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PERSONAGE IN SCIENCE

Professor V.M. Starzhinskii

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On March 10, 2008, the renowned Russian scientist in the area of mathematics and mechanics, Viacheslav Michailovich Starzhinskii, would have turned 90 years old. To commemorate Professor Starzhinskii's valuable contribution to nonlinear dynamics, the Editorial Board of the Journal presents a biographical sketch to his life and academic activities. A short review of his scientific achievements has also appeared in monograph "Advances in Stability Theory at the End of the 20th Century" (co-authored by a large team of contributors), copyrighted by Taylor and Francis, London, 2003.

1 V.M. Starzhinskii's Life

V.M. Starzhinskii was born in a family of school teachers on March 10 (February 25), 1918, in the village of Lemeshevichi of the Pinsky district belonging to the Pinsky region (now the Brest region in Belorussia).

His father, Michail Fedorovich Starzhinskii, born in 1893, was employed as a teacher until 1942. His mother, Anna Aleksandrovna Dyukova, born in 1893, had been a teacher in the village of Lemeshevich since 1928. During the Second World War (from June 22, 1941 to July 28, 1944) she stayed in the occupied territory in the town of Vysokoye in the family of her brother, the future famous astrophysicist, Ivan Aleksandrovich Dyukov. After Vysokoye was liberated, Anna Aleksandrovna worked there at the District Department of People's Education. In October of 1944 she moved to the small town of Veshnyaki in a vicinity of Moscow, where she was employed as a teacher at school number 6 almost throughout the end of her life.

Anna Aleksandrovna's family broke in 1922 and Viacheslav Michailovich stayed with his mother. In the Fall of 1935 he finished a secondary school in Veshnyaki and then was

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admitted to the Department of Mechanics and Mathematics of Moscow State University to study mechanics. During his academic years, in his search for an additional income, Viacheslav Mikhaylovich applied to the All-Union Correspondent Institute of Textil and Light Industry (A-UCITLI) and he was appointed as an adjunct faculty and he was involved in teaching Calculus courses. In 1950 he became a full-time faculty.

At that time, he got interested in automatic control systems. This influenced the topic of his upcoming PhD thesis "Some problems in the theory of tracking systems", which he successfully defended in 1948 at the Scientific Research Institute of Mechanics in Moscow State University. His graduation from Moscow State University (with distinguished grades) coincided with the beginning of the Second World War. As the result, from 1941 to 1944, he worked as a constructor engineer at the military plants in the Stupino town of the Moscow region and in the town of Verkhnyaya Salda of the Sverdlov region. From 17.08.44 to 09.09.45 he taught at the Verkhne-Salda Avia-Metallurgical Technical School of Narkomaviaprom.

In October, 1945 Viacheslav Michailovich was accepted to a full-time post-graduate school at the Scientific Research Institute of Mechanics of Moscow State University. Upon a successful graduation, he defended the above mentioned thesis in June, 1948. In the same year (from 02.02.1948) he was appointed as a senior researcher in one of the scientific research institutes of the Ministry of Industry of Communications.

From the 1st of September, 1950 he became an associate Professor of Mathematics in Calculus Program at A-UCITLI.

On August 1, 1957, Viacheslav Michailovich became an associate Professor of Theoretical Mechanics Program at A-UCITLI. After defending (in March of 1958) his habilitation thesis he became a Professor and then the Chair of the Program of Theoretical Mechanics.

Viacheslav Michailovich married Tatyana Nikolayevna Litvinenko (born on 1925) in 1949, who was a student in the Schepkin school of arts and theater. They happily lived until Viacheslav Michailovich's death (on December 5, 1993 at the age of 75).

During all this time Tatyana Nikolayevna was his best friend and a guardian angel. They raised two children: son Pavel (born in 1950 and died tragically during winter fishing at the age of 40) and daughter Vera (born in 1959). After finishing her education, Tatyana Nikolayevna Starzhinskaya (maiden name Litvinenko) was an actress at the Moscow Regional Theater of Drama.

Forty three years of work at A-UCITLI proved to be most fruitful in the life of Viacheslav Michailovich. In 1952 he published his first paper [2]. In the period of 1952-1957, the same journal published seven more papers of his on the problems of stability of periodic motions. During this time, Viacheslav Mikhailovich took a doctoral course for his habilitation degree at the Institute of Problems of Mechanics Of Academy of Sciences (his supervisor was the Corresponding Member of Ac. of Sci. of USSR, Professor N.G. Chetayev), and in 1957 he defended his habilitation thesis.

2 Main Directions of His Research

V.M. Starzhinskii published more than 150 articles and books (including 27 monographs and textbooks). His work covers the following areas:

1. The second Lyapunov method: first, second, third and fourth order equations;

2. Stability of periodic motions: estimations of characteristic constants in the second and n-th order systems; the theory of parametric resonance Maté and Hill equations;

3. Oscillations of substantially nonlinear systems, combination of the Lyapunov and Poincaré methods, oscillating chains, energy jump, damped oscillating systems, computation of normal modes; normal modes for third, fourth and sixth order systems;

4. Application of parametric resonance theory to acoustic and electromagnetic waveguides;

5. Dynamics of a solid body: dimensionless form of the Euler-Poisson equations, oscillations of a heavy body with a fixed point, exclusive cases of Kovalevskaya gyroscope motion, QP-procedure for Kovalevskaya's case.

6. Applied problems: calculation of thread tension, elastic shaft, dynamical stability of rods, problem of three bodies, torsion oscillations of crank-shafts, pendulum on spring, thread mechanics, servosystems, cyclical accelerators.

3 Teaching Activity

Viacheslav Michailovich was a skillful lecturer. He conveyed a very complex material to his students in a clear fashion, without a compromise to the depth. His teaching experience of many years was also led to the publication of the following textbooks:

1. Hertsverg, E.Ya., Starzhinskii, V.M. Statics. Moscow: A-UCITLI, 1964, 236 p.

2. Starzhinskii, V.M. Kinematics, Moscow: A-UCITLI, 1964, 115 p.

3. Starzhinskii, V.M. Dynamics, Moscow: A-UCITLI, 1962, 166 p.

4. Starzhinskii, V.M. Dynamics, Moscow: A-UCITLI, 1965, 230 p.

5. Starzhinskii, V.M. Mechanics (Section "Mechanics of solid body"), Moscow: A-UCITLI, 1968, 270 p.

6. Starzhinskii, V.M. Theoretical Mechanics, Moscow: Nauka, 1980. 464 p.

7. Starjinski V.M. Mecanique rationell. Moscow: Mir, 1984. 469 p. (in French)

8. Starjinski V.M. Mecanica teorica. Moscow: Mir, 1984. 544 p. (in Portuguese)

9. Starjinski V.M. Mecanica teorica. Moscow: Mir, 1985. 519 p. (in Spanish)

10. Starzhinskii V.M. An Advanced Course of Theoretical Mechanics for Engineering Students. Moscow: Mir, 1982. 472 p. (in English)

11. Starzhinskii, V.M. Theoretical Mechanics, Moscow: Mir, 1986. 528 p. (in Russian)

Between 1980 and 1988 Professor Starzhinskii gave a series of lectures on nonlinear oscillations and parametric resonance for post-graduate students of the Mechanical and Mathematical Department of Moscow State University. His lectures have always been a success and as they attracted many listeners who were inspired by his lectures. He worked actively with post-graduates and supervised four doctoral and five habilitation theses.

4 Scholarly Activity

Professor Starzhinskii was among actives contributors to Mathematical Encyclopedia. He also contributed two volumes:

1. Nonlinear Oscillations (Vol. III, 1982. – P. 956–958);
2. Parametric Resonance (Vol. IV, 1984. – P. 216–218).

He compiled a bibliography of Liapunov's lectures and contributed to the publication of "New Books Abroad"(see Moscow: Mir, 1979, issue 11; 1980, issue 5; 1982, issues 5, 6; 1984, issue 2). He was a member of Scientific-Methodical Council of Theoretical Mechanics of Minvuz, USSR, and a member of Mir Publisher's Editorial Board.

V.M. Starzhinskii was rewarded with three medals of honor. In 1985 he received the reward "For Successes in the Field of Higher Education".

5 List of Monographs and Books by V.M. Starzhinskii

[1] Linear Differential Equations with Periodic Coefficients. Y. Wiley, 1975, vol. 1, 386 p. (with V.A. Yakubovich)

[2] Linear Differential Equations with Periodic Coefficients. Y. Wiley, 1975, vol. 2, pp. 387–839. (with V.A. Yakubovich)

[3] Applied Methos of Nonlinear Oscillations, Moscow: Nauka, 1977, 255 p. (in Russian)

[4] Applied Methods in the Theory of Nonlinear Oscillations. Moscow: Mir, 1984, 264 p.

[5] Méthodes Appliquées en Théorie des Oscillations non Linéaires. Ђoscow: Mir, 1985, 288 p.

[6] Parametric Resonance in Linear Systems. Moscow: Nauka, 1987, 328 p. (with V.A. Yakubovich)

[7] To the Theory of Nonlinear Oscillations, Moscow: Moscow State University, 1970, Part I, 108 p.

[8] To the Theory of Nonlinear Oscillations, Moscow: Moscow State Unoversity, 1972, Part II, 60 p.

[9] To the Theory of Nonlinear Oscillations, Moscow: Moscow State University, 1974, Part III, 99 p.

[10] To the Theory of Nonlinear Oscillations, Moscow: Moscow State University, 1975, Part IV, 60 p.

[11] Linear Differential Equations with Periodic Coefficients and their Application, Moscow: Nauka, 1972, 912 p. (with V.A. Yakubovich)

[12] On Stability of Periodic Motions. Bul. Inst. Politehn., Din. Iasi, 1969, Serie nova 4–8, Part I, no. 3–4, pp. 9–68, Part II, 5 (9), no. 1–2, pp. 51–100.

[13] On the Stability of Periodic Motions. Amer. Math. Soc. Trans., 1963, Ser. 2, Part I, 33, pp. 59–121, Part II, i.d. pp. 123–187.

6 Selected Articles

[1] (1948). The effect of clearance and friction on the motion of follow-up electric drive. ONTI NII PSSM.

[2] (1952). Sufficient stability conditions for a mechanical system with one degree of freedom. *Prikl. Mat. Mekh.*, **16**(3), 369–374 (Russian).

[3] (1953). On stability of a mechanical system with one degree of freedom. *Prikl. Mat. Mekh.*, **17**(1), 117–122 (Russian).

[4] (1954). A review of work on the conditions of stability of the trivial solution of a system of linear differential equations with periodic coefficients. *Prikl. Mat. Mekh.*, **18**(4), 469–510.

[5] (1954). On stability of trivial solution of second order differential equation with periodic coefficients. *Inzhen. Sbornik*, **18**, 119–138 (Russian).

[6] (1955). A remark to the stability investigation of periodic motions. *Prikl. Mat. Mekh.*, **19**(1), 119–120 (Russian).

[7] (1957). On stability of periodic motions in a special case. *Prikl. Mat. Mekh.*, **21**(5), 720–722 (Russian).

[8] (1958). On stability of trivial solution of linear systems with periodic coefficients. *Prikl. Mat. Mekh.*, **22**(5), 646–656 (Russian).

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[19] (1964). On the stability of periodic regimes. *Nonlin. Vibrations Problems*, PWN, Warszawa, **5**, 360–369.

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Existence and Uniqueness of a Solution to a Semilinear Partial Delay Differential Equation with an Integral Condition

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Abstract: In this work we consider a semilinear delay partial differential equation with an integral condition. We apply the method of semi-discretization in time, also known as the method of lines, to establish the existence and uniqueness of solutions. We also study the the continuation of the solution to the maximal interval of existence. Finally we give examples to demonstrate the applications of our results.

Keywords: *Diffusion equation; retarded argument; weak solution; method of lines.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this paper we are concerned with the following semilinear partial delay differential equation with an integral condition,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t,u(x,t),u(x,t-\tau)), \quad (x,t) \in (0,1) \times [0,T], \tag{1}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,1), \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{3}$$

$$\int_0^1 u(x,t)dx = 0, \quad t \in [0,T], \tag{4}$$

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where $\tau > 0$, $0 < T < \infty$, the map f is defined from $(0,1) \times [0,T] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ and the history function Φ is defined from $(0,1) \times [-\tau, 0]$ into $\mathbb{R}$. Our aim is to apply the method of semi-discretization in time, also known as the method of lines or Rothe's method, to establish the existence, uniqueness of a solution and the unique continuation of a solution to the maximal interval of existence. We note that there is no loss of generality in considering the homogeneous conditions in (3) and (4) as the more general problem (1)–(4) with u, f and Φ replaced by v, g, Ψ and conditions (3) and (4) replaced by

$$\frac{\partial v(0,t)}{\partial x} = U_0(t), \quad t \in [0,T], \tag{5}$$

$$\int_0^1 v(x,t)dx = U_1(t), \quad t \in [0,T], \tag{6}$$

respectively, may be reduced to (1)–(4) using the transformations

$$u(x,t) = v(x,t) - U_0(t)\left(x - \frac{1}{2}\right) - U_1(t)$$

and

$$\begin{aligned} f(x,t,r,s) = g\Bigg(x,\, t,\, r + U_0(t)\left(x - \frac{1}{2}\right) - U_1(t),\, s + U_0(t-\tau)\left(x - \frac{1}{2}\right) - U_1(t-\tau)\Bigg) \\ - \left(x - \frac{1}{2}\right)\frac{dU_0(t)}{dt} - \frac{dU_1(t)}{dt}, \\ \Phi(x,t) = \Psi(x,t) - U_0(t)\left(x - \frac{1}{2}\right) - U_1(t), \end{aligned}$$

with $U_0(t-\tau) = U_0(0)$ and $U_1(t-\tau) = U_1(0)$ for $t \leq \tau$.

The initial work on heat equations with integral conditions has been carried out by Cannon [7]. Subsequently, similar studies have been done by Kamynin [11], Ionkin [8]. Beilin [5] has considered the wave equation with an integral condition using the method of separation of variables and Fourier series.

Pulkina [14] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of generalized solutions using the fixed point arguments.

Our analysis is motivated by the works of Bouziani and Merazga [12, 6] and Bahuguna and Shukla [4]. In [12, 6] the authors have used the method of semi-discretization to (1)-(4) without delays. In [4] the method of semigroups of bounded linear operators in a Banach space is used to study a partial differential equation involving delays arising in the population dynamics. We use the method of semi-discretizaion in time first to establish the local existence of a unique solution of (1)–(4) on a subinterval $[-\tau, T_0]$, $0 < T_0 \leq T$ and then extend it either to the whole interval $[-\tau, T]$ or to the maximal subinterval $[-\tau, T_{\max}) \subset [-\tau, T]$ of existence with $\lim_{t \to T_{\max}-} \|u(t)\| = +\infty$.

2 Preliminaries

The problem (1)–(4) may be treated as an abstract equation in the real Hilbert space $\mathbf{H} = L^2(0,1)$ of square-integrable functions defined from $(0,1)$ into $\mathbb{R}$ with the inner product

$$(u,v)=\int_0^1 u(x)v(x)\,dx, \quad u,v\in \mathbf{H},$$

and the corresponding norm

$$\|u\|^2=\int_0^1 |u(x)|^2dx.$$

For $k\in\mathbb{N}$, the Sobolev space $\mathbf{H}^k$ is the Hilbert space of all functions $u\in\mathbf{H}$ such that the distributional derivative $u^{(j)}\in\mathbf{H}$ with the inner product

$$(u,v)_k=\sum_{j=0}^{k}(u^{(j)},v^{(j)}), \quad u,v\in\mathbf{H}^k,$$

and the corresponding norm

$$\|u\|_k^2=\sum_{j=0}^{k}\|u^{(j)}\|^2.$$

We shall incorporate the integral condition (4) with the space itself under consideration by taking $\mathbf{V}\subset\mathbf{H}$ defined by

$$\mathbf{V}=\left\{u\in\mathbf{H}:\int_0^1 u(x)\,dx=0\right\}.$$

$\mathbf{V}$ is a closed subspace of $\mathbf{H}$ and hence is a Hilbert space itself with the inner product $(\cdot,\cdot)$, and the corresponding norm $\|\cdot\|$.

For any Banach space X, with the norm $\|\cdot\|_X$ and an interval $I=[a,b],\ -\infty<a<b<\infty$, we shall denote by $C(I;X)$ the space of all continuous functions u from $[a,b]$ into X with the norm

$$\|u\|_{C(I;X)}=\max_{a\le t\le b}\|u(t)\|_X.$$

The space $L^2(I;X)$ consists of all square-Bochner integrable functions (equivalent classes) u for which the norm

$$\|u\|^2_{L^2(I;X)}=\int_a^b\|u(t)\|_X^2\,dt.$$

Similarly $L^\infty(I;X)$ is the Banach space of all essentially bounded functions from I into X with the norm

$$\|u\|_{L^\infty(I;X)}=\operatorname*{ess\,sup}_{t\in I}\|u(t)\|_X,$$

and the Banach space $C^{0,1}(I;X)$ is the space of all Lipschitz continuous functions from I into X with the norm

$$\|u\|_{C^{0,1}(I;X)}=\|u\|_{C(I;X)}+\sup_{t,s\in I;\ t\neq s}\frac{\|u(t)-u(s)\|}{|t-s|}\,.$$

In addition to the spaces mentioned above, we need the space $B_2^1(0,1)$ introduced by Merazga and A. Bouziani [12] being the completion of the space $C_0(0,1)$ of all real continuous functions having compact supports in (0,1) with the inner product

$$(u,v)_{B_2^1}=\int_0^1\Im_x u.\Im_x v\,dx,$$

where $\Im_x v = \int_0^x v(\xi)\,d\xi$ for every fixed $x \in (0,1)$ and the corresponding norm

$$\|u\|^2_{B_2^1} = \int_0^1 (\Im_x u)^2 dx.$$

It follows that the following inequality

$$\|v\|^2_{B_2^1} \le \frac{1}{2}\|v\|^2$$

holds for every $v \in L^2(0,1)$, and the embedding $L^2(0,1) \to B_2^1(0,1)$ is continuous.

Given a function $h:\ (0,1)\times[a,b] \to \mathbb{R}$ such that for each $t \in [a,b]$, $h(\cdot,t):\ [a,b] \to \mathbf{H}$, we may identify it with the function $h:[a,b] \to \mathbf{H}$ given by $h(t)(x) = h(x,t)$. For a given Lipschitz continuous function $g:\ (0,1)\times[a,b]\times\mathbb{R} \to \mathbb{R}$ and h as above, we may identify it with a function $g:\ [a,b]\times\mathbf{H} \to \mathbf{H}$ by $g(t,h(t))(x) = g(x,t,h(x,t))$.

We assume the following conditions.

(A1) $f(t,u,v) \in \mathbf{H}$ for $(t,u,v) \in [0,T]\times\mathbf{H}\times\mathbf{H}$ and the Lipschitz condition

$$\|f(t_1,u_1,v_1) - f(t_2,u_2,v_2)\|_{B_2^1} \le l_f\left[|t_1-t_2| + \|u_1-u_2\|_{B_2^1} + \|v_1-v_2\|_{B_2^1}\right]$$

for all $t_i \in [0,T]$, $u_i, v_i \in \mathbf{V}$, $i = 1,2$, holds for some positive constant l_f.

(A2) For each $x \in (0,1)$, $\Phi(x,\cdot):\ [-\tau,0] \to \mathbf{H}^2 \cap C^{0,1}([-\tau,0];\mathbf{H})$ with the uniform Lipchitz constant l_Φ.

(A3) $\dfrac{d\Phi(0,x)}{dx} = 0$ and $\int_0^1 \Phi(0,x)\,dx = 0$.

Definition 2.1 By a weak solution of (1)–(4) we mean a function $u\colon [-\tau,T] \to \mathbf{H}$

(i) $u = \Phi$ on $[-\tau,0]$;

(ii) $u \in L^\infty([0,T];\mathbf{V}) \cap C^{0,1}([0,T];B_2^1(0,1))$;

(iii) u has ($a.e$ in $[0,T]$) a strong derivative $\dfrac{du}{dt} \in L^\infty([0,T];B_2^1(0,1))$;

(iv) for all $\phi \in \mathbf{V}$ and a.e. $t \in [0,T]$, the identity

$$\left(\frac{du(t)}{dt},\phi\right)_{B_2^1} + (u(t),\phi) = (f(t,u(t),u(t-\tau)),\ \phi)_{B_2^1}, \tag{7}$$

is satisfied.

Theorem 2.1 *Suppose that the conditions (A1)–(A3) are satisfied. Then problem (1)–(4) has a unique weak solution on* $[-\tau,T_0]$, *for some* $0 < T_0 \le T$. *Furthermore,* u *can be continued uniquely either on the whole interval* $[-\tau,T]$ *or there exists a maximal interval* $[0,T_{\max})$, $0 < T_{\max} < T$, *of existence with* $\lim_{t\to T_{max}-} \|u(t)\| = +\infty$.

3 Discretization Scheme and A Priori Estimates

In this section we establish existence and uniqueness of a weak solution to (1)–(4). For the the application of the method of line we proceed as follows. We choose $T_0,\ 0 < T_0 = \min\{\tau, T\}$, for $n \in \mathbb{N}$. Let $h_n = \frac{T_0}{n}$. We set $u_0^n = \Phi(0)$ for all $n \in \mathbb{N}$ and define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the time-discretized problems

$$\frac{u_j^n - u_{j-1}^n}{h_n} - \frac{d^2 u_j^n}{dx^2} = f_j^n, \quad x \in (0,1), \tag{8}$$

$$\frac{du_j^n}{dx}(0) = 0, \tag{9}$$

$$\int_0^1 u_j^n(x)\,dx = 0, \tag{10}$$

where $f_j^n = f(t_j^n, u_{j-1}^n, \Phi(t_{j-1}^n - \tau))$. The existence of unique $u_j^n \in \mathbf{H^2}$ satisfying (8),(9) is ensured as established in [13] Lemma 3.1. We first prove the estimates for u_j^n and difference quotients $\{(u_j^n - u_{j-1}^n)/h_n\}$ using (A1)–(A3). We introduce sequences $\{U^n\}$ of polygonal functions from $U^n\colon\ [-\tau, T_0] \to H^2(0,1) \cap V$ defined by

$$U^n(t) = \begin{cases} \Phi(t), & t \in [-\tau, 0], \\ u_{j-1}^n + \dfrac{t - t_{j-1}^n}{h_n}(u_j^n - u_{j-1}^n), & t \in [t_{j-1}^n, t_j^n], \end{cases} \tag{11}$$

and prove the convergence of $\{U^n\}$ to a unique solution u of (1)–(4) in $C([-\tau, T_0], B_2^1(0,1))$ as $n \to \infty$. For the notational convenience, we some time suppress the superscript n, throughout, C will represent a generic constant independent of j, h_n and n.

Lemma 3.1 *Assume that the hypotheses (A1)–(A3) are satisfied. Then there exists a positive constant C, independent of j, h and n such that.*

$$\|u_j\| \le C, \tag{12}$$

$$\|\delta u_j\|_{B_2^1} \le C, \tag{13}$$

$n \ge 1$ and $j = 1, \dots n$.

Proof Taking the inner product in $B_2^1(0,1)$ of (8) with any $\phi \in \mathbf{V}$,

$$(\delta u_j, \phi)_{B_2^1} - \left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = (f_j, \phi)_{B_2^1}. \tag{14}$$

Using (9) and integration by parts

$$\left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = \int_0^1 \frac{du_j(x)}{dx} \Im_x \phi\,dx = u_j(x)\Im_x\phi|_{x=0}^{x=1} - \int_0^1 u_j \phi\,dx.$$

Since

$$\left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = -(u_j, \phi),$$

(14) becomes

$$(\delta u_j, \phi)_{B_2^1} + (u_j, \phi) = (f_j, \phi)_{B_2^1}. \tag{15}$$

Taking $j = 1$ in (15), and $\phi = u_1$ we have

$$\begin{aligned}\frac{1}{h_n}(u_1, u_1)_{B_2^1} + (u_1, u_1) &= \left(f_1 + \frac{1}{h_n}\Phi(0), u_1\right)_{B_2^1}, \\ \|u_1\|_{B_2^1} \leq h_n \max_{t\in[0,T_0]} \|f(t_1, \Phi(0), \Phi(-\tau))\|_{B_2^1} &+ \|\Phi(0)\|_{B_2^1} = C.\end{aligned} \tag{16}$$

Again for $j = 1$ in (15) and $(\Phi(0), \phi) = -\left(\frac{d^2\Phi(0)}{dx^2}, \phi\right)_{B_2^1}$, we get

$$(\delta u_1, \phi)_{B_2^1} + h_n(\delta u_1, \phi) = \left(f_1 + \frac{d^2\Phi(0)}{dx^2}, \phi\right)_{B_2^1}$$

Testing this equality with $\phi = \delta u_1 = \frac{u_1 - \Phi(0)}{h_n} \in V$,

$$\|\delta u_1\|^2_{B_2^1} + h_n\|\delta u_1\|^2 \leq \left[\|f_1\|_{B_2^1} + \left\|\frac{d^2\Phi(0)}{dx^2}\right\|_{B_2^1}\right]\|\delta u_1\|_{B_2^1},$$

consequently we get

$$\|\delta u_1\|_{B_2^1} \leq \max_{t\in[0,T_0]} \|f(t_1, \Phi(0), \Phi(-\tau))\|_{B_2^1} + \left\|\frac{d^2\Phi(0)}{dx^2}\right\|_{B_2^1} = C. \tag{17}$$

Let $2 \leq j \leq n$. Subtracting (15) for $j-1$ from (15) for j and putting $\phi = \delta u_j$, we get

$$(\delta u_j - \delta u_{j-1},\, \delta u_j)_{B_2^1} + (u_j - u_{j-1},\, \delta u_j) = (f_j - f_{j-1},\, \delta u_j)_{B_2^1},$$

or

$$\|\delta u_j\|^2_{B_2^1} + \frac{1}{h_n}\|u_j - u_{j-1}\|^2 \leq (\|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1})\|\delta u_j\|_{B_2^1},$$

which finally gives

$$\|\delta u_j\|_{B_2^1} \leq \|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1}.$$

By assumption $(A1)$ we have for $j \geq 2$,

$$\begin{aligned}\|f_j - f_{j-1}\|_{B_2^1} &= \|f(t_j, u_{j-1}, \Phi(t_{j-1} - \tau)) - f(t_{j-1}, u_{j-2}, \Phi(t_{j-2} - \tau))\|_{B_2^1} \\ &\leq l_f[|t_j - t_{j-1}| + h_n\|\delta u_{j-1}\|_{B_2^1} + l_\Phi|t_{j-1} - t_{j-2}|] \\ &\leq Ch_n[1 + \|\delta u_{j-1}\|_{B_2^1}].\end{aligned}$$

Hence above equation becomes

$$\begin{aligned}\|\delta u_j\|_{B_2^1} &\leq (1 + Ch_n)\|\delta u_{j-1}\|_{B_2^1} + Ch_n \\ &\leq (1 + Ch_n)^2\|\delta u_{j-2}\|_{B_2^1} + Ch_n[1 + (1 + Ch_n)].\end{aligned}$$

By iterative process we obtain

$$\|\delta u_j\|_{B_2^1} \leq (1 + Ch_n)^{j-1}\left[\|\delta u_1\|_{B_2^1} + Ch_n(j-1)\right] \tag{18}$$

Replacing $(1+Ch_n)^{j-1} \leq e^{CT}$ and $Ch_n(j-1) \leq CT$ we get the second required estimate. Now for the first estimate, we take $\phi = u_j$ in (15), $j = 1, 2, \dots, n$, to get

$$\frac{1}{h_n}\|u_j\|^2_{B^1_2} + \|u_j\|^2 \leq \left(\|f_j\|_{B^1_2} + \frac{1}{h_n}\|u_{j-1}\|_{B^1_2}\right)\|u_j\|_{B^1_2},$$

which implies

$$\|u_j\|_{B^1_2} \leq h_n\|f_j\|_{B^1_2} + \|u_{j-1}\|_{B^1_2}. \tag{19}$$

By assumption $(A1)$, we have for all $j \geq 1$,

$$\begin{aligned}\|f_j\|_{B^1_2} &\leq \|f(t_j, u_{j-1}, \Phi(t_{j-1} - \tau)) - f(t_j, 0, 0)\|_{B^1_2} + \|f(t_j, 0, 0)\|_{B^1_2} \\ &\leq C(1 + \|u_{j-1}\|_{B^1_2}).\end{aligned}$$

Putting it into (19) we have

$$\|u_j\|_{B^1_2} \leq (1 + Ch_n)\|u_{j-1}\|_{B^1_2} + Ch_n.$$

Repeating the last inequality we estimate

$$\|u_j\|_{B^1_2} \leq (1 + Ch_n)^{j-1}[\|u_1\|_{B^1_2} + Ch_n(j-1)]. \tag{20}$$

Again replacing $(1 + Ch_n)^{j-1} \leq e^{CT}$ and $Ch_n(j-1) \leq CT$. We get

$$\|u_j\|_{B^1_2} \leq C. \tag{21}$$

Now taking $\phi = u_j - u_{j-1}$ in (15) and using the identity,

$$(u_j,\, u_j - u_{j-1}) = \frac{1}{2}\big(\|u_j - u_{j-1}\|^2 + \|u_j\|^2 - \|u_{j-1}\|^2\big),$$

we get

$$h_n\|\delta u_j\|^2_{B^1_2} + \frac{1}{2}\|u_j - u_{j-1}\|^2 + \frac{1}{2}\|u_j\|^2 = (f_j,\, u_j - u_{j-1})_{B^1_2} + \frac{1}{2}\|u_{j-1}\|^2.$$

Ignoring the first two terms in the left hand side, we have

$$\begin{aligned}\|u_j\|^2 &\leq 2h_n\|f_j\|_{B^1_2}\|\delta u_j\|_{B^1_2} + \|u_{j-1}\|^2 \\ &\leq Ch_n(1 + \|u_{j-1}\|_{B^1_2})\|u_{j-1}\|_{B^1_2} + \|u_{j-1}\|^2 \\ &\leq Ch_n + \|u_{j-1}\|^2.\end{aligned}$$

Repeating the above inequality we get the required estimate. This completes the proof of the lemma. □

Definition 3.1 We define Rothe's sequence $\{U^n\}$ by (11). Furthermore, we define another sequences $\{X^n\}$ of step functions from $[-h_n, T_0]$ into $\mathbf{H}^2 \cap \mathbf{V}$ given by

$$X^n(t) = \begin{cases} \Phi(0), & t \in [-h_n, 0], \\ u_j, & t \in (t_{j-1}, t_j]. \end{cases}$$

Remark 3.1 From Lemma 3.1 it follows that the function U^n is Lipschitz continuous on $[0, T_0]$. The sequences $\{U^n\}$ and $\{X^n\}$ are bounded in $C([0, T_0]; B_2^1(0,1))$ uniformly in $n \in \mathbb{N}$ and $t \in [0, T_0]$

$$\|U^n\| \leq C, \quad \|X^n\| \leq C, \quad \left\|\frac{dU^n(t)}{dt}\right\|_{B_2^1} \leq C, \quad \|U^n(t) - U^n(s)\| \leq C|t-s|,$$
$$\|X^n(t) - U^n(t)\|_{B_2^1} \leq \frac{C}{n}, \quad \text{and} \quad \|X^n(t) - X^n(t-h_n)\|_{B_2^1} \leq \frac{C}{n}.$$

For notational convenience, let

$$f^n(t) = f(t_j, X^n(t-h_n), \Phi(t_{j-1} - \tau)), \quad t \in (t_{j-1}, t_j], \quad 1 \leq j \leq n.$$

Then (15) may be rewritten as

$$\left(\frac{dU^n(t)}{dt}, \phi\right)_{B_2^1} + (X^n(t), \phi) = (f^n(t), \phi)_{B_2^1}, \tag{22}$$

for all $\phi \in \mathbf{V}$ and a.e. $t \in (0, T_0]$.

Lemma 3.2 *There exists $u \in C([0, T_0]; B_2^1(0,1))$ such that $U^n(t) \to u(t)$ uniformly on I. Moreover $u(t)$ is Lipschitz continuous on $[0, T_0]$.*

Proof From (22) for a.e. $t \in (0, T_0]$, we have

$$\begin{aligned}\left(\frac{d}{dt}(U^n(t) - U^k(t)), U^n(t) - U^k(t)\right)_{B_2^1} &+ (X^n(t) - X^k(t), U^n(t) - U^k(t)) \\ &= (f^n(t) - f^k(t), U^n(t) - U^k(t))_{B_2^1}.\end{aligned}$$

From the above equality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U^n(t) - U^k(t)\|_{B_2^1}^2 &+ \|X^n(t) - X^k(t)\|^2 \\ &= (X^n(t) - X^k(t), X^n(t) - X^k(t) - U^n(t) + U^k(t)) \\ &+ (f^n(t) - f^k(t), U^n(t) - U^k(t))_{B_2^1}.\end{aligned} \tag{23}$$

From (22), $\|f^n(t)\|_{B_2^1} \leq C$, and thus the identity

$$(X^n(t), \phi) = \left(f^n(t, X^n(t-h_n), \Phi(t_j - \tau)) - \frac{dU^n}{dt}, \phi\right)_{B_2^1}$$

gives

$$|(X^n(t), \phi)| \leq \left[\|f^n\|_{B_2^1} + \left\|\frac{dU^n}{dt}\right\|_{B_2^1}\right]\|\phi\|_{B_2^1} \leq C\|\phi\|_{B_2^1}. \tag{24}$$

Now using (24), we have the estimate

$$\begin{aligned}(X^n(t) - X^k(t),\ X^n(t) &- X^k(t) - U^n(t) + U^k(t)) \\ &\leq 2C\Big(\|X^n(t) - U^n(t)\|_{B_2^1} + \|X^k(t) - U^k(t)\|_{B_2^1}\Big) \\ &\leq 4C\left(\frac{1}{n} + \frac{1}{k}\right).\end{aligned} \tag{25}$$

By inequality $\alpha\beta \le \dfrac{\alpha^2+\beta^2}{2}$, $\alpha,\beta \in \mathbb{R}$, we may write

$$\begin{aligned}(f^n(t)-f^k(t),\, U^n(t)-U^k(t))_{B_2^1} &\le \|f^n(t)-f^k(t)\|_{B_2^1}\|U^n(t)-U^k(t)\|_{B_2^1}\\ &\le \frac{1}{2}\Big[\|f^n(t)-f^k(t)\|^2_{B_2^1}+\|U^n(t)-U^k(t)\|^2_{B_2^1}\Big].\end{aligned} \tag{26}$$

Using assumption (A1), we have

$$\begin{aligned}\|f^n(t)-f^k(t)\|_{B_2^1} &= \|f(t_j, X^n(t-h_n), \Phi(t_{j-1}-\tau)) - f(t_l, X^k(t-h_k), \Phi(t_{l-1}-\tau))\|_{B_2^1}\\ &\le \delta_{nk}(t) + l_f\|X^n(t)-X^k(t)\|_{B_2^1},\end{aligned}$$

where

$$\begin{aligned}\delta_{nk}(t) = l_f[&|t_j-t_l| + \|X^n(t-h_n)-X^n\|_{B_2^1} + \|X^k(t-h_k)-X^k(t)\|_{B_2^1}\\ &+ \|\Phi(t_{j-1}-\tau)-\Phi(t_{l-1}-\tau)\|_{B_2^1}],\end{aligned}$$

for $t \in (t_{j-1}, t_j]$ and $t \in (t_{l-1}, t_l]$, $1 \le j \le n$, $1 \le l \le k$. Clearly $\delta_{nk}(t) \to 0$ uniformly on $[0,T_0]$ as $n,k \to \infty$. Also

$$\|f^n(t)-f^k(t)\|^2_{B_2^1} \le \delta^1_{nk}(t) + l_f^2\|X^n(t)-X^k(t)\|^2_{B_2^1}.$$

Hence (26) becomes

$$\begin{aligned}(f^n(t)-f^k(t), U^n(t)-U^k(t))_{B_2^1} &\le \frac{1}{2}\delta^1_{nk} + \frac{1}{2}l_f^2\|X^n(t)-X^k(t)\|^2_{B_2^1}\\ &+ \frac{1}{2}\|U^n(t)-U^k(t)\|^2_{B_2^1}, \quad \forall\, t\in(0,T_0].\end{aligned} \tag{27}$$

Now combining (25), (26) and (27) then (23) becomes

$$\begin{aligned}\frac{d}{dt}\|U^n(t)-U^k(t)\|^2_{B_2^1} + 2\|X^n(t)-X^k(t)\|^2 &\le 2C\left(\frac{1}{n}+\frac{1}{k}\right) + l_f^2\|X^n(t)-X^k(t)\|^2\\ &+ \delta^1_{nk} + \|U^n(t)-U^k(t)\|^2_{B_2^1}, \quad \forall\, t\in(0,T_0],\end{aligned}$$

or

$$\begin{aligned}\frac{d}{dt}\|U^n(t)-U^k(t)\|^2_{B_2^1} + (2-l_f^2)\|X^n(t)-X^k(t)\|^2 &\le 2C\left(\frac{1}{n}+\frac{1}{k}\right) + \delta^1_{nk}\\ &+ \|U^n(t)-U^k(t)\|^2_{B_2^1}, \quad \forall\, t\in(0,T_0],\end{aligned}$$

where δ^1_{nk} is a sequence of numbers converging to zero as $n,k \to \infty$. Integrating over $(0,s)$, $0 < s \le t \le T_0$, taking the supremum over $(0,t)$ and using the fact that $U^n = \Phi$ on $[-\tau,0]$ for all n we get

$$\|U^n-U^k\|^2_{B_2^1} \le 2CT\left(\frac{1}{n}+\frac{1}{k}\right) + CT\delta^1_{nk} + C\int_0^t \|U^n-U^k\|^2_{B_2^1}\,ds,$$

where C is a positive constant independent of j,h and n. Applying Gronwall's inequality, we conclude that there exists a function $u \in C([0,T_0]; B_2^1(0,1))$ such that $U^n \to u$ in this space and by Remark 3.1 it follows that u is Lipschitz continuous on $[-\tau,T_0]$. This completes the proof of the lemma. $\square$

In consequence of Remark 3.1 and Lemma 3.2 we have the following remark on the weak convergence (denoted by $\rightharpoonup$) U^n and its strong derivative to the function u and its strong derivative, respectively.

Remark 3.2 (i) $u \in L^\infty([0,T_0];\mathbf{V}) \cap C^{0,1}([0,T_0];B_2^1(0,1))$;

(ii) u is strongly differentiable a.e. in $[0,T_0]$ and $\dfrac{du}{dt} \in L^\infty([0,T_0];B_2^1(0,1))$;

(iii) $U^n(t)$ and $X^n(t) \rightharpoonup u(t)$ in V for all $t \in I$;

(iv) $\dfrac{dU^n(t)}{dt} \rightharpoonup \dfrac{du}{dt}$ in $L^\infty([0,T_0];B_2^1(0,1))$.

Proof of Theorem 2.1

First we prove the existence on $[-\tau,T_0]$. Integrating the (22) over $(0,t) \subset [0,T_0]$ and invoking the fact that $U^n(0) = \Phi(0)$, we have

$$(U^n(t) - \Phi(0), \phi)_{B_2^1} + \int_0^t (X^n(s),\phi)\,ds = \int_0^t (f^n,\phi)_{B_2^1}\,ds. \tag{28}$$

Since $U^n(t) \rightharpoonup u(t)$ in $\mathbf{V}$ for all $t \in [0,T_0]$ and $\forall\,\phi \in \mathbf{V}$ and the linear functional $v \to (v,\phi)_{B_2^1}$ is bounded on $\mathbf{V}$, we have

$$(U^n(t),\phi)_{B_2^1} \to (u(t),\phi)_{B_2^1}, \quad \forall\, t \in [0,T_0]. \tag{29}$$

Now by the Lipschitz continuity of f and Remark 3.1 we get

$$f^n(s, X^n(s-h_n), \Phi(s-\tau)) \to f(s,u(s),\Phi(s-\tau)) \quad \text{in} \quad B_2^1(0,1) \tag{30}$$

as $n \to \infty$. Now from (22) and (24) the functions $|(f^n,\phi)_{B_0^1}|$ and $|(X^n,\phi)|$ are uniformly bounded. Now by bounded convergence theorem and (22), we obtain, as $n \to \infty$,

$$(u(t) - \Phi(0),\, \phi)_{B_2^1} + \int_0^t (u(s),\phi)\,ds = \int_0^t (f(s,u(s),\Phi(s-\tau)),\phi)_{B_2^1}\,ds$$

for all $\phi \in \mathbf{V}$ and $t \in [0,T_0]$. Differentiating the identity we get the required relation. Now we prove the uniqueness. Let u_1 and u_2 be two such solutions of (1)–(4). Then we have

$$\left(\frac{dU(t)}{dt}, U(t)\right)_{B_2^1} + \|U(t)\|^2 = (f(t,u_1(t),\Phi(t-\tau)) - f(t,u_2(t),\Phi(t-\tau)), U(t))_{B_2^1},$$

where $U(t) = u_1(t) - u_2(t)$. Integrating over $(0,s)$ for $0, s \le t \le T_0$ and using the fact that $U(0) = 0$, we get

$$\begin{aligned}\|U(t)\|_{B_2^1}^2 + 2\int_0^t \|U(t)\|^2 ds &= 2\int_0^t (f(s,u_1(s),\Phi(s-\tau)) \\ &\quad - f(s,u_2(s),\Phi(s-\tau)), U(s))_{B_2^1}\,ds \le 2l_f \int_0^t \|U(s)\|_{B_2^1}^2\,ds.\end{aligned}$$

Application of Gronwall's inequality implies that $U \equiv 0$ on $[-\tau,T_0]$.

Now, we prove the unique continuation of the solution u to either on whole interval $[-\tau,T]$ or to the maximal interval $[-\tau,T_{\max})$ of existence where $0 < T_{\max} < T$ and if

$T_{\max} < T$ then $\lim_{t\to T_{\max}-} \|u(t)\| = \infty$. Suppose $T_0 < T$ and $\|u(T_0)\| < \infty$. Consider the problem

$$\begin{aligned} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w(t-\tau)), \quad x \in (0,1), \quad 0 < t \le T - T_0, \\ w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1), \quad t \in [-\tau - T_0, 0], \\ \frac{\partial w(0,t)}{\partial x} &= 0, \quad t \in [0, T-T_0], \\ \int_0^1 w(x,t)\,dx &= 0, \quad x \in (0,1), \quad t \in [0, T-T_0], \end{aligned} \tag{31}$$

where $\tilde{f}(x,t,w(t),w(t-\tau)) = f(x,t+T_0,w(t),w(t-\tau)),\ x \in (0,1),\ 0 < t \le T - T_0,$

$$\tilde{\Phi}(t) = \begin{cases} \Phi(t+T_0), & t \in [-\tau - T_0, -T_0], \\ u(t+T_0), & t \in [-T_0, 0]. \end{cases}$$

Since $\tilde{\|}\Phi(0)\| = \|u(T_0)\| < \infty$ and $\tilde{f}$ satisfies $(A1)$ on $[0, T-T_0]$, we may proceed as before and prove the existence of a unique $w(t) \in C([-\tau - T_0, T_1]; B_2^1(0,1)),\ 0 < T_1 \le T - T_0$, such that w is Lipschitz continuous on $[0, T_1]$ and w satisfies

$$\begin{aligned} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w(t-\tau)), \quad x \in (0,1), \quad 0 < t \le T_1, \\ w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1), \quad t \in [-\tau - T_0, 0], \\ \frac{\partial w(0,t)}{\partial x} &= 0, \quad t \in [0, T-T_0], \\ \int_0^1 w(x,t)\,dx &= 0, \quad t \in [0, T-T_0]. \end{aligned} \tag{32}$$

Then the function

$$\overline{u}(t) = \begin{cases} u(t), & t \in [-\tau, T_0], \\ w(t-T_0), & t \in [T_0, T_0+T_1], \end{cases}$$

is Lipschitz continuous on $[0, T_0+T_1]$, $\overline{u}(t) \in C([0, T_0+T_1], B_2^1(0,1))$ for $t \in [0, T_0+T_1]$ and satisfies (1) on $[0, T_0+T_1]$. Continuing this way we may prove the existence on the whole interval $[-\tau, T]$ or there is the maximal interval $[-\tau, T_{\max})$, $0 < T_{\max} \le T$, such that u is the weak solution of (1)–(4) on every subinterval $[-\tau, \tilde{T}], 0 < \tilde{T} < T_{\max}$. In the later case, if $\lim_{t\to t_{max}} \|u(t)\| < \infty$ then we may continue the solution beyond $T_{\max}$ but this will contradict the definition of maximal interval of existence. This completes the proof of Theorem 2.1.

4 Applications

In this section we consider problems arising in the population dynamics (cf. Engel and Nagel [9]).

Example 4.1 Consider the following partial differential equation with delay,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = -d(t)u(x,t) + b(t)u(x,t-\tau), \quad (x,t) \in (0,1) \times [0,T], \tag{33}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,1), \tag{34}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{35}$$

$$\int_0^1 u(x,t)\,dx = \psi(t), \quad t \in [0,T]. \tag{36}$$

Here $u(x,t)$ denotes the size of a population at time t and at the point $x \in [0,1]$. The term $\dfrac{\partial^2 u}{\partial x^2}$ represents the internal migration. The continuous functions b and d on $[0,T]$ represent the birth and death rates and τ is the delay due to pregnancy. The function $\psi(t)$ may be viewed as a control on the average population size at time t. Thus, we have no-flux condition at the left end and the right end is free so there may be a flux at this end but the average population size is being controlled by the integral condition. Here we take $f\colon [0,T] \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ given by $f(t,\chi,\psi) = -d(t)\chi + b(t)\psi$, $t \in [0,T]$, $\chi, \psi \in \mathbf{H}$. Our analysis of the earlier sections may be applied to this problem to ensure the well-posedness of the model.

Example 4.2 In this example we consider the following problem,

$$\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = r(t)u(x,t-\tau))(1-u(x,t)), \quad (x,t) \in (0,\pi) \times [0,T], \tag{37}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,\pi), \tag{38}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{39}$$

$$\int_0^\pi u(x,t)\,dx = \psi(t), \quad t \in [0,T]. \tag{40}$$

The equation (37) arises in the study of a population density with a time delay and self-regulation (cf. Turyn [16]). In this problem we take $T = \tau$ and assume that Φ is bounded on $(0,\pi) \times [-\tau, 0]$. Also, we take $f\colon [0,\tau] \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ given by $f(t,\chi,\psi) \equiv f(t,\chi)(x) = r(t)\Phi(x,t-\tau)(1-\chi(x))$, $t \in [0,\tau]$, $\chi \in \mathbf{H}$. Here again we have considered no-flux condition on the left end and the average population size is being controlled by the function ψ in place of the Dirichlet boundary condition on u as taken in [16]. The results of the earlier sections may be used to ensure the well-posedness of this model. We shall be dealing with the problem involving the Dirichlet condition together with an integral condition in our subsequent study.

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On a Class of Manifolds for Sliding Mode Control and Observation of Induction Motor

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Abstract: The aim of this paper is to develop a general class of manifolds on which sliding mode flux observation and control of induction motors are achieved. For flux-speed tracking, we consider the case where the sliding surface is formed by the derivative of the output tracking error and a function of this error. For flux observation, the surface is a function of the estimated error. At first, we will derive the properties that must be fulfilled by the above class of manifolds in order to attain the control and observation objective. Then, we design the control law and the observer gains to make the proposed manifolds globally attractive and invariant. Simulations results are given to highlight the performances of the proposed control method.

Keywords: *Induction motor; manifold; sliding mode control; sliding observer; robustness; global stability.*

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1 Introduction

Today, the developments of electrical machine drive grow more and more in order to follow the increasing need for various fields such as industry, electric cars, actuators, etc. By means of electrical machine drive, we can get high level of productivity in industry and product quality enhancement. The induction motor is the motor of choice in many industrial applications due to its reliability, ruggedness and relatively low cost. Nevertheless, controlling induction motors has been not easy due to significant nonlinear characteristics and the imprecise knowledge of its physical parameters.

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The control of induction motors has attracted much attention in the last decades. The vector control provides decoupling control of torque and flux similar to the control of separately excited DC motor. However, this decoupling is achieved only if the instantaneous rotor flux angle is precisely known [1]. The accuracy of the knowledge of the field rotor position affects greatly the control performances. Meanwhile, this accuracy is related to the method chosen for the position field determination.

The direct field oriented control method, where the position field is known by measuring the rotor flux of motor, is robust against parameter variation due to feedback flux [2]. The manufacturers avoid this method because it requires specially prepared machines in order to install flux sensor that rises the motor cost and decreases its reliability.

In the indirect orientation field control, the position field is deduced from speed rotor and the q component stator current [8]. The latter method needs the exact machine parameters. Hence it is very sensitive to parametric variations. Many works found in the literature over the last decade are devoted to the robust field orientation in order to overcome or to compensate the increasing resistances or saturation effects [9, 10, 11].

Another way to control induction machine is to apply the nonlinear control theory that covers many aspects such as nonlinear feedback linearization, passivity approach and sliding mode control.

The nonlinear feedback linearization allows to make the dynamic of induction machine fully or partially linearized. Its major drawback comes from the fact that it requires relatively complicated differential geometry and the precise value of parameters [4, 12, 13, 14].

The passivity theory is developed for AC machines in [15] and experimental results for induction machines are given in [16]. The main idea behind the passivity based controller design is to reshape the system natural energy and inject the required damping in such a way that the control objective is achieved.

The sliding mode theory is widely applied in the field of electrical machine drive. This success is due to the fact that the design methodology is easy. Moreover, the technical constraints limits are removed, since the theoretical conditions of the sliding mode theory are actually best accomplished in practice: the new electronic power devices allow a high limit of switching frequency, and the high performance DSP ensures a weak computational time.

Furthermore, the sliding mode control of the induction machine allows obtaining excellent properties of robustness against the parametric variation [6, 17, 18, 19, 20]. This advantage is, nevertheless, attained at the expense of large control effort that produces the well known chattering phenomenon.

Beside, the flux machine is not measured but it is estimated through an observer. The problem of estimating flux has been tackled from different point of view. The classical Luenberger observer for flux estimation was first developed in [21, 22]. The extended Kalman filter is used in [23] to estimate both the flux and the rotor resistance. To cope with parameter variations, adaptive versions of the above observers are developed in [24] and [25]. Motivated by the attractive robustness properties of the sliding mode, a variable structure flux observer is proposed in [6, 20, 3].

In this paper, we consider invariant manifold technique to control flux-speed and to estimate rotor flux of induction machine. To this end, we develop a wide class of surfaces and we search the properties that must be fulfilled in order to achieve our control objective. Then, we design the control law (or the observer gain) to make the developed surfaces globally attractive and invariant. Conditions that ensure internal stability as well as the stability of the coupling between the flux observer and the control law are given. Some simulation results involving a $3.7Kw$ induction machine are also proposed.

2 Problem Formulation

Our problem consists in developing a general class of manifolds, for sliding mode control to achieve flux-speed tracking and, for flux observation, in the case of induction motor (the reader is referred to [7] for a general theory on design and control of induction motors). To do so, we give firstly the induction machine model. In the stator reference frame, the state space model of voltage fed induction machine is obtained from Park's model. The state vector is composed of the stator current components (i_α, i_β), rotor flux components $(\varphi_\alpha, \varphi_\beta)$ and rotor rotating pulsation ω_r, whereas a vector control is composed of the stator voltage components $(v_\alpha, v_\beta\ )$ and the external disturbance is represented by the load torque Γ_r. In the sequel, the state vector and the control vector are given respectively by: $x = (i_\alpha, i_\beta, \varphi_\alpha, \varphi_\beta, \omega_r)^T$, and $u = (v_\alpha, v_\beta)^T$. Using these notations, the state space model of voltage fed induction machine takes the form:

$$\begin{aligned}
\dot{x}_1 &= f_1(x) + d_1 u_1, & f_1(x) &= -a_1 x_1 + b_1 x_3 + c_1 x_4 x_5,\\
\dot{x}_2 &= f_2(x) + d_1 u_2, & f_2(x) &= -a_1 x_2 + b_1 x_4 - c_1 x_3 x_5,\\
\dot{x}_3 &= f_3(x), & f_3(x) &= a_3 x_1 - b_3 x_3 - x_4 x_5,\\
\dot{x}_4 &= f_4(x), & f_4(x) &= a_3 x_2 - b_3 x_4 + x_3 x_5,\\
\dot{x}_5 &= f_5(x), & f_5(x) &= -a_5 x_5 - b_5 x_1 x_4 + b_5 x_2 x_3 - c_5 \Gamma_r.
\end{aligned} \tag{1}$$

The coefficients $(a_1, \ldots, c_5\ )$ are given by $a_1 = \dfrac{1}{\sigma T_s} + \dfrac{1-\sigma}{\sigma T_r}$, $b_1 = \dfrac{(1-\sigma)}{\sigma M T_r}$, $c_1 = \dfrac{(1-\sigma)}{\sigma M}$, $d_1 = \dfrac{1}{\sigma L_s}$, $a_3 = \dfrac{M}{T_r}$, $b_3 = \dfrac{1}{T_r}$, $a_5 = \dfrac{k_f}{J}$, $b_5 = \dfrac{p^2 M}{J L_r}$, $c_5 = \dfrac{p}{J}$, $\sigma = 1 - \dfrac{M^2}{L_s L_r}$ where: T_r, T_s are the stator and rotor electric time constant; σ is the leakage coefficient; L_s, L_r are the stator inductance, the rotor inductance; M is the mutual inductance between stator and rotor; k_f is the friction coefficient and Γ_r is the load torque; J is the inertia; p is the number of poles pairs.

Our objective is to control rotor speed ω_r and rotor magnitude flux given by $\phi = x_3^2 + x_4^2$. In the sequel, the flux dynamic is needed so it is given by:

$$\dot{\phi} = f_\phi(x) = -2b_3\phi + 2a_3(x_3 x_1 + x_4 x_2). \tag{2}$$

Hence, the augmented plant dynamic is as follows:

$$\begin{aligned}
\dot{e}_1 &= f_5(x) - \dot{w}_{ref} \quad \text{with} \quad e_1 = w_r - w_{ref},\\
\dot{e}_2 &= f_\phi(x) - \dot{\phi}_{ref} \quad \text{with} \quad e_2 = \phi - \phi_{ref},\\
\dot{x}_1 &= f_1(x) + d_1 u_1,\\
\dot{x}_2 &= f_2(x) + d_1 u_2,\\
\dot{x}_3 &= f_3(x),\\
\dot{x}_4 &= f_4(x),\\
\dot{x}_5 &= f_5(x).
\end{aligned} \tag{3}$$

Here ϕ_{ref} , w_{ref} are the desired flux and the desired speed respectively.

To solve our control problem, we will proceed as follows:

Step 1: We characterize a class of manifolds on which flux-speed tracking (respectively flux observation) is achieved.

Step 2: We design the control law u (respectively the observer gains) that makes, the manifolds introduced in step1, attractive and invariant.

3 Design of the Control Manifold

In this section, our goal is to characterize a class of manifolds on which speed and flux tracking is achieved. Recall that, the sliding mode control objective consists in designing a suitable manifold $M(x,t) \in R^m$ defined by $M = \{x \in R^n : \ \Psi(x) = 0\}$; so that the state trajectories of the plant restricted to this manifold have a desired behavior such as tracking, regulation and stability. Then, determine a switching control law, $u(x,t)$, that is able to drive the state trajectory to this manifold and maintain it on $M(x,t)$, once intercepted, for all subsequent time. That is, $u(x,t)$ is determined such that the selected manifold $M(x,t)$ is made attractive and invariant.

Similarly, the basic sliding mode observer design procedure is performed in two steps. Firstly, design an attractive manifold $S_c(y,t) \in R^p$ so that the output estimation error trajectories restricted to $S_c(y,t)$, have a desired stable dynamics. In the second step, determine the observer gain, to stabilize the equivalent dynamic on $S_c(y,t)$.

In [26] the authors give a form of this surface which is a Hurwitz polynomial of the error and its derivative up to $r-1$, where r is the relative degree of the output.

From the fact that the outputs ϕ and ω_r are of relative degree two and in order to obtain static feedback we propose the manifolds $\Psi = (\Psi_1(e_1) \ \ \Psi_2(e_2))^T$ defined in [26]:

$$\begin{aligned} \Psi_1(e_1) &= \{x \in R^5 : \quad S_1(e_1) = \dot{e}_1 + \Lambda_1(e_1) = 0\}, \\ \Psi_2(e_2) &= \{x \in R^5 : \quad S_2(e_2) = \dot{e}_2 + \Lambda_2(e_2) = 0\} \end{aligned} \tag{4}$$

with $S = (S_1, \ S_2)^T$, and where $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are any given class C^1 functions whose properties will be derived below. One has the following result:

Proposition 3.1 *Consider the manifold Ψ defined in (4), and assume that $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are continuous functions such that $e_i\Lambda_i(e_i) > 0 \ \forall e_i \neq 0 \ (i = 1,2)$. Then, on the manifold Ψ the outputs errors e_1, e_2 converge at least asymptotically to zero.*

Proof Due to the form of manifold $\Psi(x)$, one has:

$$\dot{e}_i = -\Lambda_i(e_i), \quad i = 1,2. \tag{5}$$

Let us use the Lyapunov function given by $V_1 = \frac{1}{2}e_1^2$ and $V_2 = \frac{1}{2}e_2^2$. Their derivatives are then:

$$\dot{V}_i = -e_i\Lambda_i(e_i), \quad i = 1,2. \tag{6}$$

In order to make $\dot{V}_1$ and $\dot{V}_2$ negative definite, it is enough that $e_i\Lambda_i(e_i) > 0 \ \forall e_i \neq 0$ $(i = 1,2)$. Hence the output errors e_1 and e_2 are bounded and moreover they tend at least asymptotically to zero. □

Remark 3.1 For example, the functions $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$, can be taken as the two following functions and their linear combination with positive real coefficients: e_i^k with k odd natural number, and $\sinh(e_i)$.

Remark 3.2 From the literature, it is noticed that in [20, 21], the proposed sliding surface corresponds to the case of: $\Lambda_i(e_i) = ke_i$ with $k > 0$ and we obtain exponential convergence of the tracking errors e_i.

In the sequel, we show how to design the control signal u such that the selected manifold Ψ is attractive and invariant.

Proposition 3.2 *Consider the manifold $\Psi = (\Psi_1(e_1) \quad \Psi_2(e_2))^T$ defined in (4) and let the control signal u be given by*

$$\begin{aligned} u &= u_e + u_i, \\ u_i &= -A^{-1}(x)M\,\mathrm{sign}(\mathrm{S}), \\ u_e &= -A^{-1}(x)(B(x) + C(x)) \end{aligned} \tag{7}$$

with $m_i > 0$, $i = 1, 2$, where

$$A(x) = \begin{pmatrix} -b_5 d_1 x_4 & b_5 d_1 x_3 \\ 2a_3 d_1 x_3 & 2a_3 d_1 x_4 \end{pmatrix}, \quad B(x) = \begin{pmatrix} B_1(x) \\ B_2(x) \end{pmatrix}, \quad C(x) = \begin{pmatrix} C_1(x) \\ C_2(x) \end{pmatrix},$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \mathrm{sign}(S) = \begin{pmatrix} \mathrm{sign}(S_1) \\ \mathrm{sign}(S_2) \end{pmatrix},$$

$$\begin{aligned} B_1(x) &= -a_5 f_5(x) + b_5[-x_4 f_1(x) - x_1 f_4(x) + x_3 f_2(x) + x_2 f_3(x)] - \ddot{\omega}_{ref}, \\ B_2(x) &= -2b_3 f_\phi(x) + 2a_3[x_3 f_1(x) + x_1 f_3(x) + x_4 f_2(x) + x_2 f_4(x)] - \ddot{\phi}_{ref}, \\ C_1(x) &= (f_5(x) - \dot{\omega}_{ref})\frac{d\Lambda_1}{de_1}, \\ C_2(x) &= (f_\phi(x) - \dot{\phi}_{ref})\frac{d\Lambda_2}{de_2}, \end{aligned}$$

where $f_i(x)$ for $i = 1, \dots, 5$ are given in (1) while $f_\phi(x)$ is given in (2) and the functions $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ are characterized in Proposition 3.1. Then, Ψ is globally attractive and invariant.

Proof Let us consider the following Lyapunov function candidate $V = \frac{1}{2}S^T S$, its time derivative is then

$$\dot{V} = S^T \dot{S}, \tag{8}$$

where

$$\dot{S} = B(x) + C(x) + A(x)U. \tag{9}$$

With the control law given by

$$U = -A^{-1}(x)[B(x) + C(x) + M\,\mathrm{sign}(S)] \tag{10}$$

the surface dynamic $\dot{S}$ can be rewritten in the form

$$\dot{S} = -M\,\mathrm{sign}(S). \tag{11}$$

With relation (11), the expression (8) takes the form

$$\dot{V} = -m_1 S_1\,\mathrm{sign}(S_1) - m_2 S_2\,\mathrm{sign}(S_2). \tag{12}$$

In order to make $\dot{V}$ negative $\forall\, S \neq 0$, it is sufficient to take coefficients m_1 and m_2 as

$$m_i > 0, \quad i = 1, 2. \tag{13}$$

This condition makes $(S = 0)$, and hence Ψ is globally attractive. Furthermore since $\dot{S} = 0$, Ψ is invariant. $\square$

Remark 3.3 The determination of the input vector u is possible only if the matrix $A(x)$ has an inverse. Its determinant given by $2a_3b_5d_1(x_3^2 + x_4^2)$ is not null if the rotor flux magnitude is different from zero. The latter condition is verified, since the machine is connected to the supply, and hence the control signal is bounded.

Remark 3.4 The convergence of the output tracking error e_i $(i = 1, 2)$ to zero does not imply that the state vector $x = (x_1, x_2, x_3, x_4, x_5)^T$ of the induction motor remains bounded. However, since $e_1 = x_5 - \omega_{ref}$ and $e_2 = x_3^2 + x_4^2 - \phi_{ref}$ are asymptotically stable with w_{ref} and ϕ_{ref} bounded, one concludes that the states x_5, x_3 and x_4 are bounded. Let $\xi = (x_1, x_2)^T$ and $\eta = (x_3, x_4, x_5)^T$. We have proven that the state η is bounded and we want to prove that ξ remains bounded. From the dynamic equation (1) we can see that, since the coefficient a_1 is positive, the origin of the subsystem $\dot{\xi} = f(\xi, \eta_d)$ is stable for any fixed value η_d of the vector η. One concludes that the state ξ is bounded.

4 Flux Observer Design

In this section, the purpose is to design a current-flux sliding observer based on a general class of manifold S_c. The basic sliding mode observer design procedure is performed in two steps. Firstly, design an attractive manifold $S_c(y, t) \in R^p$ so that the output estimation error trajectories restricted to $S_c(y, t)$, have a desired stable dynamics. In the second step, determine the observer gain, to stabilize the equivalent dynamic on $S_c(y, t)$.

The (x_1, x_2) component current, the rotor speed ω_r and the control input (u_1, u_2) are assumed to be available by measurement. Furthermore, the dynamic of rotor speed ω_r is assumed to be slower than the current and flux dynamics. The observer is considered as a copy of the induction machine electric equations where the speed ω_r is taken as a time varying parameter. In the sequel, $(\hat{x}_1,\ \hat{x}_2)$ denote the observed currents, $(\hat{x}_3,\ \hat{x}_4)$ is the observed flux, (e_{r1}, e_{r2}) is the current observation error and (e_{r3}, e_{r4}) is the flux observation error. Further, we assume that the real flux is bounded as follows: $|x_3| < \rho_3$, $|x_4| < \rho_4$.

We propose an observer constituted by two subsystems, the first one concerns with the stator current observation and is given by:

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = -a_1 \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + A_0 \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} + d_1 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \Delta \begin{pmatrix} \mathrm{sign}(S_{c1}) \\ \mathrm{sign}(S_{c2}) \end{pmatrix}, \tag{14}$$

and the second subsystem concerns with the flux observation and is of the form:

$$\begin{pmatrix} \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \end{pmatrix} = a_3 \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + B_0 \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix} + K \begin{pmatrix} \mathrm{sign}(S_{c1}) \\ \mathrm{sign}(S_{c2}) \end{pmatrix}, \tag{15}$$

where the matrices A_0 and B_0 are given by

$$A_0 = \begin{pmatrix} b_1 & c_1\omega_r \\ -c_1\omega_r & b_1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} -b_3 & -\omega_r \\ \omega_r & -b_3 \end{pmatrix}, \tag{16}$$

and the gain matrices Δ and K are taken as:

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}. \tag{17}$$

Consider the sliding surfaces error $Sc = (Sc_1, Sc_2)^T$ defined by:

$$\begin{cases} S_{c1} = \Theta_1(e_{r1}) & \text{with} \quad e_{r1} = x_1 - \hat{x}_1, \\ S_{c2} = \Theta_2(e_{r2}) & \text{with} \quad e_{r2} = x_2 - \hat{x}_2. \end{cases} \tag{18}$$

Here $\Theta_1(x)$ and $\Theta_2(x)$ are class C^1 functions characterized by some properties, which will be derived later.

Proposition 4.1 *For the first subsystem (14), if the following conditions are fulfilled:*

(i) $\Theta_1(x)$ *and* $\Theta_2(x)$ *are strictly increasing function satisfying:* $\Theta_i(x) = 0$ *if and only if* $x = 0$ *for* $i = 1, 2$*;*

(ii) *the coefficients* δ_1 *and* δ_2 *satisfy*

$$\begin{aligned} \delta_1 &> a_1|e_{r1}| + b_1(|\hat{x}_3| + \rho_3) + c_1\omega_r(|\hat{x}_4| + \rho_4), \\ \delta_2 &> a_1|e_{r2}| + c_1\omega_r(|\hat{x}_3| + \rho_3) + b_1(|\hat{x}_4| + \rho_4). \end{aligned} \tag{19}$$

Then, the manifold $\Psi_c = \{x \in R^5 : \ S_c = 0\}$ *is made globally attractive and invariant, moreover the observation errors* e_{r1} *and* e_{r2} *converge at least asymptotically to zero value.*

Proof On the manifold Ψ_c one has:

$$S_{ci} = 0 \quad \text{or} \quad \Theta_i(e_{ri}) = 0, \quad i = 1, 2, \tag{20}$$

when $\Theta_1(x)$ and $\Theta_2(x)$ are chosen among functions that take zero value only at the origin $x = 0$, the condition (20) leads to $e_{r1} = e_{r2} = 0$.

Let us take the Lyapunov function $V_c = \frac{1}{2}S_c^T S_c$ with $S_c = (S_{c1},\, S_{c2})^T$ and its derivative is

$$\dot{V}_c = S_c^T \dot{S}_c, \tag{21}$$

where

$$\dot{S}_c = \begin{pmatrix} \dot{\Theta}_1(e_{r1}) \\ \dot{\Theta}_2(e_{r2}) \end{pmatrix} = \begin{pmatrix} \dot{e}r_1 \dfrac{d\Theta_1(e_{r1})}{de_1} \\ \dot{e}r_2 \dfrac{d\Theta_2(e_{r2})}{de_2} \end{pmatrix}. \tag{22}$$

From the fact that the current error dynamics $(\dot{e}r_1,\, \dot{e}r_2)$ is given by

$$\begin{pmatrix} \dot{e}r_1 \\ \dot{e}r_2 \end{pmatrix} = -a_1 \begin{pmatrix} e_{r1} \\ e_{r2} \end{pmatrix} + A_0 \begin{pmatrix} e_{r3} \\ e_{r4} \end{pmatrix} - \Delta \begin{pmatrix} \text{sign}(S_{c1}) \\ \text{sign}(S_{c2}) \end{pmatrix} \tag{23}$$

the sliding surface dynamic $(\dot{S}_{c1},\, \dot{S}_{c2})$ becomes

$$\dot{S}_{c1} = P_1(S_{c1}) \frac{d\Theta_1(e_{r1})}{de_{r1}}, \tag{24}$$

$$\dot{S}_{c2} = P_2(S_{c2}) \frac{d\Theta_2(e_{r2})}{de_{r2}}, \tag{25}$$

where

$$P_1(S_{c1}) = [-a_1 e_{r1} + b_1 e_{r3} + c_1 \omega_r e_{r4} - \delta_1 \operatorname{sign}(S_{c1})],$$
$$P_2(S_{c2}) = [-a_1 e_{r2} - c_1 \omega_r e_{r3} + b_1 e_{r4} - \delta_2 \operatorname{sign}(S_{c2})],$$

and hence $\dot{V}$ can take the form

$$\dot{V}_c = P_1(S_{c1})\Theta_1(e_{r1})\frac{d\Theta_1(e_{r1})}{de_{r1}} + P_2(S_{c2})\Theta_2(e_{r2})\frac{d\Theta_2(e_{r2})}{de_{r2}}. \tag{26}$$

The latter can be written as

$$\dot{V}_c = P_1(\Theta_1)\Theta_1(e_{r1})\frac{d\Theta_1(e_{r1})}{de_{r1}} + P_2(\Theta_2)\Theta_2(e_{r2})\frac{d\Theta_2(e_{r2})}{de_{r2}}. \tag{27}$$

In order to make $\dot{V}_c$ negative $\forall\, S_c \neq 0$, it is sufficient that the terms $\frac{d\Theta_1}{de_{r1}}$ and $\frac{d\Theta_2}{de_{r2}}$ must be positive $\forall\, e_i \neq 0$ $(i = 1, 2)$ and the gains δ_1 and δ_2 are taken as

$$\delta_1 > Max\{|-a_1 e_{r1} + b_1 e_{r3} + c_1 \omega_r e_{r4}|\} = a_1|e_{r1}| + b_1(|\hat{x}_3| + \rho_3) + c_1\omega_r(|\hat{x}_4| + \rho_4), \tag{28}$$

$$\delta_2 > Max\{|-a_1 e_{r2} - c_1 \omega_r e_{r3} + b_1 e_{r4}|\} = a_1|e_{r2}| + b_1(|\hat{x}_4| + \rho_4) + c_1\omega_r(|\hat{x}_3| + \rho_3) \tag{29}$$

Hence, the manifold Ψ_c is globally attractive and the observation errors e_{r1} and e_{r2} converge at least asymptotically to zero value. □

Remark 4.1 For example, the function Θ_i $(i = 1, 2)$ can be taken as the two following functions and their linear combination with positive real coefficients: e_i^k with k odd natural number, and $\sinh(e_{ri})$;

Remark 4.2 From the literature, it is noticed that in [20, 21], the proposed sliding surface corresponds to the case of: $\Theta_i = e_{ri}$ with $i = (1, 2) > 0$.

When the first subsystem is in sliding mode, the gain matrix K is determined in order to make the flux observation errors converge exponentially to zero. One has

Proposition 4.2 *If the first subsystem (14) satisfies Proposition 4.1 and with the gain matrix K chosen as*

$$K = \left[B_0 + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}\right]\left[(A_0)^{-1}\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}\right] \tag{30}$$

with $q_1 > 0$ and $q_2 > 0$ then, the flux observation errors (e_{r3}, e_{r4}) are uniformly exponentially stable.

Proof When the first subsystem (14) is in sliding mode ($Sc \equiv \dot{S}c \equiv 0$), then $e_{r1} = e_{r2} = \dot{e}r_1 = \dot{e}r_2 = 0$, and the terms $\operatorname{sign}(Sc_1)$, $\operatorname{sign}(Sc_2)$ are equivalent to:

$$\begin{pmatrix} \operatorname{sign}(S_{c1}) \\ \operatorname{sign}(S_{c2}) \end{pmatrix} \equiv \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}^{-1} A_0 \begin{pmatrix} e_{r3} \\ e_{r4} \end{pmatrix}. \tag{31}$$

As consequence, the second subsystem dynamic (15) is reduced to

$$\begin{pmatrix} \dot{e}r_3 \\ \dot{e}r_4 \end{pmatrix} = \left[B_0 - K \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}^{-1} A_0 \right] \begin{pmatrix} e_{r3} \\ e_{r4} \end{pmatrix} \tag{32}$$

with the gain matrix K given by

$$K = \left[B_0 + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \right] \left[(A_0)^{-1} \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \right]. \tag{33}$$

The observation error dynamic e_{r3} and e_{r4} become

$$\begin{pmatrix} \dot{e}r_3 \\ \dot{e}r_4 \end{pmatrix} = - \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} e_{r3} \\ e_{r4} \end{pmatrix}. \tag{34}$$

From expression (34) it appears clearly that the flux observation errors e_{r3} and e_{r4} converge uniformly exponentially to zero. □

As the flux components are not available by measure, we must use the observed flux in the implementation of the control law (7). Besides, the convergence of the flux observation errors (e_{r3}, e_{r4}) and the controlled output errors (e_1, e_2), defined in (3), to zero does not imply that these variables will tend to zero when the observed flux is used instead of the real flux in the control law (7). This is so, because the separation principle is no longer valid for nonlinear systems. However, since the flux observation errors are uniformly exponentially stable, a sufficient condition for the global stability of the overall system resulting from the association of the control law (7) with the flux observer is given in [27]. This condition is that functions $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ must be chosen such that the control law (7) ensures global exponential stability of the controlled output errors (e_1, e_2). One possible choice of such functions is $\Lambda_i(e_i) = ke_i$ with $i = 1, 2$.

5 Simulation Results

The three phase induction machine under test is characterized by $P = 3.7\,Kw, 220/380, 8.54/14.8\,A$, $M = 0.048\,H$, $Ls = 0.17\,H$, $Lr = 0.015\,H$, $Ts = 0.151\,s$, $Tr = 0.136\,s$, $J = 0.135\,mN/rds^{-2}$, $K_f = 0.0018\,mN/rds^{-1}$. The chattering effect, due to sliding terms contained in input control, is largely attenuated using the function sign designed by the following relation:

$$\begin{cases} \operatorname{sign}(s) = s/\varepsilon & \text{if } \ |s| \le \varepsilon, \\ \operatorname{sign}(s) = 1 & \text{if } \ s > \varepsilon, \\ \operatorname{sign}(s) = -1 & \text{if } \ s < \varepsilon. \end{cases}$$

With a view to illustrate the method, we use the following surfaces:

(i) For flux-speed tracking: $\Lambda_1(e_1) = \sinh(e_1)$ and $\Lambda_2(e_2) = \sinh(e_2)$.

(ii) For flux observation $\Theta_1(e_{r1}) = \lambda_1 e_{r1} + \sinh(e_{r1})$ and $\Theta_2(e_{r2}) = \lambda_2 e_{r2} + \sinh(e_{r2})$.

Figures 5.1 and 5.2 give the machine responses in tracking regime (for both $\omega_{ref} > 0$ and $\omega_{ref} < 0$). It appears clearly that the flux machine and speed track their references with a good accuracy. Moreover, the initial stator peak currents are attenuated by

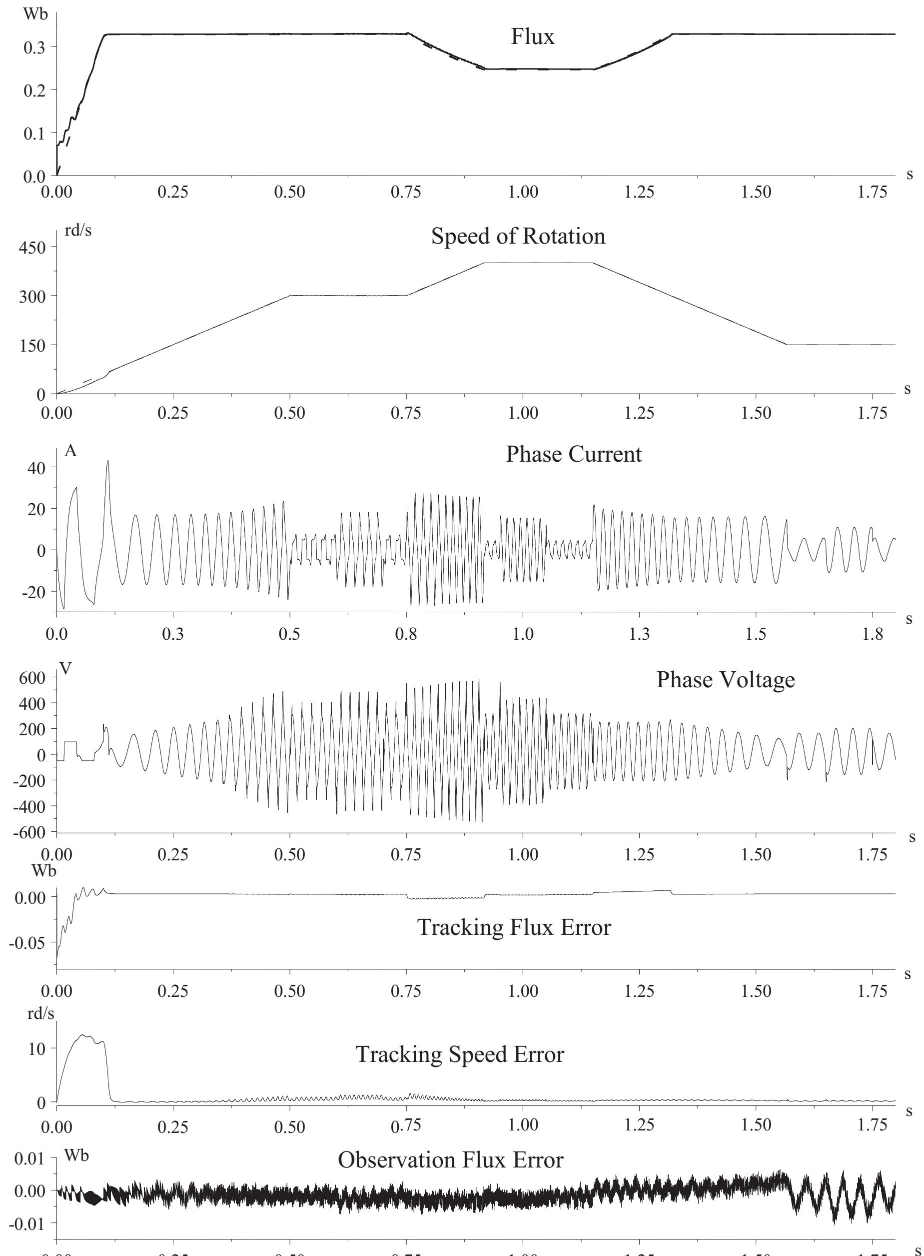

Figure 5.1: Induction machine responses in tracking regime for positive reference speed with the disturbances applied during only 0.1 s respectively at time t=0.6 s, 0.95 and 1.65 s (solid line for outputs; dashed line for references).

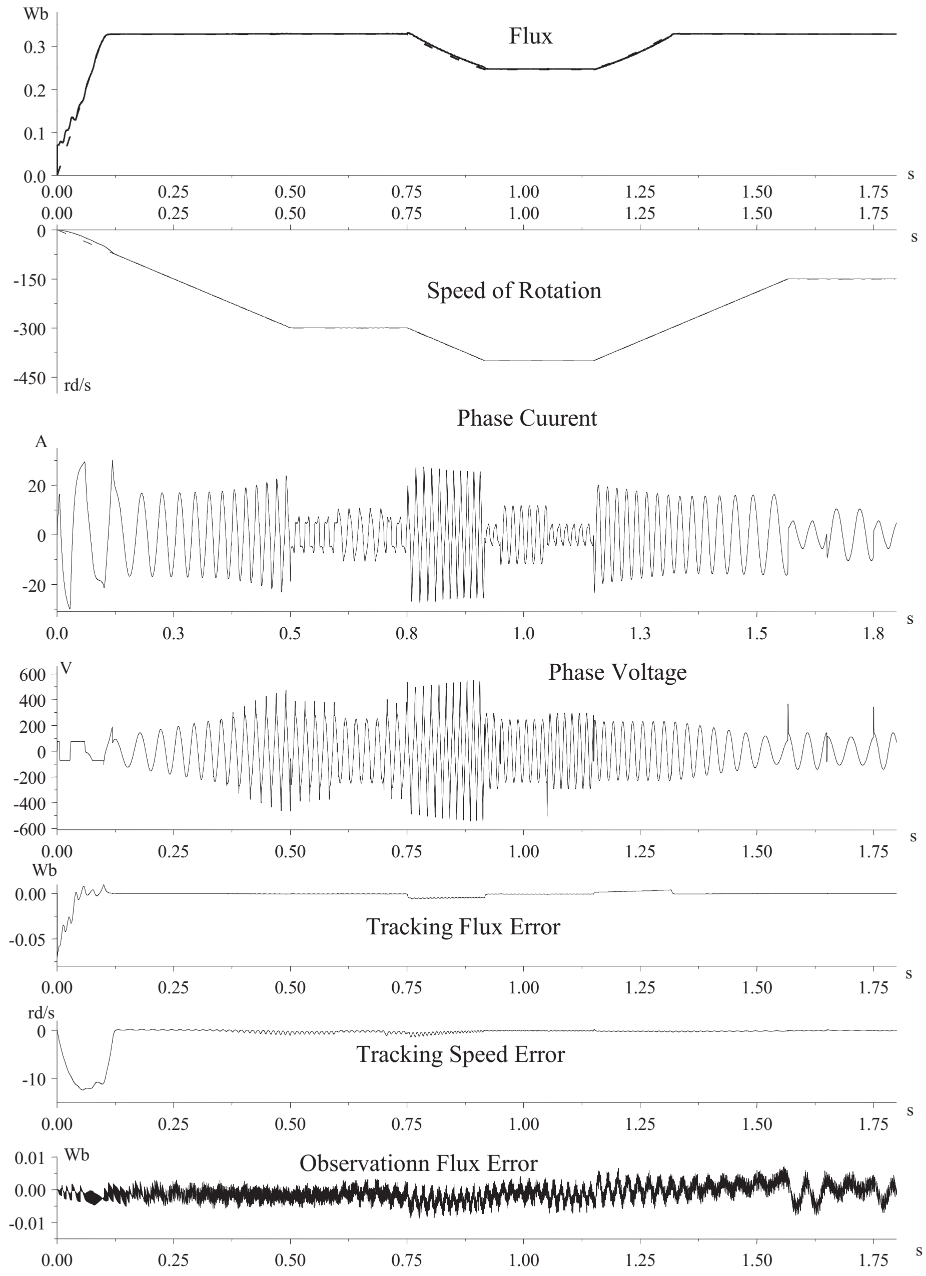

Figure 5.2: Induction machine responses in tracking regime for negative reference speed with the disturbances applied during only 0.1 s respectively at time t=0.6 s, 0.95 and 1.65 s (solid line for outputs; dashed line for references).

reducing the control inputs only in the beginning of the transient stage (for time $t \leq 0.1s$). This reduction affects the tracking speed during this interval of time. An appreciable flux tracking error (around 2%) is obtained due to an important threshold value used in function sign. In order to maintain the voltage in admissible range when the speed reference ω_{ref} grows up to nominal value $n = 300\,rd/s$, the reference flux ϕ_{ref} is reduced down to the nominal flux ϕ_n as: $\phi_{ref} = \phi_n \omega_n / \omega_{ref}$.

The machine flux tracks the desired value with a good accuracy in all speed range. Moreover, the estimated flux provided by the observer is sensibly the same as the flux machine (the error flux is around 0.001) independently of the speed value.

Further, it is noted that the speed and flux tracking and the estimated flux reveal a good robustness against disturbances represented by parametric variations and nominal load torque occurring at the same time. These disturbances are applied during $0.1s$ respectively at the time $t = 0.85\,s$, $1.35\,s$ and $t = 2.6\,s$. The robustness tests are performed for the parameter variations around nominal values as the all rotor resistances increase by an amount of 100% and, all inductances decrease by an amount of 50%. In spite of the occurring disturbances, the voltage phase value remains admissible.

6 Conclusion

In this paper, a general class of manifolds for sliding mode observation and control of induction machine, is developed. Firstly, the properties of sliding surfaces, ensuring the tracking flux-speed and observation flux, are derived. In the case of flux-speed tracking, we have studied the case when the derivative of the error control and a function of this error form the sliding surface. It has been demonstrated that this function must be odd and its derivative must be even function vanishing only at the origin. In the case of flux observation, this surface is a function of the estimated error. In later case, it has been proved that this function must be odd and it takes zero value at the origin and its derivative must be continuous even for the function taking zero value only at the origin. The simulation results have allowed obtaining the flux-speed tracking and flux observation with good accuracy. Moreover, the behavior of tracking against disturbances represented by the application of the nominal load torque in the presence of increasing rotor resistances (by an amount of 100%) and decreasing inductances (by an amount of 50%) reveals the high robustness level.

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Constrained Linear Quadratic Regulator: Continuous-Time Case

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Abstract: This paper deals with the linear quadratic regulator with constraints on the state and the input vectors. Such an optimization problem has a wide applications in industry like chemical and manufacturing industries. Our goal in this paper consists of developing an efficient numerical algorithm to solve such problem. Our technique relays on an iterative approach that uses the solution of the standard linear quadratic regulator as an initial guess for the optimal solution and then iteratively, the solution is improved by designing a controller that compensates for the violation of the constraints at each iteration. A numerical example is given to show the effectiveness of this algorithm.

Keywords: *Linear systems; linear quadratic regulator; constrained input; constrained state.*

Mathematics Subject Classification (2000): 49N10, 49N35.

1 Introduction

The linear quadratic regulator (LQR) is one of the most studied control problem in the literature. It will require many pages to cite all the works that were reported in the literature on the subject. In fact there are many variants. If we restrict ourselves to the case of LQR with constrained states and inputs, this variant consists of designing a state feedback controller that drives the state from a nonzero initial condition to zero by respecting simultaneously the constraints on the state and the control vectors.

This control problem has many applications in industry. In fact to motivate our study, let us consider a deterministic manufacturing system that produces n-items that

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can be stocked in a storage with finite size for each part and delivered to the market according to a given demand. Therefore, the inventory control problem can be stated as a constrained linear quadratic regulator problem.

This type of problem has been tackled by many authors among them we quote the works [5, 3, 4, 6, 7, 11, 9, 10, 2, 13]. In these references efficient algorithms have been developed to solve numerically the optimality conditions for the linear quadratic regulator with constraints on the states and/or the inputs. Both versions (continuous-time and discrete-time) have been tackled. Pytlak [8] presents many numerical methods for nonlinear optimal control problems with state constraints.

Our goal in this paper consists of solving the linear quadratic regulator with constrained states and inputs. To determine the control law, we develop a numerical method that uses the standard linear regular as an initial guess solution and iteratively, we improve the control law using the error at each iteration. Our idea in this paper, consists of considering the linear regulator problem as an initial guess. Based on this solution another optimization problem is formulated in which the state constraints are relaxed while the control constraints are maintained. Then, an iterative procedure is developed to solve the problem at hand while satisfying systems constraints.

The rest of this paper is organized as follows. In Section 2, the constrained linear quadratic regulator problem is stated and some results are recalled to facilitate the understanding of results. Section 3 contains the main of the paper and presents the steps of our algorithm. Section 4 provides a numerical example to show the effectiveness of the developed algorithm.

2 Problem Statement

Let us consider the class of continuous-time linear systems with the following dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ and $u(t) \in R^m$ represent respectively the state and the control of the system at time t, the matrices A and B are assumed to be known and constant, and x_0 is the initial condition.

The standard formulation of the linear quadratic regulator consists of minimizing the following cost function

$$J = \frac{1}{2}\int_0^T \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] dt, \tag{2}$$

where $Q \in R^{n\times n}$ and $R \in R^{m\times m}$ are two given matrices such that $Q \geq 0$ and $R > 0$ and $T > 0$ is a given finite time.

Under the assumption that the linear system is stabilizable and detectable it can be shown that the solution of this optimization problem is given by (see [1])

$$u^\star(t) = Kx(t), \tag{3}$$

where $K = -R^{-1}B^{\mathrm{T}}P(t)$ with $P(t)$ is the solution of the following Riccati equation

$$-\dot{P} = A^{\mathrm{T}}P(t) + P(t)A - PBR^{-1}B^{\mathrm{T}}P + Q. \tag{4}$$

As it was pointed out in the introduction, more often practical systems have constraints either on the state or the input or on both of them. Therefore, the previous formulation doesn't represent the real case and the constraints either on the state or the control or on both should be included in the previous formulation. The corresponding formulation is referred to as constrained linear quadratic regulator. For more details on this formulation either for the continuous-time or the discrete-time version, we refer the reader to [3, 4, 6, 7] and the references therein. This formulation is given by:

$$Pc\colon \begin{cases} \min J = \dfrac{1}{2}\displaystyle\int_0^T \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] dt, \quad \text{subject to:} \\ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ \underline{x} \leq x(t) \leq \bar{x}, \quad \underline{u} \leq u(t) \leq \bar{u}, \end{cases} \tag{5}$$

where $\underline{x}$, $\bar{x}$, $\underline{u}$ and $\bar{u}$ are known vectors and the other parameters keep the same definitions as before.

This optimization problem does not have an analytical solution as it is the case for the previous one and the only way to solve it is to proceed numerically. In the literature, we can find some numerical methods that solve such problem. For more details on this subject, we refer the reader to [5, 3, 4, 6, 7, 8] and the references therein. Our goal in this paper is to solve this problem and to propose a numerical algorithm that solves efficiently the optimization problem Pc. The next section will provide such algorithm and in Section 4, a numerical example is provided to show the effectiveness of this algorithm.

3 Main Results

To solve the optimization problem Pc some attempts have been proposed in the literature for more details on this topics we refer the reader to [3, 4, 6, 7] and the references therein. Here we will propose a new way that solves the problem Pc iteratively starting from an initial solution that we can get from the unconstrained optimization problem. Then, subsequently by correcting the error between desired trajectory and the one at iteration k, we can design a controller that compensates for this error which will be added to the one at iteration k. At the end of the algorithm we end up with the desired control and the trajectory that satisfy all the system constraints.

Let us denote by $\widehat{x}(t)$ and $\widehat{u}(t)$ the optimal trajectory and the optimal control for the unconstrained linear quadratic control problem. The link between the optimal control and the optimal trajectory is given by:

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + B\widehat{u}(t), \quad \widehat{x}(0) = x_0, \tag{6}$$

where $\widehat{u}(t) = -R^{-1}B^{\mathrm{T}}P(t)\widehat{x}(t)$ with $P(t)$ is the solution of the Riccati equation (4).

It is obvious that this solution is not the optimal one and some corrections are needed to be done to make it closer to the optimal solution. For this purpose by denoting by $x^\star(t)$ and $u^\star(t)$ respectively the optimal trajectory and the optimal control of the constrained linear quadratic regulator, we have:

$$\begin{aligned} \dot{x}^\star(t) &= Ax^\star(t) + Bu^\star(t), \quad x^\star(0) = x_0, \\ u^\star(t) &= \widehat{u}(t) + \Delta u^\star(t), \end{aligned}$$

with $\Delta u^\star(t)$ is a control law that we have to determine that will correct the trajectory of the system and then reduces the error.

Notice that $x(t)$ and $u(t)$ are linked to the optimal solution of the standard linear quadratic regular by the following expressions:

$$x(t) = \widehat{x}(t) + e(t), \quad u(t) = \widehat{u}(t) + \Delta u(t).$$

Using now these expressions, the cost function and the previous constraints become respectively:

$$\begin{aligned}\min J = {} & \frac{1}{2}\int_0^T \left[\widehat{x}^{\mathrm{T}}(t)Q\widehat{x}(t) + \widehat{u}^{\mathrm{T}}(t)R\widehat{u}(t)\right] dt \\ & + \int_0^T \left[\widehat{x}^{\mathrm{T}}(t)Qe(t) + \frac{1}{2}\, e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t)\right] dt\end{aligned}$$

subject to:

$$\dot{\widehat{x}}(t) + \dot{e}(t) = A\widehat{x}(t) + Ae(t) + B\widehat{u}(t) + B\Delta u(t)$$

and

$$\underline{x} \le \widehat{x}(t) + e(t) \le \bar{x}, \quad \underline{u} \le \widehat{u}(t) + \Delta u(t) \le \bar{u}.$$

Assume that we are now at the first iteration, i.e.: $k = 1$, in which $x^k(t) = \hat{x}(t)$ and $u^k(t) = \hat{u}(t)$ are known. From the constraints on the states and the knowledge of $x^k(t)$, we can determine precisely the maximum and minimum values as well as the corresponding time instant as which $x^k(t)$ trajectories violate these constraints. Let us now denote by t_{ij}^k, $j = 1, \ldots, p_p$ (where p_i is a finite positive integer) the corresponding instants at which the maximum or the minimum violations occur and by $e_i^{\star k}(t_{ij}^k)$ the value of the i-th component of the maximum and the minimum error at time t_{ij}^k that we should compensate. This imposes the following constraints which have to be satisfied in our optimization problem:

$$e_i^k(t_{ij}^k) = e_i^{\star k}(t_{ij}^k), \quad j = 1, \ldots, p_i, \quad i = 1, \ldots, n.$$

Therefore, our original problem can be transformed to the following one that has only inequality constraints on the input (for simplicity, the iteration number k will be dropped while deriving the necessary conditions for optimality)

$$\min \Delta J = \int_0^T \left[\widehat{x}^{\mathrm{T}}(t)Qe(t) + \frac{1}{2}e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t)\right] dt$$

subject to

$$\begin{aligned}\dot{e}(t) &= Ae(t) + B\Delta u(t), \quad e(0) = 0, \\ e_i(t_{ij}) &= e_i^\star(t_{ij}), \quad j = 1, \ldots, p_i, \quad i = 1, \ldots, n, \\ \underline{\Delta u(t)} &\le \Delta u(t) \le \overline{\Delta u(t)},\end{aligned}$$

with $\underline{\Delta u(t)} = \underline{u} - \widehat{u}(t)$, $\overline{\Delta u(t)} = \bar{u} - \widehat{u}(t)$.

To solve this problem, let us write the corresponding Hamiltonian:

$$\begin{aligned}H(e, \Delta u, t) = {} & \widehat{x}^{\mathrm{T}}(t)Qe(t) + \frac{1}{2}\, e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t) \\ & + \lambda^{\mathrm{T}}\left[Ae + B\Delta u\right] + \sum_{i=1}^{n}\sum_{j=1}^{p} \pi_{ij}\left[e_i(t) - e_i^\star(t)\right]\delta(t - t_{ij}),\end{aligned}$$

where λ is the costate vector, π_{ij} is the Lagrange multiplier and $\delta(t)$ is the Dirac delta function defined as follows:

$$\delta(t-t_{ij}) = \begin{cases} 1, & \text{if } t = t_{ij}, \\ 0, & \text{elsewhere.} \end{cases}$$

Based on optimization theory, the necessary conditions of optimality give

$$\frac{\partial H}{\partial \Delta u} = 0$$

which implies

$$R\widehat{u} + R\Delta u + B^{\mathrm{T}}\lambda = 0$$

that gives in turn

$$\Delta u = -\widehat{u} - R^{-1}B^{\mathrm{T}}\lambda.$$

The feasible control law, $\Delta u + \widehat{u}$ that minimizes the Hamiltonian while satisfying the previous constraints on control is given by [12]:

$$\Delta u + \widehat{u} = \begin{cases} \underline{u}, & \text{if } -R^{-1}B^{\mathrm{T}}\lambda < \underline{u}\ , \\ -R^{-1}B^{\mathrm{T}}\lambda, & \text{if } \ \underline{u} \le \Delta u(t) \le \overline{u}, \\ \overline{u}, & \text{if } -R^{-1}B^{\mathrm{T}}\lambda > \overline{u}\ . \end{cases} \tag{7}$$

The second necessary optimality condition for our problem is

$$\frac{\partial H}{\partial \lambda} = \dot{e} = Ae + B\Delta u \tag{8}$$

with $e(0) = 0$.

The third necessary optimality condition for our problem is

$$\frac{\partial H}{\partial e} = -\dot{\lambda},$$

which implies that

$$\dot{\lambda} = -Q\widehat{x} - Qe - A^{\mathrm{T}}\lambda - \pi(t) \tag{9}$$

with $\lambda(T) = 0, \quad \pi(t) = [\pi_1(t)\delta(t-t_{1j}), \dots, \pi_p(t)\delta(t-t_{p_{nj}})]$.

The last necessary optimality condition gives:

$$\frac{\partial H}{\partial \pi_{ij}} = e_i(t_{ij}) - e_i^{\star}(t_{ij}), \quad i = 1, \dots, n, \quad j = 1, \dots, p_i.$$

The error $e_i(t_{ij}) - e_i^{\star}(t_{ij})$ at iteration m is used to update π_{ij} employing the following expression:

$$\pi_{ij}^{m+1} = \pi_{ij}^{m} + \alpha\left[e_i^m(t_{ij}) - e_i^{\star}(t_{ij})\right], \tag{10}$$

where α can be chosen following the well know optimization techniques.

To solve our optimization we need to determine $\lambda(t)$ that comes from (9) that its self depends on $\pi(t)$ that we should estimate, and $e(t)$ that be can obtained from (8) that in turn depends on $\Delta u(t)$ that we should determine once we know $\lambda(t)$. All the equations are coupled and one way of obtaining a solution to this problem is numerically solve the problem. Once the optimization problem is solved, we update the trajectories, $x^k(t)$, $u^k(t)$ by $e^k(t)$ and $\Delta u^k(t)$ to get the new trajectories $x^{k+1}(t) = x^k(t) + e^k(t)$ and $u^{k+1}(t) = u^k(t) + \Delta u^k(t)$ and then repeat the whole process till the $\sup_{ij} e_i^{\star}(t_{ij}^k)$ is less than a specified given value. The steps of our algorithm are summarized by:

Algorithm 3.1

1. Initialization: Choose $\bar{\epsilon}_x > 0,\ \bar{\epsilon}_\lambda > 0,\ \bar{\epsilon}_\pi > 0,$ and let $k = 1,\ l = 1,$ and $m = 1$ (the numbers of iterations for $x(t)$, $\lambda(t)$ and $\pi(t)$), and solve the standard LQR to get $\widehat{x} = x^k$ and $\widehat{u} = u^k$.

2. Identify the values of $e_i^\star(t_{ij}^k)$ and the corresponding instants t_{ij}^k at the iteration k for each trajectory.

3. Guess $\pi_{ij}^{km}(t)$.

4. Guess $\lambda^{klm}(t)$.

5. Compute Δu^{klm} using (7), and solve (8) to determine $e^{klm}(t)$.

6. Solve (9) to get the trajectory $\lambda^{k(l+1)m}$ at the iteration k and m.

7. Compute the error on λ as follows
$$\varepsilon_\lambda = \sqrt{\int_0^T \|\lambda^{klm}(t) - \lambda^{k(l+1)m}(t)\|^2\, dt}.$$
Test: If $\varepsilon_\lambda > \bar{\epsilon}_\lambda$, use the computed $\lambda(t)$ at this iteration as a guess for $\lambda(t)$, put $l = l + 1$ and go to Step 5, otherwise continue.

8. Update $\pi_{ij}^{kl}(t)$ using for example
$$\pi_{ij}^{kl(m+1)} = \pi_{ij}^{klm} + \alpha\left(e_i^{klm}(t_{ij}^k) - e_i^\star(t_{ij}^k)\right),$$
where α can be chosen following one of the well known optimization techniques; and compute the error as:
$$\varepsilon_\pi = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{p_i} \|e_i^{klm}(t_{ij}^k) - e_i^\star(t_{ij}^k)\|^2}.$$
Test: If $\varepsilon_\pi > \bar{\epsilon}_\pi$, put $l = 1$ and $m = m + 1$ and solve (9) to get new trajectory for λ and go to Step 5, otherwise continue.

9. Calculate the new trajectory $x(t)$ and $u(t)$ at the iteration $k+1$ using the following:
$$x^{k+1} = x^k + e^k, \quad u^{k+1} = u^k + \Delta u^k.$$

10. Identify the values of $e_i^\star(t_{ij})$ and the corresponding instants t_{ij} at the iteration $k + 1$ for each trajectory and compute the error using the following
$$\varepsilon_x = \sup_{ij}\{e_i^\star(t_{ij})\} \text{ for the } e_i^\star(t_{ij}) \text{ computed at this step.}$$

11. Test: If $\varepsilon_x > \bar{\epsilon}_x$, increase k by 1, put $l = 1$, $m = 1$ and go to Step 3, else record the trajectories and the controls and stop.

In the next section a numerical example with lower bounds on the states and the control is provided to show the validness of our approach. Our algorithm has been programmed using Fortran language on Pentium PC. The computation time is very acceptable and for the one we are presenting is less than one second.

4 Numerical Example

To show the effectiveness of our algorithm, let us consider a linear system with the following data:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2.36 & -13.6 & -12.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.79 & 2.68 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \end{bmatrix}.$$

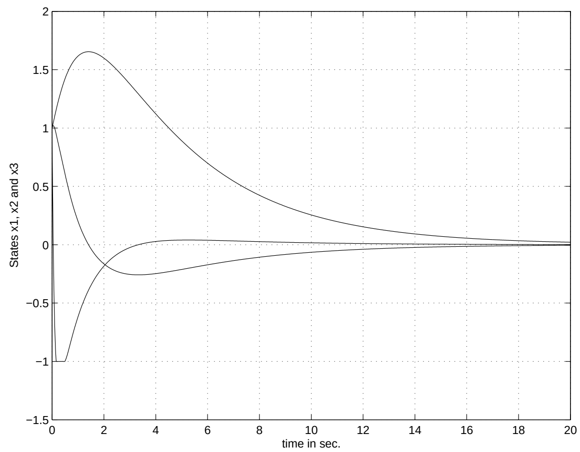

Figure 4.1: Behavior of the state variables x_1, x_2 and x_3.

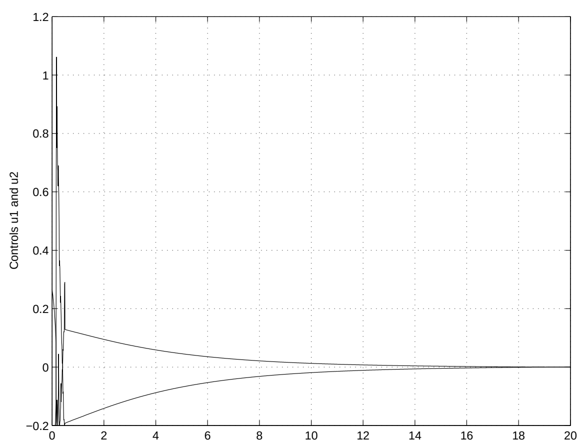

Figure 4.2: Behavior of the control variables u_1 and u_2.

As it can be seen in Figures 1 and 2, the obtained suboptimal states and control trajectories satisfy all the required system constraints.

5 Conclusion

This paper dealt with the constrained linear quadratic regulator for the class of linear continuous-time. The constraints are on both the control and the state vectors. A procedure is developed in which the original problem is converted to another one which has only constraints on control. By solving this new problem iteratively, it is possible to get the solution of the original one. The illustrative example shows the applicability of the developed technique.

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Topological Pressure of Nonautonomous Dynamical Systems ⋆

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Abstract: We define and study topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact metric space. In this paper, we obtain the basic properties and the invariant with respect to equiconjugacy of topological pressure for the non-autonomous discrete dynamical systems.

Keywords: *Topological pressure; sequence of continuous self-maps; non-autonomous system.*

Mathematics Subject Classification (2000): 58F13, 37A35, 37B55.

1 Introduction

Entropies are fundamental to our current understanding of dynamical systems. The notion of topological entropy was introduced by Adler, Konheim and Mcandrew as an invariant of topological conjugacy. Topological entropy provides a numerical measure for the complexity of an endomorphism of a compact topological space [1]. Later Bowen and Dinaburg gave a new, but equivalent, definition in the case when the space under consideration is metrizable [2]. S. Kolyada and L. Snoha studied topological entropy for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous

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self-maps of a compact topological space [3]. Topological pressure is a generalization to topological entropy for a dynamical system [4].

Our purpose is to introduce and study the notion of topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact topological space.

First, some notation and definitions are established.

Throughout the paper, (X,d) will be a compact metric space and $C(X,X)$ be the set of continuous maps from (X,d) into itself, $C(X,R)$ be the functional space containing all continuous, real-valued functions on X.

Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps from X to X. The identity map on X will be denoted by id_X or shortly by id. Let N, R, Z be the set of all positive integers, real and integers, respectively. For any $i \in N$ let $f_i^0 = f_i^{-0} = id_X$ and for any $i, n \in N$ set $f_i^n = f_{i+(n-1)} \circ \cdots f_{i+1} \circ f_i$ (first apply f_i) and $f_i^{-n} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots f_{i+(n-1)}^{-1}$ (the last notations will be applied to sets, we do not assume that the maps f_i are invertible). Finally, denote by $f_{1,\infty}^n$ the sequence of maps $\{f_{in+1}^n\}_{i=0}^{\infty}$ and by $f_{1,\infty}^{-1}$ the sequence $\{f_i^{-1}\}_{i=0}^{\infty}$.

Now we are going to describe the main results of the paper and how it is organized. For the precise statements of the results and for the definitions used see corresponding sections.

Let $f_{1,\infty} \in C(X,X)$ and $\varphi \in C(X,R)$. In this paper we will define and study the topological pressure $P(f_{1,\infty}, \varphi)$ of non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ with respect to φ.

In Section 1, we give the basic definition of topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact metric space. In Section 2, we study the basic properties of topological pressure for the non-autonomous discrete dynamical systems.

2 Topological Pressure of a Sequence of Maps on a Compact Metric Space

We are going to define the topological pressure of a non-autonomous dynamical system $(X; \{f_i\}_{i=1}^{\infty})$ analogously to the topological pressure of a autonomous dynamical system $(X; f)$ ([4]). Of course, for $f_1 =_2 = \cdots = f$ we get the classical definition.

For each $n \geq 1$ there is a positive integer. Define the metric in X by $d_n(x,y) = \max_{0 \leq j \leq n-1} d(f_1^j(x), f_1^j(y))$. A subset E of the space X is called (n,ε)–separated if for any two distinct points $x, y \in E$, $d_n(x,y) > \varepsilon$. Let $C(X,R)$ be the space of real-valued continuous functions of X. For $\varphi \in C(X,R)$ and $n \in N$ we denote $\sum\limits_{i=0}^{n-1} \varphi(f_1^i(x))$ by $(S_n\varphi)(x)$. For $\varepsilon > 0,\ x \in X$, we put

$$P_n(f_{1,\infty}, \varphi, \varepsilon) := \sup \left\{ \sum_{x \in E} e^{(S_n\varphi)(x)} \mid E \text{ is a (} n, \varepsilon) \text{ separated set for } X \right\}.$$

Then we put

$$P(f_{1,\infty}, \varphi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(f_{1,\infty}, \varphi, \varepsilon)$$

and we define the topological pressure of $f_{1,\infty}$ with respect to φ as

$$P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \to 0} P(f_{1,\infty}, \varphi, \varepsilon).$$

It is clear that $P(f_{1,\infty}, 0) = h(f_{1,\infty})$.

A set $F \subset X$ $(n, \varepsilon) -$ *spans* another set $K \subset X$ provided that for each $x \in K$ there is $y \in F$ for which $d_n(x, y) \leq \varepsilon$. For $\varepsilon > 0, x \in X$, we put

$$Q_n(f_{1,\infty}, \varphi, \varepsilon) := \inf\{\sum_{x \in E} e^{(S_n\varphi)(x)} \mid E \text{ is a } (\ n, \varepsilon) \text{ spanning set for } X\}.$$

Remark 2.1 $Q_n(f_{1,\infty}, \varphi, \varepsilon) \leq P_n(f_{1,\infty}, \varphi, \varepsilon)$.

Proof It follows from the fact that $e^{(S_n\varphi)(x)} > 0$ and a (n, ε) separated set which cannot be enlarge to a (n, ε) separated set must be a (n, ε) spanning set of X. $\square$

Remark 2.2 *If $\delta > 0$ is such that $d(x, y) < \frac{\varepsilon}{2}$ implies that $\mid \varphi(x) - \varphi(y) \mid < \delta$ then $P_n(f_{1,\infty}, \varphi, \varepsilon) \leq e^{n\delta} Q_n(f_{1,\infty}, \varphi, \varepsilon)$.*

Proof Let E be a (n, ε) separated set and F is a $(n, \frac{\varepsilon}{2})$ spanning set. Define $\phi : E \to F$ by choosing, for each $x \in E$, some point $\phi(x) \in F$ with $d_n(x, \phi(x)) \leq \frac{\varepsilon}{2}$. Then ϕ is injective and

$$\sum_{y \in F} e^{(S_n\varphi)(y)} \geq \sum_{y \in \phi(E)} e^{(S_n\varphi)(y)} \geq \left(\min_{x \in E} e^{(S_n\varphi)(\phi(x)) - (S_n\varphi)(x)} \right) \sum_{x \in E} e^{(S_n\varphi)(x)}$$
$$\geq e^{-n\delta} \sum_{x \in E} e^{(S_n\varphi)(x)}.$$

Therefore $Q_n(f_{1,\infty}, \varphi, \varepsilon) \leq e^{-n\delta} P_n(f_{1,\infty}, \varphi, \varepsilon)$. $\square$

Remark 2.3 *By (1) and (2), if we put*

$$P(f_{1,\infty}, \varphi, \varepsilon) := \limsup_{n \to \infty} \frac{1}{n} \log Q_n(f_{1,\infty}, \varphi, \varepsilon)$$

we will have

$$P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \to 0} Q(f_{1,\infty}, \varphi, \varepsilon).$$

Let α be an open cover of X. For $x \in X$, we put

$$q_n(f_{1,\infty}, \varphi, \alpha) := \inf \left\{ \sum_{B \in \beta} \inf_{x \in B} e^{(S_n\varphi)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} f_1^{-i}\alpha \right\}$$

and put

$$p_n(f_{1,\infty}, \varphi, \alpha) := \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{(S_n\varphi)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} f_1^{-i}\alpha \right\}.$$

Clearly $q_n(f_{1,\infty}, \varphi, \alpha) \leq p_n(f_{1,\infty}, \varphi, \alpha)$. In addition similar to the case of the autonomous systems we have the following Proposition.

Proposition 2.1 *Let $f_{1,\infty} \in C(X, X)$ and $\varphi \in C(X, R)$.*

(1) *If α is an open cover of X with Lebesgue δ then $q_n(f_{1,\infty}, \varphi, \alpha) \leq Q_n(f_{1,\infty}, \varphi, \varepsilon)$.*

(2) *If* $\varepsilon > 0$ *and* γ *is an open cover with* $\operatorname{diam}(\gamma) \le \varepsilon$ *then* $P_n(f_{1,\infty}, \varphi, \varepsilon) \le p_n(f_{1,\infty}, \varphi, \gamma)$.

(3) *If* α *is an open cover of* X, *then*

$$\lim_{n\to\infty} \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha)$$

exists and equals to $\inf_n \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha)$.

(4) *If* α, γ *are open covers of* X *and* $\alpha \prec \gamma$ *(i.e. for each* $C \in \gamma$, *there is an* $A \in \alpha$ *such that* $C \subset A$*), then* $q_n(f_{1,\infty}, \varphi, \alpha) \le q_n(f_{1,\infty}, \varphi, \gamma)$.

(5) *If* $d(x, y) < \operatorname{diam}(\alpha)$ *implies* $\mid f(x) - f(y) \mid \le \delta$ *then* $p_n(f_{1,\infty}, \varphi, \alpha) \le e_{n\delta} q_n(f_{1,\infty}, \varphi, \gamma)$.

(6) $P(f_{1,\infty}, \varphi) = \lim\limits_{k\to\infty} [\lim\limits_{n\to\infty} \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha_k)] = \lim\limits_{k\to\infty} [\limsup\limits_{n\to\infty} \frac{1}{n} \log q_n(f_{1,\infty}, \varphi, \alpha_k)]$ *if* α_k *is a sequence of open covers with* $diam(\alpha_k) \to 0$.

(7) $P(f_{1,\infty}, \varphi) = \lim\limits_{\varepsilon\to 0} \liminf\limits_{n\to\infty} \frac{1}{n} \log P_n(f_{1,\infty}, \varphi, \varepsilon)$.

(8) $P(f_{1,\infty}, \varphi) = \lim\limits_{\varepsilon\to 0} \liminf\limits_{n\to\infty} \frac{1}{n} \log Q_n(f_{1,\infty}, \varphi, \varepsilon)$.

The proof of Proposition 2.1 is similar to the case of the autonomous systems (for detailed proof see [4]), we omitted it.

3 Properties of Pressure of a Sequence of Maps on a Compact Metric Space

We now study the properties of $P(f_{1,\infty} \cdot) : C(X, X) \to R \cup \infty$. In particular we see that either $P(f_{1,\infty} \cdot)$ never takes the value ∞ or is identical to ∞.

Theorem 3.1 *Let* $f_{1,\infty} : X \to X$ *be a continuous maps of a compact metric space* X *and* $\varphi \in C(X, R), \varepsilon > 0$. *Then* $P(f^k_{1,\infty}, S_k\varphi) \le kP(f_{1,\infty}, \varphi)$ *(here* $(S_k\varphi)(x) = \sum\limits_{i=0}^{k-1} \varphi(f^i_1(x))$*) for any* $k \ge 1$.

Proof If F is (nk, ε) spanning for $f_{1,\infty}$ then F is (n, ε) spanning for $f^k_{1,\infty}$. Here $Q_n(f^k_{1,\infty}, S_k\varphi, \varepsilon) \le Q_{nk}(f_{1,\infty}, \varphi, \varepsilon)$ so that $P(f^k_{1,\infty}, S_k\varphi) \le kP(f_{1,\infty}, \varphi)$. $\square$

Remark 3.1 *In general we cannot claim that* $P(f^k_{1,\infty}, S_k\varphi) = kP(f_{1,\infty}, \varphi)$ *for any* $k \ge 1$.

Example 3.1 Indeed, on $X = I = [0, 1]$ take the standard tent map $g(x) = 1 - |2x - 1|$, $\varphi = 0$ and

$$f_{1,\infty} = \left\{ g, \frac{1}{2} id_{S^1}, g^2, \frac{1}{4} id_{S^1}, \ldots, g^n, \frac{1}{2^n} id_{S^1}, \ldots \right\}.$$

Since $f^{2n-1}_1 = g^n$ for every n, we have $s(f_{1,\infty}, 2n, \varepsilon) = s(g, n, \varepsilon)$ and therefore $P(f_{1,\infty}, \varphi) = h(f_{1,\infty}) \ge \frac{1}{2} h(g) = \frac{1}{2} \log 2$. On the other hand, $f^2_{1,\infty} = \{f^2_1, f^2_3, \cdots, f^2_{2n-1}, \cdots\}$, where for any $n \in N$ and for any $x \in I$, $f^2_{2n-1}(x) \le \frac{1}{2^n}$. Therefore $\limsup\limits_{n\to\infty} \frac{1}{n} \log s(f^2_{1,\infty}, n, \varepsilon) = 0$ for every $\varepsilon > 0$ and so $P(f^2_{1,\infty}, S_2\varphi) = h(f^2_{1,\infty}) = 0$. Thus $h(f^2_{1,\infty}) < 2h(f_{1,\infty})$, i.e. $P(f^2_{1,\infty}, S_2\varphi) < 2P(f_{1,\infty}, \varphi)$.

So if we wish to have the equality instead of the inequality in Theorem 3.1, we need additional assumptions. We present here one result of this kind, we restrict ourselves to compact metric spaces and sequences of equicontinuous maps.

Theorem 3.2 *Let $f_{1,\infty}: X \to X$ be a sequence of equicontinuous self-maps of the compact metric space X. $P(f_{1,\infty}^k, S_k\varphi) = kP(f_{1,\infty}, \varphi)$ (here $(S_k\varphi)(x) = \sum_{i=0}^{k-1} \varphi(f_1^i(x))$) for any $k \geq 1$.*

Proof For $k = 1$ this is trivial. Take any $k \geq 2$. In view of Theorem 3.1 it suffices to prove that $P(f_{1,\infty}^k, S_k\varphi) \geq kP(f_{1,\infty}, \varphi)$. To this end, for every $\varepsilon > 0$ take $\delta(\varepsilon) \geq \varepsilon$ such that $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$ and $d(f_i^m(x), f_i^m(y)) \leq \delta(\varepsilon)$ whenever $i \geq 1, m \in \{1, 2, \cdots, k-1\}$ and $d(x, y) \leq \varepsilon$. Take any positive integer n, then any $(nk, \delta(\varepsilon))$-separated set for $f_{1,\infty}$ is (n, ε)-separated set for $f_{1,\infty}^k$ and so $P_n(f_{1,\infty}^k, S_k\varphi, \delta(\varepsilon)) \geq P_{nk}(f_{1,\infty}, \varphi, \varepsilon)$. Therefore $P(f_{1,\infty}^k, S_k\varphi) \geq kP(f_{1,\infty}, \varphi)$. $\square$

In the sequel, let us consider the following situation: (X, d) and (Y, ρ) are compact metric spaces, $f_{1,\infty}$ is a sequence of continuous maps from X into itself and $g_{1,\infty}$ is a sequence of continuous maps from Y into itself.

Kolyada and Snoha proved the topological entropy of sequence of continuous maps is invariant with equiconjugacy [3]. Now we mainly show the topological pressure of sequence of continuous maps is invariant with equiconjugacy.

Suppose that $\pi_{1,\infty}$ is a sequence of continuous maps from X into Y such that $\pi_{i+1} \circ f_i = g_i \circ \pi_i$ for every $i \geq 1$. There are two special cases when we can compare the pressure of $f_{1,\infty}$ and $g_{1,\infty}$. They are the following.

(i) When $\pi_{1,\infty}$ is a sequence of equicontinuous surjective (i.e., onto) maps from X onto Y. In this case we say that $\pi_{1,\infty}$ topologically equisemiconjuates $f_{1,\infty}$ with $g_{1,\infty}$, $\pi_{1,\infty}$is a topological equisemiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and the dynamical systems $(X, f_{1,\infty})$ is topologically equisemiconjugate with $(Y, g_{1,\infty})$. The system $(Y, g_{1,\infty})$ is an equifactor of $(X, f_{1,\infty})$.

(ii) When $\pi_{1,\infty}$ is an equicontinuous sequence of homeomorphisms such that the sequence $\pi_{1,\infty}^{-1} = \{\pi_i^{-1}\}_{i=1}^{\infty}$ of inverse homeomorphisms is also equicontinuous. In this case we say that $\pi_{1,\infty}$ topologically equiconjugates $f_{1,\infty}$ with $g_{1,\infty}$, $\pi_{1,\infty}$ is a topological equiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and the dynamical systems $(X, f_{1,\infty})$ is topologically equiconjugate with $(Y, g_{1,\infty})$.

Theorem 3.3 *Let (X, d) and (Y, ρ) be compact metric spaces, $f_{1,\infty}$ be a sequence of continuous maps from X into itself and $g_{1,\infty}$ be a sequence of continuous maps from Y into itself. If the system $(X, f_{1,\infty})$ is topologically equisemiconjugate with $(Y, g_{1,\infty})$ (denote the equisemiconjugacy by $\pi_{1,\infty}$) then*

$$P(g_{1,\infty}, \varphi) \leq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty}),$$

for any $\varphi \in C(Y, R)$.

Proof Since $\pi_{1,\infty}$ is a sequence of equicontinuous maps from X to Y, given $\varepsilon > 0$ there exists $\varepsilon > \delta(\varepsilon) > 0$ such that if $\rho(\pi_i(x), \pi_i(y)) > \varepsilon$ for some $i \geq 1$, then $d(x, y) > \delta(\varepsilon)$. Let $F \subset Y$ be a $(n, \varepsilon, g_{1,\infty}, \rho)$-separated set, then $\pi_1^{-1}(F)$ is an $(n, \delta(\varepsilon), f_{1,\infty}, d)$-separated set. Thus

$$\sum_{x \in F} e^{\varphi(x)+\varphi(g_1(x))+\cdots+\varphi(g_1^{n-1}(x))} = \sum_{y \in \pi_1^{-1}(F)} e^{\varphi(\pi_1(y))+\varphi(\pi_1 f_1(y))+\cdots+\varphi(\pi_1 f_1^{n-1}(y))}.$$

Therefore $P(g_{1,\infty},\varphi,\varepsilon) \le P(f_{1,\infty},\varphi\circ\pi_{1,\infty},\delta(\varepsilon))$. It follows that

$$P(g_{1,\infty},\varphi) \le P(f_{1,\infty},\varphi\circ\pi_{1,\infty}).$$

□

Corollary 3.1 *Let (X,d) and (Y,ρ) be compact metric spaces, $f_{1,\infty}$ be a sequence of continuous maps from X into itself and $g_{1,\infty}$ is a sequence of continuous maps from Y into itself. If the system $(X,f_{1,\infty})$ is topologically equiconjugate with $(Y,g_{1,\infty})$ then*

$$P(g_{1,\infty},\varphi) = P(f_{1,\infty},\varphi\circ\pi_{1,\infty}).$$

Proof Denote the conjugacy by $\pi_{1,\infty}$. We have $P(g_{1,\infty},\varphi) \le P(f_{1,\infty},\varphi\circ\pi_{1,\infty})$ since $\pi_{1,\infty}$ is a semiequiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and $P(g_{1,\infty},\varphi) \ge P(f_{1,\infty},\varphi\circ\pi_{1,\infty})$ since $\pi_{1,\infty}^{-1}$ is a semiequiconjugacy between $g_{1,\infty}$ and $f_{1,\infty}$. □

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Strange Attractors and Classical Stability Theory

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Abstract: Definitions of global attractor, B-attractor and weak attractor are introduced. Relationships between Lyapunov stability, Poincare stability and Zhukovsky stability are considered. Perron effects are discussed. Stability and instability criteria by the first approximation are obtained. Lyapunov direct method in dimension theory is introduced. For the Lorenz system necessary and sufficient conditions of existence of homoclinic trajectories are obtained.

Keywords: *Attractor, instability, Lyapunov exponent, stability, Poincaré section, Hausdorff dimension, Lorenz system, homoclinic bifurcation.*

Mathematics Subject Classification (2000): 34C28, 34D45, 34D20.

1 Introduction

In almost any solid survey or book on chaotic dynamics, one encounters notions from classical stability theory such as Lyapunov exponent and characteristic exponent. But the effect of sign inversion in the characteristic exponent during linearization is seldom mentioned. This effect was discovered by Oscar Perron [1], an outstanding German mathematician. The present survey sets forth Perron's results and their further development, see [2]–[4]. It is shown that Perron effects may occur on the boundaries of a flow of solutions that is stable by the first approximation. Inside the flow, stability is completely determined by the negativeness of the characteristic exponents of linearized systems.

It is often said that the defining property of strange attractors is the sensitivity of their trajectories with respect to the initial data. But how is this property connected with the classical notions of instability? For continuous systems, it was necessary to remember the almost forgotten notion of Zhukovsky instability. Nikolai Egorovich Zhukovsky, one of the founders of modern aerodynamics and a prominent Russian scientist, introduced

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his notion of stability of motion in 1882, see [5, 6] — ten years before the publication of Lyapunov's investigations [7]. The notion of Zhukovsky instability is adequate to the sensitivity of trajectories with respect to the initial data for continuous dynamical systems. In this survey we consider the notions of instability according to Zhukovsky, Poincaré, and Lyapunov, along with their adequacy to the sensitivity of trajectories on strange attractors with respect to the initial data.

In order to investigate Zhukovsky stability, a new research tool — a moving Poincaré section — is introduced. With the help of this tool, extensions of the widely-known theorems of Andronov–Witt and Demidovic are carried out.

At the present time, the problem of justifying nonstationary linearizations for complicated, nonperiodic motions on strange attractors bears a striking resemblance to the situation that occurred 120 years ago.

J.C. Maxwell [8] and I.A. Vyshnegradskii [9], the founders of automatic control theory, courageously used linearization in a neighborhood of stationary motions, leaving the justification of such linearization to H. Poincaré [10] and A.M. Lyapunov [7]. Now many specialists in chaotic dynamics believe that the positiveness of the largest characteristic exponent of a linear system of the first approximation implies the instability of solutions of the original system. Moreover, there is a great number of computer experiments in which various numerical methods for calculating characteristic exponents and Lyapunov exponents of linear systems of the first approximation are used. As a rule, authors largely ignore the justification of the linearization procedure and use the numerical values of exponents thus obtained to construct various numerical characteristics of attractors of the original nonlinear systems (Lyapunov dimensions, metric entropies, and so on). Sometimes computer experiments serve as arguments for partial justification of the linearization procedure. For example, computer experiments in [11, 12] show the coincidence of the Lyapunov and Hausdorff dimensions of the attractors of Henon, Kaplan–Yorke, and Zaslavskii. But for B-attractors of Henon and Lorenz, such coincidence does not hold, see [13, 14]).

So linearizations along trajectories on strange attractors require justification. This problem gives great impetus to the development of the nonstationary theory of instability by the first approximation. The present survey describes the contemporary state of the art of the problem of justifying nonstationary linearizations.

The method of Lyapunov functions — Lyapunov's so-called direct method — is an efficient research device in classical stability theory. It turns out that even in the dimension theory of strange attractors one can progress by developing analogs of this method. This interesting line of investigation is also discussed in the present survey.

When the parameters of a dynamical system are varied, the structure of its minimal global attractor can change as well. Such changes are the subject of bifurcation theory. Here we describe one of these, namely the homoclinic bifurcation.

The first important results concerning homoclinic bifurcation in dissipative systems were obtained in 1933 by the outstanding Italian mathematician Francesco Tricomi [15]. Here we give Tricomi's results along with similar theorems for the Lorenz system.

For the Lorenz system, necessary and sufficient conditions for the existence of homoclinic trajectories are obtained.

2 Definitions of Attractors

The attractor of a dynamical system is an attractive closed invariant set in its phase space.

Consider the dynamical systems generated by the differential equations

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1,\ x \in \mathbb{R}^n, \tag{2.1}$$

and by the difference equations

$$x(t+1) = f\left(x(t)\right), \quad t \in \mathbb{Z},\ x \in \mathbb{R}^n. \tag{2.2}$$

Here $\mathbb{R}^n$ is a Euclidean space, $\mathbb{Z}$ is the set of integers, and $f(x)$ is a vector-function: $\mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.1. We say that (2.1) or (2.2) *generates a dynamical system* if for any initial data $x_0 \in \mathbb{R}^n$ the trajectory $x(t, x_0)$ is uniquely determined for $t \in [0, +\infty)$. Here $x(0, x_0) = x_0$.

It is well known that the solutions of dynamical system (2.1) satisfy the semigroup property

$$x(t+s, x_0) = x\left(t, x(s, x_0)\right) \tag{2.3}$$

for all $t \geq 0$, $s \geq 0$.

For equation (2.1) on $[0, +\infty)$ there are many existence and uniqueness theorems [16]–[19] that can be used for determining the corresponding dynamical system with the phase space $\mathbb{R}^n$. The partial differential equations, generating dynamical systems with different infinite-dimensional phase spaces, can be found in [20]–[24]. The classical results of the theory of dynamical systems with a metric phase space are given in [25].

For (2.2) it is readily shown that in all cases the trajectory, defined for all $t = 0, 1, 2, \ldots$, satisfying (2.3), and having initial condition x_0, is unique. Thus (2.2) always generates a dynamical system with phase space $\mathbb{R}^n$.

A dynamical system generated by (2.1) is called *continuous*. Equation (2.2) generates a *discrete dynamical system.*

The definitions of attractors are, as a rule, due to [14, 23, 24, 26].

Definition 2.2. We say that K *is invariant* if $x(t, K) = K$, $\forall t \geq 0$. Here

$$x(t, K) = \{x(t, x_0) \mid x_0 \in K\}.$$

Definition 2.3. We say that the invariant set K is *locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of K the relation

$$\lim_{t \to +\infty} \rho\left(K, x(t, x_0)\right) = 0, \quad \forall\, x_0 \in K(\varepsilon)$$

is satisfied. Here $\rho(K, x)$ is the distance from the point x to the set K, defined by

$$\rho(K, x) = \inf_{z \in K} |z - x|.$$

Recall that $|\cdot|$ is a Euclidean norm in $\mathbb{R}^n$, and $K(\varepsilon)$ is the set of points x such that $\rho(K, x) < \varepsilon$.

Definition 2.4. We say that the invariant set K is *globally attractive* if

$$\lim_{t \to +\infty} \rho\big(K, x(t, x_0)\big) = 0, \quad \forall\, x_0 \in \mathbb{R}^n.$$

Definition 2.5. We say that the invariant set K is *uniformly locally attractive* if for a certain ε-neighborhood $K(\varepsilon)$ of it and for any $\delta > 0$ and bounded set B there exists $t(\delta, B) > 0$ such that

$$x\big(t, B \cap K(\varepsilon)\big) \subset K(\delta), \quad \forall t \geq t(\delta, B).$$

Here

$$x\big(t, B \cap K(\varepsilon)\big) = \{x(t, x_0) \mid x_0 \in B \cap K(\varepsilon)\}.$$

Definition 2.6. We say that the invariant set K is *uniformly globally attractive* if for any $\delta > 0$ and bounded set $B \subset \mathbb{R}^n$ there exists $t(\delta, B) > 0$ such that

$$x(t, B) \subset K(\delta), \quad \forall t \geq t(\delta, B).$$

Definition 2.7. We say that the invariant set K is *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x(t, K(\delta)) \subset K(\varepsilon), \quad \forall t \geq 0.$$

Note that if K consists of one trajectory, then the last definition coincides with the classical definitions of the Lyapunov stability of solution. If such K is locally attractive, then we have asymptotic stability in the sense of Lyapunov.

Definition 2.8. We say that K is

1) an *attractor* if it is an invariant closed and locally attractive set;

2) a *global attractor* if it is an invariant closed and globally attractive set;

3) a *B-attractor* if it is an invariant, closed, and uniformly locally attractive set;

4) a *global B-attractor* if it is an invariant, closed, and uniformly globally attractive set.

A trivial example of an attractor is the whole phase set $\mathbb{R}^n$ if the trajectories are defined for all $t \geq 0$. This shows that it is sensible to introduce the notion of a *minimal attractor*, namely the minimal invariant set possessing the attractive property.

We give the simplest examples of attractors.

Example 2.1. Consider the equations of pendulum motion:

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \beta \sin\theta, \end{aligned} \tag{2.4}$$

where α and β are positive. The trajectories have a well-known asymptotic behavior (Figure 2.1).

Any solution of (2.4) tends to a certain equilibrium as $t \to +\infty$. Therefore the minimal global attractor of (2.4) is a stationary set.

Consider now a ball B of small radius centered on the separatrix of the saddle. As $t \to +\infty$ the image $x(t, B)$ of this small ball tends to the set consisting of a saddle equilibrium and of two separatrices, leaving this point and tending to an asymptotically stable equilibrium (Figure 2.2) as $t \to +\infty$.

Thus, a global minimal B-attractor is a union of a stationary set and the separatrices, leaving the saddle points (unstable manifolds of the saddle points) (Figure 2.3).

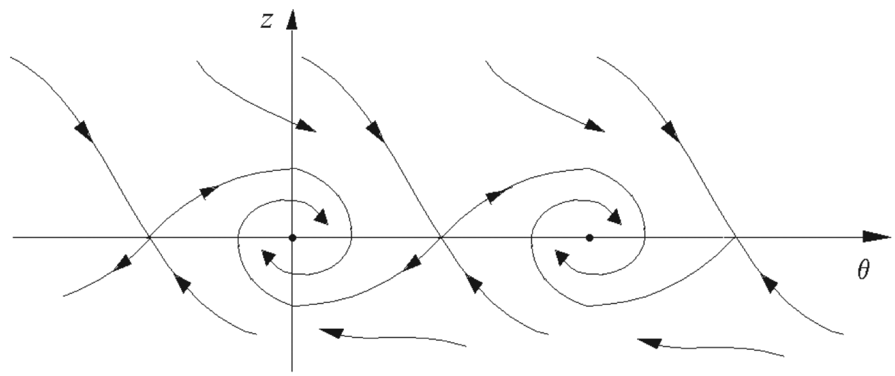

Figure 2.1:

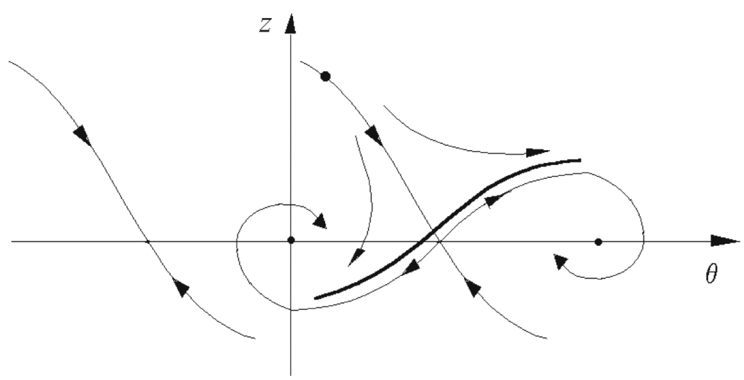

Figure 2.2:

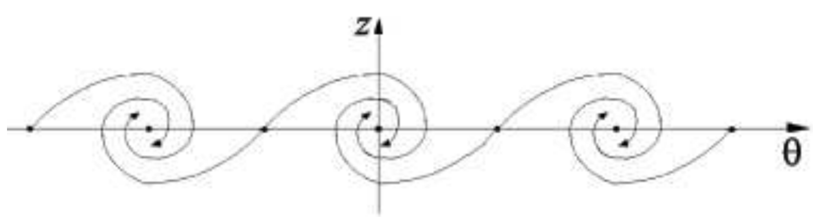

Figure 2.3:

In more general situations the B-attractor involves the unstable manifolds of saddle (hyperbolic) points. This fact is often used for estimating the topological dimension of attractors from below [20]. □

We remark that the natural generalization of the notion of attractor is to weaker requirements of attraction: on sets of positive Lebesgue measure, almost everywhere, and so on. As an illustration of such an approach we give a definition of weak attractor [26].

Definition 2.9. We say that K is a *weak attractor* if K is an invariant closed set for which there exists a set of positive Lebesgue measure $U \subset \mathbb{R}^n$ satisfying the following relation:

$$\lim_{t\to+\infty} \rho(K, x(t, x_0)) = 0, \quad \forall x_0 \in U.$$

Note that for each concrete system it is necessary to detail the set U.

3 Strange Attractors and the Classical Definitions of Instability

One of the basic characteristics of a strange attractor is the sensitivity of its trajectories to the initial data.

We consider the correlation of such "sensitivity" with a classical notion of instability. We recall first the basic definitions of stability.

Consider the system

$$\frac{dx}{dt} = F(x, t), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where $F(x, t)$ is a continuous vector-function, and

$$x(t + 1) = F(x(t), t), \quad t \in \mathbb{Z}, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Denote by $x(t, t_0, x_0)$ the solution of (3.1) or (3.2) with initial data t_0, x_0:

$$x(t_0, t_0, x_0) = x_0.$$

Definition 3.1. The solution $x(t, t_0, x_0)$ is said to be *Lyapunov stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta(\varepsilon, t_0)$ such that

1. all the solutions $x(t, t_0, y_0)$, satisfying the condition

$$|x_0 - y_0| \leq \delta,$$

   are defined for $t \geq t_0$,

2. for these solutions the inequality

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \leq \varepsilon, \quad \forall t \geq t_0$$

   is valid.

If $\delta(\varepsilon, t_0)$ is independent of t_0, the Lyapunov stability is called *uniform.*

Definition 3.2. The solution $x(t, t_0, x_0)$ is said to be *asymptotically Lyapunov stable* if it is Lyapunov stable and for any $t_0 \geq 0$ there exists $\Delta(t_0) > 0$ such that the solution $x(t, t_0, y_0)$, satisfying the condition $|x_0 - y_0| \leq \Delta$, has the following property:

$$\lim_{t\to+\infty} |x(t, t_0, x_0) - x(t, t_0, y_0)| = 0.$$

Definition 3.3. The solution $x(t, t_0, x_0)$ is said to be *Krasovsky stable* if there exist positive numbers $\delta(t_0)$ and $R(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t, t_0, y_0)$ is defined for $t \ge t_0$ and satisfies

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le R(t_0)|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ and R are independent of t_0, then Krasovsky stability is called *uniform*.

Definition 3.4. The solution $x(t, t_0, x_0)$ is said to be *exponentially stable* if there exist the positive numbers $\delta(t_0)$, $R(t_0)$, and $\alpha(t_0)$ such that for any y_0, satisfying the condition

$$|x_0 - y_0| \le \delta(t_0),$$

the solution $x(t, t_0, y_0)$ is defined for all $t \ge t_0$ and satisfies

$$|x(t, t_0, x_0) - x(t, t_0, y_0)| \le R(t_0) \exp(-\alpha(t_0)(t- t_0))|x_0 - y_0|, \quad \forall t \ge t_0.$$

If δ, R, and α are independent of t_0, then exponential stability is called *uniform*.

Consider now dynamical systems (3.1) and (3.2). We introduce the following notation:

$$L^+(x_0) = \big\{x(t, x_0) \mid \ t \in [0, +\infty)\big\}.$$

Definition 3.5. The trajectory $x(t, x_0)$ of a dynamical system is said to be *Poincaré stable* (or *orbitally stable*) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the relation

$$\rho\big(L^+(x_0),\, x(t, y_0)\big) \le \varepsilon, \quad \forall\, t \ge 0$$

is satisfied. If, in addition, for a certain number δ_0 and for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta_0$, the relation

$$\lim_{t \to +\infty} \rho\big(L^+(x_0),\, x(t, y_0)\big) = 0$$

holds, then the trajectory $x(t, x_0)$ is said to be *asymptotically Poincaré stable* (or *asymptotically orbitally stable*).

Note that for continuous dynamical systems we have $t \in \mathbb{R}^1$, and for discrete dynamical systems $t \in \mathbb{Z}$.

We now introduce the definition of Zhukovsky stability for continuous dynamical systems. For this purpose we must consider the following set of homeomorphisms:

$$\mathrm{Hom} = \{\tau(\cdot) \mid \tau : [0, +\infty) \to [0, +\infty), \tau(0) = 0\}.$$

The functions $\tau(t)$ from the set Hom play the role of the reparametrization of time for the trajectories of system (3.1).

Definition 3.6 [5, 6, 27, 28]. The trajectory $x(t, x_0)$ of system (3.1) is said to be *Zhukovsky stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the function $\tau(\cdot) \in \mathrm{Hom}$ can be found such that the inequality

$$|x(t, x_0) - x(\tau(t), y_0)| \le \varepsilon, \quad \forall t \ge 0$$

is valid. If, in addition, for a certain number $\delta_0 > 0$ and any y_0 from the ball $\{y|\,|x_0-y| \le \delta_0\}$ the function $\tau(\cdot) \in \mathrm{Hom}$ can be found such that the relation

$$\lim_{t\to+\infty} |x(t,x_0) - x(\tau(t),y_0)| = 0$$

holds, then the trajectory $x(t,x_0)$ is *asymptotically stable in the sense of Zhukovsky.*

This means that Zhukovsky stability is Lyapunov stability for the suitable reparametrization of each of the perturbed trajectories.

Recall that, by definition, Lyapunov instability is the negation of Lyapunov stability. Analogous statements hold for Krasovsky, Poincaré, and Zhukovsky instability.

The following obvious assertions can be formulated.

Proposition 3.1. *For continuous dynamical systems, Lyapunov stability implies Zhukovsky stability, and Zhukovsky stability implies Poincaré stability.*

Proposition 3.2. *For discrete dynamical systems, Lyapunov stability implies Poincaré stability.*

Proposition 3.3. *For equilibria, all the above definitions due to Lyapunov, Zhukovsky, and Poincaré are equivalent.*

Proposition 3.4. *For periodic trajectories of discrete dynamical systems with continuous $f(x)$, the definitions of Lyapunov and Poincaré stability are equivalent.*

Proposition 3.5. *For the periodic trajectories of continuous dynamical systems with differentiable $f(x)$, the definitions of Poincaré and Zhukovsky stability are equivalent.*

Also well known are examples of periodic trajectories of continuous systems that happen to be Lyapunov unstable but Poincaré stable.

Now we proceed to compare the definitions given above with the effect of trajectory sensitivity to the initial data for strange attractors.

Lyapunov instability cannot characterize the "mutual repulsion" of continuous trajectories due to small variations in initial data. Neither can Poincaré instability characterize this repulsion. In this case, the perturbed solution can leave the ε-neighborhood of a certain segment of the unperturbed trajectory (the effect of repulsion) while simultaneously entering the ε-neighborhood of another segment (the property of Poincaré stability). Thus, mutually repulsive trajectories can be Poincaré stable. Let us consider these effects in more detail.

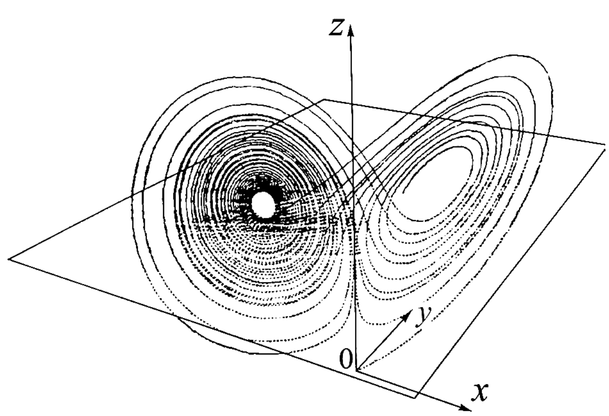

Figure 3.1: Unstable manifold of the saddle of the Lorenz system. The first fifty turns.

In computer experiments it often happens that the trajectories, situated on the unstable manifold of a saddle singular point, everywhere densely fill the B-attractor (or that portion of it consisting of the bounded trajectories). This can be observed on the B-attractor of the Lorenz system [29]

$$\begin{aligned}\dot{x} &= -\sigma(x-y),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned}$$

where $\sigma = 10$, $r = 28$, and $b = 8/3$ (Figure 2.1).

Example 3.1. Consider the linearized equations of two decoupled pendula:

$$\begin{aligned}\dot{x}_1 &= y_1, \quad \dot{y}_1 = -\omega_1^2 x_1,\\ \dot{x}_2 &= y_2, \quad \dot{y}_2 = -\omega_2^2 x_3.\end{aligned} \tag{3.3}$$

The solutions are

$$\begin{aligned}x_1(t) &= A\sin(\omega_1 t + \varphi_1(0)),\\ y_1(t) &= A\omega_1\cos(\omega_1 t + \varphi_1(0)),\\ x_2(t) &= B\sin(\omega_2 t + \varphi_2(0)),\\ y_2(t) &= B\omega_2\cos(\omega_2 t + \varphi_2(0)).\end{aligned}$$

For fixed A and B, the trajectories of system are situated on two-dimensional tori

$$\omega_1^2 x_1^2 + y_1^2 = A^2, \qquad \omega_2^2 x_2^2 + y_2^2 = B^3.$$

When ω_1/ω_2 is irrational, the trajectories are everywhere densely situated on the tori for any initial data $\varphi_1(0)$ and $\varphi_2(0)$.

This implies asymptotic Poincaré stability of the trajectories of the dynamical system on tori. However, the motion of the points $x(t, x_0)$ and $x(t, y_0)$ along the trajectories occurs in such a way that they do not tend toward each other as $t \to +\infty$. Neither are the trajectories "pressed" toward each other. Hence the intuitive conception of asymptotic stability as a convergence of objects toward each other is in contrast to the formal definition of Poincaré.

It is clear that a similar effect is lacking for the notion of Zhukovsky stability: in the case under consideration, asymptotic Zhukovsky stability does not occur. □

Example 3.2. We reconsider the dynamical system (3.3) with ω_1/ω_2 irrational. Change the flow of trajectories on the tori as follows. Cut the toroidal surface along a certain segment of the fixed trajectory from the point z_1 to the point z_2. Then the surface is stretched diffeomorphically along the torus so that a cut is mapped into the circle with the fixed points z_1 and z_2 (Fig. 3.2). Denote by H the interior of the circle.

Change the dynamical system so that z_1 and z_2 are saddle stationary points and the semicircles connecting z_1 and z_2 are heteroclinic trajectories, tending as $t \to +\infty$ and $t \to -\infty$ to z_2 and z_1, respectively (Figure 3.2).

Outside the "hole" H, after the diffeomorphic stretching, the disposition of trajectories on the torus is the same.

Consider the behavior of the system trajectories from the Poincaré and Zhukovsky points of view.

Outside the hole H, the trajectories are everywhere dense on torus. They are therefore, as before, asymptotically Poincaré stable.

Now we consider a certain δ-neighborhood of the point z_0, situated on the torus and outside the set H. The trajectory leaving z_0 is either everywhere dense or coincides with

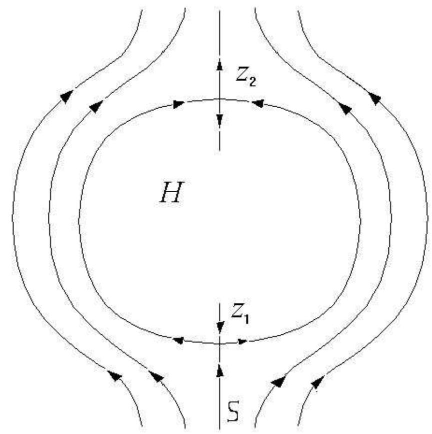

Figure 3.2:

the separatrix S of the saddle z_1, tending to z_1 as $t \to +\infty$ (Figure 3.2). Then there exists a time t such that some trajectories, leaving the δ-neighborhood of z_0, are situated in a small neighborhood of z_1 to the right of the separatrix S. At time t the remaining trajectories, leaving this neighborhood of z_0, are situated to the left of S. It is clear that in this case the trajectories, situated to the right and to the left of S, envelope the hole H on the right and left, respectively. It is also clear that these trajectories are repelled from each other; hence, the trajectory leaving z_0 is Zhukovsky unstable.

Thus, a trajectory can be asymptotically Poincaré stable and Zhukovsky unstable.

This example shows that the trajectories are sensitive to the initial data and can diverge considerably after some time. The notion of Zhukovsky instability is adequate to such a sensitivity.

Note that the set of such sensitive trajectories is situated on the smooth manifold, named “a torus minus the hole H”. Thus, the bounded invariant set of trajectories, which are sensitive to the initial data, do not always have a noninteger Hausdorff dimension or the structure of the Cantor set.

Hence, from among the classical notions of instability for studying strange attractors, the most adequate ones are Zhukovsky instability (in the continuous case) and Lyapunov instability (in the discrete case).

4 Characteristic Exponents and Lyapunov Exponents

Definition 4.1. The number (or the symbol $+\infty, -\infty$), defined by the formula

$$\lambda = \limsup_{t \to +\infty} \frac{1}{t} \ln |f(t)|,$$

is called a *characteristic exponent* of the vector-function $f(t)$.

Definition 4.2. The characteristic exponent λ of the vector-function $f(t)$ is said to be *sharp* if there exists the following finite limit:

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \ln |f(t)|.$$

The value

$$\lambda = \liminf_{t \to +\infty} \frac{1}{t} \ln |f(t)|$$

is often called a *lower characteristic exponent* of $f(t)$.

Consider the linear system

$$\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n, \tag{4.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0, +\infty)$. Let $X(t) = (x_1(t), \dots, x_n(t))$ be a fundamental matrix of (4.1) (i.e. $\det X(0) \neq 0$). It is well known that under the above conditions the characteristic exponents λ_j of the solutions $x_j(t)$ are numbers.

Definition 4.3. Fundamental matrix $X(t)$ is said to be *normal* if the sum $\sum\limits_{j=1}^{n} \lambda_j$ of the characteristic exponents of the vector-functions $x_j(t)$ is minimal in comparison to other fundamental matrices.

The following substantial and almost obvious results are well-known.

Theorem 4.1. *For all normal fundamental matrices* $(x_1(t), \dots, x_n(t))$ *the number of solutions* $x_j(t)$ *having the same characteristic exponent is the same.*

We can now introduce the following definitions.

Definition 4.4. The set of characteristic exponents $\lambda_1, \dots, \lambda_n$ of the solutions $x_1(t), \dots, x_n(t)$ of certain normal fundamental matrices $X(t)$ is called the *complete spectrum* of linear system (4.1), and the numbers λ_j are called the *characteristic exponents* of (4.1).

Thus, any normal fundamental matrix realizes the complete spectrum of the system (4.1).

In the sequel, by $\Sigma = \sum\limits_{j=1}^{n} \lambda_j$ is denoted the sum of characteristic exponents of system (4.1).

The Lyapunov inequality

$$\Sigma \geq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau \tag{4.3}$$

is well known. Here Tr is a spur of the matrix A.

Definition 4.5. If the relation

$$\Sigma = \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau$$

is satisfied, then system (4.1) is called *regular.*

It is well-known that each characteristic exponent of a regular system is sharp.

Definition 4.6. The number

$$\Gamma = \Sigma - \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{Tr} A(\tau)\, d\tau$$

is called the *coefficient of irregularity* for (4.1).

We assume further that $\lambda_1 \geq \cdots \geq \lambda_n$. The number λ_1 is called a *largest characteristic exponent*.

Let $X(t)$ be a fundamental matrix of system (4.1). We introduce the singular values $\alpha_1(X(t)) \geq \cdots \geq \alpha_n(X(t)) \geq 0$ of $X(t)$. Recall that the singular values $\alpha_j(X(t))$ of a matrix $X(t)$ are square roots of eigenvalues of the matrix $X(t)^*X(t)$. Geometrically, the $\alpha_j(X(t))$ coincide with the principal axes of the ellipsoid $X(t)B$, where B is the unit ball.

Definition 4.7 [22]. The *Lyapunov exponent* μ_j is the number

$$\mu_j = \limsup_{t\to+\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

We say that μ_j is *sharp* if there exists the finite limit

$$\lim_{t\to+\infty} \frac{1}{t} \ln \alpha_j(X(t)).$$

Proposition 4.1. *The largest characteristic exponent λ_1 and the Lyapunov exponent μ_1 coincide.*

5 Perron Effects

In 1930, O. Perron [1] showed that the negativity of the largest characteristic exponent of the system of the first approximation does not always imply the stability of the zero solution of the original system. In addition, in an arbitrary small neighborhood of zero the solutions of the original system with positive characteristic exponent can exist. Perron's results impressed the specialists in the theory of motion stability.

The effect of sign reversal for the characteristic exponent of solutions of the system of the first approximation, and of the original system under the same initial data, we shall call the *Perron effect*.

We cite the outstanding result of Perron. Consider a system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1, \\ \frac{dx_2}{dt} &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_2 + x_1^2, \end{aligned} \tag{5.1}$$

where a satisfies

$$1 < 2a < 1 + \frac{1}{2}\exp(-\pi). \tag{5.2}$$

The solution of the equation of the first approximation takes the form

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_2(0). \end{aligned}$$

It is obvious that for the system of the first approximation under condition (5.2) we have $\lambda_1 < 0$.

Now we write the solution of (5.1):

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]\Big(x_2(0)+ \\ &+x_1(0)^2 \int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau\Big). \end{aligned} \tag{5.3}$$

Letting $t = \exp[(2k + \frac{1}{2})\pi] - 1$, where k is an integer, we obtain

$$\begin{aligned} &\exp[(t+1)\sin(\ln(t+1)) - 2at] = e(\exp[(1- \\ &- 2a)t]),\ (1+t)e^{-\pi} - 1 > 0, \end{aligned}$$

and

$$\begin{aligned} &\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]d\tau > \\ &\quad > \int_{f(k)}^{g(k)} \exp[-(\tau+1)\sin(\ln(\tau+1))]\, d\tau \\ &\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\right]\, d\tau \\ &\quad > \int_{f(k)}^{g(k)} \exp\left[\frac{1}{2}(\tau+1)\exp(-\pi)\right]\, d\tau \\ &\quad = \exp\left[\frac{1}{2}(t+1)\exp(-\pi)\right](t+ \\ &\quad + 1)\left(\exp\left(-\frac{2\pi}{3}\right) - \exp(-\pi)\right), \end{aligned}$$

where

$$\begin{aligned} f(k) &= (1+t)\exp[-\pi] - 1, \\ g(k) &= (1+t)\exp\left[-\frac{2\pi}{3}\right] - 3. \end{aligned}$$

This implies the estimate

$$\begin{aligned} &\exp[(t+1)\sin(\ln(t+1)) - 2at]\int_0^t \exp[-(\tau+ \\ &+ 1)\sin(\ln(\tau+1))]d\tau \\ &> \exp\left[\frac{1}{2}(2+\exp(-\pi))\right]\left(\exp\left(-\frac{2\pi}{3}\right) - \right. \\ &\left. - \exp(-\pi)\right)\cdot \exp\left[\left(1 - 2a + \frac{1}{2}\exp(-\pi)\right)t\right]. \end{aligned} \tag{5.4}$$

This and condition (5.2) imply that the characteristic exponent λ of the solutions of system (5.1) for $x_1(0) \neq 0$ is positive.

Thus, all characteristic exponents of the system of the first approximation are negative, and almost all solutions of the original system (5.1) tend exponentially to infinity as $k \to +\infty$. □

We consider the similar effect of the sign reversal of characteristic exponents but "on the contrary", namely the solution of the system of the first approximation has a positive characteristic exponent while the solution of the original system with the same initial data has a negative exponent [2, 3, 4]. Consider a system

$$\begin{aligned} &\dot{x}_1 = -ax_1, \\ &\dot{x}_2 = -2ax_2, \\ &\dot{x}_3 = [\sin(\ln(t+1)) + \cos(\ln(t+1)) - \\ &-2a]x_3 + x_2 - x_1^2, \end{aligned} \tag{5.5}$$

on the invariant manifold

$$M = \{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2\}.$$

Here a satisfies (5.2). The solutions of (5.5) on the set M are

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[-2at]x_2(0), \\ x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]x_3(0), \\ x_1(0)^2 &= x_2(0). \end{aligned}$$

Obviously, these have negative characteristic exponents.

Consider now the system of the first approximation in the neighborhood of the zero solution of system (5.5):

$$\begin{aligned} \dot{x}_1 &= -ax_1, \\ \dot{x}_2 &= -2ax_2, \\ \dot{x}_3 &= [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2a]x_3 + x_2. \end{aligned} \tag{5.6}$$

Its solutions have the form

$$\begin{aligned} x_1(t) &= \exp[-at]x_1(0), \\ x_2(t) &= \exp[-2at]x_2(0), \\ x_3(t) &= \exp[(t+1)\sin(\ln(t+1)) - 2at]\Big(x_3(0)+ \\ &\quad + x_2(0)\int_0^t \exp[-(\tau+1)\sin(\ln(\tau+1))]\,d\tau\Big). \end{aligned} \tag{5.7}$$

Comparing (5.7) with (5.3) and applying (5.4), we find that for $x_2(0) \neq 0$ the relation

$$\limsup_{t\to+\infty} \frac{1}{t} \ln|x_3(t)| > 0$$

holds. It is easily shown that for the solutions of systems (5.5) and (5.6) we have

$$(x_1(t)^2 - x_2(t))^\bullet = -2a(x_1(t)^2 - x_2(t)).$$

Then

$$x_1(t)^2 - x_2(t) = \exp[-2at](x_1(0)^2 - x_2(0)).$$

It follows that M is a global attractor for the solutions of (5.5) and (5.6). This means that the relation $x_1(0)^2 = x_2(0)$ yields $x_1(t)^2 = x_2(t)$ for all $t \in \mathbb{R}^1$ and that for any initial data we have

$$|x_1(t)^2 - x_2(t)| \leq \exp[-2at]|x_1(0)^2 - x_2(0)|.$$

Thus, systems (5.5) and (5.6) have the same global attractor M on which almost all the solutions of the system of the first approximation (5.6) have a positive characteristic exponent and all the solutions of original system (5.5) have negative characteristic exponents.

Here the Perron effect occurs on the two-dimensional manifold, namely

$$\{x_3 \in \mathbb{R}^1,\ x_2 = x_1^2 \neq 0\}.$$

To construct the exponentially stable system for which the first approximation has a positive characteristic exponent, we change (5.5) to

$$\begin{aligned} &\dot{x}_1 = F(x_1, x_2), \\ &\dot{x}_2 = G(x_1, x_2), \\ &\dot{x}_3 = [\sin\ln(t+1) + \cos\ln(t+1) - \\ &-2a]x_3 + x_2 - x_1^3. \end{aligned} \tag{5.8}$$

Here the functions $F(x_1, x_2)$ and $G(x_1, x_2)$ have the form

$$\begin{aligned} F(x_1, x_2) &= \pm 2x_2 - ax_1, \\ G(x_1, x_2) &= \mp x_1 - \varphi(x_1, x_2), \end{aligned}$$

in which case the upper sign is taken for $x_1 > 0$, $x_2 > x_1^2$ and for $x_1 < 0$, $x_2 < x_1^2$, the lower one for $x_1 > 0$, $x_2 < x_1^2$ and for $x_1 < 0$, $x_2 > x_1^2$. The function $\varphi(x_1, x_2)$ is defined as

$$\varphi(x_1, x_2) = \begin{cases} 4ax_2, & |x_2| > 2x_1^2, \\ 2ax_2, & |x_2| < 2x_1^3. \end{cases}$$

The solutions of system (5.8) are credited to A.F. Filippov [19]. Then for the given functions F and G, on the lines of discontinuity $\{x_1 = 0\}$ and $\{x_2 = x_1^2\}$ the system

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2), \\ \dot{x}_2 &= G(x_1, x_2), \end{aligned} \tag{5.9}$$

has the sliding solutions, which are defined as

$$x_1(t) \equiv 0, \quad \dot{x}_2(t) = -4ax_2(t),$$

and

$$\begin{aligned} &\dot{x}_1(t) = -ax_1(t), \quad \dot{x}_2(t) = -2ax_2(t), \\ &x_2(t) \equiv x_1(t)^2. \end{aligned}$$

In this case the solutions of system (5.9) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$ attain the curve $\{x_2 = x_1^2\}$ in a finite time, which does not exceed 2π. The phase picture of such a system is shown in Figure 5.1.

From the above it follows that for the solutions of system (5.8) with the initial data $x_1(0) \neq 0$, $x_2(0) \in \mathbb{R}^1$, $x_3(0) \in \mathbb{R}^1$ for $t \geq 2\pi$ we have the relations $F(x_1(t), x_2(t)) =$

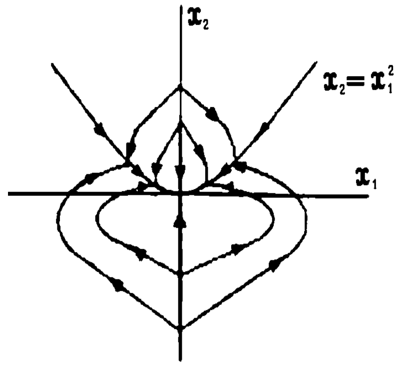

Figure 5.1:

$-ax_1(t)$, $G(x_1(t), x_2(t)) = -2ax_2(t)$. Therefore on these solutions, for $t \geq 2\pi$ system (5.6) is the system of the first approximation.

This system, as we have shown earlier, has a positive characteristic exponent. At the same time all the solutions of system (5.8) tend exponentially to zero. □

The technique considered here permits us to construct the different classes of nonlinear systems for which Perron effects occur.

6 Stability Criteria by the First Approximation

We now describe the most famous stability criteria by the first approximation for the system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{6.1}$$

Here $A(t)$ is a continuous $n \times n$ matrix bounded for $t \geq 0$, and $f(t,x)$ is a continuous vector-function, satisfying in some neighborhood $\Omega(0)$ of the point $x = 0$ the condition

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0). \tag{6.2}$$

Here κ and ν are certain positive numbers, $\nu \geq 1$.

We refer to

$$\frac{dx}{dt} = A(t)x \tag{6.3}$$

as the *system of the first approximation*. Suppose that there exist $C > 0$ and a piecewise continuous function $p(t)$ such that Cauchy matrix $X(t)X(\tau)^{-1}$ of (6.3) satisfies

$$|X(t)X(\tau)^{-1}| \leq C \exp \int_\tau^t p(s)\, ds, \quad \forall t \geq \tau \geq 0.$$

Theorem 6.1. *If condition (6.2) with $\nu = 1$ and the inequality*

$$\limsup_{t \to +\infty} \frac{1}{t} \int_0^t p(s)\, ds + C\kappa < 0$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Theorem 6.1 shows that for the equation of the first order the negativity of the characteristic exponent of the system of the first approximation implies the asymptotic Lyapunov stability of the zero solution. (Here $\nu > 1$ or $\nu = 1$ and κ is sufficiently small.)

Let us now assume that $X(t)X(\tau)^{-1}$ satisfies

$$\begin{aligned} &|X(t)X(\tau)^{-1}| \leq C \exp[-\alpha(t-\tau) + \gamma\tau], \\ &\forall t \geq \tau \geq 0, \end{aligned} \tag{6.4}$$

where $\alpha > 0$ and $\gamma \geq 0$.

Theorem 6.2 [30]. *If conditions (6.4) with $\gamma = 0$ and (6.2) with $\nu = 1$ and sufficiently small κ are valid, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.*

Theorem 6.2 results from Theorem 6.1 for $p(t) \equiv -\alpha$.

Theorem 6.3 [31, 32, 33]. *If conditions (6.4), (6.2), and the inequality*

$$(\nu - 1)\alpha - \gamma > 0 \tag{6.5}$$

hold, then the solution $x(t) \equiv 0$ of (6.1) is asymptotically Lyapunov stable.

Consider a system

$$\frac{dx}{dt} = F(x,t), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{6.6}$$

where $F(x,t)$ is a twice continuously differentiable vector-function. Suppose that for the solutions of system (6.6) with the initial data $y = x(0,y)$ from a certain domain Ω, the following condition is satisfied. The maximal singular value $\alpha_1(t,y)$ of the fundamental matrix $X(t,y)$ of the linear system

$$\frac{dz}{dt} = A(t)z \tag{6.7}$$

satisfies the inequality

$$\alpha_1(t,y) \leq \alpha(t), \quad \forall t \geq 0, \quad \forall y \in \Omega.$$

Here

$$A(t) = \left.\frac{\partial F(x,t)}{\partial x}\right|_{x=x(t,y)}$$

is the Jacobian matrix of the vector-function $F(x,t)$ on the solution $x(t,y)$, $X(0,y) = I$.

Theorem 6.4 [34]. *Let the function $\alpha(t)$ be bounded on the interval $(0, +\infty)$. Then the solution $x(t,y)$, $y \in \Omega$, is Lyapunov stable. If, in addition, we have*

$$\lim_{t \to +\infty} \alpha(t) = 0,$$

then the solution $x(t,y)$, $y \in \Omega$, is asymptotically Lyapunov stable.

Consider now the hypotheses of Theorem 6.4. The theorem establishes the asymptotic Lyapunov stability of solutions with the initial data from Ω if the corresponding equations (6.7) have negative Lyapunov exponents (or negative characteristic exponents). In this case the requirement that the negativity of Lyapunov exponents is uniform by Ω replaces the requirement in Theorem 6.3 that the coefficient of irregularity is small.

Thus, the Perron effects, considered in Section 5, are possible on the boundaries of the flow stable by the first approximation only.

7 Instability Criteria

Consider a system

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.1}$$

where the $n \times n$ matrix $A(t)$ is continuous and bounded on $[0,\infty)$. We assume that the vector-function $f(t,x)$ is continuous and in some neighborhood $\Omega(0)$ of the point $x = 0$ the inequality

$$|f(t,x)| \leq \kappa |x|^\nu, \quad \forall t \geq 0, \quad \forall x \in \Omega(0) \tag{7.2}$$

holds. Here $\kappa > 0$ and $\nu > 1$.

Consider the normal fundamental matrix

$$Z(t) = (z_1(t), \ldots, z_n(t)), \tag{7.3}$$

consisting of the linearly independent solutions $z_j(t)$ of the following linear system of the first approximation:

$$\frac{dz}{dt} = A(t)z. \tag{7.4}$$

Theorem 7.1 [35]. *If*

$$\begin{gathered}\sup_k \liminf_{t \to +\infty} \left[\frac{1}{t} \left(\int_0^t TrA(s)\ ds - \right.\right. \\ \left.\left. - \sum_{j \neq k} \ln |z_j(t)| \ \right)\right] > 0,\end{gathered} \tag{7.5}$$

then the solution $x(t) \equiv 0$ of system (7.1) is Krasovsky unstable.

The condition for Krasovsky instability (7.5) was obtained by [31] under the additional requirement of the analyticity of $f(t,x)$.

Theorem 7.2 [4]. *Assume that for some numbers $C > 0$, $\beta > 0$, and $\alpha_j < \beta$ $(j = 1, \ldots, n-1)$ the following conditions are valid:*

1. *for $n > 2$*

$$\prod_{j=1}^n |z_j(t)| \leq C \exp \int_0^t TrA(s)\, ds, \ \ \forall t \geq 0.$$

2.

$$|z_j(t)| \leq C \exp(\alpha_j (t-\tau)) |z_j(\tau)|,$$
$$\forall\, t \geq \tau \geq 0, \quad \forall j = 1, \ldots, n-1.$$

3.

$$\frac{1}{(t-\tau)} \int_\tau^t TrA(s)\, ds > \beta + \sum_{j=1}^{n-1} \alpha_j,$$
$$\forall\, t \geq \tau \geq 0.$$

Then the zero solution of system (7.1) is Lyapunov unstable.

Let us reconsider the ensemble of solutions $x(t,t_0,x_0)$ of the system

$$\frac{dx}{dt} = F(x,t), \quad t \geq 0, \quad x \in \mathbb{R}^n, \tag{7.6}$$

where $F(x,t)$ is a continuously differentiable function. Here $x_0 \in \Omega$, where Ω is a certain bounded open set in $\mathbb{R}^n$, and t_0 is a certain fixed nonnegative number.

Assume that for the fundamental matrix $X(t,t_0,x_0)$ of the system

$$\frac{dz}{dt} = \left(\frac{\partial F(x,t)}{\partial x} \bigg|_{x=x(t,t_0,x_0)} \right) z$$

with the initial data $X(t_0,t_0,x_0) = I$ and a certain vector-function $\xi(t)$ the relations

$$|\xi(t)| = 1, \quad \inf_{\Omega} |X(t,t_0,x_0)\xi(t)| \geq \alpha(t), \quad \forall t \geq t_0$$

are valid.

Theorem 7.3 [4]. *Suppose that the function $\alpha(t)$ satisfies*

$$\limsup_{t \to +\infty} \alpha(t) = +\infty.$$

Then any solution $x(t,t_0,x_0)$ with the initial data $x_0 \in \Omega$ is Lyapunov unstable.

Conclusion

Let us summarize the investigations of stability by the first approximation, considered in Sections 5–7.

Theorems 6.4 and 7.3 give a complete solution to the problem for the flows of solutions in the noncritical case when for small variations of the initial data of the original system, the system of the first approximation preserves its stability (or the instability in the certain "direction" $\xi(t)$).

Thus, here the classical problem on the stability by the first approximation of time-varying motions is completely proved in the generic case [32].

The Perron effects, described in Section 5, are possible on the boundaries of flows that are either stable or unstable by the first approximation only. From this point of view here we have a nongeneric case.

Progress in the generic case became possible since the theorem on finite increments permits us to reduce the estimate of the difference between perturbed and unperturbed solutions to the analysis of the system of the first approximation, linearized along a certain "third" solution of the original system. Such an approach renders the proof of the theorem "almost obvious".

8 Zhukovsky Stability

Zhukovsky stability is simply the Lyapunov stability of reparametrized trajectories. To study it, we may apply the arsenal of methods and devices that were developed for the study of Lyapunov stability.

The reparametrization of trajectories permits us to introduce another tool for investigation, the *moving Poincaré section*. The classical Poincaré section is the transversal $(n-1)$-dimensional surface S in the phase space $\mathbb{R}^n$, which possesses a recurring property. The latter means that for the trajectory of a dynamical system $x(t,x_0)$ with the initial data $x_0 \in S$, there exists a time instant $t = T > 0$ such that $x(T,x_0) \in S$. The transversal property means that

$$n(x)^* f(x) \neq 0, \quad \forall x \in S.$$

Here $n(x)$ is a normal vector of the surface S at the point x, and $f(x)$ is the right-hand side of the differential equation

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1, \quad x \in \mathbb{R}^n, \tag{8.1}$$

generating a dynamical system.

We now "force" the Poincaré section to move along the trajectory $x(t, x_0)$. We assume further that the vector-function $f(x)$ is twice continuously differentiable and that the trajectory $x(t, x_0)$, whose the Zhukovsky stability (or instability) will be considered, is wholly situated in a certain bounded domain $\Omega \subset \mathbb{R}^n$ for $t \geq 0$. Suppose also that $f(x) \neq 0$, $\forall x \in \overline{\Omega}$. Here $\overline{\Omega}$ is a closure of the domain Ω. Under these assumptions there exist positive numbers δ and ε such that

$$f(y)^* f(x) \geq \delta, \quad \forall y \in S(x, \varepsilon), \quad \forall x \in \overline{\Omega}.$$

Here

$$S(x, \varepsilon) = \{y \mid (y - x)^* f(x) = 0, \quad |x - y| < \varepsilon\}.$$

Definition 8.1. The set $S(x(t, x_0), \varepsilon)$ is called a *moving Poincaré section*.

Note that for small ε it is natural to restrict oneself to the family of segments of the surfaces $S(x(t, x_0), \varepsilon)$ rather than arbitrary surfaces. From this point of view a more general consideration does not give new results. It is possible to consider the moving Poincaré section more generally by introducing the set

$$S(x, q(x), \varepsilon) = \{y \mid (y - x)^* q(x) = 0, \quad |x - y| < \varepsilon\},$$

where the vector-function $q(x)$ satisfies the condition $q(x)^* f(x) \neq 0$. Such a consideration can be found in [27]. We treat the most interesting and descriptive case $q(x) \equiv f(x)$.

The classical Poincaré section allows us to clarify the behavior of trajectories using the information at their discrete times of crossing the section. Reparametrization makes it possible to organize the motion of trajectories so that at time t all trajectories are situated on the same moving Poincaré section $S(x(t, x_0), \varepsilon)$:

$$x(\varphi(t), y_0) \in S(x(t, x_0), \varepsilon). \tag{8.2}$$

Here $\varphi(t)$ is a reparametrization of the trajectory $x(t, y_0)$, $y_0 \in S(x_0, \varepsilon)$. This consideration has, of course, a local property and is only possible for t satisfying

$$|x(\varphi(t), y_0) - x(t, x_0)| < \varepsilon. \tag{8.3}$$

Let as consider system of the first approximation

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))w \tag{8.4}$$

System (8.4) has the one null characteristic exponent λ_1. Denote by $\lambda_2 \geq \cdots \geq \lambda_n$ the remaining characteristic exponents, γ is coefficient of irregularity.

Theorem 8.1.[36] *If for system (8.4) the inequality*

$$\lambda_2 + \gamma < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable.

This result generalizes the well-known Andronov–Witt theorem.

Theorem 8.2 (Andronov, Witt). *If the trajectory $x(t, x_0)$ is periodic, differs from equilibria, and for system (8.4) the inequality*

$$\lambda_2 < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable (asymptotically Poincaré stable).

Theorem 8.2 is a corollary of Theorem 8.1 since system (8.4) with the periodic matrix

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

is regular.

Recall that for periodic trajectories, asymptotic stability in the senses of Zhukovsky and Poincaré are equivalent.

The theorem of Demidovich is also a corollary of Theorem 8.3.

Theorem 8.3 [37]. *If system (8.4) is regular (i.e. $\gamma = 0$) and $\lambda_2 < 0$, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable.*

9 Lyapunov Functions in the Estimates of Attractor Dimension

Harmonic oscillations are characterized by an amplitude, period, and frequency, and periodic oscillations by a period. Numerous investigations have shown that more complex oscillations have also numerical characteristics. These are the dimensions of attractors, corresponding to ensembles of such oscillations.

The theory of topological dimension [38, 39], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor.

For investigating attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value, and on such customary objects in Euclidean space as a smooth curve, a surface, or a countable set of points, it coincides with the topological dimension. Let us proceed to the definition of Hausdorff dimension.

Consider a compact metric set X with metric ρ, a subset $E \subset X$, and numbers $d \geq 0$, $\varepsilon > 0$. We cover E by balls of radius $r_j < \varepsilon$ and denote

$$\mu_H(E, d, \varepsilon) = \inf \sum_j r_j^d,$$

where the infimum is taken over all such ε-coverings E. It is obvious that $\mu_H(E, d, \varepsilon)$ does not decrease with decreasing ε. Therefore there exists the limit (perhaps infinite), namely

$$\mu_H(E, d) = \lim_{\varepsilon \to 0} \mu_H(E, d, \varepsilon).$$

Definition 9.1. The function $\mu_H(\cdot, d)$ is called the *Hausdorff d-measure.*

For fixed d, the function $\mu_H(E, d)$ possesses all properties of outer measure on X. For a fixed set E, the function $\mu_H(E, \cdot)$ has the following property. It is possible to find $d_{kp} \in [0, \infty]$ such that

$$\mu_H(E, d) = \infty, \quad \forall\, d < d_{kp},$$
$$\mu_H(E, d) = 0, \quad \forall\, d > d_{kp}.$$

If $X \subset \mathbb{R}^n$, then $d_{kp} \leq n$. Here $\mathbb{R}^n$ is an Euclidean n-dimensional space.

We put

$$\dim_H E = d_{kp} = \inf\{d \mid \mu_H(E,d) = 0\}.$$

Definition 9.2. We call $\dim_H E$ the *Hausdorff dimension* of the set E.

Example 9.1. Consider the Cantor set

$$E = \bigcap_{j=0}^{\infty} E_j,$$

where $E_0 = [0,1]$ and E_j consists of 2^j segments of length 3^{-j}, obtained from the segments belonging to E_{j-1} by eliminating from them the open middle segments of length 3^{-j}. In the classical theory of topological dimension it is well known that $\dim_T E = 0$. From the definitions of Hausdorff dimension we deduce easily that $\mu_H(E,d) = 1$ for $d = \log 2/\log 3 = 0.63010\ldots$ and, therefore,

$$\dim_H E = \frac{\log 2}{\log 3}. \qquad \square$$

Topological dimension is invariant with respect to homeomorphisms. Hausdorff dimension is invariant with respect to diffeomorphisms, and noninteger Hausdorff dimension is not invariant with respect to homeomorphisms [38].

In studying the attractors of dynamical systems in phase space, the smooth change of coordinates is often used. Therefore, in such considerations it is sufficient to assume invariance with respect to diffeomorphisms.

It is well known that $\dim_T E \leq \dim_H E$. The Cantor set E shows that this inequality can be strict.

We give now two equivalent definitions of fractal dimension. Denote by $\mathcal{N}_\varepsilon(E)$ the minimal number of balls of radius ε needed to cover the set $E \subset X$. Consider the numbers $d \geq 0$, $\varepsilon > 0$ and put

$$\mu_F(E,d,\varepsilon) = \mathcal{N}_\varepsilon(E)\varepsilon^d,$$
$$\mu_F(E,d) = \limsup_{\varepsilon\to 0} \mu_F(E,d,\varepsilon).$$

Definition 9.3. The *fractal dimension* of the set E is the value

$$\dim_F E = \inf\{d \mid \mu_F(E,d) = 0\}.$$

Note that this definition is patterned after that for Hausdorff dimension. However in this case the covering is by the balls of the same radius ε only.

Definition 9.4. The fractal dimension of E is the value

$$\dim_F E = \limsup_{\varepsilon\to 0} \frac{\log \mathcal{N}_\varepsilon(E)}{\log(1/\varepsilon)}.$$

It is easy to see that

$$\dim_H E \leq \dim_F E.$$

Example 9.2. For $X = [0,1]$ and $E = \{0, 1, 2^{-1}, 3^{-1}, \ldots\}$ we have

$$\dim_H E = 0, \quad \dim_F E = \frac{1}{2}. \qquad \square$$

The extension of the scheme for introducing the Hausdorff and fractal measures and dimensions and the definitions of different metric dimensional characteristics can be found in [40]. It turns out [41]–[44] that the upper estimate of the Hausdorff and fractal dimension of invariant sets is the Lyapunov dimension, which will be defined below.

Consider the continuously differentiable map F of the open set $U \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Denote by T_xF the Jacobian matrix of the map F at the point x. The continuous differentiability of F gives

$$F(x+h) - F(x) = (T_xF)h + o(h).$$

We shall assume further that the set $K \subset U$ is invariant with respect to the transformation F: $F(K) = K$.

Consider the singular values of the $n \times n$ matrix A

$$\alpha_1(A) \geq \cdots \geq \alpha_n(A).$$

Recall that a singular value of A is a square root of an eigenvalue of the matrix A^*A. Here the asterisk denotes either transposition (in the real case) or Hermitian conjugation. Further we shall often write

$$\omega_d(A) = \alpha_1(A)\cdots\alpha_j(A)\alpha_{j+1}(A)^s,$$

where $d = j + s$, $s \in [0,1]$, j is an integer from the interval $[1,n]$.

Definition 9.5. The *local Lyapunov dimension* of the map F at the point $x \in K$ is the number

$$\dim_L(F,x) = j + s,$$

where j is the largest integer from the interval $[1,n]$ such that

$$\alpha_1(T_xF)\cdots\alpha_j(T_xF) \geq 1$$

and s is such that $s \in [0,1]$ and

$$\alpha_1(T_xF)\cdots\alpha_j(T_xF)\alpha_{j+1}(T_xF)^s = 1.$$

By definition in the case $\alpha_1(T_xF) < 1$ we have $\dim_L(F,x) = 0$ and in the case

$$\alpha_1(T_xF)\cdots\alpha_n(T_xF) \geq 1$$

we have $\dim_L(F,x) = n$.

Definition 9.6. The *Lyapunov dimension* of the map F of the set K is the number

$$\dim_L(F,K) = \sup_K \dim_L(F,x).$$

Definition 9.7. The local Lyapunov dimension of the sequence of the maps F^i at the point $x \in K$ is the number

$$\dim_L x = \limsup_{i\to+\infty} \dim_L(F^i,x).$$

Definition 9.8. The Lyapunov dimension of the sequence of the maps F^i of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

For the maps F_t, depending on the parameter $t \in \mathbb{R}^1$, we can introduce the following analog of Definitions 9.7 and 9.8.

Definition 9.9. The local Lyapunov dimension of the map F_t at the point $x \in K$ is the number

$$\dim_L x = \limsup_{t \to +\infty} \dim_L(F_t, x).$$

Definition 9.10. The Lyapunov dimension of the map F_t of the set K is the number

$$\dim_L K = \sup_K \dim_L x.$$

Again, the inequality[41]–[44] $\dim_F K \le \dim_L K$ is an important property of Lyapunov dimension. Its proof can be found in [43, 44].

Thus, we have

$$\dim_T K \le \dim_H K \le \dim_F K \le \dim_L K.$$

Note that the Lyapunov dimension can be used as the characteristic of the inner instability of the dynamical system, defined on the invariant set K and generated by the family of the maps F^i or F_t.

The Lyapunov dimension is not a dimensional characteristic in the classical sense. However, it does permit us to estimate from above a topological, Hausdorff, or fractal dimension. It is also the characteristic of instability of dynamical systems. Finally, it is well "adapted" for investigations by the methods of classical stability theory. We shall demonstrate this, introducing the Lyapunov functions in the estimate of Lyapunov dimension. The idea of introducing Lyapunov functions in the estimate of dimensional characteristics first appeared in [45], and was subsequently developed in [46]–[60]. Here we follow, in the main, these ideas.

Consider the $n \times n$ matrices $Q(x)$, depending on $x \in \mathbb{R}^n$. We assume that

$$\det Q(x) \neq 0, \quad \forall\, x \in U,$$

and that there exist c_1 and c_2 such that

$$\sup_K \omega_d(Q(x)) \le c_1, \qquad \sup_K \omega_d(Q^{-1}(x)) \le c_2.$$

Theorem 9.1. *Let $F(K) = K$ and suppose that for some matrix $Q(x)$*

$$\sup_K \omega_d\big(Q(F(x)) T_x F Q^{-1}(x)\big) < 1. \tag{9.1}$$

Then

$$\dim_L(F^i, K) \le d \tag{9.2}$$

for sufficiently large natural numbers i.

Proof For the matrix $T_x F^i$ we have

$$T_x F^i = (T_{F^{i-1}(x)} F)(T_{F^{i-2}(x)} F) \cdots (T_x F).$$

This relation can be represented as

$$\begin{aligned}
&T_x F^i = Q(F^i(x))^{-1} \big(Q(F^i(x)) T_{F^{i-1}(x)} F Q(F^{i-1}(x))^{-1}\big) \cdot \\
&\cdot (Q(F^{i-1}(x)) T_{F^{i-2}(x)} F Q(F^{i-2}(x))^{-1}) \cdot \\
&\cdots (Q(F(x)) T_x F Q(x)^{-1}) Q(x).
\end{aligned}$$

From this and the well-known property [60]

$$\omega_d(AB) \le \omega_d(A)\omega_d(B)$$

we obtain

$$\omega_d(T_x F^i) \le c_1 c_2 \left[\sup_K \omega_d(Q(F(x))T_x F Q(x)^{-1} \right]^i.$$

This estimate, the condition (10.1) of the theorem, and the definitions of Lyapunov dimension imply the estimate (9.2).

Condition (9.1) is easily seen to be invariant with respect to the linear nonsingular change $y = Sx$, where S is a constant $n \times n$-matrix. It is clear that in the new basis condition (9.1) is also satisfied with the new matrix $Q_1(y)$:

$$Q_1(y) = Q(F(S^{-1}y))S.$$

Consider the important special case

$$Q(x) = p(x)S,$$

where S is a constant nondegenerate $n \times n$ matrix, $p(x)$ is the continuous function $\mathbb{R}^n \to \mathbb{R}^1$ for which

$$p_1 \le p(x) \le p_2, \quad \forall\, x \in K.$$

Here p_1 and p_2 are positive. In this case inequality (9.1) takes the form

$$\sup_K \omega_d \left(\frac{p(F(x))}{p(x)} \, S T_x F S^{-1} \right) < 1. \tag{9.3}$$

As will be shown below in condition (9.3) the multipliers of the type $p(F(x))/p(x)$ play the role of the Lyapunov type functions. This becomes especially clear in the case of the passage to the dynamical systems generated by differential equations.

Consider the system

$$\frac{dx}{dt} = f(t,x), \quad t \in \mathbb{R}^1,\ x \in \mathbb{R}^n, \tag{9.4}$$

where $f(t,x)$ is the continuously differentiable T-periodic vector-function $\mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t+T,x) = f(t,x)$. We assume that the solutions $x(t,x_0)$ of system (9.4) with the initial data $x(0,x_0) = x_0$ are defined on the interval $[0,T]$ and denote by G_T a shift operator along the solutions of system (9.4):

$$G_T q = x(T,q).$$

Suppose that the bounded set $K \subset \mathbb{R}^n$ is invariant with respect to the operator G_T, namely

$$G_T K = K.$$

Denote by $J(t,x)$ the Jacobian matrix of the vector-function $f(t,x)$:

$$J(t,x) = \frac{\partial f(t,x)}{\partial x}$$

and consider the nondegenerate $n \times n$ matrix S. Denote by $\lambda_1(t,x,S) \geq \cdots \geq \lambda_n(t,x,S)$ the eigenvalues of

$$\frac{1}{2}\left[SJ(t,x)S^{-1} + (SJ(t,x)S^{-1})^*\right].$$

Here the asterisk denotes transposition.

Theorem 9.2. *Suppose that for the integer $j \in [1,n]$ and $s \in [0,1]$ there exists a function $v(x)$, continuously differentiable on $\mathbb{R}^n$, and a nondegenerate $n \times n$ matrix S such that*

$$\sup_K \int_0^T [\lambda_1(t,x(t,q),S) + \ldots + \lambda_j(t,x(t,q),S) + \\ + s\lambda_{j+1}(t,x(t,q),S) + \dot{v}(x(t,q))]\,dt < 0. \tag{9.5}$$

Then for sufficiently large i the inequality

$$\dim_L(G_T^i, K) \leq j + s. \tag{9.6}$$

holds.

Proof Denote the Jacobian matrix by

$$H(t,q) = \frac{\partial x(t,q)}{\partial q}.$$

Substituting $x(t,q)$ in (9.4) and differentiating both sides of (9.4) with respect to q, we obtain

$$\frac{dH(t,q)}{dt} = J(t,x(t,q))H(t,q).$$

Represent this relation as

$$\frac{d}{dt}[SH(t,q)S^{-1}] = [SJ(t,x(t,q))S^{-1}][SH(t,q)S^{-1}].$$

For the singular values $\sigma_1(t) \geq \cdots \geq \sigma_n(t)$ of the matrix $SH(t,q)S^{-1}$ we have the inequality [60]

$$\sigma_1 \cdots \sigma_k \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_k)\,d\tau\right)$$

for any $k = 1, \ldots, n$. From this and the relation

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s = (\sigma_1 \cdots \sigma_j)^{1-s}(\sigma_1 \cdots \sigma_{j+1})^s$$

we obtain the estimate

$$\sigma_1 \cdots \sigma_j \sigma_{j+1}^s \leq \exp\left(\int_0^t (\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1})\,d\tau\right). \tag{9.7}$$

Put

$$p(x) = \big(\exp v(x)\big)^{1/(j+s)}$$

and multiply both sides of (9.7) by the relation

$$\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s} = \exp\left[v(x(t,q) - v(q)\right] = \exp\left(\int_0^t \dot{v}(x(\tau,q))\,d\tau\right).$$

As a result we obtain

$$\begin{aligned}&\left(\frac{p(x(t,q))}{p(q)}\right)^{j+s}\sigma_1\dots\sigma_j\sigma_{j+1}^s\\&\quad\le\exp\left(\int_0^t(\lambda_1+\dots+\lambda_j+s\lambda_{j+1}+\dot v(x(\tau,q))\,d\tau\right).\end{aligned}$$

This implies the estimate

$$\begin{aligned}&\alpha_1(t,q)\dots\alpha_j(t,q)\alpha_{j+1}(t,q)^s\\&\quad\le\exp\left(\int_0^t\big(\lambda_1(\tau,x(\tau,q),S)+\dots+\lambda_j(\tau,x(\tau,q),S)\right.\\&\qquad\left.+\,s\lambda_{j+1}(\tau,x(\tau,q),S)+\dot v(x(\tau,q))\big)\,d\tau\right),\end{aligned}\tag{9.8}$$

where $\alpha_k(t,q)$ are the singular values of the matrix

$$\frac{p(x(t,q))}{p(q)}\,SH(t,q)S^{-1}.$$

From estimate (9.8) and condition (9.5) of Theorem 9.2 it follows that there exists $\varepsilon>0$ such that

$$\alpha_1(T,q)\cdots\alpha_j(T,q)\alpha_{j+1}(T,q)^s\le\exp(-\varepsilon)$$

for all $q\in K$. Thus, in this case condition (9.3) with $F=G_T$

$$T_qF=T_qG_T=H(T,q)$$

is satisfied and, therefore, estimate (9.6) is valid.

The following simple assertions will be useful in the sequel.

Lemma 9.1. *Suppose that the real matrix A can be reduced to the diagonal form*

$$SAS^{-1}=\begin{pmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{pmatrix},$$

where S is a real nonsingular matrix. Then there exist positive numbers c_1 and c_2 such that

$$c_1|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i\le\omega_d(A^i)\le c_2|\lambda_1\cdots\lambda_j\lambda_{j+1}^s|^i.$$

Proof It is sufficient to note that the singular values of the matrix

$$\begin{pmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{pmatrix}$$

are the numbers $|\lambda_j|$, and for the singular values $\alpha_1\ge\dots\ge\alpha_n$ the inequalities

$$\alpha_n(C)\alpha_j(B)\le\alpha_j(CB)\le\alpha_1(C)\alpha_j(B)$$

are satisfied.

Lemma 9.2. *Let $F(x) = x$ and the Jacobian matrix T_xF of the map F have the simple real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then the local Lyapunov dimension of the sequence of maps F^i at the point x is equal to $j + s$, where j and s are determined from*

$$|\lambda_1 \cdots \lambda_j \lambda_{j+1}^s| = 1.$$

Lemma 9.2 is a direct corollary of Lemma 9.1. A similar result holds for the map F_t.

Lemma 9.3. *Let $T_xF_t = e^{At}$ and the matrix A satisfy the condition of Lemma 9.1. Then the local Lyapunov dimension of the map F_t at the point x is equal to $j + s$, where j and s are determined from*

$$\lambda_1 + \cdots + \lambda_j + s\lambda_{j+1} = 0.$$

Lemma 9.3 is also a corollary of Lemma 9.1.

Now we apply Theorems 9.1 and 9.2 to the Henon and Lorenz systems in order to construct Lyapunov functions $p(x)$ (for the Henon system) and $v(x)$ (for the Lorenz system). Consider the Henon map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} &x \to a + by - x^2, \\ &y \to x, \end{aligned} \tag{9.9}$$

where $a > 0$, $b \in (0, 1)$ are the parameters of mapping. Consider the bounded invariant set K of map (9.9), $FK = K$, involving stationary points of this map:

$$\begin{aligned} x_+ &= \frac{1}{2}\left[b - 1 + \sqrt{(b-1)^2 + 4a}\,\right], \\ x_- &= \frac{1}{2}\left[b - 1 - \sqrt{(b-1)^2 + 4a}\,\right]. \end{aligned}$$

Theorem 9.3. *For the map F we have*

$$\dim_L K = 1 + \frac{1}{1 - \ln b / \ln \alpha_1(x_-)},$$

where

$$\alpha_1(x_-) = \sqrt{x_-^2 + b} - x_-.$$

Proof Denote $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$. The Jacobian matrix $T_\xi F$ of the map F takes the form

$$\begin{pmatrix} -2x & b \\ 1 & 0 \end{pmatrix}.$$

We introduce the matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

In this case

$$ST_\xi FS^{-1} = \begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}. \tag{9.10}$$

We shall show that the singular values of (9.10) are

$$\begin{aligned} \alpha_1(x) &= \sqrt{x^2+b} + |x|, \\ \alpha_2(x) &= \sqrt{x^2+b} - |x| = \frac{b}{\alpha_1(x)}. \end{aligned} \tag{9.11}$$

It is obvious that

$$\begin{aligned} \alpha_1(x)^2 &= 2x^2 + b + 2|x|\sqrt{x^2+b}\,, \\ \alpha_2(x)^2 &= 2x^2 + b - 2|x|\sqrt{x^2+b}\,. \end{aligned}$$

It is clear that $\alpha_k(x)^2$ are zeros of the polynomial

$$\lambda^2 - (4x^2 + 2b)\lambda + b^2,$$

which is the characteristic polynomial of the matrix

$$\begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix} \begin{pmatrix} -2x & \sqrt{b} \\ \sqrt{b} & 0 \end{pmatrix}.$$

Thus, formulas (9.11) are proved.

From Theorem 9.1 it follows that if there exist $s \in [0,1)$ and a continuously differentiable function $p(\xi)$, positive on K and such that

$$\sup_{\xi \in K} \alpha_1(x)\alpha_2(x)^s \left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s} < 1, \tag{9.12}$$

then

$$\dim_L K \le 1 + s.$$

Put

$$p(\xi)^{1+s} = e^{\gamma(1-s)(x+by)},$$

where γ is a positive parameter. It is not hard to prove that

$$\left(\frac{p(F(\xi))}{p(\xi)}\right)^{1+s} = e^{\gamma(1-s)(a+(b-1)x-x^2)}.$$

This implies that after taking the logarithm, condition (9.12) becomes

$$\begin{aligned} &\sup_K \left[\ln \alpha_1(x) + s \ln \alpha_2(x) + \gamma(1-s)(a + (b-1)x - x^2)\right] \\ &\quad = \sup_K \left[(1-s)\ln \alpha_1(x) + s \ln b + \gamma(1-s)(a + (b-1)x - x^2)\right] < 0. \end{aligned}$$

This inequality is satisfied if

$$s \ln b + (1-s)\varphi(x) < 0, \quad \forall\, x \in (-\infty, +\infty),$$

where

$$\varphi(x) = \ln\left[\sqrt{x^2+b} + |x|\right] + \gamma(a + (b-1)x - x^2).$$

The inequalities $\gamma > 0$, $b - 1 < 0$ result in the estimate

$$\varphi(-|x|) \ge \varphi(|x|).$$

Therefore it suffices to consider the extremum point of the functions $\varphi(x)$ for $x \in (-\infty, 0]$. It is clear that on this set we have

$$\varphi'(x) = \frac{-1}{\sqrt{x^2+b}} + \gamma\big[(b-1) - 2x\big], \quad \varphi''(x) < 0.$$

Letting

$$\gamma = \frac{1}{(b-1-2x_-)\sqrt{x_-^2+b}}\,,$$

we find that $\varphi'(x_-) = 0$ and therefore, for such a choice of γ,

$$\varphi(x) \le \ln\left(\sqrt{x_-^2+b} + |x_-|\right) = \ln \alpha_1(x_-).$$

Thus, inequality (9.12) holds for all s satisfying

$$s > \frac{\ln \alpha_1(x_-)}{\ln \alpha_1(x_-) - \ln b}. \tag{9.13}$$

Hence the estimate

$$\dim_L K \le 1 + s$$

is valid for all s satisfying (9.13). Passing to the limit, we obtain

$$\dim_L K \le 1 + \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}. \tag{9.14}$$

Note that the point $x = x_-$, $y = x_-$ is stationary for the map F. Then

$$\alpha_1(x_-)\alpha_2(x_-)^s = 1, \tag{9.15}$$

where

$$s = \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}.$$

It is easily shown that $\alpha_1(x_-)$ and $\alpha_2(x_-)$ are the eigenvalues of the Jacobian matrix $T_\xi F$ of the map F at the fixed point $y = x = x_-$:

$$T_\xi F = \begin{pmatrix} -2x_- & b \\ 1 & 0 \end{pmatrix}.$$

From relation (9.15) by Lemma 9.2 we conclude that the local Lyapunov dimension of the sequence of maps F^i at this stationary point is equal to

$$1 + \frac{1}{1 - \ln b/\ln \alpha_1(x_-)}. \tag{9.16}$$

By inequality (9.14) we obtain the assertion of Theorem 9.3. □

Note that for $a = 1.4$, $b = 0.3$ from Theorem 9.3 we have

$$\dim_L K = 1.49532\ldots$$

Consider a Lorenz system

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned} \tag{9.17}$$

where r, b, σ are positive. Suppose that the inequalities $r > 1$,

$$\sigma + 1 \geq b \geq 2, \tag{9.18}$$

are valid. Consider the shift operator along the trajectory of system (9.17) G_T, where T is an arbitrary positive number. Let K be an invariant set with respect to this operator G_T. Suppose that K involves the stationary point $x = y = z = 0$. Such a set is represented in Fig. 3.1. We provide a formula for the Lyapunov dimension $\dim_L K$ of the set K with respect to the sequence of maps $(G_T)^i$.

Theorem 9.4. *Suppose the inequalities* (9.18) *and*

$$r\sigma^2(4-b) + 2\sigma(b-1)(2\sigma - 3b) > b(b-1)^2 \tag{9.19}$$

are valid. Then

$$\dim_L K = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{9.20}$$

Proof The Jacobian matrix of the right-hand side of system (9.17) has the form

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}.$$

Introduce the matrix

$$S = \begin{pmatrix} -a^{-1} & 0 & 0 \\ -\sigma^{-1}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a = \dfrac{\sigma}{\sqrt{r\sigma + (b-1)(\sigma - b)}}$. In this case we obtain

$$SJS^{-1} = \begin{pmatrix} b - \sigma - 1 & \sigma/a & 0 \\ \dfrac{\sigma}{a} - az & -b & -x \\ ay + \dfrac{a(b-1)}{\sigma}x & x & -b \end{pmatrix}.$$

Therefore the characteristic polynomial of the matrix

$$\frac{1}{2}((SJS^{-1})^* + (SJS^{-1})) =$$
$$\begin{pmatrix} b - \sigma - 1 & \dfrac{\sigma}{a} - \dfrac{az}{2} & \dfrac{1}{2}\left(ay + \dfrac{a(b-1)}{\sigma}x\right) \\ \dfrac{\sigma}{a} - \dfrac{az}{2} & -b & 0 \\ \dfrac{1}{2}\left(ay + \dfrac{a(b-1)}{\sigma}x\right) & 0 & -b \end{pmatrix}$$

takes the form

$$(\lambda+b)\left\{\left[\lambda^2+(\sigma+1)\lambda+b(\sigma+1-b)-\left(\frac{\sigma}{a}-\frac{az}{2}\right)^2\right]-\left[\frac{a(b-1)}{2\sigma}x+\frac{ay}{2}\right]^2\right\}.$$

This implies that eigenvalues of the matrix

$$\frac{1}{2}[(SJS^{-1})^*+(SJS^{-1})]$$

are the values

$$\lambda_2=-b,$$

and

$$\lambda_{1,3}=-\frac{\sigma+1}{2}\pm\frac{1}{2}\left[(\sigma+1-2b)^2+\left(\frac{2\sigma}{a}-az\right)^2\right.$$
$$\left.+\left(\frac{a(b-1)}{\sigma}x+ay\right)^2\right]^{1/2}.$$

From relations (9.18) it follows easily that $\lambda_1\geq\lambda_2\geq\lambda_3$.

Consider the Lyapunov type function

$$v(x,y,z)=\frac{1}{2}a\theta^2(1-s)\left(\gamma_1x^2+\gamma_2\left(y^2+z^2-\frac{(b-1)^2}{\sigma^2}x^2\right)+\gamma_3z\right),$$

where $s\in(0,1)$,

$$\theta^2=\left(2\sqrt{(\sigma+1-2b)^2+\left(\frac{2\sigma}{a}\right)^2}\right)^{-1},$$
$$\gamma_3=-\frac{4\sigma}{ab},\quad \gamma_2=\frac{a}{2},$$
$$\gamma_1=-\frac{1}{2\sigma}\left[2\gamma_2\,\frac{r\sigma-(b-1)^2}{\sigma}+\gamma_3+2\,\frac{a(b-1)}{\sigma}\right].$$

Consider the relation

$$2\big[\lambda_1+\lambda_2+s\lambda_3+\dot v\big]=-(\sigma+1+2b)-s(\sigma+1)+(1-s)\varphi(x,y,z),$$

where

$$\varphi(x,y,z)=\left((\sigma+1-2b)^2+\left(\frac{2\sigma}{a}-az\right)^2\right.$$
$$\left.+\left(\frac{a(b-1)}{\sigma}x+ay\right)^2\right)^{1/2}$$
$$+\theta^2\left\{\left(-2a\gamma_1\sigma+2\gamma_2\frac{a(b-1)^2}{\sigma}\right)x^2-2a\gamma_2y^2\right.$$
$$\left.-2a\gamma_2bz^2+a\left(2\sigma\gamma_1+2\gamma_2\frac{r\sigma-(b-1)^2}{\sigma}+\gamma_3\right)xy-\gamma_3abz\right\}.$$

By using the obvious inequality

$$\sqrt{u} \le \frac{1}{4\theta^2} + \theta^2 u,$$

we obtain the estimate

$$\begin{aligned}\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2 \Big\{ & (\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 \\ & + \left[-2a\gamma_1\sigma + 2\gamma_2 \frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2}\right] x^2 \\ & + [a^2 - 2a\gamma_2]y^2 + [a^2 - 2\gamma_2 ab]z^2 \\ & + \left[a\left(2\sigma\gamma_1 + 2\gamma_2 \frac{r\sigma - (b-1)^2}{\sigma} + \gamma_3\right)\right. \\ & \left. + 2a^2 \frac{b-1}{\sigma}\right] xy - [\gamma_3 ab + 4\sigma]z \Big\}.\end{aligned}$$

Note that the parameters γ_1, γ_2, and γ_3 are chosen in such a way that

$$\begin{aligned}\varphi(x,y,z) \le \frac{1}{4\theta^2} + \theta^2 \Big\{ & (\sigma+1-2b)^2 + \left(\frac{2\sigma}{a}\right)^2 \\ & + \left[-2a\gamma_1\sigma + 2\gamma_2 \frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2}\right] x^2 \Big\}.\end{aligned}$$

It is not hard to prove that for the above parameters $\gamma_1, \gamma_2, \gamma_3$ under condition (9.19) we have

$$-2a\gamma_1\sigma + 2\gamma_2 \frac{a(b-1)^2}{\sigma} + \frac{a^2(b-1)^2}{\sigma^2} \le 0.$$

Thus, for all x, y, z we have

$$\varphi(x,y,z) \le \sqrt{4r\sigma + (\sigma-1)^2}\ .$$

This implies that for any number

$$s < s_0 = \frac{\sqrt{4r\sigma + (\sigma-1)^2} - 2b - \sigma - 1}{\sqrt{4r\sigma + (\sigma-1)^2} + \sigma + 1}$$

there exists $\varepsilon > 0$ such that for all x, y, z the estimate

$$\lambda_1(x,y,z) + \lambda_2(x,y,z) + s\lambda_3(x,y,z) + \dot{v}(x,y,z) < -\varepsilon$$

is satisfied. Letting $s \to s_0$ on the right, by Theorem 9.2 we obtain

$$\dim_L K \le 3 - \frac{2(\sigma+b+1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{9.21}$$

By Lemma 9.3 we see that the local Lyapunov dimension of the stationary point $x = y = z = 0$ of system (9.17) is equal to

$$3 - \frac{2(\sigma+b+1)}{\sigma + 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}. \tag{9.22}$$

Relations (9.21) and (9.22) yield the formula (9.20).

By using a similar approach to the construction of the Lyapunov functions, we can obtain formulas for the Lyapunov dimension of the attractors of the dissipative Chirikov map [56].

10 Homoclinic Bifurcation

When the parameters of a dynamical system are varied, the structure of the minimal global attractor can vary as well. Such changes are the subject of bifurcation theory. Here we describe one particular phenomenon: the homoclinic bifurcation.

The first important results, concerning homoclinic bifurcations in dissipative dynamical systems, were obtained in 1933 by the outstanding Italian mathematician Franchesko Tricomi [15]. Here we give Tricomi's results and similar theorems for the Lorenz systems.

Consider the second-order differential equation

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = \gamma, \tag{10.1}$$

where α and γ are positive. This describes the motion of a pendulum with a constant moment of force, the operation of a synchronous electrical machine, and the phase-locked loop [61, 62]. For $\gamma < 1$ the equivalent system

$$\begin{aligned} \dot{\theta} &= z, \\ \dot{z} &= -\alpha z - \sin\theta + \gamma, \end{aligned} \tag{10.2}$$

has the saddle equilibria $z = 0,\ \theta = \theta_0 + 2k\pi$. Here θ_0 is a number for which $\sin\theta_0 = \gamma$ and $\cos\theta_0 < 0$.

Consider the trajectory $z(t), \theta(t)$ of (10.2) for which

$$\lim_{t\to+\infty} z(t) = 0, \qquad \lim_{t\to+\infty} \theta(t) = \theta_0, z(t) > 0, \qquad \forall\, t \geq T.$$

Here T is a certain number. In Fig. 1, such a trajectory is denoted by S. It is often called a separatrix of the saddle.

Fix $\gamma > 0$ and vary the parameter α. For $\alpha = 0$ the system (10.2) is integrable. It is easily shown that in this case, for the trajectory $S = \{z(t), \theta(t)\}$ there exists τ such that

$$\begin{aligned} &z(\tau) = 0, \quad \theta(\tau) \in (\theta_0 - 2\pi, \theta_0) \\ &z(t) > 0, \quad \forall\, t > \tau. \end{aligned} \tag{10.3}$$

Consider now the line segment $z = -K(\theta - \theta_0),\ \theta \in [\theta_0 - 2\pi, \theta_0]$. It is not hard to prove that on this segment for system (10.2) the relations

$$\begin{aligned} (z + K(\theta - \theta_0))^{\bullet} &= -\alpha z + Kz - \sin\theta + \gamma \\ &= (\theta - \theta_0)\left(-K(K - \alpha) + \frac{\gamma - \sin\theta}{\theta - \theta_0}\right) \end{aligned}$$

are valid. We make use of the obvious inequality

$$\left|\frac{\gamma - \sin\theta}{\theta - \theta_0}\right| \leq 1, \quad \forall\, \theta \neq \theta_0.$$

If the conditions

$$\alpha > 2, \qquad \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - 1} < K < \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - 1},$$

are satisfied, we obtain the estimate

$$(z + K(\theta - \theta_0))^{\bullet} < 0$$

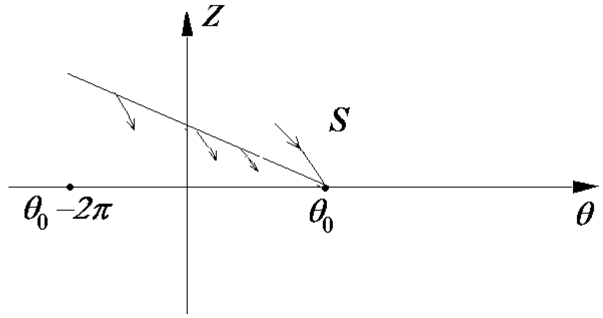

Figure 10.1: Estimate of separatrix.

for $z = -K(\theta - \theta_0),\ \theta \in (\theta_0 - 2\pi, \theta_0)$. See Fig. 10.1.

The figure shows that there does not exist τ such that conditions (10.3) are satisfied (Fig. 2).

It is well known that the piece of the trajectory $S : \{z(t), \theta(t) \mid t \geq \tau\}$ is continuously dependent on the parameter α. Here τ satisfies (10.3).

Then from the disposition of the trajectory S for $\alpha > 2$ (Fig. 10.2) it follows that there exists $\alpha_0 \in (0, 2)$ such that the trajectory S of system (10.2) with $\alpha = \alpha_0$ satisfies the relation

$$\lim_{t\to-\infty} z(t) = 0, \quad \lim_{t\to-\infty} \theta(t) = \theta_0 - 2\pi. \tag{10.4}$$

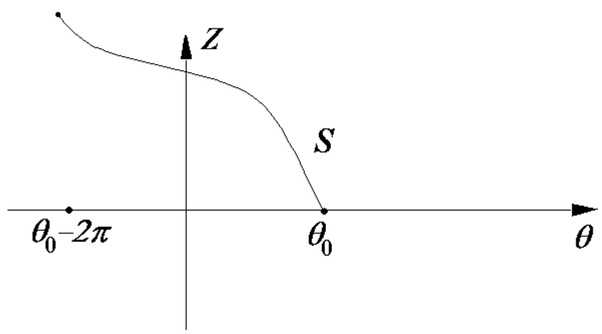

Figure 10.2: Behavior of separatrix.

Thus, $\alpha = \alpha_0$ is a bifurcational parameter. To this parameter corresponds the heteroclinic trajectory $S = \{z(t), \theta(t) \mid t \in \mathbb{R}^1\}$. Recall that the trajectory $x(t)$ of the system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{10.5}$$

is said to be *heteroclinic* if

$$\lim_{t\to+\infty} x(t) = c_1, \quad \lim_{t\to-\infty} x(t) = c_2, \quad c_1 \neq c_2.$$

In the case $c_1 = c_2$, the trajectory $x(t)$ is called *homoclinic*.

Sometimes for systems involving angular coordinates, the cylindrical phase space is introduced. We do this for system (10.2).

It is obvious that the properties of system (10.2) are invariant with respect to the shift $x + d$. Here

$$x = \begin{pmatrix} \theta \\ z \end{pmatrix}, \quad d = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}.$$

In other words, if $x(t)$ is a solution of system (10.2), then so is $x(t) + d$.

Consider a discrete group

$$\Gamma = \{x = kd \mid k \in Z\}.$$

We consider the factor group R^2/Γ, the elements of which are the classes of the residues $[x] \in R^2/\Gamma$. They are defined as

$$[x] = \{x + u \mid u \in \Gamma\}.$$

We introduce the so-called plane metric

$$\rho([x],[y]) = \inf_{\substack{u \in [x] \\ v \in [y]}} |u - v|.$$

Here, as above, $|\cdot|$ is a Euclidean norm in $\mathbb{R}^2$.

It is obvious that $[x(t)]$ is a solution and the metric space R^2/Γ is a phase space of system (10.2). This space is partitioned into the nonintersecting trajectories $[x(t)]$, $t \in \mathbb{R}^1$.

It is easy to establish the following diffeomorphism between R^2/Γ and a surface of the cylinder $R^1 \times C$. Here C is a circle of unit radius.

Consider the set $\Omega = \{x \mid \theta \in (0, 2\pi],\ z \in \mathbb{R}^1\}$, in which exactly one representer of each class $[x] \in \mathbb{R}^2/\Gamma$ is situated. Cover the surface of cylinder by the set Ω, winding Ω round this surface (Fig. 3)

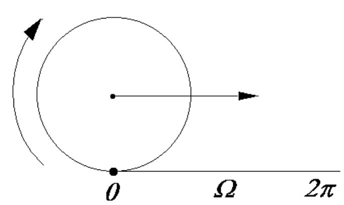

Figure 10.3: Cylindrical space.

It is obvious that the map constructed is a diffeomorphism. Therefore, the surface of the cylinder is also partitioned into nonintersecting trajectories. Such a phase space is called *cylindrical*.

Note that heteroclinic trajectory (10.4) in the phase space $\mathbb{R}^2$ becomes homoclinic in the cylindrical phase space and in the phase space R^2/Γ since we have

$$\lim_{t\to+\infty}[x(t)] = \lim_{t\to-\infty}[x(t)] = \left[\begin{array}{c}\theta_0\\0\end{array}\right],$$

$$\left[\begin{array}{c}\theta_0\\0\end{array}\right] = \left\{\left(\begin{array}{c}\theta_0+2k\pi\\0\end{array}\right) \mid k\in Z\right\}.$$

Now the assertion obtained is formulated in the following way. Consider the smooth path $\alpha(s)$ ($s\in[0,1]$) such that $\alpha(0)=0$, $\alpha(s)>0, \forall\, s\in(0,1)$, $\alpha(1)>2$.

Theorem 10.1 (Tricomi). *For any $\gamma>0$ there exists $s_0\in(0,1)$ such that system (10.2) with the parameters $\gamma,\alpha(s_0)$ has a homoclinic trajectory in the phase space R^2/Γ.*

We proceed to obtain a similar assertion for the Lorenz system

$$\begin{aligned}\dot x &= \sigma(x-y),\\ \dot y &= rx-y-xz,\\ \dot z &= -bz+xy,\end{aligned} \tag{10.6}$$

where σ,b,r are positive. The function

$$V(x,y,z) = y^2+z^2+\frac{1}{\sigma}x^2$$

satisfies

$$\dot V(x(t),y(t),z(t)) = -2bz(t)^2-2y(t)^2-2x(t)^2+2(r+1)x(t)y(t).$$

From this we easily find that for $r\le 1$, all the solutions of system (10.6) tend to zero as $t\to+\infty$. Therefore we consider further the case $r>1$.

Using the transformation

$$\begin{aligned}&\theta=\tfrac{\varepsilon x}{\sqrt{2\sigma}},\ \ \eta=\varepsilon^2\sqrt{2}(y-x),\ \ \xi=\varepsilon^2(z-\tfrac{x^2}{b}),\\ &t=t_1\tfrac{\sqrt{\sigma}}{\varepsilon},\ \ \varepsilon=\tfrac{1}{\sqrt{r-1}},\end{aligned} \tag{10.7}$$

we reduce system (10.6) to the form

$$\begin{aligned}\dot\theta &= \eta,\\ \dot\eta &= -\mu\eta-\xi\theta-\varphi(\theta),\\ \dot\xi &= -\alpha\xi-\beta\theta\eta.\end{aligned} \tag{10.8}$$

Here

$$\varphi(\theta)=-\theta+\gamma\theta^3,\quad \mu=\frac{\varepsilon(\sigma+1)}{\sqrt{\sigma}},\quad \alpha=\frac{\varepsilon b}{\sqrt{\sigma}},\quad \beta=2\left(\frac{2\sigma}{b}-1\right),\quad \gamma=\frac{2\sigma}{b}.$$

It follows easily that if the conditions

$$\begin{aligned}&\lim_{t\to+\infty}\theta(t)=\lim_{t\to-\infty}\theta(t)=\lim_{t\to+\infty}\eta(t)=\\ &=\lim_{t\to-\infty}\eta(t)=\lim_{t\to+\infty}\xi(t)=\lim_{t\to-\infty}\xi(t)=0\end{aligned}$$

are satisfied, then

$$\lim_{t\to+\infty} x(t) = \lim_{t\to-\infty} x(t) = \lim_{t\to+\infty} y(t) =$$
$$= \lim_{t\to-\infty} y(t) = \lim_{t\to+\infty} z(t) = \lim_{t\to-\infty} z(t) = 0.$$

Thus, a homoclinic trajectory of system (10.8) corresponds to a homoclinic trajectory of system (10.6).

Denote by $\theta^+(t), \eta(t)^+, \xi(t)^+$ a separatrix of the saddle $\theta = \eta = \xi = 0$, outgoing in the half-plane $\{\theta > 0\}$. See Fig. 4.

In other words, we consider a solution of system (10.6) such that

$$\lim_{t\to-\infty} \theta(t)^+ = \lim_{t\to-\infty} \eta(t)^+ = \lim_{t\to-\infty} \xi(t)^+ = 0$$

and $\theta(t)^+ > 0$ for $t \in (-\infty, T)$. Here T is a certain number or $+\infty$.

Consider the smooth path $b(s), \sigma(s), r(s)$ $(s \in [0,1])$ in a space of the parameters $\{b, \sigma, r\}$. It is clear that in this case the parameters $\alpha, \beta, \gamma, \mu$ are also smooth functions of $s \in [0,1]$.

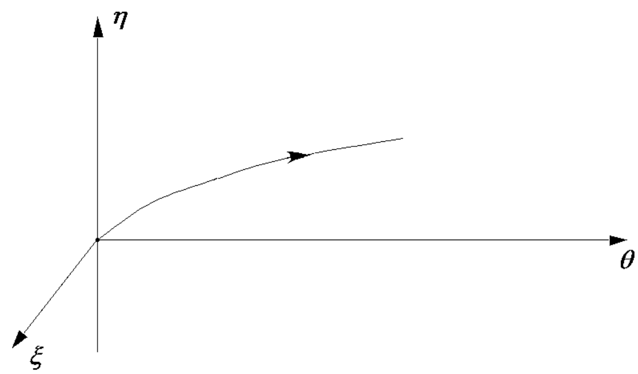

Figure 10.4: Separatrix of the Lorenz system.

Theorem 10.2. *Let $\beta(s) > 0$, $\forall\, s \in [0,1]$ and for $s \in [0, s_0)$ suppose there exist $T(s) > \tau(s)$ such that the relations*

$$\theta(T)^+ = \eta(\tau)^+ = 0 \tag{10.9}$$

$$\theta(t)^+ > 0,\ \forall\, t < T, \tag{10.10}$$

$$\eta(t)^+ \neq 0, \quad \forall\, t < T,\ \ t \neq \tau \tag{10.11}$$

are satisfied. Suppose also that for $s = s_0$ there does not exist the pair $T(s_0) > \tau(s_0)$ such that relations (10.9)–(10.11) are satisfied. Then system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$:

$$\lim_{t\to+\infty} \theta(t)^+ = \lim_{t\to+\infty} \eta(t)^+ = \lim_{t\to+\infty} \xi(t)^+ = 0.$$

To prove this theorem we need the following

Lemma 10.1. *If for system (10.8) the conditions*

$$\eta(\tau)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall\, t \in (-\infty, \tau)$$

are valid, then $\dot\eta(\tau)^+ < 0$.

Proof Suppose to the contrary that $\dot\eta(\tau)^+ = 0$. In this case from the last two equations of system (10.8) we obtain

$$\ddot\eta(\tau)^+ = \alpha\xi(\tau)^+\theta(\tau)^+. \tag{10.12}$$

From the relations $\eta(t)^+ > 0$, $\theta(t)^+ > 0$, $\forall\, t \in (-\infty, \tau)$ and the last equation of system (10.8) we obtain $\xi(t)^+ < 0$ $\forall\, t \in (-\infty, \tau]$. Then (10.12) yields the inequality $\ddot\eta(\tau)^+ < 0$, which contradicts the assumption $\dot\eta(\tau)^+ = 0$ and the hypotheses of the lemma.

Lemma 10.2. *Let* $\beta(s) > 0$, $\forall\, s \in [0, 1]$. *Suppose that for system (10.8), relations (10.9), (10.10) and the inequalities*

$$\begin{aligned} &\eta(t)^+ > 0, \quad \forall\, t \in (-\infty, \tau) \\ &\eta(t)^+ \le 0, \quad \forall\, t \in (\tau, T) \end{aligned} \tag{10.13}$$

are valid. Then inequality (10.11) is also valid.

Proof Assuming the contrary, we see that there exists $\rho \in (\tau, T)$ such that the relations

$$\begin{aligned} &\eta(\rho)^+ = \dot\eta(\rho)^+ = 0, \\ &\ddot\eta(\rho)^+ = \alpha\theta(\rho)^+\xi(\rho)^+ < 0, \\ &\eta(t)^+ < 0, \quad \forall\, t \in (\rho, T), \end{aligned}$$

are satisfied. Then from conditions (10.9), (10.10) and from the fact that the trajectories $\theta(t) = \eta(t) = 0$, $\xi(t) = \xi(0)\exp(-\alpha t)$ belong to a stable manifold of the saddle $\theta = \eta = \xi = 0$ we obtain the crossing of the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ and this stable manifold. Therefore the separatrix belongs completely to the stable manifold of the saddle. In addition, the condition $\theta(t)^+ > 0$, $\forall\, t \ge \rho$ holds. The latter is in the contrast to condition (10.9). This contradiction proves Lemma 10.2.

It is possible to give the following geometric interpretation of the proof of Lemma 10.2 in the phase space with coordinates θ, η, ξ. "Under" the set $\{\theta > 0, \eta = 0, \xi \le 1 - \gamma\theta^2\}$ is situated the piece of stable two-dimensional manifold of the saddle $\theta = \eta = \xi = 0$. This does not allow the trajectories with the initial data from this set to attain the plane $\{\theta = 0\}$ if they remain in the quadrant $\{\theta \ge 0, \eta \le 0\}$.

Consider the polynomial

$$p^3 + ap^2 + bp + c, \tag{10.14}$$

where a, b, c are positive.

Lemma 10.3. *Either all zeros of (10.14) have negative real parts, or two zeros of (10.14) have nonzero imaginary parts.*

Proof It is well known [62] that all zeros of (10.14) have negative real parts if and only if $ab > 0$. For $ab = c$, polynomial (10.14) has two pure imaginary zeros.

Suppose now that for the certain a, b, c such that $ab < c$, polynomial (10.14) has real zeros only. From the positiveness of coefficients it follows that these zeros are negative. The latter yields $ab > c$, which contradicts the assumption.

Proof of Theorem 10.2. We shall show that to the values of the parameters $b(s_0), \sigma(s_0), r(s_0)$ there corresponds a homoclinic trajectory.

First note that for these parameters for the certain τ the relations

$$\begin{array}{l} \eta(t)^+ > 0, \quad \forall\, t < \tau, \quad \eta(t)^+ \le 0, \quad \forall\, t \ge \tau \\ \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, +\infty), \end{array} \tag{10.15}$$

hold. Actually, if there exist $T_2 > T_1 > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \quad \forall\, t \in (-\infty, T_2); \;\; \theta(T_2)^+ = 0, \;\; \eta(T_1)^+ > 0 \\ \eta(t)^+ > 0, \;\; \forall\, t < \tau; \;\; \eta(\tau)^+ = 0, \;\; \dot\eta(\tau)^+ < 0, \end{gathered}$$

then for the values $s < s_0$ and for the values s sufficiently close to s_0 the inequality $\eta(T_1)^+ > 0$ holds true. This is in the contrast to the definition of s_0. If there exist $T_1 > \tau$ such that

$$\begin{gathered} \eta(T_1)^+ > 0, \;\; \eta(t)^+ > 0, \;\; \forall\, t < \tau \\ \eta(\tau)^+ = 0, \;\; \dot\eta(\tau)^+ < 0, \;\; \theta(t)^+ > 0, \;\; \forall\, t \in (-\infty, +\infty) \end{gathered}$$

then for $s < s_0$, which is sufficiently closed to s_0, the inequality $\eta(T_1)^+ > 0$ holds true, which is in contrast to the definition of s_0. If there exist $T > \tau$ such that

$$\begin{gathered} \theta(t)^+ > 0, \; \forall\, t < T, \;\; \theta(T)^+ = 0, \;\; \eta(t)^+ > 0, \; \forall\, t < \tau \\ \eta(t)^+ \le 0, \quad \forall\, t \in [\tau, T], \end{gathered}$$

then by Lemma 10.2 inequality (10.11) holds. Therefore for $s = s_0$ relations (10.9)–(10.11) are valid, which is in the contrast to the hypotheses of the theorem. This contradiction proves inequality (10.15).

From (10.15) it follows that only one of the equilibria can be the ω-limit set of the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ for $s = s_0$. We shall show that the equilibrium $\theta = 1/\sqrt{\gamma}$, $\eta = \xi = 0$ cannot be the ω-limit point of the considered trajectory.

Having performed the linearization in the neighborhood of this equilibrium, we obtain the characteristic polynomial

$$p^3 + (\alpha + \mu)p^2 + (\alpha\mu + 2/\gamma)p + 2\alpha.$$

Suppose, for $s = s_0$ the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ has in its ω-limit set the point $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$. By Lemma 10.3 and from a continuous dependence of the semitrajectories $\{\theta(t)^+, \eta(t)^+, \xi(t)^+ | \, t \in (-\infty, t_0)\}$ on the parameter s we obtain that for the values s sufficiently close to s_0, the separatrices $\theta(t)^+, \eta(t)^+, \xi(t)^+$ either tend to the equilibrium $\theta = 1/\sqrt{\gamma}, \eta = \xi = 0$ as $t \to +\infty$ or oscillate on the certain time interval with the sign reversal of the coordinate η. Both possibilities are in contrast to properties (10.9)–(10.11).

Thus, for system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ the trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$ tends to zero equilibrium as $t \to +\infty$. $\square$

Remark 10.1. It is well known that the semitrajectory

$$\{\theta(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$$

depends continuously on the parameter s. Here t_0 is a certain fixed number. Then Lemma 10.1 implies that if for system (10.8) with the parameters $b(s_1), \sigma(s_1), r(s_1)$ relations

(10.9)–(10.11) are satisfied, then these relations are also satisfied for $b(s), \sigma(s), r(s)$. Here $s \in (s_1 - \delta, s_1 + \delta)$ where δ is sufficiently small. □

Theorem 10.2 and Remark 10.1 result in the following

Theorem 10.3. *Let be $\beta(s) > 0$, $\forall s \in [0,1]$. Suppose, for system (10.8) with the parameters $b(0), \sigma(0).r(0)$ there exist $T > \tau$ such that relations (10.9)–(10.11) are valid. Suppose also that for system (10.8) with the parameters $b(1), \sigma(1), r(1)$ the inequality*

$$\theta(t)^+ > 0, \quad \forall t \in (-\infty, +\infty)$$

holds. Then there exists $s_0 \in [0,1]$ such that system (10.8) with the parameters $b(s_0), \sigma(s_0), r(s_0)$ has the homoclinic trajectory $\theta(t)^+, \eta(t)^+, \xi(t)^+$.

We shall show that if

$$3\sigma - 2b > 1, \tag{10.16}$$

then for sufficiently large r the relations (10.9)–(10.11) are valid. Consider the system

$$\begin{aligned} Q\frac{dQ}{d\theta} &= -\mu Q - P\theta - \varphi(\theta), \\ Q\frac{dP}{d\theta} &= -\alpha P - \beta Q\theta, \end{aligned} \tag{10.17}$$

which is equivalent to (10.8) in the sets $\{\theta \geq 0, \eta > 0\}$ and $\{\theta \geq 0, \eta < 0\}$. Here P and Q are the solutions of system (10.17). It is clear that P and Q are functions of $\theta : P(\theta), Q(\theta)$.

We perform the asymptotic integration of the solutions of system (10.17) with the small parameter ε, which corresponds to the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$. For this purpose we transform (10.17) as follows:

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\theta}(Q(\theta))^2 = -\mu Q(\theta) - P(\theta)\theta - \varphi(\theta), \\ &\frac{dP(\theta)}{d\theta} = -\alpha\frac{P(\theta)}{Q(\theta)} - \beta\theta. \end{aligned}$$

Here α and μ are small parameters. In the first approximation the solutions considered can be represented in the form

$$\begin{aligned} &Q_1(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_0^\theta \theta\sqrt{1 - \tfrac{\theta^2}{2}}d\theta - \\ &-2\alpha\beta\int_0^\theta \theta\left(1 - \sqrt{1 - \tfrac{\theta^2}{2}}\right)d\theta, \\ &Q_1(\theta) \geq 0, \;\; P_1(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1 - \sqrt{1 - \tfrac{\theta^2}{2}}\right), \\ &Q_2(\theta)^2 = \theta^2 - \tfrac{\theta^4}{2} - 2\mu\int_\theta^{\sqrt{2}} \theta\sqrt{1 - \tfrac{\theta^2}{2}}\,d\theta - \tfrac{4}{3}\mu + \\ &+2\alpha\beta\int_\theta^{\sqrt{2}} \theta\left(1 + \sqrt{1 - \tfrac{\theta^2}{2}}\right)d\theta - \tfrac{2}{3}\alpha\beta \\ &Q_2(\theta) \leq 0, \;\; P_2(\theta) = -\left(\tfrac{\beta}{2}\right)\theta^2 + \alpha\beta\left(1 + \sqrt{1 - \tfrac{\theta^2}{2}}\right). \end{aligned}$$

This implies that if inequality (10.16) is satisfied, then for the certain $T > \tau$ relations (10.9)–(10.11) are valid. In addition we have

$$\begin{aligned} &\xi(T)^+ = P_2(0) = 2\alpha\beta, \\ &\eta(T)^+ = Q_2(0) = -\sqrt{8(\alpha\beta - \mu)/3} = -\sqrt{8\varepsilon(3\sigma - 2b - 1)/3\sqrt{\sigma}}. \end{aligned}$$

Thus, if inequality (10.16) is satisfied, then for sufficiently large r relations (10.9)–(10.11) are valid . $\square$

We now obtain conditions such that relations (10.9)–(10.11) do not hold and

$$\theta(t)^+ > 0, \ \ \forall\, t \in (-\infty, +\infty). \tag{10.18}$$

Consider first the case $\beta < 0$. Here for the function

$$V(\theta, \eta, \xi) = \eta^2 - \frac{1}{\beta}\xi^2 + \int_0^\theta \varphi(\theta) d\theta$$

we have

$$\dot{V}(\theta(t), \eta(t), \xi(t)) = -2\left(\mu\eta(t)^2 - \frac{\alpha}{\beta}\xi(t)^2\right). \tag{10.19}$$

Thus, for $\beta < 0$ the function V is the Lyapunov function for system (10.8). From the conditions (10.19) and $\beta < 0$ we obtain

$$V(\theta(t)^+, \eta(t)^+, \xi(t)^+) < V(\theta(-\infty)^+, \eta(-\infty)^+, \xi(-\infty)^+) = V(0,0,0) = 0, \\ \forall\, t \in (-\infty, +\infty).$$

This implies (10.18). In this case the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ does not tend to zero as $t \to +\infty$. For $\beta = 0$ we have $\xi(t)^+ \equiv 0$ and from the first two equations of system (10.8) we obtain at once (10.18). In this case the separatrix $\theta(t)^+, \eta(t)^+, \xi(t)^+$ does not tend to zero as $t \to +\infty$. $\square$

Consider the case

$$\beta = \frac{2}{b}(2\sigma - b) > 0.$$

In this case by using the change of variables

$$\eta = \sigma(x - y), \quad Q = z - x^2/(2\sigma),$$

we can reduce system (10.6) to the form

$$\begin{aligned} &\dot{x} = \eta, \\ &\dot{\eta} = -(\sigma+1)\eta + \sigma\{(r-1) - Q - \tfrac{x^2}{2\sigma}\}x, \\ &\dot{Q} = -bQ + (1 - \tfrac{b}{2\sigma})x^2. \end{aligned} \tag{10.20}$$

Consider the separatrix $x(t)^+, \eta(t)^+, Q(t)^+$ of zero saddle equilibrium such that

$$\begin{aligned} &\lim_{t\to-\infty} x(t)^+ = \lim_{t\to-\infty} \eta(t)^+ = \lim_{t\to-\infty} Q(t)^+ = 0, \\ &x(t)^+ > 0, \quad \forall\, t \in (-\infty, T). \end{aligned} \tag{10.21}$$

Find the estimates of this separatrix.

Lemma 10.3. *The estimate*

$$Q(t)^+ \geq 0, \quad \forall\, t \in (-\infty, +\infty) \tag{10.22}$$

is valid.

Proof From the inequality $2\sigma > b$ and from the last equation of (10.20) we have

$$\dot{Q}(t) \geq -bQ(t).$$

This implies that

$$Q(t) \geq \exp(-bt)Q(0).$$

Therefore (10.22) holds.

Lemma 10.4. *From condition (10.21) follows the inequality*

$$\eta(t)^+ \leq Lx(t)^+, \quad \forall\, t \in (-\infty, T), \tag{10.23}$$

where

$$L = -\frac{\sigma+1}{2} + \sqrt{\frac{(\sigma+1)^2}{4} + \sigma(r-1)}.$$

Proof Relation (10.22) and the first two equations of system (10.20) give

$$\eta(t)^+ \leq \widetilde{\eta}(t)^+, \quad \forall\, t \in (-\infty, T). \tag{10.24}$$

Here $\widetilde{\eta}(t)^+, \widetilde{x}(t)^+$ is the separatrix of zero saddle of the system

$$\begin{aligned}
&\dot{x} = \eta, \\
&\dot{\eta} = -(\sigma+1)\eta + \sigma(r-1)x.
\end{aligned}$$

Obviously, $\widetilde{\eta}(t)^+ = L\widetilde{x}(t)^+$. The lemma follows from (10.24).

Lemma 10.5. *From condition (10.21) follows the estimate*

$$Q(t)^+ \geq a(x(t)^+)^2, \quad \forall\, t \in (-\infty, T), \tag{10.25}$$

where

$$a = \frac{(2\sigma - b)}{(2\sigma(2L+b))}.$$

Proof Estimate (10.23) gives the differential inequality

$$\begin{aligned}
&\left(Q(t)^+ - a(x(t)^+)^2\right)^{\bullet} + b(Q(t)^+ - a(x(t)^+)^2) \geq \\
&\geq \left[(1 - \tfrac{b}{2\sigma}) - 2aL - ab\right](x(t)^+)^2 = 0.
\end{aligned}$$

This implies (10.25).

Consider now the Lyapunov-type function introduced in [63]:

$$V(x,\eta,Q) = \eta^2 + \sigma x^2 \left(\frac{x^2}{4\sigma} + Q - (r-1)\right) + (\sigma+1)x\eta. \tag{10.26}$$

It can easily be checked that for the solutions $x(t), \eta(t), Q(t)$ of system (10.20) we have

$$\begin{aligned}
&\dot{V}(x(t),\eta(t),Q(t)) = -(\sigma+1)V(x(t),\eta(t),Q(t)) + \\
&+\tfrac{3}{4}\left(\sigma - \tfrac{2b+1}{3}\right)x(t)^4 - b\sigma Q(t)x(t)^2.
\end{aligned} \tag{10.27}$$

Lemma 10.6. *Let the inequality*

$$3\sigma - (2b+1) < \frac{2b(2\sigma - b)}{2L+b} \tag{10.28}$$

be valid. Then condition (10.21) results in the estimate

$$\begin{aligned}&\dot V(x(t)^+,\eta(t)^+,Q(t)^+)+(\sigma+1)V(x(t)^+,\eta(t)^+,Q(t)^+)<0,\\&\forall\, t\in(-\infty,T).\end{aligned}\tag{10.29}$$

Proof From (10.28) and (10.25) we have

$$\frac{3}{4}\left(\sigma+\frac{2b+1}{3}\right)(x(t)^+)^4-b\sigma Q(t)(x(t)^+)^2<0,\quad \forall\, t\in(-\infty,T).$$

Then (10.27) yields estimate (10.29). Note that relation (10.29) results in the inequality

$$V(x(T)^+,\eta(T)^+,Q(T)^+)<0.$$

It is easy to see that

$$V(0,\eta,Q)\geq 0,\quad \forall \eta\in R^1,\ \forall Q\in R^1.$$

Therefore, if (10.28) is satisfied, then (10.29) is satisfied for all $T\in\mathbb{R}^1$.

Thus, we can formulate

Theorem 10.4 [63]. *If inequality (10.28) holds, then so does (10.18) and the separatrix $x(t)^+,\eta(t)^+,Q(t)^+$ does not tend to zero as $t\to+\infty$.*

This implies

Theorem 10.5. *If*

$$2b+1\geq 3\sigma,$$

then for any $r>1$ the homoclinic trajectory of system (10.6) does not exist.

Theorem 10.6. *If*

$$2b+1<3\sigma,$$

then for the values $r>1$ and sufficiently close to 1 the conditions (10.9)–(10.11) are not valid.

Theorems 10.3, 10.5, 10.6 imply the following

Theorem 10.7. *Given b and σ fixed, for the existence of $r\in(1,+\infty)$, corresponding to the homoclinic trajectory of the saddle $x=y=z=0$, it is necessary and sufficient that*

$$2b+1<3\sigma.\tag{10.30}$$

The sufficiency of condition (10.30) was first proved in [64, 65]. It was proved by another method (the shooting method [66]–[68]) in [69]. The papers [68, 69] involve the notes, added in the proof, about a priority of the assertion from [64].

In the papers [64, 65] the conjecture was asserted that (10.30) is a necessary condition for the existence of a homoclinic trajectory. This conjecture is proved in [69] on the basis of constructing the Lyapunov-type function (10.26).

We remark that the consideration of the smooth paths, in the space of parameters of nonlinear dynamic systems, on which there exist the points of homoclinic bifurcation, is a fruitful direction in the development of the analytic theory of global bifurcations.

We formulate now one more assertion of the same type, obtained for the Lorenz system in the paper [55].

Theorem. *Let be $\sigma=10, r=28$. Then there exists $b\in(0,+\infty)$ such that (10.6) has a homoclinic trajectory.*

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Nonlinear Dynamics of a Two-Degrees of Freedom Hamiltonian System: Bifurcations and Integration ⋆

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Abstract: In this paper we treat the motion induced by a starting pulse on a system of two-degrees of freedom s, θ. Decoupling the motion equations, we obtain the s-nonlinear ordinary differential equation

$$\ddot{s} = c^2 \frac{s}{(d^2+s^2)^2} - \lambda^2 s,$$

where $(c, d, \lambda) > 0$, and the dots mean time derivatives. A bifurcation analysis has revealed the onset of periodic motions for $\lambda \neq 0$ (presence of elastic forces inside the system), whilst for $\lambda = 0$ nonperiodic motions will appear. Almost all the cases (five for $\lambda \neq 0$, three for $\lambda = 0$) have been integrated by obtaining $t = t(s)$ by means of the Jacobi elliptic functions.
The other (angle) coordinate θ has been in any case brought to the quadratures by knowing s.

Keywords: *Nonlinear differential equations; Hamiltonian systems; bifurcations; elliptic functions.*

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1 Introduction

The cases in which the ordinary differential equations (ode) can be integrated in closed form, or even reduced to the quadratures, are quite limited.

Since much time we are involved in the integrable systems with and without friction, contributing with closed form integrations of ode by means of higher transcendental functions [3, 4, 5]. In this frame we treated single degree of freedom systems.

This article tackles a system[1] whose planar, frictionless motion depends upon *two* Lagrangian coordinates (the displacement s and the angle θ). The couple of nonlinear ode with the coordinates tied, has been de-coupled and integrated. A bifurcation analysis has been carried out on the basis of the values of a certain parameter λ which is controlling the elastic force inside the system: its description follows. An homogeneous straight pipe of mass M and length ℓ can rotate on a horizontal plane, around its middle fixed point O, without friction.

A punctual body P of mass m can flow frictionless inside it, acted by a spring which is at rest only when P≡O. Consequently the deformation of the spring entering its elastic potential, will coincide with the particle's coordinate s.

The movement is induced by a starting instantaneous pulse (s_0, $\dot{s}_0$, θ_0, $\dot{\theta}_0$): the absence of any propelling force, drag, friction is assumed then it will persist indefinitely. Let the angle θ be the pipe axis inclination, and $s = \mathrm{OP}$ be the instantaneous distance

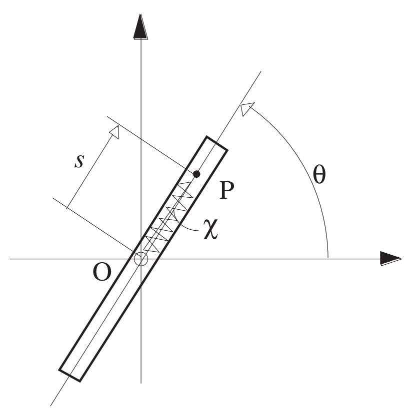

Figure 1.1: The elastic pendulum: a system's geometrical sketch.

of P from the pivot O. For the system $\mathcal{L}$-function is

$$\mathcal{L} = -\chi\frac{s^2}{2} + \frac{1}{2}J\dot{\theta}^2 + \frac{m}{2}\dot{s}^2 + \frac{m}{2}s^2\dot{\theta}^2,$$

being $J = \dfrac{1}{12}M\ell^2$ the pipe moment of inertia, and $\chi > 0$ a measure of the spring elastic stiffness.

[1] The problem has been introduced -and only sketched- on pages 279–280 of [6], where the Lagrange equations (1) and (2) are obtained, and the second order ode (6) integrated a first time. But a mistake occurred ($\dot{s}$ instead of $\dot{s}^2$). The second integration has not been carried out there, nor the motion any way analyzed, nor qualitatively discussed.

The above system could be termed as *elastic pendulum*, namely a non-circular pendulum obtained from the classic one, by replacing its unextensible, weightless rod between the body P and the suspension O, with the deformable constraint of a linear elastic spring OP. Of course the weight force on P is perpendicular to the motion's plane shown in Figure 1.1. The Lagrange equation for s gives

$$\begin{aligned} &m\,\ddot{s} + \chi\, s - m\, s\,\dot{\theta}^2 = 0, \\ &s(0) = s_0, \quad \dot{s}(0) = \dot{s}_0, \end{aligned} \tag{1}$$

while, for $\mathcal{L}$ not depending upon θ, we get the other one:

$$\begin{aligned} &\frac{d}{dt}\left(J\,\dot{\theta} + m\,s^2\,\dot{\theta}\right) = 0, \\ &\theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0. \end{aligned} \tag{2}$$

By (2), putting $\ell = 2\sqrt{3}b$ and then $J = Mb^2$, we have immediately a first integral:

$$\left(Mb^2 + m\,s^2\right)\dot{\theta} = \dot{\theta}_0\left(Mb^2 + m\,s_0^2\right) \equiv c_1 \tag{3}$$

for c_1 being a positive constant depending on both the system characteristic and the initial conditions.

2 Bifurcation Analysis in the Presence of Elastic Force

Starting from (3), we have

$$\theta(t) = \theta_0 + \frac{c_1}{m}\int_0^t \frac{d\tau}{\gamma^2 + s^2(\tau)} \tag{4}$$

with

$$\gamma^2 = \frac{M}{m}b^2 > 0, \tag{5}$$

then θ is known if we succeed in evaluating $s(t)$. Moreover, if (3) is replaced in (1), we can get rid of $\ddot{\theta}$, obtaining

$$\ddot{s} = c^2\frac{s}{\left(d^2 + s^2\right)^2} - \lambda^2 s = f(s), \tag{6}$$

where all the constants below are positive, i.e.

$$c = \frac{c_1}{m},\ d = \frac{b}{m},\ \lambda^2 = \frac{\chi}{m}. \tag{7}$$

Of course the meaning of the parameter λ is the presence, or the absence, $\lambda = 0$, of the elastic force inside the system. This will induce the system to bifurcate.

Following the Weierstraß method [8], we write the relevant time equation as

$$t = \pm\int_{s_0}^{s}\frac{du}{\sqrt{\Phi(u)}},$$

with

$$\Phi(s) = 2\int_{s_0}^{s} f(u)\,du + \dot{s}_0^2,$$

the sign has to be taken, according to the sign of the initial speed $\dot{s}_0^2$, or, if $\dot{s}_0^2 = 0$ according to the $f(s_0)$ sign, as it is well known, see e.g. [7] page 114 or [1] pages 287–292.

The $\Phi = 0$ roots' existence and kind, marks completely the motion, deciding its periodic or aperiodic nature. Obviously the reality condition $\Phi(s) \geq 0$ must be met: it is always satisfied in a neighborhood of s_0^2. We have

$$\Phi(s) = h^2 - \lambda^2 s^2 - \frac{c^2}{d^2 + s^2}, \tag{8}$$

where

$$h^2 = h^2(c, d, \lambda; s_0, \dot{s}_0) = \lambda^2 s_0^2 + \frac{c^2}{d^2 + s_0^2} + \dot{s}_0^2 > 0. \tag{9}$$

Therefore the motion reality condition stems from the positivity of the 4^{th} degree polynomial

$$p(s) = -\lambda^2 s^4 + (h^2 - d^2\lambda^2)s^2 + d^2 h^2 - c^2.$$

Such a problem is an elementary, but quite tedious exercise of Calculus. First notice that in any case, by construction, we have $p(s_0) = \dot{s}_0^2(d^2 + s_0^2) > 0$. The discussion is pivoted on the number of roots of its first derivative

$$p'(s) = 2s\left(h^2 - d^2\lambda^2 - 2s^2\lambda^2\right).$$

Therefore all the treatment is centered on the motion (6) which takes place along a rotating straight-line of variable inclination (4) during the time.

Degeneracy

If $\dot{\theta}_0 = 0$, then $c_1 = 0$, and (6) would return the elementary harmonic movement with period $2\pi/\lambda$. Then the presence of a nonzero initial pulse of angular speed $\dot{\theta}_0 \neq 0$ is essential: from now on our analysis will deal with non-degenerate cases only.

2.1 First case: $p'(s)$ has three real roots ($h^2 > d^2\lambda^2$)

Suppose first that

$$h^2 - d^2\lambda^2 > 0. \tag{10}$$

If (10) holds, the first derivative of $p(s)$ has three real roots, say

$$\hat{s} = 0, \quad \hat{s}_+ = \frac{\sqrt{h^2 - d^2\lambda^2}}{\lambda\sqrt{2}}, \quad \hat{s}_- = -\hat{s}_+.$$

Moreover (10) implies that $\hat{s} = 0$ is for $p(s)$ a relative minimum given by

$$p(0) = h^2 d^2 - c^2, \tag{11}$$

and at $\hat{s}_+$ and $\hat{s}_-$ $p(s)$ has two relative maxima whose common value is

$$p(\hat{s}_-) = p(\hat{s}_+) = \frac{\left(h^2 + d^2\,\lambda^2\right)^2}{4\lambda^2} - c^2.$$

Three following sub-cases are possible.

First, $p(s)$ has two real zeros if the condition

$$h^2 d^2 > c^2 \tag{12}$$

holds. This means that, as far as it concerns the s coordinate, the particle oscillates periodically along the pipe between the symmetrical extremes

$$s_{\pm} = \pm\frac{1}{\lambda\sqrt{2}}\sqrt{h^2 - d^2\,\lambda^2 + \sqrt{\left(h^2 + d^2\,\lambda^2\right)^2 - 4\,c^2\,\lambda^2}}. \tag{13}$$

Second, if, vice-versa, we have

$$h^2 d^2 < c^2 \tag{14}$$

and if we have also

$$\frac{\left(h^2 + d^2\lambda^2\right)^2}{4\lambda^2} > c^2, \tag{15}$$

we are in the presence of four real zeros of $p(s)$ placed symmetrically on the real axis, i.e.: $-s_2 < -s_1 < 0 < s_1 < s_2$; the motion will be periodic between the two positive or the two negative roots, according to the sign of s_0. We have

$$s_2 = \frac{1}{\lambda\sqrt{2}}\sqrt{h^2 - d^2\lambda^2 + \sqrt{\left(h^2 + d^2\lambda^2\right)^2 - 4c^2\lambda^2}}, \tag{16}$$

$$s_1 = \frac{1}{\lambda\sqrt{2}}\sqrt{h^2 - d^2\lambda^2 - \sqrt{\left(h^2 + d^2\lambda^2\right)^2 - 4c^2\lambda^2}}. \tag{17}$$

The range of coordinates $(-s_1,\, s_1)$ is forbidden to the motion for the reality condition $p(s) > 0$ being not met.

Notice that the situation

$$h^2 d^2 < c^2, \quad \frac{\left(h^2 + d^2\lambda^2\right)^2}{4\lambda^2} < c^2$$

would be against the reality of the motion, as implying all the roots of $p(s)$ to be complex, the negativity of $p(s)$, and so forth.

Finally, third, if

$$h^2 d^2 = c^2 \tag{18}$$

we are faced with a double root at the origin for $p(s)$, whose form is

$$p(s) = s^2(h^2 - d^2\lambda^2 - \lambda^2 s^2).$$

Then it will be $p(s) \geq 0$ for

$$-\frac{1}{\lambda}\sqrt{h^2 - d^2\lambda^2} \leq s \leq \frac{1}{\lambda}\sqrt{h^2 - d^2\lambda^2},$$

but the double zero at the origin implies an asymptotic motion, not a periodic one. The motion will take place for positive or negative values of s according to the sign of s_0. If $s_0 = 0$, the sign of $\dot{s}_0$ will determine the region of motion. Finally, if $s_0 = \dot{s}_0 = 0$, there will be no motion at all.

2.2 Second case: $p'(s)$ with one real root ($h^2 < d^2\lambda^2$)

Suppose that
$$h^2 - d^2\lambda^2 < 0. \tag{19}$$
Now $p(s)$ has only one stationary point in $\hat{s} = 0$ which is a maximum. The relative extremum is once more given by (11), and the motion reality is ensured again by (12).

The particle's movement is periodic between the roots singled out by (13).

2.3 Third case: $p'(s)$ has a real triple root ($h^2 = d^2\lambda^2$)

We now have
$$h^2 - d^2\lambda^2 = 0. \tag{20}$$
This means that $p(s) = -\lambda^2 s^4 + \lambda^2 d^4 - c^2,\ p'(s) = -4\lambda^2 s^3$. In order to meet the reality condition we must require that
$$d^2 > \frac{c}{\lambda}. \tag{21}$$
So the motion is periodic between the two real symmetric roots of $p(s) = 0$.

3 Integration: $\lambda \neq 0$

The time equation
$$t = \pm \int_{s_0}^{s} \frac{du}{\sqrt{h^2 - \lambda^2 u^2 - c^2(d^2 + u^2)^{-1}}} \tag{22}$$
will be solved by transforming the integral (22) in a form studied in [2], involving the I and III kind canonical elliptic integrals. For the purpose, let we pass from u to the new variable ζ defined by $u = \sqrt{\zeta^2 - d^2}$. Discarding the problem of the sign, we can take both s_0 and s positive without loss of generality. Then (22) will be transformed into
$$t = \frac{1}{2} \int_{d^2+s_0^2}^{d^2+s^2} \sqrt{\frac{\zeta}{[-\lambda^2\zeta^2 + (h^2 + d^2\lambda^2)\zeta + c^2](\zeta - d^2)}}\, d\zeta. \tag{23}$$

The discriminant Δ of the second degree polynomial
$$q(\zeta) = -\lambda^2\zeta^2 + (h^2 + d^2\lambda^2)\zeta + c^2, \tag{24}$$
appearing in (23) is
$$\Delta = (h^2 - 2c\lambda + d^2\lambda^2)(h^2 + 2c\lambda + d^2\lambda^2), \tag{25}$$
and its positivity depends on the sign of the first factor $h^2 - 2c\lambda + d^2\lambda^2$. If we consider it as a function of λ, its discriminant is
$$\Delta_1 = c^2 - d^2h^2. \tag{26}$$

Now we have to go back to the discussion about the Weierstraß function Φ introduced in (8).

1. Assume (10) and (12) hold.
2. Assume (10), (14) and (15) hold.
3. Assume (10) and (18) hold.
4. Assume (19) and (12) hold.
5. Assume (20) and (21) hold.

Case 1

The quantity Δ_1 introduced in (26) is negative and so (25) is positive. This implies that $q(\zeta)$, see (24), has two real roots $\zeta_1 < \zeta_2$, and recalling that $\zeta_1 \le d^2 + s_0^2 \le d^2 + s^2 \le \zeta_2$, we infer $s_0^2 \le \zeta - d^2 \le s^2$. We have now to locate the position of d^2 with respect to ζ_1 and ζ_2; this can be done because for (12) we have $q(d^2) = h^2 d^2 - c^2 > 0$ and this means that $\zeta_1 < d^2 < \zeta_2$. Now if we write (23) as

$$t = \frac{1}{2}\int_{d^2+s_0^2}^{d^2+s^2} \sqrt{\frac{\zeta}{(\zeta_2-\zeta)(\zeta-d^2)(\zeta-\zeta_1)}}\, d\zeta, \tag{27}$$

we can use first the integrals 256.13 page 122, and then 339.01 page 203, of [2] to evaluate (27). In fact, first we write (27) as

$$t = \frac{1}{2}\left\{\int_{d^2}^{d^2+s^2} R(\zeta)\, d\zeta - \int_{d^2}^{d^2+s_0^2} R(\zeta)\, d\zeta\right\},$$

where

$$R(\zeta) = \sqrt{\frac{\zeta}{(\zeta_2-\zeta)(\zeta-d^2)(\zeta-\zeta_1)}}.$$

In such a way the time is expressed by

$$t(s) = A(d^2+s^2) - A(d^2+s_0^2), \tag{28}$$

where

$$A(y) = \frac{1}{d\sqrt{(\zeta_2-\zeta_1)}}\left[\zeta_1 F(\varphi_1(y), k_1) + (d^2-\zeta_1)\Pi(\varphi_1(y), \alpha_1^2, k_1)\right]$$

and

$$\varphi_1(y) = \arcsin\sqrt{\frac{(\zeta_2-\zeta_1)(y-d^2)}{(\zeta_2-d^2)(y-\zeta_1)}}$$

is the amplitude of the elliptic integrals of I and III kind $F(\varphi_1, k_1)$ and $\Pi(\varphi_1, \alpha_1^2, k_1)$ of modulus k_1 and parameter α_1^2:

$$k_1^2 = \frac{(\zeta_2-d^2)\zeta_1}{(\zeta_2-\zeta_1)d^2}, \quad \alpha_1^2 = \frac{\zeta_2-d^2}{\zeta_2-\zeta_1}.$$

Of course the oscillation period T will be given by

$$\frac{T}{2} = A(d^2+s_+^2) - A(d^2+s_-^2).$$

Case 2

In such a case the inequality (15) ensures that the discriminant of $q(\zeta)$ is positive, in fact from (15) we infer that

$$h^2 + d^2\lambda^2 > 2c\lambda$$

and by this we get that the first factor of (25) is positive:

$$h^2 - 2c\lambda + d^2\lambda^2 > 2c\lambda - 2c\lambda = 0.$$

As before, we have to single out the location of d^2 with respect to the roots ζ_1 and ζ_2 of the polynomial $q(\zeta)$ introduced in (24). Taking into account that the condition (14) holds, we find out $q(d^2) = h^2 d^2 - c^2 < 0$ and then $d^2 \notin [\zeta_1, \zeta_2]$. To establish if d^2 lies on the left or on the right of $[\zeta_1, \zeta_2]$, we evaluate the half-sum Σ of ζ_1 and ζ_2 and, by (10), we find

$$\Sigma - d^2 = \frac{h^2 - d^2\lambda^2}{2\lambda^2} > 0.$$

Therefore the inequality holds:

$$d^2 < \zeta_1 < \zeta_2.$$

Henceforth the integral (27) is again evaluated by means of the formulae 256.13 page 122, and 339.01 page 203, of [2], but now the lower extreme of integration is ζ_1:

$$t(s) = \frac{1}{2}\left\{ \int_{\zeta_1}^{d^2+s^2} R(\zeta)\, d\zeta - \int_{\zeta_1}^{d^2+s_0^2} R(\zeta)\, d\zeta \right\}.$$

The time is then expressed by

$$t(s) = B(d^2+s^2) - B(d^2+s_0^2), \tag{29}$$

where

$$B(y) = \frac{1}{\sqrt{\zeta_1(\zeta_2 - d^2)}}[d^2 F(\varphi_2(y), k_2) + (\zeta_1 - d^2)\Pi(\varphi_2(y), \alpha_2^2, k_2)],$$

with

$$k_2^2 = \frac{(\zeta_2 - \zeta_1)d^2}{(\zeta_2 - d^2)\zeta_1}, \quad \alpha_2^2 = \frac{\zeta_2 - \zeta_1}{\zeta_2 - d^2}$$

and

$$\varphi_2(y) = \arcsin\sqrt{\frac{(\zeta_2 - d^2)(y - \zeta_1)}{(\zeta_2 - \zeta_1)(y - d^2)}}.$$

Of course the oscillation period T will be

$$\frac{T}{2} = B(d^2 + s_2^2) - B(d^2 + s_1^2).$$

Case 3

In this occurrence (asymptotic motion), the integration of (22) does not require elliptic integrals any longer, but elementary functions only. First, notice that solving with respect to h in (18), the condition (10) becomes:

$$c^2 - d^4\lambda^2 > 0. \tag{30}$$

Therefore by (18), (22) gives

$$t(s) = \frac{1}{\lambda}\int_{s_0}^{s} \frac{1}{u}\sqrt{\frac{d^2+u^2}{\Lambda^2 - u^2}}\, du, \tag{31}$$

where, for (30)

$$\Lambda^2 = \frac{c^2 - d^4\lambda^2}{d^2\lambda^2} > 0.$$

The integration of (31) is elementary: $t(s) = \frac{1}{\lambda}[C(s) - C(s_0)]$, where

$$C(s) = \arctan\sqrt{\frac{d^2+s^2}{\Lambda^2 - s^2}} - \frac{d}{\Lambda}\operatorname{arctanh}\sqrt{\frac{\Lambda^2(d^2+s^2)}{d^2(\Lambda^2 - s^2)}}. \tag{32}$$

Case 4 and Case 5

In these last two situations the analytical treatment is the same as in the case 1, because we find $q(d^2) > 0$. This means that $\zeta_1 < d^2 < \zeta_2$ allowing us to repeat the integration seen in the case 1.

4 Bifurcation Analysis: $\lambda = 0$

The case of the absence of the elastic force, is a *free* motion of a m-particle pulsed by some speed on a rotating straight line. Putting $\lambda = 0$ in (6), we obtain

$$\ddot{s} = c^2 \frac{s}{(d^2 + s^2)^2} \tag{33}$$

with $c = \frac{c_1}{m}, \quad d = \frac{b}{m}$. Neither (27), nor the conclusions expressed by formulae (28) and (29) involving the real roots ζ_1 and ζ_2 of $q(\zeta)$, (24), can be used for this occurrence. It is now necessary to go back to the Weierstraß method in order to write the $\lambda = 0$ time equation[2]

$$t = \pm \int_{s_0}^{s} \frac{du}{\sqrt{\Phi^*(u)}} = \pm \int_{s_0}^{s} \sqrt{\frac{d^2 + u^2}{((h^*)^2 d^2 - c^2) + (h^*)^2 u^2}}\, du,$$

where

$$h^2(c, d, \lambda; s_0, \dot{s}_0)\big|_{\lambda=0} = (h^*)^2 = \frac{c^2}{d^2 + s_0^2} + \dot{s}_0^2 > 0$$

and with the usual cautions about the sign's choice. We can see three different situations

(a) $(h^*)^2 d^2 > c^2$, which implies $\Phi^*(s) > 0$ for any s (aperiodic motion for any allowable s);

(b) $(h^*)^2 d^2 < c^2$, which implies $\Phi^*(s) > 0$ for $s^2 > c^2 (h^*)^{-2} - d^2$ and $\Phi^*(s) = 0$ for $s^2 = c^2 (h^*)^{-2} - d^2$ (simple root), (aperiodic motion with forbidden region);

(c) $(h^*)^2 d^2 = c^2$, which implies $\Phi^*(s) > 0$ for any $s > 0$ and $\Phi^*(0) = 0$, double root (asymptotic motion towards the origin).

The reader should be aware that the physical sense is fully met by the analytical discussion just done: in fact the elastic force disappearance is the physical cause leaving any periodicity from the straight-linear motions.

5 Integration: $\lambda = 0$

After the former discussion, we perform the relevant integration.

[2] We mark by a star $(*)$ the quantities Φ and h in the case $\lambda = 0$. On the contrary, the same symbols s_0 and $\dot{s}_0$ have been kept for meaning the initial conditions also in the $\lambda = 0$ motion. If a $\lambda \neq 0$ motion previously took place, the last computed values by (6), will provide the initial conditions input for (33).

Case (a)

Let us write the numerator of Φ^* as

$$(h^*)^2\left(d^2 - \frac{c^2}{(h^*)^2}\right) + (h^*)^2 s^2 = (h^*)^2(\Gamma^2 + s^2),$$

where hypothesis (a) ensures that $0 < \Gamma^2 = d^2 - \frac{c^2}{(h^*)^2} < d^2$. Discarding the sign (i.e. we can take $\dot{s}_0 > 0$ with no loss of generality) we find

$$t(s) = \frac{1}{h^*}\left\{\int_0^s \sqrt{\frac{d^2+u^2}{\Gamma^2+u^2}}\,du - \int_0^{s_0} \sqrt{\frac{d^2+u^2}{\Gamma^2+u^2}}\,du\right\}. \tag{34}$$

Both integrals at the right hand side of (34) are once more evaluated in [2]: first we use integral 221.03 page 61, and then integral 321.02 page 198, obtaining

$$t = \frac{1}{h^*}[A_0(s) - A_0(s_0)], \tag{35}$$

where

$$\begin{aligned} A_0(y) = d\,[&F(\varphi_3(y), k_3) - E(\varphi_3(y), k_3) \\ &+ \mathrm{dn}(F(\varphi_3(y), k_3), k_3)\,\mathrm{tn}(F(\varphi_3(y), k_3), k_3)] \end{aligned} \tag{36}$$

is a function depending on y through the amplitude $\varphi_3(y)$

$$\varphi_3(y) = \arctan\frac{y}{\Gamma}, \quad k_3^2 = \frac{d^2 - \Gamma^2}{d^2} = \frac{c^2}{(h^*)^2 d^2},$$

and where $u = F(\varphi_3, k_3)$ and $E(\varphi_3, k_3)$ are the Legendre elliptic integrals of I and II kind with modulus k_3 and amplitude φ_3; $\mathrm{dn}\, u$, $\mathrm{tn}\, u$ are two Jacobian elliptic functions of argument u and modulus k_3.

Case (b)

Once again let the numerator of Φ^* be written as $(h^*)^2(s^2 - \Theta^2)$, where

$$\Theta^2 = \frac{c^2}{(h^*)^2} - d^2 > 0. \tag{37}$$

In such a way, the relevant time equation, defined for $0 < \Theta \le s_0 \le s$, taking the square root's positive determination and minding (37), becomes

$$t = \frac{1}{h^*}\left\{\int_\Theta^s \sqrt{\frac{u^2+d^2}{u^2-\Theta^2}}\,du - \int_\Theta^{s_0} \sqrt{\frac{u^2+d^2}{u^2-\Theta^2}}\,du\right\}. \tag{38}$$

To evaluate the integrals in (38), we refer for the last time in this paper, to [2], integrals 211.03 page 82 and 321.02 page 198. We find

$$t = \frac{1}{h^*}[B_0(s) - B_0(s_0)], \tag{39}$$

where

$$\begin{aligned} B_0(y) = \frac{c}{(h^*)^2}\,[&F(\varphi_4(y), k_4) - E(\varphi_4(y), k_4) \\ &+ \mathrm{dn}(F(\varphi_4(y), k_4), k_4)\,\mathrm{tn}(F(\varphi_4(y), k_4), k_4)]\,, \end{aligned} \tag{40}$$

is a function of y through the amplitude $\varphi_4(y)$:

$$\varphi_4(y) = \arccos \frac{\sqrt{c^2 - (h^*)^2 d^2}}{h^* y}, \quad k_4^2 = \frac{d^2 (h^*)^2}{c^2}. \tag{41}$$

As usually, $u = F(\varphi_4, k_4)$ and $E(\varphi_4, k_4)$ denote the Legendre elliptic integrals of I and II kind with modulus k_4 and amplitude φ_4. Furthermore $\operatorname{dn} u$, $\operatorname{tn} u$ are two Jacobi elliptic functions of argument u and modulus k_4. Finally, notice that in (41) and in (40) we used the identity

$$\sqrt{d^2 + \Theta^2} = \sqrt{\frac{c^2}{(h^*)^2}} = \frac{c}{h^*}.$$

Case (c)

In such asymptotic sub-case, the time equation, for positive initial speed and for the spatial coordinate s, is

$$t(s) = \frac{1}{h^*} \int_{s_0}^{s} \frac{\sqrt{u^2 + d^2}}{u} \, du. \tag{42}$$

The integral in (42) is elementary:

$$t(s) = \frac{1}{h^*} [C_0(s) - C_0(s_0)], \tag{43}$$

where

$$C_0(y) = \sqrt{y^2 + d^2} - d \ln \frac{2\left(d + \sqrt{y^2 + d^2}\right)}{d^2 y}.$$

6 Conclusions

We summarize five points, without degeneracy, $c \neq 0$, i.e. with $\dot{\theta}_0 \neq 0$.

(i) s-motions under elastic force

The s-motions we examined in the presence of the elastic force ($\lambda \neq 0$) are five, as grasped by the table

Conditions	$p(s)$ behavior	s-motion	case
(10) & (12)	2 real roots	symmetric oscillation	1
(10) & (14)	4 real roots	asymmetric oscillation	2
(10) & (18)	double root $s = 0$	asymptotic behavior	3
(19) & (12)	2 real roots	symmetric oscillation	4
(20) & (21)	2 real roots	symmetric oscillation	5

which is self-explanatory.

Almost all of the s-motions with $\lambda \neq 0$ are oscillatory, except the case 3, which is asymptotic, and whose time law is depending upon elementary functions.

In the cases 1, 2, 4 and 5, time is linked to the coordinate s by means of the I and III kind elliptic integrals, whose upper bound is algebraically tied to s. Each oscillatory motion, according to its initial conditions, can have a double nature: either symmetric or not symmetric, namely centered or not around the origin O of the reference.

(ii) s-motions with no forces

An alternative situation is that of the spring cut-off ($\lambda = 0$): further nonlinear (but in no way oscillatory) s-motions have been so found (whose nature is decided by $(h^*)^2 d^2 \gtreqless c^2$) and ruled by different elliptic functions.

(iii) The angle θ

The time equation concerning the angle θ is given by (4), a formula which needs to know s as a function of the time, and then the 5+3 analytical solutions linking t to s. Even if for each case we gave the relevant plots of s versus the time, it should be clear that nobody can invert *formally* the relevant functions[3] $t = t(s)$; and then the θ-integral (4) requiring $s = s(t)$ cannot be in any way evaluated in closed form.

However our explicit formulae for $t = t(s)$ allow an easy tabulation of $s = s(t)$, and therefore one might implement some numerical integration algorithm for getting θ as a (tabular) function of the time.

The θ time-behavior will be always growing: rotations cannot in fact extinguish ever, because neither friction nor drag are consuming the initial pulse.

(iv) Trajectory

The planar *trajectory* of P, see Figure 1.1, might be obtained in a polar reference, 0 being the pole, assuming θ as anomaly, and the absolute value of s as radius. For the purpose, one should try to eliminate the time between $s = s(t)$ and $\theta = \theta(t)$. No hope this could be accomplished in closed form.

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[3] The functions we mean are expressed by the formulae: (28), (29), (32), (35), (39), (43), and those strictly tied to them.

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