in \mathbb{T}$ and define $f(t) = \exp(\alpha(t-s))\|e_B(t,s)\|$. We thus obtain the following estimate

$$\begin{aligned}\exp(-\alpha(t-s))f(t) \quad &\le \quad \|e_A(t,s)\| + \|A - B\| \cdot \\ &\qquad \int_s^t \|e_A(t, u + \mu^*(u))\| \exp(-\alpha(u-s)) f(u)\,\Delta u.\end{aligned}$$

This implies with (7) that

$$f(t) \le K + K\|A - B\| \int_s^t \exp(\alpha\mu^*(u)) f(u)\,\Delta u \quad \text{for all } t \ge s. \tag{8}$$

Due to Theorem 3.1 the graininess of $\mathbb{T}$ is bounded. Fix $H > 0$ such that $\mu^*(t) \le H$ for all $t \in \mathbb{T}$. Hence, we get from (8) that

$$f(t) \le K + K\|A - B\| \exp(\alpha H) \int_s^t f(u)\,\Delta u \quad \text{for all } t \ge s.$$

Applying Gronwall's inequality (see Bohner and Peterson [4, Corollary 6.7]) and with $f(s) = 1$ we obtain

$$f(t) \le K e_M(t,s) \quad \text{for all } t \ge s,$$

where $M = K\|A - B\| \exp(\alpha H)$. By virtue of Lemma 3.1 and the definition of the function $f(t)$ we get

$$\|e_B(t,s)\| \le K \exp((-\alpha + M)(t-s)) \quad \text{for all } t \ge s. \tag{9}$$

Choose and fix $\varepsilon > 0$ such that $K\varepsilon \exp(\alpha H) < \alpha$. Now for any $B \in \mathbb{R}^{d\times d}$ with $\|A-B\| \leq \varepsilon$, we obtain from (9) that

$$\|e_B(t,s)\| \leq K \exp((-\alpha + K\varepsilon \exp(\alpha H))(t-s)) \quad \text{for all } t \geq s.$$

Since $-\alpha + K\varepsilon \exp(\alpha H) < 0$, the claim follows.

The robustness of uniform exponential stability of a time-varying system is also investigated in DaCuhna [5, Theorem 5.1]. However, the notion of uniform exponential stability and the type of perturbation in DaCuhna [5] are different to those here. Precisely, he used the exponential functions to define uniform exponential stability. For a more details, we refer the reader to DaCuhna [5], Du and Tien [6] and the references therein.

Corollary 3.1 (Uniform implies robust exponential stability) *Let* $\mathbb{T}$ *be a time scale which is unbounded above and with bounded graininess. Suppose that the system*

$$x^\Delta = Ax, \quad A \in \mathbb{R}^{d\times d}, \tag{10}$$

is uniformly exponentially stable. Then system (10) *is also robustly exponentially stable.*

Next we construct an example which asserts that, in general, robust exponential stability does not imply weak-uniform exponential stability. As a consequence, robust exponential stability does not imply uniform exponential stability.

Example 3.1 Let $d = 1$. We define a sequence s_k recursively by

$$s_1 := 0, \qquad s_{k+1} = s_k + 4k, \quad k \in \mathbb{N}$$

and

$$\mathbb{T}_k := \{s_k + i | i = 0, 1, \cdots, k-1\} \cup \{s_k + k + 3i | i = 0, 1, \cdots, k-1\}, \quad k \in \mathbb{N},$$

and the time scale $\mathbb{T}$ by the discrete set

$$\mathbb{T} := \bigcup_{k=1}^{\infty} \mathbb{T}_k.$$

Clearly, $\mathbb{T}$ is unbounded above and has a bounded graininess. Consider on $\mathbb{T}$ the scalar equation

$$x^\Delta = -x. \tag{11}$$

For $k \geq 1$ an elementary calculation yields for $x_0 \in \mathbb{R}$ that

$$\varphi(s_k + 4k, s_k + k, x_0) = (-2)^k x_0.$$

This shows that the system (11) is not weak-uniformly exponentially stable, as a solution starting in $x_0 = 1$ may become arbitrarily large depending on the initial time $t_0 \in \mathbb{T}$. Now we are going to show that, on the other hand, the system (11) is robustly exponentially stable. To verify this claim we show that the perturbed system

$$x^\Delta = (-1 + \alpha)x \tag{12}$$

is exponentially stable for all $\alpha \in (-\frac{1}{10}, \frac{1}{10})$. Let $x_0 \in \mathbb{R}$ be an arbitrary initial value and $t_0 \in \mathbb{T}$. Denote by k_0 the smallest integer such that $t_0 \leq s_{k_0}$. Now we are going to prove inductively the following estimate

$$|\varphi(t, s_{k_0}, x_0)| \leq \left(\frac{1}{2}\right)^{\frac{t-s_{k_0}}{3}} |x_0| \quad \text{for all } s_k < t \leq s_{k+1} \text{ and } k_0 \leq k. \tag{13}$$

We first prove (13) in case $k = k_0$. Indeed, a straightforward computation yields that

$$\varphi(t, s_{k_0}, x_0) = \begin{cases} \alpha^m x_0 & \text{if } t = s_{k_0} + m, \text{for } m = 1, \dots k_0, \\ \alpha^{k_0}(-2+3\alpha)^m x_0 & \text{if } t = s_{k_0} + k_0 + 3m, \text{for } m = 0, 1, \dots k_0. \end{cases}$$

This implies with the inequality $|\alpha(-2+3\alpha)| \leq \frac{1}{4}$ the inequality (13) in case $k = k_0$. Suppose that the inequality (13) holds for $k = n-1$. We will show that this also holds for $k = n+1$. Indeed, an elementary computation gives

$$\varphi(t, s_{k_0}, x_0) = \begin{cases} \alpha^m \varphi(s_n, s_{k_0}, x_0) & \text{if } t = s_n + m, \text{for } m = 1, \dots n, \\ \alpha^n(-2+3\alpha)^m \varphi(s_n, s_{k_0}, x_0) & \text{if } t = s_n + n + 3m, \\ & \qquad \text{for } m = 0, 1, \dots n. \end{cases}$$

This implies with the inequality $|\alpha(-2+3\alpha)| \leq \frac{1}{4}$ the inequality (13) in case $k = n$ and then the claim follows. Define $K(t_0) = |\Phi(s_{k_0}, t_0, x_0)|, \beta = \frac{\log 2}{3}$ and by (13) we get

$$|\varphi(t, t_0 x_0)| \leq K(t_0) \exp(-\beta(t - t_0)) \quad \text{for all } t_0 \leq t.$$

Corollary 3.2 *Let T be a time scale which is unbounded above and with bounded graininess. For $\lambda \in \mathbb{C}$ consider the Jordan block $J_\lambda \in \mathbb{C}^{d\times d}$ given by*

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

The scalar equation

$$x^\Delta = \lambda x \tag{14}$$

is uniformly exponentially stable if and only if the system

$$x^\Delta = J_\lambda x \tag{15}$$

is uniformly exponentially stable.

Proof ($\Rightarrow$) Assume that (14) is uniformly stable. Hence, the equation

$$x^\Delta = \lambda I_d \, x$$

is also uniformly exponentially stable. So, by virtue of Proposition 3.1 there exists $\varepsilon > 0$ such that the system

$$x^\Delta = Bx \tag{16}$$

is uniformly exponentially stable for all $B \in \mathbb{C}^{d\times d}$ such that $\|B - \lambda I_d\| \le \varepsilon$. Define $P_\varepsilon = \mathrm{diag}(1, \varepsilon^{-1}, \dots, \varepsilon^{-d})$. A straightforward computation yields that

$$B_\varepsilon := P_\varepsilon J_\lambda P_\varepsilon^{-1} = \begin{pmatrix} \lambda & \varepsilon & 0 & \dots & 0 \\ & \lambda & \varepsilon & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

Consequently,

$$e_{J_\lambda}(t,s) = P_\varepsilon e_{B_\varepsilon}(t,s) \qquad \text{for all} \quad t \ge s. \tag{17}$$

On the other hand, $\|B_\varepsilon - \lambda\, I_d\| \le \varepsilon$. Hence, by (16) there exists $K, \alpha > 0$ such that

$$\|e_{B_\varepsilon}(t,s)\| \le K e^{-\alpha(t-s)} \qquad \text{for all} \quad t \ge s.$$

This implies with (17) that

$$\|e_{J_\lambda}(t,s)\| \le K\|P_\varepsilon\| e^{-\alpha(t-s)} \qquad \text{for all} \quad t \ge s.$$

Therefore, (15) is uniformly exponentially stable and it completes the proof.

($\Leftarrow$) The converse direction is trivial.

In the next theorem, we will show that uniform exponential stability of a linear system depends only on the eigenvalues of its matrix.

Theorem 3.2 *The system*

$$x^\Delta = Ax, \qquad A \in \mathbb{R}^{d\times d},$$

is uniformly exponentially stable if and only if the system

$$x^\Delta = \lambda x$$

is uniformly exponentially stable for every $\lambda \in \sigma(A)$.

Proof Without loss of generality, we deal with the norm $\|M\| = \max_{1\le i,j\le n} |m_{ij}|$ for all $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. Let P be the transformation such that $P^{-1}AP = \mathrm{diag}(J_1, J_2, \dots, J_p)$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ & \lambda_k & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda_k \end{pmatrix}, \qquad \text{for } k = 1, 2, \dots, p,$$

are the Jordan blocks of A. Clearly, $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$. A straightforward computation yields that

$$e_A(t,s) = P\mathrm{diag}(e_{J_1}(t,s), e_{J_2}(t,s), \dots, e_{J_p}(t,s)).$$

Therefore,

$$\frac{1}{\|P^{-1}\|} \max_{1\le k\le p} \|e_{J_k}(t,s)\| \le \|e_A(t,s)\| \le \|P\| \max_{1\le k\le p} \|e_{J_k}(t,s)\|.$$

This implies that (3.2) is exponentially stable if and only if the systems

$$x^\Delta = J_k x, \qquad k = 1, 2, \dots, p,$$

are exponentially stable. Then by virtue of Corollary 3.2 the claim follows.

Remark 3.1 The robust exponential stability depends also only on the eigenvalues of the matrix of a system.

4 Exponential Stability and Weak-uniform Exponential Stability

In view of Corollary 3.2, the question arises whether the exponential stability of a time-invariant linear system could be characterized by the spectrum of its matrix. In general, this is not the case, since in the subsequent example two systems are presented whose matrices have the same spectrum, one of them is exponentially stable and the other is not.

Example 4.1 There exists a time scale $\mathbb{T}$, which has bounded graininess such that the system

$$x^{\Delta} = -x \tag{18}$$

is exponentially stable and the system

$$x^{\Delta} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x \tag{19}$$

is not exponentially stable.

Indeed, denote by α_n the positive number such that

$$\alpha_k(1+\alpha_k) = 2^{1-4^{k+1}}, \qquad \text{for all } k \in \mathbb{N}. \tag{20}$$

Equivalently,

$$\alpha_k := \frac{-1+\sqrt{1+2^{3-4^{k+1}}}}{2}, \qquad \text{for all } k \in \mathbb{N}.$$

We define a discrete time scale $\mathbb{T}$ as follows

$$\mathbb{T} = \bigcup_{k=1}^{\infty} \mathbb{T}_k,$$

where

$$\mathbb{T}_k := \{4^k\} \cup \{4^k + 1 - \alpha_k + 3i : i = 0, 1, \cdots, 4^k - 1\}. \tag{21}$$

To verify exponential stability of system (18) we show that

$$|e_{-1}(t,4)| \leq \left(\frac{1}{2}\right)^{t-4} \qquad \text{for all } t \in \mathbb{T}, t \geq 4. \tag{22}$$

Indeed, let t be an arbitrary but fixed element in $\mathbb{T}$. Define

$$n_0 := \max\{n : n \in \mathbb{N}, 4^n < t\}.$$

A straightforward computation together with (20) yields that

$$e_{-1}(4^{n+1}, 4^n) = \alpha_n(1+\alpha_n)2^{4^n-1} = \left(\frac{1}{2}\right)^{4^{n+1}-4^n}, \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

Therefore,

$$\begin{aligned} e_{-1}(t,4) &= e_{-1}(t,4^{n_0})e_{-1}(4^{n_0},4) \\ &= e_{-1}(t,4^{n_0})\left(\frac{1}{2}\right)^{4^{n_0}-4}. \end{aligned}$$

Clearly, if $t = 4^{n_0+1}$ then (22) follows. Hence, it remains to deal with the case $t < 4^{n_0+1}$. By definition of $\mathbb{T}$, see (21), we obtain

$$|e_{-1}(t,4^{n_0})| = \alpha_{n_0}2^k,$$

where $t = 4^{n_0} + 1 - \alpha_{n_0} + 3k$ for $k \in \{0,1,\dots,4^{n_0}-1\}$. This implies with (20) that

$$\begin{aligned} |e_{-1}(t,4^{n_0})| &\leq 2^{1-4^{n_0+1}+k} \\ &\leq \left(\frac{1}{2}\right)^{3k+1-\alpha_{n_0}} \qquad \text{for all } k \in \{0,\dots,4^{n_0}-1\}, \end{aligned}$$

proving (22). As a consequence, system (18) is exponentially stable. It remains to show that system (19) is not exponentially stable. System (19) can be represented in the following form

$$\begin{aligned} x_1^\Delta &= -x_1, \\ x_2^\Delta &= -x_2 + x_1, \end{aligned}$$

where $x = (x_1, x_2)$. Denote by $(x_1(t), x_2(t))$ the solution of this system starting at $t_0 = 4$ in $(1,1)$. A straightforward computation yields

$$\begin{aligned} \begin{pmatrix} x_1(4^{k+1}) \\ x_2(4^{k+1}) \end{pmatrix} &= \begin{pmatrix} -1-\alpha_k & 0 \\ 1 & -1-\alpha_k \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix}^{4^k-1} \times \\ &\qquad \times \begin{pmatrix} \alpha_k & 0 \\ 1 & \alpha_k \end{pmatrix} \begin{pmatrix} x_1(4^k) \\ x_2(4^k) \end{pmatrix}, \end{aligned}$$

which gives

$$x_2(4^{k+1}) = (2^{4^k-1} - 2^{4^k-4^{k+1}} - 1)x_1(4^k) + 2^{4^k-4^{k+1}}x_2(4^k).$$

This implies together with $x_1(4^k) = 2^{4-4^k}$ that

$$x_2(4^{k+1}) = 8 - 2^{3-4^{k+1}} + 2^{4^k-4^{k+1}}x_2(4^k) \quad \text{for all } k \in \mathbb{N}.$$

Hence, $\lim_{k\to\infty} x_2(4^k) = 8$. As a consequence, system (19) is not exponentially stable.

Proposition 4.1 *Let $\mathbb{T}$ be a time scale which is unbounded above. For $\lambda \in \mathbb{C}$ consider the Jordan block $J_\lambda \in \mathbb{C}^{d\times d}$ given by*

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

The scalar equation

$$x^{\Delta} = \lambda x \tag{24}$$

is weak-uniformly exponentially stable if and only if the system

$$x^{\Delta} = J_{\lambda} x \tag{25}$$

is weak-uniformly exponentially stable.

Proof Clearly, if $\lambda = 0$ then $e_0(t,\tau) = 1$ for all $t \geq \tau$. Hence, neither the system (24) nor the system (25) is weak-uniformly exponentially stable. Therefore we are only interested in the case $\lambda \neq 0$.

($\Rightarrow$) Assume that system (24) is weak-uniformly exponentially stable. Fix $s \in \mathbb{T}$. Hence, there exist $\alpha > 0$ and $K = K(s) \geq 1$ such that

$$|e_{\lambda}(t,\tau)| \leq K \exp(-\alpha(t-\tau)) \qquad \text{for all } t \geq \tau \geq s. \tag{26}$$

To verify the assertion we construct explicit bounds for the solution of (25) with initial condition $x(\tau) = \xi \in \mathbb{C}^d$ for a fixed $\tau \in \mathbb{T}$ with $\tau \geq s$. Without loss of generality we use the norm $\|x\| := \max\{|x_1|, \dots, |x_d|\}$ for $x = (x_1, \dots, x_d) \in \mathbb{C}^d$ in our consideration.

Choose and fix $\varepsilon > 0$ such that $\alpha > d\varepsilon$. Define $\beta_j = \alpha + j\varepsilon - d\varepsilon$ for $j = 1, \dots, d$. Clearly, $\beta_j > 0$ and we will prove by induction on $j = d, \dots, 1$ that there exist constants K_j such that the j-th component of the solution of (25) is exponentially bounded by

$$|x_j(t)| \leq K_j \exp(-\beta_j(t-\tau))\|\xi\| \qquad \text{for all} \quad t \geq \tau. \tag{27}$$

For $j = d$ the assertion follows from the assumption as the d-th entry of $x(t)$ is a solution of (24) and hence by (26) we have

$$|x_d(t)| = |e_{\lambda}(t,\tau)\xi_d| \leq K_d \exp(-\beta_j(t-\tau))\|\xi\| \quad \text{for all } t \geq \tau,$$

where $K_d := K$. Assume that the assertion (27) is shown for some index $d \geq j+1 \geq 2$, i.e. there exists K_{j+1} with

$$|x_{j+1}(t)| \leq K_{j+1} \exp(-\beta_{j+1}(t-\tau))\|\xi\| \quad \text{for all } t \geq \tau. \tag{28}$$

By construction, the j-th component of the solution satisfies the equation

$$x_j^{\Delta}(t) = \lambda x_j(t) + x_{j+1}(t) \quad \text{for } t \in \mathbb{T}.$$

Using the variation of constants formula (see Bohner and Peterson [4, pp 77]) we have the representation

$$x_j(t) = e_{\lambda}(t,\tau)\xi_j + \int_{\tau}^{t} e_{\lambda}(t, u + \mu^*(u)) x_{j+1}(u)\, \Delta u.$$

Fix $t \in \mathbb{T}$. Using the exponential bound of $e_{\lambda}(t,\tau)$ and (28) we obtain

$$\begin{aligned} |x_j(t)| &\leq |e_{\lambda}(t,\tau)\xi_j| + \int_{\tau}^{t} |e_{\lambda}(t, u+\mu^*(u))||x_{j+1}(u)|\, \Delta u \\ &\leq K\|\xi\| \exp(-\alpha(t-\tau)) + K_{j+1}\|\xi\| \int_{\tau}^{t} g(u)\, \Delta u, \end{aligned} \tag{29}$$

where $g(u) := |e_\lambda(t, u + \mu^*(u))| \exp(-\beta_{j+1}(u - \tau))$. Denote by $t_1 < t_2 < \cdots < t_n$ the right scattered points in $[\tau, t]$ with

$$|1 + \lambda\mu^*(t_i)| \geq 2, \quad i = 1, 2, \dots, n.$$

Now we are going to estimate $g(u)$ for $u \in [\tau, t]$. If $u = t_k$ for $k = 1, \dots, n$ we get

$$\begin{aligned} e_\lambda(t, u) &= e_\lambda(t, u + \mu^*(u)) e_\lambda(u + \mu^*(u), u) \\ &= e_\lambda(t, u + \mu^*(u))(1 + \lambda\mu^*(u)). \end{aligned}$$

This implies with (26) that

$$|e_\lambda(t, u + \mu^*(u))| \leq \frac{K}{2} \exp(-\alpha(t - u)).$$

Therefore,

$$g(u) \leq \frac{K}{2} \exp(-\beta_{j+1}(t - \tau)) \qquad \text{if } u \in \{t_1, \dots, t_n\}. \tag{30}$$

If $u \notin \{t_1, \dots, t_n\}$, we get $\mu^*(u) \leq \frac{3}{|\lambda|}$. Applying (26) to $e_\lambda(t, u + \mu^*(u))$, we obtain

$$g(u) \leq K \exp(-\alpha(t - u)) \exp(\alpha\mu^*(u))) \exp(-\beta_{j+1}(u - \tau)).$$

Therefore,

$$g(u) \leq K \exp\left(\frac{3\alpha}{|\lambda|}\right) \exp(-\beta_{j+1}(t - \tau)) \quad \text{for } u \notin \{t_1, \dots, t_n\}. \tag{31}$$

Combining (30) and (31), there exists $M > 0$ such that

$$g(u) \leq M \exp(-\beta_{j+1}(t - \tau)) \quad \text{for all } u \in [\tau, t].$$

This implies with (29) that

$$|x_j(t)| \leq K\|\xi\| \exp(-\alpha(t - \tau)) + MK_{j+1}\|\xi\|(t - \tau) \exp(-\beta_{j+1}(t - \tau)).$$

On the other hand, $\varepsilon(t - \tau) \leq \exp(\varepsilon(t - \tau))$ for all $t \geq \tau$. We thus obtain

$$|x_j(t)| \leq (K + \frac{MK_{j+1}}{\varepsilon})\|\xi\| \exp(-\beta_j(t - \tau)) \quad \text{for all } t \geq \tau,$$

proving (27) with $K_j := K + \frac{MK_{j+1}}{\varepsilon}$. As we have exponential decay of all components of the solution $x(t)$, we obtain the assertion.

We now construct an example which ensures that in general weak-uniform exponential stability does not imply uniform exponential stability.

Example 4.2 We define a discrete time scale $\mathbb{T}$ by

$$\mathbb{T} = \bigcup_{k=1}^{\infty} \mathbb{T}_k \cup [0, \infty),$$

where

$$\mathbb{T}_k := \{-k + \frac{-i}{k} : i = 0, 1, \dots, k - 1\} \qquad \text{for all } k \in \mathbb{N}.$$

Consider on $\mathbb{T}$ the scalar system

$$x^\Delta = -x. \tag{32}$$

We first show that the system (32) is weak-uniformly exponentially stable. Obviously, for any $s \in \mathbb{T}$ with $s \geq 0$ we have

$$|e_{-1}(t,\tau)| = \exp(-(t-\tau)) \quad \text{for all } t \geq \tau \geq 0. \tag{33}$$

For an arbitrary but fixed $s \in \mathbb{T}$ with $s < 0$, we are going to estimate $|e_{-1}(t,\tau)|$ for $t \geq \tau \geq s$. A straightforward computation yields that

$$|e_{-1}(t,\tau)| = \begin{cases} 0, & \text{if } t \geq 0 > \tau, \\ \exp(-(t-\tau)), & \text{if } t \geq \tau \geq 0. \end{cases}$$

We thus obtain

$$|e_{-1}(t,\tau)| \leq K(s)\exp(-(t-\tau)) \quad \text{for all } t \geq \tau \geq s,$$

where

$$K(s) := \max_{0 \geq t \geq \tau \geq s} |e_{-1}(t,\tau)| \exp(t-\tau).$$

Hence system (32) is weak-uniformly exponentially stable. On the other hand, a direct computation gives that

$$|e_{-1}(-k,-k-1)| = (1-\frac{1}{k})^k \quad \text{for all } k \in \mathbb{N},$$

which implies that system (32) is not uniformly exponentially stable.

Remark 4.1 Observe that Proposition 4.1 in combination with example 4.2 provides a negative answer to the question mentioned in the conclusion of Pötzsche et al [16] whether the uniform exponential stability of system (24) is a necessary condition for the exponential stability of system (25). Moreover, the time scale $\mathbb{T}$ in Pötzsche et al [16] is assumed to have bounded graininess. This assumption is dropped in Proposition 4.1.

By virtue of Proposition 4.1 in combination with an analogous argument as in the proof of Theorem 3.2 we get the following corollary to characterize weak-uniform exponential stability.

Corollary 4.1 *The system*

$$x^\Delta = Ax, \qquad A \in \mathbb{R}^{d\times d},$$

is weak-uniformly exponentially stable if and only if the system

$$x^\Delta = \lambda x$$

is weak-uniformly exponentially stable for every $\lambda \in \sigma(A)$.

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Oscillation Criteria for Half-Linear Delay Dynamic Equations on Time Scales

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Abstract: This paper is concerned with oscillation of the second-order half-linear delay dynamic equation

$$(r(t)(x^{\Delta})^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$ is the quotient of odd positive integers, $p(t)$, and $\tau : \mathbb{T} \to \mathbb{T}$ are positive rd-continuous functions on $\mathbb{T}$, $r(t)$ is positive and (delta) differentiable, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Our results in the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ involve and improve some oscillation results for second-order differential and difference equations; and when $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0}$ and $\mathbb{T} = \mathbb{N}^2$ our oscillation results are essentially new. Some examples illustrating the importance of our results are also included.

Keywords: *oscillation; delay half-linear dynamic equations; time scales.*

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1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph. D. Thesis in 1988 in order to unify continuous and discrete analysis, see [10]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [3]). This new theory of these so-called "dynamic equations" not only unifies the corresponding theories for the differential equations and difference equations cases, but it also extends these classical cases to cases "in between". That is, we are able to treat the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory (see [12])) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the set of the harmonic numbers. The books on the subject of time scales by Bohner and Peterson [3], [4] summarize and organize much of time scale calculus. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [1], [5], [7], [8], [9], [15] and the references cited therein. In this paper, we are concerned with oscillation behavior of the second-order half-linear delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma \geq 1$ is a quotient of odd positive integers, p is a positive $rd-$continuous function on $\mathbb{T}$, $r(t)$ is positive and (delta) differentiable and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real–valued function $x \in C^1_{rd}[T_x, \infty)$, $T_x \geq t_0$ which has the property that $r(t)(x^{\Delta}(t))^{\gamma} \in C^1_{rd}[T_x, \infty)$ and satisfies equation (1.1) on $[T_x, \infty)$, where C_{rd} is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Note that if $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and (1.1) becomes the half-linear delay differential equation

$$\left(r(t)\left(x'(t)\right)^{\gamma}\right)' + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.2}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the half-linear delay difference equation

$$\Delta(r(t)\left(\Delta x(t)\right)^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.3}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, $y^{\Delta}(t) = \Delta_h y(t) := \frac{y(t+h)-y(t)}{h}$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h$, and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_h(r(t)\left(\Delta_h x(t)\right)^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.4}$$

If $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q - 1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(q\,t) - x(t))/(q - 1)\,t$, $\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f(q^k)\mu(q^k)$, where $t_0 = q^{n_0}$, and (1.1) becomes the second-order $q-$half-linear delay difference equation

$$\Delta_q(r(t)\,(\Delta_q x(t))^\gamma) + p(t)x^\gamma(\tau(t)) = 0. \tag{1.5}$$

If $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$, $\mu(t) = 1 + 2\sqrt{t}$, $\Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}}$, and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_N(r(t)\,(\Delta_N x(t))^\gamma) + p(t)x^\gamma(\tau(t)) = 0. \tag{1.6}$$

If $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the so-called n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^n \frac{1}{k}$, $n \in \mathbb{N}_0$, then $\sigma(H_n) = H_{n+1}$, $\mu(H_n) = \frac{1}{n+1}$, $y^\Delta(t) = \Delta_{H_n} y(H_n) = (n+1)\Delta y(H_n)$ and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_{H_n}(r(H_n)\,(\Delta_{H_n} x(H_n))^\gamma) + p(H_n)x^\gamma(\tau(H_n)) = 0. \tag{1.7}$$

Recall that for a discrete time scale

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_\mathbb{T}} f(t)\mu(t).$$

In the following, we state some oscillation results for differential and difference equations that will be related to our oscillation results for (1.1) on time scales and explain the important contributions of this paper. We will see that our results not only unify some of the known oscillation results for differential and difference equations but also give new oscillation criteria which include the equations (1.3)–(1.7), where in many cases the oscillatory behavior of their solutions was not known. In 1948 Hille [11] considered the linear differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.8}$$

where $p(t)$ is a positive function, and proved that if

$$\liminf_{t\to\infty} t\int_t^\infty p(s)ds > \frac{1}{4}, \tag{1.9}$$

then every solution of (1.8) oscillates. In 1957 Nehari [13] proved that if

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t s^2 p(s)ds > \frac{1}{4}, \tag{1.10}$$

then every solution of (1.8) oscillates. We note that the inequalities (1.9) and (1.10) are exact and can not be weakened. Indeed, let $p(t) = 1/4t^2$ for $t \geq 1$. Then we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^t s^2 p(s)ds = \liminf_{t\to\infty} t\int_t^\infty p(s)ds = \frac{1}{4},$$

and the second-order Euler–Cauchy differential equation

$$x''(t) + \frac{1}{4t^2}x(t) = 0, \quad t \geq 1, \tag{1.11}$$

has a nonoscillatory solution $x(t) = \sqrt{t}$. In other words the constant 1/4 is the lower bound for oscillation for all solutions of (1.11). In 1971 Wong [17] generalized the Hille-type condition (1.9) for the delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \tag{1.12}$$

where $\tau(t) \geq \alpha t$ with $0 < \alpha < 1$, and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4\alpha}, \tag{1.13}$$

then every solution of (1.12) is oscillatory. In 1973 Erbe [6] improved the condition (1.13) and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}ds > \frac{1}{4}, \tag{1.14}$$

then every solution of (1.12) oscillates where $\tau(t) \leq t$. In 1984 Ohriska [14] proved that, if

$$\limsup_{t\to\infty} t \int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right) ds > 1, \tag{1.15}$$

then every solution of (1.12) oscillates. Note that when $p(t) = \frac{\lambda}{t\tau(t)}$, with $\tau(t) \leq t$, (1.12) reduces to the second-order delay differential equation

$$x''(t) + \frac{\lambda}{t\tau(t)}x(\tau(t)) = 0, \quad t \geq t_0. \tag{1.16}$$

From (1.14) we see that every solution of (1.16) is oscillatory if $\lambda > \frac{1}{4}$ and nonoscillatory if $\lambda \leq \frac{1}{4}$, with oscillation constant 1/4 (see [1]). For oscillation of half-linear differential equations, Agarwal et al [2] considered the equation

$$\left((x'(t))^\gamma\right)' + p(t)x^\gamma(\tau(t)) = 0, \tag{1.17}$$

and extended the condition (1.15) of Ohriska and proved that if

$$\limsup_{t\to\infty} t^\gamma \int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds > 1, \tag{1.18}$$

then every solution of (1.17) oscillates. It is clear that the condition (1.18) reduces to (1.15) when $\gamma = 1$. From which, we can easily see that the oscillation condition (1.18) that has been established by Agarwal et al [2] for (1.17) is not a sharp sufficient condition for oscillation of (1.17), since the condition (1.15) that has been established by Ohriska [14] is not sharp. For oscillation of half-linear difference equations, Thandapani et al [16] considered the difference equation

$$\Delta((\Delta x(n)))^\gamma) + p(n)x^\gamma(n) = 0, \quad n \geq n_0, \tag{1.19}$$

where $\gamma > 0$, $p(n)$ is a positive sequence, and proved that every solution is oscillatory, if

$$\sum_{n=n_0}^\infty p(n) = \infty. \tag{1.20}$$

We note that the condition (1.20) can not be applied to the difference equation

$$\Delta((\Delta x(n)))^\gamma) + \frac{\beta}{n^\gamma}x^\gamma(n) = 0, \qquad \text{for } \gamma > 1. \tag{1.21}$$

In view of the above comments, we shall establish oscillation criteria for the dynamic equation (1.1) on a time scale $\mathbb{T}$ which as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$:

(i) involve the oscillation conditions (1.9) and (1.10) that have been given by Hille [11] and Nehari [13] for equation (1.8)

(ii) involve the oscillation condition (1.14) that was given by Erbe [6] for delay equation (1.12),

(iii) improve the oscillation condition (1.18) that was given by Agarwal et al [2] for half-linear differential equation (1.17),

(iv) improve the oscillation condition (1.20) that was established by Thandapani et al [16] for half-linear difference equation (1.19).

This paper is organized as follows: In Section 2, we establish some sufficient conditions for oscillation of (1.1) when $r(t) = 1$, which partially anwers the above question. Also, by using the Riccati transformation technique we will establish some new oscillation criteria for (1.1) in its general form when $r(t) \not\equiv 1$, which can be considered as a generalization of the results that have been established by Saker [15] and as a special case involve some results established by Agarwal et al [2] for half-linear differential equations. In Section 3, we give several examples which illustrate the importance of our main results. Note that the results are essentially new for equations (1.2)–(1.7). To the best of our knowledge nothing is known regarding the oscillatory behavior of half-linear delay dynamic equations on time scales until now so this paper initiates this study.

2 Main Results

Thoughout the paper we assume that $r^{\Delta}(t) \geq 0$ and

$$\int_{t_0}^{\infty} \tau^{\gamma}(t) p(t) \Delta t = \infty \tag{2.1}$$

is satisfied. Before stating our main results, we begin with the following lemma which will play an important role in the proof of our main results.

Lemma 2.1 *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty \tag{2.2}$$

holds and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad x(t) > t x^{\Delta}(t), \quad \left(\frac{x(t)}{t}\right)^{\Delta} < 0 \quad \text{on } [T, \infty)_{\mathbb{T}}.$$

Proof Let x be as in the statement of this theorem. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since x is a positive solution of (1.1), we have $(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} = -p(t) x^{\gamma}(\tau(t)) < 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. Then $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We claim that $r(t)\left(x^{\Delta}(t)\right)^{\gamma} > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} =: c < 0$. Therefore, $r(t)\left(x^{\Delta}(t)\right)^{\gamma} \leq r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} = c$, for $t \in [t_2, \infty)_{\mathbb{T}}$, and therefore $x^{\Delta}(t) \leq \frac{a}{r^{\frac{1}{\gamma}}(t)}$, for $t \in [t_2, \infty)_{\mathbb{T}}$ where $a := c^{\frac{1}{\gamma}} < 0$. Integrating, we find that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s) \Delta s \leq x(t_2) + a \int_{t_2}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \to -\infty \quad \text{as } t \to \infty,$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)\left(x^{\Delta}(t)\right)^{\gamma} > 0$ on $[t_1,\infty)_{\mathbb{T}}$ and so $x^{\Delta}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. We now show that $x^{\Delta\Delta}(t) < 0$. Since $(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0$ on $[t_1,\infty)_{\mathbb{T}}$, we have after differentiation that

$$r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma} + r^{\sigma}(t)\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0. \tag{2.3}$$

Using the Pötzsche chain rule ([3, Theorem 1.90])

$$(f\circ g)^{\Delta}(t) = \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh\; g^{\Delta}(t), \tag{2.4}$$

we have

$$\begin{aligned}\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} &= \gamma\int_0^1 \left[x^{\Delta}(t) + h\mu(t)x^{\Delta\Delta}(t)\right]^{\gamma-1}\; dh\; x^{\Delta\Delta}(t)\\ &= \gamma x^{\Delta\Delta}(t)\int_0^1 \left[x^{\Delta}(t) + h[x^{\Delta\sigma}(t) - x^{\Delta}(t)]\right]^{\gamma-1} dh\\ &= \gamma x^{\Delta\Delta}(t)\int_0^1 \left[hx^{\Delta\sigma}(t) + (1-h)x^{\Delta}(t)\right]^{\gamma-1} dh. \end{aligned}\tag{2.5}$$

From (2.3) we have that

$$r^{\sigma}(t)\left((x^{\Delta}(t))^{\gamma}\right)^{\Delta} < -r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma} \le 0,$$

since $r^{\Delta}(t) \ge 0$ and $x^{\Delta}(t) > 0$ and so it follows that

$$r^{\sigma}(t)\left((x^{\Delta}(t))^{\gamma}\right)^{\Delta} < 0.$$

This shows by (2.5) that $x^{\Delta\Delta}(t) < 0$, since the integral in (2.5) is positive. Next, we show that $\left(\frac{x(t)}{t}\right)^{\Delta} < 0$. To see this, let $U(t) := x(t) - tx^{\Delta}(t)$, then $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. This implies that $U(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. We claim there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Assume not, then $U(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.6}$$

Pick $t_3 \in [t_1,\infty)_{\mathbb{T}}$ so that $\tau(t) \ge \tau(t_1)$, for $t \ge t_3$. Then

$$x(\tau(t))/\tau(t) \ge x(\tau(t_1))/\tau(t_1) =: d > 0,$$

so that $x(\tau(t)) \ge d\tau(t)$ for $t \ge t_3$. Now by integrating both sides of the dynamic equation (1.1) from t_3 to t we have

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} - r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} + \int_{t_3}^{t} p(s)x^{\gamma}(\tau(s))\Delta s = 0.$$

This implies that

$$r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} \ge \int_{t_3}^{t} p(s)x^{\gamma}(\tau(s))\Delta s \ge d^{\gamma}\int_{t_3}^{t} p(s)\tau^{\gamma}(s)\Delta s. \tag{2.7}$$

Letting $t \to \infty$ we obtain a contradiction to assumption (2.1). Hence there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_2, \infty)_{\mathbb{T}}, \tag{2.8}$$

and we have that $\left(\frac{x(t)}{t}\right)^{\Delta} < 0$ on $[t_2, \infty)_{\mathbb{T}}$. This completes the proof of Lemma 2.1.

In the following, we consider the equation (1.1) in the special case $r(t) \equiv 1$, namely,

$$((x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{2.9}$$

where $\gamma \geq 1$ is the quotient of odd positive integers and $p(t)$ is an $rd-$continuous and positive function and $\tau(t) \leq t$. We introduce the following notation.

$$p_* := \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} P(s)\Delta s, \quad q_* := \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^{\gamma+1} P(s)\Delta s, \tag{2.10}$$

where $P(t) := p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}$ and assume that $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. Note that $0 \leq l \leq 1$. In order for the definition of p_* to make sense we assume that

$$\int_{t_0}^{\infty} P(t)\Delta t < \infty. \tag{2.11}$$

Theorem 2.1 *Assume that* $l > 0$ *and* (2.11) *holds. Let* $x(t)$ *be an eventually positive solution of* (2.9) *such that* $x(t)$ *and* $x(\tau(t)) > 0$ *for* $t \geq t_1 > t_0$. *Let* $w(t) = \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}$ *and*

$$r := \liminf_{t\to\infty} t^{\gamma} w^{\sigma}(t), \quad \text{and} \quad R := \limsup_{t\to\infty} t^{\gamma} w^{\sigma}(t), \tag{2.12}$$

then

$$p_* \leq r - l^{\gamma} r^{1+\frac{1}{\gamma}} \quad \text{and} \quad p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.13}$$

Proof From Lemma 2.1 we get there is a $T \in [t_1, \infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemma 2.1. This implies that $w(t)$ is positive. Using the quotient rule and equation (2.9), we get

$$w^{\Delta}(t) = -\left(\frac{x(\tau(t))}{x^{\sigma}(t)}\right)^{\gamma} p(t) - \frac{(x^{\Delta}(t))^{\gamma}\,(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}}.$$

Since

$$\frac{x(\tau(t))}{\tau(t)} \geq \frac{x(t)}{t} \geq \frac{x^{\sigma}(t)}{\sigma(t)} \quad \text{and} \quad x^{\Delta}(t) \geq x^{\Delta\sigma}(t),$$

we get the inequality

$$w^{\Delta}(t) \leq -\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} p(t) - \frac{(x^{\Delta\sigma}(t))^{\gamma}\,(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}}, \tag{2.14}$$

since $x^{\Delta\Delta}(t) < 0$. By the Pötzsche chain rule, and the fact that $x^{\Delta}(t) > 0$, we obtain

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1 \left[x(t)+h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t)\\ &\geq \gamma\int_0^1 (x(t))^{\gamma-1}\, dh\, x^{\Delta}(t)\\ &= \gamma(x(t))^{\gamma-1}x^{\Delta}(t).\end{aligned} \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned}w^{\Delta}(t) &\leq -P(t)-\frac{\left(x^{\Delta\sigma}(t)\right)^{\gamma}\gamma(x(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}\\ &= -P(t)-\gamma w^{\sigma}(t)w^{\frac{1}{\gamma}}(t).\end{aligned}$$

Then $w(t)$ satisfies the dynamic Riccati inequality

$$w^{\Delta}(t)+P(t)+\gamma w^{\sigma}(t)w^{\frac{1}{\gamma}}(t)\leq 0, \quad \text{for} \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.16}$$

Since $P(t) > 0$ and $w(t) > 0$ for $t \geq t_1$, it follows from (2.16) that $w^{\Delta}(t) < 0$ and hence $w(t)$ is strictly decreasing for $t \in [T,\infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that

$$w(t)=\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}<\frac{1}{t^{\gamma}}, \quad t\in[T,\infty)_{\mathbb{T}}, \tag{2.17}$$

which implies that $\lim_{t\to\infty} w(t) = 0$ and that

$$0\leq r\leq R\leq 1. \tag{2.18}$$

Now, we prove that the first inequality in (2.13) holds. Let $\epsilon > 0$, then by the definition of p_* and r we can pick $t_2 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$t^{\gamma}\int_{\sigma(t)}^{\infty}P(s)\Delta s\geq p_*-\epsilon, \quad \text{and} \quad t^{\gamma}w^{\sigma}(t)\geq r-\epsilon,$$

for $t \in [t_2,\infty)_{\mathbb{T}}$. Integrating (2.16) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^{\sigma}(t)\geq\int_{\sigma(t)}^{\infty}P(s)\Delta s+\gamma\int_{\sigma(t)}^{\infty}w^{\frac{1}{\gamma}}(s)w^{\sigma}(s)\Delta s, \quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{2.19}$$

It follows from (2.19) that

$$\begin{aligned}t^{\gamma}w^{\sigma}(t) &\geq t^{\gamma}\int_{\sigma(t)}^{\infty}P(s)\Delta s+\gamma t^{\gamma}\int_{\sigma(t)}^{\infty}w^{\frac{1}{\gamma}}(s)w^{\sigma}(s)\Delta s\\ &\geq (p_*-\epsilon)+\gamma t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{s\left(w^{\sigma}(s)\right)^{\frac{1}{\gamma}}s^{\gamma}w^{\sigma}(s)}{s^{\gamma+1}}\Delta s\\ &\geq (p_*-\epsilon)+(r-\epsilon)^{1+\frac{1}{\gamma}}\,t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\gamma}{s^{\gamma+1}}\Delta s.\end{aligned} \tag{2.20}$$

Using the Pötzsche chain rule we get

$$\begin{aligned} \left(\frac{-1}{s^{\gamma}}\right)^{\Delta} &= \gamma \int_0^1 \frac{1}{[s+h\mu(s)]^{\gamma+1}} dh \\ &\leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh = \frac{\gamma}{s^{\gamma+1}}. \end{aligned} \tag{2.21}$$

Then from (2.20) and (2.21), we have

$$t^{\gamma} w^{\sigma}(t) \geq (p_* - \epsilon) + (r-\epsilon)^{1+\frac{1}{\gamma}} \left(\frac{t}{\sigma(t)}\right)^{\gamma}.$$

Taking the lim inf of both sides as $t \to \infty$ we get that

$$r \geq p_* - \epsilon + (r-\epsilon)^{1+\frac{1}{\gamma}} l^{\gamma}.$$

Since $\epsilon > 0$ is arbitrary, we get the desired result

$$r \geq p_* + (r)^{1+\frac{1}{\gamma}} l^{\gamma}.$$

To complete the proof it remains to prove the second inequality in (2.13). Since $w^{\Delta}(t) \leq 0$, we have $w(t) \geq w^{\sigma}(t)$, and (2.16) becomes

$$w^{\Delta}(t) + P(t) + \gamma (w^{\sigma})^{\lambda} \leq 0, \quad t \in [T, \infty)_{\mathbb{T}}, \tag{2.22}$$

where $\lambda = 1 + \frac{1}{\gamma}$. Multiplying both sides (2.22) by $t^{\gamma+1}$, and integrating from T to t $(t \geq T)$ we get

$$\int_T^t s^{\gamma+1} w^{\Delta}(s) \Delta s \leq -\int_T^t s^{\gamma+1} P(s) \Delta s - \gamma \int_T^t s^{\gamma+1} (w^{\sigma}(s))^{\lambda} \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned} t^{\gamma+1} w(t) &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s - \gamma \int_T^t s^{\gamma+1} (w^{\sigma}(s))^{\lambda} \Delta s \\ &+ \int_T^t \left(s^{\gamma+1}\right)^{\Delta_s} w^{\sigma}(s) \Delta s. \end{aligned}$$

But, by the Pötzsche chain rule,

$$\begin{aligned} \left(s^{\gamma+1}\right)^{\Delta} &= (\gamma+1) \int_0^1 [s+h\mu(s)]^{\gamma} dh \\ &\leq (\gamma+1) \int_0^1 [\sigma(s)]^{\gamma} dh \\ &= (\gamma+1)[\sigma(s)]^{\gamma}. \end{aligned} \tag{2.23}$$

Hence

$$\begin{aligned} t^{\gamma+1} w(t) &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s + \int_T^t (\gamma+1)(\sigma(s))^{\gamma} w^{\sigma}(s) \Delta s \\ &- \gamma \int_T^t s^{\gamma+1} [w^{\sigma}(s)]^{\lambda} \Delta s. \end{aligned}$$

Let $0 < \epsilon < l$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that

$$\frac{s}{\sigma(s)} > l - \epsilon, \quad s \geq T.$$

It follows that

$$\sigma(s) \leq Ks, \qquad s \geq T \quad \text{where} \quad K := \frac{1}{l-\epsilon}.$$

We then get that

$$\begin{aligned} t^{\gamma+1} w(t) \ &\leq \ T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \ \int_T^t [(\gamma+1) K^\gamma s^\gamma w^\sigma(s) - \gamma s^{\gamma+1} [w^\sigma(s)]^\lambda] \Delta s. \end{aligned}$$

Let

$$u(s) := s^\gamma w^\sigma(s),$$

then

$$(u(s))^{\frac{\gamma+1}{\gamma}} = s^{\gamma+1} [w^\sigma(s)]^\lambda.$$

It follows that

$$\begin{aligned} t^{\gamma+1} w(t) \ &\leq \ T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \ \int_T^t [(\gamma+1) K^\gamma u(s) - \gamma [u(s)]^\lambda] \Delta s. \end{aligned}$$

Using the inequality

$$Bu - Au^\lambda \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \tag{2.24}$$

where A, B are constants, we get

$$\begin{aligned} t^{\gamma+1} w(t) \ &\leq \ T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \ \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1) K^\gamma]^{\gamma+1}}{\gamma^\gamma} \Delta s \\ &\leq \ T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} (t - T). \end{aligned}$$

It follows from this that

$$t^\gamma w(t) \leq \frac{T^{\gamma+1} w(T)}{t} - \frac{1}{t} \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} \frac{(t-T)}{t}.$$

Since $w^\sigma(t) \leq w(t)$, we get

$$t^\gamma w^\sigma(t) \leq \frac{T^{\gamma+1} w(T)}{t} - \frac{1}{t} \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} \frac{(t-T)}{t}.$$

Taking the lim sup of both sides as $t \to \infty$ we obtain

$$R \leq -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}.$$

Since $\epsilon > 0$ is arbitrary, we get that

$$R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Using this and the first inequality in (2.13) we get

$$p_* \leq r - l^{\gamma} r^{1+\frac{1}{\gamma}} \leq r \leq R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}},$$

which gives us the desired second inequality in (2.13).

Using Theorem 2.1 we can now easily prove the following oscillation result.

Theorem 2.2 *If*

$$p_* = \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} p(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} \Delta s > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.25}$$

then (2.9) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Assume (2.9) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$, then the hypotheses of Theorem 2.1 hold. From the first inequality in (2.13) we have that

$$p_* \leq r - l^{\gamma} r^{\frac{\gamma+1}{\gamma}}.$$

Using the inequality (2.24), with $B = 1$ and $A = l^{\gamma}$ we get that

$$p_* \leq \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.25).

Using the second inequality in Theorem 2.1 we easily get the following result

Theorem 2.3 *If*

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{2.26}$$

then (2.9) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Remark 2.1 Note that when $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and the condition (2.25) becomes

$$\liminf_{t\to\infty} t^{\gamma} \int_{t}^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} ds > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{2.27}$$

which is a sufficient condition for oscillation of (1.17). We note that the condition (2.27) generalizes the condition (1.14) that has been established by Erbe [6]. Also when $\mathbb{T} = \mathbb{N}$, $\sigma(t) = t + 1$ and condition (2.25) becomes

$$\liminf_{t\to\infty} t^{\gamma} \sum_{s=t+1}^{\infty} p(s) \left(\frac{\tau(s)}{s+1}\right)^{\gamma} > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{2.28}$$

which is a sufficient condition for oscillation of (1.19). We note that the condition (2.28) may be viewed as an extension of the oscillation condition (1.20) that has been established by Thandapani et al [16]. As a special case when $\tau(t) = t$, the condition (2.27) becomes the Hille condition (1.9).

Remark 2.2 We give an example which shows that the inequality (2.27) and hence the inequality (2.19) can not be weakened. To see this let $\mathbb{T} = [1, \infty)$, and

$$p(t) := \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{1}{t^{\gamma+1}}, \quad t \geq 1,$$

we have that

$$p_* = \liminf_{t\to\infty} t^{\gamma} \int_t^{\infty} p(s)ds = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}},$$

and the second-order half-linear differential equation

$$\left((x'(t))^{\gamma}\right)' + p(t)x^{\gamma}(t) = 0,$$

has a nonoscillatory solution $x(t) = t^{\frac{\gamma}{\gamma+1}}$. This shows that the constant $\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}$ is sharp for the oscillation for all solutions of this equation. Note in the case when $\gamma = 1$ this constant is $\frac{1}{4}$.

Theorem 2.4 *Assume that* (2.2) *holds and that*

$$\limsup_{t\to\infty} \frac{t^{\gamma}}{r(t)} \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s > 1. \tag{2.29}$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Assume x is an eventually positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Using Lemma 2.1 there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad \frac{x(t)}{t} > x^{\Delta}(t),$$

on $[t_1, \infty)_{\mathbb{T}}$ and $\frac{x(t)}{t}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then integrating both sides of the dynamic equation (1.1) from t to T, $T \geq t \geq t_1$ we obtain

$$\int_t^T p(s)x^{\gamma}(\tau(s))\Delta s = r(t)(x^{\Delta}(t))^{\gamma} - r(T)(x^{\Delta}(T))^{\gamma}.$$

Since $x^{\Delta}(t) > 0$, we get that

$$\frac{1}{r(t)} \int_t^T p(s)x^{\gamma}(\tau(s))\Delta s \leq (x^{\Delta}(t))^{\gamma}.$$

Since $\frac{x(t)}{t}$ is strictly decreasing and using $x^{\Delta}(t) < \frac{x(t)}{t}$, we obtain

$$\frac{1}{r(t)} \int_t^T p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} x^{\gamma}(s)\Delta s \leq \frac{x^{\gamma}(t)}{t^{\gamma}}.$$

Since $x(t)$ is increasing, we get

$$\frac{t^\gamma}{r(t)}\int_t^T p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1,$$

which implies that

$$\frac{t^\gamma}{r(t)}\int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1,$$

which gives us the contradiction

$$\limsup_{t\to\infty}\frac{t^\gamma}{r(t)}\int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1.$$

Remark 2.3 When $\mathbb{T}=\mathbb{R}$, Theorem 2.4 improves the results established by Ohriska [14] and Agarwal et al [2] for differential equations. In the case when $\mathbb{T}=\mathbb{N}$ and $r(t)=1$ the condition (2.29) becomes

$$\limsup_{t\to\infty} t^\gamma \sum_{s=t}^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)^\gamma > 1, \tag{2.30}$$

which is a new oscillation condition for (1.19).

Motivated by Theorem 3.1 in [15], we can prove the following result which is a new oscillation result for equation (1.1).

Theorem 2.5 *Assume that* (2.2) *holds. Furthermore, assume that there exists a positive* $\Delta-$*differentiable function* $\delta(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^t \left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^\gamma - \frac{r(s)((\delta^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s = \infty, \tag{2.31}$$

where $d_+(t) := \max\{d(t),0\}$ *is the positive part of any function* $d(t)$*. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_\mathbb{T}$.

Proof Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_\mathbb{T}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_\mathbb{T}$ such that $x(t)$ satisfies the conclusions of Lemma 2.1 on $[t_1,\infty)_\mathbb{T}$ with $x(\tau(t)) > 0$ on $[t_1,\infty)_\mathbb{T}$. Let $\delta(t)$ be a positive Δ differentiable function and consider the generalized Riccati substitution

$$w(t) = \delta(t)r(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\gamma.$$

Then by Lemma 2.1, we see that the function $w(t)$ is positive on $[t_1,\infty)_\mathbb{T}$. By the product rule and then the quotient rule (suppressing arguments)

$$\begin{aligned} w^\Delta &= \delta^\Delta\left(\frac{r(x^\Delta)^\gamma}{x^\gamma}\right)^\sigma + \delta\left(\frac{r(x^\Delta)^\gamma}{x^\gamma}\right)^\Delta \\ &= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma + \delta\frac{x^\gamma(r(x^\Delta)^\gamma)^\Delta - r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma x^{\gamma\sigma}} \\ &= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma + \frac{\delta x^\gamma(-px^{\tau\gamma})}{x^\gamma(x^\sigma)^\gamma} - \frac{\delta r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma} \\ &= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma - p\delta\left(\frac{x^\tau}{x^\sigma}\right)^\gamma - \delta\frac{r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}. \end{aligned}$$

Using the fact that $\frac{x(t)}{t}$ and $r(t)(x^{\Delta}(t))^{\gamma}$ are strictly decreasing (from Lemma 2.1) we get

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)} \geq \frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad r(t)(x^{\Delta}(t))^{\gamma} \geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}.$$

From these last two inequalites we obtain

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \delta \frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.$$

Using (2.15) and the definition of w we have that

$$\begin{aligned} w^{\Delta} &\leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{x^{\Delta}}{x} w^{\sigma} \\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{r^{\frac{1}{\gamma}} x^{\Delta}}{r^{\frac{1}{\gamma}} x} w^{\sigma}. \end{aligned}$$

Since

$$r(t)(x^{\Delta}(t))^{\gamma} \geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}, \quad \text{and} \quad x^{\sigma}(t) \geq x(t),$$

we get that

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{(r^{\frac{1}{\gamma}} x^{\Delta})^{\sigma}}{r^{\frac{1}{\gamma}} x^{\sigma}} w^{\sigma}.$$

Using the definition of w we finally obtain

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \tag{2.32}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. It follows from (2.32) that

$$w^{\Delta} \leq \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}. \tag{2.33}$$

Define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := \frac{\gamma\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \quad B^{\lambda-1} := \frac{r^{\frac{1}{\gamma+1}}}{\lambda(\gamma\delta)^{\frac{1}{\lambda}}} (\delta^{\Delta})_{+}.$$

Then, using the inequality ($\lambda \geq 1$)

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda - 1) B^{\lambda},$$

we get that

$$\begin{aligned} \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda} &= \lambda A B^{\lambda-1} - A^{\lambda} \\ &\leq (\lambda - 1) B^{\lambda} \\ &\leq \frac{r \left(\delta^{\Delta}\right)_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^{\gamma}}. \end{aligned}$$

From this last inequality and (2.33) we get

$$w^{\Delta} \leq \frac{r((\delta^{\Delta})_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}.$$

Integrating both sides from t_1 to t we get

$$-w(t_1) \leq w(t) - w(t_1) \leq \int_{t_1}^{t} \left[\frac{r((\delta^{\Delta})_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s,$$

which leads to a contradiction, since the right hand side tends to $-\infty$ by (2.31).

By Theorem 2.5, by choosing $\delta(t) = 1$, $t \geq t_0$ we have the following oscillation result which as a special case gives the oscillation theorem established by Agarwal et al [2, Theorem 2.8].

Corollary 2.6 *Assume that* (2.2) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} p(s)\Delta s = \infty, \tag{2.34}$$

hold. Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Similarly letting $\delta(t) = t$ in Theorem 2.5 we get the following result.

Corollary 2.7 *Assume that* (2.2) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sp(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right] \Delta s = \infty, \tag{2.35}$$

hold. Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Note that again when $\mathbb{T} = \mathbb{N}$, Theorem 2.5 and Corollaries 2.6 and 2.7 improve the oscillation results that have been established by Thandapani et al [16]. In the following, we assume that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} < \infty, \tag{2.36}$$

holds and establish some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or converges to zero. The proof is similar to the proof of Theorem 3.3 in [15] and hence is omitted.

Theorem 2.8 *Assume that* (2.36) *and*

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty, \tag{2.37}$$

hold. If one of the conditions (2.31), (2.34), *and* (2.35) *holds, then every solution of* (1.1) *oscillates or converges to zero.*

3 Examples

In this section we give some examples to illustrate our main results.

Example 3.1 Consider the half-linear delay dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)x^{\gamma}(\tau(t))=0, \tag{3.1}$$

where $p(t):=\frac{\beta}{t^{\gamma+1}}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$, where β is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. It is clear that

$$\int_{t_0}^{\infty}\tau^{\gamma}(t)p(t)\Delta t=\beta\int_{t_0}^{\infty}\left(\frac{\sigma(t)}{t}\right)^{\gamma}\frac{1}{t}\Delta t\geq\beta\int_{t_0}^{\infty}\frac{\Delta t}{t}=\infty,$$

(i.e., (2.1) holds). For equation (3.1), we have

$$\begin{aligned}p_{*} &= \liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty}p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\Delta s\\ &= \beta\liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\Delta s}{s^{\gamma+1}}.\end{aligned}$$

But, by the Pötzsche chain rule

$$\left(-\frac{1}{t^{\gamma}}\right)^{\Delta}=\gamma\int_0^1\frac{dh}{(t+h\mu(t))^{\gamma+1}}\leq\gamma\int_0^1\frac{dh}{t^{\gamma+1}}=\frac{\gamma}{t^{\gamma+1}},$$

so we get that

$$p_{*}\geq\frac{\beta}{\gamma}\liminf_{t\to\infty}\left(\frac{t}{\sigma(t)}\right)^{\gamma}=\frac{\beta}{\gamma}l^{\gamma}.$$

So if

$$\beta>\frac{\gamma^{\gamma+1}}{l^{\gamma(\gamma+1)}(\gamma+1)^{\gamma+1}},$$

then (2.25) holds and we have by Theorem 2.2 that (3.1) is oscillatory.

Note that in the case $\mathbb{T}=\mathbb{R}$, $\tau(t)=t$, $\gamma=1$, we get that $l=1$ and we see that $\beta>\frac{1}{4}$ which is the sharp condition for the Euler–Cauchy differential equation to be oscillatory (see [1] for related results for the delay case). Also, note that the results by Agarwal et al [2] and Thandapani et al [16] can not be applied to equation (3.1) in the cases of differential and difference equations.

Example 3.2 Consider the half-linear delay dynamic equation

$$\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\alpha}{t^2}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}x^{\gamma}(\tau(t))=0, \tag{3.2}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where α is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. Here $p(t)=\frac{\alpha}{t^2}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$ and $r(t)=t^{\gamma-1}$. It is clear that condition (2.1) holds and condition (2.2) is satisfied, since

$$\int_{t_0}^{\infty}\frac{\Delta t}{t^{\frac{\gamma-1}{\gamma}}}=\infty,\quad\text{for}\quad\gamma\geq 1,$$

by Example 5.60 in [4]. To apply Corollary 2.7, it remains to prove that condition (2.35) holds. To see this note that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sp(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s$$
$$=\left(\alpha-\frac{1}{(\gamma+1)^{\gamma+1}}\right)\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\Delta s}{s}=\infty,$$

if $\alpha>\frac{1}{(\gamma+1)^{\gamma+1}}$. We conclude, by Corollary 2.7, that if

$$\alpha>\frac{1}{(\gamma+1)^{\gamma+1}},$$

then every solution of (3.2) is oscillatory.

Example 3.3 Consider the half-linear delay dynamic equation

$$\left(t^{\gamma+1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\beta\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}x^{\gamma}(\tau(t))=0, \tag{3.3}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where β is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. In this case $p(t)=\beta\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$ and $r(t)=t^{\gamma+1}$. It is clear that (2.1) holds and $r(t)$ satisfies condition (2.36) since

$$\int_{t_0}^{\infty}\frac{\Delta t}{t^{\frac{\gamma+1}{\gamma}}}<\infty,\quad \gamma\geq 1,$$

for those time scales $[t_0,\infty)_{\mathbb{T}}$, where $\int_{t_0}^{\infty}\frac{1}{t^p}\Delta t<\infty$ when $p>1$. This holds for many time scales (see Theorems 5.64 and 5.65 in [4] and see Example 5.63 where this result does not hold). To see that (2.37) holds note that

$$\begin{aligned}\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t &= \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\beta\left(\frac{\sigma(s)}{\tau(s)}\right)^{\gamma}\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &\geq \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\beta\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &= \beta^{\frac{1}{\gamma}}\int_{t_0}^{\infty}\left(\frac{t-t_0}{t}\right)^{\frac{1}{\gamma}}\frac{\Delta t}{t}=\infty.\end{aligned}$$

To apply Theorem 2.8, it remains to prove that the condition (2.31) holds.To see this note that if $\delta(t)=1$, then

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)((\delta^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s$$
$$=\beta\int_{t_0}^{\infty}\Delta t=\infty.$$

We conclude that $[t_0,\infty)_{\mathbb{T}}$ is a time scale where $\int_{t_0}^{\infty}\frac{1}{t^p}\Delta t<\infty$ when $p>1$, then, by Theorem 2.8, every solution of (3.3) is oscillatory or converges to zero.

Example 3.4 One can use Theorem 2.4 to show that if $\beta > 1$, then the equation

$$\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{\beta t^{\gamma-1}}{\tau^{\gamma}(t)\sigma(t)} x^{\gamma}(\tau(t)) = 0,$$

is oscillatory for any time scale where $\int_{t_0}^{\infty} \frac{t^{\gamma-1}}{\sigma(t)} \Delta t = \infty$. We leave the details to the reader.

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On Solutions of a Nonlinear Boundary Value Problem on Time Scales

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Abstract: We study a boundary value problem (BVP) for second order nonlinear dynamic equations on time scales. A condition is established that ensures existence and uniqueness of solutions to the BVP under consideration.

Keywords: *time scale; delta and nabla derivatives; eigenvalue; fixed point theorem.*

Mathematics Subject Classification (2000): 34B15.

1 Introduction

Let $\mathbf{T}$ be a time scale and $a, b \in \mathbf{T}$ be fixed points with $a < b$ such that the time scale interval

$$(a, b) = \{t \in \mathbf{T} : a < t < b\}$$

is not empty. Throughout, all the intervals are time scale intervals. For standard notions and notations related to time scales calculus see [1, 2].

In this paper, we deal with the nonlinear boundary value problem (BVP)

$$y^{\Delta\nabla}(t) + f(t, y(t)) = 0, \quad t \in (a, b), \tag{1}$$

$$y(a) = y(b) = 0. \tag{2}$$

A function $y : [a, b] \to \mathbf{R}$ is called a *solution* of the BVP (1), (2) if the following conditions are satisfied:

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(a) y is continuous on $[a,b]$ and delta differentiable on (a,b) and such that there exist (finite) limits

$$y^{\Delta}(a) := \lim_{t\to a^+} y^{\Delta}(t) \quad \text{and} \quad y^{\Delta}(b) := \lim_{t\to b^-} y^{\Delta}(t).$$

(b) y^{Δ} is ∇-differentiable on $(a,b]$.

(c) y satisfies equation (1) and boundary conditions (2).

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose $f : [a,b]\times\mathbf{R} \to \mathbf{R}$ is continuous, $f(b,0) = 0$ in the case b is left-scattered, and suppose f satisfies the Lipschitz condition*

$$|f(t,\xi) - f(t,\eta)| \le L\,|\xi - \eta| \tag{3}$$

for all $t \in [a,b]$ and $\xi,\eta \in \mathbf{R}$, where $L > 0$ is a constant (Lipschitz constant), $\mathbf{R}$ denotes the set of real numbers. Suppose further that

$$L < \lambda_1, \tag{4}$$

where λ_1 is the least positive eigenvalue of the problem

$$y^{\Delta\nabla}(t) + \lambda y(t) = 0, \quad t \in (a,b), \tag{5}$$

$$y(a) = y(b) = 0. \tag{6}$$

Then the BVP (1), (2) has a unique solution.

Proof of Theorem 1.1 is presented in Section 2 and it uses a Hilbert space technique.

In Section 3, we compute the eigenvalues of (5), (6) explicitly in the cases $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$ (the set of integers) and show that

$$\lambda_1 = \frac{\pi^2}{(b-a)^2} \quad \text{if} \quad \mathbf{T} = \mathbf{R},$$

and

$$\lambda_1 = 4\sin^2\frac{\pi}{2(b-a)} \ge \frac{8}{(b-a)^2} \quad \text{if} \quad \mathbf{T} = \mathbf{Z}.$$

Finally, in Section 4, we show that in the general case of arbitrary time scale $\mathbf{T}$ the estimation

$$\lambda_1 \ge \frac{4}{(b-a)^2}$$

holds and therefore the more explicit condition of the form

$$L < \frac{4}{(b-a)^2}$$

implies condition (4) of Theorem 1.1.

2 Proof of Theorem 1.1

Denote by $\mathcal{H}$ the Hilbert space of all real ∇-measurable functions $y : (a,b] \to \mathbf{R}$ such that $y(b) = 0$ in the case b is left-scattered, and that

$$\int_a^b y^2(t)\nabla t < \infty,$$

with the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\| = \sqrt{\langle y, y\rangle} = \left\{\int_a^b y^2\,(t)\,\nabla t\right\}^{\frac{1}{2}}.$$

Next denote by $\mathcal{D}$ the set of all functions $y \in \mathcal{H}$ satisfying the following three conditions:

(i) y is continuous on $(a,b]$, $y(b) = 0$, there exists $y(a) := \lim_{t\to a^+} y(t)$ and $y(a) = 0$.

(ii) y is continuously Δ-differentiable on (a,b), there exist (finite) limits

$$y^\Delta(a) := \lim_{t\to a^+} y^\Delta(t) \quad \text{and} \quad y^\Delta(b) := \lim_{t\to b^-} y^\Delta(t).$$

(iii) y^Δ is ∇-differentiable on $(a,b]$ and $y^{\Delta\nabla} \in \mathcal{H}$.

Define the operators $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ by

$$(Ay)(t) = -y^{\Delta\nabla}(t) \quad \text{for} \quad y \in \mathcal{D},$$

$$(Fy)(t) = f(t, y(t)) \quad \text{for} \quad y \in \mathcal{H}.$$

Note that the operator A is linear, while F is nonlinear in general. The eigenvalues of problem (5), (6) coincide with the eigenvalues of the operator A.

As is shown in [3], the operator A is symmetric and positive:

$$\langle Ay, z\rangle = \langle y, Az\rangle \quad \text{for all} \quad y, z \in \mathcal{D},$$

$$\langle Ay, y\rangle > 0 \quad \text{for all} \quad y \in \mathcal{D},\ y \neq 0.$$

Further, A has $N = \dim \mathcal{H}$ (where $N \le \infty$) orthonormal eigenfunctions φ_k which form a basis for $\mathcal{H}$ and the corresponding eigenvalues are simple and positive:

$$A\varphi_k = \lambda_k \varphi_k,$$

$$\langle \varphi_k, \varphi_l\rangle = 0 \text{ if } k \neq l \text{ and } \langle \varphi_k, \varphi_l\rangle = 1 \text{ if } k = l,$$

$$0 < \lambda_1 < \lambda_2 < \dots \quad .$$

For any function $u \in \mathcal{H}$ we have (expansion formula and Parseval's equality)

$$u = \sum_{k=1}^{N} c_k \varphi_k, \quad c_k = \langle u, \varphi_k\rangle\,, \tag{7}$$

$$\|u\|^2 = \langle u, u\rangle = \sum_{k=1}^{N} c_k^2.$$

In the case $N = \infty$ the sum in (7) becomes an infinite series and it converges to the function u in metric of the space $\mathcal{H}$. Since the operator A is positive, it is invertible. We have

$$Au = \sum_{k=1}^{N} c_k \lambda_k \varphi_k \quad \text{for all} \quad u \in \mathcal{D},$$

$$A^{-1}u = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \varphi_k \quad \text{for all} \quad u \in \mathcal{H},$$

where c_k are defined in (7). Hence

$$\left\|A^{-1}u\right\|^2 = \sum_{k=1}^{N} \frac{c_k^2}{\lambda_k^2} \le \frac{1}{\lambda_1^2} \sum_{k=1}^{N} c_k^2 = \frac{1}{\lambda_1^2} \|u\|^2 .$$

Thus we have established the following result: The operator A is invertible and

$$\left\|A^{-1}u\right\| \le \frac{1}{\lambda_1} \|u\| \quad \text{for all} \quad u \in \mathcal{H}. \tag{8}$$

The BVP (1), (2) is equivalent to the vector equation

$$Ay = Fy \quad \text{for} \quad y \in \mathcal{D},$$

which can be written in the form

$$y = A^{-1}Fy \quad \text{for} \quad y \in \mathcal{H}. \tag{9}$$

Note that the inverse operator A^{-1} maps $\mathcal{H}$ onto $\mathcal{D}$ and therefore if $y \in \mathcal{H}$ satisfies (9) then $y \in \mathcal{D}$. Let us set $S = A^{-1}F$. Then we get that the BVP (1), (2) is equivalent to the equation

$$y = Sy \quad (y \in \mathcal{H}).$$

The last equation is a fixed point problem.

We will use the following well-known contraction mapping theorem: *Let $\mathcal{H}$ be a Banach space and suppose that $S : \mathcal{H} \to \mathcal{H}$ is a contraction mapping, i.e., there is an α, $0 < \alpha < 1$, such that $\|Su - Sv\| \le \alpha \|u - v\|$ for all $u, v \in \mathcal{H}$. Then S has a unique fixed point in $\mathcal{H}$.*

It will be sufficient to show that the operator $S = A^{-1}F$ is a contraction mapping on the space $\mathcal{H}$. We have, using (8),

$$\begin{aligned} \|Su - Sv\| &= \left\|A^{-1}Fu - A^{-1}Fv\right\| \\ &= \left\|A^{-1}(Fu - Fv)\right\| \\ &\le \frac{1}{\lambda_1} \|Fu - Fv\| . \end{aligned} \tag{10}$$

Next, making use of the Lipschitz condition (3), we get

$$\begin{aligned} \|Fu - Fv\|^2 &= \int_a^b |f(t, u(t)) - f(t, v(t))|^2 \nabla t \\ &\leq L^2 \int_a^b |u(t) - v(t)|^2 \nabla t \\ &= L^2 \|u - v\|^2 \end{aligned}$$

so that

$$\|Fu - Fv\| \leq L \|u - v\| \quad \text{for all} \quad u, v \in \mathcal{H}.$$

Thus, from (10) we obtain

$$\|Su - Sv\| \leq \frac{L}{\lambda_1} \|u - v\| \quad \text{for all} \quad u, v \in \mathcal{H}.$$

Consequently, we see that under the condition (4), S is a contraction mapping and hence it has a unique fixed point in $\mathcal{H}$ by the contraction mapping theorem. Theorem 1.1 is proved.

Remark 2.1 The condition that functions $y \in \mathcal{H}$ satisfy $y(b) = 0$ in the case b is left-scattered guarantees the density of $\mathcal{D}$ in $\mathcal{H}$ (this is needed for the operator theory) and the condition that $f(b, 0) = 0$ in the case b is left-scattered guarantees $Fy \in \mathcal{H}$ for $y \in \mathcal{H}$.

3 Examples

In the case $\mathbf{T} = \mathbf{R}$, problem (1), (2) takes the form

$$y''(t) + f(t, y(t)) = 0, \quad t \in (a, b),$$

$$y(a) = y(b) = 0,$$

and eigenvalue problem (5), (6) takes the form

$$y''(t) + \lambda y(t) = 0, \quad t \in (a, b), \tag{11}$$

$$y(a) = y(b) = 0. \tag{12}$$

The eigenvalues of (11), (12) are

$$\lambda_k = \frac{\pi^2 k^2}{(b-a)^2} \quad (k = 1.2, \ldots)$$

with the corresponding orthonormal eigenfunctions

$$\varphi_k(t) = \alpha_k \sin \frac{\pi k(t-a)}{b-a} \quad (k = 1, 2, \ldots),$$

where α_k are normirating constants. Therefore in this case

$$\lambda_1 = \frac{\pi^2}{(b-a)^2}$$

and condition (4) becomes

$$L < \frac{\pi^2}{(b-a)^2}.$$

In the case $\mathbf{T} = \mathbf{Z}$, problem (1), (2) takes the form

$$y(t-1) - 2y(t) + y(t+1) + f(t, y(t)) = 0, \quad t \in [a+1, b-1],$$

$$y(a) = y(b) = 0,$$

and eigenvalue problem (5), (6) takes the form

$$y(t-1) - 2y(t) + y(t+1) + \lambda y(t) = 0, \quad t \in [a+1, b-1], \tag{13}$$

$$y(a) = y(b) = 0. \tag{14}$$

The eigenvalues of (13), (14) are (cf. [4, Chap.7])

$$\lambda_k = 4\sin^2 \frac{\pi k}{2(b-a)} \quad (1 \leq k \leq b-a-1)$$

with the corresponding orthonormal eigenfunctions

$$\varphi_k(t) = \alpha_k \sin \frac{\pi k(t-a)}{b-a} \quad (1 \leq k \leq b-a-1),$$

where α_k are normirating constants. Therefore

$$\lambda_1 = 4\sin^2 \frac{\pi}{2(b-a)}$$

and condition (4) becomes

$$L < 4\sin^2 \frac{\pi}{2(b-a)}. \tag{15}$$

Since $b - a \geq 2$, using the inequality

$$\sin x \geq \frac{2\sqrt{2}}{\pi} x \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{4},$$

we have that

$$\sin^2 \frac{\pi}{2(b-a)} \geq \frac{8}{\pi^2} . \frac{\pi^2}{4(b-a)^2} = \frac{2}{(b-a)^2}$$

and, therefore, the condition of the form

$$L < \frac{8}{(b-a)^2}$$

implies condition (15).

4 An Estimation for λ_1 in General Case

In the case of arbitrary time scale $\mathbf{T}$ we have (8). Besides, from $A\varphi_1 = \lambda_1\varphi_1$ we have

$$\left\|A^{-1}\varphi_1\right\| = \left\|\frac{1}{\lambda_1}\varphi_1\right\| = \frac{1}{\lambda_1}.$$

Consequently,

$$\left\|A^{-1}\right\| = \frac{1}{\lambda_1}. \tag{16}$$

On the other hand, the inverse operator A^{-1} has the form (see [3])

$$(A^{-1}u)(t) = \int_a^b G(t,s)u(s)\nabla s \quad \text{for any} \quad u \in \mathcal{H},$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases} (t-a)(b-s) & \text{if} \quad t \le s, \\ (s-a)(b-t) & \text{if} \quad t \ge s, \end{cases} \tag{17}$$

Hence

$$\begin{aligned} \left\|A^{-1}u\right\|^2 &= \int_a^b \left|\int_a^b G(t,s)u(s)\nabla s\right|^2 \nabla t \\ &\le \|u\|^2 \int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \end{aligned}$$

so that

$$\left\|A^{-1}\right\| \le \left\{\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t\right\}^{\frac{1}{2}}.$$

Therefore, taking into account (16), we get

$$\lambda_1 \ge \left\{\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t\right\}^{-\frac{1}{2}}. \tag{18}$$

Next, from (17) it follows that

$$0 \le G(t,s) \le \frac{1}{b-a}(s-a)(b-s)$$

for all t and s in $[a,b]$. Therefore

$$\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \le \frac{1}{(b-a)^2} \int_a^b \int_a^b (s-a)^2(b-s)^2 \nabla s \nabla t$$

and observing that

$$0 \le (s-a)(b-s) \le \frac{(b-a)^2}{4} \quad \text{for} \quad s \in [a,b],$$

we find

$$\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \le \frac{(b-a)^4}{16}.$$

Comparing this with (18), we obtain

$$\lambda_1 \ge \frac{4}{(b-a)^2}.$$

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On Expansions in Eigenfunctions for Second Order Dynamic Equations on Time Scales

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Abstract: In this study, we explore an eigenvalue problem on a bounded time scales interval for self-adjoint second order dynamic equations with self-adjoint separated boundary conditions. Existence of the eigenvalues and eigenfunctions is shown. Next, mean square convergent and uniformly convergent expansions in eigenfunctions are established.

Keywords: *time scale; delta and nabla derivatives; Green's function; eigenvalue; eigenfunction.*

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1 Introduction

The first papers on eigenvalue problems for linear Δ-differential equations on time scales were fulfilled by Agarwal, Bohner, and Wong in [1] and Chyan, Davis, Henderson, and Yin in [6]. In [1], an oscillation theorem is offered for Sturm-Liouville eigenvalue problem on time scales with separated boundary conditions and Rayleigh's principle is established for the eigenvalues. In [6], the theory of positive operators with respect to a cone in a Banach space is applied to eigenvalue problems associated with the second order linear Δ-differential equations on time scales to prove existence of a smallest positive eigenvalue and then a theorem is established comparing the smallest positive eigenvalues for two problems of that type.

Recently, Guseinov [7] investigated eigenfunction expansions for the simple Sturm–Liouville eigenvalue problem

$$-y^{\Delta\nabla}(t) = \lambda y(t), \quad t \in (a, b), \tag{1}$$
$$y(a) = y(b) = 0, \tag{2}$$

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where a and b are some fixed points in a time scale $\mathbf{T}$ with $a < b$ and such that the time scale interval (a,b) is not empty. In that paper [7], existence of the eigenvalues and eigenfunctions for problem (1), (2) is proved and mean square convergent and uniformly convergent expansions in eigenfunctions are established.

In the present paper, we extend the results of [7] to the more general eigenvalue problem

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in (a,b], \tag{3}$$

$$y(a) - hy^{[\Delta]}(a) = 0, \quad y(b) + Hy^{[\Delta]}(b) = 0, \tag{4}$$

where $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$ is the so-called quasi Δ-derivative of $y(t)$.

We will assume that the following two conditions are satisfied.

(C1) $p(t)$ is continuous on $[a,b]$ and continuously ∇-differentiable on $(a,b]$, $q(t)$ is piecewise continuous on $[a,b]$, h and H are given real numbers.

(C2) $p(t) > 0$, $q(t) \geq 0$ for $t \in [a,b]$, and $h \geq 0$, $H \geq 0$.

The paper is organized as follows. In Section 2, the Hilbert-Schmidt theorem on self-adjoint completely continuous operators is applied to show that the eigenvalue problem (3), (4) has a system of eigenfunctions that forms an orthonormal basis for an appropriate Hilbert space. This yields mean square convergent (that is, convergent in an L^2-metric) expansions in eigenfunctions. In Section 3, uniformly convergent expansions in eigenfunctions are obtained when the expanded functions satisfy some smoothness conditions. In the last Section 4, two special cases are described.

Finally, for easy reference, we state here two integration by parts formulas on time scales which are employed in the subsequent sections.

Let $\mathbf{T}$ be a time scale and $a, b \in \mathbf{T}$ be fixed points with $a < b$ such that the time scale interval

$$(a,b) = \{t \in \mathbf{T} : a < t < b\}$$

is not empty. Throughout, all the intervals are time scale intervals. For standard notions and notations connected to time scales calculus we refer to [4, 5].

Theorem 1.1 (see [7, Theorem 2.4]). *Let f and g be continuous functions on $[a,b]$. Suppose that f is Δ-differentiable on $[a,b)$ with the continuous Δ-derivative f^{Δ} that is Δ-integrable over $[a,b)$ and g is ∇-differentiable on $(a,b]$ with the continuous ∇-derivative g^{∇} that is ∇-integrable over $(a,b]$. Then:*

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t)\mid_a^b - \int_a^b f(t)g^{\nabla}(t)\nabla t, \tag{5}$$

$$\int_a^b g^{\nabla}(t)f(t)\nabla t = g(t)f(t)\mid_a^b - \int_a^b g(t)f^{\Delta}(t)\Delta t. \tag{6}$$

2 L²-convergent Expansions

Denote by $\mathcal{H}$ the Hilbert space of all real ∇-measurable functions $y : (a,b] \to \mathbf{R}$ such that $y(b) = 0$ in the case b is left-scattered and $H = 0$, and that

$$\int_a^b y^2(t)\nabla t < \infty,$$

with the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\| = \sqrt{\langle y, y\rangle} = \left\{\int_a^b y^2(t)\,\nabla t\right\}^{1/2}.$$

Next denote by $\mathcal{D}$ the set of all functions $y \in \mathcal{H}$ satisfying the following three conditions:

(i) y is continuous on $[a, \sigma(b)]$, where σ denotes the forward jump operator.

(ii) $y^{\Delta}(t)$ is defined for $t \in [a, b]$ and

$$y(a) - hy^{[\Delta]}(a) = 0, \quad y(b) + Hy^{[\Delta]}(b) = 0, \tag{7}$$

where $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$.

(iii) $p(t)y^{\Delta}(t)$ is ∇-differentiable on $(a, b]$ and $[p(t)y^{\Delta}(t)]^{\nabla} \in \mathcal{H}$.

Obviously $\mathcal{D}$ is a linear subset dense in $\mathcal{H}$. Now we define the operator $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ as follows. The domain of definition of A is $\mathcal{D}$ and we put

$$(Ay)(t) = -[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t), \quad t \in (a, b],$$

for $y \in \mathcal{D}$.

Definition 2.1 A complex number λ is called an eigenvalue of problem (3), (4) if there exists a nonidentically zero function $y \in \mathcal{D}$ such that

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in (a, b].$$

The function y is called an eigenfunction of problem (3), (4), corresponding to the eigenvalue λ.

We see that the eigenvalue problem (3), (4) is equivalent to the equation

$$Ay = \lambda y, \quad y \in \mathcal{D},\ y \neq 0. \tag{8}$$

Theorem 2.1 *Under the condition (C1) we have, for all* $y, z \in \mathcal{D}$,

$$\langle Ay, z\rangle = \langle y, Az\rangle\,, \tag{9}$$

$$\langle Ay, y\rangle = h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t. \tag{10}$$

Proof Using integration by parts formulas (5), (6), we have for all $y, z \in \mathcal{D}$,

$$\begin{aligned}
\langle Ay, z\rangle &= \int_a^b \{-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t)\}z(t)\nabla t \\
&= -p(t)y^{\Delta}(t)z(t)\mid_a^b + \int_a^b p(t)y^{\Delta}(t)z^{\Delta}(t)\Delta t + \int_a^b q(t)y(t)z(t)\nabla t \\
&= -p(t)y^{\Delta}(t)z(t)\mid_a^b + y(t)p(t)z^{\Delta}(t)\mid_a^b \\
&\quad - \int_a^b y(t)[p(t)z^{\Delta}(t)]^{\nabla}\nabla t + \int_a^b q(t)y(t)z(t)\nabla t \\
&= \int_a^b y(t)\{-[p(t)z^{\Delta}(t)]^{\nabla} + q(t)z(t)\}\nabla t = \langle y, Az\rangle\,,
\end{aligned}$$

where we have used the boundary conditions (7) for functions $y, z \in \mathcal{D}$.

Simultaneously we have also got

$$\begin{aligned} \langle Ay, y\rangle &= -p(t)y^{\Delta}(t)y(t)\mid_a^b + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t \\ &= h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t. \end{aligned}$$

The theorem is proved.

Relation (9) shows that the operator A is symmetric (self-adjoint), while (10) shows that, under the additional condition (C2), it is positive:

$$\langle Ay, y\rangle > 0 \quad \text{for all} \quad y \in \mathcal{D},\ y \neq 0.$$

Therefore all eigenvalues of the operator A are real and positive and any two eigenfunctions corresponding to the distinct eigenvalues are orthogonal. Besides, it can easily be seen that eigenvalues of problem (3), (4) are simple, that is, to each eigenvalue there corresponds a single eigenfunction up to a constant factor (equation (3) cannot have two linearly independent solutions satisfying the condition $y(a) - hy^{[\Delta]}(a) = 0$).

Now we are going to prove the existence of eigenvalues for problem (3), (4).

Note that

$$\ker A = \{y \in \mathcal{D} : Ay = 0\}$$

consists only of the zero element. Indeed, if $y \in \mathcal{D}$ and $Ay = 0$, then from (10) we have by the condition (C2) that $y^{\Delta}(t) = 0$ for $t \in [a, b)$ and hence $y(t) =$*constant* on $[a, b]$. Then using boundary conditions (7) we get that $y(t) \equiv 0$.

It follows that the inverse operator A^{-1} exists. To present its explicit form we introduce the Green function (see [2, 3])

$$G(t, s) = -\frac{1}{\omega}\begin{cases} u(t)v(s), & \text{if} \quad t \leq s, \\ u(s)v(t), & \text{if} \quad t \geq s, \end{cases} \tag{11}$$

where $u(t)$ and $v(t)$ are solutions of the equation

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = 0, \quad t \in (a, b],$$

satisfying the initial conditions

$$u(a) = h,\ u^{[\Delta]}(a) = 1; \quad v(b) = H,\ v^{[\Delta]}(b) = -1,$$

and

$$\omega = W_t(u, v) = u(t)v^{[\Delta]}(t) - u^{[\Delta]}(t)v(t),$$

the Wronskian of the solutions u, v, is constant so that

$$\omega = -v(a) + hv^{[\Delta]}(a) = -u(b) - Hu^{[\Delta]}(b).$$

Note that $\omega \neq 0$. Otherwise we would have $u \in \mathcal{D}$ and $Au = 0$ so that $u \in \ker A$. But this is a contradiction, since we showed above that $\ker A = \{0\}$, while u is not equal to the zero element (we have $u^{[\Delta]}(a) = 1$).

Then

$$(A^{-1}f)(t) = \int_a^b G(t,s)f(s)\nabla s \quad \text{for any} \quad f \in \mathcal{H}. \tag{12}$$

The equations (11) and (12) imply that A^{-1} is a completely continuous (or compact) self-adjoint linear operator in the Hilbert space $\mathcal{H}$.

The eigenvalue problem (8) is equivalent (note that $\lambda = 0$ is not an eigenvalue of A) to the eigenvalue problem

$$Bg = \mu g, \quad g \in \mathcal{H}, \ g \neq 0,$$

where

$$B = A^{-1} \quad \text{and} \quad \mu = \frac{1}{\lambda}.$$

In other words, if λ is an eigenvalue and $y \in \mathcal{D}$ is a corresponding eigenfunction for A, then $\mu = \lambda^{-1}$ is an eigenvalue for B with the same corresponding eigenfunction y; conversely, if $\mu \neq 0$ is an eigenvalue and $g \in \mathcal{H}$ is a corresponding eigenfunction for B, then $g \in \mathcal{D}$ and $\lambda = \mu^{-1}$ is an eigenvalue for A with the same eigenfunction g.

Note that $\mu = 0$ cannot be an eigenvalue for B. In fact, if $Bg = 0$, then applying A to both sides we get that $g = 0$.

Next we use the following well-known Hilbert-Schmidt theorem (see, for example, [8, Section 24.3]): *For every completely continuous self-adjoint linear operator B in a Hilbert space $\mathcal{H}$ there exists an orthonormal system $\{\varphi_k\}$ of eigenvectors corresponding to eigenvalues $\{\mu_k\}$ ($\mu_k \neq 0$) such that each element $f \in \mathcal{H}$ can be written uniquely in the form*

$$f = \sum_k c_k \varphi_k + \psi,$$

where $\psi \in \ker B$, that is, $B\psi = 0$. Moreover,

$$Bf = \sum_k \mu_k c_k \varphi_k$$

and if the system $\{\varphi_k\}$ is infinite, then $\lim \mu_k = 0$ ($k \to \infty$).

As a corollary of the Hilbert-Schmidt theorem we have: *If B is a completely continuous self-adjoint linear operator in a Hilbert space $\mathcal{H}$ and if $\ker B = \{0\}$, then the eigenvectors of B form an orthogonal basis of $\mathcal{H}$.*

Applying the corollary of the Hilbert-Schmidt theorem to the operator $B = A^{-1}$ and using the above described connection between the eigenvalues and eigenfunctions of A and the eigenvalues and eigenfunctions of B we obtain the following result.

Theorem 2.2 *Under the conditions (C1) and (C2), for the eigenvalue problem (3), (4) there exists an orthonormal system $\{\varphi_k\}$ of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$. Each eigenvalue λ_k is positive and simple. The system $\{\varphi_k\}$ forms an orthonormal basis for the Hilbert space $\mathcal{H}$. Therefore the number of the eigenvalues is equal to $N = \dim \mathcal{H}$. Any function $f \in \mathcal{H}$ can be expanded in eigenfunctions φ_k in the form*

$$f(t) = \sum_{k=1}^{N} c_k \varphi_k(t), \tag{13}$$

where c_k are the Fourier coefficients of f defined by

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t. \tag{14}$$

In the case $N = \infty$ the sum in (13) becomes an infinite series and it converges to the function f in metric of the space $\mathcal{H}$, that is, in mean square metric:

$$\lim_{n\to\infty} \int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = 0. \tag{15}$$

Note that since

$$\int_a^b \left[f(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right]^2 \nabla t = \int_a^b f^2(t) \nabla t - \sum_{k=1}^{n} c_k^2,$$

we get from (15) the Parseval equality

$$\int_a^b f^2(t) \nabla t = \sum_{k=1}^{N} c_k^2. \tag{16}$$

Remark 2.1 Above in the definition of the Hilbert space $\mathcal{H}$ we required the condition $y(b) = 0$ for functions $y : (a, b] \to \mathbf{R}$ in $\mathcal{H}$ in the case b is left-scattered and $H = 0$. This is needed to ensure that $\mathcal{D}$ is dense in $\mathcal{H}$ which in turn is essential for the theory of operators.

Remark 2.2 It is easy to see that the dimension of the space $\mathcal{H}$ is finite if and only if the time scale interval $(a, b]$ consists of a finite number of points and in this case $\dim \mathcal{H}$ is equal to the number of points in the interval $(a, b]$ if $H \neq 0$, and to the number of points in the interval (a, b) if $H = 0$.

Remark 2.3 If we denote by $\varphi(t, \lambda)$ the solution of equation (3) satisfying the initial conditions

$$\varphi(a, \lambda) = h, \quad \varphi^{[\Delta]}(a, \lambda) = 1,$$

then the eigenvalues of problem (3), (4) will coincide with the zeros of the function $\chi(\lambda) = \varphi(b, \lambda) + H\varphi^{[\Delta]}(b, \lambda)$, the characteristic function of problem (3), (4). So we have proved existence of zeros of $\chi(\lambda)$ by proving existence of eigenvalues of problem (3), (4). It is possible (see [1]) to prove existence of zeros of $\chi(\lambda)$ directly and to get in this way existence of the eigenvalues.

3 Uniformly Convergent Expansions

In this section, assuming that the conditions (C1) and (C2) formulated in Section 1 are satisfied, we prove the following result (we assume that $\dim \mathcal{H} = \infty$, since in the case $\dim \mathcal{H} < \infty$ the series becomes a finite sum).

Theorem 3.1 *Let $f : [a, b] \to \mathbf{R}$ be a function such that it has a Δ-derivative $f^{\Delta}(t)$ everywhere on $[a, b]$, except at a finite number of points $t_1, t_2, \ldots, t_m$ belonging to (a, b), the Δ-derivative being continuous everywhere except at these points, at which f^{Δ} has finite limits from the left and right. Besides assume that f satisfies the boundary conditions*

$$f(a) - hf^{[\Delta]}(a) = 0, \quad f(b) + Hf^{[\Delta]}(b) = 0, \tag{17}$$

where $f^{[\Delta]}(t) = p(t)f^{\Delta}(t)$. *Then the series*

$$\sum_{k=1}^{\infty} c_k \varphi_k(t), \tag{18}$$

where

$$c_k = \int_a^b f(t)\varphi_k(t)\nabla t, \tag{19}$$

converges uniformly on $[a, b]$ *to the function* f.

Proof We employ a method applied in the case of the usual ($\mathbf{T} = \mathbf{R}$) Sturm-Liouville problem by V. A. Steklov (see [9, Section 182]). First for simplicity we assume that the function f is Δ-differentiable everywhere on $[a, b]$ and that f^{Δ} is continuous on $[a, b]$. Consider the functional

$$J(y) = h[y^{[\Delta]}(a)]^2 + H[y^{[\Delta]}(b)]^2 + \int_a^b p(t)\left[y^{\Delta}(t)\right]^2 \Delta t + \int_a^b q(t)y^2(t)\nabla t$$

so that we have $J(y) \geq 0$. Substituting in the functional $J(y)$

$$y = f(t) - \sum_{k=1}^{n} c_k \varphi_k(t),$$

where c_k are defined by (19), we obtain

$$\begin{aligned}
&J\left(f - \sum_{k=1}^{n} c_k\varphi_k\right) \\
&= h\left[f^{[\Delta]}(a) - \sum_{k=1}^{n} c_k\varphi_k^{[\Delta]}(a)\right]^2 + H\left[f^{[\Delta]}(b) - \sum_{k=1}^{n} c_k\varphi_k^{[\Delta]}(b)\right]^2 \\
&\quad + \int_a^b p\left(f^{\Delta} - \sum_{k=1}^{n} c_k\varphi_k^{\Delta}\right)^2 \Delta t + \int_a^b q\left(f - \sum_{k=1}^{n} c_k\varphi_k\right)^2 \nabla t \\
&= h[f^{[\Delta]}(a)]^2 + H[f^{[\Delta]}(b)]^2 - 2\sum_{k=1}^{n} c_k[hf^{[\Delta]}(a)\varphi_k^{[\Delta]}(a) + Hf^{[\Delta]}(b)\varphi_k^{[\Delta]}(b)] \\
&\quad + \sum_{k,l=1}^{n} c_k c_l[h\varphi_k^{[\Delta]}(a)\varphi_l^{[\Delta]}(a) + H\varphi_k^{[\Delta]}(b)\varphi_l^{[\Delta]}(b)] \\
&\quad + \int_a^b pf^{\Delta 2}\Delta t + \int_a^b qf^2\nabla t - 2\sum_{k=1}^{n} c_k\left(\int_a^b pf^{\Delta}\varphi_k^{\Delta}\Delta t + \int_a^b qf\varphi_k\nabla t\right) \\
&\quad + \sum_{k,l=1}^{n} c_k c_l\left(\int_a^b p\varphi_k^{\Delta}\varphi_l^{\Delta}\Delta t + \int_a^b q\varphi_k\varphi_l\nabla t\right).
\end{aligned} \tag{20}$$

Next, applying integration by parts formula (5), we get

$$\begin{aligned}
&\int_a^b pf^{\Delta}\varphi_k^{\Delta}\Delta t+\int_a^b qf\varphi_k\nabla t\\
&= p(t)f(t)\varphi_k^{\Delta}(t)\mid_a^b+\int_a^b f\left[-(p\varphi_k^{\Delta})^{\nabla}+q\varphi_k\right]\nabla t\\
&= f(b)\varphi_k^{[\Delta]}(b)-f(a)\varphi_k^{[\Delta]}(a)+\lambda_k\int_a^b f\varphi_k\nabla t\\
&= -Hf^{[\Delta]}(b)\varphi_k^{[\Delta]}(b)]-hf^{[\Delta]}(a)\varphi_k^{[\Delta]}(a)+\lambda_k c_k,
\end{aligned}$$

$$\begin{aligned}
&\int_a^b p\varphi_k^{\Delta}\varphi_l^{\Delta}\Delta t+\int_a^b q\varphi_k\varphi_l\nabla t\\
&= p(t)\varphi_k(t)\varphi_l^{\Delta}(t)\mid_a^b+\int_a^b \varphi_k[-(p\varphi_l^{\Delta})^{\nabla}+q\varphi_l]\nabla t\\
&= \varphi_k(b)\varphi_l^{[\Delta]}(b)-\varphi_k(a)\varphi_l^{[\Delta]}(a)+\lambda_l\int_a^b \varphi_k\varphi_l\nabla t\\
&= -h\varphi_k^{[\Delta]}(a)\varphi_l^{[\Delta]}(a)-H\varphi_k^{[\Delta]}(b)\varphi_l^{[\Delta]}(b)+\lambda_l\delta_{kl},
\end{aligned}$$

where δ_{kl} is the Kronecker symbol and where we have used the boundary conditions (17),

$$\varphi_k(a)-h\varphi_k^{[\Delta]}(a)=0,\quad \varphi_k(b)+H\varphi_k^{[\Delta]}(b)=0, \tag{21}$$

and the equation

$$-[p(t)\varphi_k^{\Delta}(t)]^{\nabla}+q(t)\varphi_k(t)=\lambda_k\varphi_k(t).$$

Therefore we have from (20)

$$\begin{aligned}
J\left(f-\sum_{k=1}^{n}c_k\varphi_k\right) &= h[f^{[\Delta]}(a)]^2+H[f^{[\Delta]}(b)]^2\\
&\quad+\int_a^b(pf^{\Delta 2}+qf^2)\Delta t-\sum_{k=1}^{n}\lambda_k c_k^2.
\end{aligned}$$

Since the left-hand side is nonnegative, we get the inequality

$$\sum_{k=1}^{\infty}\lambda_k c_k^2\le h[f^{[\Delta]}(a)]^2+H[f^{[\Delta]}(b)]^2+\int_a^b(pf^{\Delta 2}+qf^2)\Delta t \tag{22}$$

analogous to Bessel's inequality, and the convergence of the series on the left follows. All the terms of this series are nonnegative, since $\lambda_k>0$.

Note that the proof of (22) is entirely unchanged if we assume that the function f satisfies only the conditions stated in the theorem. Indeed, when integrating by parts, it is sufficient to integrate over the intervals on which f^{Δ} is continuous and then add all these integrals (the integrated terms vanish by (17), (21), and the fact that f, φ_k, and φ_k^{Δ} are continuous on $[a,b]$).

We now show that the series

$$\sum_{k=1}^{\infty}|c_k\varphi_k(t)| \tag{23}$$

is uniformly convergent on the interval $[a, b]$. Obviously from this the uniform convergence of series (18) will follow.

Using the integral equation

$$\varphi_k(t) = \lambda_k \int_a^b G(t,s)\varphi_k(s)\nabla s$$

which follows from $\varphi_k = \lambda_k A^{-1}\varphi_k$ by (12), we can rewrite (23) as

$$\sum_{k=1}^{\infty} \lambda_k \left| c_k g_k(t) \right|, \tag{24}$$

where

$$g_k(t) = \int_a^b G(t,s)\varphi_k(s)\nabla s$$

can be regarded as the Fourier coefficient of $G(t,s)$ as a function of s. By using inequality (22), we can write

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k g_k^2(t) \quad &\leq \quad h[p(a)G^{\Delta_s}(t,a)]^2 + H[p(b)G^{\Delta_s}(t,b)]^2 \\ &\quad + \int_a^b \left[p(s)G^{\Delta_s 2}(t,s) + q(s)G^2(t,s) \right] \Delta s, \end{aligned} \tag{25}$$

where $G^{\Delta_s}(t,s)$ is the delta derivative of $G(t,s)$ with respect to s. The function appearing under the integral sign is bounded (see (11)), and it follows from (25) that

$$\sum_{k=1}^{\infty} \lambda_k g_k^2(t) \leq M,$$

where M is a constant. Now replacing λ_k by $\sqrt{\lambda_k}\sqrt{\lambda_k}$, we apply the Cauchy-Schwarz inequality to the segment of series (24):

$$\begin{aligned} \sum_{k=m}^{m+p} \lambda_k \left| c_k g_k(t) \right| \quad &\leq \quad \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{\sum_{k=m}^{m+p} \lambda_k g_k^2(t)} \\ &\leq \quad \sqrt{\sum_{k=m}^{m+p} \lambda_k c_k^2} \sqrt{M}, \end{aligned}$$

and this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$ (see (22)), at once implies that series (24), and hence series (23) is uniformly convergent on the interval $[a, b]$.

Denote the sum of series (18) by $f_1(t)$:

$$f_1(t) = \sum_{k=1}^{\infty} c_k \varphi_k(t). \tag{26}$$

Since the series in (26) is convergent uniformly on $[a,b]$, we can multiply both sides of (26) by $\varphi_l(t)$ and then ∇ integrate it term-by-term to get

$$\int_a^b f_1(t)\varphi_l(t)\nabla t = c_l.$$

Therefore the Fourier coefficients of f_1 and f are the same. Then the Fourier coefficients of the difference $f_1 - f$ are zero and applying the Parseval equality (16) to the function $f_1 - f$ we get that $f_1 - f = 0$, so that the sum of series (18) is equal to $f(t)$.

4 Examples

1. In the case $\mathbf{T} = \mathbf{R}$ of reals, for functions $y : \mathbf{T} \to \mathbf{R}$ we have

$$y^{\Delta}(t) = y^{\nabla}(t) = y'(t), \quad t \in \mathbf{R},$$

and problem (3), (4) becomes

$$-[p(t)y'(t)]' + q(t)y(t) = \lambda y(t), \quad t \in (a,b], \tag{27}$$

$$y(a) - hy^{[1]}(a) = 0, \quad y(b) + Hy^{[1]}(b) = 0, \tag{28}$$

where $y^{[1]}(t) = p(t)y'(t)$, $p(t)$ is continuously differentiable and $q(t)$ is piecewise continuous on $[a,b]$, and

$$p(t) > 0,\ q(t) \geq 0 \quad \text{for } t \in [a,b], \quad \text{and} \quad h \geq 0,\ H \geq 0.$$

Theorem 2.2 and Theorem 3.1 give expansion results for the ordinary Sturm-Liouville problem (27), (28). Such results for problem (27), (28) in the case $h = H = 0$ were established earlier by V. A. Steklov (see [9, Section 182]).

2. In the case $\mathbf{T} = \mathbf{Z}$ of integers, for functions $y : \mathbf{T} \to \mathbf{Z}$ we have

$$y^{\Delta}(t) = y(t+1) - y(t), \quad y^{\nabla}(t) = y(t) - y(t-1), \quad t \in \mathbf{Z},$$

and problem (3), (4) can be written in the form

$$-p(t-1)y(t-1) + q_1(t)y(t) - p(t)y(t+1) = \lambda y(t), \quad t \in [a+1,b], \tag{29}$$

$$[1+hp(a)]y(a) - hp(a+1)y(a+1) = 0, \quad [1-Hp(b)]y(b) + Hp(b+1)y(b+1) = 0, \tag{30}$$

where $[a+1,b] = \{a+1, a+2, \ldots, b\}$ is a discrete interval, $\{y(t)\}_{t=a}^{b+1}$ is a desired solution, $q_1(t) = p(t-1) + p(t) + q(t)$,

$$p(t) > 0 \text{ for } t \in [a,b+1], \quad q(t) \geq 0 \text{ for } t \in [a+1,b], \quad \text{and} \quad h \geq 0,\ H \geq 0.$$

Consider two possible cases separately.

(i) Let $H \neq 0$. Then we have, from (30),

$$y(a) = \frac{hp(a+1)}{1+hp(a)}y(a+1), \quad y(b+1) = -\frac{1-Hp(b)}{Hp(b+1)}y(b). \tag{31}$$

Taking (31) into account in (29), we can rewrite problem (29), (30) in the form

$$Jy = \lambda y,$$

where J is the $(b-a)\times(b-a)$ matrix and y is the $(b-a)\times 1$ column vector of the form

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ldots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{b-a-2} & \beta_{b-a-2} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{b-a-2} & \alpha_{b-a-1} & \beta_{b-a-1} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{b-a-1} & \alpha_{b-a} \end{bmatrix}, \tag{32}$$

$$y = [y(a+1),\ y(a+2),\ \ldots\ y(b-1),\ y(b)]^T,$$

where

$$\alpha_1 = q_1(a+1) - \frac{hp(a)p(a+1)}{1+hp(a)}, \quad \alpha_{b-a} = q_1(b) + \frac{p(b)[1-Hp(b)]}{Hp(b+1)}, \tag{33}$$

$$\alpha_i = q_1(a+i), \quad i \in \{2,3,\ldots,b-a-1\}, \tag{34}$$

$$\beta_i = -p(a+i), \quad i \in \{1,2,\ldots,b-a-1\}, \tag{35}$$

T denotes the transpose. Therefore Theorem 2.2 expresses simply an expansion in eigenvectors of the matrix J defined by (32).

(ii) If $H=0$, then from (30) we have

$$y(b) = 0 \tag{36}$$

and equation (29) gives, for $t=b$,

$$-p(b-1)y(b-1) - p(b)y(b+1) = 0,$$

whence

$$y(b+1) = -\frac{p(b-1)}{p(b)}y(b-1). \tag{37}$$

Therefore, in the case $H=0$, problem (29), (30) is equivalent to the eigenvalue problem

$$Jy = \lambda y, \tag{38}$$

where J is the $(b-a-1)\times(b-a-1)$ matrix and y is the $(b-a-1)\times 1$ column vector of the form

$$J = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ldots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{b-a-3} & \beta_{b-a-3} & 0 \\ 0 & 0 & 0 & \cdots & \beta_{b-a-3} & \alpha_{b-a-2} & \beta_{b-a-2} \\ 0 & 0 & 0 & \cdots & 0 & \beta_{b-a-2} & \alpha_{b-a-1} \end{bmatrix},$$

$$y = [y(a+1),\ y(a+2),\ \ldots\ y(b-2),\ y(b-1)]^T, \tag{39}$$

where the numbers α_i, β_i are defined as in (33)–(35). In the case $H=0$, the equivalence of problem (29), (30) to the problem (38) means that if $\{y(t)\}_{t=a}^{b+1}$ is a solution of problem (29), (30), then the column vector y of the form (39), formed by using that solution, is a solution of equation (38), and conversely, if a column vector y of the form (39) is a solution of (38) then $\{y(t)\}_{t=a}^{b+1}$ in which the values $y(a+1), y(a+2), \ldots\ y(b-2), y(b-1)$ are taken from the vector (39) and the components $y(a)$, $y(b)$, and $y(b+1)$ are defined by (31), (36), and (37), respectively, is a solution of problem (29), (30).

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The Connection Between Boundedness and Periodicity in Nonlinear Functional Neutral Dynamic Equations on a Time Scale

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Abstract: Let $\mathbb{T}$ be a time scale that is unbounded above. We use a direct mapping and then utilize a Krasnosel'skiĭ fixed point theorem to show the existence of solutions of the nonlinear functional neutral dynamic system with delay

$$x^{\Delta}(t) = f(t, x(t), x^{\Delta}(t - h(t))) + g(t, x(t), x(t - h(t))),\ t, t - h(t) \in \mathbb{T}.$$

Then, we consider a special form of the above mentioned system and use the contraction mapping principle and show the existence of a uniform bound on all solutions and then conclude the existence of a unique periodic solution. Finally, the connection between the boundedness of solutions and the existence of periodic solutions leads us to the extension of Massera's theorem to functional differential equations on general periodic time scales.

Keywords: *connection between boundedness and periodic solutions; existence; functional; neutral; time scale.*

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1 Introduction

We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a review of this topic we direct the reader to the monographs [5], [6] and [10].

Let $\mathbb{T}$ be a time scale that is unbounded above. By the notation $[a, b]$ we mean

$$[a, b] = \{t \in \mathbb{T} : a \le t \le b\}$$

unless otherwise specified. The intervals $[a, b), (a, b]$, and (a, b) are defined similarly. In this note we examine the existence of solutions of the nonlinear functional neutral dynamical equation

$$x^{\Delta}(t) = f\big(t, x(t), x^{\Delta}(t - h(t))\big) + g\big(t, x(t), x(t - h(t))\big); \ t, t - h(t) \in \mathbb{T}, \qquad (1)$$

where f, g and h are continuous, $h : \mathbb{T} \to [0, h_0]$ for some positive constant $h_0 \in \mathbb{T}$, and $f, g : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The first and third authors have considered a variation of (1); namely, in [11] they studied the existence of periodic solutions of the neutral dynamical system

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\Delta}(t - h(t)) + g\big(x(t), x(t - h(t))\big), \ t, t - h(t) \in \mathbb{T}, \qquad (2)$$

where $\mathbb{T}$ is a periodic time scale and a, b and h are periodic. In [12], the first and third authors showed

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t), x(t - g(t))))^{\Delta} + G\big(t, x(t), x(t - g(t))\big), t \in \mathbb{T}, \qquad (3)$$

has a periodic solution. In both papers, the authors obtained the existence of a periodic solution using a Krasnosel'skiĭ fixed point theorem. Moreover, under a slightly more stringent inequality they showed that the periodic solution is unique using the contraction mapping principle. The authors also showed that the zero solution was asymptotically stable using the contraction mapping principle provided that $Q(t, 0, 0) = G(t, 0, 0) = 0$.

In obtaining the existence of a periodic solution of (2) and (3) and the stability of the zero solution of (3), the authors inverted (2) and (3) and generated a variation of parameters-like formula. This formula was the sum of two mappings; one mapping was shown to be compact and the other was shown to be a contraction. We remark that the inversion of either (2) or (3) was made possible by the linear term $-a(t)x^{\sigma}(t)$, a luxury that (1) does not enjoy.

A neutral differential equation is an equation where the immediate growth rate is affected by the past growth rate. This can be observed in the behavior of a stock price or the growth of a child. Also, in the case $\mathbb{T} = \mathbb{R}$, neutral equations arise in circuit theory (see [3]) and in the study of drug administration and populations, (see [4], [13], [14]). This paper extends the results of [7] to time scales. Also, it is worth mentioning that the book [8] contains a wealth of information regarding stability and periodicity using fixed point theory.

Now we state Krasnosel'skiĭ's fixed point theorem which enables us to prove the existence of a solution. For its proof we refer the reader to [16].

Theorem 1.1 (Krasnosel'skiĭ) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathbb{B}, \|\cdot\|\big)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

2 Existence

For emphasis, we consider

$$x^{\Delta}(t) = f\big(t, x(t), x^{\Delta}(t - h(t))\big) + g\big(t, x(t), x(t - h(t))\big);\ t, t - h(t) \in \mathbb{T}, \tag{4}$$

where f, g and h are continuous, $h : \mathbb{T} \to [0, h_0]$ for some positive constant $h_0 \in \mathbb{T}$, $f, g : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Since our equation has a delay and the derivative enters nonlinearly on the right side, we must ask for an initial function $\eta \in C^1([-h_0, 0), \mathbb{R})$ whose Δ-derivative at 0 satisfies the expression

$$\eta^{\Delta}(0) = f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big) + g\big(0, \eta(0), \eta(-h(0))\big). \tag{5}$$

In the next definition, we state what we mean by a solution for (4) in terms of a given initial function.

Definition 2.1 Let $\eta \in C^1([-h_0, 0), \mathbb{R})$ be a given bounded initial function that satisfies (5). We say $x(t, \eta)$ is a solution of (4) on an interval $[-h_0, r), r > 0, r \in \mathbb{T}$, if $x(t, \eta) = \eta(t)$ on $[-h_0, 0]$ and satisfies (4) on $[0, r)$.

The above definition allows us to continue the solution on $[r, r_1)$, for some $r_1 > r$ under the requirement of certain conditions.

Let $C_{rd} = C_{rd}([-h_0, r], \mathbb{R})$ be the space of all rd-continuous functions and define the set S by

$$S = \left\{\varphi \in C_{rd} :\ \varphi(t) = \eta^{\Delta}(t) \text{ on } [-h_0, 0]\right\}.$$

Then $\big(S, \nu\big)$ is a complete metric space, where $\nu(\varphi, \psi) = \|\varphi - \psi\| = \sup_{t\in[0,r]} \{|\varphi(t) - \psi(t)|\}$.

If $\varphi \in S$, we define

$$\Phi(t) = \begin{cases} \eta(t), & t \in [-h_0, 0], \\ \eta(0) + \int_0^t \varphi(s)\Delta s, & t \in [0, r]. \end{cases}$$

It is clear that $\Phi^{\Delta}(t) = \varphi(t)$ on $[0, r]$ and $\Phi \in C_{rd}$. Next, we suppose there is an $a > 0$ and define a subset S^* of S by

$$S^* = \{\varphi \in S : |\varphi(t) - \eta^{\Delta}(0)| \le a\},$$

such that there are an $\alpha > 0$ and $\beta < 1/2$ so that for $\varphi, \psi \in S^*$ we have

$$\begin{aligned} &\Big|f\big(t, \Phi(t), \varphi(t - h(t))\big) - f\big(t, \Psi(t), \psi(t - h(t))\big)\Big| \\ &\qquad \le \alpha|\Phi(t) - \Psi(t)| + \beta\big|\varphi(t - h(t)) - \psi(t - h(t))\big|. \end{aligned} \tag{6}$$

To be able to use Krasnosel'skiĭ's fixed point theorem, we define the two required mappings as follow. For $\varphi, \psi \in S^*$:

$$(A\varphi)(t) = g\big(t, \Phi(t), \varphi(t - h(t))\big), \quad (B\varphi)(t) = f\big(t, \Phi(t), \varphi(t - h(t))\big).$$

Then (6) implies that

$$\big|(B\varphi)(t) - (B\psi)(t)\big| \le \alpha|\Phi(t) - \Psi(t)| + \beta|\varphi(t - h(t)) - \psi(t - h(t))|. \tag{7}$$

It is obvious from the constructions of sets S and S^*, that fixed points of S^* are solutions of (4).

Theorem 2.1 *If η satisfies* (5) *and* (7)*, then there is an $r > 0$ such that the solution $x(t, \eta)$ of* (4) *exists on* $[0, r)$.

Proof Let a be given as in the set S^*. Since the functions f and g are continuous in their respective arguments, then for $\varphi \in S^*$, we can find a positive constant $L(a)$ depending on a, such that

$$\big|(B\varphi)(t)\big| + \big|(A\varphi)(t)\big| \le L(a),\ t \in \mathbb{T}.$$

Moreover, AS^* is equicontinuous. Now, for a fixed $a > 0$, we claim that there is an $r > 0$ such that and for all $t \in [0, r]_{\mathbb{T}}$,

$$\Big|f\big(t, \Phi(t), \varphi(t - h(t))\big) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\Big| \le a/2. \tag{8}$$

To see this

$$\begin{aligned}
&\Big|f\big(t, \Phi(t), \varphi(t - h(t))\big) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\Big| \\
&\qquad \le \alpha|\Phi(t) - \eta(0)| + \beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| \\
&\qquad \le \alpha \sup_{t\in[0,r]_{\mathbb{T}}} \left|\eta(0) + \int_0^t \varphi(s)\,\Delta s - \eta(0)\right| + \beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| \\
&\qquad \le \alpha\, t|\varphi(\xi)| + \beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big|,\ \xi \in (0, t)_{\mathbb{T}},
\end{aligned}$$

where $|\varphi(\xi)| = \sup\limits_{\xi\in(0,t)} |\varphi(\xi)|$.

On the one hand, suppose $h(0) = 0$. Then $0 \le t - h(t) \le t$. As a consequence, for $\varphi \in S^*$, we have

$$\beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| = \beta\big|\varphi(t - h(t)) - \eta^{\Delta}(0)\big| \le \beta a < a/2.$$

On the other hand, if $h(0) > 0$, then there is an $r_1 > 0$ such that $t-h(t) \le 0$ for $t \in [0, r_1]$. Hence, $\varphi(t-h(t)) = \eta^{\Delta}(t-h(t))$ for $t \in [0, r_1]$. Since η^{Δ} is continuous, there is an $r_2 > 0$ so that $|\varphi(t - h(t)) - \eta^{\Delta}(0)| < a$, for $t \in [0, r_2]$. Thus, if we choose $r^* \in (0, \min\{r_1, r_2\})$, then we can set $r = r^*$, so that for $t \in [0, r^*]$, we have

$$\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| \le a. \tag{9}$$

Due to inequality (9) and since $\beta < 1/2$, we can find a positive number $q < a/2$ so that

$$\beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| \le q.$$

Moreover, since $\varphi \in S^*$, we can choose an r, $r \in (0, r^*)$ so that for $t \in [0, r]$, we have

$$\alpha\, t|\varphi(\xi)| + \beta\big|\varphi(t - h(t)) - \eta^{\Delta}(-h(0))\big| \le \alpha\, t|\varphi(\xi)| + q \le a/2,$$

which proves (8).

This shows that A is compact.

Finally, we claim that we can make r small enough so that for $\varphi \in S^*$, $t \in [0, r]$, we have

$$\big|g(t, \Phi(t), \Phi(t - h(t))) - g(0, \eta(0), \eta(-h(0)))\big| \le a/2. \tag{10}$$

The proof of the claim follows from the fact g is uniformly continuous on any bounded set. For $0 \le t - h(t)$, we have

$$\begin{aligned}
&\Big|g\big(t, \Phi(t), \Phi(t-h(t))\big) - g\big(0, \eta(0), \eta(-h(0))\big)\Big| \\
&\quad \le |t-0| + |\Phi(t) - \eta(0)| + \big|\Phi(t-h(t)) - \eta(-h(0))\big| \\
&\quad \le t + t|\varphi(\xi)| + \left|\eta(0) + \int_0^{t-h(t)} \varphi(s)\,\Delta s - \eta(-h(0))\right| \\
&\quad \le t + t|\varphi(\xi)| + (t-h(t)|\varphi(\xi)| \\
&\quad \le r[1 + 2|\varphi(\xi)|],
\end{aligned}$$

where $|\varphi(\xi)| = \sup\limits_{\xi \in (0,r)} |\varphi(\xi)|$, which can be made arbitrary small. Due to the continuity of η, the case $t - h(t) < 0$ readily follows. This completes the proof of (10).

We now go back to the proof of the theorem. It readily follows form (7) that for $\varphi, \psi \in S^*$, there is a $\lambda < 1$ so that the

$$\|B\varphi - B\psi\| \le \lambda \|\varphi - \psi\|. \tag{11}$$

Next we show that if $\varphi, \psi \in S^*$, then $A\psi + B\varphi \in S^*$. We remark that $(A\psi)(0) + (B\psi)(0) = \eta^\Delta(0)$, where $\eta^\Delta(0)$ is given by (5). As a consequence, we have by (8) and (10)

$$\begin{aligned}
|(A\psi)(t) \quad &+ \quad (B\varphi)(t) - \eta^\Delta(0)| \\
&= \quad |(A\psi)(t) - (A\psi)(0) + (B\varphi)(t) - (B\psi)(0)| \\
&= \quad \big|(A\psi)(t) - g\big(0, \eta(0), \eta(-h(0))\big) + (B\varphi)(t) - f\big(0, \eta(0), \eta^\Delta(-h(0))\big)\big| \\
&\le \quad \big|(A\psi)(t) - g\big(0, \eta(0), \eta(-h(0))\big)\big| + \big|(B\varphi)(t) - f\big(0, \eta(0), \eta^\Delta(-h(0))\big)\big| \\
&\le \quad a/2 + a/2 = a.
\end{aligned}$$

This completes the proof of $A\psi + B\varphi \in S^*$.

Also, (11) shows that B is a contraction. Hence all the conditions of Theorem 1.1 are satisfied, which imply that there is $\varphi \in S^*$, such that $\varphi = A\varphi + B\varphi$. $\Box$

3 Connection Between Boundedness and Periodicity

Intuitively, in the study of stability or periodic solutions in dynamical systems one will have to ask for the existence of solutions in the sense that solutions exist for all time or remain bounded. Thus, we may look at boundedness of solutions as a necessary condition before studying stability or attempt to search for a periodic solution. In this section, we examine the relationship between the boundedness of solutions and the existence of a periodic solution of the nonlinear non-autonomous delay dynamical system of the form

$$x^\Delta(t) = -a(t)x^\sigma(t) + b(t)g\big(x(t - r(t))\big) + q(t), \tag{12}$$

where $\mathbb{T}$ is unbounded above and below.

We assume that $a, b : \mathbb{T} \to \mathbb{R}$ are continuous and $q : [0, \infty) \to \mathbb{R}$ is continuous. In order for the function $x(t - r(t))$ to be well-defined over $\mathbb{T}$, we assume that $r : \mathbb{T} \to \mathbb{R}$ and that $id - r : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

The proof of Lemma 3.2 below can be easily deduced from [11], and hence we omit the proof. But first, we state some facts about the exponential function. A function

$p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\,\Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma, [5, Theorem 2.36].

Lemma 3.1 *Let $p, q \in \mathcal{R}$. Then*
(i) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ where, $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lemma 3.2 *x is a solution of equation* (12) *if and only if*

$$x(t) = x(0)e_{\ominus a}(t,0) + \int_0^t b(s)g(x(s - r(s))e_{\ominus a}(t,s)\,\Delta s + \int_0^t q(s)e_{\ominus a}(t,s)\,\Delta s.$$

Let $\psi : (-\infty, 0] \to \mathbb{R}$ be a given bounded Δ-differentiable initial function. We say $x := x(\cdot, 0, \psi)$ is a solution of (12) if $x(t) = \psi(t)$ for $t \leq 0$ and satisfies (12) for $t \geq 0$. If $\psi : (-\infty, 0] \to \mathbb{R}$, then we set

$$\|\psi\| = \sup_{s \in (-\infty,0]} |\psi(s)|.$$

Definition 3.1 Let ψ be as defined above. We say solutions of (12) are uniformly bounded if for each $B_1 > 0$, there exists $B_2 > 0$ such that

$$[t_0 \geq 0,\ \|\psi\| \leq B_1,\ t \leq t_0] \Rightarrow |x(\cdot, 0, \psi)| < B_2.$$

For the next theorem we assume the following. There is a positive constant Q so that

$$\int_0^t |q(s)|e_{\ominus a}(t,s)\,\Delta s \leq Q, \tag{13}$$

$$\int_0^t a(s)\,\Delta s \to \infty, \tag{14}$$

there is an $\alpha < 1$ so that

$$\int_0^t |b(s)|e_{\ominus a}(t,s)\,\Delta s < \alpha, \tag{15}$$

$$0 \leq r(t),\ t - r(t) \to \infty \text{ as } t \to \infty, \tag{16}$$

and if $x, y \in \mathbb{R}$, then

$$g(0) = 0 \text{ and } \ |g(x) - g(y)| < |x - y|. \tag{17}$$

Theorem 3.1 *If* (13)–(17) *hold, then solutions of* (12) *are uniformly bounded at* $t_0 = 0$.

Proof First by (14), there is a constant $k > 1$ so that $e_{\ominus a}(t,0) \le k$. Let B_1 be given so that if $\psi : (-\infty, 0] \to \mathbb{R}$ be a given bounded initial function, $\|\psi\| \le B_1$. Define the constant B_2 by $B_2 = \frac{kB_1+Q}{1-\alpha}$. Let

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \in (-\infty, 0],\ \|\varphi\| \le B_2\}.$$

Then $\big(S, \|\cdot\|\big)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

For $\varphi \in S$, define the mapping P

$$\big(P\varphi\big)(t) = \psi(t), t \le 0$$

and

$$\begin{aligned}\big(P\varphi\big)(t) \quad &= \quad \psi(0)e_{\ominus a}(t,0) + \int_0^t b(s)g(\varphi(s - r(s))e_{\ominus a}(t,s)\,\Delta s \\ &\qquad + \int_0^t q(s)e_{\ominus a}(t,s)\,\Delta s,\ t \ge 0.\end{aligned}$$

It follows from (17) that

$$|g(x)| = |g(x) - g(0) + g(0)| \le |g(x) - g(0)| + |g(0)| \le |x|.$$

This implies that

$$|(P\varphi)(t)| \le kB_1 + \alpha B_2 + Q = B_2.$$

Thus, $P : S \to S$. It is easy to show, using (17), that P is a contraction with contraction constant α. Hence there is a unique fixed point in S, which solves (12). □

We end this paper by examining the existence of a periodic solution of (12). We must first define what we mean by a periodic time scale.

Definition 3.2 We say that a time scale $\mathbb{T}$ is *periodic* if there exists a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

The above definition is due to Kaufmann and Raffoul [12]. Other definitions of periodic time scales are due to Atici *et. al* [2], C. D. Ahlbrandt and J. Ridenhour [1], and J. J. DaCunha and J. M. Davis [9].

Example 3.1 The following time scales are periodic.

1. $\mathbb{T} = \displaystyle\bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih]$, $h > 0$ has period $p = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $p = 1$.

Remark 3.1 Using the above definition, all periodic time scales are unbounded above and below.

Definition 3.3 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 3.2 If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. Define $P_T = \{\varphi \in C(\mathbb{T}, R) : \varphi(t + T) = \varphi(t)\}$, where $C(\mathbb{T}, R)$ is the space of all real valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

Here we let the function $q : (-\infty, \infty) \to \mathbb{R}$. Since we are searching for a periodic solution, we must ask that

$$a(t+T) = a(t),\ b(t+T) = b(t),\ r(t+T) = r(t),\ \text{and } q(t+T) = q(t). \tag{18}$$

Lemma 3.3 *Suppose* (14)–(18) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (12) *if and only if*

$$x(t) = \int_{-\infty}^{t} b(s) g(x(s - r(s))) e_{\ominus a}(t,s)\,\Delta s + \int_{-\infty}^{t} q(s) e_{\ominus a}(t,s)\,\Delta s. \tag{19}$$

Proof Due to condition (14) and the fact that $p(t)$ is periodic, condition (13) is satisfied. Thus, by Theorem 3.1, solutions of (12) are bounded for all $t \in (-\infty, \infty)$. As a consequence, if we multiply both sides of (12) by $e_a(s, 0)$, and then integrate from $-\infty$ to t we obtain (19). By taking the Δ-derivative on both sides of (19) we obtain (12). □

Theorem 3.2 *Assume the hypothesis of Lemma 3.3. Then* (12) *has a unique* T*-periodic solution.*

Proof For $\phi \in P_T$, define a mapping $H : P_T \to P_T$ by

$$(H\phi)(t) = \int_{-\infty}^{t} b(s) g(\phi(s - r(s))) e_{\ominus a}(t,s)\,\Delta s + \int_{-\infty}^{t} p(s) e_{\ominus a}(t,s)\,\Delta s.$$

It is easy to verify that H is periodic and defines a contraction on P_T. Thus, H has a unique fixed point in P_T by the contraction mapping principle, which solves (12) by Lemma 3.8. □

We remark that Lemma 3.3 and Theorem 3.2 show a clear connection between boundedness and the existence of a periodic solution. In the case $\mathbb{T} = \mathbb{R}$, this result is known as Massera's theorem, see [15]. Below we state and prove Massera's theorem for the general case of $\mathbb{T}$ being a periodic time scale. We begin with a lemma that will be needed in the proof.

Lemma 3.4 *Let $\mathbb{T}$ be a periodic time scale with period $T > 0$. Let $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $F(t+T, x) = F(t, x)$ and that $F(t, x)$ satisfy a local Lipschitz condition with respect to x.*

1. If $x(t)$ is a solution of $x^\Delta = F(t, x)$, then $x(t+T)$ is also a solution of $x^\Delta = F(t, x)$.

2. The equation $x^\Delta = F(t, x)$ has a T-periodic solution if and only if there is a (t_0, x_0) with $x(t_0+T; t_0, x_0) = x_0$ where $x(t; t_0, x_0)$ is the unique solution of $x^\Delta = F(t, x), x(t_0) = x_0$.

Proof For part (1), let $q(t) = x(t + T)$. Then,

$$q^\Delta(t) = x^\Delta(t + T) = F(t + T, x(t + T)) = F(t, q(t)),$$

and the proof of part (1) is complete.

For part (2), first suppose that $x(t; t_0, x_0)$ is T-periodic. Then, $x(t_0 + T; t_0, x_0) = x(t_0; t_0, x_0) = x_0$.

Now suppose that there exists (t_0, x_0) such that $x(t_0+T; t_0, x_0) = x_0$. From part (1), $q(t) \equiv x(t + T; t_0, x_0)$ is also a solution of $x^\Delta = F(t, x)$. Since $q(t_0) = x(t_0 + T; t_0, x_0) = x_0$, then by the uniqueness of solutions of initial value problems, $x(t+T; t_0, x_0) = q(t) = x(t; t_0, x_0)$. This completes the proof of part (2). □

Theorem 3.3 *Let $\mathbb{T}$ be a periodic time scale with period $T > 0$. Let $F : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $F(t+T, x) = F(t, x)$ and that $F(t, x)$ satisfies a local Lipschitz condition with respect to x. If the equation*

$$x^\Delta = F(t, x) \tag{20}$$

has a solution bounded in the future, then it has a T-periodic solution.

Proof Let $x(t)$ be the solution of (20) such that $|x(t)| \le M$ for all $t \in \mathbb{T}, t \ge 0$. Define the sequence $\{x_n(t)\}$ by $x_n(t) = x(t + nT), n = 0, 1, 2, \ldots$. By Lemma 3.4, $x_n(t)$ is a solution of (20) for each n and furthermore, $|x_n(t)| \le M$ for $t \ge 0$. There are two cases to consider.

Case 1: Suppose that for some n, $x_n(0) = x_{n+1}(0)$. By uniqueness of solutions for initial value problems we have $x(t + nT) = x(t + (n + 1)T)$ for all $t \in \mathbb{T}$. Thus, $x(t)$ is a T-periodic solution of (20).

Case 2: Suppose that $x_n(0) \ne x_{n+1}(0)$ for all n. We may assume, without loss of generality, that $x(0) < x_1(0)$. By uniqueness, we have $x(t) < x_1(t)$ for all $t \in \mathbb{T}, t \ge 0$. In particular, $x_n(0) = x(0 + nT) < x_1(0 + nT) = x_{n+1}(0)$. Hence, $x_n(t) < x_{n+1}(t)$ for all $t \in \mathbb{T}, t \ge 0$. Thus, $\{x_n(t)\}$ is an increasing sequence bounded above by M. Thus $x_n(t) \to x^*(t)$ for each $t \in \mathbb{T}, t \ge 0$ as $n \to \infty$. Since $|F(t, x)| \le J$ for $t \in \mathbb{T}$ and $|x| \le M$, then $|x^\Delta(t)| \le J, t \in \mathbb{T}$.

¿From the Mean Value Theorem (see [5, Corollary 1.68]) we have $|x_n(t) - x_n(s)| \le \sup_{r\in[s,t]^\kappa} |F^\Delta(r, x_n)|\,|t-s| \le J|t-s|$ for $t, s \in \mathbb{T}$ with $0 \le s \le t, n \ge 0$. Using the Arzela-Ascoli Theorem, we get that on any compact subinterval of $\mathbb{T}$ there exists a subsequence of $\{x_n(t)\}$ that converges uniformly. We know that the original sequence is monotone, and so, the original sequence is convergent on any compact interval. Since for each n,

$$x_n(t) = x_n(0) + \int_0^t F(s, x_n(s))\,\Delta s,$$

then the limiting function $x^*(t)$ is a solution of (20). Finally, since $x^*(T) = \lim x_n(T) = \lim x_{n+1}(0) = x^*(0)$, then by Lemma 3.4, the limiting function is a T-periodic solution of (20) and the proof is complete. □

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Limit-Point Criteria for a Second Order Dynamic Equation on Time Scales

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Abstract: In this paper, we establish some criteria under which the second order formally self-adjoint dynamic equation

$$(p(t)x^{\Delta})^{\nabla} + q(t)x = 0$$

is of limit-point type on a time scale $\mathbb{T}$. As a special case when $\mathbb{T} = \mathbb{R}$, our results include those of Wong and Zettl [11] and Coddington and Levinson [5]. Our results are new in a general time scale setting and can be applied to difference and q-difference equations.

Keywords: *time scales; limit-point; limit-circle; second-order equation.*

Mathematics Subject Classification (2000): 34A99.

1 Introduction

In this paper, assume that $\inf \mathbb{T} = t_0$, and $\sup \mathbb{T} = \infty$. We will sometimes refer to $\mathbb{T}$ as $[t_0, \infty)$ which we mean to be the real interval $[t_0, \infty)$ intersected with $\mathbb{T}$. Assume that $p(t) \neq 0$ and $q(t) \neq 0$ for $t \in \mathbb{T}$ are continuous functions on $\mathbb{T}$. We will consider the formally self-adjoint equations

$$Lx = \left(p(t)x^{\Delta}\right)^{\nabla} + q(t)x = 0 \tag{1.1}$$

and

$$\widetilde{L}y = \left(\frac{1}{q(t)}y^{\nabla}\right)^{\Delta} + \frac{1}{p(t)}y = 0. \tag{1.2}$$

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Let $\mathbb{D}$ be the set of functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^\Delta : \mathbb{T} \to \mathbb{R}$ is continuous, and $(px^\Delta)^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ is continuous. Let $\widetilde{\mathbb{D}}$ be the set of functions $x : \mathbb{T} \to \mathbb{R}$ such that $x^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ is continuous, and $\left(\frac{1}{q}x^\nabla\right)^\Delta : \mathbb{T}_\kappa \to \mathbb{R}$ is continuous. We say (1.1) and (1.2) are *reciprocal equations* of each other. See [7] for more on reciprocal equations and [8], [9] and [10] for other results dealing with second-order equations, and [6] and [4] for more on general theories used in this paper.

These equations are said to be formally self-adjoint because they satisfy the following Lagrange identity.

Theorem 1.1 (Lagrange identity)

(i) Let $u, v \in \mathbb{D}$. Then

$$u(t)Lv(t) - v(t)Lu(t) = \{u; v\}^\nabla(t)$$

for $t \in \mathbb{T}_\kappa$, where the Lagrange bracket $\{u; v\}$ is defined by

$$\{u; v\}(t) := p(t)W(u, v)(t),$$

where

$$W(u, v)(t) := \begin{vmatrix} u(t) & v(t) \\ u^\Delta(t) & v^\Delta(t) \end{vmatrix}.$$

(ii) Let $\widetilde{u}, \widetilde{v} \in \widetilde{\mathbb{D}}$. Then

$$\widetilde{u}(t)\widetilde{L}\widetilde{v}(t) - \widetilde{v}(t)\widetilde{L}\widetilde{u}(t) = \left(\frac{1}{q(t)}\widetilde{W}(\widetilde{u}, \widetilde{v})(t)\right)^\Delta$$

for $t \in \mathbb{T}_\kappa$, where

$$\widetilde{W}(\widetilde{u}, \widetilde{v})(t) := \begin{vmatrix} \widetilde{u}(t) & \widetilde{v}(t) \\ \widetilde{u}^\nabla(t) & \widetilde{v}^\nabla(t) \end{vmatrix}.$$

For a proof of Theorem 1.1 (i), see Theorem 4.33 in [3].

Corollary 1.1 (Abel's formula)
(i) If x and y both solve (1.1) *then*

$$p(t)W(x, y)(t) = a \quad t \in \mathbb{T},$$

where a is a constant.
(ii) If x and y both solve (1.2) *then*

$$\frac{1}{q(t)}\widetilde{W}(x, y)(t) = a \quad t \in \mathbb{T}_\kappa,$$

where a is a constant.

For a proof of Corollary 1.1 for the case of (1.1), see Corollary 4.34 in [3].

Definition 1.1 The set $L^2[t_0, \infty)$ is defined to be the set of all functions $f(t)$ such that the Lebesgue integral

$$\int_{t_0}^\infty f^2(t)\Delta t < \infty.$$

We define the L^2-norm of a function $f \in L^2[t_0, \infty)$ by

$$\|f\|_{L^2} = \|f\| := \left(\int_{t_0}^{\infty} f^2(t)\Delta t \right)^{1/2}.$$

Definition 1.2 We say that the operator L is (Δ-)*limit-circle type* if for every solution x of $Lx = 0$, we have the Lebesgue integral

$$\int_{t_0}^{\infty} x^2(t)\Delta t < \infty.$$

If not, we say that the operator L is (Δ-)*limit-point type.*

Refer to Wong and Zettl, [11], and Coddington and Levinson, [5], for an analysis of the differential equations case.

2 Preliminary Lemmas

Lemma 2.1 *If there exists a function* $\beta(t)$ *with* $\frac{1}{\beta} \notin L^2[t_0, \infty)$ *such that* $px^{\Delta}(t) = O(\beta(t))$ *as* $t \to \infty$ *for every solution* x *of* (1.1)*, then* L *is limit-point type.*

Proof Suppose (1.1) is limit-circle type, and let x_1, x_2 be linearly independent solutions of (1.1), so we have by Corollary 1.1 part (i)

$$p(t)(x_1(t)x_2^{\Delta}(t) - x_2(t)x_1^{\Delta}(t)) \equiv a \quad t \in \mathbb{T}.$$

Then there exist constants $c, d \geq 0$ such that

$$\begin{aligned} a &\leq |x_1(t)||p(t)x_2^{\Delta}(t)| + |x_2(t)||p(t)x_1^{\Delta}(t)| \\ &\leq c\beta(t)|x_1(t)| + d\beta(t)|x_2(t)| \quad \text{ for large } t \in \mathbb{T}. \end{aligned}$$

Thus, for large $t \in \mathbb{T}$,

$$\frac{a}{\beta(t)} \leq c|x_1(t)| + d|x_2(t)|.$$

It follows that for T large,

$$\begin{aligned} a\int_T^t \frac{1}{\beta^2(s)}\Delta s &\leq \int_T^t [c^2x_1^2(s) + 2cdx_1(s)x_2(s) + d^2x_2^2(s)]\Delta s \\ &\leq c^2\|x_1\|^2 + 2cd\|x_1\|\|x_2\| + d^2\|x_2\|^2 \\ &< \infty \end{aligned}$$

by the Cauchy-Schwarz inequality (Theorem 6.15, [2]). This contradicts the fact that $\frac{1}{\beta} \notin L^2[t_0, \infty)$, so L is limit-point type.

Lemma 2.2 *Suppose* $q \in C^1[t_0, \infty)$*. If there exists a positive function* β *with* $\frac{1}{\beta} \notin L^2[t_0, \infty)$ *such that* $y(t) = O(\beta(t))$ *as* $t \to \infty$ *for every solution* y *of* (1.2)*, then* L *is limit-point type.*

Proof Let x be a solution of (1.1), and put $y = px^\Delta$. Then $y^\nabla = -qx$ and

$$\left(\frac{1}{q}y^\nabla\right)^\Delta = -x^\Delta = -\frac{y}{p}.$$

Hence, y solves (1.2). Thus,

$$y(t) = (px^\Delta)(t) = O(\beta(t)) \text{ as } t \to \infty.$$

Thus, by Lemma 2.1, L is limit-point type.

A useful corollary to these lemmas is obtained by letting $\beta(t) \equiv 1$.

Corollary 2.1 *If $(px^\Delta)(t)$ is bounded for every solution x of* (1.1)*, or if every solution y of* (1.2) *is bounded, then L is limit-point type.*

3 Riccati Substitution

Suppose y is a solution of (1.2) with $q(t)y(t)y^\sigma(t) > 0$ for $t \geq t_0$. We can then make the Riccati substitution

$$z(t) = \frac{y^\nabla(t)}{q(t)y(t)} \quad \text{for} \quad t \in [t_0, \infty).$$

Then, we have

$$\begin{aligned} z^\Delta(t) &= \left(\left(\frac{y^\nabla(t)}{q(t)}\right)\left(\frac{1}{y(t)}\right)\right)^\Delta \\ &= \left(\frac{y^\nabla(t)}{q(t)}\right)^\Delta\left(\frac{1}{y(t)}\right) + \left(\frac{y^\nabla(t)}{q(t)}\right)^\sigma\left(\frac{1}{y(t)}\right)^\Delta \\ &= -\frac{1}{p(t)} + \left(\frac{y^\nabla(t)}{q(t)}\right)^\sigma\left(\frac{-y^\Delta(t)}{y(t)y^\sigma(t)}\right) \\ &= -\frac{1}{p(t)} - \frac{z^\sigma(t)y^\Delta(t)}{y(t)}. \end{aligned}$$

We now use the following lemma, due to Atici and Guseinov [1]:

Lemma 3.1 *If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is ∇-differentiable on $\mathbb{T}_\kappa$ and*

$$f^\nabla(t) = f^{\Delta\rho}(t) \quad t \in \mathbb{T}_\kappa.$$

If $g : \mathbb{T} \to \mathbb{R}$ is ∇-differentiable on $\mathbb{T}_\kappa$ and if g^∇ is continuous on $\mathbb{T}_\kappa$, then g is Δ-differentiable on $\mathbb{T}^\kappa$ and

$$g^\Delta(t) = g^{\nabla\sigma}(t) \quad t \in \mathbb{T}^\kappa.$$

See also Corollary 4.11 and Theorem 4.8 and Corollary 4.10 in [3] for a generalization of this result.

Thus, we get

$$\frac{z^\sigma(t)y^\Delta(t)}{y(t)} = \frac{z^\sigma(t)y^\Delta(t)}{y^\sigma(t)-\mu(t)y^\Delta(t)} = \frac{z^\sigma(t)\frac{y^{\nabla\sigma}(t)}{y^\sigma(t)}}{1-\mu(t)\frac{y^{\nabla\sigma}(t)}{y^\sigma(t)}}$$
$$= \frac{q^\sigma(t)(z^\sigma(t))^2}{1-\mu(t)q^\sigma(t)z^\sigma(t)} = \frac{(z^\sigma(t))^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma(t)}.$$

Hence, we get that $z(t)$ solves the so-called Riccati equation associated with (1.2)

$$z^\Delta + \frac{1}{p(t)} + \frac{(z^\sigma)^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma} = 0. \tag{3.1}$$

Notice that $\dfrac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) > 0$ for all $t \geq t_0$:

$$\frac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) = \frac{1}{q^\sigma(t)} - \mu(t)\frac{y^\Delta(t)}{q^\sigma(t)y^\sigma(t)}$$
$$= \frac{1}{q^\sigma(t)y^\sigma(t)}[y^\sigma(t)-\mu(t)y^\Delta(t)]$$
$$= \frac{y(t)}{q^\sigma(t)y^\sigma(t)} > 0.$$

Hence, we have proven the following lemma:

Lemma 3.2 *If $y(t)$ is a solution of (1.2) with $q(t)y(t)y^\sigma(t) > 0$ then $z(t) := \dfrac{y^\nabla(t)}{q(t)y(t)}$ is a solution of (3.1) that satisfies $\dfrac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t) > 0$ for all $t \in \mathbb{T}$.*

4 Main Results

Theorem 4.1 *Suppose that $p(t) > 0$ and $q(t) > 0$ on $[t_0,\infty)$, and $\displaystyle\int_{t_0}^\infty \frac{1}{p(t)}\Delta t = \infty$.*
(a) If (1.2) is nonoscillatory, then L is limit-point.
(b) If (1.1) is nonoscillatory, then L is limit-point.

Proof Suppose (1.2) is nonoscillatory. Let y be a positive solution of (1.2) on $[t_0,\infty)$, and make the Riccati substitution $z(t) = \dfrac{y^\nabla(t)}{q(t)y(t)}$. Then z solves

$$z^\Delta = -\frac{1}{p(t)} - \frac{(z^\sigma)^2}{\frac{1}{q^\sigma(t)}-\mu(t)z^\sigma}.$$

Integrate both sides from t_0 to t:

$$z(t)-z(t_0) = -\int_{t_0}^t \frac{1}{p(s)}\Delta s - \int_{t_0}^t \frac{(z^\sigma(s))^2}{\frac{1}{q^\sigma(s)}-\mu(s)z^\sigma(s)}\Delta s. \tag{4.1}$$

Since

$$\frac{(z^\sigma(t))^2}{\frac{1}{q^\sigma(t)} - \mu(t)z^\sigma(t)} \geq 0$$

for all $t \geq t_0$, we get that the right hand side of (4.1) goes to $-\infty$ as t goes to ∞. Thus, $z(t) \to -\infty$ as $t \to \infty$, so z, and hence y^∇, is eventually negative. Thus, eventually $y(t) > 0$ and $y^\nabla(t) < 0$, hence y is bounded. Thus, by Corollary 2.1 we get that L is limit-point.

Now suppose (1.1) is nonoscillatory. Let x be a positive solution of (1.1) on $[t_0, \infty)$. Since $q(t) > 0$, we have $\left(p(t)x^\Delta(t)\right)^\nabla = -q(t)x(t) < 0$ on $[t_0, \infty)$.
Claim: $p(t)x^\Delta(t) \geq 0$ on $[t_0, \infty)$.
To see this, suppose not. Then there exists $t_1 \geq t_0$ with $p(t_1)x^\Delta(t_1) < 0$. Since $p(t)x^\Delta(t)$ is decreasing, $p(t)x^\Delta(t) \leq p(t_1)x^\Delta(t_1) < 0$ on $[t_1, \infty)$. Then, dividing by $p(t)$ and integrating, we get

$$x(t) - x(t_1) \leq p(t_1)x^\Delta(t_1) \int_{t_1}^t \frac{1}{p(s)} \Delta s.$$

Thus, $\lim_{t\to\infty} x(t) = -\infty$. This contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. Hence the claim holds and we see then that $p(t)x^\Delta(t)$ is bounded, so by Corollary 2.1, L is limit-point.

Definition 4.1 The set $L^2_\nabla[t_0, \infty)$ is defined to be the set of all functions $f(t)$ such that the Lebesgue integral

$$\int_{t_0}^\infty f^2(t) \nabla t < \infty.$$

We define the L^2_∇-norm of a function $f \in L^2_\nabla[t_0, \infty)$ by

$$\|f\|_{L^2_\nabla} := \left(\int_{t_0}^\infty f^2(t) \nabla t\right)^{1/2}.$$

Definition 4.2 The operator L is said to be ∇-*limit-circle* if all solutions of $Lx = 0$ satisfy $x, x^\rho \in L^2_\nabla[t_0, \infty)$. We say L is ∇-*limit-point* if there is a solution $x(t)$ of $Lx = 0$ such that $x \notin L^2_\nabla[t_0, \infty)$ or $x^\rho \notin L^2_\nabla[t_0, \infty)$.

Theorem 4.2 *Let M be a positive ∇-differentiable function and $k_1, k_2 > 0$ such that there is a $T \in \mathbb{T}$, sufficiently large such that*
(i) $q(t) \leq k_1 M(t)$ for $t \in [T, \infty)$,
(ii) $\int_T^\infty (p^\rho M^\rho)^{-1/2} \nabla s = \infty$,
(iii) $\left| \left(\frac{p^\rho(t)}{M^\rho(t)}\right)^{1/2} \frac{M^\nabla(t)}{M(t)} \right| \leq k_2$ for $t \in [T, \infty)$.
Then L is ∇-limit-point.

Proof Suppose x is a solution of $Lx = 0$ and $x, x^\rho \in L^2_\nabla[t_0, \infty)$. Since $(px^\Delta)^\nabla = -qx$, we get that for some $c > 0$,

$$\int_c^t \frac{(px^\Delta)^\nabla x}{M} \nabla s = -\int_c^t \frac{q}{M} x^2 \nabla s \geq -k_1 \int_c^t x^2 \nabla s. \tag{4.2}$$

Using the integration by parts formula ([2], Theorem 8.47 (vi))

$$\int_a^b f(s)g^{\nabla}(s)\nabla s = f(s)g(s)\Big|_a^b - \int_a^b f^{\nabla}(s)g^{\rho}(s)\nabla s,$$

we get from (4.2)

$$\begin{aligned}\frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t (px^{\Delta})^{\rho}\left(\frac{x}{M}\right)^{\nabla}\nabla s &= \frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t p^{\rho}x^{\Delta\rho}\left(\frac{x^{\nabla}M - xM^{\nabla}}{MM^{\rho}}\right)\nabla s\\ &= \frac{x}{M}px^{\Delta}\Big|_c^t - \int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s + \int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s\\ &\geq -k_1\int_c^t x^2\nabla s.\end{aligned}$$

Thus, multiplying by -1, we get

$$-\frac{x}{M}px^{\Delta}\Big|_c^t + \int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s - \int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s \leq k_1\|x\|^2 < k_3$$

for some $k_3 > 0$.

Let $H(t) = \displaystyle\int_c^t \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_c^t \frac{p^{\rho}xx^{\nabla}M^{\nabla}}{MM^{\rho}}\nabla s\right|^2 &= \left|\int_c^t \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} M^{-1}M^{\nabla}\left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} xx^{\nabla}\nabla s\right|^2\\ &\leq k_2^2\left(\int_c^t \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} xx^{\nabla}\nabla s\right)^2 \qquad \text{by } (iii)\\ &\leq k_2^2 H(t)\int_c^t x^2\nabla s.\end{aligned}$$

Thus, there exists a constant $k_4 > 0$ such that

$$-\frac{px^{\Delta}x}{M} + H - k_4H^{1/2} < k_3.$$

If $H(t) \to \infty$ as $t \to \infty$, then for all large t, $\dfrac{px^{\Delta}x}{M} > \dfrac{H}{2}$. Then x and x^{Δ} have the same sign for all large t, which contradicts $x \in L^2_{\nabla}[t_0, \infty)$. Thus,

$$H(\infty) = \int_{t_0}^{\infty} \frac{p^{\rho}}{M^{\rho}}(x^{\nabla})^2\nabla s < \infty.$$

Now suppose L is ∇-limit-circle. Let ϕ, ψ be two linearly independent solutions of $Lx = 0$ with $p(t)\left(\phi(t)\psi^{\Delta}(t) - \psi(t)\phi^{\Delta}(t)\right) = 1$ and ϕ, ϕ^{ρ}, ψ, $\psi^{\rho} \in L^2_{\nabla}[t_0, \infty)$. Then

$$\begin{aligned}1 &= p^{\rho}(t)\left(\phi^{\rho}(t)\psi^{\Delta\rho}(t) - \psi^{\rho}(t)\phi^{\Delta\rho}(t)\right)\\ &= p^{\rho}(t)\left(\phi^{\rho}(t)\psi^{\nabla}(t) - \psi^{\rho}(t)\phi^{\nabla}(t)\right).\end{aligned}$$

So, if we divide both sides by $(p^{\rho} M^{\rho})^{1/2}$, we get

$$\frac{1}{(p^{\rho} M^{\rho})^{1/2}} = \phi^{\rho}(t) \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \psi^{\nabla}(t) - \psi^{\rho}(t) \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \phi^{\nabla}(t). \tag{4.3}$$

If we integrate both sides of (4.3) from t_0 to ∞, we get

$$\int_{t_0}^{\infty} \frac{1}{(p^{\rho} M^{\rho})^{1/2}} \nabla s = \int_{t_0}^{\infty} \phi^{\rho} \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \psi^{\nabla} \nabla s - \int_{t_0}^{\infty} \psi^{\rho} \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \phi^{\nabla} \nabla s. \tag{4.4}$$

By assumption, the left-hand side of (4.4) is infinite. But, by the Cauchy-Schwarz inequality, the right-hand side becomes

$$\begin{aligned}
&\left| \int_{t_0}^{\infty} \phi^{\rho} \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \psi^{\nabla} \nabla s - \int_{t_0}^{\infty} \psi^{\rho} \left(\frac{p^{\rho}}{M^{\rho}}\right)^{1/2} \phi^{\nabla} \nabla s \right| \\
&\leq \|\phi^{\rho}\|_{L^2_{\nabla}} \left(\int_{t_0}^{\infty} \frac{p^{\rho}}{M^{\rho}} (\psi^{\nabla})^2 \nabla s \right)^{1/2} + \|\psi^{\rho}\|_{L^2_{\nabla}} \left(\int_{t_0}^{\infty} \frac{p^{\rho}}{M^{\rho}} (\phi^{\nabla})^2 \nabla s \right)^{1/2} \\
&< \infty.
\end{aligned}$$

This is a contradiction to the assumption that L is ∇-limit-circle. Thus, we have that L is ∇-limit-point

5 Example

Fix $q > 1$. Let $\mathbb{T} = \{q^n : n \in \mathbb{N}_0\}$. Consider the dynamic equation

$$x^{\Delta\nabla} + (t \ln t)^2 x = 0.$$

Here, we have $p(t) \equiv 1$, and $q(t) = (t \ln t)^2$. We need to show that the three assumptions in Theorem 4.2 hold. Fix $N > 0$ sufficiently large and let $T = q^N$. Also, let $M(t) = (t \ln t)^2$. For (i), if we take $k_1 = 1$, we get that $q(t) = M(t) = (t \ln t)^2$ for all $t \in \mathbb{T}$, so certainly $q(t) \leq M(t)$ for $t \geq T$.

For (ii), consider

$$\begin{aligned}
\int_{T}^{\infty} (p^{\rho}(s) M^{\rho}(s))^{-1/2} \nabla s &= \int_{T}^{\infty} \frac{1}{(M^{\rho}(s))^{1/2}} \nabla s = \int_{T}^{\infty} \frac{1}{((\rho(s) \ln \rho(s))^2)^{1/2}} \nabla s \\
&= \int_{T}^{\infty} \frac{1}{\rho(s) \ln \rho(s)} \nabla s = \sum_{k=N+1}^{\infty} \frac{1}{q^{k-1} \ln q^{k-1}} \nu(q^k) \\
&= \sum_{k=N+1}^{\infty} \frac{1}{q^{k-1} \ln q^{k-1}} (q^k - q^{k-1}) = \sum_{k=N+1}^{\infty} \frac{q^{k-1}(q-1)}{q^{k-1} \ln q^{k-1}} \\
&= \frac{q-1}{\ln q} \sum_{k=N+1}^{\infty} \frac{1}{k-1} = \frac{q-1}{\ln q} \sum_{k=N}^{\infty} \frac{1}{k} = \infty.
\end{aligned}$$

Notice,

$$
\begin{aligned}
M^{\nabla}(t) &= \frac{(q^k \ln q^k)^2 - (q^{k-1}\ln q^{k-1})^2}{q^k - q^{k-1}} \\
&= \frac{(q^k \ln q^k - q^{k-1}\ln q^{k-1})(q^k \ln q^k + q^{k-1}\ln q^{k-1})}{q^k - q^{k-1}} \\
&= \frac{(q^{k-1})^2(\ln q)^2(qk-(k-1))(qk+(k-1))}{q^{k-1}(q-1)} \\
&= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{q-1}.
\end{aligned}
$$

Thus, for part (iii), we have for $k \geq N$

$$
\begin{aligned}
\left| \left(\frac{p^{\rho}(t)}{M^{\rho}(t)} \right)^{1/2} \frac{M^{\nabla}(t)}{M(t)} \right| &= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{(q-1)q^{k-1}\ln(q^{k-1})\, q^{2k}(\ln q^k)^2} \\
&= \frac{q^{k-1}(\ln q)^2(q^2k^2-(k-1)^2)}{(q-1)q^{k-1}q^{2k}(k-1)k^2\ln q(\ln q)^2} \\
&= \frac{q^2k^2-(k-1)^2}{(q-1)k^2(k-1)q^{2k}\ln q} \\
&\leq \frac{q^2k^2}{(q-1)k^2(k-1)q^{2k}\ln q} \\
&\leq \frac{q^2}{(q-1)(k-1)q^{2k}\ln q} \\
&\leq \frac{1}{(q-1)(k-1)q^{2k-2}\ln q} \\
&\leq \frac{1}{(q-1)(N-1)q^{2N-2}\ln q} := k_2
\end{aligned}
$$

Thus, the assumptions of Theorem 4.2 hold, so we get that

$$x^{\Delta\nabla} + (t\ln t)^2 x = 0$$

is ∇-limit-point.

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