Volume 3, Number 1, 2003 ISSN 1562-8353

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

Academic Periodical Press

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

 ISSN 1562-8353 Printed in Ukraine

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published since 2001

Volume 3 Number 1 2003

CONTENTS

Founded by A.A.Martynyuk in 2001.
Registered in Ukraine Number: KB №5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562-8353) is an international journal published under the auspices of the S.P.Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and the Laboratory for Industrial and Aplied Mathematics (LIAM) at York University (Toronto, Canada). It is aimed at publishing high quality original scientific papers and surveys in area of nonlinear dynamics and systems theory and technical reports on solving practical problems. The scope of the journal is very broad covering:

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PUBLICATION AND SUBSCRIPTION INFORMATION

The Nonlinear Dynamics and Systems Theory (ISSN 1562-8353) is published two times per year in 2003.

Base list subscription price per volume: US$99.00. This price is available only to individuals whose library subscribes to the journal OR who warrant that the Journal is for their own use and provide a home address for mailing. Separate rates apply to academic and corporate/ government institutions. Postage and handling charges are extra. Our charge includes postage, packing, handling and airmail delivery of all issues.

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ABSTRACTING INFORMATION

Nonlinear Dynamics and Systems Theory is currently abstracted/indexed by Zentralblatt MATH and Mathematical Reviews.

Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 1–22

Asymptotic Stability for a Conducting Electromagnetic Material with a Dissipative Boundary Condition

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Received: February 14, 2002; Revised: December 13, 2002

Abstract: In this work we study the existence and uniqueness of the solution for a linear electromagnetic material characterized by the memory effects due to a rate-type equation for the electric conduction when a general dissipative boundary condition is assumed on the boundary of the solid. We show the existence of a domain of dependence and we give some limitations of the values of the material constants which assure the asymptotic stability of the solution.

Keywords: *Linear electromagnetism; asymptotic stability.*

Mathematics Subject Classification (2000): 78A25, 35B35, 35B40.

1 Introduction

The purpose of this paper is to keep on studying memory effects in electromagnetic systems, which occur through a rate-type equation for the electric conduction. Its presence in the system of equations has been recently considered in another work [5], where we have supposed that the boundary of the solid is a perfect conductor.

In the present work a homogeneous, isotropic and conducting solid, characterized also by linear constitutive equations for the electric displacement and the magnetic induction, is considered on supposing that a general dissipative boundary condition holds on its boundary.

After introducing the field equations, the thermodynamic restrictions on the constitutive equations and the free energy in Section 2, we formulate the initial-boundary value problem. Thus, we show that a domain of dependence inequality exists for these bodies and we derive a useful energy estimate.

†Work performed under the auspices of C.N.R. and M.U.R.S.T..

In Section 4 we prove the existence and uniqueness theorems for the weak and the strong solution of this evolutive problem; then, we study the asymptotic stability, which holds under suitable hypotheses on the values of the material constants of the medium.

2 Basic Equations and Thermodynamic Restrictions

Let $\mathcal{B}$ be an electromagnetic solid, which occupies at time t a bounded and regular domain $\Omega \subset R^3$ with a smooth boundary $\partial\Omega$.

The electromagnetic phenomena of $\mathcal{B}$ are described by Maxwell's equations

$$\dot{\mathbf{D}}(x,t) = \nabla \times \mathbf{H}(x,t) - \mathbf{J}(x,t) - \mathbf{J}_f(x,t), \quad \dot{\mathbf{B}}(x,t) = -\nabla \times \mathbf{E}(x,t), \tag{2.1}$$

$$\nabla \cdot \mathbf{D}(x,t) = 0, \qquad \nabla \cdot \mathbf{B}(x,t) = 0, \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic fields, $\mathbf{J}$ is the electric current density, $\mathbf{D}$ is the electric displacement, $\mathbf{B}$ is the magnetic induction; moreover, $\mathbf{J}_f$ is a forced current density which must be considered as a given function of the position $x \in \Omega$ and $t \in R^+$. In $(2.2)_1$ we have supposed that the free charge density is zero.

Besides Maxwell's equations we must consider the thermodynamic principles $[1-2]$.

The Dissipation Principle states that for any cyclic process the following inequality

$$\oint [\dot{\mathbf{D}}(x,t) \cdot \mathbf{E}(x,t) + \dot{\mathbf{B}}(x,t) \cdot \mathbf{H}(x,t) + \mathbf{J}(x,t) \cdot \mathbf{E}(x,t)]\, dt \geq 0 \tag{2.3}$$

holds, the equality sign referring to reversible processes.

The Second Law for smooth isothermal processes yields

$$\dot{\psi}(x,t) \leq \dot{\mathbf{D}}(x,t) \cdot \mathbf{E}(x,t) + \dot{\mathbf{B}}(x,t) \cdot \mathbf{H}(x,t) + \mathbf{E}(x,t) \cdot \mathbf{J}(x,t), \tag{2.4}$$

where ψ is the free energy.

Let us assume that $\mathcal{B}$ is a homogeneous and isotropic conductor, whose constitutive equations are linear and given by

$$\mathbf{D}(x,t) = \varepsilon \mathbf{E}(x,t), \qquad \mathbf{B}(x,t) = \mu \mathbf{H}(x,t), \tag{2.5}$$

where both the dielectric constant ε and the permeability μ are positive constants. For the electric conduction we suppose that the following rate-type equation

$$\alpha \dot{\mathbf{J}}(x,t) + \mathbf{J}(x,t) = \sigma \mathbf{E}(x,t) \tag{2.6}$$

holds, where α is a positive parameter and σ denotes the conductivity, which is assumed constant too.

Using (2.5) and the relation derived from (2.6) for $\mathbf{E}$, inequality (2.4) becomes [5]

$$\oint \frac{d}{dt} \frac{1}{2} \left(\varepsilon \mathbf{E}^2 + \mu \mathbf{H}^2 + \frac{\alpha}{\sigma} \mathbf{J}^2 \right) dt + \oint \frac{1}{\sigma} \mathbf{J}^2\, dt \geq 0, \tag{2.7}$$

which, taking into account that the integration is made on cycles, yields

$$\sigma > 0. \tag{2.8}$$

Finally, we can introduce the free energy

$$\psi(x,t) = \frac{1}{2}\left[\varepsilon \mathbf{E}^2(x,t) + \mu \mathbf{H}^2(x,t) + \frac{\alpha}{\sigma}\,\mathbf{J}^2(x,t)\right], \tag{2.9}$$

which satisfies (2.4) on account of (2.6).

3 Formulation of the Problem and Domain of Dependence

Maxwell's equations (2.1), taking account of the constitutive equations (2.5), and (2.6) take the form

$$\nabla \times \mathbf{H}(x,t) - \varepsilon \dot{\mathbf{E}}(x,t) - \mathbf{J}(x,t) = \mathbf{f}(x,t), \tag{3.1}$$

$$\nabla \times \mathbf{E}(x,t) + \mu \dot{\mathbf{H}}(x,t) = \mathbf{g}(x,t), \tag{3.2}$$

$$\alpha \dot{\mathbf{J}}(x,t) + \mathbf{J}(x,t) - \sigma \mathbf{E}(x,t) = \mathbf{l}(x,t), \tag{3.3}$$

on introducing the source terms $\mathbf{g}$ and $\mathbf{l}$, which are two known functions of x and t as well as $\mathbf{f} \equiv \mathbf{J}_f$; the other two equations (2.2) reduce to

$$\nabla \cdot \mathbf{E}(x,t) = 0, \qquad \nabla \cdot \mathbf{H}(x,t) = 0, \tag{3.4}$$

in $\Omega \times R^+$.

The initial conditions are

$$\mathbf{E}(x,0) = \mathbf{E}_0(x), \quad \mathbf{H}(x,0) = \mathbf{H}_0(x), \quad \mathbf{J}(x,0) = \mathbf{J}_0(x) \quad \forall\, x \in \Omega, \tag{3.5}$$

while on $\partial\Omega$ we consider a linear dissipative boundary condition, characterized by the following definition [8].

We first denote by Σ the set of the states, to which $\mathbf{E}$ and $\mathbf{H}$ belong together $\mathbf{J}$, and introduce the function space

$$I(\Omega) = \left\{ \mathbf{E} \in L^2(\Omega) \colon \int_\Omega \mathbf{E}(x,t) \cdot \nabla \phi(x,t)\, dx = 0 \quad \forall\, \phi \in C_0^\infty(\Omega, R) \right\},$$

which allows us to consider equations (3.4) automatically satisfied if both $\mathbf{E}$ and $\mathbf{H}$ belong to it.

Definition 3.1 A *linear* and *dissipative boundary condition* $\Sigma' \subset \Sigma$ is a *linear closed subset* of $I(\Omega) \times I(\Omega)$ such that $C_0^1(\Omega) \times C_0^1(\Omega) \subset \Sigma'$ and

$$\int_{\partial\Omega} \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x)\, da \geq 0 \quad \forall\, (\mathbf{E}, \mathbf{H}) \in \Sigma' \tag{3.6}$$

with the static condition

$$\int_{\partial\Omega} |\mathbf{H} \cdot \mathbf{n}|^2\, da = 0,$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

We shall denote by P the problem (3.1)–(3.6).

Lemma 3.1 *The electromagnetic fields* $\mathbf{E}$ *and* $\mathbf{H}$ *satisfy this inequality*

$$\chi\psi(x,t) + \mathbf{E}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{u}(x) \geq 0, \qquad \chi = 2(\varepsilon\mu)^{-1/2}, \tag{3.7}$$

for any unit vector $\mathbf{u}(x)$.

Proof From the definition of the free energy, given by (2.9), it follows that

$$|\mathbf{E}(x,t)| \leq [2\psi(x,t)/\varepsilon]^{1/2}, \qquad |\mathbf{H}(x,t)| \leq [2\psi(x,t)/\mu]^{1/2} \tag{3.8}$$

and hence we obtain (3.7) easily.

Let

$$E(A,t) = \int_A \psi(x,t)\, dx \tag{3.9}$$

be the total energy for every domain $A \subset \Omega$, where ψ is given by (2.9), we have the following theorem.

Theorem 3.1 *If the triplet* $(\mathbf{E}, \mathbf{H}, \mathbf{J})$ *is a solution of the problem* P, *for every* $(x_0, T) \in \Omega \times R^+$ *the total energy satisfies*

$$\begin{aligned} E(B(x_0,\rho),T) \leq E(B(x_0,\rho+\chi T),0) + \int_0^T \int_{\Omega\cap B(x_0,\rho+\chi(T-t))} [\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma \\ + \mathbf{g}(x,t)\cdot\mathbf{H}(x,t) - \mathbf{f}(x,t)\cdot\mathbf{E}(x,t)]\, dx\, dt, \end{aligned} \tag{3.10}$$

where χ *is given by* $(3.7)_2$ *and* $B(x_0,\rho) = \{x \in \Omega\colon |x - x_0| \leq \rho\}$.

Proof We introduce the weighted energy

$$E_\phi(\Omega,t) = \int_\Omega \psi(x,t)\phi(x,t)\, dx, \tag{3.11}$$

where ψ is expressed by (2.9) in terms of the solution $(\mathbf{E}, \mathbf{H}, \mathbf{J})$ of the problem P and $\phi(x,t) \in C_0^\infty(\Omega, R^+)$, and we consider its derivative with respect to time, where $\dot{\mathbf{E}}$, $\dot{\mathbf{H}}$ and $\dot{\mathbf{J}}$ can be eliminated by means of (3.1)–(3.3). Using the identity $\nabla\times\mathbf{E}\cdot\mathbf{H} - \nabla\times\mathbf{H}\cdot\mathbf{E} = \nabla\cdot(\mathbf{E}\times\mathbf{H})$, we get

$$\begin{aligned} \dot{E}_\phi(\Omega,t) = \int_\Omega [\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma + \mathbf{g}(x,t)\cdot\mathbf{H}(x,t) - \mathbf{f}(x,t)\cdot\mathbf{E}(x,t) \\ - \mathbf{J}^2(x,t)/\sigma]\phi(x,t)\, dx + \int_\Omega [\mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\nabla\phi(x,t) + \psi(x,t)\dot{\phi}(x,t)]dx \\ - \int_{\partial\Omega} \mathbf{E}(x,t)\times\mathbf{H}(x,t)\cdot\mathbf{n}(x)\phi(x,t)da. \end{aligned} \tag{3.12}$$

Following [4], we put $\phi(x,t) = \phi_\delta(x,t) = \phi_\delta(y) \in C_0^\infty(R)$, a monotonic decreasing function of $y = |x - x_0| - \rho - \chi(T-t)$, with $\rho > 0$, $(x_0,T) \in \Omega\times\mathbf{R}^+$, χ given by

$(3.7)_2$, such that $\phi_\delta(y) = 1$ for all $y \le -\delta$, $\phi_\delta(y) = 0$ for all $y > \delta$, $\phi'_\delta(y) \le 0$, $\dot\phi_\delta(x,t) = \chi\phi'_\delta(y)$, $\nabla\phi_\delta(x,t) = \phi'_\delta(y)\nabla|x-x_0|$, for any $(x,t) \in \Omega\times(0,T)$. Thus, from (3.12), using (3.7) and the properties of ϕ_δ, it follows an inequality, which, integrated over $(0,T)$, yields

$$\begin{aligned} E_{\phi_\delta}(\Omega,T) - E_{\phi_\delta}(\Omega,0) \le \int_0^T\int_\Omega [\mathbf{l}(x,t)\cdot\mathbf{J}(x,t)/\sigma + \mathbf{g}(x,t)\cdot\mathbf{H}(x,t) \\ - \mathbf{f}(x,t)\cdot\mathbf{E}(x,t)]\phi_\delta(x,t)\,dx\,dt, \end{aligned} \tag{3.13}$$

whose limit as δ tends to zero gives (3.10), since ϕ_δ tends to the characteristic function of the subset $B(x_0, \rho + \chi(T-t))$.

From this theorem a useful estimate of the energy can be derived as follows.

Corollary 3.1 *For any solution* $(\mathbf{E},\mathbf{H},\mathbf{J})$ *of the problem* P *we have this inequality*

$$E(\Omega,t) \le e^T\Big\{E(\Omega,0) + M\int_0^T\int_\Omega [\mathbf{f}^2(x,t) + \mathbf{g}^2(x,t) + \mathbf{l}^2(x,t)]\,dx\,dt\Big\}, \tag{3.14}$$

where $M = \max\{2/\varepsilon, 2/\mu, 2/(\alpha\sigma)\}$ *and* $t \in (0,T)$.

Proof If ρ is large enough, (3.10) yields

$$E(\Omega,t) - E(\Omega,0) \le \int_0^t\int_\Omega [\mathbf{l}(x,\tau)\cdot\mathbf{J}(x,\tau)/\sigma + \mathbf{g}(x,\tau)\cdot\mathbf{H}(x,\tau) - \mathbf{f}(x,\tau)\cdot\mathbf{E}(x,\tau)]\,dx\,d\tau, \tag{3.15}$$

where $t\in(0,T)$.

Applications of Schwarz's inequality allow us to increase the integral as follows

$$\begin{aligned} &\int_0^t\int_\Omega (\mathbf{l}\cdot\mathbf{J}/\sigma + \mathbf{g}\cdot\mathbf{H} - \mathbf{f}\cdot\mathbf{E})\,dx\,d\tau \le \int_0^t \Big(\int_\Omega \frac{1}{\sigma}\mathbf{l}^2\,dx\Big)^{1/2}\Big(\int_\Omega \frac{1}{\sigma}\mathbf{J}^2\,dx\Big)^{1/2} d\tau \\ &+ \int_0^t \Big(\int_\Omega \mathbf{g}^2\,dx\Big)^{1/2}\Big(\int_\Omega \mathbf{H}^2\,dx\Big)^{1/2} d\tau + \int_0^t \Big(\int_\Omega \mathbf{f}^2\,dx\Big)^{1/2}\Big(\int_\Omega \mathbf{E}^2\,dx\Big)^{1/2} d\tau \\ &\le \Big(\int_0^t \frac{2}{\alpha\sigma}\int_\Omega \mathbf{l}^2\,dx\,d\tau\Big)^{1/2}\Big(\int_0^t \frac{1}{2}\int_\Omega \frac{\alpha}{\sigma}\mathbf{J}^2\,dx\,d\tau\Big)^{1/2} + \Big(\int_0^t \frac{2}{\mu}\int_\Omega \mathbf{g}^2\,dx\,d\tau\Big)^{1/2} \\ &\times \Big(\int_0^t \frac{1}{2}\int_\Omega \mu\mathbf{H}^2\,dx\,d\tau\Big)^{1/2} + \Big(\int_0^t \frac{2}{\varepsilon}\int_\Omega \mathbf{f}^2\,dx\,d\tau\Big)^{1/2}\Big(\int_0^t \frac{1}{2}\int_\Omega \varepsilon\mathbf{E}^2\,dx\,d\tau\Big)^{1/2} \\ &\quad \le \Big[\int_0^t \frac{1}{2}\int_\Omega \Big(\varepsilon\mathbf{E}^2 + \mu\mathbf{H}^2 + \frac{\alpha}{\sigma}\mathbf{J}^2\Big)dx\,d\tau\Big]^{1/2}\Big[\Big(\int_0^t \frac{2}{\varepsilon}\int_\Omega \mathbf{f}^2\,dx\,d\tau\Big)^{1/2} \\ &\qquad + \Big(\int_0^t \frac{2}{\mu}\int_\Omega \mathbf{g}^2\,dx\,d\tau\Big)^{1/2} + \Big(\int_0^t \frac{2}{\alpha\sigma}\int_\Omega \mathbf{l}^2\,dx\,d\tau\Big)^{1/2}\Big]. \end{aligned} \tag{3.16}$$

Using the elementary inequality $2ab \le a^2 + b^2$, we have $2(ab)^{1/2} \le a + b$; then, $(a^{1/2} + b^{1/2} + c^{1/2})^2 = a + b + c + 2(ab)^{1/2} + 2(ac)^{1/2} + 2(bc)^{1/2} \le 4(a + b + c)$ and hence $a^{1/2} + b^{1/2} + c^{1/2} \le 2(a + b + c)^{1/2}$.

Therefore, (3.16) becomes

$$\int_0^t \int_\Omega (\mathbf{l} \cdot \mathbf{J}/\sigma + \mathbf{g} \cdot \mathbf{H} - \mathbf{f} \cdot \mathbf{E}) dx\, d\tau \tag{3.17}$$

$$\le \Big[\int_0^t E(\Omega, \tau)\, d\tau \Big]^{1/2} 2 \Big(\int_0^t \frac{2}{\varepsilon} \int_\Omega \mathbf{f}^2\, dx\, d\tau + \int_0^t \frac{2}{\mu} \int_\Omega \mathbf{g}^2\, dx\, d\tau + \int_0^t \frac{2}{\alpha\sigma} \int_\Omega \mathbf{l}^2\, dx\, d\tau \Big)^{1/2}$$

$$\le \int_0^t E(\Omega, \tau)\, d\tau + \int_0^t \int_\Omega \Big(\frac{2}{\varepsilon} \mathbf{f}^2 + \frac{2}{\mu} \mathbf{g}^2 + \frac{2}{\alpha\sigma} \mathbf{l}^2 \Big) dx\, d\tau$$

and (3.15), with

$$\xi(t) = \int_0^t E(\Omega, \tau)\, d\tau, \quad \xi'(t) = E(\Omega, t), \quad \xi'(0) = E(\Omega, 0), \tag{3.18}$$

can be written as follows

$$\xi'(t) - \xi'(0) \le \xi(t) + M \int_0^T \int_\Omega (\mathbf{f}^2 + \mathbf{g}^2 + \mathbf{l}^2)\, dx\, dt. \tag{3.19}$$

Putting

$$a = \xi'(0) + M \int_0^T \int_\Omega (\mathbf{f}^2 + \mathbf{g}^2 + \mathbf{l}^2)\, dx\, dt, \tag{3.20}$$

(3.19) reduces to

$$\xi'(t) \le \xi(t) + a \quad \forall\, t \in (0, T). \tag{3.21}$$

From the last inequality, integrating with $\xi(0) = 0$, we have

$$\xi(t) \le a(e^t - 1), \tag{3.22}$$

which allows us to derive from (3.21)

$$\xi'(t) \le a e^t \tag{3.23}$$

and hence to obtain (3.14).

4 Existence and Uniqueness Theorem

To study the existence and uniqueness of the solution to the problem P, we consider the following function spaces

$$
\begin{aligned}
I_1(\Omega) &= \{\mathbf{E} \in I(\Omega) \colon \nabla \times \mathbf{E} \in L^2(\Omega)\}, \\
H^1_\alpha(\Omega,(0,T)) &= L^2((0,T); I_1(\Omega)) \cap H^1((0,T); L^2(\Omega)), \\
H^1_\beta(\Omega,(0,T)) &= H^1((0,T); L^2(\Omega)), \\
\mathcal{H}(\Omega,(0,T)) &= \{(\mathbf{E},\mathbf{H},\mathbf{J}) \in H^1_\alpha(\Omega,(0,T)) \times H^1_\alpha(\Omega,(0,T)) \times H^1_\beta(\Omega,(0,T)) \colon \\
&\qquad (\mathbf{E},\mathbf{H}) \ \textit{satisfies (3.6) on} \ \ \partial\Omega \times (0,T)\}, \\
\mathcal{W}(\Omega,(0,T)) &= \{(\mathbf{E},\mathbf{H},\mathbf{J}) \in L^2((0,T); I(\Omega)) \times L^2((0,T); I(\Omega)) \\
&\qquad \times L^2((0,T); L^2(\Omega)) \colon (\mathbf{E},\mathbf{H}) \ \textit{satisfies (3.6) on} \ \ \partial\Omega \times (0,T)\},
\end{aligned}
$$

together with

$$
\begin{aligned}
\mathcal{W}_0(\Omega,(0,T)) &= L^2((0,T); L^2(\Omega)) \times L^2((0,T); I(\Omega)) \times L^2((0,T); L^2(\Omega)), \\
\mathcal{W}_1(\Omega,(0,T)) &= L^2((0,T); I_1(\Omega)) \times L^2((0,T); I_1(\Omega)) \times L^2((0,T); L^2(\Omega)),
\end{aligned}
$$

where $(0,T) \subset R^+$.

Definition 4.1 We call *strong solution of* P with sources $(\mathbf{f},\mathbf{g},\mathbf{l}) \in \mathcal{W}_0(\Omega,(0,T))$ and *initial data* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I(\Omega) \times I(\Omega) \times L^2(\Omega)$ any triplet $(\mathbf{E},\mathbf{H},\mathbf{J}) \in \mathcal{H}(\Omega,(0,T))$ which satisfies almost everywhere (3.1)–(3.3) in $\Omega \times (0,T)$ and (3.5) in Ω.

Definition 4.2 We call *weak solution of* P with sources $(\mathbf{f},\mathbf{g},\mathbf{l}) \in \mathcal{W}_0(\Omega,(0,T))$ and *initial data* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I(\Omega) \times I(\Omega) \times L^2(\Omega)$ any triplet $(\mathbf{E},\mathbf{H},\mathbf{J}) \in \mathcal{W}(\Omega,(0,T))$ such that the following identity

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big\{ [\varepsilon \dot{\mathbf{e}}(x,t) - \nabla \times \mathbf{h}(x,t) + \mathbf{p}(x,t)] \cdot \mathbf{E}(x,t) + [\mu \dot{\mathbf{h}}(x,t) + \nabla \times \mathbf{e}(x,t)] \\
&\quad \cdot \mathbf{H}(x,t) + \left[\frac{\alpha}{\sigma}\, \dot{\mathbf{p}}(x,t) - \frac{1}{\sigma}\, \mathbf{p}(x,t) - \mathbf{e}(x,t) \right] \cdot \mathbf{J}(x,t) - \mathbf{f}(x,t) \cdot \mathbf{e}(x,t) \\
&\quad + \mathbf{g}(x,t) \cdot \mathbf{h}(x,t) + \mathbf{l}(x,t) \cdot \frac{1}{\sigma}\, \mathbf{p}(x,t) \Big\} dx\, dt - \int_0^T \int_{\partial\Omega} [\mathbf{e}(x,t) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x) \\
&\quad + \mathbf{E}(x,t) \times \mathbf{h}(x,t) \cdot \mathbf{n}(x)]\, dx\, dt + \int_\Omega [\varepsilon \mathbf{E}_0(x) \cdot \mathbf{e}(x,0) + \mu \mathbf{H}_0(x) \cdot \mathbf{h}(x,0) \\
&\quad + \frac{\alpha}{\sigma}\, \mathbf{J}_0(x) \cdot \mathbf{p}(x,0)]\, dx = 0,
\end{aligned}
\tag{4.1}
$$

holds for any $(\mathbf{e},\mathbf{h},\mathbf{p}) \in \mathcal{H}(\Omega,(0,T))$ such that

$$
\mathbf{e}(x,T) = \mathbf{0}, \quad \mathbf{h}(x,T) = \mathbf{0}, \quad \mathbf{p}(x,T) = \mathbf{0}. \tag{4.2}
$$

We now prove the uniqueness and the existence of the weak solution.

Theorem 4.1 (Uniqueness) *The problem* P *has at most one solution in the sense of Definition 4.2.*

Proof The identity (4.1) must hold for any $(\mathbf{e},\mathbf{h},\mathbf{p}) \in \mathcal{H}(\Omega,(0,T))$; therefore, following [6], we can choose

$$
\begin{aligned}
\mathbf{e}(x,t) &= \begin{cases} (\tau - t)\mathbf{a}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases}, \\
\mathbf{h}(x,t) &= \begin{cases} (\tau - t)\mathbf{b}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases}, \\
\mathbf{p}(x,t) &= \begin{cases} (\tau - t)\mathbf{c}(x) & 0 \le t \le \tau \\ \mathbf{0} & \tau \le t \le T \end{cases},
\end{aligned}
\tag{4.3}
$$

where τ is a fixed value of $(0,T)$ and $(\mathbf{a},\mathbf{b},\mathbf{c})$ is an arbitrary triplet of $I_1(\Omega) \times I_1(\Omega) \times L^2(\Omega)$.

Substituting (4.3) into (4.1), we obtain

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \Big\{ (\tau - t)[\nabla \times \mathbf{a}(x) \cdot \mathbf{H}(x,t) - \nabla \times \mathbf{b}(x) \cdot \mathbf{E}(x,t) - \mathbf{J}(x,t) \cdot \mathbf{c}(x)/\sigma \\
&- (\mathbf{J}(x,t) + \mathbf{f}(x,t)) \cdot \mathbf{a}(x) + \mathbf{g}(x,t) \cdot \mathbf{b}(x) + (\sigma \mathbf{E}(x,t) + \mathbf{l}(x,t)) \cdot \mathbf{c}(x)/\sigma] \\
&\quad - \left[\varepsilon \mathbf{a}(x) \cdot \mathbf{E}(x,t) + \mu \mathbf{b}(x) \cdot \mathbf{H}(x,t) + \frac{\alpha}{\sigma}\, \mathbf{c}(x) \cdot \mathbf{J}(x,t) \right] \Big\} dx\, dt \\
&\quad - \int_0^\tau \int_{\partial\Omega} (\tau - t)[\mathbf{a}(x) \times \mathbf{H}(x,t) \cdot \mathbf{n}(x) + \mathbf{E}(x,t) \times \mathbf{b}(x) \cdot \mathbf{n}(x)] da\, dt \\
&\quad + \tau \int_\Omega \left[\varepsilon \mathbf{E}_0(x) \cdot \mathbf{a}(x) + \mu \mathbf{H}_0(x) \cdot \mathbf{b}(x) + \frac{\alpha}{\sigma}\, \mathbf{J}_0(x) \cdot \mathbf{c}(x) \right] dx = 0.
\end{aligned}
\tag{4.4}
$$

Hence, differentiating with respect to τ, we have an identity, which, on introducing the following notation for the fields in (4.4)

$$
\Phi_1(x,\tau) = \int_0^\tau \Phi(x,t)\, dt, \tag{4.5}
$$

becomes

$$
\begin{aligned}
&\int_\Omega \{ \nabla \times \mathbf{a}(x) \cdot \mathbf{H}_1(x,\tau) - \nabla \times \mathbf{b}(x) \cdot \mathbf{E}_1(x,\tau) - [\mathbf{J}_1(x,\tau) + \mathbf{f}_1(x,\tau)] \cdot \mathbf{a}(x) \\
&\quad + \mathbf{g}_1(x,\tau) \cdot \mathbf{b}(x) + [\sigma \mathbf{E}_1(x,\tau) - \mathbf{J}_1(x,\tau) + \mathbf{l}_1(x,\tau)] \cdot \mathbf{c}(x)/\sigma \} dx \\
&\quad - \int_{\partial\Omega} [\mathbf{a}(x) \times \mathbf{H}_1(x,\tau) \cdot \mathbf{n}(x) + \mathbf{E}_1(x,\tau) \times \mathbf{b}(x) \cdot \mathbf{n}(x)] da
\end{aligned}
\tag{4.6}
$$

$$+\int_\Omega [\varepsilon \mathbf{E}_0(x)\cdot \mathbf{a}(x) + \mu \mathbf{H}_0(x)\cdot \mathbf{b}(x) + \frac{\alpha}{\sigma}\,\mathbf{J}_0(x)\cdot \mathbf{c}(x)]dx$$

$$-\int_\Omega \left[\varepsilon \mathbf{a}(x)\cdot \frac{d}{d\tau}\,\mathbf{E}_1(x,\tau) + \mu \mathbf{b}(x)\cdot \frac{d}{d\tau}\,\mathbf{H}_1(x,\tau) + \frac{\alpha}{\sigma}\,\mathbf{c}(x)\cdot \frac{d}{d\tau}\,\mathbf{J}_1(x,\tau)\right]dx = 0.$$

This identity must be applied to the case with $(\mathbf{f},\mathbf{g},\mathbf{l}) = (\mathbf{0},\mathbf{0},\mathbf{0})$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) = (\mathbf{0},\mathbf{0},\mathbf{0})$, which corresponds to the homogeneous system with zero initial data.

We observe that the relation derived in such a case must hold for every $(\mathbf{a},\mathbf{b},\mathbf{c}) \in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$; therefore, in particular, it follows that both $\mathbf{E}_1$ and $\mathbf{H}_1$ belong to $I_1(\Omega)$.

Thus, we can put

$$\mathbf{a}(x) = \mathbf{E}_1(x,\tau), \quad \mathbf{b}(x) = \mathbf{H}_1(x,\tau), \quad \mathbf{c}(x) = \mathbf{J}_1(x,\tau), \tag{4.7}$$

in the modified relation (4.6), which reduces to

$$\begin{aligned}&\frac{d}{d\tau}\frac{1}{2}\int_\Omega \left[\varepsilon \mathbf{E}_1^2(x,\tau) + \mu \mathbf{H}_1^2(x,\tau) + \frac{\alpha}{\sigma}\,\mathbf{J}_1^2(x,\tau)\right]dx \\ &= -\int_\Omega \frac{1}{\sigma}\,\mathbf{J}_1^2(x,\tau)\,dx - \int_{\partial\Omega} \mathbf{E}_1(x,\tau)\times \mathbf{H}_1(x,\tau)\cdot \mathbf{n}(x)\,da \le 0,\end{aligned} \tag{4.8}$$

on account of (3.6) too.

Since $\mathbf{E}_1(x,0) = \mathbf{0}$, $\mathbf{H}_1(x,0) = \mathbf{0}$, $\mathbf{J}_1(x,0) = \mathbf{0}$, by integrating (4.8) over $(0,\tau)$ we get

$$\int_\Omega \left[\varepsilon \mathbf{E}_1^2(x,\tau) + \mu \mathbf{H}_1^2(x,\tau) + \frac{\alpha}{\sigma}\,\mathbf{J}_1^2(x,\tau)\right]dx \le 0, \tag{4.9}$$

from which we have

$$\mathbf{E}_1(x,\tau) = \mathbf{0}, \quad \mathbf{H}_1(x,\tau) = \mathbf{0}, \quad \mathbf{J}_1(x,\tau) = \mathbf{0} \tag{4.10}$$

for almost all $\tau \in (0,T)$; therefore, it follows that

$$\mathbf{E}(x,t) = \mathbf{0}, \quad \mathbf{H}(x,t) = \mathbf{0}, \quad \mathbf{J}(x,t) = \mathbf{0} \tag{4.11}$$

in $\Omega \times (0,T)$, i.e. the uniqueness of the weak solution.

For the existence of the weak solution we first give this theorem.

Theorem 4.2 *Let us consider the sets*

$$\begin{aligned}\mathcal{R} = \big\{(\mathbf{f},\mathbf{g},\mathbf{l}) \in \mathcal{W}_0(\Omega,(0,T))\colon \mathbf{f} = \nabla\times\mathbf{H} - \varepsilon\dot{\mathbf{E}} - \mathbf{J},\ \mathbf{g} = \nabla\times\mathbf{E} \\ + \mu\dot{\mathbf{H}},\ \mathbf{l} = \alpha\dot{\mathbf{J}} + \mathbf{J} - \sigma\mathbf{E} \quad \forall\,(\mathbf{E},\mathbf{H},\mathbf{J}) \in \mathcal{H}(\Omega,(0,T))\big\},\end{aligned} \tag{4.12}$$

$$\mathcal{S} = \{(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)\}, \tag{4.13}$$

$$\mathcal{T} = \{(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0) \in I(\Omega)\times I(\Omega)\times L^2(\Omega)\}, \tag{4.14}$$

$\mathcal{R}\times\mathcal{S}$ is dense in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$.

Proof To prove the density of $\mathcal{R}\times\mathcal{S}$ in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$, we consider its closure $\overline{\mathcal{R}\times\mathcal{S}}$, which is a closed linear subspace of $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$, and we prove that its orthogonal complement $\mathcal{C}$ contains the null element only.

If we suppose that a non-zero element $((\mathbf{f}^*,\mathbf{g}^*,\mathbf{l}^*),(\mathbf{E}_0^*,\mathbf{H}_0^*,\mathbf{J}_0^*))\in\mathcal{C}$ exists, it follows that for any $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}$ the following equality

$$\begin{aligned}\int_0^T\int_\Omega\Big[(\nabla\times\mathbf{H}-\varepsilon\dot{\mathbf{E}}-\mathbf{J})\cdot\mathbf{f}^*+(\nabla\times\mathbf{E}+\mu\dot{\mathbf{H}})\cdot\mathbf{g}^*+\frac{1}{\sigma}(\alpha\dot{\mathbf{J}}+\mathbf{J}-\sigma\mathbf{E})\cdot\mathbf{l}^*\Big]dx\,dt\\ +\int_\Omega\Big(\varepsilon\mathbf{E}_0^*\cdot\mathbf{E}_0+\mu\mathbf{H}_0^*\cdot\mathbf{H}_0+\frac{\alpha}{\sigma}\mathbf{J}_0^*\cdot\mathbf{J}_0\Big)dx=0.\end{aligned}\tag{4.15}$$

must hold.

In this identity the arbitrariness of $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}$ allows us to take first $\mathbf{H}\equiv\mathbf{0}$, $\mathbf{J}\equiv\mathbf{0}$ and $\mathbf{E}_0=\mathbf{0}$, then $\mathbf{E}\equiv\mathbf{0}$, $\mathbf{J}\equiv\mathbf{0}$ and $\mathbf{H}_0=\mathbf{0}$ and finally $\mathbf{E}\equiv\mathbf{0}$, $\mathbf{H}\equiv\mathbf{0}$ and $\mathbf{J}_0=\mathbf{0}$ and to obtain

$$\int_0^T\int_\Omega(\varepsilon\dot{\mathbf{E}}\cdot\mathbf{f}^*-\nabla\times\mathbf{E}\cdot\mathbf{g}^*+\mathbf{E}\cdot\mathbf{l}^*)dx\,dt=0,\quad \mathbf{E}_0=0,\tag{4.16}$$

$$\int_0^T\int_\Omega(\mu\dot{\mathbf{H}}\cdot\mathbf{g}^*+\nabla\times\mathbf{H}\cdot\mathbf{f}^*)dx\,dt=0,\qquad \mathbf{H}_0=0,\tag{4.17}$$

$$\int_0^T\int_\Omega\Big[\frac{1}{\sigma}(\alpha\dot{\mathbf{J}}+\mathbf{J})\cdot\mathbf{l}^*-\mathbf{J}\cdot\mathbf{f}^*\Big]dx\,dt=0,\qquad \mathbf{J}_0=0,\tag{4.18}$$

respectively.

The initial conditions, which must be considered in these three identities, suggest to proceed as we have done for the uniqueness theorem, assuming now

$$\begin{aligned}\mathbf{E}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{A}(x) & \tau\le t\le T\end{cases},\\ \mathbf{H}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{B}(x) & \tau\le t\le T\end{cases},\\ \mathbf{J}(x,t)&=\begin{cases}\mathbf{0} & 0\le t\le\tau\\ (t-\tau)\mathbf{C}(x) & \tau\le t\le T\end{cases},\end{aligned}\tag{4.19}$$

where τ is a fixed value in $(0,T)$, for every $(\mathbf{A},\mathbf{B},\mathbf{C})\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$, such that (3.6) holds; therefore, with this choice $(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T))$.

Substituting (4.19) into (4.16)−(4.18), we get three relations where the range of integration $(0,T)$ reduces to (τ,T). Then, differentiating with respect to τ the relations so derived and putting

$$\mathbf{f}_i^*(x,\tau)=\int_\tau^T \mathbf{f}^*(x,t)dt \quad \text{with} \quad \frac{d}{d\tau}\,\mathbf{f}_i^*(x,\tau)=-\mathbf{f}^*(x,\tau) \tag{4.20}$$

and analogous expressions for $\mathbf{g}^*$ and $\mathbf{l}^*$, we obtain the following system

$$\begin{aligned}\varepsilon\int_\Omega \mathbf{A}(x)\cdot\frac{d}{d\tau}\,\mathbf{f}_i^*(x,\tau)\,dx+\int_\Omega [\nabla\times\mathbf{A}(x)\cdot\mathbf{g}_i^*(x,\tau)&\\ -\mathbf{A}(x)\cdot\mathbf{l}_i^*(x,\tau)]\,dx=0,&\end{aligned} \tag{4.21}$$

$$\mu\int_\Omega \mathbf{B}(x)\cdot\frac{d}{d\tau}\,\mathbf{g}_i^*(x,\tau)\,dx-\int_\Omega \nabla\times\mathbf{B}(x)\cdot\mathbf{f}_i^*(x,\tau)\,dx=0, \tag{4.22}$$

$$\frac{\alpha}{\sigma}\int_\Omega \mathbf{C}(x)\cdot\frac{d}{d\tau}\,\mathbf{l}_i^*(x,\tau)\,dx+\int_\Omega \mathbf{C}(x)\cdot\left[\mathbf{f}_i^*(x,\tau)-\frac{1}{\sigma}\,\mathbf{l}_i^*(x,\tau)\right]dx=0. \tag{4.23}$$

We observe that this system must hold for every $(\mathbf{A},\mathbf{B},\mathbf{C})\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$; hence, in particular, it follows that $\mathbf{f}_i^*$ and $\mathbf{g}_i^*$ belong to $I_1(\Omega)$ and are equal to zero on $\partial\Omega$ for the absence of any surface integral in the system.

Thus, we can put

$$\mathbf{A}(x)=\mathbf{f}_i^*(x,\tau),\quad \mathbf{B}(x)=\mathbf{g}_i^*(x,\tau),\quad \mathbf{C}(x)=\mathbf{l}_i^*(x,\tau) \tag{4.24}$$

and, adding (4.21)–(4.23), we get

$$\begin{aligned}&\frac{d}{d\tau}\frac{1}{2}\int_\Omega\left\{\varepsilon[\mathbf{f}_i^*(x,\tau)]^2+\mu[\mathbf{g}_i^*(x,\tau)]^2+\frac{\alpha}{\sigma}[\mathbf{l}_i^*(x,\tau)]^2\right\}dx\\ &=\frac{1}{\sigma}\int_\Omega[\mathbf{l}_i^*(x,\tau)]^2\,dx-\int_\Omega[\nabla\times\mathbf{f}_i^*(x,\tau)\cdot\mathbf{g}_i^*(x,\tau)-\nabla\times\mathbf{g}_i^*(x,\tau)\cdot\mathbf{f}_i^*(x,\tau)]\,dx.\end{aligned} \tag{4.25}$$

Hence, taking account of the previous observation, since from $(4.20)_1$ we have $\mathbf{f}_i^*(x,T)=\mathbf{0}$, $\mathbf{g}_i^*(x,T)=\mathbf{0}$, $\mathbf{l}_i^*(x,T)=\mathbf{0}$, the integral over (τ,T) yields

$$\frac{1}{2}\int_\Omega\left\{\varepsilon[\mathbf{f}_i^*(x,\tau)]^2+\mu[\mathbf{g}_i^*(x,\tau)]^2+\frac{\alpha}{\sigma}[\mathbf{l}_i^*(x,\tau)]^2\right\}dx+\int_\tau^T\int_\Omega\frac{1}{\sigma}[\mathbf{l}_i^*(x,\xi)]^2\,dx\,d\xi=0. \tag{4.26}$$

Therefore, we get

$$\mathbf{f}^*(x,t)=\mathbf{0},\quad \mathbf{g}^*(x,t)=\mathbf{0},\quad \mathbf{l}^*(x,t)=\mathbf{0}. \tag{4.27}$$

Thus, (4.15) reduces to

$$\int_\Omega\left(\varepsilon\mathbf{E}_0^*\cdot\mathbf{E}_0+\mu\mathbf{H}_0^*\cdot\mathbf{H}_0+\frac{\alpha}{\sigma}\,\mathbf{J}_0^*\cdot\mathbf{J}_0\right)dx=0\quad\forall\,(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}, \tag{4.28}$$

from which, choosing $\mathbf{E}(t)\equiv\mathbf{E}_0$, $\mathbf{H}(t)\equiv\mathbf{0}$, $\mathbf{J}(t)\equiv\mathbf{0}$, then $\mathbf{H}(t)\equiv\mathbf{H}_0$, $\mathbf{E}(t)\equiv\mathbf{0}$, $\mathbf{J}(t)\equiv\mathbf{0}$ and finally $\mathbf{J}(t)\equiv\mathbf{J}_0$, $\mathbf{E}(t)\equiv\mathbf{0}$, $\mathbf{H}(t)\equiv\mathbf{0}$, we get

$$\mathbf{E}_0^*=\mathbf{0},\quad \mathbf{H}_0^*=\mathbf{0},\quad \mathbf{J}_0^*=\mathbf{0}. \tag{4.29}$$

Equations (4.27) and (4.29) are contrary to the assumed hypothesis and hence $\mathcal{R}\times\mathcal{S}$ is dense in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$.

Theorem 4.3 (Existence) *There exists the solution of the problem P in the sense of Definition 4.2 for all data* $(\mathbf{f},\mathbf{g},\mathbf{l})\in\mathcal{W}_0(\Omega,(0,T))$ *and* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{T}$.

Proof To show the theorem we must prove that $\mathcal{R}\times\mathcal{S}$ is closed in $\mathcal{W}_0(\Omega,(0,T))\times\mathcal{T}$. Let $(\mathbf{f}^{(n)},\mathbf{g}^{(n)},\mathbf{l}^{(n)})\in\mathcal{R}$ and $(\mathbf{E}_0^{(n)},\mathbf{H}_0^{(n)},\mathbf{J}_0^{(n)})\in\mathcal{S}$, $n=1,2,...,$ be two sequences convergent to $(\mathbf{f},\mathbf{g},\mathbf{l})\in\mathcal{W}_0(\Omega,(0,T))$ and $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in\mathcal{S}$, respectively; we denote by $(\mathbf{E}^{(n)},\mathbf{H}^{(n)},\mathbf{J}^{(n)})\in\mathcal{H}(\Omega,(0,T))$, $n=1,2...,$ the corresponding solutions.

Applying Corollary 3.1 to the differences $\mathbf{E}^{(n)}-\mathbf{E}^{(m)}$, $\mathbf{H}^{(n)}-\mathbf{H}^{(m)}$, $\mathbf{J}^{(n)}-\mathbf{J}^{(m)}$ yields

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\left[\varepsilon|\mathbf{E}^{(n)}-\mathbf{E}^{(m)}|^2+\mu|\mathbf{H}^{(n)}-\mathbf{H}^{(m)}|^2+\frac{\alpha}{\sigma}\,|\mathbf{J}^{(n)}-\mathbf{J}^{(m)}|^2\right]dx\\
&\le e^T\left\{\left[\frac{1}{2}\int_\Omega\left[\varepsilon|\mathbf{E}_0^{(n)}-\mathbf{E}_0^{(m)}|^2+\mu|\mathbf{H}_0^{(n)}-\mathbf{H}_0^{(m)}|^2+\frac{\alpha}{\sigma}\,|\mathbf{J}_0^{(n)}-\mathbf{J}_0^{(m)}|^2\right]dx\right.\right.\\
&\left.\quad+M\int_0^T\int_\Omega[|\mathbf{f}^{(n)}-\mathbf{f}^{(m)}|^2+|\mathbf{g}^{(n)}-\mathbf{g}^{(m)}|^2+|\mathbf{l}^{(n)}-\mathbf{l}^{(m)}|^2]dx\,dt\right\}
\end{aligned}\tag{4.30}$$

and hence it follows that $(\mathbf{E}^{(n)},\mathbf{H}^{(n)},\mathbf{J}^{(n)})$, $n=1,2...,$ is a Cauchy sequence; thus, there exists the limit

$$\lim_{n\to\infty}(\mathbf{E}^{(n)},\mathbf{H}^{(n)},\mathbf{J}^{(n)})=(\mathbf{E},\mathbf{H},\mathbf{J})\in\mathcal{H}(\Omega,(0,T)).\tag{4.31}$$

Substituting the solutions and the corresponding sources into equations (3.1) – (3.3) gives a sequence of identities; the limit as $n\to+\infty$ is an analogous identity expressed in terms of $(\mathbf{f},\mathbf{g},\mathbf{l})$ and $(\mathbf{E},\mathbf{H},\mathbf{J})$, which is the solution of our problem.

We can now prove the uniqueness and the existence of the strong solution.

Theorem 4.4 *There exists a unique strong solution of the problem P in the sense of Definition 4.1 for all data* $(\mathbf{f},\mathbf{g},\mathbf{l})\in\mathcal{W}_1(\Omega,(0,T))$ *and* $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$.

Proof We observe that a strong solution, when it exists, coincides with the weak solution of the problem P. In fact, let $(\tilde{\mathbf{E}},\tilde{\mathbf{H}},\tilde{\mathbf{J}})\in\mathcal{H}(\Omega,(0,T))$ be such a strong solution, corresponding to given initial conditions $(\mathbf{E}_0,\mathbf{H}_0,\mathbf{J}_0)\in I_1(\Omega)\times I_1(\Omega)\times L^2(\Omega)$ and sources $(\mathbf{f},\mathbf{g},\mathbf{l})\in\mathcal{W}_1(\Omega,(0,T))$, then it satisfies the system (3.1) – (3.3) almost everywhere. It is enough to take the integrals over Ω and $(0,T)$ of the inner product of each equation of the system with any $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{p}/\sigma$, respectively, such that $(\mathbf{e},\mathbf{h},\mathbf{p})\in\mathcal{H}(\Omega,(0,T))$ and $\mathbf{e}(\mathbf{x},T)=\mathbf{0}$, $\mathbf{h}(\mathbf{x},T)=\mathbf{0}$, $\mathbf{p}(\mathbf{x},T)=\mathbf{0}$, and subtract the second and the third relations from the first one, to arrive at (4.1), which characterizes weak solutions.

Therefore, applying Theorem 4.1, the uniqueness of the strong solution follows at once.

For the existence of the strong solution, let $(\mathbf{E}',\mathbf{H}',\mathbf{J}')\in\mathcal{W}(\Omega,(0,T))$ be the weak solution to the problem P, whose existence and uniqueness have been already proved, corresponding to suitable data $(\mathbf{f}',\mathbf{g}',\mathbf{l}')\in\mathcal{W}_0(\Omega,(0,T))$ and $(\mathbf{E}_0',\mathbf{H}_0',\mathbf{J}_0')\in I(\Omega)\times I(\Omega)\times L^2(\Omega)$. This solution satisfies (4.1) for any $(\mathbf{e},\mathbf{h},\mathbf{p})\in\mathcal{H}(\Omega,(0,T))$ such that $\mathbf{e}(\mathbf{x},T)=\mathbf{0}$, $\mathbf{h}(\mathbf{x},T)=\mathbf{0}$, $\mathbf{p}(\mathbf{x},T)=\mathbf{0}$; therefore, as we have done to prove the uniqueness theorem, we can choose the form (4.3) for $(\mathbf{e},\mathbf{h},\mathbf{p})$ and derive a relation analogous to (4.6), which allows us to conclude that both $\mathbf{E}_1'$ and $\mathbf{H}_1'$, defined by (4.5), belong to $I_1(\Omega)$ and that $(\mathbf{E}', \mathbf{H}', \mathbf{J}')$ satisfies the following system

$$\varepsilon\dot{\mathbf{E}}_1' = \nabla \times \mathbf{H}_1' - \mathbf{J}_1' - \mathbf{f}_1' + \varepsilon\mathbf{E}_0', \tag{4.32}$$

$$\mu\dot{\mathbf{H}}_1' = -\nabla \times \mathbf{E}_1' + \mathbf{g}_1' + \mu\mathbf{H}_0', \tag{4.33}$$

$$\alpha\dot{\mathbf{J}}_1' = -\mathbf{J}_1' + \sigma\mathbf{E}_1' + \mathbf{l}_1' + \alpha\mathbf{J}_0'. \tag{4.34}$$

We can now fix the suitable data as follows

$$\mathbf{f}' = -\frac{1}{\mu}\nabla \times \mathbf{g} + \frac{1}{\alpha}\mathbf{l}, \qquad \mathbf{g}' = \frac{1}{\varepsilon}\nabla \times \mathbf{f}, \qquad \mathbf{l}' = -\frac{\sigma}{\varepsilon}\mathbf{f} - \frac{1}{\alpha}\mathbf{l}, \tag{4.35}$$

$$\mathbf{E}_0' = \frac{1}{\varepsilon}(\nabla \times \mathbf{H}_0 - \mathbf{J}_0), \qquad \mathbf{H}_0' = -\frac{1}{\mu}\nabla \times \mathbf{E}_0, \qquad \mathbf{J}_0' = \frac{1}{\alpha}(\sigma\mathbf{E}_0 - \mathbf{J}_0). \tag{4.36}$$

Then, we put

$$\tilde{\mathbf{E}} = \mathbf{E}_1' - \frac{1}{\varepsilon}\mathbf{f}_1 + \mathbf{E}_0, \quad \tilde{\mathbf{H}} = \mathbf{H}_1' + \frac{1}{\mu}\mathbf{g}_1 + \mathbf{H}_0, \quad \tilde{\mathbf{J}} = \mathbf{J}_1' + \frac{1}{\alpha}\mathbf{l}_1 + \mathbf{J}_0, \tag{4.37}$$

which yield

$$\dot{\mathbf{E}}_1' = \dot{\tilde{\mathbf{E}}} + \frac{1}{\varepsilon}\mathbf{f}, \qquad \dot{\mathbf{H}}_1' = \dot{\tilde{\mathbf{H}}} - \frac{1}{\mu}\mathbf{g}, \qquad \dot{\mathbf{J}}_1' = \dot{\tilde{\mathbf{J}}} - \frac{1}{\alpha}\mathbf{l} \tag{4.38}$$

and

$$\tilde{\mathbf{E}}(x,0) = \mathbf{E}_0(x), \quad \tilde{\mathbf{H}}(x,0) = \mathbf{H}_0(x), \quad \tilde{\mathbf{J}}(x,0) = \mathbf{J}_0(x). \tag{4.39}$$

Substituting (4.38), (4.36) and the expressions of $(\mathbf{E}_1', \mathbf{H}_1', \mathbf{J}_1')$, derived from (4.37), together with the expressions of $(\mathbf{f}_1', \mathbf{g}_1', \mathbf{l}_1')$, which follow from (4.35), we have

$$\varepsilon\dot{\tilde{\mathbf{E}}} = \nabla \times \tilde{\mathbf{H}} - \tilde{\mathbf{J}} - \mathbf{f}, \tag{4.40}$$

$$\mu\dot{\tilde{\mathbf{H}}} = -\nabla \times \tilde{\mathbf{E}} + \mathbf{g}, \tag{4.41}$$

$$\alpha\dot{\tilde{\mathbf{J}}} = -\tilde{\mathbf{J}} + \sigma\tilde{\mathbf{E}} + \mathbf{l} \tag{4.42}$$

and hence see that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}, \tilde{\mathbf{J}})$ is the strong solution of the problem P.

5 Asymptotic Stability

The problem P can be transformed into an equivalent one characterized by zero initial data, by putting

$$\breve{\mathbf{E}}(x,t) = \mathbf{E}(x,t) - \mathbf{u}(x,t), \qquad \breve{\mathbf{H}}(x,t) = \mathbf{H}(x,t) - \mathbf{v}(x,t),$$
$$\breve{\mathbf{J}}(x,t) = \mathbf{J}(x,t) - \mathbf{w}(x,t),$$

where $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ are regular fields with support compact in $\Omega \times \mathbf{R}^+$ such that

$$\nabla \cdot \mathbf{u}(x,t) = 0, \qquad \nabla \cdot \mathbf{v}(x,t) = 0$$

and

$$\mathbf{u}(x,0)=\mathbf{E}_0(x), \qquad \mathbf{v}(x,0)=\mathbf{H}_0(x), \qquad \mathbf{w}(x,0)=\mathbf{J}_0(x).$$

On substituting the expressions of $(\mathbf{E},\mathbf{H},\mathbf{J})$ in terms of $(\breve{\mathbf{E}},\breve{\mathbf{H}},\breve{\mathbf{J}})$ and $(\mathbf{u},\mathbf{v},\mathbf{w})$, equations (3.1)–(3.3) assume a similar form but with the terms

$$\begin{aligned}\mathbf{F}(x,t)&=-\mathbf{f}(x,t)-\mathbf{w}(x,t)+\nabla\times\mathbf{v}(x,t)-\varepsilon\dot{\mathbf{u}}(x,t),\\ \mathbf{G}(x,t)&=\mathbf{g}(x,t)-\nabla\times\mathbf{u}(x,t)-\mu\dot{\mathbf{v}}(x,t),\\ \mathbf{I}(x,t)&=\mathbf{l}(x,t)+\sigma\mathbf{u}(x,t)-\mathbf{w}(x,t)-\alpha\dot{\mathbf{w}}(x,t)\end{aligned}$$

to be considered as three new sources in the corresponding equations.

Therefore, without changing the notation of the fields $(\mathbf{E},\mathbf{H},\mathbf{J})$, the new problem is given by

$$\varepsilon\dot{\mathbf{E}}(x,t)-\nabla\times\mathbf{H}(x,t)+\mathbf{J}(x,t)=\mathbf{F}(x,t), \tag{5.1}$$

$$\mu\dot{\mathbf{H}}(x,t)+\nabla\times\mathbf{E}(x,t)=\mathbf{G}(x,t), \tag{5.2}$$

$$\alpha\dot{\mathbf{J}}(x,t)+\mathbf{J}(x,t)-\sigma\mathbf{E}(x,t)=\mathbf{I}(x,t) \tag{5.3}$$

with (3.4), the new initial conditions

$$\mathbf{E}_0(x)=\mathbf{0}, \quad \mathbf{H}_0(x)=\mathbf{0}, \quad \mathbf{J}_0(x)=\mathbf{0} \tag{5.4}$$

and the boundary condition (3.6), which holds because of the hypotheses on $\mathbf{u}$ and $\mathbf{v}$, equal to zero on $\partial\Omega$.

We introduce the Fourier transform of any $f\colon \mathbf{R}^+\to\mathbf{R}^n$, identified with the causal extension on $(-\infty,0)$, where f is put equal to zero, i.e.

$$\hat{f}(\omega)=\int_{-\infty}^{+\infty} f(t)\exp[-i\omega t]dt, \tag{5.5}$$

and recall that if $f,\ f'\in L^2(\mathbf{R}^+)$ then $\hat{f},\ \hat{f}'\in L^2(\mathbf{R})$ and we have

$$\hat{f}'(\omega)=i\omega\hat{f}(\omega)-f(0), \qquad f(0)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\,d\omega. \tag{5.6}$$

We denote by P' the new problem (5.1)–(5.4) with the boundary condition (3.6), for which, since it holds for any $t\in\mathbf{R}^+$, Plancherel's theorem justifies the assumption that

$$\int_{\partial\Omega}\hat{\mathbf{E}}(x,\omega)\times\hat{\mathbf{H}}^*(x,\omega)\cdot\mathbf{n}(x)\,da\geq 0 \quad \forall\,(\hat{\mathbf{E}},\hat{\mathbf{H}})\in\hat{\Sigma}', \quad \forall\,(x,\omega)\in\partial\Omega\times\mathbf{R}, \tag{5.7}$$

where * denotes the complex conjugate and $\hat{\Sigma}'$ is the set of Fourier's transforms of the electromagnetic fields $(\mathbf{E},\mathbf{H})\in\Sigma'$.

We consider the function spaces of Section 4, where $(0,T)$ is changed into $\mathbf{R}^+$, and, in particular, we introduce

$$\mathcal{W}_2(\Omega,\mathbf{R}^+) = \Big\{(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_0(\Omega,\mathbf{R}^+)\colon \frac{\partial^{n+1}}{\partial t^{n+1}}(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_0(\Omega,\mathbf{R}^+), \quad \Big[\frac{\partial^n}{\partial t^n}(\mathbf{F},\mathbf{G},\mathbf{I})\Big]_{t=0} = 0 \quad (n=0,1,2,3)\Big\},$$

where the last conditions, on the initial values of the new sources and of their derivatives with respect to time, are satisfied by choosing the derivatives of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ at $t=0$ opportunely.

When the Fourier transforms with respect to time are considered, the function spaces can be distinguished with a superposed $\hat{}$; in particular $\mathcal{W}(\Omega,\mathbf{R}^+)$ becomes

$$\hat{\mathcal{W}}(\Omega,\mathbf{R}) = \big\{(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}}) \in L^2(\mathbf{R};I(\Omega)) \times L^2(\mathbf{R};I(\Omega)) \times L^2(\mathbf{R};L^2(\Omega))\colon i\omega\hat{\mathbf{E}}, \quad i\omega\hat{\mathbf{H}} \in L^2(\mathbf{R};I(\Omega)) \text{ and } (\hat{\mathbf{E}},\hat{\mathbf{H}}) \text{ satisfies (5.7) on } \partial\Omega\times\mathbf{R}\big\}$$

and analogously for $\hat{\mathcal{W}}_2(\Omega,\mathbf{R})$.

Theorem 5.1 *If*

$$\mathcal{I}(\omega) = \int_\Omega \big(|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{H}}|^2 + |\hat{\mathbf{J}}|^2 + |\nabla\times\hat{\mathbf{E}}|^2 + |\nabla\times\hat{\mathbf{H}}|^2\big)\,dx, \tag{5.8}$$

under suitable conditions on the material constants the following inequality

$$(\min\{\varepsilon,\mu\})^2\mathcal{I}(\omega) \le \delta^2(\omega)\int_\Omega \big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx \tag{5.9}$$

holds with $\delta(\omega)$ a positive function of the material constants for any $\omega\in\mathbf{R}$.

Proof Application of Fourier's transform to the system (5.1) – (5.3), taking account of $(5.6)_1$ and (5.4), yields

$$i\omega\varepsilon\hat{\mathbf{E}}(x,\omega) - \nabla\times\hat{\mathbf{H}}(x,\omega) + \hat{\mathbf{J}}(x,\omega) = \hat{\mathbf{F}}(x,\omega), \tag{5.10}$$

$$i\omega\mu\hat{\mathbf{H}}(x,\omega) + \nabla\times\hat{\mathbf{E}}(x,\omega) = \hat{\mathbf{G}}(x,\omega), \tag{5.11}$$

$$(1+i\omega\alpha)\hat{\mathbf{J}}(x,\omega) - \sigma\hat{\mathbf{E}}(x,\omega) = \hat{\mathbf{I}}(x,\omega). \tag{5.12}$$

From this system, the integrals over Ω of the inner products of the first equation with $\hat{\mathbf{E}}^*$, $\hat{\mathbf{J}}^*$ and $\nabla\times\hat{\mathbf{H}}^*$ yield

$$i\omega\varepsilon\int_\Omega |\hat{\mathbf{E}}|^2\,dx - \int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{E}}^*\,dx + \int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^*\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx, \tag{5.13}$$

$$i\omega\varepsilon\int_\Omega \hat{\mathbf{E}}\cdot\hat{\mathbf{J}}^*\,dx - \int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{J}}^*\,dx + \int_\Omega |\hat{\mathbf{J}}|^2\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx, \tag{5.14}$$

$$i\omega\varepsilon\int_\Omega \hat{\mathbf{E}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx - \int_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx + \int_\Omega \hat{\mathbf{J}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx = \int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx; \tag{5.15}$$

analogously, the inner products of the conjugate of the second equation with $\hat{\mathbf{H}}$ and $\nabla \times \hat{\mathbf{E}}$ give

$$-i\omega\mu \int_\Omega |\hat{\mathbf{H}}|^2\, dx + \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx = \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx, \tag{5.16}$$

$$-i\omega\mu \int_\Omega \hat{\mathbf{H}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx + \int_\Omega |\nabla \times \hat{\mathbf{E}}|^2\, dx = \int_\Omega \hat{\mathbf{G}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx, \tag{5.17}$$

and finally from the inner products of the conjugate of the third equation with $\hat{\mathbf{J}}$ and $\hat{\mathbf{E}}$ it follows that

$$(1 - i\omega\alpha) \int_\Omega |\hat{\mathbf{J}}|^2\, dx - \sigma \int_\Omega \hat{\mathbf{E}}^* \cdot \hat{\mathbf{J}}\, dx = \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}}\, dx, \tag{5.18}$$

$$(1 - i\omega\alpha) \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx - \sigma \int_\Omega |\hat{\mathbf{E}}|^2\, dx = \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{E}}\, dx. \tag{5.19}$$

Let $\omega \neq 0$. The real parts of (5.19), (5.18), (5.17) and (5.15) yield

$$\sigma \int_\Omega |\hat{\mathbf{E}}|^2\, dx = -\operatorname{Re} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{E}}\, dx + \operatorname{Re} \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx + \omega\alpha \operatorname{Im} \int_\Omega \hat{\mathbf{J}}^* \cdot \hat{\mathbf{E}}\, dx, \tag{5.20}$$

$$\int_\Omega |\hat{\mathbf{J}}|^2\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{I}}^* \cdot \hat{\mathbf{J}}\, dx + \sigma \operatorname{Re} \int_\Omega \hat{\mathbf{E}}^* \cdot \hat{\mathbf{J}}\, dx, \tag{5.21}$$

$$\int_\Omega |\nabla \times \hat{\mathbf{E}}|^2\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx - \omega\mu \operatorname{Im} \int_\Omega \hat{\mathbf{H}}^* \cdot \nabla \times \hat{\mathbf{E}}\, dx, \tag{5.22}$$

$$\begin{aligned} \int_\Omega |\nabla \times \hat{\mathbf{H}}|^2\, dx &= -\operatorname{Re} \int_\Omega \hat{\mathbf{F}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx \\ &\quad - \omega\varepsilon \operatorname{Im} \int_\Omega \hat{\mathbf{E}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx + \operatorname{Re} \int_\Omega \hat{\mathbf{J}} \cdot \nabla \times \hat{\mathbf{H}}^*\, dx; \end{aligned} \tag{5.23}$$

while the imaginary part of (5.16) gives

$$\mu \int_\Omega |\hat{\mathbf{H}}|^2\, dx = \frac{1}{\omega}\Big(-\operatorname{Im} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx + \operatorname{Im} \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx \Big). \tag{5.24}$$

In (5.20) – (5.24) we have some quantities to be derived.

First, from the real part of (5.16) we have at once

$$\operatorname{Re} \int_\Omega \nabla \times \hat{\mathbf{E}}^* \cdot \hat{\mathbf{H}}\, dx = \operatorname{Re} \int_\Omega \hat{\mathbf{G}}^* \cdot \hat{\mathbf{H}}\, dx, \tag{5.25}$$

which, taking account of (5.7), allows us to derive from the real part of (5.13)

$$\operatorname{Re}\int_\Omega \hat{\mathbf{J}}\cdot\hat{\mathbf{E}}^*\,dx = \operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da; \tag{5.26}$$

therefore, (5.21) becomes

$$\begin{aligned}\int_\Omega |\hat{\mathbf{J}}|^2\,dx = \operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx + \sigma\Big(&\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ &+ \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big).\end{aligned} \tag{5.27}$$

Then, we consider (5.19), whose imaginary part, on account of (5.26), yields

$$\begin{aligned}\operatorname{Im}\int_\Omega \hat{\mathbf{J}}^*\cdot\hat{\mathbf{E}}\,dx = \operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\alpha\Big(&\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ &+ \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),\end{aligned} \tag{5.28}$$

whence, taking account of (5.26) too, (5.20) assumes the following form

$$\begin{aligned}\sigma\int_\Omega |\hat{\mathbf{E}}|^2\,dx = &-\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\alpha\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ &+ (1+\omega^2\alpha^2)\Big[\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big].\end{aligned} \tag{5.29}$$

Substituting this relation into the imaginary part of (5.13) we get

$$\begin{aligned}\operatorname{Im}\int_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{E}}^*\,dx = &-\operatorname{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \Big(\omega^2\frac{\alpha\varepsilon}{\sigma} - 1\Big)\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ &- \omega\,\frac{\varepsilon}{\sigma}\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \omega\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2) - \alpha\Big]\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ &+ \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)\end{aligned} \tag{5.30}$$

useful to rewrite (5.24) and (5.22) as follows

$$\begin{aligned}\mu\int_\Omega |\hat{\mathbf{H}}|^2\,dx = &-\frac{1}{\omega}\Big(\operatorname{Im}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx + \operatorname{Im}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\Big) - \frac{\varepsilon}{\sigma}\operatorname{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\ &+ \frac{1}{\omega}\Big(\omega^2\frac{\alpha\varepsilon}{\sigma} - 1\Big)\operatorname{Im}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx + \Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2) - \alpha\Big]\Big(\operatorname{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx \\ &+ \operatorname{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx - \int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),\end{aligned} \tag{5.31}$$

$$
\begin{aligned}
\int\limits_\Omega |\nabla\times\hat{\mathbf{E}}|^2\,dx &= \mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx+\omega\mu\Big\{-\mathrm{Im}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\\
&-\omega\,\frac{\varepsilon}{\sigma}\,\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\Big(\omega^2\,\frac{\alpha\varepsilon}{\sigma}-1\Big)\mathrm{Im}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\omega\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2)-\alpha\Big]\\
&\times\Big(\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int\limits_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)\Big\}.
\end{aligned}
\tag{5.32}
$$

Finally, for (5.23) we must derive its last term, which follows from the real part of (5.14), using (5.27) and (5.28), i.e.

$$
\begin{aligned}
\mathrm{Re}\int\limits_\Omega \nabla\times\hat{\mathbf{H}}\cdot\hat{\mathbf{J}}^*\,dx &= -\,\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx-\omega\varepsilon\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx\\
&+(\sigma-\omega^2\varepsilon\alpha)\Big(\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int\limits_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big),
\end{aligned}
\tag{5.33}
$$

hence, we have

$$
\begin{aligned}
\int\limits_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx &= -\,\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx+\omega\varepsilon\Big[-\mathrm{Im}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx\\
&+\omega\,\frac{\varepsilon}{\sigma}\Big(-\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\omega\alpha\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx\Big)\Big]-\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx\\
&-2\omega\varepsilon\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx+\Big\{\sigma+\omega^2\varepsilon\Big[\frac{\varepsilon}{\sigma}(1+\omega^2\alpha^2)-2\alpha\Big]\Big\}\\
&\times\Big(\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx-\int\limits_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big).
\end{aligned}
\tag{5.34}
$$

Thus, from (5.29) multiplied by ε/σ, (5.31), (5.27), (5.32) and (5.34) we have

$$
\begin{aligned}
\int\limits_\Omega \big(\varepsilon|\hat{\mathbf{E}}|^2+\mu|\hat{\mathbf{H}}|^2+|\hat{\mathbf{J}}|^2+|\nabla\times\hat{\mathbf{E}}|^2+|\nabla\times\hat{\mathbf{H}}|^2\big)dx &= \Big\{\frac{\varepsilon}{\sigma}(1+\alpha^2\omega^2)[2+(\varepsilon+\mu)\omega^2]\\
&+2\sigma-\alpha[1+(2\varepsilon+\mu)\omega^2]\Big\}\Big(\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx\\
&-\int\limits_{\partial\Omega}\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da\Big)-\frac{1}{\omega}[1+(\varepsilon+\mu)\omega^2]\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx-\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx\\
&-\mathrm{Re}\int\limits_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx-\frac{1}{\omega}\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx+\mathrm{Re}\int\limits_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx
\end{aligned}
\tag{5.35}
$$

$$-\frac{\varepsilon}{\sigma}[2+(\varepsilon+\mu)\omega^2]\,\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+\frac{1}{\omega}\Big\{\frac{\varepsilon\alpha}{\sigma}\,\omega^2[2+(\varepsilon+\mu)\omega^2]$$

$$-[1+(2\varepsilon+\mu)\omega^2]\Big\}\,\mathrm{Im}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx+2\,\mathrm{Re}\int\limits_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx.$$

In this equality we have the presence of a surface integral, which must satisfy the boundary condition (5.7). This term can be neglected if its coefficient is not positive, that is when

$$\varphi(\xi)\equiv\frac{\varepsilon}{\sigma}\,\alpha^2(\varepsilon+\mu)\xi^2+\left[\frac{\varepsilon}{\sigma}(\varepsilon+\mu+2\alpha^2)-\alpha(2\varepsilon+\mu)\right]\xi+2\frac{\varepsilon}{\sigma}+2\sigma-\alpha\geq 0 \qquad (5.36)$$

for all $\omega\in\mathbf{R}$, with $\xi=\omega^2$.

We give some sufficient conditions to can neglect this boundary term in (5.35).

We first examine the case when all the coefficients in (5.36) are positive or null; thus, we impose that the following system

$$\begin{cases} 2\varepsilon\alpha^2-(2\varepsilon+\mu)\sigma\alpha+\varepsilon(\varepsilon+\mu)\geq 0, \\ 2\sigma^2-\alpha\sigma+2\varepsilon\geq 0 \end{cases} \qquad (5.37)$$

must be satisfied for all positive values of the material constants.

If we consider the first inequality in function of α and the second of σ, the system is always satisfied if the discriminants are not greater than zero, i.e. when σ and α satisfy these inequalities

$$\sigma\leq\frac{2\varepsilon}{2\varepsilon+\mu}\sqrt{2(\varepsilon+\mu)}, \qquad \alpha\leq 4\sqrt{\varepsilon}. \qquad (5.38)$$

Moreover, some other particular cases can be considered by imposing that be positive or null the sum of the first two terms or the sum of the second and the third term of each inequality in (5.37). Thus, we see that if one of the following conditions, relative to $(5.37)_1$,

$$\frac{\alpha}{\sigma}\geq\frac{2\varepsilon+\mu}{2\varepsilon} \quad \text{or} \quad \alpha\sigma\leq\frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu} \qquad (5.39)$$

is satisfied together with one of the other two conditions, corresponding now to the second inequality of (5.37),

$$\alpha\leq 2\sigma \quad \text{or} \quad \alpha\sigma\leq 2\varepsilon, \qquad (5.40)$$

then the system (5.37) holds and the boundary term is negligible in (5.35).

Finally, another interesting condition on the parameters can be easily derived by neglecting the boundary terms in (5.27) and (5.29), since their coefficients are negative for all $\omega\in\mathbf{R}$, and by assuming $\varepsilon/\sigma\geq\alpha$, which allows us to neglect the boundary terms also in (5.31) and (5.32), while in (5.34) we can consider $\varepsilon/\sigma\geq 2\alpha$. Therefore, the other sufficient condition is the following one

$$\alpha\sigma\leq\frac{\varepsilon}{2}. \qquad (5.41)$$

This inequality is a simpler restriction on the product of α and σ for it is expressed in term of ε only. However, if we consider that the previous case, when $(5.39)_2$ holds together with $(5.40)_2$, since $2\varepsilon > \frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}$, is expressed by the unique condition

$$\alpha\sigma \leq \frac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}, \tag{5.42}$$

we see that (5.41) is more restrictive than (5.42), where also μ is interested, being $\dfrac{\varepsilon}{2} < \dfrac{\varepsilon(\varepsilon+\mu)}{2\varepsilon+\mu}$.

Thus, the equality (5.35) becomes an inequality whenever the boundary term can be neglected. In these cases, let us consider the sum of the moduli of the coefficients of the real and imaginary parts of the same integral; then we denote by $\gamma(\omega)$ the maximum of these quantities and we get

$$\begin{aligned}
\min\{\varepsilon,\mu\}\mathcal{I}(\omega) &\leq \gamma(\omega)\Bigg[\Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{E}}|^2\,dx\Big)^{1/2} \\
&+ \Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{J}}|^2\,dx\Big)^{1/2} + \Big(\int_\Omega |\hat{\mathbf{F}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla\times\hat{\mathbf{H}}|^2\,dx\Big)^{1/2} \\
&+ \Big(\int_\Omega |\hat{\mathbf{G}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{H}}|^2\,dx\Big)^{1/2} + \Big(\int_\Omega |\hat{\mathbf{G}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla\times\hat{\mathbf{E}}|^2\,dx\Big)^{1/2} \\
&+ \Big(\int_\Omega |\hat{\mathbf{I}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{E}}|^2\,dx\Big)^{1/2} + \Big(\int_\Omega |\hat{\mathbf{I}}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |\hat{\mathbf{J}}|^2\,dx\Big)^{1/2}\Bigg] \\
&\leq 7\gamma(\omega)\Big[\int_\Omega \big(|\hat{\mathbf{F}}|^2 + |\hat{\mathbf{G}}|^2 + |\hat{\mathbf{I}}|^2\big)\,dx\Big]^{1/2}[\mathcal{I}(\omega)]^{1/2}.
\end{aligned} \tag{5.43}$$

Hence we have (5.9) immediately.

Let $\omega = 0$. In this case we are interested in finding static solutions, consequently (5.10) − (5.12) must be considered with $\omega = 0$, as well as (5.13) − (5.19). Proceeding as we have done previously, we see that (5.35) reduces to

$$\begin{aligned}
\mathcal{I}_0(0) &\equiv \int_\Omega \big(|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{J}}|^2 + |\nabla\times\hat{\mathbf{E}}|^2 + |\nabla\times\hat{\mathbf{H}}|^2\big)\,dx \\
&= \Big(\frac{1}{\sigma} + 2\sigma\Big)\Big(\mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{E}}^*\,dx + \mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\hat{\mathbf{H}}\,dx\Big) - \mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\hat{\mathbf{J}}^*\,dx \\
&- \mathrm{Re}\int_\Omega \hat{\mathbf{F}}\cdot\nabla\times\hat{\mathbf{H}}^*\,dx + \mathrm{Re}\int_\Omega \hat{\mathbf{G}}^*\cdot\nabla\times\hat{\mathbf{E}}\,dx - \frac{1}{\sigma}\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{E}}\,dx \\
&+ 2\,\mathrm{Re}\int_\Omega \hat{\mathbf{I}}^*\cdot\hat{\mathbf{J}}\,dx - \Big(2\sigma + \frac{1}{\sigma}\Big)\int_{\partial\Omega} \hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\cdot\mathbf{n}\,da
\end{aligned} \tag{5.44}$$

$$\leq c\Big[\int_\Omega \big(|\hat{\mathbf{F}}|^2+|\hat{\mathbf{G}}|^2+|\hat{\mathbf{I}}|^2\big)\,dx\Big]^{1/2}[\mathcal{I}_0(0)]^{1/2},$$

where all the fields are functions of $(x,0)$ and c is a constant [9]. Hence it follows that

$$\mathcal{I}_0(0) \leq C\int_\Omega \big[|\hat{\mathbf{F}}(x,0)|^2+|\hat{\mathbf{G}}(x,0)|^2+|\hat{\mathbf{I}}(x,0)|^2\big]\,dx, \tag{5.45}$$

i.e. a relation similar to (5.9) with a constant C.

Theorem 5.2 *Let the sources be* $(\mathbf{F},\mathbf{G},\mathbf{I}) \in \mathcal{W}_2(\Omega,\mathbf{R}^+)$*, then the inverse Fourier transforms of* $(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}}) \in \hat{H}(\Omega,\mathbf{R})$ *exist and are* L^2*-functions with zero initial data.*

Proof In (5.9) $\delta(\omega)$ is a positive function of $\omega \in \mathbf{R}$ and approaches infinity as ω^4; such a condition, together with the hypotheses on the sources, states that the integral on $\mathbf{R}$ exists for the right-hand side of (5.9), that is

$$\int_{-\infty}^{+\infty}\int_\Omega \delta^2(\omega)\big(|\hat{\mathbf{F}}|^2+|\hat{\mathbf{G}}|^2+|\hat{\mathbf{I}}|^2\big)\,dx\,d\omega < +\infty. \tag{5.46}$$

Therefore, (5.9) gives

$$\int_{-\infty}^{+\infty}\mathcal{I}(\omega)\,d\omega \leq \int_{-\infty}^{+\infty}\int_\Omega \left(\frac{\delta(\omega)}{\min\{\varepsilon,\mu\}}\right)^2\big(|\hat{\mathbf{F}}|^2+|\hat{\mathbf{G}}|^2+|\hat{\mathbf{I}}|^2\big)\,dx\,d\omega, \tag{5.47}$$

i.e. there exists finite the integral over $\mathbf{R}$ of $\mathcal{I}(\omega)$.

Application of Plancherel's theorem yields the existence of the inverse Fourier transforms of $(\hat{\mathbf{E}},\hat{\mathbf{H}},\hat{\mathbf{J}})$; moreover, these solutions have the asymptotic behaviour which follows by belonging to the space $\hat{H}(\Omega,\mathbf{R})$.

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Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 23–41

Existence and Uniqueness for the Weak Solutions of the Ginzburg-Landau Equations

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Received: February 14, 2002; Revised: December 13, 2002

Abstract: In this paper we study a gauge-invariant Ginzburg-Landau model which describes the phenomenon of the superconductivity, characterizing the state of the material by means of observable variables. We give a definition of weak solutions for the steady and the time-dependent Ginzburg-Landau equations and prove theorems of existence and uniqueness.

Keywords: *Superconductivity, Ginzburg-Landau equations.*

Mathematics Subject Classification (2000): 82D55, 35A05.

1 Introduction

The Ginzburg-Landau theory gives a macroscopic model which explains the main experimental phenomena related to the superconductivity, i.e. the absence of electrical resistance and the Meissner effect ([5, 13]). In their model, Ginzburg and Landau describe the behaviour of a superconducting material in steady conditions, through the introduction of a free energy functional and assume that the state of the system minimizes such a functional. They identify the state of the superconductor with the pair $(\psi, \mathbf{A})$, where ψ is a complex order parameter, whose squared modulus coincides with the number density of the superconducting electrons and $\mathbf{A}$ is the vector magnetic potential.

Later, the model was extended to the non stationary case by Gor'kov and Èliashberg [8], who deduce the time-dependent Ginzburg-Landau equations from the microscopic theory BCS. Such equations constitute a non linear differential system for which theorems of existence and uniqueness are proved ([4, 12, 15]).

Recently, Fabrizio [6, 7] has proposed a macroscopic model which characterizes the state of the material by means of real and observable variables. Therefore, while in the

[†]Research performed under the auspices of G.N.F.M. - I.N.D.A.M. and partially supported by Italian M.I.U.R. through the project "*Mathematical Methods in Continuum Mechanics*".

classical formulation the unknown quantities are defined up to a gauge transformation, the variables involved in this model have a well determined physical meaning, so that they are gauge-invariant.

In this paper, the real Ginzburg-Landau equations are studied both in the steady and in the time-dependent case. The two models are presented in Section 2. In Section 3, we introduce a new definition of weak solutions which allows to prove existence and uniqueness theorems. In the steady case, the uniqueness of the weak solution is shown, provided that the coefficients of the equations and the domain occupied by the superconducting material are sufficiently small. In the time-dependent case, the uniqueness is proved in two-dimensional domains, with L^2 initial data.

Both in the stationary and in the time-dependent problem the results are obtained with the same method used in [1] and [2], namely by introducing a suitable decomposition of the unknown variables and reducing the original system to an equivalent one.

2 Ginzburg-Landau Model of Superconductivity

The electromagnetic behaviour of a superconducting material is described by Maxwell equations

$$\varepsilon\,\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \quad \nabla \cdot \mathbf{E} = \rho, \tag{2.1}$$

$$\mu\,\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{H} = 0, \tag{2.2}$$

where ε, μ, ρ are respectively the dielectric constant, the magnetic permeability and the charge density. For simplicity ε and μ are assumed constant.

According to London theory, the electrons in a superconductor behave like a fluid which may appear either in the normal or in the superconducting phase. Therefore, the current density $\mathbf{J}$ inside the material can be expressed through the sum

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s \tag{2.3}$$

of the normal and the superconducting current. The conduction current $\mathbf{J}_n$ is required to obey Ohm's law

$$\mathbf{J}_n = \sigma \mathbf{E}, \tag{2.4}$$

while the superconducting current satisfies London constitutive equation

$$\nabla \times (\Lambda \mathbf{J}_s) = -\mu \mathbf{H}, \qquad L = \frac{m}{e^2 f^2}, \tag{2.5}$$

where m, e, f^2 denote respectively the mass, the charge and the number density of the superconducting electrons.

By means of the equation (2.5), London theory describes the superconducting features of a material in the hypothesis that the parameter Λ is constant, so that the density of superconducting electrons is uniform. However, near the transition temperature there occurs a mixed state consisting of alternating domains of normal and superconducting phase. Therefore the material cannot be considered spatially homogeneous.

The Ginzburg-Landau model extends London theory since it allows spatial variations of the density of superconducting electrons. In the stationary case and without free charge, Maxwell equations $(2.1)_2$, $(2.2)_1$ together with the boundary condition

$$\mathbf{E}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{2.6}$$

imply $\mathbf{E}=\mathbf{0}$, so that the system (2.1)–(2.2) reduces to

$$\nabla\times\mathbf{H}=\mathbf{J}_s, \qquad \nabla\cdot\mathbf{H}=0. \tag{2.7}$$

Since the electric field can be neglected, the state of the material is identified with the pair $(f,\mathbf{J}_s)$.

According to the Ginzburg-Landau theory, the material is in a state which minimizes the free energy. If we denote by Ω the domain occupied by the superconductor and by $\partial\Omega$ its boundary, the free energy can be written as a functional of the variables $(f,\mathbf{J}_s)$ in the form ([6, 7])

$$\begin{aligned}\mathcal{E}(f,\mathbf{J}_s)&=\int\limits_{\Omega}\Big[-\alpha f^2+\frac{\beta}{2}f^4+\frac{1}{2\mu}|\nabla\times(\Lambda(f)\mathbf{J}_s)|^2+\frac{\hbar^2}{2m}|\nabla f|^2\Big]dx\\&\quad+\int\limits_{\Omega}\frac{1}{2}\Lambda(f)\mathbf{J}_s^2\,dx-2\int\limits_{\partial\Omega}\Lambda(f)\mathbf{J}_s\times\mathbf{H}_{ex}^{\tau}\cdot\mathbf{n}\,d\sigma,\end{aligned} \tag{2.8}$$

where α, β are positive constants depending on the temperature, $\hbar$ is the Planck constant and $\mathbf{H}_{ex}^{\tau}$ is the tangential component of the external magnetic field.

Henceforth, we consider external magnetic fields $\mathbf{H}_{ex}$ which satisfy the relation

$$\int\limits_{\partial\Omega}\nabla\varphi\times\mathbf{H}_{ex}\cdot\mathbf{n}\,d\sigma=0, \tag{2.9}$$

where φ is the trace on $\partial\Omega$ of an arbitrary function φ.

By introducing the quantity

$$\mathbf{p}_s=\Lambda(f)\mathbf{J}_s=\frac{m\mathbf{v}_s}{e}, \tag{2.10}$$

identified with the linear momentum of the superconducting electrons per unit charge, the free energy (2.8) can be expressed in terms of the variables $(f,\mathbf{p}_s)$ in the form

$$\begin{aligned}\mathcal{E}(f,\mathbf{p}_s)&=\int\limits_{\Omega}\Big[-\alpha f^2+\frac{\beta}{2}f^4+\frac{1}{2\mu}|\nabla\times\mathbf{p}_s|^2+\frac{\hbar^2}{2m}|\nabla f|^2+\frac{e^2}{2m}f^2\mathbf{p}_s^2\Big]dx\\&\quad-2\int\limits_{\partial\Omega}\mathbf{p}_s\cdot\mathbf{H}_{ex}^{\tau}\times\mathbf{n}\,d\sigma.\end{aligned} \tag{2.11}$$

The stationariety of the functional (2.11) with respect to $(f,\mathbf{p}_s)$ leads to the system

$$\frac{\hbar^2}{2m}\nabla^2 f-\frac{e^2}{2m}f\mathbf{p}_s^2+\alpha f-\beta f^3=0, \tag{2.12}$$

$$\nabla\times\nabla\times\mathbf{p}_s+\frac{\mu e^2}{m}f^2\mathbf{p}_s=\mathbf{0}, \tag{2.13}$$

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega}=0, \quad (\nabla\times\mathbf{p}_s)\times\mathbf{n}|_{\partial\Omega}=\mu\mathbf{H}_{ex}\times\mathbf{n}|_{\partial\Omega}. \tag{2.14}$$

In order to reduce our notations, we introduce the following non-dimensional quantities

$$f = \left(\frac{\alpha}{\beta}\right)^{1/2} f', \qquad \mathbf{p}_s = \left(\frac{2m\alpha}{e^2}\right)^{1/2} \mathbf{p}'_s, \tag{2.15}$$

$$x = \left(\frac{m\beta}{e^2\alpha\mu}\right)^{1/2} x', \qquad \mathbf{H}_{ex} = \left(\frac{2\alpha^2}{\beta\mu}\right)^{1/2} \mathbf{H}'_{ex}, \tag{2.16}$$

$$\mathcal{E} = \left(\frac{m\alpha^3}{\beta^2\mu}\right)^{1/2} \mathcal{E}', \qquad k = \left(\frac{2m^2\beta}{\hbar^2 e^2\mu}\right)^{1/2}. \tag{2.17}$$

With such positions, dropping the primes, the free energy (2.11) assumes the form

$$\begin{aligned} \mathcal{E}(f, \mathbf{p}_s) = \int_\Omega \left[\frac{1}{2}(f^2-1)^2 + |\nabla \times \mathbf{p}_s|^2 + \frac{1}{k^2}|\nabla f|^2 + f^2 \mathbf{p}_s^2\right] dx \\ - 2\int_{\partial\Omega} \mathbf{p}_s \cdot \mathbf{H}_{ex}^\tau \times \mathbf{n}\, d\sigma \end{aligned} \tag{2.18}$$

and the ensuing Ginzburg-Landau system is

$$\frac{1}{k^2}\nabla^2 f - f\mathbf{p}_s^2 + f - f^3 = 0, \tag{2.19}$$

$$\nabla \times \nabla \times \mathbf{p}_s + f^2 \mathbf{p}_s = \mathbf{0}, \tag{2.20}$$

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{p}_s) \times \mathbf{n}|_{\partial\Omega} = \mathbf{H}_{ex}^\tau \times \mathbf{n}|_{\partial\Omega}. \tag{2.21}$$

Moreover, we assume the boundary condition

$$\mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

The generalization of the Ginzburg-Landau model to the time-dependent case is obtained by introducing a further variable ϕ_s which is related to the charge density ρ. In non-dimensional variables the time-dependent Ginzburg-Landau equations are ([6, 7])

$$\frac{\partial f}{\partial t} - \nabla^2 f + k^2(f^2-1)f + f\mathbf{p}_s^2 = 0, \tag{2.22}$$

$$\eta\,\frac{\partial \mathbf{p}_s}{\partial t} + \nabla \times \nabla \times \mathbf{p}_s + \eta\nabla\phi_s + f^2\mathbf{p}_s = \mathbf{0}, \tag{2.23}$$

with boundary conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{p}_s - \mathbf{H}_{ex}^\tau) \times \mathbf{n}|_{\partial\Omega} = 0 \tag{2.24}$$

and initial data

$$f(x,0) = f_0(x), \quad \mathbf{p}_s(x,0) = \mathbf{p}_{s0}(x). \tag{2.25}$$

As in the steady model, the equation (2.23) can be obtained from Maxwell equation $(2.1)_1$. However, (2.23) coincides with $(2.1)_1$ only if the time derivative of the electric field is negligible. In such a case, the equation $(2.1)_1$ assumes the form

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_s \tag{2.26}$$

and, by using London equation (2.5), we get

$$\nabla \times \nabla \times \mathbf{p}_s + \mu\sigma \mathbf{E} + \frac{\mu e^2}{m} f^2 \mathbf{p}_s = \mathbf{0}. \tag{2.27}$$

On the other hand, the equation $(2.2)_1$ yields

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{p}_s) - \nabla \times \mathbf{E} = \mathbf{0},$$

hence there exists a function ϕ_s such that

$$\mathbf{E} = \frac{\partial}{\partial t} \mathbf{p}_s + \nabla \phi_s. \tag{2.28}$$

Substitution in (2.23) leads to the equation

$$\mu\sigma \frac{\partial}{\partial t} \mathbf{p}_s + \nabla \times \nabla \times \mathbf{p}_s + \mu\sigma \nabla \phi_s + \frac{\mu e^2}{m} f^2 \mathbf{p}_s = \mathbf{0},$$

which coincides (in non-dimensional form) with (2.23).

We assume the following constitutive equation for ϕ_s

$$f^2 \phi_s = -\frac{\hbar^2}{2m\tau} \nabla \cdot (f^2 \mathbf{p}_s), \tag{2.29}$$

or in non-dimensional form

$$f^2 \phi_s = -\nabla \cdot (f^2 \mathbf{p}_s). \tag{2.30}$$

The choice of the equation (2.29) corresponds to a particular choice of the charge density ρ. Indeed, the relation (2.26) implies

$$\nabla \cdot \mathbf{J}_s = -\nabla \cdot \mathbf{J}_n = -\sigma \nabla \cdot \mathbf{E} = -\frac{\sigma}{\varepsilon} \rho,$$

so that, by substituting in (2.29), we obtain

$$f^2 \phi_s = -\frac{\hbar^2}{2e^2\tau} \nabla \cdot \mathbf{J}_s = \frac{\sigma \hbar^2}{2e^2 \varepsilon \tau} \rho.$$

3 Existence and Uniqueness of Solutions

3.1 The stationary case

In this section we prove that the functional (2.18) admits at least a minimizer and, under suitable hypotheses on the coefficients of the equations, such a minimizer is unique.

Let $\mathcal{D}(\Omega)$ be the domain of the functional (2.18), constituted by the pairs $(f, \mathbf{p}_s)$ such that the free energy is finite, namely

$$\mathcal{D}(\Omega) = \big\{(f, \mathbf{p}_s)\colon f \in \mathrm{H}^1(\Omega),\ \nabla \times \mathbf{p}_s \in \mathbf{L}^2(\Omega),\ f\mathbf{p}_s \in \mathbf{L}^2(\Omega)\big\}.$$

We introduce a new variable $\mathbf{p}_1$ which satisfies

$$\nabla\times\mathbf{p}_1=\nabla\times\mathbf{p}_s,\qquad \nabla\cdot\mathbf{p}_1=0,\qquad \mathbf{p}_1\cdot\mathbf{n}|_{\partial\Omega}=0. \tag{3.1}$$

Therefore $\mathbf{p}_s$ can be decomposed as

$$\mathbf{p}_s=\mathbf{p}_1+\frac{1}{k}\,\nabla\theta, \tag{3.2}$$

with $\mathbf{p}_1\in\mathcal{R}_0(\Omega)=\{\mathbf{v}\colon \nabla\times\mathbf{v}\in\mathbf{L}^2(\Omega),\ \nabla\cdot\mathbf{v}=0,\ \mathbf{v}\cdot\mathbf{n}|_\Omega=0\}$.

Observe that for each $\mathbf{p}_s$ such that $\nabla\times\mathbf{p}_s\in\mathbf{L}^2(\Omega)$, there exist unique $\mathbf{p}_1$ and $\nabla\theta$ which satisfy (3.1), (3.2) and $\nabla\theta\cdot\mathbf{n}|_\Omega=0$. Moreover, the condition $\mathbf{p}_1\in\mathcal{R}_0(\Omega)$ implies $\mathbf{p}_1\in\mathbf{H}^1(\Omega)$.

With such positions, the functional (2.11) can be expressed in terms of the variables $(f,\theta,\mathbf{p}_1)$ as

$$\begin{aligned}\mathcal{E}(f,\theta,\mathbf{p}_1)=\int_\Omega\left[\frac{1}{2}f^4-f^2+|\nabla\times\mathbf{p}_1|^2+\frac{1}{k^2}|\nabla f|^2+f^2\left|\mathbf{p}_1+\frac{1}{k}\,\nabla\theta\right|^2\right]dx\\ -2\int_{\partial\Omega}\mathbf{p}_1\cdot\mathbf{H}_{ex}^\tau\times\mathbf{n}\,d\sigma\end{aligned} \tag{3.3}$$

and the Ginzburg-Landau system (2.19) – (2.21) assumes the form

$$\frac{1}{k^2}\,\nabla^2 f-f\left|\mathbf{p}_1+\frac{1}{k}\,\nabla\theta\right|^2+f-f^3=0, \tag{3.4}$$

$$\nabla\times\nabla\times\mathbf{p}_1+f^2\left(\mathbf{p}_1+\frac{1}{k}\,\nabla\theta\right)=\mathbf{0}, \tag{3.5}$$

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega}=0,\qquad (\nabla\times\mathbf{p}_1)\times\mathbf{n}|_{\partial\Omega}=\mathbf{H}_{ex}^\tau\times\mathbf{n}|_{\partial\Omega}. \tag{3.6}$$

Lemma 3.1 *A pair $(f,\mathbf{p}_s)\in\mathcal{D}(\Omega)$ is a minimizer of the functional (2.18) if and only if the triplet $(f,\theta,\mathbf{p}_1)\in\mathcal{D}_1(\Omega)$, with*

$$\mathcal{D}_1(\Omega)=\{(f,\theta,\mathbf{p}_1)\colon f\in\mathrm{H}^1(\Omega),\ \mathbf{p}_1\in\mathcal{R}_0(\Omega),\ f\nabla\theta\in\mathbf{L}^2(\Omega)\}$$

and θ, $\mathbf{p}_1$ satisfying (3.2), is a minimizer of the functional (3.3).

Proof It suffices to prove that $(f,\mathbf{p}_s)\in\mathcal{D}(\Omega)$ if and only if $(f,\theta,\mathbf{p}_1)\in\mathcal{D}_1(\Omega)$. Let $(f,\mathbf{p}_s)\in\mathcal{D}(\Omega)$. In view of the embedding $\mathrm{H}^1(\Omega)\hookrightarrow\mathrm{L}^4(\Omega)$, we have $f\in\mathrm{L}^4(\Omega)$ and $\mathbf{p}_1\in\mathbf{L}^4(\Omega)$, thus $f\mathbf{p}_1\in\mathbf{L}^2(\Omega)$. In this way, $f\nabla\theta=k(f\mathbf{p}_s-f\mathbf{p}_1)\in\mathbf{L}^2(\Omega)$, so that $(f,\theta,\mathbf{p}_1)\in\mathcal{D}_1(\Omega)$.

Conversely, if $(f,\theta,\mathbf{p}_1)\in\mathcal{D}_1(\Omega)$, it results $f\mathbf{p}_1\in\mathbf{L}^2(\Omega)$. Therefore, by (3.2), we obtain $f\mathbf{p}_s\in\mathbf{L}^2(\Omega)$.

As shown in [1], the functional (3.3) admits at least a minimizer, so that, in view of Lemma 3.1, the existence theorem can be proved.

Theorem 3.1 *For each $\mathbf{H}_{ex}^\tau\in\mathbf{H}^{-1/2}(\partial\Omega)$, satisfying the relation (2.9), there exists at least a pair $(f,\mathbf{p}_s)\in\mathcal{D}(\Omega)$ which minimizes the free energy (2.18).*

Definition 3.1 A *triplet* $(f,\theta,\mathbf{p}_1) \in \mathcal{D}_1(\Omega)$ is a *weak solution of the Ginzburg-Landau problem* if it satisfies

$$\begin{aligned}\int_\Omega \cos(\theta-\varphi)\Big[\frac{1}{k^2}\nabla f\cdot\nabla h + fh\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\varphi\Big)+h(f^3-f)\Big]\,dx \\ +\frac{1}{k}\int_\Omega \sin(\theta-\varphi)\Big[h\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\varphi\Big)-f\nabla h\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]dx=0,\end{aligned} \tag{3.7}$$

$$\int_\Omega \Big[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\mathbf{q}_1\Big]dx+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}^\tau_{ex}\cdot\mathbf{n}\,d\sigma=0, \tag{3.8}$$

for each $(h,\varphi,\mathbf{q}_1)\in\mathcal{D}_1(\Omega)$.

Remark It is possible to give a different definition of weak solution for the problem (3.4) – (3.6), by replacing the equations (3.7) – (3.8) with the following

$$\int_\Omega \Big[\frac{1}{k^2}\nabla f\cdot\nabla g+fg\Big|\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big|^2+(f^3-f)g\Big]dx=0, \tag{3.9}$$

$$\int_\Omega \Big[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\mathbf{q}_1\Big]dx+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}^\tau_{ex}\cdot\mathbf{n}\,d\sigma=0. \tag{3.10}$$

Though the equations (3.9) – (3.10) can be obtained by (3.7) – (3.8), choosing suitably the functions $(g,\mathbf{q}_1)$, the definitions are not equivalent, since the spaces of test functions are different.

Proposition 3.1 *If $(f,\theta,\mathbf{p}_1)$ is a regular solution of the Ginzburg-Landau problem (3.4) – (3.6), then it is a weak solution in the sense of Definition 3.1.*

Proof By taking the divergence of (3.5), we obtain

$$\nabla\cdot\Big[f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]=0.$$

Hence

$$f\Big[2\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)+f\nabla\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]=0,$$

which leads to

$$\int_\Omega \Big[2\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)+f\nabla\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]h\sin(\theta-\varphi)dx=0$$

for each (h,φ) such that $h\in H^1(\Omega)$, $h\nabla\varphi\in\mathbf{L}^2(\Omega)$. An integration by parts yields

$$\frac{1}{k}\int_\Omega [(h\nabla f-f\nabla h)\sin(\theta-\varphi)-fh(\nabla\theta-\nabla\varphi)\cos(\theta-\varphi)]\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)dx=0. \tag{3.11}$$

Moreover, multiplying (3.4) by $h\cos(\theta-\varphi)$ and integrating on Ω, it results

$$\begin{aligned}&\int_\Omega \frac{1}{k^2}\,\nabla f\cdot(\nabla\varphi-\nabla\theta)\sin(\theta-\varphi)\,dx\\&+\int_\Omega\left[\frac{1}{k^2}\,\nabla f\cdot\nabla h+fh\left|\mathbf{p}_1+\frac{1}{k}\,\nabla\theta\right|^2-fh(1-f^2)\right]\cos(\theta-\varphi)\,dx=0.\end{aligned}\tag{3.12}$$

By adding (3.11) and (3.12), we obtain (3.7). Finally, the relation (3.8) can be proved by multiplying (3.5) by an arbitrary function $\mathbf{q}_1\in\mathcal{R}_0(\Omega)$ and integrating by parts.

Denoting by $f_c=f\cos\theta$, $f_s=f\sin\theta$, $h_c=h\cos\varphi$, $h_s=h\sin\varphi$, the equations (3.7) and (3.8) can be written in the form

$$\begin{aligned}&\int_\Omega\left[\frac{1}{k^2}(\nabla f_c\cdot\nabla h_c+\nabla f_s\cdot\nabla h_s)+\frac{1}{k}(h_c\nabla f_s-f_s\nabla h_c+f_c\nabla h_s)\cdot\mathbf{p}_1\right]dx\\&\quad-\int_\Omega\left[\frac{1}{k}\,h_s\nabla f_c\cdot\mathbf{p}_1-(\mathbf{p}_1^2+f_c^2+f_s^2-1)(f_ch_c+f_sh_s)\right]dx=0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\int_\Omega\left[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+(f_c^2+f_s^2)\mathbf{p}_1\cdot\mathbf{q}_1+\frac{1}{k}(f_c\nabla f_s-f_s\nabla f_c)\cdot\mathbf{q}_1\right]dx\\&\qquad+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}_{ex}^\tau\cdot\mathbf{n}\,d\sigma=0.\end{aligned}\tag{3.14}$$

It is easy to verify that $(f,\theta,\mathbf{p}_1)\in\mathcal{D}(\Omega)$ if and only if $(f_c,f_s,\mathbf{p}_1)\in H^1(\Omega)\times H^1(\Omega)\times\mathcal{R}_0(\Omega)$. Moreover the equations (3.13)–(3.14) can be obtained by writing the free energy (3.3) as a functional of the variables $(f_c,f_s,\mathbf{p}_1)$

$$\begin{aligned}\mathcal{E}(f_c,f_s,\mathbf{p}_1)=&\int_\Omega\left[\frac{1}{2}(f_c^2+f_s^2)^2-f_c^2-f_s^2+\frac{1}{2}(|\nabla f_c|^2+|\nabla f_s|^2)+|\nabla\times\mathbf{p}_1|^2\right]dx\\&+\int_\Omega\left[\mathbf{p}_1^2(f_c^2+f_s^2)+\frac{2}{k}\,\mathbf{p}_1\cdot(f_c\nabla f_s-f_s\nabla f_c)\right]dx+2\int_{\partial\Omega}\mathbf{p}_1\cdot\mathbf{H}_{ex}^\tau\times\mathbf{n}\,d\sigma\end{aligned}$$

and then by computing the first variation with respect to such variables.

To reduce our notations, we put $\Pi=(f_c,f_s,\mathbf{p}_1)$, $\Sigma=(g_c,g_s,\mathbf{r}_1)$, $\Theta=(h_c,h_s,\mathbf{q}_1)$ and define

$$a(\Pi,\Theta)=\int_\Omega\left[\frac{1}{k^2}(\nabla f_c\cdot\nabla h_c+\nabla f_s\cdot\nabla h_s)+\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1\right]dx,\tag{3.15}$$

$$\begin{aligned}l(\Sigma,\Theta)=&-\int_\Omega\frac{1}{k}(h_c\nabla g_s-g_s\nabla h_c+g_c\nabla h_s-h_s\nabla g_c)\cdot\mathbf{r}_1\,dx\\&-\int_\Omega[(\mathbf{r}_1^2+g_c^2+g_s^2-1)(g_ch_c+g_sh_s)+(g_c^2+g_s^2)\mathbf{r}_1\cdot\mathbf{q}_1]dx\\&-\int_\Omega\frac{1}{k}(g_c\nabla g_s-g_s\nabla g_c)\cdot\mathbf{q}_1\,dx-\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}_{ex}^\tau\cdot\mathbf{n}\,d\sigma=0.\end{aligned}\tag{3.16}$$

Hence (3.13) and (3.14) can be written as

$$a(\Pi,\Theta) = l(\Pi,\Theta). \tag{3.17}$$

Consider now the equation

$$a(\Pi,\Theta) = l(\Sigma,\Theta), \tag{3.18}$$

for each $\Theta \in \mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega) \subset \mathrm{H}^1(\Omega)\times \mathrm{H}^1(\Omega)\times \mathcal{R}_0(\Omega)$ is a closed subspace which does not contain triplets $(h_c, h_s, \mathbf{q}_1)$ with constant h_c and h_s. It can be proved ([10]) that $\mathcal{V}(\Omega)$ is a Hilbert space with respect to the norm

$$\|(h_c,h_s,\mathbf{q}_1)\|^2_{\mathcal{V}} = \frac{1}{k^2}(\|\nabla h_c\|_2^2 + \|\nabla h_s\|_2^2) + \|\mathbf{q}_1\|^2_{\mathcal{R}_0}, \tag{3.19}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ and $\|\mathbf{q}_1\|_{\mathcal{R}_0} = \|\nabla\times\mathbf{q}_1\|_2$.

We will prove that the equation (3.18) admits a unique solution $\Pi = T(\Sigma) \in \mathcal{V}(\Omega)$. Moreover, with suitable hypotheses, T is a contraction, whose fixed point satisfies the relation (3.17).

Lemma 3.2 *For each $\Sigma \in \mathcal{V}(\Omega)$ there exists a unique $T(\Sigma)\in\mathcal{V}(\Omega)$ such that*

$$a(T(\Sigma),\Theta) = l(\Sigma,\Theta) \tag{3.20}$$

for all $\Theta\in\mathcal{V}(\Omega)$.

Proof In view of the definition (3.16), for each $\Sigma\in\mathcal{V}(\Omega)$, the map

$$l(\Sigma,\cdot) : \mathcal{V}(\Omega) \to \mathbb{R}$$

is linear. Moreover, the following inequalities can be easily proved

$$\Big|\int_\Omega \mathbf{r}_1\cdot(h_c\nabla g_s - g_s\nabla h_c + g_c\nabla h_s - h_s\nabla g_c)\,dx\Big|$$

$$\le \|\mathbf{r}_1\|_4(\|h_c\|_4\|\nabla g_s\|_2 + \|g_s\|_4\|\nabla h_c\|_2 + \|g_c\|_4\|\nabla h_s\|_2 + \|h_s\|_4\|\nabla g_c\|_2)$$

$$\le (c_1+1)\|\mathbf{r}_1\|_4[(\|\nabla g_s\|_2 + \|g_s\|_4)\|h_c\|_{\mathrm{H}^1(\Omega)} + (\|\nabla g_c\|_2 + \|g_c\|_4)\|h_s\|_{\mathrm{H}^1(\Omega)}],$$

$$\Big|\int_\Omega (g_ch_c + g_sh_s)\mathbf{r}_1^2\,dx\Big| \le c_1\|\mathbf{r}_1\|_4^2(\|g_c\|_4\|h_c\|_{\mathrm{H}^1(\Omega)} + \|g_s\|_4\|h_s\|_{\mathrm{H}^1(\Omega)}),$$

$$\Big|\int_\Omega (g_ch_c + g_sh_s)(1-g_c^2-g_s^2)\,dx\Big| \le (\|g_c\|_2 + \|g_c\|_6^3$$

$$+ c_1\|g_c\|_4\|g_s\|_4^2)\|h_c\|_{\mathrm{H}^1(\Omega)} + (\|g_s\|_2 + c_1\|g_c\|_4^2\|g_s\|_4 + \|g_s\|_6^3)\|h_s\|_{\mathrm{H}^1(\Omega)},$$

$$\Big|\int_\Omega (g_c^2+g_s^2)\mathbf{r}_1\cdot\mathbf{q}_1\,dx\Big| \le c_2(\|g_c\|_4^2 + \|g_s\|_4^2)\|\mathbf{r}_1\|_4\|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

$$\Big|\int_\Omega (g_c\nabla g_s - g_s\nabla g_c)\cdot\mathbf{q}_1\,dx\Big| \le c_2(\|g_c\|_4\|\nabla g_s\|_2 + \|g_s\|_4\|\nabla g_c\|_2)\|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

$$\left|\int\limits_{\partial\Omega} \mathbf{q}_1 \times \mathbf{H}^\tau_{ex} \cdot \mathbf{n}\, d\sigma\right| \le \|\mathbf{H}^\tau_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \|\mathbf{q}_1 \times \mathbf{n}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$$
$$\le c_3 \|\mathbf{H}^\tau_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

where c_1, c_2, c_3 satisfy respectively the inequalities

$$\|f\|_4 \le c_1 \|f\|_{\mathrm{H}^1(\Omega)}, \qquad \|\mathbf{p}_1\|_{\mathbf{H}^1(\Omega)} \le c_2 \|\mathbf{p}_1\|_{\mathcal{R}_0}, \qquad \|\mathbf{p}_1 \times \mathbf{n}\|_{\mathbf{H}^{1/2}(\Omega)} \le c_3 \|\mathbf{p}_1\|_{\mathcal{R}_0}. \tag{3.21}$$

Therefore, keeping the definition (3.19) into account, we get

$$l(\Sigma, \Theta) \le C(\Sigma) \|\Theta\|_{\mathcal{V}},$$

where $C(\Sigma)$ denotes a positive constant depending on Σ.

Since $l(\Sigma, \cdot)$ is continuous, the Riesz theorem ensures the existence of a unique $\Theta(\Sigma) \in \mathcal{V}(\Omega)$ such that

$$a(T(\Sigma), \Theta) = l(\Sigma, \Theta), \qquad \forall\, \Theta \in \mathcal{V}(\Omega). \tag{3.22}$$

Lemma 3.3 *For each $M > 0$ and Σ_1, $\Sigma_2 \in \mathcal{V}(\Omega)$ satisfying $\|\Sigma_i\|_{\mathcal{V}} \le M$, $i = 1, 2$, there exists a constant $\delta_M > 0$ such that*

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le \delta_M \|\Sigma_1 - \Sigma_2\|_{\mathcal{V}}.$$

Proof From the equation (3.22) we obtain the identity

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} = \sup_{\|\Theta\|_{\mathcal{V}}=1} [a(T(\Sigma_1), \Theta) - a(T(\Sigma_2), \Theta)] = \sup_{\|\Theta\|_{\mathcal{V}}=1} [l(\Sigma_1, \Theta) - l(\Sigma_2, \Theta)]$$
$$= \sup_{\|\Theta\|_{\mathcal{V}}=1} [I_1(\Theta) + I_2(\Theta) + I_3(\Theta) + I_4(\Theta) + I_5(\Theta) + I_6(\Theta)],$$

where

$$I_1(\Theta) = \frac{1}{k} \int\limits_\Omega [(\nabla g_{c1} \cdot \mathbf{r}_{11} - \nabla g_{c2} \cdot \mathbf{r}_{12}) h_s - (\nabla g_{s1} \cdot \mathbf{r}_{11} - \nabla g_{s2} \cdot \mathbf{r}_{12}) h_c] dx,$$

$$I_2(\Theta) = \frac{1}{k} \int\limits_\Omega [(g_{s1}\mathbf{r}_{11} - g_{s2}\mathbf{r}_{12}) \cdot \nabla h_c - (g_{c1}\mathbf{r}_{11} - g_{c2}\mathbf{r}_{12}) \cdot \nabla h_s] dx,$$

$$I_3(\Theta) = -\int\limits_\Omega (\mathbf{r}_{11}^2 g_{c1} + g_{c1}^3 + g_{c1} g_{s1}^2 - g_{c1} - \mathbf{r}_{12}^2 g_{c2} - g_{c2}^3 - g_{c2} g_{s2}^2 + g_{c2}) h_c dx$$

$$I_4(\Theta) = -\int\limits_\Omega (\mathbf{r}_{11}^2 g_{s1} + g_{s1}^3 + g_{s1} g_{c1}^2 - g_{s1} - \mathbf{r}_{12}^2 g_{s2} - g_{s2}^3 - g_{s2} g_{c2}^2 + g_{s2}) h_s dx$$

$$I_5(\Theta) = -\int\limits_\Omega (g_{c1}^2 \mathbf{r}_{11} + g_{s1}^2 \mathbf{r}_{11} - g_{c2}^2 \mathbf{r}_{12} - g_{s2}^2 \mathbf{r}_{12}) \cdot \mathbf{q}_1 dx$$

$$I_6(\Theta) = \frac{1}{k} \int\limits_\Omega (g_{c1} \nabla g_{s1} - g_{s1} \nabla g_{c1} - g_{c2} \nabla g_{s2} + g_{s2} \nabla g_{c2}) \cdot \mathbf{q}_1\, dx.$$

Since $\|\Sigma_i\|_{\mathcal{V}} \le M,\ i = 1,2$, we have the inequalities

$$\|\nabla g_{ci}\|_2 \le kM, \qquad \|\nabla g_{si}\|_2 \le kM, \qquad \|\mathbf{r}_{1i}\|_{\mathcal{R}_0} \le M$$

Therefore, by the definition of $\mathcal{V}(\Omega)$, g_{ci} and g_{si} satisfy the estimates

$$\|g_{ci}\|_{\mathrm{H}^1(\Omega)} \le c_4 kM, \qquad \|g_{si}\|_{\mathrm{H}^1(\Omega)} \le c_4 kM.$$

Let $\delta g_c = g_{c1} - g_{c2},\ \delta g_s = g_{s1} - g_{s2},\ \delta\mathbf{r}_1 = \mathbf{r}_{11} - \mathbf{r}_{12}$. We deduce the estimates

$$\begin{aligned}
|I_1(\Theta)| &\le Mc_1c_2[(\|h_c\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)})\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} \\
&\quad + \frac{1}{k}(\|h_c\|_{\mathrm{H}^1(\Omega)}\|\nabla(\delta g_s)\|_2 + \|h_s\|_{\mathrm{H}^1(\Omega)}\|\nabla(\delta g_c)\|_2)], \\
|I_2(\Theta)| &\le Mc_1c_2[c_4(\|h_c\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)})\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} \\
&\quad + \frac{1}{k}(\|h_c\|_{\mathrm{H}^1(\Omega)}\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)}\|\delta g_c\|_2)] \\
|I_3(\Theta)| &\le M^2c_1^2c_2[c_2\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + 2c_2c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + 2c_1c_4^2k^2\|\delta g_s\|_{\mathrm{H}^1(\Omega)} \\
&\quad + 4c_1c_4^2k^2\|\delta g_c\|_{\mathrm{H}^1(\Omega)}]\|h_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_c\|_2\|h_c\|_2 \\
|I_4(\Theta)| &\le M^2c_1^2c_2[c_2\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + 2c_2c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + 2c_1c_4^2k^2\|\delta g_c\|_{\mathrm{H}^1(\Omega)} \\
&\quad + 4c_1c_4^2k^2\|\delta g_s\|_{\mathrm{H}^1(\Omega)}]\|h_s\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_2\|h_s\|_2 \\
|I_5(\Theta)| &\le 2M^2c_1^2c_2^2c_4k(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)} + c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0})\|\mathbf{q}_1\|_{\mathcal{R}_0} \\
|I_6(\Theta)| &\le Mc_1c_2k(1+c_4)(\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + \|\delta g_c\|_{\mathrm{H}^1(\Omega)})\|\mathbf{q}_1\|_{\mathcal{R}_0}.
\end{aligned}$$

Thus

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le A\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + B(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)}),$$

where

$$\begin{aligned}
A &= 2Mc_1c_2c_4k(1 + c_4 + 3Mc_1c_2c_4k) \\
B &= Mc_1c_2(Mc_2c_4k + 6Mc_1^2c_4^3k^3 + 2Mc_1c_2c_4k + 1 + 3c_4) + c_4k.
\end{aligned}$$

By using the inequality $2xy \le \varepsilon x^2 + \frac{1}{\varepsilon}y^2$, we obtain

$$\begin{aligned}
\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}}^2 &\le [A\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + B(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)})]^2 \\
&\le A^2(1+\varepsilon)\|\delta\mathbf{r}_1\|_{\mathcal{R}_0}^2 + B^2\,\frac{1+\varepsilon}{\varepsilon}\,(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)})^2 \\
&\le A^2(1+\varepsilon)\|\delta\mathbf{r}_1\|_{\mathcal{R}_0}^2 + 2B^2c^2\,\frac{1+\varepsilon}{\varepsilon}\,(\|\nabla(\delta g_c)\|_2^2 + \|\nabla(\delta g_s)\|_2^2).
\end{aligned}$$

Hence, the choice

$$\delta_M^2 = \max\left\{A^2(1+\varepsilon), 2B^2c^2k^2\,\frac{1+\varepsilon}{\varepsilon}\right\}$$

yields

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le \delta_M\|\Sigma_1 - \Sigma_2\|_{\mathcal{V}}.$$

Lemma 3.4 *If the following inequalities*

$$2Mc_4k[2Mc_1c_2 + M^2c_1^2c_4k(c_2^2 + 2c_1^2c_4^2k^2) + c_4k] \le M, \tag{3.23}$$

$$4M^2c_1c_2c_4k(Mc_1c_2k + 1) + 2c_3\|\mathbf{H}_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \le M, \tag{3.24}$$

hold, T defined through (3.20), maps $\mathcal{B}_M = \{\Sigma \in \mathcal{V}(\Omega) \colon \|\Sigma\|_{\mathcal{V}} \le M\}$ in itself.

Proof By the definition (3.22) we have

$$\|T(\Sigma)\|_{\mathcal{V}}^2 = l(\Sigma, T(\Sigma)).$$

Therefore, proceeding as in the proof of Lemma 3.4, it results

$$\|T(\Sigma)\|_{\mathcal{V}}^2 \le \frac{D}{k}(\|\nabla f_c\|_2 + \|\nabla f_s\|_2) + E\|\mathbf{p}_1\|_{\mathcal{R}_0},$$

where

$$D = Mc_4k[2Mc_1c_2 + M^2c_1^2c_4k(c_2^2 + 2c_1^2c_4^2k^2) + c_4k],$$
$$E = 2M^2c_1c_2c_4k(Mc_1c_2k + 1) + c_3\|\mathbf{H}_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)}.$$

The hypotheses (3.23) and (3.24) imply

$$\|T(\Sigma)\|_{\mathcal{V}}^2 \le M\|T(\Sigma)\|_{\mathcal{V}},$$

so that $T(\Sigma) \in \mathcal{B}_M$.

By applying the previous lemmas, we get the uniqueness result.

Theorem 3.2 *Let $(f_c, f_s, \mathbf{p}_1) \in \mathcal{B}_M$ and $(g_c, g_s, \mathbf{q}_1) \in \mathcal{B}_M$, satisfying the Definition 3.1. If the inequalities (3.23) and (3.24) hold and there exists $\varepsilon > 0$ such that $\delta_M < 1$, then $(f_c, f_s, \mathbf{p}_1) = (g_c, g_s, \mathbf{q}_1)$.*

The Theorem 3.2 ensures the uniqueness of weak solutions, provided that the parameter k, the external field and the domain Ω are sufficiently small. A non-uniqueness result can be obtained with the same method used in [9, 11], if the domain Ω is sufficiently large.

3.2 The time-dependent case

The weak formulation of the evolution problem (2.22) – (2.25) is obtained by introducing the functional space

$$\begin{aligned}\mathcal{H}(Q) = \Big\{&(f, \mathbf{p}_s, \phi_s) \colon f \in \mathrm{L}^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))'),\\ &\nabla \times \mathbf{p}_s \in \mathrm{L}^2(0, T; \mathbf{L}^2(\Omega)),\ \dot{\mathbf{p}}_s + \nabla\phi_s \in \mathrm{L}^2(0, T; (\mathbf{H}^1_{\mathbf{n}}(\Omega))'),\\ &f\mathbf{p}_s \in \mathrm{L}^2(0, T; \mathbf{L}^2(\Omega)),\ f\phi_s \in \mathrm{L}^2(0, T; (H^1(\Omega))'),\ \mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0\Big\},\end{aligned}$$

where $Q = \Omega \times (0, T)$, $\mathbf{H}^1_{\mathbf{n}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \colon \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$ and X' denotes the dual space of X.

Proceeding as in the stationary case, in order to prove existence and uniqueness theorems, we decompose $\mathbf{p}_s$ and ϕ_s in the following form

$$\mathbf{p}_s = \mathbf{p}_1 - \nabla\theta, \tag{3.25}$$

$$\phi_s = \phi + \dot{\theta}, \tag{3.26}$$

with $\mathbf{p}_1 \in \mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))$ and

$$\phi = -\frac{1}{\eta}\,\nabla\cdot\mathbf{p}_1. \tag{3.27}$$

By means of the positions (3.25)–(3.26), the system (2.22)–(2.25) can be written in the form

$$\dot{f} - \nabla^2 f + k^2(f^2-1)f + f(\mathbf{p}_1 - \nabla\theta)^2 = 0, \tag{3.28}$$

$$\eta\dot{\mathbf{p}}_1 + \nabla\times\nabla\times\mathbf{p}_1 + \eta\nabla\phi + f^2(\mathbf{p}_1 - \nabla\theta) = \mathbf{0}, \tag{3.29}$$

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{p}_1 + \mathbf{H}^\tau_{ex})\times\mathbf{n}|_{\partial\Omega} = 0, \tag{3.30}$$

$$\nabla\theta\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \mathbf{p}_1\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.31}$$

$$f(x,0) = f_0(x), \qquad \mathbf{p}_1(x,0) - \nabla\theta(x,0) = \mathbf{p}_{s0}(x). \tag{3.32}$$

Let

$$\begin{aligned}\mathcal{K}(Q) = \Big\{(f,\theta,\mathbf{p}_1)\colon\ & f\in\mathrm{L}^2(0,T;H^1(\Omega))\cap H^1(0,T;(H^1(\Omega))'),\\ & f\nabla\theta\in\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega)),\ \ f\dot{\theta}\in\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))'),\\ & \mathbf{p}_1\in\mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))\cap H^1(0,T;(\mathbf{H}^1_{\mathbf{n}}(\Omega))')\Big\}.\end{aligned}$$

From the definition of $\mathcal{H}(Q)$, it follows that if $\mathbf{p}_1$ satisfies (3.25)–(3.27), then the following equation holds

$$\dot{\mathbf{p}}_1 - \frac{1}{\eta}\,\nabla^2\mathbf{p}_1 = \mathbf{F}, \tag{3.33}$$

where $\mathbf{F} = \dot{\mathbf{p}}_s + \nabla\phi_s + \frac{1}{\eta}\,\nabla\times\nabla\times\mathbf{p}_s \in \mathrm{L}^2(0,T;(\mathrm{H}^1_{\mathbf{n}}(\Omega))')$. The equation (3.33), with the boundary and initial conditions $(3.30)_2$, $(3.31)_2$, $(3.32)_2$ admits a unique solution $\mathbf{p}_1\in\mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))\cap H^1(0,T;(\mathbf{H}^1_{\mathbf{n}}(\Omega))')$ (see [14]). Hence, by putting $\phi = -\frac{1}{\eta}\,\nabla\cdot\mathbf{p}_1$, we have that for each $(f,\mathbf{p}_s,\phi_s)\in\mathcal{H}(Q)$, there exists a unique triplet $(f,\theta,\mathbf{p}_1)\in\mathcal{K}(Q)$ which satisfies (3.25)–(3.27).

Definition 3.2 A *triplet* $(f,\theta,\mathbf{p}_1)\in\mathcal{K}(Q)$ is a *weak solution* of the problem (3.28)–(3.32), with $\mathbf{H}^\tau_{ex}\in\mathbf{H}^{-1/2}(\partial\Omega)$ if

$$\begin{aligned}&\int_\Omega [\dot{f}g + k^2(f^2-1)fg + \nabla f\cdot\nabla g + fg(\mathbf{p}_1-\nabla\theta)\cdot(\mathbf{p}_1-\nabla\varphi)]\cos(\theta-\varphi)\,dx \\ &- \int_\Omega [fg\dot{\theta} - \frac{1}{\eta}\,fg\nabla\cdot\mathbf{p}_1 + g\nabla f\cdot(\mathbf{p}_1-\nabla\varphi) - f\nabla g\cdot(\mathbf{p}_1-\nabla\theta)]\sin(\theta-\varphi)\,dx = 0,\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\int_\Omega \big\{\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1 + \nabla\cdot\mathbf{p}_1\nabla\cdot\mathbf{q}_1 + [\eta\dot{\mathbf{p}}_1 + f^2(\mathbf{p}_1-\nabla\theta)]\cdot\mathbf{q}_1\big\}\,dx \\ &- \int_{\partial\Omega}\mathbf{H}^\tau_{ex}\times\mathbf{q}_1\cdot\mathbf{n}\,d\sigma = 0,\end{aligned} \tag{3.35}$$

for each $(g, \varphi, \mathbf{q}_1) \in \mathcal{K}(Q)$, a.e. $t \in (0, T)$ and

$$f(x, 0) = f_0(x), \qquad \mathbf{p}_1(x, 0) - \nabla\theta(x, 0) = \mathbf{p}_{s0}(x).$$

By means of the decomposition (3.25) – (3.26), the constitutive equation (2.30) yields

$$f^2\dot{\theta} + \left(1 - \frac{1}{\eta}\right) f^2 \nabla \cdot \mathbf{p}_1 + 2f\nabla f \cdot (\mathbf{p}_1 - \nabla\theta) - f^2\nabla^2\theta = 0. \tag{3.36}$$

Proposition 3.2 *Every regular solution $(f, \theta, \mathbf{p}_1)$ of the Ginzburg-Landau equations (3.28) – (3.32) is a weak solution in the sense of Defintion 3.2.*

Proof By multiplying (3.28) by $g\cos(\theta - \varphi)$ and integrating on Ω, it results

$$\begin{aligned} &\int_\Omega [\dot{f}g + \nabla f \cdot \nabla g + k^2 fg(f^2 - 1) + fg|\mathbf{p}_1 - \nabla\theta|^2] \cos(\theta - \varphi)\, dx \\ &\qquad - \int_\Omega g\nabla f \cdot (\nabla\theta - \nabla\varphi) \sin(\theta - \varphi)\, dx = 0. \end{aligned} \tag{3.37}$$

On the other hand, from (3.36), we obtain

$$\begin{aligned} &\int_\Omega [f\dot{\theta} + \left(1 - \frac{1}{\eta}\right) fg\nabla \cdot \mathbf{p}_1 + 2g\nabla f \cdot (\mathbf{p}_1 - \nabla\theta)] \sin(\theta - \varphi)\, dx \\ &+ \int_\Omega [(f\nabla g + g\nabla f) \cdot \nabla\theta \sin(\theta - \varphi) + fg(\nabla\theta - \nabla\varphi) \cdot \nabla\theta \cos(\theta - \varphi)]\, dx = 0. \end{aligned} \tag{3.38}$$

By subtracting (3.37) and (3.38), we get (3.34).
Finally, inner multiplication of (3.29) by $\mathbf{q}_1$ and an integration by parts, lead to (3.35).

Let $(f, \theta, \mathbf{p}_1)$ be a weak solution of the problem (3.28) – (3.32). The choice $g = f$, $\varphi = \theta$ in the Definition 3.2 yields

$$\int_\Omega \left[\frac{1}{2}\frac{\partial f^2}{\partial t} + |\nabla f|^2 + f^2|\mathbf{p}_1 - \nabla\theta|^2 + k^2 f^4\right] dx = \int_\Omega k^2 f^2\, dx. \tag{3.39}$$

Therefore, by applying Gronwall's inequality, we obtain

$$\|f(t)\|_2 \le \|f_0\|_2 \exp(2k^2 t) \quad 0 \le t \le T. \tag{3.40}$$

By integrating (3.39) in the interval $(0, t)$, with $0 \le t \le T$, it follows[1]

$$\int_0^t \int_\Omega [|\nabla f|^2 + f^2|\mathbf{p}_1 - \nabla\theta|^2 + k^2 f^4]\, dx\, d\tau \le k_1 \|f_0\|_2^2, \tag{3.41}$$

[1] Henceforth k_i denotes a function of the variable t, belonging to $L^1(0, T)$.

so that $\|f\|^2_{\mathrm{H}^1(\Omega)}$ and $\|f(\mathbf{p}_1 - \nabla\theta)\|_2^2$ are $\mathrm{L}^1(0,T)$-functions.

Analogously, by choosing $\mathbf{q}_1 = \mathbf{p}_1$ in (3.35) and integrating on $(0,t)$, we obtain the identity

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + \int_0^t\int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$= \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 - \int_0^t\int_\Omega f^2\mathbf{p}_1\cdot(\mathbf{p}_1 - \nabla\theta)\,dx\,d\tau - \int_0^t\int_{\partial\Omega} \mathbf{H}^\tau_{ex}\times\mathbf{p}_1\cdot\mathbf{n}\,d\sigma\,d\tau,$$

so that

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + \int_0^t\int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$\le \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 + \int_0^t \|\mathbf{H}^\tau_{ex}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Omega)}\|\mathbf{p}_1\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Omega)}\,d\tau$$

$$+ \int_0^t \left[\frac{k_2}{2}\|f\|_4^2 + \frac{1}{2k_2}\|\mathbf{p}_1\|_4^2 + \|f(\mathbf{p}_1-\nabla\theta)\|_2^2\right]d\tau.$$

The inequality (3.41), implies

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + k_3\int_0^t\int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$\le \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 + \int_0^t \left[\frac{k_2}{2}\|f\|_4^2 + k_4\|\mathbf{p}_1\|_2^2 + \|f(\mathbf{p}_1-\nabla\theta)\|_2^2\right]d\tau \tag{3.42}$$

$$+ \int_0^t k_5\|\mathbf{H}^\tau_{ex}\times\mathbf{n}\|^2_{\mathbf{H}^{-1/2}(\Omega)}\,d\tau \le k_6 + k_4\int_0^t \|\mathbf{p}_1\|_2^2\,d\tau.$$

Thus Gronwall's inequality yields

$$\|\mathbf{p}_1(t)\|_2 \le k_6(1 + k_4 t\exp(k_4 t)), \quad 0\le t\le T. \tag{3.43}$$

Therefore $\|\mathbf{p}_1\|^2_{\mathrm{H}^1(\Omega)} \in \mathrm{L}^1(0,T)$.

Theorem 3.3 *If* $\Omega\subset\mathbb{R}^2$, $\mathbf{H}^\tau_{ex}\in\mathbf{H}^{-1/2}(\partial\Omega)$, $f_0\in\mathrm{L}^2(\Omega)$ *and* $\mathbf{p}_{s0} = \mathbf{p}_{10} - \nabla\theta_0$ *with* $\mathbf{p}_{10}\in\mathbf{L}^2(\Omega)$*, there exists a unique triplet* $(f,\theta,\mathbf{p}_1)$ *satisfying the Definition 3.2.*

Proof Let $(f_1,\theta_1,\mathbf{p}_{11})$ and $(f_2,\theta_2,\mathbf{p}_{12})$ be weak solutions of the problem (3.28)–(3.32).

Denote by $\delta f_c = f_{c1} - f_{c2} = f_1 \cos\theta_1 - f_2 \cos\theta_2$, $\delta f_s = f_{s1} - f_{s2} = f_1 \sin\theta_1 - f_2 \sin\theta_2$ and $\delta \mathbf{p}_1 = \mathbf{p}_{11} - \mathbf{p}_{12}$. Write the equation (3.34), first with $(f, \theta, \mathbf{p}_1) = (f_1, \theta_1, \mathbf{p}_{11})$, then with $(f, \theta, \mathbf{p}_1) = (f_2, \theta_2, \mathbf{p}_{12})$ and subtract the ensuing relations. By choosing (g, φ) such that $g \cos\varphi = \delta f_c$, $g \sin\varphi = \delta f_s$, we obtain

$$\begin{aligned} &\int_\Omega \left[\frac{1}{2}\frac{d}{dt}(\delta f_c)^2 + \frac{1}{2}\frac{d}{dt}(\delta f_s)^2 + |\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2\right] dx \\ &\qquad = \int_\Omega k^2[(\delta f_c)^2 + (\delta f_s)^2]\, dx + J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.44}$$

where

$$\begin{aligned} J_1 &= -\frac{1}{\eta}\int_\Omega (\nabla\cdot \mathbf{p}_{11} f_1 \sin\theta_1 - \nabla\cdot\mathbf{p}_{12} f_2 \sin\theta_2)\delta f_c\, dx \\ &\quad + \frac{1}{\eta}\int_\Omega (\nabla\cdot\mathbf{p}_{11} f_1 \cos\theta_1 - \nabla\cdot\mathbf{p}_{12} f_2\cos\theta_2)\delta f_s\, dx \\ &= -\frac{1}{\eta}\int_\Omega [f_1 \sin\theta_1 \nabla\cdot(\delta\mathbf{p}_1)\delta R - f_1\cos\theta_1 \nabla\cdot(\delta\mathbf{p}_1)\delta f_s]\, dx, \\ J_2 &= -k^2 \int_\Omega [(f_1^3\cos\theta_1 - f_2^3\cos\theta_2)\delta f_c + (f_1^3\sin\theta_1 - f_2^3 \sin\theta_2)\delta f_s]\, dx \\ &= -k^2\int_\Omega \{f_1^2[(\delta f_c)^2 + (\delta f_s)^2] + (f_1^2 - f_2^2)(f_2\cos\theta_2\delta f_c + f_2\sin\theta_2 \delta f_s)\}\, dx \\ J_3 &= -\int_\Omega [(\mathbf{p}_{11}^2 f_1 \cos\theta_1 - \mathbf{p}_{12}^2 f_2\cos\theta_2)\delta f_c + (\mathbf{p}_{11}^2 f_1\sin\theta_1 - \mathbf{p}_{12}^2 f_2 \sin\theta_2)\delta f_s]\, dx \\ &\quad + \int_\Omega [\nabla(f_1\sin\theta_1)\cdot\mathbf{p}_{11} - \nabla(f_2\sin\theta_2)\cdot\mathbf{p}_{12}]\delta f_c\, dx \\ &\quad - \int_\Omega [\nabla(f_1\cos\theta_1)\cdot\mathbf{p}_{11} - \nabla(f_2\cos\theta_2)\cdot\mathbf{p}_{12}]\delta f_s\, dx \\ &= -\int_\Omega (f_1\cos\theta_1\delta f_c + f_1\sin\theta_1\delta f_s)(\mathbf{p}_{11} + \mathbf{p}_{12})\cdot\delta\mathbf{p}_1\, dx \\ &\quad + \int_\Omega \{\mathbf{p}_{12}^2[(\delta R)^2 + (\delta I)^2] + [\nabla(f_1\sin\theta_1)\delta f_c - \nabla(f_1\cos\theta_1)]\cdot\delta\mathbf{p}_1\}\, dx \\ &\quad + \int_\Omega \mathbf{p}_{12}\cdot[\nabla(\delta f_s)\delta f_c - \nabla(\delta f_c)\delta f_s]\, dx, \\ J_4 &= -\int_\Omega (\mathbf{p}_{11} f_1 \sin\theta_1 - \mathbf{p}_{12} f_2 \sin\theta_2)\cdot\nabla(\delta R)\, dx \end{aligned}$$

$$+\int_\Omega (\mathbf{p}_{11} f_1 \cos\theta_1 - \mathbf{p}_{12} f_2 \cos\theta_2)\cdot\nabla(\delta I)\,dx$$

$$= -\int_\Omega (\delta f_s \mathbf{p}_{11} + f_2 \sin\theta_2 \delta\mathbf{p}_1)\cdot\nabla(\delta f_c)$$

$$-\int_\Omega (\delta f_c \mathbf{p}_{11} + f_2 \cos\theta_2 \delta\mathbf{p}_1)\cdot\nabla(\delta f_s)\,dx.$$

Keeping the estimates (3.40) and (3.43) into account, we deduce the following inequality

$$\begin{aligned}
|J_1| &\le \frac{1}{\eta}\,\|f_1\|_4\|\nabla\cdot\delta\mathbf{p}_1\|_2(\|\delta f_c\|_4 + \|\delta f_s\|_4)\\
&\le \frac{\varepsilon}{2}\,\|\nabla\cdot\delta\mathbf{p}_1\|_2^2 + \frac{C}{2\varepsilon}\,\|f_1\|_4^2(\|\delta f_c\|_2\|\delta f_c\|_{\mathrm{H}^1(\Omega)} + \|\delta f_s\|_2\|\delta f_s\|_{\mathrm{H}^1(\Omega)})\\
&\le C(\varepsilon)[\|\nabla\cdot(\delta\mathbf{p}_1)\|_2^2 + \|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2] + C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2],
\end{aligned}$$

where $\varepsilon > 0$, $C(\varepsilon)$ is a positive constant and $C(t)$ is a L^1-function.

Analogously, we can prove the estimates

$$\begin{aligned}
|J_2| &\le C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2\big) + C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2\big),\\
|J_3| &\le C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2\big)\\
&\quad + C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla\cdot(\delta\mathbf{p}_1)\|_2^2 + \|\nabla\times(\delta\mathbf{p}_1)\|_2^2\big),\\
|J_4| &\le C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2\big)\\
&\quad + C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla\cdot(\delta\mathbf{p}_1)\|_2^2 + \|\nabla\times(\delta\mathbf{p}_1)\|_2^2\big).
\end{aligned}$$

Therefore, from (3.44) we get

$$\begin{aligned}
&\int_\Omega \Big[\frac{1}{2}\,\frac{d}{dt}\,(\delta f_c)^2 + \frac{1}{2}\,\frac{d}{dt}\,(\delta f_s)^2 + |\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2\Big]dx\\
&\le C(\varepsilon)\big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla\cdot(\delta\mathbf{p}_1)\|_2^2 + \|\nabla\times(\delta\mathbf{p}_1)\|_2^2\big]\\
&\qquad + C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2].
\end{aligned} \tag{3.45}$$

With the same technique, from (3.35), we have

$$\int_\Omega \Big[\frac{\eta}{2}\,\frac{d}{dt}\,(\delta\mathbf{p}_1)^2 + |\nabla\times(\delta\mathbf{p}_1)|^2 + |\nabla\cdot(\delta\mathbf{p}_1)|^2\Big]dx = -J_5 + J_6, \tag{3.46}$$

with

$$J_5 = \int_\Omega (f_1^2 \mathbf{p}_{11} - f_2^2 \mathbf{p}_{12}) \cdot \delta\mathbf{p}_1 \, dx$$
$$= \int_\Omega [f_1^2 \delta\mathbf{p}_1 + \mathbf{p}_{12}(f_1 - f_2)(f_1 + f_2)] \cdot \delta\mathbf{p}_1 \, dx,$$
$$J_6 = \int_\Omega (f_1^2 \nabla\theta_1 - f_2^2 \nabla\theta_2) \cdot \delta\mathbf{p}_1 \, dx$$
$$= \int_\Omega [f_1(f_1 \nabla\theta_1 - f_2 \nabla\theta_2) + f_2 \nabla\theta_2 (f_1 - f_2)] \cdot \delta\mathbf{p}_1 \, dx.$$

Therefore

$$\begin{aligned} &\int_\Omega \left[\frac{\eta}{2} \frac{d}{dt} (\delta\mathbf{p}_1)^2 + |\nabla \times (\delta\mathbf{p}_1)|^2 + |\nabla \cdot (\delta\mathbf{p}_1)|^2 \right] dx \\ &\le C(\varepsilon) \big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta\mathbf{p}_1)\|_2^2 + \|\nabla \times (\delta\mathbf{p}_1)\|_2^2 \big] \\ &\quad + C(t) \big[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2 \big]. \end{aligned} \tag{3.47}$$

From the relations (3.45) and (3.47) we have

$$\begin{aligned} &\int_\Omega \frac{1}{2} \frac{d}{dt} (\delta f_c)^2 + (\delta I)^2 + \eta(\delta\mathbf{p}_1)^2] \, dx \\ &+ \int_\Omega [|\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2 + |\nabla \times (\delta\mathbf{p}_1)|^2 + |\nabla \cdot (\delta\mathbf{p}_1)|^2] \, dx \\ &\le C(\varepsilon) \big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta\mathbf{p}_1)\|_2^2 \\ &+ \|\nabla \times (\delta\mathbf{p}_1)\|_2^2 \big] + C(t) [\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2] \end{aligned}$$

so that, by a suitable choice of the constant $C(\varepsilon)$, we conclude

$$\int_\Omega \frac{1}{2} \frac{d}{dt} [(\delta f_c)^2 + (\delta I)^2 + \eta(\delta\mathbf{p}_1)^2] \, dx \le C(t) [\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta\mathbf{p}_1\|_2^2].$$

Gronwall's inequality yields $\delta f_c = 0$, $\delta f_s = 0$, $\delta\mathbf{p}_1 = 0$. Hence $f_1 = f_2$, $\nabla\theta_1 = \nabla\theta_2$, $\mathbf{p}_{11} = \mathbf{p}_{12}$.

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Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 43–63

Generic Emergence of Cognitive Behaviour in Self-Generating Neural Networks

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Received: May 3, 2002; Revised: December 13, 2002

Abstract: We discuss the design and behaviour of families of neural networks which grow out of a small set of "mother" neurons in response to external stimuli and to the activities present in various parts of the net at a given time. The growth process is subject to a few fundamental rules, like

- the ability of neurons to grow new neurons or connections is gradually exhausted with the number of generations
- neurons are either of excitatory or inhibitive type
- inhibitive neurons have a tendency to form long-range connections, whereas excitatory neurons "prefer" short-range connections.

In addition, there are a number of free parameters in the equations driving the time evolution of the neural activities. Our design is implemented using Matlab, such that the growth process of the network and its activity can be observed and controlled interactively on the computer screen.

Once the networks are grown both periodic attractors and fixed points are observed generically in response to external input. The inputs used in the network's formation are typically distinguished by characteristic responses, but the resulting networks are capable of other behaviour in response to other inputs.

Keywords: *Neural networks; growth rules.*

Mathematics Subject Classification (2000): 92B20, 68T05, 37N25.

1 Introduction

Standard artificial neural networks are either feed-forward nets of perceptron type, or they allow feedback between the various layers of neurons. The latter type is usually described as a dynamical system, and the equations for the membrane potential (or firing rate, or activity) of the individual neurons form a coupled system of nonlinear differential equations. For surveys on the theory of these nets, we refer the reader to the books by Arbib [2], Hertz, *et al.* [11], Amit [1], or the (older) papers by Grossberg [8] or Lippmann [12].

What all these nets have in common is that their graph structure is fixed once and for all. Connections exist or not, and training or learning of the net is done by adjusting the connection weights. This imitates the biological process of changing synaptic efficacies; various learning rules are used, the most common being variants of the Hebb Rule ([9]). In essence, a neural net consists of the following three elements:

1. A set of neurons at certain locations; each neuron can send output or receive input via connections. A connection is a one-way street; from the point of view of the sending neuron, we will call it an axonal (output) connection; from the point of view of the receiving neuron, we will call it a synaptic (input) connection. In simple additive networks, the rate of change of a neuron's membrane potential is determined by a sum of synaptic inputs, each of which is a nonlinear function of a presynaptic neuron's membrane potential (which may be interpreted as a firing rate), modulated by a "synaptic efficacy" (connection strength).
2. A set of existing connections between the various neurons, which gives the whole object the structure of a directed graph.
3. A matrix of connectivity (synaptic) strengths, which can change in the learning process.

In these standard nets, the first two of these elements are fixed; only the third is subject to change. Because of this, the traditional method of developing an artificial network is to connect every neuron to every other neuron with some initial connection weight (if the network is layered then typically only neighbouring layers are fully connected) and then to evolve the connection weights according to some training process. For nets that consist of a large number of neurons, this training process can become quite awkward.

In any case, biological evidence points in a different direction. Complicated biological neural networks, i.e., brains and nervous systems, evolved from simpler ones in an evolutionary process, a process which is repeated following a genetic blueprint with some accuracy in each individual as the brain develops. The end results are large brains, with as many as billions of neurons but a relatively sparse connection matrix. Even small subsystems of the brain are far from fully connected. These natural neural networks display a wide variety of principles of organization ([14]) including "layered" structures (e.g., separation between "processing" neurons and data gathering neurons, with relatively few intermediate layers, cortical layers and columns, or different 'nuclei' or structures in the brain), interplay between excitation and inhibition (sometimes "short-range excitation, long-range inhibition" to produce "cell assemblies", sometimes pairs with negative feedback, such as motor neurons and Renshaw cells), and dynamic responses (e.g., oscillations) to inputs.

Neurons differ from other body cells in the sense that they do not split. As a consequence, a mature brain is a more stable biological structure than, say, a muscle or any other organ, where the individual cells are not so important. Neurons, once they die, can possibly be replaced, but the functionality of the brain is mostly maintained by the sharing of tasks by many neurons. This redundancy is indeed a central feature of biological neural nets.

It is easy to think of reasons why neurons in the mature brain might not have been designed to split. Brain functions are more complex than the functions of other organs, and the way that neurons interact is crucial for this function. Neural cell division on a large scale would presumably alter brain function in ways incompatible with maintaining the multiple specific tasks assigned to the brain by nature.

In the process of evolution, however, such splitting must have occurred in one way or the other, because otherwise there would be no sophisticated brains at all (when we use the word "brain" here, we mean any biological neural structure with at least a few dozen interconnected neurons). Splitting of some kind must also occur in the growing brain, albeit controlled by the genetic code.

The objective of the project described in this article is to attempt a simulation of an evolutionary growth processes for neural networks. Although not truly evolution, the process used has been inspired by the biological idea. Our purpose is to develop a network displaying characteristic responses to inputs used to guide the growth. We start with a network of very simple type, namely a few neurons which are present from the outset and whose activity is interpreted as the information processing done by the emerging network (we call these neurons the "mother neurons"). The mother neurons have the ability to split, or, more accurately in our implementation, "sprout" new neurons, which can themselves sprout again, or alternatively, form connections to already existing neurons. The resulting "child" neurons, "grandchild" neurons etc. are connected to their parent by a synaptic connection, and they can, with some randomness in the process, grow axonal connections to other neurons and "share" their activity, as measured by their membrane potentials. The whole process is driven by external inputs and by the activity levels of the existing neurons at any given time. The decision as to when and where to connect or sprout is deterministic. However, when the "best" receiving location is not unique, one is chosen randomly. Synaptic connection strengths change over time and can be excitatory or inhibitive, but a given neuron can grow either excitatory or inhibitive connections, not both. We shall therefore classify neurons as either excitatory or inhibitive, depending on what type of connections they grow. Inhibitive neurons are never allowed to sprout, because the activity of a "child" of such a neuron would be immediately suppressed and therefore would contribute little to the information processing ability of the net. Instead, when an inhibitive neuron grows a connection, this connection must lead to an already existing neuron.

We implemented limits on the size of the net by setting low upper limits on how many axonal connections to new or existing neurons a given neuron can form and how many synaptic connections it can receive. For all the details of the process, see Sections 2 and 3.

The growth process of the neural networks presented here is dependent on the current spatial configuration. As a courtesy to the human visual system, the system evolves on a two-dimensional grid, and the direction in which a new neuron sprouts from an existing one depends on a "potential" created by the neurons already in place. Neurons sprout downward from a top layer (occupied by the mother neurons), and the generation number of a neuron is identical to the number of its layer. In this sense, all of the mother neurons are first generation neurons. The potential has the effect of allowing neurons to "feel" each others presence even without being connected. As well as depending on the location and type of the neurons in the network, the potential is also dependent upon both the number of available synaptic connections left in other neurons and the generation number of each neuron. This latter dependence was introduced after a number of numerical experiments indicated that a sufficient number of inhibitive back-connections are required for the network to exhibit interesting behaviour. Depending upon the neuron-type (excitatory vs. inhibitive) the neuron can either sprout or connect. Recall that inhibitive neurons can only connect. Rather than searching the complete two-dimensional grid for viable locations, the sprouting/connecting is only allowed inside a certain window whose size depends on the generation (level) and on the neuron-type. Connections are allowed anywhere inside this window but sprouting is further restricted by allowing each neuron to only sprout down one level. Inhibitive neurons differ from excitatory neurons in the sense that they connect preferably over longer distances, and preferably to neurons of earlier generations.

We dwell further on the problem of why the mathematical analysis of complex neural networks seems so hard if attempted directly. A network of, say, 100 neurons, in which each neuron is connected to 10 others, will be described by 100 coupled nonlinear ordinary differential equations, with 10 coupling terms on the right hand side of each equation. If equations for the connectivities (on a slower time scale) are added, the number of equations will grow by another thousand. Even if we disregard these latter equations for the modeling of the neural dynamics, the state space of the system has 100 dimensions, and it seems nearly hopeless to predict the dynamical behaviour of such a system in the general case without some structural principle, or its response to external inputs (modeled as force terms on the right-hand sides of the equations). Even for much simpler systems of differential equations, with only 3 or 4 dependent variables, prediction of the dynamic response to inputs is a real challenge (for a case study on the Lorenz system of equations, see Evans, *et al.* [7]). Attempts to simplify the description of such systems by nonlinear diffusion equations were made by Cottet [4] and, in greater generality, by Edwards [5], but the analysis done in these papers shows that such mean field approximations are only feasible under very special assumptions on the connection matrix (in Edwards [5]), connections are local, predominantly excitatory, and nearby connections cannot have very different strengths). Other structural principles, such as symmetry of connections (Hopfield [10]), tend to be overly restrictive and un-biological.

Yet nature has found ways to design much larger networks, described by systems of equations of unimaginable complexity, which display highly complex yet highly organized behaviour. Our paper is an attempt to repeat such an organic design process in a computer.

2 The Model

We now describe, in a series of steps, how our network starts from the mother neurons and grows in response to external stimuli and, eventually, internal activity. The activity of a given neuron is identified with the membrane potential of that neuron. External inputs are identified with force terms on the right-hand sides of the equations. For a given size of the net (say n neurons), the evolution of the membrane potentials with respect to time is given by network equations of the form

$$\dot{u}_i = -u_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} T_{ij} \ \mathrm{sgm}(\mu(u_j - w_j)) + I_i, \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

where u_i is the membrane potential of the i-th neuron. The three terms on the right hand side are respectively: a leakage term, the contribution from all the neurons connected to the i-th neuron and an external force term. The connection strength between the neurons with label j and i is denoted as T_{ij}, where j is the label of the sender and i is the label of the receiver. The external force term is denoted as I_i and is interpreted as a temporary input upon neuron i. This can be switched on or off, and can take any real value inside an interval $[-I_{\max}, I_{\max}]$, thereby enhancing or inhibiting the activity of the i-th neuron. The symbol sgm denotes a sigmoid function and for this paper, we take

$$\mathrm{sgm}(x) = \frac{1}{2}\,(1 + \tanh x) = \frac{1}{1 + e^{-2x}}, \tag{2.2}$$

which assumes values in (0,1). Other sigmoids are certainly possible. The constants w_j are inserted as firing thresholds which further differentiate the activity of inhibitive from that of excitatory neurons. In our experiments, we chose $w_j = 0$ for all excitatory neurons, $w_j = 0.5$ for all inhibitive neurons. The rationale for this, in a discrete time context, is explained in [6] and [13]. In essence, by first choosing the constant μ sufficiently large and then combining inhibitive neurons with a positive threshold with excitatory neurons with threshold at zero creates conditions that are likely to lead to oscillatory and perhaps even chaotic behaviour of the net in the absence of external input. The experiments discussed in Section 4 confirm this. The constant μ is known as the gain, and

$$v_i = \mathrm{sgm}(\mu(u_i - w_i)) \tag{2.3}$$

is the firing rate of the ith neuron. By construction, the membrane potential takes values in $\mathbb{R}$, whereas the firing rate takes only values in (0,1).

We will also need a rule to initialize the connection strengths T_{ij}. Once a connection strength is determined, we could apply a learning rule to adjust it dynamically, but we chose to leave it fixed for the lifetime of the network so that effects of the growth process would not be confounded by subsequent learning effects.

The basic plan behind the development of the network can be encapsulated with a few simple rules.

1) Neurons are grown in layers.
2) Excitatory neurons form in locations of low activity.
3) Inhibitive neurons form in locations of high activity.
4) Input that changes in time modulates the activity.
5) Input is distributed on the x-axis, while layers are formed in the y direction.

Because of this design plan, the network is grown on an $M \times M$ grid in response to a series of inputs which are visualized as being applied from the bottom. The position of a neuron is therefore given by a pair of numbers (a, b), $a \in 1, \dots, M$, $b \in 1, \dots, M$. The horizontal coordinate a is cyclic in that the position $(a + kM, b)$ is identified with (a, b) for all $k \in \mathbb{Z}$. The vertical coordinate is grouped into a set of equally spaced disjoint intervals the size and number of which depends on the size of the grid, M, and the maximum number of neurons that could be placed. These vertical intervals are referred to as layers, levels or even generations. If we denote the width of a level as Δl and the width of the gap between levels as Δg then level $k+1$ is defined as the points (a, b) with

$$M - k(\Delta g + \Delta l) \le b < M - k(\Delta g + \Delta l) + \Delta l.$$

In the special case of level one, $k = 0$, we take $b = M$. One additional (arbitrary) constraint is that the bottom of the last layer must lie at $b = 2$. This implies that the number of layers is $1 + (M-2)/(\Delta g + \Delta l)$. For the numerical experiments we chose $M = 30$ and because of the maximum number of expected neurons, the vertical position is partitioned into five disjoint subsets each with a width of $\Delta l = 3$ grid points and separated by a gap of $\Delta g = 4$ grid points. This is illustrated in Figure 2.1. To start the net, we place N mother neurons equidistant from each other on level one. In the case of $N = 3$ these are placed at (5,30), (15,30), (25,30). The growth process is then driven by inputs located in the lowest layer. When the system is first defined, no neurons exist at this input layer. Therefore during the training process each neuron experiences a weighted average of the signal applied at the input layer according to the expression

$$I_i = \frac{1}{n_i}\sum_{j=1}^{n_i} f(x_j), \qquad x_j = x_i - \frac{n_i - 1}{2}, \ldots, x_i + \frac{n_i - 1}{2}.$$

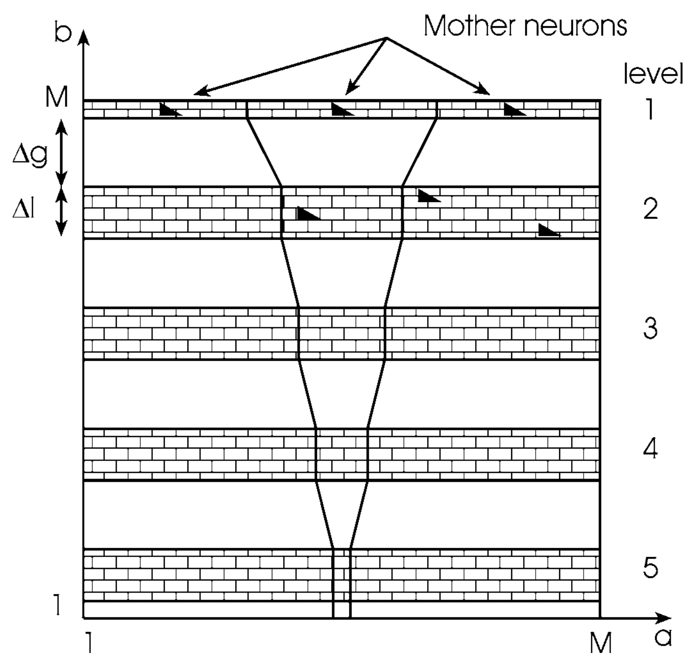

Figure 2.1. Shown here is the location of the various layers of the network used in the numerical experiments. It depicts the location of the mother neurons, the various levels and the location of the input layer. Also included is the vertical cone extending from an external input located at $(a, b) = (15, 1)$.

Here I_i is the external force that appears in equation (2.1) and is applied to neuron i by the input vector f. The number of points over which to average, n_i, is governed by the layer of the i-th neuron and in the case of our 30×30 array we chose $\{n_i\}_{i=1}^{5} = 11$, 7, 5, 3, 1 for layers 1 thru 5 respectively. Consequently, the inputs applied at the bottom layer are *felt* by all of the neurons that lie inside a vertical cone that emanates from the location where the input is applied. Once again, refer to Figure 2.1. After the training process, the input level for all of the layers is divided by a factor $5 - l + 1$, where l is the layer of the neuron under consideration. Ultimately, this means that the external input I to the mother neurons is suppressed by a factor of one over the maximum number of layers while the neurons at the bottom of the network, $l = 5$, feel the external input directly.

3 The Growth Process

3.1 The mother neurons

The mother neurons are really just ordinary neurons, but they have the ability to sprout new neurons, to which they will be connected via axonal connections, in response to their own activity. Specifically, the mother neurons can initially receive an input, i.e., be stimulated externally from the bottom, and if their activities rise above a predetermined threshold τ, they will sprout an axon with a child neuron. The mother neurons are by definition excitatory neurons, meaning that the axonal connection which connects them with children is excitatory. The child neurons can be either inhibitive or excitatory; the choice is made randomly, and the probability that a sprouted neuron is inhibitive is a free parameter in the program, denoted by ρ. We varied ρ from layer to layer, starting with $\rho = 0.2$ for the second layer and rising to $\rho = 0.6$ for the bottom layer.

If a child neuron is excitatory, it has again the ability to sprout new neurons in response to sufficient activity. Recall that the sprouting takes place when the *activity* is above the threshold; for a sufficiently large network, this can happen even when the neuron receives no external input. If the child is inhibitive, we do not allow it to sprout "grandchildren"; its only way to build connections is to connect back to the mother neuron or, later, to other neurons already in existence.

We require that the mother neurons can sprout child neurons, or connect via axonal connections to other neurons (e.g., grandchildren), only a finite number of times; the mechanism which we use to enforce this is to define an integer G which is the maximal possible number of generations that can sprout from the mother neurons. So that we generate a nontrivial network, we require that $G > 1$. The total number of axonal connections for any neuron is $G+1$ so that even if a mother neuron has sprouted G times, there is still one axonal connection available to connect to an already existing neuron. For the next generation, only $G-1$ of the total $G+1$ potential axonal connections are allowed to sprout, etc. This rule implies that every subsequent generation can have one fewer sprouted connection than the previous generation and is a convenient (and natural) way to limit the size of the net. Indeed for our 30×30 array we have five layers so that $G = 4$ and therefore using the above conditions, the network can have a maximum of 195 neurons. Since $\rho \neq 0$, we usually have inhibitive neurons and as a result the expected number of total neurons is less.

If g_i is the generation label of the i-th neuron, then g_i is an integer between 1 and $G+1$. By $a_i(t)$ we denote the number of remaining potential axonal connections of the i-th neuron at time t. If t_* is the moment of creation of this neuron, then $a_i(t_*) = G+1$ and the number of these potential connections that can sprout is $G - g_i + 1$.

The third characteristic number associated with each mother neuron is again an integer, which we denote by S, the maximal number of synaptic connections which the mother neuron (or any other neuron) can potentially accommodate. We choose $S = G$.

We summarize: In the beginning, we create N mother neurons which are characterized by three parameters, τ (threshold for sprouting or connecting), G, the number of admissible axons that can sprout from the mother, and S, the number of possible synaptic

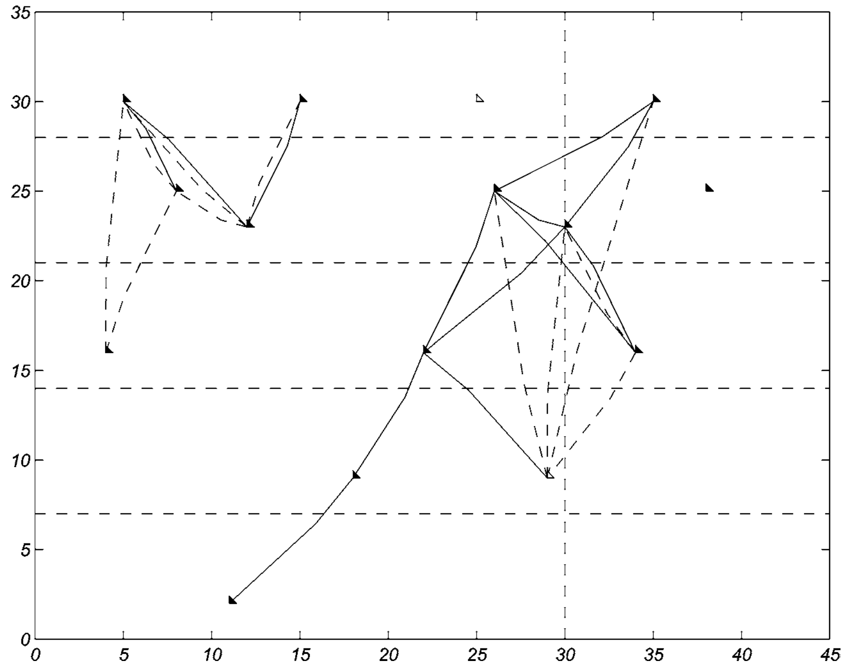

Figure 3.1. Illustrated above is a sample network showing a few mother neurons along with their children. When a new neuron is created in the i-th layer (generation $g_i = i$) it is assigned $a_i = G + 1$ potential axonal connections ($G + 1 - g_i$ can sprout) and S potential synaptic connections. As the network evolves both of these values can decrease. The horizontal dashed lines divide the net into the five levels while the vertical dashed line indicates, where the net wraps around so that the neuron at (35,30) is another image of the neuron at (5,30). Solid connections are excitatory; dashed connections are inhibitive.

connections. Both S and τ are the same for all neurons while the number of admissible axonal connections that can sprout depends on the layer in which a particular neuron finds itself. Figure 3.1 gives a simple network that has begun to evolve under these assumptions.

3.2 The next generations

The neurons of the next generation (the children of the mother neurons) are set to have essentially the same properties as the mother neurons: They have again room for S synaptic connections, and their threshold for sprouting or connecting is again τ. However, we allow them to sprout only $G - 1$ times, and the next generation will be allowed to sprout only $G - 2$ times, etc. This is a built-in mechanism to limit the size of the net, and it follows that the net can never grow to more than

$$N_G = N \sum_{k=0}^{G} \frac{G!}{(G-k)!} = N \sum_{k=0}^{G} \binom{G}{k} k!$$

neurons, where N is the number of mother neurons (which are also thought of as the information processing neurons). For the situation $N = 3$ mother neurons and $G = 4$, there is a total of $N_G = 195$ possible neurons.

Once a few neurons are in place, neurons which receive an input above the chosen threshold do not necessarily have to sprout a new neuron. We also give them the option to build a connection to an already existing neuron, e.g., to a mother neuron, a sister neuron, or any other neuron with room for synaptic connections. In fact, as already mentioned above for the first child of the mother neuron, inhibitive neurons have only this option.

We have so far only briefly discussed the geometric structure of the emerging net. In fact, it is conceivable to ignore this aspect completely and simply not assign the new neurons a spatial location at all; in such a model, every neuron will consider every other one as a candidate for a possible connection, and may make a random choice among these candidates. Numerical and graphical experiments with such a setup proved messy and showed that it was all but impossible to explore interesting correlations between the given input, the structure of the net and the output.

Therefore, we postulated that our neurons actually had to have a spatial location as described above. As a simple mechanism to record which sites in the array are occupied and by which type of neuron, we introduce two matrices. The first is an occupation matrix O_{ij} whose entries are 0 at vacant sites and are the labels of the neurons at occupied sites (neurons are labeled in the order in which they are created). The second matrix Z_{ij} encodes the neuron type and has entries 1, -1, or 0, depending on whether the site is occupied by an excitatory or inhibitive neuron, or vacant. These two matrices could be easily combined into a single matrix.

We further postulate that by their mere presence, the neurons create a potential field which determines the direction of new axonal connections. Such connections may terminate in new neurons (sprouting) or at neurons already present (connecting). Inhibitive neurons have only the second choice. The potential is a function of the state of the system at a given time. We define this state by the neurons which exist at this time and their properties. The neuron with label k at a location (site) $(x_k, y_k) \in \mathbb{Z}^2$ in level g_k of type Z_{x_k,y_k} has activity (membrane potential) $u_k(t)$, $s_k(t) \le S$ free synapses and $a_k(t) \le G + 1$ free axons. We sometimes use the shorthand $Z_k = Z_{x_k,y_k}$.

The state of the system at time t, when K neurons have been created, is then given by the set of coordinates $\{(x_i, y_i, g_i, s_i(t), a_i(t), Z_i);\ i = 1, \dots, K\}$. Notice that neither the membrane potential, nor the number of existing connections play a part in the definition of current state.

Modifications of our model could certainly include these other characteristics into the definition of state. However, the current potential depends only on the variables defined above. We tried various candidates for potentials; the one which seemed to lead to satisfactory patterns is the following:

$$V(x, y) = \sum_{i=1}^{N} \frac{S/2 - s_i + \gamma|G - g_i|}{1 + |x - x_i| + |y - y_i|}, \tag{3.1}$$

where γ is a free parameter. We found $\gamma = 1/2$ to work quite well for our numerical experiments. There are two effects that are modelled in the potential. The $S/2 - s_i$ term has the effect that a neuron at site (x_i, y_i) will make a negative contribution to the potential while it has many free synapses; as its number of free synapses, s_i, decreases, this potential will gradually increase and eventually become positive. The $|G - g_i|$ term makes neurons which are vertically far away more attractive. We will see below that this tends to create longer back connections.

If our potential were a potential in the usual physical sense, forces between various neurons would act in the direction of the gradient of the potential. We loosely follow this idea in using the potential to give directions for sprouting or connecting. To do this, we need one more ingredient, namely the idea that neurons will look for sites to sprout to, or connection partners, in a neighborhood rather than far away. We implicitly work under the assumption that there is a natural scale of length for the typical connection, i.e., that the axonal connection cannot exceed a certain length. This length defines a "window" around each neuron already in place. The size and shape of the window depends on both the generation number and the neuron-type. Inhibitive neurons are permitted to connect further, so their windows are larger than those associated with excitatory neurons. For a given neuron at position (i_0, j_0) the excitatory and inhibitive windows about this point are given by

$$W^E(i_0, j_0) = \{(i,j) \mid |i - i_0| \le w_g^E,\ -h_g^E \le j - j_0 \le h_g^E - |i - i_0|\},$$
$$W^I(i_0, j_0) = \{(i,j) \mid |i - i_0| \le w_g^I,\ -h_g^I + |i - i_0| \le j - j_0 \le h_g^I\},$$

where the width and height of the excitatory/inhibitive window, $w_g^{E,I}$ and $h_g^{E,I}$ depend on the generation g and are listed in the following table.

Generation g	w_g^E	h_g^E	w_g^I	h_g^I
1	10	7	10	2
2	8	9	8	7
3	8	9	6	14
4	8	9	6	21
5	8	3	6	28

Typical windows for each of the levels are depicted in Figure 3.2. Notice that the excitatory window is diamond shaped at the top and rectangular at the bottom while the inhibitive window is inverted. The excitatory window is chosen so that a neuron can connect no further than adjacent levels (and only sprout into the next level down) while the inhibitive window allows connections back to the mother neurons from any generation.

If the neuron at site (x_j, y_j) is inhibitive, it can connect to another neuron inside its window. If it is excitatory, it can connect to another neuron in its (smaller) window, or it can sprout a new neuron at an empty site inside this window. The details of these sprouting and connecting routines are described in what follows.

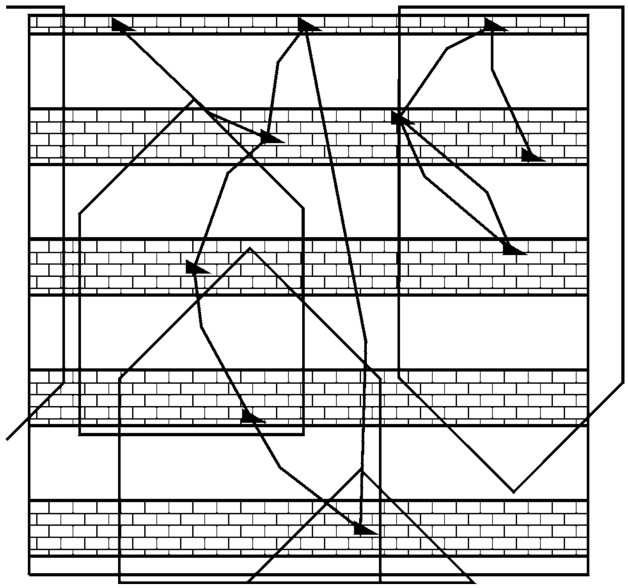

Figure 3.2. Depicted above are examples of the various excitatory and inhibitive windows or target sets. Specifically shown are excitatory target sets for levels three, four and five and an inhibitive window for level three. The characteristics of the window depend on the type and level of the neuron on which it is based.

3.3 The sprouting and connecting routines

These procedures are outlined here in detail, but they are also the content of the flow chart in Figure 3.3. For the purposes of this section, assume that we have arrived at a certain net after a number of steps, and that the activity of some of the existing neurons, either by external or internal stimulation, is above the threshold. We then loop through all the neurons and make the following tests and choices.

A. For the i-th neuron, check whether the activity is above the threshold. If no, go to the next neuron, if yes, check whether the neuron is excitatory or inhibitive. In the first case, go to B. In the second case, go to C.

B. At this point we know that the i-th neuron can either connect or sprout. Therefore define the size of the window, $W^E(x_i, y_i)$, in which to search for a suitable site. The size and shape of the window is based on the neuron type and location as described in Section 3.2. There are now three subcases.
   i) If this neuron has no axons left, $a_i(t) = 0$ then go to the next neuron.
   ii) If this neuron only has one axon left, $a_i(t) = 1$ then this neuron can only connect. A neuron cannot sprout if it only has one axon left. Go to step C.
   iii) If $a_i(t) > 1$ and the neuron is excitatory then the neuron can either sprout a new neuron or connect. Go to step D.

C. We get to this step if we are only allowed to connect which always happens for inhibitive neurons. If the i-th neuron is inhibitive, it looks at those neurons with at least one free synapse inside its target set, $W^I(x_i, y_i)$, and chooses the location that has the *maximal* potential. Alternatively, if the i-th neuron is excitatory, the target set becomes $W^E(x_i, y_i)$ and instead of maximizing the potential, it

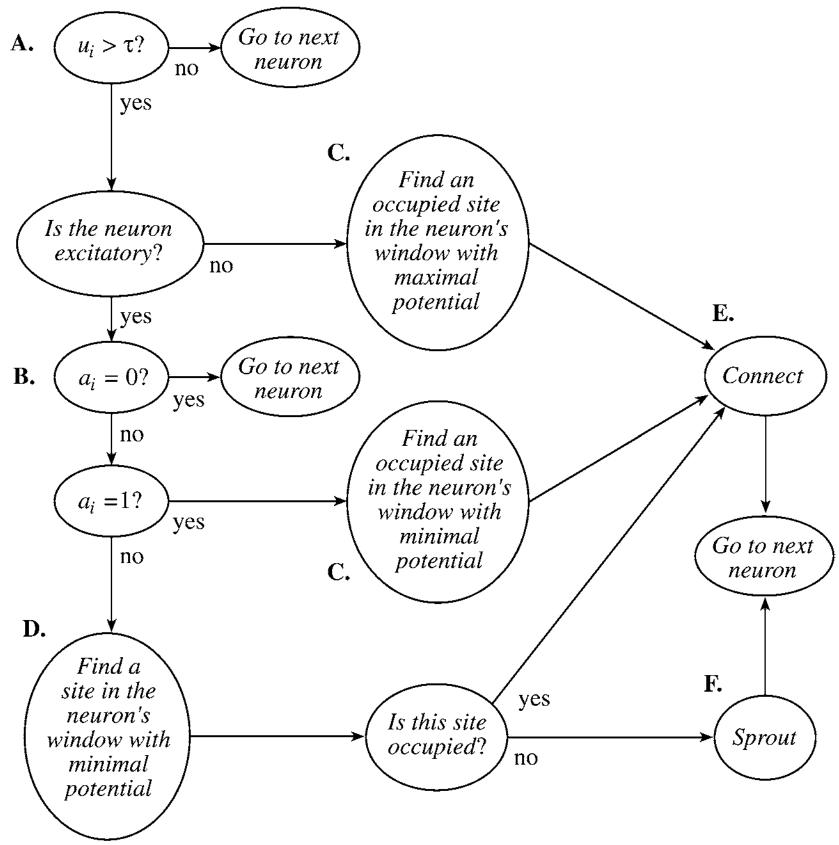

Figure 3.3. Shown here is the algorithm for the decision to sprout or connect. This is also outlined in Section 3.3.

looks for the location where the potential is *minimal*. In this search, a number of neurons are disregarded according to the following criteria:

i) We cannot already be connected to this neuron.
ii) The location to which we want to connect must have synapses left.
iii) We cannot connect to ourselves.
iv) If we are on the bottom layer we cannot connect to the bottom layer.

From the way that our potential was defined, the inhibitive neurons will typically be targeted towards neurons that already have many synaptic connections and are therefore likely to be highly active. If there are several sites which have the same maximal potential and are occupied by connection candidates, the neuron chooses one randomly for connection. Denote the suitable candidate as the j-th neuron. If no suitable neuron is found then return to step A otherwise continue to step E.

D. We get here if we can either connect or sprout. In addition, we know that the i-th neuron is excitatory. In this case we are looking for a minimum of the potential inside the target set $W^E(x_i, y_i)$. If the site under consideration is vacant we can sprout a new neuron. In this case there are two extra criteria:

i) The i-th neuron must be allowed to sprout. Recall that a neuron on level l can only sprout $G - l + 1$ times.

ii) If the position under consideration is above the current neuron's position, $y_j > y_i$, then the position is not eligible since we cannot sprout upwards.

If the site is occupied then the i-th neuron must connect. In this case we have the four criteria already listed in item C.

If there are several sites which have the same minimal potential, the neuron chooses one randomly for connection. Denote the suitable candidate as the j-th neuron. If the i-th neuron sprouts go to step F. If it connects, go to step E and if no viable site is found go back to step A.

E. Connect the i-th neuron to the j-th neuron. Section 3.4 will detail how the connection strength is determined. In the process, the number of available axonal connections for the i-th (sending) neuron is reduced by one, $a_i \to a_i - 1$, and the number of available synapses of the j-th (receiving) neuron is reduced by one, $s_j \to s_j - 1$.

F. Sprout a new neuron to position (x_j, y_j). Section 3.4 will describe how the connection weight is determined. As well, the number of free axonal connections of the i-th neuron is reduced by one, $a_i \to a_i - 1$. For the new neuron, we assign a generation number that is one less than its parent. The choice whether it be excitatory or inhibitive is made randomly, with a probability determined by the parameter ρ described in Section 3.1.

Notice that the sprouting process will automatically come to an end with the last generation, and the structure of the net will have been created by its own activity, and hence, to some degree, by the inputs which were active during the process.

3.4 The connection weights

We used a connectivity function which creates a connection weight between the i-th (sending) and j-th (receiving) neuron as

$$C(i,j) = \lambda Z_i \ \mathrm{sgm}(\mu u_i) \ \mathrm{sgm}(\mu u_j), \tag{3.2}$$

where Z_i is the neuron type of the sending neuron (-1 for inhibitive and $+1$ for excitatory neurons). In terms of the firing rates v_i (2.3),

$$C(i,j) = \lambda Z_i v_i v_j$$

and we set the connection matrix entry

$$T_{ij} = C(i,j). \tag{3.3}$$

3.5 The dynamics of the system

The dynamics of the system happen on two different time scales: A fast one for the evolution of the individual neurons' membrane potentials, given by equations (2.1), and a slow one for alterations of the net, involving the growth process, the creation of new connections and the setting of new connection strengths in accordance with (3.2) – (3.3). Once defined, connection strengths are *not* allowed to update. In practice, we solved equations (2.1) by the Matlab routine for two time units, and then stopped to update the system according to the algorithm described in Section 3.3. Any connections created at this time get the connection strength defined by (3.3). We emphasize that the already existing connectivities are not updated. Rather, they remain fixed throughout the existence of the network. It would be possible to allow synaptic plasticity via a generalized Hebb rule where already existing connectivities are updated according to

$$T_{ij}^{\text{new}} = (1-\epsilon)T_{ij}^{\text{old}} + \epsilon C(i,j)$$

with ϵ is a free parameter which determines how rapidly the connection strengths change. We did not do this as we wished to separate the effects of network growth in response to inputs from those of subsequent synaptic change. However, experiments with the learning rule included led to very similar results.

3.6 Input application

Input signals are applied directly to the lowest level of the net and felt at the upper layers through the averaging process described in Section 2. The following describes the implementation that we used. We emphasize that this is only one of many possible choices. Our first input is zero everywhere except at the grid position (5,1) (under the first mother neuron) where it has a nonzero value u_0. In our case we chose the value $u_0 = 11$. The input remains at this position for eight time units which is four of the long time cycles. This gives the network a chance to grow and alter itself in response to this input. After four long cycles the input is moved to the position (15,1) under the second mother neuron. It remains here for four cycles then moves to (25,1) completing one pass of the training cycle. This process is continued until no further growth in the net is observed. The goal of this process is to train the network to classify clearly these three input locations; in addition, the emerging networks display characteristic responses to many other inputs.

3.7 Input-output representation, graphical depiction of the net and of the potential

This is done in three windows brought up by Matlab. In one of these windows, the network window, the neurons are visible at grid-points, and their connections are displayed as piecewise straight lines (solid green for excitatory connections, dashed red for inhibitive connections). The second window, the potential window, shows the potential as a colour map. The only purpose of this map is to indicate in which direction the net will grow at a given time. The third window, the activity window, displays the activities of the mother neurons (in our case three) as a function of time; what we are looking for in this window is characteristic responses of the top layer neurons to classes of inputs.

Almost all the relevant information is contained in the network window, where the whole net and its activity is shown. In this window neurons are represented as triangles and we found it most convenient to display the connections as piecewise straight lines with one “kink” that has a rational and irrational component. This serves two purposes. The direction of the connection is from the side of the shorter segment to the longer segment. In addition, because the location of the kink has both a rational and irrational component, it is unlikely that the graphical representations of two connections will fall on top of each other.

4 Results and Discussion

We conducted extensive tests with the program and made the following observations.

4.1 The effect of ρ

In most cases, the net will grow rapidly if there is sufficient initial stimulation. The input leads to high activity of the mother neurons, to sprouting, to high activity of the next generation, which sprouts or reconnects, and so on. The process can end prematurely if too many inhibitive neurons are created early in the game. These must connect to other neurons, whose activity then gets inhibited to the point where they may not sprout new neurons anymore. High stimulation of such neurons can lead to new growth.

Typically, however, when the number of inhibitors is kept low, once a few neurons are in place, growth continues even when input is turned off. The chosen growth rule, which directs new excitatory (inhibitive) neurons to locations of lowest (highest) potential, appears to lead naturally to the formation of subnets (clusters). These clusters will usually form inhibitive connections to other clusters and back to the upper layers. The reason is the longer range of inhibitive connections. It follows that the inhibitive neurons in the lower layers are the most sensitive neurons to inputs.

We found that interesting nets, in the sense of having characteristics oscillations, were more likely to emerge if the fraction of inhibitive neurons increases from layer to layer. To this end, the parameter ρ, which is the probability that a sprouted neuron be inhibitive, was set at 0.2 for the second layer, at 0.3 and 0.4 for the next two layers, and 0.6 for the last layer which is not allowed to sprout at all, merely connect.

This description indicates that ρ is one of the most important parameters. If there is too little inhibition, such mostly excitatory nets tend to freeze in high activity states, whereas too much inhibition leads to a freeze in low activity states. Such nets still respond to input, but in a fairly trivial way-neurons directly affected by input, or directly connected to a neuron receiving input, show a response, but the rest of the net seems rather unaffected. It follows that an appropriate ratio between inhibitive and excitatory neurons is important.

4.2 The effect of the gain

The other crucial parameter appears to be the gain μ. This is not very surprising, as the gain is well-known to influence the behaviour of a neural network in sensitive ways. For low values of μ, the net tends to grow towards one with nonoscillatory stable rest states; inputs can force the net from one such steady state to another one, but it is questionable whether this is really an optimal way of information processing. First, it is not consistent with biological observations, where chaotic activity is seen in rest states ([15]) and periodic attractors seem the typical responses to inputs ([3]). Second, it may well be that chaotic rest state activity increases the information capacity of a network in ways which we do not (yet) understand.

Be that as it may, for larger values of μ, $(2 \leq \mu \leq 10)$ and for the choices of ρ described earlier we observed that many of our emerging nets showed characteristic *oscillatory* responses towards specific inputs; i.e., if for a grown net an input is applied at one of the 30 locations at the bottom, one, two or even all three of the mother neurons will display oscillations with characteristic amplitudes and frequencies. For most inputs, only one or two of the mother neurons will oscillate, while the others will show steady activity. This observation is satisfying inasmuch as it is consistent with biological observations;

considering oscillations as acceptable output also increases the differentiating capacity of the net dramatically, the main advantage being a vastly larger set of possible attractors, with the ability to respond characteristically to many different types of input.

4.3 What is cognitive behaviour anyway?

We have so far dodged this question, in spite of the fact that it is of central importance for our project. We admit from the outset that we have no clear definition of cognitive behaviour; we let ourselves be guided by what is seen in nature.

There are at least two different types of information-processing ability which are observed in natural neural networks. The first kind is a rather straightforward method of image-projection, in which signals are received by receptors (e.g., the retina cells in the back of the eye) and forwarded via neural connections towards a subunit of the brain, where the observed pattern is then reproduced with some accuracy. For example, this is how the visual cortex operates.

We refer to this type of information processing as trivial cognitive behaviour, for two reasons: a) it is immediate how to connect input and output neurons for the objective, namely, by high-fidelity bundles of connections which transfer an input essentially unmodulated to the output layer, and b) there is no attempt or potential for interpretation; the process is simply one of data transmission. Clearly, to produce this kind of information processing, we could have done no growing at all in our net: The three input neurons will certainly respond with different steady levels of activity to various inputs, simply by the established rule that the applied inputs are to be felt initially (with some averaging) by the mother neurons. This process will certainly define a (trivial) input-output relationship like the mentioned example of the visual system.

The second, non-trivial kind of information processing is harder to define, and we follow natural observation for a tentative definition. In biological observations, a typical reaction to a stimulus in a subunit of a neural net is a transformation from a rest state which may be noisy or chaotic ([15]) to oscillations with characteristic frequency and amplitude. It is this type of transformation which must be interpreted as a transition from a rest state to a cognitive state. Presumably, information processing via oscillatory states defines a much larger volume of all “cognitive” states than just steady states of neuron arrays ([3]), thus enhancing cognitive capacity; besides, oscillatory states may be easy to attain if certain parameter ranges and connectivity types are permitted, thus setting a natural path for evolution.

Our experiments indicate that this is a feasible path; as the results from the next section show, the mother neurons tend to respond with characteristic oscillations for most of the stimuli one can apply at the input layer; the option of a steady output is still there, but it is supplemented by a multitude of characteristic oscillations with varying frequencies and amplitudes for each neuron. The fact that this happens for an apparently open set of choices in parameter space led to the term “generic” in the title of this paper.

4.4 Results

We describe one particular realization of our network-building algorithm. Applying input to the three mother neurons in the sequence described in Section 3.6 resulted in a network that stabilized after $t = 64$ time intervals or 32 iterations with a total of 27 neurons. Snapshots of this growth process are depicted in Figure 4.1. A summary of the parameters for each of these 27 neurons at the end of this process is listed in Table 4.1.

i	(x_i, y_i)	l_i	a_i	s_i	r_i	Z_i	i	(x_i, y_i)	l_i	a_i	s_i	r_i	Z_i
1	5, 30	1	1	0	4/4	1	15	3, 24	2	0	0	1/3	1
2	15, 30	1	2	0	1/4	1	16	13, 16	3	0	0	-	-1
3	25, 30	1	1	0	4/4	1	17	23, 23	2	2	0	3/3	1
4	30, 23	2	2	0	2/3	1	18	28, 16	3	0	1	-	-1
5	12, 23	2	1	0	-	-1	19	17, 17	3	0	0	2/2	1
6	22, 16	3	0	0	2/2	1	20	1, 18	3	2	0	0/2	1
7	8, 25	2	4	0	-	-1	21	9, 9	4	0	0	-	-1
8	4, 16	3	0	0	-	-1	22	24, 9	4	2	1	1/1	1
9	18, 9	4	0	0	1/1	1	23	2, 2	5	0	2	-	-1
10	26, 25	2	1	0	0/3	1	24	9, 18	3	2	0	2/2	1
11	29, 9	4	0	0	-	-1	25	14, 10	4	2	1	-	-1
12	11, 2	5	1	0	0/0	1	26	5, 9	4	2	3	-	-1
13	19, 23	2	0	0	-	-1	27	21, 25	2	5	3	-	-1
14	16, 25	2	0	0	1/3	1							

Table 4.1. This table lists the parameters for all of the 27 neurons in our particular realization. Recall that a_i and s_i denote the number of unused axonal and synaptic connections for the i-th neuron respectively. The values in the r_i column are of the form p/q where p is the number of sprouts actually made out of a total possible of q. For example neuron 14 is located in level $l_i = 2$ at position (16,25). It is an excitatory neuron that has used all of its synapses, sprouted once and made four other connections since $a_{14} = 0$.

From the table one can see that this example network is fairly saturated in that there are very few synapses left for a neuron to connect to and very few axons to connect to these synapses. In addition, there are 15 excitatory and 12 inhibitive neurons. With few exceptions it is the inhibitive neurons that have a significant number of potential axonal connections remaining once the network stops growing. Recall that inhibitive neurons are not allowed to sprout.

The last image in Figure 4.1 shows the fully grown network. Two structures that seem to be essential to the development of oscillations are that all the mother neurons have at least one inhibitive connection and that there are various inhibitive neurons that have fan-like connections into the levels above them. Four of these latter structures extend from neurons 8, 16 and 18 in level 3 and neuron 11 in level 4. While similar in appearance, neuron 12 in level 5 is an excitatory neuron that has 3 inhibitive synaptic connections.

Once the network is fully grown, the time evolution of the membrane potential of the three mother neurons is observed when a delta function of strength $u_0 = 11$ is applied

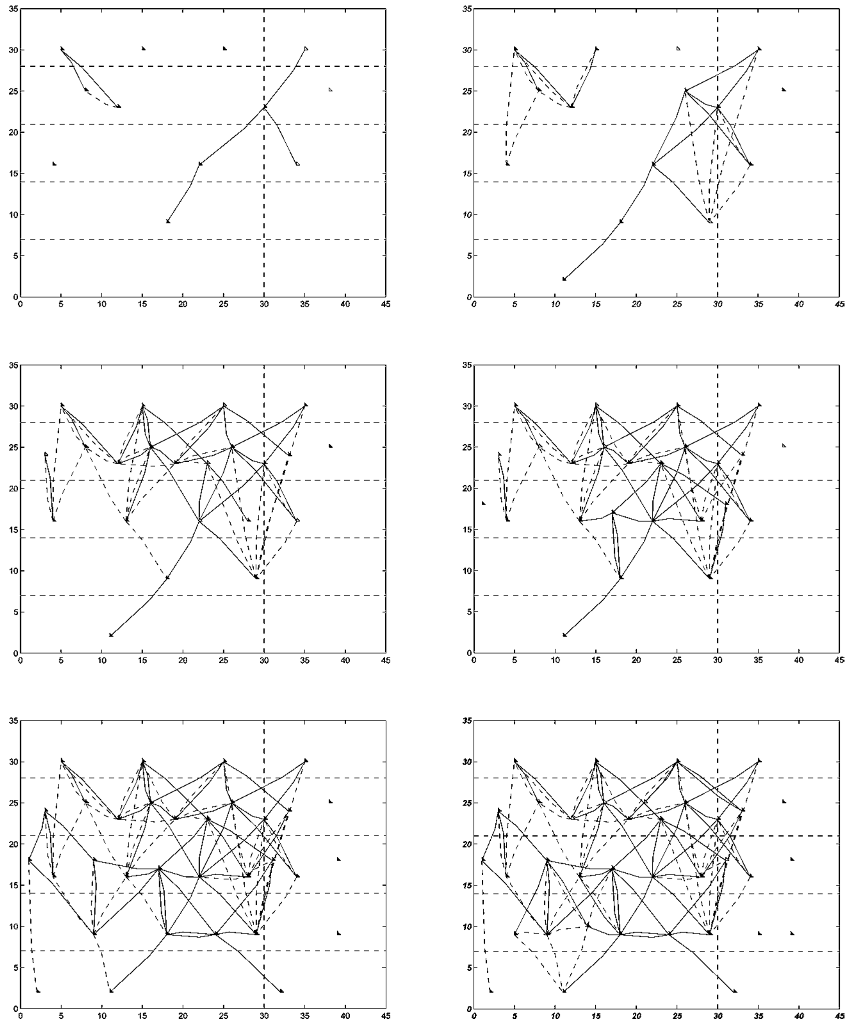

Figure 4.1. Shown here is the evolution of the network. The six figures correspond to times $t = 6$, 16, 40, 46, 56 and 64. Each iteration of the growth process takes up two time units. After $t = 64$ the network experiences no further growth.

to the bottom of the network. For a given input location the set of neurons that directly experience the input lie inside a cone as described in Section 2. Neurons outside the cone can also receive input but only through their connections. Figure 4.2 shows the membrane potentials of the three mothers with no input applied. The network tends to return to this state whenever the input application ceases.

We refer to Figure 4.3 for typical responses of the mother neurons upon stimulation at the input locations. We observed that there are large basins of attraction that extend from input locations $a = 1$ to $a = 7$ and from $a = 20$ to $a = 30$. When input is applied from $a = 8$ to $a = 19$ the network responds in a complicated fashion so that the three training locations have been effectively encoded. We emphasize that this without a Hebbian learning rule.

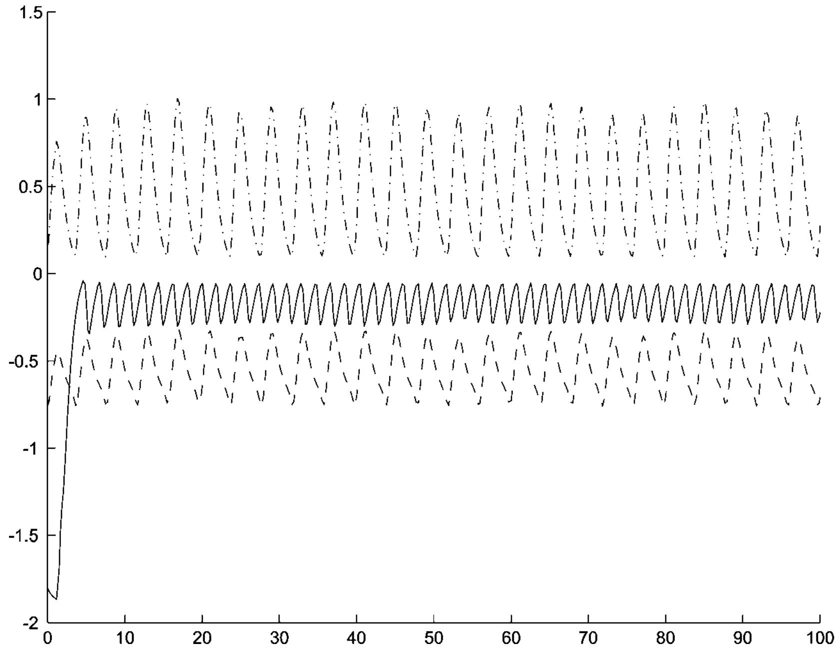

Figure 4.2. Illustrated is the response of the three mother neurons as a function of time when no input is applied to the network. The membrane potential of the neuron at position (5,30) is the solid line. The second and third neurons are at (15,30) and (25,30) and their membrane potentials are the dashed and dashed-dot lines respectively. Without any input, the activities of these two neurons oscillate in phase with each other.

By having various oscillatory responses the dimension of output space is increased. One can think of time as the extra dimension, or the frequency of each oscillation; an extra dimension is added for each output neuron. This has the clear advantage that the network grows "smarter" without adding many more neurons; the extra "intelligence" is implicitly encoded in the network structure. The main point of our work lies in the fact that this type of enhanced network performance arises generically for a range of growth rules, and it is therefore reasonable that the type of oscillatory behaviour we observe is commonplace in nature.

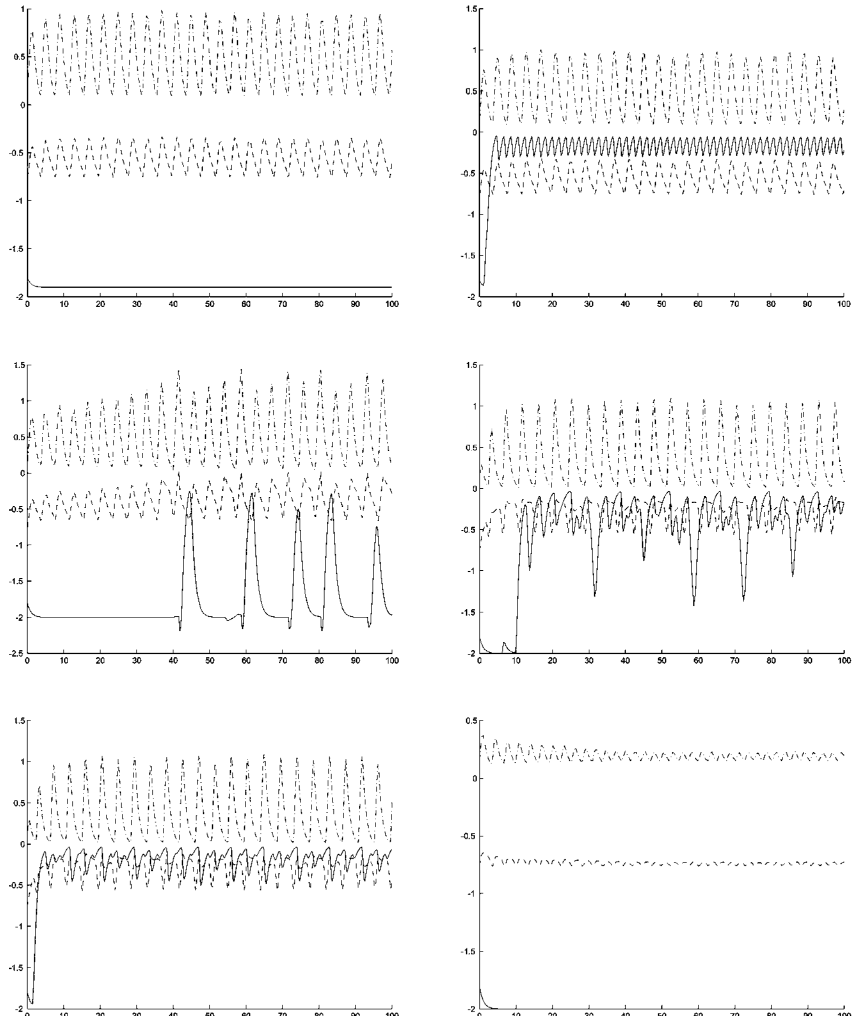

Figure 4.3. The above six figures show the response of the three mother neurons to an input of strength $u_0 = 11$ applied at locations $a = 7$, 9, 11, 13, 15 and 20. Applying input to any of the locations $a \in [1, 7]$ gives the same response as the first figure and is the same response as the rest state. As well, input applied to any $a \in [20, 30]$ results in the same response as the last figure.

5 Conclusions

We demonstrated that there are simple growth rules for a multi-layer neural network consisting of excitatory and inhibitive neurons such that the finished network will display characteristic oscillatory responses corresponding to classes of inputs. This behaviour is consistent with observed biological behaviour. We conclude that the emergence of oscillatory responses is generic in the sense that it will occur for growth rules and parameter ranges that are sufficiently general to be accessible to an evolutionary process. While our model does not directly simulate the development of biological neural systems, it demonstrates the feasibility of such a process in an idealized context and the ability to grow networks in 'sensible' ways.

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Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 65–74

Design of Stable Controllers for Takagi-Sugeno Systems with Concentric Characteristic Regions

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Received: May 3, 2002; Revised: December 8, 2002

Abstract: The design of a fuzzy Takagi-Sugeno system with concentric regions and the use of discontinuous piecewise Lyapunov functions allows to relax stability conditions which can be expressed very easily as a set of Linear Matrix Inequalities. An adaptive algorithm allows to determine gradually the embedded sets and the corresponding local models.

Keywords: *Fuzzy control; linear matrix inequalities; Lyapunov functions; spherical coordinates.*

Mathematics Subject Classification (2000): 93D05, 93D15.

1 Introduction

The Takagi-Sugeno (TS) fuzzy model allows to represent a wide class of non-linear systems by a set of fuzzy rules for which the consequent parts are linear state models [10]. Using aggregation of rules, which induce a polyhedral partition of the state-space, a weighted sum of the linear state models is able to describe accurately the non-linear system. The so-called parallel distributed compensation (PDC) technique is an intuitive algorithm which consists of designing a fuzzy control rule according to each model rule of a TS fuzzy system. The premise part of the model rule and its corresponding control rule are identical. A sufficient condition to ensure the stability of a TS fuzzy plant model controlled with the corresponding PDC is to find a common quadratic Lyapunov function for all subsystems [11, 12]. The search of the Lyapunov function can be viewed as a convex optimization problem in terms of linear matrix inequalities (LMI) for which efficient solvers exist [1, 4]. The main drawback of this method is the conservativeness of the results which grows with the number of subsystems which must be taken into account.

The use of multiple (and in particular piecewise quadratic) Lyapunov functions is an alternative method to prove the stability of TS fuzzy controllers [6 – 9]. The quadratic

Lyapunov functions can be designed to ensure the continuity of the overall Lyapunov function at the boundaries of the cells which map the state space; the condition requiring the continuity of the Lyapunov function can be relaxed if the energy decreases when the trajectory moves from a cell into another [6,7]. Another related method is to drive gradually the state space trajectory through a series of embedded sets, where an attractor of a set is included into the next set of the series [2,3]. This algorithm reproduces the intuitive characteristic of fuzzy control for which the trajectory is smoothly driven from one region into another closer region (in terms of distance) of the origin, until it reaches the equilibrium. The attractors may be computed using comparison systems methods and vector norms, which leads, however, to conservative results [2,3].

A TS fuzzy structure which uses generalized spherical coordinates in the premise part is proposed in this paper, for which some characteristic regions can be put more easily under the form of quadratic inequalities than the general polyhedral scheme. The design of discontinuous Lyapunov functions together with appropriate embedded sets will allow to derive relaxed stability conditions for a TS fuzzy system controlled by PDC techniques.

2 Design of Takagi-Sugeno Systems with Ellipsoidal Domains

A. Takagi-Sugeno systems with generalized spherical coordinates

1) The basic model

Let us consider the fuzzy dynamic model of the Takagi and Sugeno system described by the following IF-THEN rules $R_i,\ i = 1,\dots,r$:

$$\text{IF } z_1 \text{ is } M_{i,1} \text{ AND } \dots z_n \text{ is } M_{i,n} \text{ THEN } \dot{x} = A_i x + B_i u,$$

where $x = (x_1,\dots,x_n)^T$ is the state vector, $u \in \mathbb{R}$ is the control vector, $z = (z_1,\dots,z_n)^T$ are the premise variables and $M_{i,j}(\cdot)$ are the membership functions of the fuzzy sets $M_{i,j}$. We suppose that $\operatorname{card}(z) = \operatorname{card}(x) = n$. The state equation can be defined as follows [10]:

$$\dot{x} = \sum_{i=1}^{r} \lambda_i (A_i x + B_i u),$$

where $\lambda_i = \dfrac{\omega_i(z)}{\sum\limits_{j=1}^{r} \omega_j(z)}$ with $\omega_i(z) = \prod\limits_{j=1}^{n} M_{i,j}(z_j)$.

Let us introduce a basis of n-dimensional generalized coordinates which consists of one radius and $n-1$ angles,

$$z = (\rho, \theta_1, \dots, \theta_{n-1})^T \in \mathbb{R}^n,$$

where $\rho = \sqrt{\sum\limits_{i=1}^{n} \left(\dfrac{x_i}{\alpha_i}\right)^2}$, $\alpha_i \in \mathbb{R}$. In the case where $\alpha_i^2 = 1$ for all $i = 1,\dots,n$, $z = (\rho, \theta_1, \dots, \theta_{n-1})^T$ will correspond to the generalized spherical coordinates basis; if moreover, the dimension is 2, $z = (\rho, \theta)$ will reduce to polar coordinates, where ρ and θ are respectively the radial and the angular coordinate.

The Takagi-Sugeno system using $z = (\rho, \theta_1, \dots, \theta_{n-1})^T$ as variable for premises is described by the set of rules:

$$R_i\text{: IF } \rho \text{ is } \rho_i \text{ AND } \theta_i \text{ is } \Theta_{i,1} \text{ AND ... } \theta_{n-1} \text{ is } \Theta_{i,n-1} \text{ THEN } \dot{x} = A_i x + B_i u. \tag{1}$$

2) The overlapping condition

In most fuzzy control applications, the input membership functions $M_{i,j}(\cdot)$ and $M_{i+1,j}$ of every variable z_j overlap pairwise in an interval $\left]\hat{z}_{k_j,j}, \hat{z}_{k_j+1,j}\right[$, where the other membership functions are zero. Consider the region $\Delta_k = \bigcup_{j=1}^{n} \left]\hat{z}_{k_j,j}, \hat{z}_{k_j+1,j}\right[,\ k = 1, \dots, K$, where $1 \le k_j \le n,\ n_j$ is the number of predicates for the variable $z_j,\ K$ is the number of possible regions. Only a limited number of rules are activated in Δ_k since, for every premise z_j, only the membership functions $M_{k_j,j}$ and $M_{k_j+1,j}$ are nonzero, the rules which involve other fuzzy sets fire.

In the case where the TS system is described by equation (1), the regions $\Delta_k,\ k = 1, \dots, K$, can be represented by the following inequalities:

$$\rho_k \le \rho \le \rho_{k+1}, \quad \text{or} \quad \rho_k \le x^T P x \le \rho_{k+1}, \tag{2}$$

where $P = \operatorname{diag}\left(\dfrac{1}{\alpha_i^2}\right)_{i=1,\dots,n}$, and

$$0 \le \Psi_k \theta_k, \tag{3}$$

where Ψ_k is a constant vector.

The set of regions where $\rho_m \le \rho \le \rho_{m+1}$ will be called $\Omega_m,\ m = 1, \dots, M$. A region which encloses the origin belongs to the set Ω_1, for which $\rho_m = 0$. From the preceding hypotheses, rules which are active in Ω_m are also active either in Ω_{m-1} or in Ω_{m+1}, and are not active elsewhere. Note that rules which are active in Ω_1 are also active in Ω_2.

In the rest of the paper, these conditions will be referred to as the “overlapping conditions”.

B. Design of a control structure

Two kind of controllers will be examined:

– the simple linear state feedback control with regionwise valued parameters:

$$u = F_k x \quad \text{if} \quad x \in \text{region}\, \Delta_k; \tag{4}$$

– the Parallel Distributed Compensation controller, the most popular and natural control for TS systems, which consists of designing each control rule from the corresponding rule of a TS system, with which it shares its premise parts. In a PDC, a rule R_i of the TS system to be controlled [11, 12] corresponds to a dual regulator rule $\hat{R}_i$:

$$\hat{R}_i\text{: IF } \rho \text{ is } \rho_i \text{ AND } \theta_i \text{ is } \Theta_{i,1} \text{ AND ... } \theta_{n-1} \text{ is } \Theta_{i,n-1} \text{ THEN } u = F_i x. \tag{5}$$

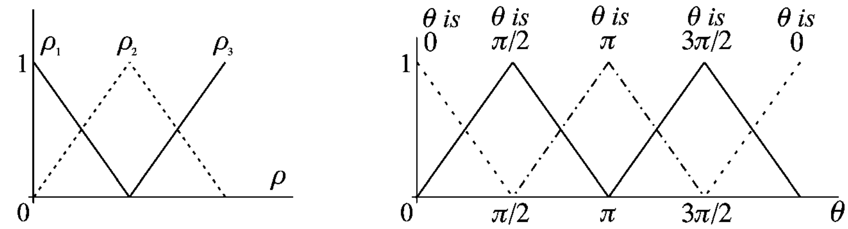

Figure 2.1. Membership functions for generalized coordinates.

C. A 2-D example

Consider a system described by the following set of rules:

$$\text{IF } \rho \text{ is } \rho_i \text{ AND } \theta \text{ is } \Theta_i \text{ THEN } \dot{x} = A_i x + B_i u,$$

where $x = (x_1, x_2)^T$ is the state vector, u is the control vector, $X_1 = x_1/a$, $X_2 = x_2/b$, $X = (X_1, X_2)^T$, $a, b \in \mathbb{R}$. ρ and θ are polar coordinates in the plane $X = (X_1, X_2)^T$, $\rho = \|X\|_2$, $\theta = \arg(X)$.

The following triangular membership functions are given on Figure 2.1.

Fuzzy sets for θ are "θ is 0", "θ is $\pi/2$", "θ is π", "θ is $3\pi/2$". Figure 2.2 shows the regions where rules are active. For example, in region Δ_1 of Figure 2.2, only rules

"ρ is ρ_1 AND θ is 0", "ρ is ρ_1 AND θ is $\pi/2$",

"ρ is ρ_2 AND θ is 0", "ρ is ρ_2 AND θ is $\pi/2$"

are activated, the remaining rules fire. Region Δ_1 can be described by the following constraints: $x_1 \geq 0$, $x_2 \geq 0$ and $x^T P x \leq c$, where $P = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}$ and $c = 1$. The regions Δ_k are a sector of a cone (when enclosing the origin) or of an annulus, for which only 4 rules are active, the other ones fire. The main differences with respect to classical TS-fuzzy controllers are now clear: the state space partition is not polyhedral, but the local models are distributed following the distance and orientation with respect to the origin, in the state-space. The notion of distance (from the equilibrium) respects the intuitive nature of fuzzy predicates such as "FAR" or "NEAR", and some of the constraints on regions where parameters are constant can be expressed as quadratic inequalities, which shows quite useful in Lyapunov stability techniques. In the general case $(n > 2)$, the characteristic regions of the TS system can be chosen as ellipsoidal.

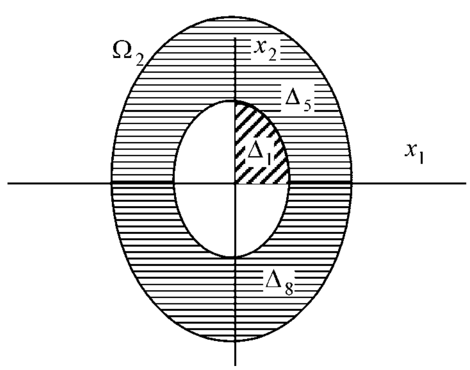

Figure 2.2. Example of regions where a limited number of rules are active.

3 LMI-based Stability Analysis of the Fuzzy Controller

A. Stability theorems based on multiple Lyapunov functions

Suppose that the original system is described by equation (1) – the premise variables are not necessarily spherical coordinates – and verifies the overlapping condition. It will thus be possible to find N disjoint regions Υ_m for which a scalar energy function V_m can be defined. Let the switching boundary Λ_{ml} for which the trajectory $x(t)$ passes from some neighboring regions Υ_m to Υ_l, i.e.

$$\Lambda_{ml} = \big\{x \backslash x(t^-) \in \Upsilon_m,\ x(t) \in \Upsilon_l\big\}.$$

Theorem 3.1 [6] *Suppose that there exist class K functions α and β such that, for all $m, l = 1, \dots, M$,*

(i) $\alpha(\|x\|) \le V_m(x) \le \beta(\|x\|)$ *for all* $x \in \Upsilon_m$,
(ii) $\dot V_m(x) \le 0$ *for all* $x \in \Upsilon_m$,
(iii) $V_l(x) \le V_m(x)$ *for all* $x \in \Lambda_{ml}$,

then the origin is (uniformly) stable in the sense of Lyapunov.

Theorem 3.1 allows to relax the continuity condition for the Lyapunov function, and a companion theorem exists for exponential stability [7]. A corollary has been given in [6] for quadratic Lyapunov functions. We propose a simplified criterion using the special structure given in (1), which will allow the control problems to be expressed as a simple set of LMIs.

Theorem 3.2 *Consider a regionwise valued fuzzy system defined in (1). If there exists a series of positive definite matrices Z_m, $m = 1, \dots, M$, such that:*

$$x^T Z_m x \le 0 \quad \text{for all} \quad x \in \Omega_m,$$
$$Z_{m-1} - Z_m \le 0 \quad \text{for all } m,$$

then the origin is (uniformly) stable in the sense of Lyapunov.

Proof of the Theorem 3.1 — ??? In the domain Ω_m, for which $\rho_m \le x^T P x \le \rho_{m+1}$, condition (i) can always be fulfilled, since $V_m = x^T Z_m x$. Condition $x^T Z_m x \le x^T Z_{m-1} x$ must be satisfied at the boundary $\rho_m = x^T P x$, for which the radius is fixed and the angles θ_i are any. If condition (iii) is satisfied at the boundary $\rho_m = x^T P x$, it should then also be satisfied for any $z = (\rho, \theta_1, \dots, \theta_n)^T$ and thus for any of the state space.

Remark 3.1 It possible to choose independent Lyapunov functions for every ring Ω_m, provided that these Lyapunov functions are always decreasing. The search for Lyapunov matrices should thus start from $m = M$ down to $m = 1$. If $Z = Z_m$, $\forall\, m = 1, \dots, M$, then the problem is reduced to the more general case of finding a common Lyapunov function.

B. LMI-based control of TS-systems with concentric regions

As in [6 – 9], control of TS-systems under a combination of piecewise-linear controls can be seen as a convex optimization problem with constraints that can be solved using powerful numerical tools, using Linear Matrix Inequalities [1, 4].

1) Application to piecewise linear control

Theorem 3.3 *Consider the TS-system defined in (1) with the piecewise linear controller defined in (4). Define* Ω_m*,* $m = 1, \ldots, M$*, as the set of regions* Δ_k *for which* $\rho_m \leq x^T P x \leq \rho_{m+1}$*. If there exist a series of positive-definite matrices* Z_m*,* $m = 1, \ldots, M$ *and a positive constant number* τ_m *such that, for every region* $\Delta_k \subset \Omega_m$ *and for every rule* R_i *which is active in* Δ_k*:*

$$A_i^T Z_m + Z_m A_i + Z_m B_i F_k + F_k^T B_i^T Z_m + \tau_m P < 0, \tag{6}$$

$$Z_{m-1} - Z_m \leq 0. \tag{7}$$

The origin is (uniformly) stable in the sense of Lyapunov.

Proof Consider the Lyapunov function $V = x^T Z_m x$.
In region Δ_k

$$\dot{V}_m = \dot{x}^T Z_m x + x^T Z_m \dot{x} + = \sum_{i=1}^{\delta_k} \lambda_i \big(x^T (A_i^T Z_m + F_k^T B_i^T Z_m) x + x^T (Z_m A_i + Z_m B_i F_k) x \big).$$

$\dot{V}_m < 0$ if $\forall\, i,\, k,$

$$x^T (A_i^T P + F_k^T B_i^T P m) x + x^T (P A_i + P B_i F_k) x < 0.$$

The LMI can be relaxed by considering the regionwise constraints, which can be written, according to the concentric nature of regions:

$$\Psi_k x < 0, \qquad \rho_m - x^T P x < 0, \qquad x^T P x - \rho_{m+1} < 0,$$

and, by the S-procedure [1], a sufficient condition for $\dot{V}_m < 0$ if the existence of positive constants $\tau_{1,m}$, $\tau_{2,m}$, $\tau_{3,k}$ such that:

$$\begin{gathered} x^T (A_i^T Z_m + F_k^T B_i^T Z_m) x + x^T (Z_m A_i + Z_m B_i F_k) x \\ - \tau_{3,k} \Psi_k x - \tau_{1,m} (\rho_m - x^T P x) - \tau_{2,m} (x^T P x - \rho_{m+1}) < 0. \end{gathered}$$

If condition $\Psi_k x < 0$ is not taken into account,

$$x^T (A_i^T Z_m + F_k^T B_i^T Z_m + \tau_{1,m} P - \tau_{2,m} P + Z_m A_i + Z_m B_i F_k) x - \tau_{1,m} \rho_m + \tau_{2,m} \rho_{m+1} < 0$$

which is satisfied if $A_i^T Z_m + F_k^T B_i^T Z_m + (\tau_{1,m} - \tau_{2,m}) P + Z_m A_i + Z_m B_i F_k < 0$ and

$$-\tau_{1,m} \rho_m + \tau_{2,k} \rho_{m+1} \leq 0.$$

Taking $\tau_{2,m} = \tau_{1,m} \dfrac{\rho_m}{\rho_{m+1}}$ and $\tau_m = \tau_{1,m} - \tau_{2,m} = \tau_{1,m} \left(1 - \dfrac{\rho_m}{\rho_{m+1}} \right)$ gives condition (6).

If the conditions in (6) are fulfilled, then $\dot{V}_m < 0$ in Ω_m. From equations (6), (7), applying Theorem 3.2, the origin is uniformly stable.

2) PDC control of TS-systems using concentric Lyapunov surfaces

Theorem 3.4 *Consider the TS-system defined in (1) with the Parallel Distributed Compensation controller defined in (5). If there exist a series of positive-definite matrices* Z_m, $m = 1, \dots, M$, *and a positive constant number* τ_m *such that, for every region* $\Delta_k \subset \Omega_m$, *and for every rules* R_i, R_j *which are active in* Δ_k,

$$\begin{gathered} G_{ii}^T Z_m + Z_m G_{ii} + \tau_m P < 0, \quad \forall\, i = 1, \dots, \delta_k, \\ \left(\frac{G_{ij} + G_{ji}}{2}\right)^2 Z_m + Z_m \left(\frac{G_{ij} + G_{ji}}{2}\right) + \tau_m P \le 0, \quad \forall\, i < j, \end{gathered} \tag{8}$$

$$Z_{m-1} - Z_m \le 0, \tag{9}$$

where $G_{ij} = A_i + B_i F_j$, *and* δ_k *is the number of active rules in* Δ_k, *the origin is (uniformly) stable in the sense of Lyapunov.*

Proof The proof follows the same sketch as in [12] and in Theorem 3.3.

3) TS-system with adaptive rule selection

The algorithm in Theorem 3.1 allows to check the stability of a TS-controller with PDC with relaxed stability conditions, for which the membership functions and validity domains are defined a priori by the user. In general, there is little guideline to help to determine these crucial parameters of fuzzy controllers. As an alternative, it is proposed to build gradually the domains Ω_m (and thus the corresponding rules and local models) from the Lyapunov function found in the previous subset Ω_{m+1}.

Rules are designed in a first time only in the outer set Ω_M. The upper boundary of the new set Ω_{M-1} will be chosen as the smallest Lyapunov surface (from the common Lyapunov function which matches the stability conditions in Ω_M) which contains the lower boundary of Ω_M. The same method will apply for next subsets.

Consider a TS-system defined in (1), using $z = (\rho^M, \theta_1, \dots, \theta_{n-1})^T$ as variable for premises, for which

$$\rho_M = x^T Z_{M+1} x.$$

The set of rules, which are only active in $\Omega_M = \left\{\rho_M^- \le x^T Z_{M+1} x \le \rho_M^+\right\}$ is:

R_i: IF ρ^M is ρ_i AND θ_1 is $\Theta_{i,1}$ AND ... θ_{n-1} is $\Theta_{i,n-1}$ THEN $\dot{x} = A_i x + B_i u$,

where the membership functions $\rho_{M-1}(\cdot)$ and $\rho_M(\cdot)$ fully overlap in the domain $\lfloor \rho_M^-, \rho_M^+ \rfloor$. The membership functions of the other premise variables verify the overlapping condition.

Let us introduce the piecewise linear controller: $u = F_k x$ if $x \in$ region Δ_k, with $\Delta_k \subset \Omega_M$.

Theorem 3.5 *If there exists a series of positive definite matrices* Z_m, $m = 1, \dots, M$, *and positive numbers* ρ_m^- *and* ρ_m^+ *such that:*

(i) *Define*

$$\Omega_m = \left\{\rho_m^- \le x^T Z_{m+1} x \le \rho_m^+\right\}, \qquad \Omega_m^+ = \left\{x^T Z_{m+1} x = \rho_m^+\right\},$$
$$\Omega_m^- = \left\{x^T Z_{m+1} x = \rho_m^-\right\},$$

where Ω^+_{m-1} is the biggest domain that includes Ω^-_m, and Ω^+_{m-1} is enclosed into Ω^+_m;

(ii) *Rules in Ω_m take the form:*

$$R_i\text{: IF } \rho^m \text{ is } \rho_i \text{ AND } \theta_1 \text{ is } \Theta_{i,1} \text{ AND ... } \theta_{n-1} \text{ is } \Theta_{i,n-1} \text{ THEN } \dot{x} = A_i x + B_i u,$$

where $\rho^m = x^T Z_{m+1} x$, and the corresponding local controller is designed in the appropriate regions Δ_k;

(iii) $A_i^T Z_m + Z_m A_i + Z_m B_i F_k + F_k^T B_i^T Z_m + \tau_m Z_m < 0,\ \forall\, m = 1, \dots, M,$

then the overall system is asymptotically stable.

Proof Taking $V = x^T Z_{m+1} x$ in Ω_m, condition (iii) ensures that if a trajectory crosses a surface $x^T Z_m x = c$, where c is some constant, then the trajectory stays in the domain $x^T Z_m x \leq c$ [5]. Hence, if condition (iii) is verified, all trajectories that start in Ω_m will reach Ω_{m-1}. The trajectory converges thus towards the equilibrium (see Figure 3.3).

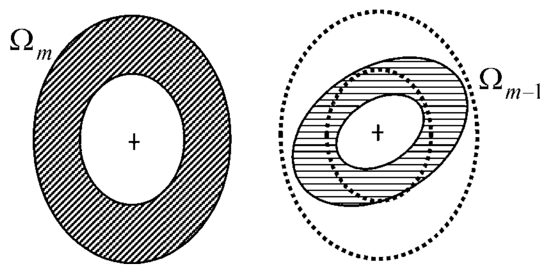

Figure 3.1. Gradual determination of domains.

Remark 3.2 The main advantage of the method is to allow a wide flexibility in the construction of regions. The original system and controllers are not "frozen", since rules and local models are adapted from the stability conditions found for the former set. The counterpart is that, in general, a new set of local models should be determined (and identified) for every domain Ω_m.

4 Example

Consider the 2-D system described in Section 2(C) with $P = I$ (see Figure 2.1), to be controlled by the piecewise linear controller in (4). Suppose that, for $\theta \in \left\{0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}\right\}$, the consequent part is described by:

$$\dot{x} = A_{\rho,\theta} x + Bu,$$

where

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad A_{\rho_2,\theta} = \begin{pmatrix} -1 & \cos(\theta) - 1 \\ -2 + \sin(\theta) & -1 \end{pmatrix},$$

$$A_{\rho_3,\theta} = \begin{pmatrix} -1 & 2\sin(\theta) - 1 \\ -2 + 2\cos(\theta) & -1 \end{pmatrix}, \qquad A_{\rho_1,\theta} = \begin{pmatrix} -2 & -2 \\ 3 & 0 \end{pmatrix}.$$

The regionwise valued controllers for every region Δ_k are given in Table 4.1.

Region	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
F_k^T	$(-1\ \ 1)$	$(-0.5\ \ 0.5)$	$(-2\ \ 1.5)$	$[-0.5\ \ 1.5]$	$(-1\ \ 0)$	$(-2\ \ 2)$	$(-2\ \ 3)$	$[-3\ \ 3]$

Table 4.1. Regionwise valued controllers.

It is impossible to find a common Lyapunov matrix for all controlled systems (actually 28 equations, which would be the same in a corresponding rectangular partition). However, the application of Theorem 3.3 gives

$$Z_1 = \begin{pmatrix} 11.79 & 1.64 \\ 1.64 & 5.11 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 16.05 & -4.89 \\ -4.89 & 18.34 \end{pmatrix}$$

and the overall controller is now stable.

5 Conclusion

Introducing generalized spherical coordinates in the premise part of TS fuzzy systems, it has been shown that an appropriate choice of membership functions allows to separate the state space into a number of concentric regions in which only a limited number of rules are active. PDC techniques can be used to control the TS fuzzy system. A piecewise quadratic Lyapunov function has been designed for every concentric region; the stability of the controlled system is ensured if the piecewise Lyapunov function is decreasing in the corresponding region and if it is smaller than that of the previous domain. Since the regions can be viewed as constraints which can be described with the help of quadratic inequalities, it is easy to include these into a set of inequalities which derives from the Lyapunov stability analysis, which relax LMI conditions. An adaptive algorithm has then be proposed which allows to choose the embedded sets and the corresponding local models and rules.

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Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 75–86

Robustness Analysis of a Class of Discrete-Time Systems with Applications to Neural Networks

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Received: January 11, 2000; Revised: December 30, 2002

Abstract: In this paper we study the robust stability properties of a large class of nonlinear discrete-time systems by addressing the following question: given a nonlinear discrete-time system with specified exponentially stable equilibria, under what conditions will a perturbed model of the discrete-time system possess exponentially stable equilibria that are close (in distance) to the exponentially stable equilibria of the unperturbed discrete-time system? In arriving at our results, we establish robust stability results for the perturbed discrete-time systems considered herein. We apply the above results in the robustness analysis of a large class of discrete-time recurrent neural networks.

Keywords: *Discrete-time systems; robust stability; neural networks.*

Mathematics Subject Classification (2000): 34C35, 34D05, 34D20, 34D45, 34H05, 54H20, 93C10, 93C15, 93C50, 93C60, 93D05, 93D20, 93D30.

1 Introduction

We consider discrete-time systems described by first-order ordinary difference equations of the form

$$x(k+1) = f(x(k)) + h(x(k)), \tag{1}$$

where $x(k)$ is a real n-vector, $k \in Z_+$ (the set of nonnegative integers) and f and h are continuously differentiable n-vector valued functions. We view (1) as a perturbation model of systems described by

$$x(k+1) = f(x(k)). \tag{2}$$

*Anthony N. Michel was supported in part by an Alexander von Humboldt Foundation Senior Research Award, Institut für Nachrichtentechnik, Ruhr-Universität Bochum, Germany.

Thus, $h(x(k))$ in (1) represents uncertainties or perturbation terms.

In the present paper we study robustness properties of system (2) with respect to perturbations. Of particular interest to us will be the robust stability of equilibria and estimates of the perturbations of the equilibrium locations. To demonstrate applicability, we apply these results in the qualitative analysis of a large class of discrete-time recurrent neural networks.

Qualitative robustness results for linear and nonlinear dynamical systems abound (refer to the references cited in pp. II-144 − II-147 of [1] concerning robustness for linear systems and pp. II-147 − II-148 of [1] concerning robustness for nonlinear systems). Although several of these works are tangentially related to the present work, to the best of our knowledge the present results are new. In particular, results involving perturbations of equilibrium locations for discrete-time systems do not seem to have received much attention. Rather, the present results are more in the spirit of those established in [24] for the case of continuous-time systems. We emphasize, however, that the present results are not straightforward translations of the results given in [24] to the case of discrete-time systems.

In Section 2 we provide the necessary notation and definitions used throughout the paper. Given an exponentially stable equilibrium x_e for (2), we establish in Section 3 sufficient conditions for the exponential stability of an equilibrium $\bar{x}_e$ for (1) with the property the $\bar{x}_e$ is near x_e, i.e., $|x_e - \bar{x}_e|_\infty < \epsilon$, where ϵ is sufficiently small. To establish these results, we require several preliminary results which are established in the appendix.

In Section 4, we apply the above results in a perturbation analysis of a large class of discrete-time recurrent neural networks described by systems of first-order ordinary difference equations

$$x_i(k+1) = b_i x_i(k) + c_i s_i\left(\sum_{j=1}^{n} T_{ij} x_j(k) + I_i\right), \qquad i = 1, \cdot, n, \tag{3}$$

where x_i represents the state of the i-th neuron, $T = (T_{ij})_{n\times n}$ is the real-valued matrix of the synaptic connection weights, I_i is a constant external input to the i-th neuron, $s_i(\cdot)$ is the i-th nonlinear activation function, and the self-feedback constant and the neural gain are assumed to satisfy $-1 \le b_i \le 1$ and $c_i \ne 0$, $k \in Z_+$, respectively.

The paper is concluded with some pertinent remarks in Section 5.

2 Notation and Definitions

Let R denote the set of real numbers, let $R_+ = [0, \infty)$, and let R^n denote real n-dimensional vector space. If $x \in R^n$, then $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. Let Z and Z_+ denote the set of integers and the set of nonnegative integers, respectively.

If X and Y are subsets of R^n and R^m, respectively, we let $C[X, Y]$ denote the set of all continuous functions from X to Y. When X is an open subset of R^n, we let $C^N[X, Y]$ denote the set of all functions from X to Y whose partial derivatives up to order N are continuous, $N \ge 1$.

In R^n, we let $|\cdot|$ denote any equivalent norm if we do not specify a particular norm. The norms $|\cdot|_p$, $p \ge 1$, are defined by $|x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, and, in particular, when

$p = 1,\ p = 2$, and $p = \infty$, then $|x|_1 = \sum_{i=1}^{n} |x_i|,\ |x|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, and $|x|_\infty = \max_{1\le i\le n} |x_i|$, respectively.

Let $A = [a_{ij}]$ denote an $n \times n$ matrix and let A^T denote the transpose of A. The matrix norms $|\cdot|_p,\ 1 \le p \le \infty$, induced by the norms $|\cdot|_p$ on $R^n,\ 1 \le p \le \infty$, are defined as $|A|_p = \sup_{0\ne x\in R^n} [|Ax|_p/|x|_p],\ 1 \le p \le \infty$. In particular, we have $|A|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$, $|A|_2 = \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{1/2}$, and $|A|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$.

Let $x_e \in R^n$ and $\epsilon_0 > 0$ be an appropriate positive number. We define $B(x_e, \epsilon_0)$ by $B(x_e, \epsilon_0) = \{x\colon |x - x_e| < \epsilon_0\}$. In this paper, we assume that $f,\, h \in C^2[B(x_e, \epsilon_0), R^n]$.

Definition 2.1 A *square matrix* A is said to be *Schur stable*, if all eigenvalues of A are located within the unit circle.

Definition 2.2 For $f\colon R^n \to R^n$ and $x_e \in R^n$, $\frac{\partial f}{\partial x_i}(x_e)$ is defined by $\frac{\partial f}{\partial x_i}(x_e) = \left(\frac{\partial f_1}{\partial x_i}(x_e), \dots, \frac{\partial f_n}{\partial x_i}(x_e)\right)^T$ and $Df(x_e)$ is defined by the Jacobian matrix $\frac{\partial f}{\partial x}(x)\big|_{x=x_e}$.

In the present paper we use E to denote the $n \times n$ identity matrix.

3 Robustness Analysis of Perturbed Discrete-Time Systems

This section consists of three parts.

3.1 Robust stability: Perturbed discrete-time systems with fixed equilibria

In this subsection we first consider the special case where an equilibrium x_e of the original system (2) is unchanged in the resulting perturbed system (1).

In order to establish our first result, we consider the discrete-time systems with uncertainties and perturbations of the form

$$x(k+1) = (A + \Delta A)x(k) + m(x(k)), \tag{4}$$

where $x(k) \in R^n$, A and ΔA are constant and uncertain $n\times n$ matrices, respectively, $k \in Z_+$, $x(k) \equiv 0$ is an equilibrium of (4), $m \in C[U, R^n]$ satisfies the condition $\lim_{x\to 0} \frac{|m(x)|}{|x|} = 0$, $U \subset R^n$ is an open subset containing x_e.

Lemma 3.1 *In addition to the assumptions $x_e = 0$ and $\lim_{x\to 0} \frac{|m(x)|}{|x|} = 0$, we assume for system (4) that*

(i) *A is Schur stable;*

(ii) *$|\Delta A|_\infty < \sigma$, where $\sigma \in \left(0, -|A|_2 + \left(|A|_2^2 + \frac{1}{|P|_2}\right)^{1/2}\right)$, and P is a symmetric and positive definite matrix determined by $A^T PA - P = -E$.*

Then the equilibrium $x(k) \equiv 0$ of (4) is exponentially stable.

Proof In applying the second method of Lyapunov, we choose the Lyapunov function given by $v(x(k)) = x^T(k)Px(k)$. Let $\Delta v(x(k))_{(4)} = v(x(k+1)) - v(x(k))$, where $x(k+1)$

satisfies the difference equation (4). For all $x(k) \in U$, we have, using condition (i) and (ii) as well as the relation $|\Delta A|_\infty \leq |\Delta A|_2$,

$$
\begin{aligned}
\Delta v(x(k))_{(4)} &= [Ax(k) + (\Delta A)x(k) + m(x(k))]^T P[Ax(k) \\
&\quad + (\Delta A)x(k) + m(x(k))] - x^T(k)Px(k) \\
&= x^T(k)[A^T PA - P]x(k) + x^T(k)[(\Delta A)^T PA \\
&\quad + A^T P(\Delta A) + (\Delta A)^T P(\Delta A)]x(k) \\
&\quad + 2[Ax(k) + (\Delta A)x(k)]^T Pm(x(k)) \\
&\quad + m(x(k))^T Pm(x(k)) \\
&= x^T(k)[-E + (\Delta A)^T PA + A^T P(\Delta A) \\
&\quad + (\Delta A)^T P(\Delta A)]x(k) + 2[Ax(k) \\
&\quad + (\Delta A)x(k)]^T Pm(x(k)) + m(x(k))^T Pm(x(k)) \\
&\leq [-1 + 2\sigma|P|_2|A|_2 + \sigma^2|P|_2]x^T(k)x(k) \\
&\quad + 2x^T(k)[A + \Delta A]^T Pm(x(k)) + m(x(k))^T Pm(x(k)) \\
&< -4\epsilon x^T(k)x(k) + 2x^T(k)[A + \Delta A]^T Pm(x(k)) \\
&\quad + m(x(k))^T Pm(x(k)),
\end{aligned}
\tag{5}
$$

where $-4\epsilon = -1+2\sigma|P|_2|A|_2+\sigma^2|P|_2 < 0$ by condition (ii). Since $\lim_{x\to 0}(|m(x)|/|x|) = 0$, it is clear that there exists an open subset of the origin, $V \subset U$, such that for all $x \in V$, $2x^T[A + (\Delta A)]^T Pm(x) < 2\epsilon x^T x$ and $m(x)^T Pm(x) < \epsilon x^T x$. Therefore, from (5) we obtain for $x(k) \in V$, $\Delta v(x(k))_{(4)} < -\epsilon x^T(k)x(k)$. By the basic stability theorem of Lyapunov, the equilibrium $x(k) \equiv 0$ of (4) is exponentially stable.

Remark 3.1 The existence and uniqueness of solutions of the Lyapunov equation $A^T PA - P = -E$ are guaranteed by the assumption that A is Schur stable (see, e.g., [3]).

We will require the following assumption.

Assumption 3.1 *For systems (1) and (2), it is true that*

(i) *x_e is an equilibrium of both (1) and (2);*
(ii) *$A = Df(x_e)$ is Schur stable;*
(iii) *$|\Delta A|_\infty < \sigma$, where $\Delta A = Dh(x_e)$, $\sigma \in \left(0, -|A|_2 + \left(|A|_2^2 + \frac{1}{|P|_2}\right)^{1/2}\right)$, and P is a symmetric and positive definite matrix determined by $A^T PA - P = -E$.*

Theorem 3.1 *Under Assumption 3.1, the equilibrium $x(k) \equiv x_e$ of system (1) is exponentially stable.*

Proof By the assumption that $f, h \in C^2[B(x_e, \epsilon_0), R^n]$ and $x(k) \equiv x_e$ is an equilibrium of (1), we can express (1) in the following equivalent form

$$x(k + 1) - x_e = f(x(k)) - f(x_e) + h(x(k)) - h(x_e). \tag{6}$$

The right-hand side of (6) can be rewritten in the form

$$
\begin{aligned}
&f(x(k)) - f(x_e) + h(x(k)) - h(x_e) \\
&= Df(x_e)(x(k) - x_e) + Dh(x_e)(x(k) - x_e) + m(x(k) - x_e) \\
&= (A + \Delta A)(x(k) - x_e) + m(x(k) - x_e),
\end{aligned}
\tag{7}
$$

where $m(\cdot)$ denotes the remaining higher-order terms with respect to $(x(k)-x_e)$.

Let $y(k)=x(k)-x_e$. Then system (1) can be rewritten in the following equivalent form

$$y(k+1)=(A+\Delta A)y(k)+m(y(k)). \tag{8}$$

It is clear that $y(k)\equiv 0$ is an equilibrium of (8) and all conditions of Lemma 3.1 are satisfied. Therefore, the equilibrium $y(k)\equiv 0$ of (8) is exponentially stable and thus the equilibrium $x(k)\equiv x_e$ of (1) is exponentially stable.

3.2 Robust stability: Perturbed discrete-time systems with perturbed equilibria

In this subsection, we will consider the case where the equilibrium $\bar{x}_e$ of the perturbed discrete-time system (1) differs from the equilibrium x_e of the unperturbed discrete-time system (2).

Assumption 3.2 *Let $\bar{x}_e$ and x_e denote the equilibrium of (1) and (2), respectively. Assume that*

(i) *$A=Df(x_e)$ is Schur stable;*

(ii) *$|Dh(x_e)|_\infty < a_1$,*

where $a_1=\frac{\sigma}{2}$, $\sigma\in\left(0,-|A|_2+\left(|A|_2^2+\frac{1}{|P|_2}\right)^{\frac{1}{2}}\right)$, A is given in (i) and P is a positive definite and symmetric matrix which is determined by $A^TPA-A=-E$; and

(iii) *$|\bar{x}_e-x_e|_\infty<\epsilon$,*

where $0<\epsilon<\epsilon_1$, $\epsilon_1=\min\left\{\frac{\sigma}{2M_2},\epsilon_0\right\}$, $M_2=\sup\limits_{x,y\in B(x_e,\epsilon_0)}|Q_2(x,y)|_\infty$, σ is given in part (ii), and $Q_2(x,y)$ satisfies the properties of Lemma A.1 with respect to $q=f+h$ (see the Appendix).

Theorem 3.2 *If Assumption 3.2 is satisfied, then the equilibrium $x(k)\equiv\bar{x}_e$ of the perturbed system (1) is exponentially stable.*

Proof Since $x(k)\equiv\bar{x}_e$ is an equilibrium of (1), we can rewrite (1) as

$$x(k+1)-\bar{x}_e=f(x(k))+h(x(k))-(f(\bar{x}_e)+h(\bar{x}_e)) \tag{9}$$

or its equivalent form

$$x(k+1)-\bar{x}_e=(Df(\bar{x}_e)+Dh(\bar{x}_e))(x(k)-\bar{x}_e)+m(x(k)-\bar{x}_e). \tag{10}$$

Let $A=Df(x_e)$ and $\Delta A=Df(\bar{x}_e)+Dh(\bar{x}_e)-Df(x_e)$. Then we can rewrite (10) as

$$x(k+1)-\bar{x}_e=(A+\Delta A)(x(k)-\bar{x}_e)+m(x(k)-\bar{x}_e). \tag{11}$$

Letting $y(k)=x(k)-\bar{x}_e$, (11) can be rewritten as

$$y(k+1)=(A+\Delta A)y(k)+m(y(k)). \tag{12}$$

Using Lemma A.1 in the Appendix and Remark A.1, we have

$$\begin{aligned}\Delta A&=Df(\bar{x}_e)+Dh(\bar{x}_e)-Df(x_e)\\&=Df(\bar{x}_e)+Dh(\bar{x}_e)-(Df(x_e)+Dh(x_e))+Dh(x_e)\\&=Q_2(\bar{x}_e,x_e)\Lambda(\bar{x}_e-x_e)+Dh(x_e),\end{aligned} \tag{13}$$

where Q_2 and Λ satisfy the properties of Lemma A.1 with respect to $q = f + h$ (see the Appendix).

Using parts (ii) and (iii) of Assumption 3.2, we have

$$\begin{aligned} |\Delta A|_\infty &\leq |Q_2(\bar{x}_e, x_e)|_\infty \cdot |\bar{x}_e - x_e|_\infty + |Dh(x_e)|_\infty \\ &\leq M_2|\bar{x}_e - x_e|_\infty + |Dh(x_e)|_\infty \leq M_2\epsilon + a_1 < \frac{1}{2}\sigma + \frac{1}{2}\sigma = \sigma. \end{aligned} \tag{14}$$

It is clear that all conditions of Lemma 3.1 are satisfied for (12). We conclude that the equilibrium $y(k) \equiv 0$ of (12) is exponentially stable and thus the equilibrium $x(k) \equiv \bar{x}_e$ of (1) is exponentially stable.

3.3 Example

In the following, we utilize a specific example to demonstrate the applicability of Theorem 3.1. In the next section, we consider a general class of problems.

In (1) and (2), let $x = [x_1, x_2]^T$, $f(x) = [f_1(x), f_2(x)]^T$, $h(x) = [h_1(x), h_2(x)]^T$, $f_1(x) = x_1 - \frac{1}{2}\arctan x_1$, $f_2(x) = x_2 - \frac{1}{2}\arctan(x_1 + x_2)$, $h_1(x) = \delta_1 \arctan x_1$, and $h_2(x) = \delta_2 \arctan(x_1 + x_2)$, where δ_1 and δ_2 are perturbation parameters. Presently, systems (1) and (2) assume the form

$$\begin{aligned} x_1(k+1) &= x_1(k) - \frac{1}{2}\arctan x_1(k) + \delta_1 \arctan x_1(k), \\ x_2(k+1) &= x_2(k) - \frac{1}{2}\arctan(x_1(k) + x_2(k)) + \delta_2 \arctan(x_1(k) + x_2(k)) \end{aligned} \tag{15}$$

and

$$\begin{aligned} x_1(k+1) &= x_1(k) - \frac{1}{2}\arctan x_1(k), \\ x_2(k+1) &= x_2(k) - \frac{1}{2}\arctan(x_1(k) + x_2(k)), \end{aligned} \tag{16}$$

respectively.

$x_e = 0$ is an equilibrium for both (15) and (16). We have

$$A = Df(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

which is Schur stable. Also, $A^T P A - P = -E$ with $P = P^T$ yields

$$P = \begin{bmatrix} \frac{56}{27} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{3}{4} \end{bmatrix}.$$

In our result we have $\sigma_M = -|A|_2 + \left(|A|_2^2 + \frac{1}{|P|_2}\right)^{1/2} = 0.2432$ and $\Delta A = Dh(0) = \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & \delta_2 \end{bmatrix}$, $|\Delta A|_\infty| = \min\{|\delta_1|, 2|\delta_2|\}$. If $\min\{|\delta_1|, 2|\delta_2|\} < \sigma < 0.2030$. Theorem 3.1 implies that the state $x_e = 0$ is an exponentially stable equilibrium of (15).

4 Applications to Neural Networks

This section consists of three parts.

4.1 Model of discrete-time recurrent neural networks

In the present section we consider discrete-time recurrent neural networks described by systems of nonlinear difference equations of the form

$$x_i(k+1) = b_i x_i(k) + c_i s_i\left(\sum_{j=1}^{n} T_{ij} x_j(k) + I_i\right), \qquad i = 1, \cdots, n, \tag{17}$$

where x_i represents the state of the i-th neuron, $T = (T_{ij})_{n\times n}$ is the real-valued matrix of the synaptic connection weights, I_i is a constant external input to the i-th neuron, $s_i(\cdot)$ is the i-th nonlinear activation function, and the self-feedback constant and the neural gain are assumed to be $-1 \le b_i \le 1$ and $c_i \ne 0$, $k \in Z_+$, respectively.

In (17), the neural activation function $s_i(\cdot)$ is chosen to be a continuously differentiable nonlinear sigmoidal function (i.e., $s_i(\cdot)$ maps the real axis R into the real interval $(-1, 1)$, it is smooth and monotonically increasing, and its graph in the plane is symmetric with respect to the origin). Typical examples of activation functions include: $s_i(y_i) = \frac{2}{\pi}\arctan\left(\frac{\pi}{2}\, y_i\right)$, $s_i(y_i) = \frac{1-e^{-y_i}}{1+e^{-y_i}}$, and $s_i(y_i) = \tanh(y_i) = \frac{e^{y_i} - e^{-y_i}}{e^{y_i} + e^{-y_i}}$.

We can represent the neural network (17) in vector form as

$$x(k+1) = Bx(k) + Cs(Tx(k) + I), \tag{18}$$

where $x = (x_1, \cdots, x_n)^T$ is the state vector and $s(y) = (s_1(y_1), \cdots, s_n(y_n))^T$ for $y = (y_1, \cdots, y_n)^T \in R^n$. Also, $B = \operatorname{diag}[b_1, \cdots, b_n]$, $C = \operatorname{diag}[c_1, \cdots, c_n]$, $T = (T_{ij})_{n\times n}$, and $I = (I_1, \cdots, I_n)^T$.

Stability properties of recurrent discrete-time neural networks have been widely studied (see, e.g., [4, 10, 16, 18, 19, 21, 22]). Some of the most important applications of such networks concern associative memories (see, e.g., [4, 16, 18]).

For system (18) we consider the perturbation model

$$x(k+1) = (B + \Delta B)x(k) + (C + \Delta C)s[(T + \Delta T)x(k) + (I + \Delta I)], \tag{19}$$

where ΔB, ΔC, ΔT, and ΔI are the uncertain or perturbation matrices with the same dimension as B, C, T, and I, respectively.

In Feng and Michel [5], a robustness analysis for the neural network (18) is given. In the present section, we will consider the neural network (18) as a special case of (2) and apply the robustness results in Section 3 to the discrete-time system (2) to establish robustness results for the neural network (18).

4.2 Stability of perturbed neural networks with unperturbed equilibria

In this subsection we first consider the special case where an equilibrium x_e of the original system (18) is unchanged in the resulting perturbed system (19).

Let x_e be an equilibrium of system (18), let ϵ_0 be an appropriate fixed positive number, and let R_0, L_1, and L_2 denote positive real numbers satisfying $R_0 \ge |x_e|_\infty$, $L_1 \ge$

$\sup_{x \in B(x_e,\epsilon_0)} |s'(x)|$, and $L_2 \geq \sup_{x \in B(x_e,\epsilon_0)} |s''(x)|$, where $s'(x) = \mathrm{diag}\,[s_1'(x_1), \cdots, s_n'(x_n)]$, and $s''(x) = \mathrm{diag}\,[s_1''(x_1), \cdots, s_n''(x_n)]$, $s_i'(\cdot)$ and $s_i''(\cdot)$ denote the first-order and the second-order derivatives of $s_i(\cdot)$, respectively. In practice, L_1 and L_2 can frequently be chosen independently of x_e and ϵ_0. For example, if $s_j(x_j) = \arctan(\lambda_j x_j)$ with $\lambda_j > 0$, $1 \leq j \leq n$, then for all $x \in R^n$ we have $|s'(x)|_\infty \leq \max_{1\leq j\leq n}\{\lambda_j\}$ and $|s''(x)|_\infty \leq \max_{1\leq j\leq n}\{\lambda_j^2\}$. Therefore, in the present example, we may choose $L_1 = \max_{1\leq j\leq n}\{\lambda_j\}$ and $L_2 = \max_{1\leq j\leq n}\{\lambda_j^2\}$.

We will require the following assumption.

Assumption 4.1 *For systems (18) and (19), it is true that*

(i) x_e *is an equilibrium of both (18) and (19);*
(ii) $A = B + Cs'(Tx_e + I)T$ *is Schur stable;*
(iii) $\max\{|\Delta B|_\infty, |\Delta C|_\infty, |\Delta T|_\infty, |\Delta I|_\infty\} < K_0$, *where K_0 is given by*

$$K_0 = \frac{1}{2L_1}\big[-\beta + (\beta^2 + L_1\sigma)^{1/2}\big],$$

where

$$\beta = 1 + L_1|T|_\infty + L_1|C|_\infty + L_2|C|_\infty|T|_\infty(R_0+1),$$
$$\sigma \in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big),$$

and where $P = P^T$ is a positive definite matrix that is determined by $A^TPA - P = -E$, and A is defined in (ii) above.

We note that in Assumption 4.1, K_0 is a positive number determined by system (18) and is independent of the system perturbations. The following result shows that K_0 is an admissible bound for robust stability.

Proposition 4.1 *Under Assumption 4.1, the equilibrium $x = x_e$ of system (18) is exponentially stable.*

Proof Let

$$f(x) = Bx + Cs(Tx + I) \tag{20}$$

and

$$h(x) = (B+\Delta B)x + (C+\Delta C)s[(T+\Delta T)x + (I+\Delta I)] - [Bx + Cs(Tx+I)]. \tag{21}$$

Then (19) can be expressed in the form of $x(k+1) = f(x(k)) + h(x(k))$, or in the form of (1). We have that

$$Df(x_e) = B + Cs'(Tx_e + I)T \tag{22}$$

and

$$Dh(x_e) = (\Delta B) + (C+\Delta C)s'[(T+\Delta T)x_e + (I+\Delta I)](T+\Delta T) - Cs'(Tx_e+I)T. \tag{23}$$

To show that the equilibrium $x = x_e$ of (19) is exponentially stable, we only need to verify that all conditions of Theorem 3.1 are satisfied, or to verify that all statements in

Assumption 3.1 are true. By part (ii) of Assumption 4.1, $B + Cs'(Tx_e + I)T$ is Schur stable and thus part (ii) of Assumption 3.1 is satisfied.

To show that part (iii) of Assumption 3.1 is also satisfied, it suffices to show that $|Dh(x_e)|_\infty < \sigma$, where $\sigma \in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big)$, and where P is given by $A^T PA - P = -E$. Using part (iii) of Assumption 4.1, we have

$$
\begin{aligned}
Dh(x_e) &= \Delta B + (C + \Delta C)s'[(T + \Delta T)x_e \\
&\quad + (I + \Delta I)](T + \Delta T) - Cs'(Tx_e + I)T \\
&= \Delta B + Cs'[(T + \Delta T)x_e + (I + \Delta I)](\Delta T) \\
&\quad + (\Delta C)s'[(T + \Delta T)x_e + (I + \Delta I)]T \\
&\quad + (\Delta C)s'[(T + \Delta T)x_e + (I + \Delta I)](\Delta T) \\
&\quad + CQ_2((T + \Delta T)x_e + (I + \Delta I), Tx_e + I)\Lambda((\Delta T)x_e + \Delta I)T,
\end{aligned} \tag{24}
$$

where Q_2 and Λ satisfy the properties of Lemma A.1 in the Appendix with respect to $q = s$. Using part (iii) of Assumption 4.1 and noticing that

$$
\sup_{x,y \in B(x_e,\epsilon_0)} |Q_2(x, y)|_\infty \le L_2 = \sup_{x \in B(x_e,\epsilon_0)} |s''(x)|_\infty,
$$

we obtain

$$
\begin{aligned}
|Dh(x_e)|_\infty &\le |\Delta B|_\infty + L_1|C|_\infty|\Delta T|_\infty + L_1|\Delta C|_\infty|T|_\infty + L_1|\Delta C|_\infty|\Delta T|_\infty \\
&\quad + L_2R_0|C|_\infty|\Delta T|_\infty|T|_\infty + L_2|C|_\infty|\Delta I|_\infty|T|_\infty \le L_1K_0^2 + \beta K_0 < \sigma.
\end{aligned} \tag{25}
$$

This shows that part (iii) of Assumption 3.1 is satisfied. Therefore, the results follow from Theorem 3.1.

4.3 Stability of perturbed neural networks with perturbed equilibria

In this subsection, we will consider the case where the equilibrium $\bar{x}_e$ of the perturbed neural network (19) differs from the equilibrium x_e of the original neural network (18).

Assumption 4.2 *Let x_e and $\bar{x}_e$ denote equilibria of systems (18) and (19), respectively. Assume that*

(i) $A = B + Cs'(Tx_e + I)T$ *is Schur stable and therefore there exists a positive definite matrix* $P = P^T$ *determined by the matrix equation* $A^T PA - P = -E$*;*

(ii) $\max\{|\Delta B|_\infty, |\Delta C|_\infty, |\Delta T|_\infty, |\Delta I|_\infty\} < K_1$*, where* K_1 *is given by*

$$
K_1 = \frac{1}{2L_1}\Big[-\beta + \Big(\beta^2 + \frac{L_1\sigma}{2}\Big)^{1/2}\Big],
$$

where

$$
\begin{aligned}
\beta &= 1 + L_1(|T|_\infty + |C|_\infty) + L_2|C|_\infty|T|_\infty(R_0 + 1), \\
\sigma &\in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big);
\end{aligned}
$$

and

(iii) $|\bar{x}_e - x_e| \le \epsilon$*, where* $0 < \epsilon < \bar{\epsilon}_1$*,* $\bar{\epsilon}_1 = \min\Big\{\frac{\sigma}{2\alpha_1 L_2}, \epsilon_0\Big\}$*, where* $\alpha_1 = (|C|_\infty + K_1)L_2(|T|^2 + 2|T|K_1 + K_1^2)$ *and* ϵ_0 *is given in the previous section.*

Proposition 4.2 *If Assumption 4.2 is true, then the equilibrium $\bar{x}_e$ of the perturbed system (19) is exponentially stable.*

Proof Let $f(x)$ and $h(x)$ be given by (20) and (21), respectively. To prove the result, it suffices to verify all conditions in Assumption 3.2.

From part (i) of Assumption 4.2, it follows that $A = Df(x_e)$ is Schur stable and thus part (i) of Assumption 3.2 is true.

Using similar statements as in the proof of Proposition 4.1 (see (24) and (25)), we can prove that part (ii) of Assumption 4.1 implies part (ii) of Assumption 3.1.

To show part (iii) of Assumption 3.1 is also satisfied, it suffices to verify that $\bar{\epsilon}_1 \le \epsilon_1$, where $\epsilon_1 = \min\left\{\frac{\sigma}{2M_2}, \epsilon_0\right\}$ or $M_2 \le \alpha_1 L_2$, where $M_2 = \sup\limits_{x,y\in B(x_e,\epsilon_0)} |Q_2(x,y)|$, where Q_2 is a function satisfying the properties of Lemma A.1 with respect to $f+h$ with $f(x)+h(x) = (B+\Delta B)x + (C+\Delta C)s[(T+\Delta T)x + (I+\Delta I)]$. Using part (iii) of Assumption 4.2 and the definition of L_2, we have

$$\begin{aligned}
M_2 = \sup_{x,y\in B(x_e,\epsilon_0)} |Q_2(x,y)|_\infty &= \sup_{x,y\in B(x_e,\epsilon_0)} \Bigg| \int_0^1 (C+\Delta C) \\
&\times s''[(T+\Delta T)(x+t(y-x)) + (I+\Delta I)](T+\Delta T)^2\, dt \Bigg|_\infty \\
&\le |C+\Delta C|_\infty \sup_{x\in B(x_e,\epsilon_0)} |s''(x)|_\infty |T+\Delta T|^2 \\
&\le (|C|_\infty + |\Delta C|_\infty) L_2 (|T|_\infty^2 + 2|T|_\infty|\Delta T|_\infty + |\Delta T|_\infty^2) \le \alpha_1 L_2
\end{aligned} \tag{26}$$

which implies $\bar{\epsilon}_1 \le \epsilon_1$.

This shows that all conditions of Assumption 3.2 are satisfied. Therefore, the result of Proposition 4.2 follows from Theorem 3.2.

Remark 4.1 It should be noted that in Assumption 4.2, the existence of an equilibrium of the perturbed system (19) is hypothesized to be not far away from the corresponding equilibrium of the unperturbed system (18). It is reasonable to expect that when the perturbations of the system in question are sufficiently small, this assumption will be satisfied.

5 Concluding Remarks

A robustness analysis was conducted for a large class of nonlinear discrete-time systems described by ordinary difference equations under perturbations. The results presented aimed to give an answer to the following question: given a nonlinear discrete-time system with specified exponentially stable equilibria, under what conditions will a perturbed model of the discrete-time system possess exponentially stable equilibria that are close (in distance) to the exponentially stable equilibria of the unperturbed model? Robustness stability results for perturbed nonlinear discrete-time systems were established. Using these results, a set of sufficient conditions was established for robust stability of a large class of discrete-time recurrent neural networks for associative memories under perturbations of system parameters.

Appendix

We require the following result in the proofs of Theorem 3.2 and Proposition 4.2.

Lemma A.1 *Let $q \in C^2[\bar{U}, R^n]$, where $U \subset R^n$ is a convex open set and $\bar{U}$ denotes the closure of U. Then there exists a $Q_1 \in C^1[U \times U, R^{n\times n}]$ and $Q_2 \in C^1[U \times U, R^{n\times n^2}]$ satisfying the following properties for all $x, y \in U$:*

(i) *$q(x) - q(y) = Q_1(x,y)(x-y)$, where $Q_1(x,y)$ is given by*

$$Q_1(x,y) = \int_0^1 Dq^T(x + t(y-x))\,dt; \tag{A.1}$$

(ii) *$Dq(x) - Dq(y) = Q_2(x,y)\Lambda(x-y)$, where $Q_2(x,y)$ and $\Lambda(x-y)$ are given by*

$$Q_2(x,y) = [Q_{21}(x,y), \cdots, Q_{2n}(x,y)] \tag{A.2}$$

with

$$Q_{2i}(x,y) = \int_0^1 D\left(\frac{\partial q}{\partial x_i}\right)(x + t(y-x))\,dt, \tag{A.3}$$

and

$$\Lambda(x-y) = \begin{bmatrix} x-y & 0 & \dots & 0 \\ 0 & x-y & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x-y \end{bmatrix}, \tag{A.4}$$

respectively.

Proof Part (i) can be proved by using the following formula from the calculus (refer to pp. 48–49 in Chapter 2 of [3]):

$$q(x) - q(y) = \left(\int_0^1 Dq(x + t(y-x))\,dt\right)(x-y). \tag{A.5}$$

Part (ii) can be obtained by using part (i) for every column of Dq:

$$Dq(x) - Dq(y) = \left[\int_0^1 D\left(\frac{\partial q}{\partial x_1}(x + t(y-x)\right)dt, \cdots, \int_0^1 D\left(\frac{\partial q}{\partial x_n}(x + t(y-x)\right)dt\right]. \tag{A.6}$$

Remark A.1 In the following we assume that $U = B(x_e, \epsilon_0)$, where $x_e \in R^n$, $\epsilon_0 > 0$. As a consequence of Lemma A.1, for any $x, y \in U$, if $U \in R^n$ is bounded, then we have

$$|q(x) - q(y)|_\infty \le |Q_1(x,y)|_\infty \cdot |x-y|_\infty \le M_1|x-y|_\infty \tag{A.7}$$

and

$$|Dq(x) - Dq(y)|_\infty \le |Q_2(x,y)|_\infty \cdot |x-y|_\infty \le M_2|x-y|_\infty, \tag{A.8}$$

where $M_1 = \sup\limits_{x\in U} |Dq(x)|_\infty$ and $M_2 = \sup\limits_{x,y\in U} |Q_2(x,y)|_\infty$.

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Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 87–103

Nonlinear Relations to Final Semi-Major Axis in Continuous Orbital Transfers

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Received: October 26, 2002; Revised: January 24, 2003

Abstract: We studied nonimpulsive orbital transfers under thrust errors through algebraic analysis method. We analyzed the relationship among final semi-major axis and mean deviations in the thrust vector. The nonlinear (near parabolic) relations were found, confirming the Monte-Carlo simulations realized in the numerical phase this investigation. These results suggest and partially characterize the progressive deformation of the final semi-major axis along the propulsive arc, turning 3sigma ellipsoids into banana shaped volumes curved to the center of attraction (we call them "bananoids") due to the loss of optimality of the actual (with errors) trajectories with respect to the nominal (no errors) trajectory.

Keywords: *Orbits; transfer; nonimpulsive; thrust errors; algebraic analysis.*

Mathematics Subject Classification (2000): 70M20, 65C05, 62L70.

1 Introduction

According to Marec [1], the orbital transfer of a space vehicle under the gravitational attraction of a celestial body is one of the classical and important problems of Astronautics. Since the early decades of XX century, many researchers dedicated much attention and interest to this problem. Ideally, we can say that, to transfer a space vehicle from one orbit to other consists of changing its initial state, defined by its position, velocity and mass $(\vec{r_0}, \vec{v_0}, m_0)$in a certain initial instant t_0, to another state, defined by its respective state variables $(\vec{r_f}, \vec{v_f}, m_f)$in a final instant $t_f > t_0$. When the transfer is done aiming to minimize the fuel spent, we define the "Fundamental Problem of Astronautics", that is,

†This work has the partial financial support from CAPES (Brazilian agency).

to transfer a vehicle, changing its initial state to a final state with the smallest possible fuel spent $(m_0 - m_f)$. According to Jesus [2], orbital transfers are done to meet some objectives under some restrictions that can be put into the problem to obtain the best possible performance for a particular mission. Examples of objectives are minimum fuel consumption, minimum transfer time, maximum final velocity, etc. Examples of restrictions are rendezvous of two vehicles, the transfer between two given points in a fixed time (Lambert problem), etc. In this way, we can consider that the orbital transfer problem, despite basic in nature, present challenges with respect to its general characteristics since diverse natural, modeling, economic, operational limitations need to be considered in its formulation. So, to find the desired optimal solutions, different problem formulations and optimization criteria can be used to best approximate it to the actual case. The actuator models are normally used in the orbital transfer or correction problems. The infinite propulsion, where the source is modeled as being capable of applying a large magnitude but small duration force (with respect to the orbital period) is the most used, but the non-impulsive hypothesis (finite propulsion) is also found in the literature under many different constraints. The applications of these maneuvers include: small orbital corrections of an Earth artificial satellite, to put a satellite in geostationary orbit, the rendezvous or intercept missions, the long interplanetary travels (like "Voyager" and "Pioneer" missions), the transportation and assembly of the International Space Station. In Brazil, they include: to put and to keep in orbit the Remote Sensing Satellites 1,2 and China-Brazil Earth Resources Satellites 1,2.

Most space missions need orbit transfers to reach their goals. These orbits are reached sequentially through transfers between them, by changing at least one component of the vehicle velocity or position vectors, that is, at least one of corresponding Keplerian elements by firing thrusts, apogee motors, or other force sources. The actual thrust vector has errors in magnitude and/or in direction with respect to the ideal thrust vector.

The magnitude errors are caused by many and unpredictable reasons as: limitations in the manufacturing process (mechanical imprecision due to mechanical and chemical machining, tolerances in the components, etc.) in the loading process (tolerances in the physical and chemical characteristics of the used substances, etc.), in the thrust operation (pulsed, blow-down, under the actual conditions, etc.). They can be modeled as: 1) a constant but random deviation ("random bias") with respect to the ideal magnitude, resulting in a constant actual mean magnitude with a certain probability density function (uniform in the worst case, Gaussian in the best case); and/or 2) random fluctuations around this actual mean magnitude with little or no correlation in time ("pink or white noise") and with a certain probability density function (uniform in the worst case, Gaussian in the best case).

The direction errors of misalignment's errors with respect to its nominal action line are caused by many and unpredictable reasons as: linear and angular displacements during the vehicle assembly and particularly, during the thrusts assembly; displacements of the center of mass during the injection in orbit/trajectory, and during the vehicle operation, due to movable parts as solar panels, antennas, booms, pendulums, etc., or due to fuel consumption, specially during their firing; many and asymmetric thrusts firing at the same time, dead zones existing in all attitude controls used during the firing; partial deviation of some jet plumes by the vehicle structure (plume impingement); etc. They can be modeled like the magnitude errors. These are the magnitude and direction errors models used in the work of Jesus [2].

So, most space missions need trajectory/orbit transfers and they have linear and/or angular misalignments that displace the vehicle with respect its nominal directions. The mathematical treatment for these deviations can be done under many approaches (deterministic, probabilistic, minimax, etc.)

In the deterministic approach: we highlight Schwende and Strobl [3], Tandon [4], Rodrigues [5], Santos-Paulo [6], Rocco [7] and Schultz [8], among others.

In the probabilistic approach Porcelli and Vogel [9] presented an algorithm for the determination of the orbit insertion errors in biimpulsive noncoplanar orbital transfers (perigee and apogee), using the covariance matrices of the sources of errors. Adams and Melton [10] extended such algorithm to ascent transfers under a finite thrust, modeled as a sequence of impulsive burns. They developed an algorithm to compute the propagation of the navigation and direction errors among the nominal trajectory, with finite perigee burns. Rao [11] built a semi-analytic theory to extend covariance analysis to long-term errors on elliptical orbits. Howell and Gordon [12] also applied covariance analysis to the orbit determination errors and they develop a station-keeping strategy of Sun-Earth L1 libration point orbits. Junkins, *et al.* [13] and Junkins [14] discussed the precision of the error covariance matrix method through nonlinear transformations of coordinates. He also found a progressive deformation of the initial ellipsoid of trajectory distribution (due to Gaussian initial condition errors), that was not anticipated by the covariance analysis of linearized models with zero mean errors. Its main results also characterize how close or how far are Monte-Carlo analysis and covariance analysis for those examples. Carlton-Wippern [15] proposed differential equations in polar coordinates for the growth of the mean position errors of satellites (due to errors in the initial conditions or in the drag), by using an approximation of Langevin's equation and a first order perturbation theory. Alfriend [16] studied the effects of drag uncertainty via covariance analysis.

In the minimax approach: see Russian authors, mainly.

However, all these analyses are approximated. This motivated an exhaustive numerical (see [17, 18]) but exact analysis (by Monte-Carlo), and a partial algebraic analysis done by Jesus [2]. In this work we present the first part of the algebraic analysis of nonimpulsive orbital transfers under thrust errors. The results were obtained for two transfers: the first, a low thrust transfer between high coplanar orbits (we call it "theoretical transfer"), used by Biggs [19, 20] and Prado [21]; the second, a high thrust transfer between middle noncoplanar orbits (the first transfer of the EUTELSAT II-F2 satellite, we call it "practical transfer") implemented by Kuga, *et al.* [22]. The simulations were done for both transfers with minimum fuel consumption. The "pitch" and "yaw" angles were taken as control variables such that the overall minimum fuel consumption defines each burn of the thrusts. The errors sources that we considered were the magnitude errors, the "pitch" and "yaw" direction errors of the thrust vector, as causes of the deviations found in the several Keplerian elements of the transfer trajectory. These errors sources (random-bias and white-noise errors) introduced in the orbital transfer dynamics produce effects in the final orbit Keplerian elements in the final instant.

In this work we present an algebraic analysis of the effects of these errors on the mean of the deviations of the Keplerian elements of the final orbit with respect to the reference orbit (final orbit without errors in the thrust vector) for both transfers. The approach that we used in this work for the treatment of the errors was the probabilistic one, assuming these as having zero mean Gaussian probability density function or having zero mean Uniform probability density function.

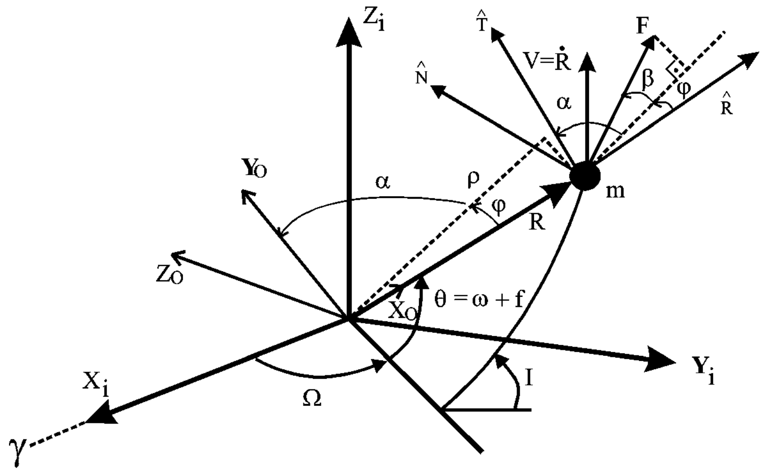

Figure 2.1. Reference systems used in this work.

2 Mathematical Formulation and Coordinates Systems

The orbital transfer problem studied can be formulated in the following way:

1) Globally minimize the performance index: $J = m(t_0) - m(t_f)$;
2) With respect to α: $[t_0, t_f] \to R$ ("pitch" angle) and β: $[t_0, t_f] \to R$ ("yaw" angle) with α, $\beta \in C^{-1}$ in $[t_0, t_f]$;
3) Subject to the dynamics in inertial coordinates X_i, Y_i, Z_i of Figure 2.1: $\forall t \in [t_0, t_f]$;

$$m(t)\,\frac{d^2X}{dt^2} = -\mu m(t)\,\frac{X}{R^3} + F_x, \tag{1}$$

$$m(t)\,\frac{d^2Y}{dt^2} = -\mu m(t)\,\frac{Y}{R^3} + F_y, \tag{2}$$

$$m(t)\,\frac{d^2Z}{dt^2} = -\mu m(t)\,\frac{Z}{R^3} + F_z, \tag{3}$$

$$\begin{aligned} F_x = F\big[&\cos\beta\sin\alpha(\cos\Omega\cos\theta - \sin\Omega\cos I\sin\theta) + \sin\beta\sin\Omega\sin I \\ &- \cos\beta\cos\alpha(\cos\Omega\sin\theta + \sin\Omega\cos I\cos\theta)\big] \end{aligned} \tag{4}$$

$$\begin{aligned} F_y = F\big[&\cos\beta\sin\alpha(\sin\Omega\cos\theta + \cos\Omega\cos I\sin\theta) - \sin\beta\cos\Omega\sin I \\ &- \cos\beta\cos\alpha(\sin\Omega\sin\theta - \cos\Omega\cos I\cos\theta)\big], \end{aligned} \tag{5}$$

$$F_z = F(\cos\beta\sin\alpha\sin I\sin\theta + \cos\beta\cos\alpha\sin I\cos\theta + \sin\beta\cos I), \tag{6}$$

$$m(t) = m(t_0) + \dot{m}(t - t_0) \tag{7}$$

with $\dot{m} < 0$

$$F \cong |\dot{m}|c. \tag{8}$$

Or in orbital coordinates (radial R, transverse T, and binormal N) of Figure 2.1:

$$m(t)a_R(t) = F\cos\beta(t)\sin\alpha(t) - \frac{\mu m(t)}{R^2(t)}, \tag{9}$$

$$m(t)a_T(t) = F\cos\beta(t)\cos\alpha(t), \tag{10}$$

$$m(t)a_N(t) = F\sin\beta(t), \tag{11}$$

$$a_R(t) = \dot V_R - \frac{V_T^2}{R} - \frac{V_N^2}{R}, \tag{12}$$

$$a_T(t) = \dot V_T + \frac{V_R V_T}{R} - V_N \dot I\cos\theta - V_N\dot\Omega\sin I\sin\theta, \tag{13}$$

$$a_N(t) = \dot V_N + \frac{V_R V_N}{R} + V_T \dot I\cos\theta + V_T\dot\Omega\sin I\sin\theta, \tag{14}$$

$$V_R = \dot R, \tag{15}$$

$$V_T = R(\dot\Omega\cos I + \dot\theta), \tag{16}$$

$$V_N = R(-\dot\Omega\sin I\cos\theta + \dot I\sin\theta), \tag{17}$$

$$\theta = \omega + f; \tag{18}$$

4) Given the initial and final orbits, and the parameters of the problem $(m(t_0), c, \dots)$. These equations were obtained by: 1) writing in coordinates of the dexterous rectangular reference system with inertial directions $0X_iY_iZ_i$ the Newton's laws for the motion of a satellite S with mass m, with respect to this reference system, centered in the Earth's center of mass 0 with X_i axis toward the Vernal point, X_iY_i plane coincident with Earth's Equator, and Z_i axis toward the Polar Star approximately; 2) rewriting them in coordinates of the dexterous rectangular reference system with radial, transverse, binormal directions $SRTN$, centered in the satellite center of mass S; helped by 3) a parallel system with $0X_0Y_0Z_0$ directions, centered in the Earth's center of mass 0, X_0 axis toward the satellite S, X_0Y_0 plane coincident with the plane established by the position $\vec R$ and velocity $\vec V$ vectors of the satellite, and Z_0 axis perpendicular to this plane; and helped by 4) the instantaneous Keplerian coordinates $(\Omega, I, \omega, f, a, e)$. These equations were later rewritten and simulated by using 5) 9 state variables, defined and used by Biggs [19, 20] and Prado [21], as functions of an independent variable s shown in Figure 2.2.

The non-ideal thrust vector, with magnitude and direction errors, is given by:

$$\vec{F_E} = \vec F + \Delta\vec F, \tag{19}$$

$$\vec{F_E} = \vec{F_R} + \vec{F_T} + \vec{F_N}, \tag{20}$$

$$|\vec{F_E}| = F_E, \quad |\vec F| = F, \tag{21}$$

$$F_R = (F+\Delta F)\cos(\beta+\Delta\beta)\sin(\alpha+\Delta\alpha), \tag{22}$$

$$F_T = (F+\Delta F)\cos(\beta+\Delta\beta)\cos(\alpha+\Delta\alpha), \tag{23}$$

$$F_N = (F+\Delta F)\sin(\beta+\Delta\beta), \tag{24}$$

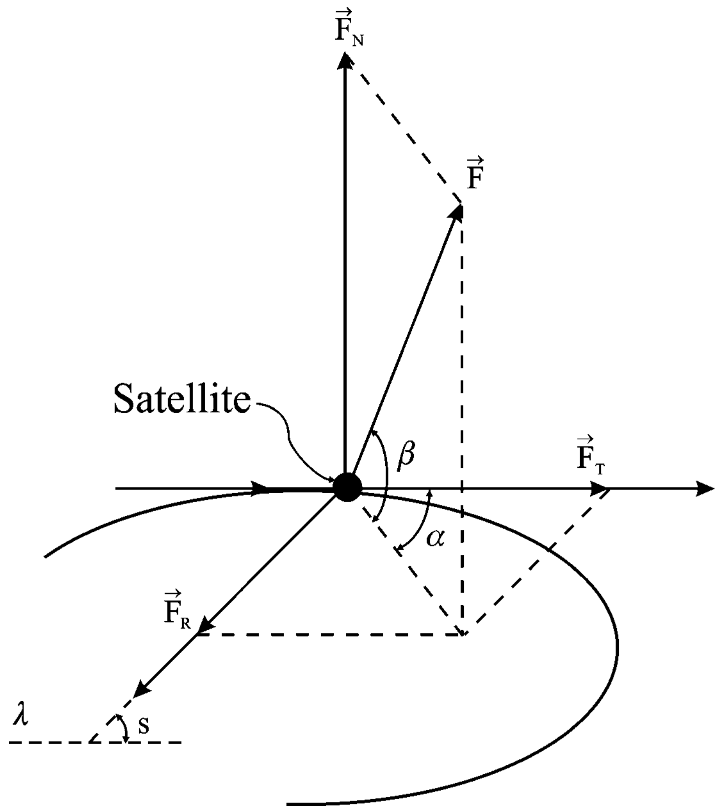

Figure 2.2. Thrust vector applied to the satellite and the s variable.

where: $\vec{F}$, $\vec{F_E}$ and $\Delta\vec{F}$ are: the thrust vector without errors, the thrust vector with errors, and the error in the thrust vector, respectively; $\Delta\alpha$ and $\Delta\beta$ are the errors in the "pitch" and in the "yaw" angles, respectively; F_R, F_T and F_N are the components of the thrust vector with errors $\vec{F_E}$ in the radial, transverse and normal directions, respectively. In this way, for each implementation of the orbital transfer arc, values of α and β are chosen, whose errors are inside the range, that produce the direction for the overall minimum fuel consumption.

3 Transfers with Errors in the Thrust Vector: Algebraic Analysis

We start our algebraic analysis by planar $(\alpha \neq 0)$ and $(\beta = 0)$ transfer maneuvers. We also choose F and m constants. Under these hypotheses, Equations (9) − (14) become:

$$F_t = m\dot{v}_t(t) = F\cos(\alpha(t)) - mv_r(t)\dot{f}(t), \tag{25}$$

$$F_r = m\dot{v}_r(t) = F\sin(\alpha(t)) + mv_t(t)\dot{f}(t) - \frac{\mu m(t)}{r^2(t)}, \tag{26}$$

$$\dot{f}(t) = \frac{v_t(t)}{r(t)}, \tag{27}$$

$$\dot{r}(t) = v_r(t), \tag{28}$$

with, F_t and F_r the transverse and radial components of the thrust vector, respectively; $\dot{v}_t(t)$, $\dot{v}_r(t)$ the transverse and radial components of the accelerations, respectively; $v_t(t)$, $v_r(t)$ the transverse and radial components of the velocities, respectively; $\dot{f}(t)$ the angular velocity; $r(t)$ the vector position between satellite and central body.

Our algebraic approach for the semi-major axis deviations is done through the rate of change of the satellite mechanical energy, which is equal the power applied by forces components in the transverse and radial directions. Their energy rate of change are:

$$\frac{d[E_c(t)]_r}{dt} = mv_r(t)\dot{v}_r(t), \tag{29}$$

$$\frac{d[E_c(t)]_t}{dt} = mv_t(t)\dot{v}_t(t), \tag{30}$$

$$\frac{dE_p(t)}{dt} = \frac{\mu m(t)v_r(t)}{r^2(t)}. \tag{31}$$

Adding these equations we obtain the rate of change of the satellite mechanical energy, E_M without "pitch" error,

$$\frac{dE_M(t)}{dt} = F\cos\alpha(t)v_t(t) + F\sin\alpha(t)v_r(t) \tag{32}$$

or, during the time interval Δt,

$$\begin{aligned} \Delta E_M(t_1,t_2) &= E_M(t_2) - E_M(t_1) \\ &= \int_{t_1}^{t_2} F\cos\alpha(t)v_t(t) + \sin\alpha(t)v_r(t)\,dt = \frac{-\mu m}{2a(t_2)} + \frac{\mu m}{2a(t_1)}, \end{aligned} \tag{33}$$

with $a(t_i)$ the semi-major axis of the satellite orbit of the instant i.

Equation (33) for one transfer under "pitch" error, $\Delta\alpha(t)$ is,

$$\begin{aligned} \Delta E'_M(t_1,t_2) &= E'_M(t_2) - E'_M(t_1) \\ &= \int_{t_1}^{t_2} F(\cos[\alpha(t)+\Delta\alpha(t)]v'_t(t))\,dt + \int_{t_1}^{t_2} F(\sin[\alpha(t)+\Delta\alpha(t)]v'_r(t))\,dt. \end{aligned} \tag{34}$$

Taking the difference between Equations (33) and (34), we obtain,

$$\begin{aligned} \Delta_2 E_M(t_1,t_2) &\equiv \Delta E'_M(t_1,t_2) - \Delta E_M(t_1,t_2) \\ &= \frac{-\mu m}{2a'(t_2)} + \frac{\mu m}{2a'(t_1)} + \frac{\mu m}{2a(t_2)} + \frac{-\mu m}{2a(t_1)} \\ &= \int_{t_1}^{t_2} F(\cos[\alpha(t)+\Delta\alpha(t)]v'_t(t) - \cos\alpha(t)v_t(t))\,dt \\ &+ \int_{t_1}^{t_2} F(\sin[\alpha(t)+\Delta\alpha(t)]v'_r(t) - \sin\alpha(t)v_r(t))\,dt. \end{aligned} \tag{35}$$

If we use the fact that the semi-major axis of the initial and final orbits in the initial instant are equal, and doing some algebraic manipulation, taking the expectation ($\mathcal{E}$), of the final equation, we have,

$$\begin{aligned}\mathcal{E}[\Delta_2 E_M(t_1,t_2)] &= \mathcal{E}\left[\int_{t_1}^{t_2} F\{\cos\alpha(t)[\cos\Delta\alpha(t)-1] - \sin\alpha(t)\sin\Delta\alpha(t)\}v_t'(t)\,dt\right] \\ &+ \mathcal{E}\left[\int_{t_1}^{t_2} F\{\sin\alpha(t)[\cos\Delta\alpha(t)-1] + \cos\alpha(t)\sin\Delta\alpha(t)\}v_r'(t)\,dt\right] \\ &+ \mathcal{E}\left[\int_{t_1}^{t_2} F\{\cos\Delta\alpha(t)\}\{v_t'(t)-v_t(t)\}dt + F\{\sin\Delta\alpha(t)\}\{v_r'(t)-v_r(t)\}dt\right].\end{aligned} \tag{36}$$

Now, we consider that the stochastic processes are ergodic. So, the expectation operator (mean in the ensemble) commutes with the integral operator (in time). Besides this, the function F and the trigonometric functions are deterministic in time. Therefore, we evaluate the mean through the ensemble for equation (36),

$$\begin{aligned}\mathcal{E}[\Delta_2 E_M(t_1,t_2)] &= \int_{t_1}^{t_2} F\cos\alpha(t)\mathcal{E}[[\cos\Delta\alpha(t)-1]v_t'(t)]\,dt \\ &- \int_{t_1}^{t_2} F\sin\alpha(t)\mathcal{E}[\sin\Delta\alpha(t)v_t'(t)]\,dt + \int_{t_1}^{t_2} F\sin\alpha(t)\mathcal{E}[[\cos\Delta\alpha(t)-1]v_r'(t)]\,dt \\ &+ \int_{t_1}^{t_2} F\cos\alpha(t)\mathcal{E}[\sin\Delta\alpha(t)v_r'(t)]\,dt + \int_{t_1}^{t_2} F\cos\alpha(t)\mathcal{E}[v_t'(t)-v_t(t)]dt \\ &+ \int_{t_1}^{t_2} F\sin\alpha(t)\mathcal{E}[v_r'(t)-v_r(t)]\,dt.\end{aligned} \tag{37}$$

Equation (37) is general for any probability distribution function to $\Delta\alpha(t)$ and for any kind of noise, that is, “white-noise”, “‘pink-noise”, or other. But, we must define if the variables inside the integral in equation (37) are correlated or not, to evaluate the expectation, as follows:

4 Case 1: $\Delta\alpha(t)$ Not Correlated with Transverse and Radial Velocities (White-Noise), Uniform Errors

In this case, we decompose the expectation operator as one product of the individual expectations for the trigonometric functions of the $\Delta\alpha(t)$ and the velocities components, because they are not correlated. For the $\Delta\alpha(t)$ with uniform distribution inside the interval $[-\Delta\alpha_{max}, \alpha_{max}]$, we have,

$$\begin{gathered}
\mathcal{E}\{[\cos\Delta\alpha(t_1) - 1]v_t'(t_1)\} = \mathcal{E}\{[\cos\Delta\alpha(t_1) - 1]\}\mathcal{E}\{v_t'(t_1)\} \\
= v_t(t_1)\mathcal{E}\{[\cos\Delta\alpha(t_1) - 1]\} = \{\mathcal{E}[\cos\Delta\alpha(t_1)] - \mathcal{E}(1)\}v_t(t_1) \\
= v_t(t_1)\,\frac{1}{2\Delta\alpha_{max}}\left[\int\limits_{-\Delta\alpha_{max}}^{\Delta\alpha_{max}} d(\Delta\alpha)\cos\Delta\alpha - 1\right] \\
= \frac{1}{2\Delta\alpha_{max}}\,[\sin\Delta\alpha\big|_{-\Delta\alpha_{max}}^{\Delta\alpha_{max}} - 1]v_t(t_1) = v_t(t_1)\left[\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\right]
\end{gathered} \tag{38}$$

and,

$$\mathcal{E}\{[\cos\Delta\alpha(t_1) - 1]v_r'(t_1)\} = v_r(t_1)\left[\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\right] \tag{39}$$

with,

$$\begin{gathered}
\mathcal{E}\{[\sin\Delta\alpha(t_1)]v_t'(t_1)\} = \mathcal{E}\{[\sin\Delta\alpha(t_1)]\mathcal{E}[v_t'(t_1)]\} \\
= v_t(t_1)\,\frac{1}{2\Delta\alpha_{max}}\int\limits_{-\Delta\alpha_{max}}^{\Delta\alpha_{max}} d(\Delta\alpha)\sin\Delta\alpha \\
= v_t(t_1)\left[\frac{1}{2\Delta\alpha_{max}}\right][\cos\Delta\alpha]_{-\Delta\alpha_{max}}^{\Delta\alpha_{max}} = 0
\end{gathered} \tag{40}$$

and,

$$\mathcal{E}[\sin\Delta\alpha(t_1)]v_r'(t_1) = 0. \tag{41}$$

We consider that the velocities effects inside the internal $[-\Delta\alpha_{max}, \Delta\alpha_{max}]$ in the same time are, practically, balanced, because the deviations occur between values maxima and minima inside them. That is, the velocities with errors and without them are very close values. So,

$$\mathcal{E}\{v_{t,r}'(t)\} = v_{t,r}(t_1). \tag{42}$$

With these results equation (37) becomes,

$$\begin{aligned}
\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = &\int\limits_{t_1}^{t_2} F\cos\alpha(t)v_t(t)\left\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\right\}dt \\
&+ \int\limits_{t_1}^{t_2} F\sin\alpha(t)v_r(t)\left\{\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\right\}dt.
\end{aligned} \tag{43}$$

In other hand, we have,

$$\mathcal{E}\{\Delta_2 E_M(t_1,t_2)\} = \mathcal{E}\left\{\frac{\mu m}{2a(t_2)} - \frac{\mu m}{2a'(t_2)}\right\} = \frac{\mu m}{2}\,\frac{1}{a(t_2)}\,\mathcal{E}\left\{\frac{\Delta a(t_2)}{a'(t_2)}\right\} \tag{44}$$

with,

$$\Delta a(t_2) = a'(t_2) - a(t_2). \tag{45}$$

If we expand equation (44) about the rate $\dfrac{\Delta a(t_2)}{a(t_2)}$, we get:

$$\begin{aligned}
&\frac{\mu m}{2}\left[\frac{1}{a^2(t_2)}\,\mathcal{E}\{\Delta a(t_2)\} - \frac{1}{a^3(t_2)}\,\mathcal{E}\{\Delta^2 a(t_2)\}\right.\\
&\left.+\frac{1}{a^4(t_2)}\,\mathcal{E}\{\Delta^3 a(t_2)\} - \frac{1}{a^5(t_2)}\,\mathcal{E}\{\Delta^4 a(t_2)\} + \ldots\right]\\
&= \frac{\mu m}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\,\mathcal{E}\{\Delta^n a(t_2)\}.
\end{aligned} \tag{46}$$

We can compare equations (46) and (43), getting:

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\,\mathcal{E}\{\Delta^n a(t_2)\} = K_1\left[\frac{\sin\Delta\alpha_{max}}{\Delta\alpha_{max}} - 1\right]\\
&= K_1\left[-\frac{1}{3!}\,\Delta^2\alpha_{max} + \frac{1}{5!}\,\Delta^4\alpha_{max} - \frac{1}{7!}\,\Delta^6\alpha_{max} + \ldots\right]\\
&= K_1\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\,\Delta^{2n}\alpha_{max}
\end{aligned} \tag{47}$$

with,

$$K_1 = \frac{2}{\mu}\frac{(Q_1+Q_2)}{m}, \tag{48}$$

where Q_1 and Q_2 are quadratures.

Equation (47) describes a sequence of even power terms for the maximum deviation in "pitch" with respect the expected values of the semi-major axis. For $n = 1$, we have,

$$\mathcal{E}\{\Delta a(t_2)\} = -\frac{1}{3!}\,\Delta^2\alpha_{max}K_1 a^2(t_2) = -\frac{1}{3!}\,\Delta^2\alpha_{max}K_2, \tag{49}$$

$$K_2 = K_1 a^2(t_2). \tag{50}$$

This result shows that in first order the cause/effect relationship is parabolic. But that the general curve would be a composition of all even power terms.

5 Case 2: $\boldsymbol{\Delta\alpha(t)}$ Not Correlated with Transverse and Radial Velocities (White-Noise), Gaussian Errors

The procedures for the $\Delta\alpha(t)$ with Gaussian distribution inside the interval $[-\Delta\alpha_{max}, \Delta\alpha_{max}]$ are the same for the uniform distribution. So,

$$
\begin{aligned}
[\mathcal{E}\{\cos\Delta\alpha(t_1)\}-1]v_t(t_1) &= v_t(t_1)\left\{\int_{-\infty}^{\infty}\cos[\Delta\alpha]\,d(\Delta\alpha)\,\frac{e^{-\frac{(\Delta\alpha)}{2\sigma_\alpha}}}{\sqrt{2\pi}\sigma_\alpha}-1\right\} \\
&= v_t(t_1)\left\{e^{\frac{-\sigma_\alpha^2}{2}}-1\right\} = v_t(t_1)\left\{-\frac{\sigma_\alpha^2}{2}+\frac{\sigma_\alpha^4}{8}-\frac{\sigma_\alpha^6}{48}+\dots\right\},
\end{aligned}
\tag{51}
$$

$$
\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\,\mathcal{E}\{\Delta^n a(t_2)\} = K_1\sum_{n=1}^{\infty}(-1)^n\frac{\sigma_\alpha^{2n}}{2^n n!}. \tag{52}
$$

The form of the curve in equation (52) is similar that in equation (47). That is, there is a clear nonlinear relationship between cause $(\Delta\alpha_{max}=\sqrt{3}\sigma_\alpha)$ and effect $(\Delta a(t_2))$. For $n=1$, we have,

$$
\mathcal{E}\{\Delta a(t_2)\} = -\frac{1}{6}\,\sigma_\alpha^2 K_2. \tag{53}
$$

6 Case 3: $\boldsymbol{\Delta\alpha(t)}$ Correlated with Transverse and Radial Velocities (Pink-Noise), Uniform Errors

In this case, we cannot decompose the expectation operator as a product of the individual expectations for the trigonometric functions of the $\Delta\alpha(t)$ and the velocities components, because now they are correlated. The procedures are the same done until this point, except that we must evaluate the expectation of the products of the different variables, without decomposing them. Besides this, we consider the $\Delta\alpha(t)$ random-bias deviations, that is, $\Delta\alpha(t)=\text{constant}=\Delta\alpha(t_1)=\Delta\alpha$. After many mathematical manipulations we found the following equation, for both cases, uniform and Gaussian distribution,

$$
I_{r,t} = \int_{t_2}^{t_1}\mathcal{E}\{(\cos\Delta\alpha)\dot{v}'_{r,t}(t)\dot{f}'(t)\}\,dt. \tag{54}
$$

We know that the integral of the odd functions for symmetrical distributions is null. But equation (54) has an even product of the functions. The odd functions inside the product are not known, but we can modeled its product as one even function, for example, $\cos\Delta\alpha$.

Other important approach in this way is to consider for equation (26) that the expectation of the product is equal the product of the expectations of the functions, so that,

$$
\mathcal{E}\left\{\frac{\cos(\Delta\alpha)}{r'^2(t)}\right\} = \mathcal{E}\left\{\cos(\Delta\alpha)\,\frac{1}{r'^2(t)}\right\} \cong \mathcal{E}\{\cos(\Delta\alpha)\}\mathcal{E}\left\{\frac{1}{r'^2(t)}\right\} = \frac{\mathcal{E}\{\cos(\Delta\alpha)\}}{r^2(t)}. \tag{55}
$$

The final forms are:

$$
\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\,\mathcal{E}\{\Delta^n a(t_2)\} = \lambda_1-\lambda_2\Delta^2\alpha_{max}+\lambda_3\Delta^4\alpha_{max}-\dots \tag{56}
$$

for the uniform case and,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}\,\mathcal{E}\{\Delta^n a(t_2)\} = \lambda_4 - \lambda_5\sigma_\alpha^2 + \lambda_6\sigma_\alpha^4 - \lambda_7\sigma_\alpha^6 + \dots \tag{57}$$

for the Gaussian case, where the coefficients are

$$\lambda_1 = Q_8[Q_5 + Q_6 - v_t(t_1)] + Q_{12}[Q_{10} + Q_3 - v_r(t_1)], \tag{58}$$

$$\lambda_2 = \left\{[2Q_3Q_{12} - Q_8Q_4 + Q_8Q_5]\left[\frac{1}{2}\frac{1}{2!} + \frac{1}{2}\frac{1}{3!}\right] + [Q_{10}Q_{12} + Q_8Q_6 + Q_8Q_5]\frac{1}{3!}\right\}, \tag{59}$$

$$\lambda_3 = \left\{[-Q_8Q_4 + Q_8Q_5]\left[\frac{1}{2}\frac{1}{2!}\frac{1}{3!} + \frac{1}{2}\frac{1}{4!}\right] + \frac{1}{3!}[Q_6 + Q_{10}Q_{12}] + \frac{Q_8Q_5}{7!} - \frac{Q_{12}Q_3}{2}\frac{1}{5!}\right\}, \tag{60}$$

$$\lambda_4 = Q_8[Q_6 - v_t(t_1)] + Q_{12}[Q_{11} - v_r(t_1)], \tag{61}$$

$$\lambda_5 = Q_{12} + \frac{Q_8Q_6}{2} - Q_8Q_4 + Q_{12}Q_{11,1}, \tag{62}$$

$$\lambda_6 = Q_{12} + \frac{Q_8Q_6}{8} - Q_8Q_4 + Q_{12}Q_{11,2}, \tag{63}$$

$$\lambda_7 = \frac{2}{3}Q_{12} + \frac{Q_8Q_6}{48} - \frac{2}{3}Q_8Q_4 + Q_{12}Q_{11,3}. \tag{64}$$

The Q_{ij} functions are quadratures. The first order for both cases are:

$$\mathcal{E}\{\Delta a(t_2)\} = \lambda_1 a^2(t_2) - \lambda_2 a^2(t_2)\Delta^2\alpha_{max} \tag{65}$$

for the uniform case and,

$$\mathcal{E}\{\Delta a(t_2)\} = \lambda_4 a^2(t_2) - \lambda_5 a^2(t_2)\sigma_\alpha^2, \tag{66}$$

for the Gaussian case.

These results show once more the nonlinear relationship between cause and effect. The terms $\lambda_1 a^2(t_2)$ and $\lambda_4 a^2(t_2)$ are constants and do not change the general form of the curves. We can compare both results of the deviations (uniform and Gaussian) by relating,

$$\Delta\alpha_{max} = \sqrt{3}\sigma_\alpha. \tag{67}$$

If we replace this equation inside equation (47), we conclude that:

(a) for the first order the results are the same, for the same σ_α;

(b) for other orders, the Gaussian semi-major axis deviations are $\dfrac{(2n+1)!}{6^n n!}$ greater than the uniform deviations, for the same σ_α.

7 Transfers with Errors in the Thrust Vector: Numerical Analysis

The numerical results confirm the algebraic results obtained. We simulated (Monte-Carlo) 1000 ensembles of the transfer trajectories for both kind of deviations (uniform-U, Gaussian-G), for both maneuvers ("theoretical"-T, "practical"-P), for random bias (S) and white noise (O) deviations. Figures 7.1 and 7.2 show $\mathcal{E}[a(t_2)]$ for cases TUS, TUO, TGS, TGO, and PUS, PUO, PGS, PGO, respectively.

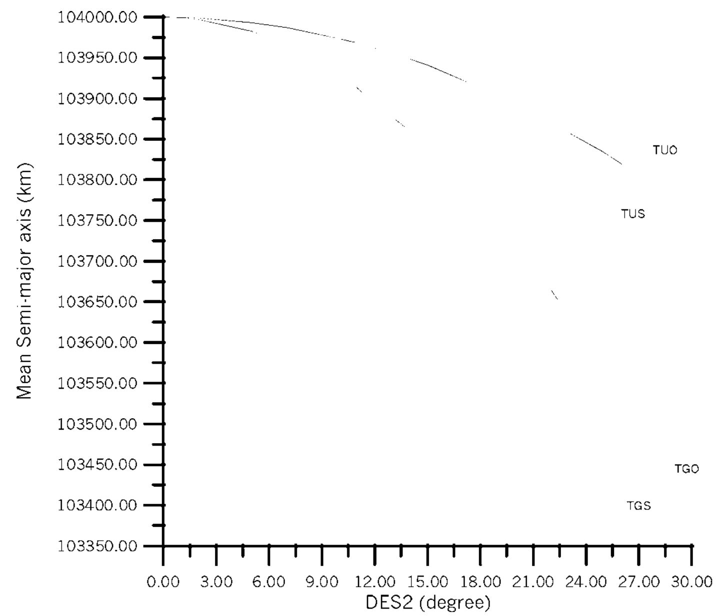

Figure 7.1. Mean semi-major axis × DES2, Theoretical Orbits.

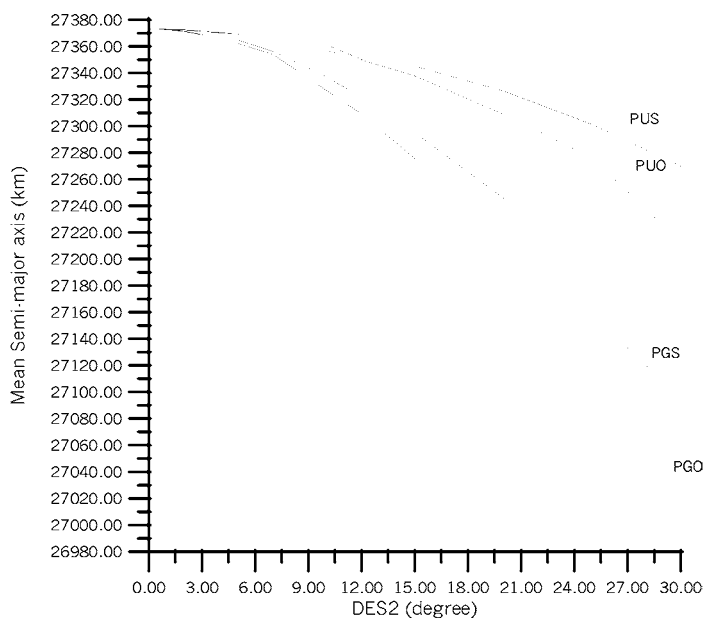

Figure 7.2. Mean semi-major axis × DES2, Practical Orbits.

In these figures $DES2 = \sqrt{3}\sigma_{\Delta\alpha}$, where $\sigma_{\Delta\alpha}$ is the pitch angle standard deviation for zero mean. We can observe clearly the nonlinear shapes of the curves like parabolas. The numerical results for the relation between uniform and Gaussian deviations confirms equation (67). Figures 7.3, 7.4, 7.5 and 7.6 show that the Gaussian deviations (ΔG) are more than the uniform deviations (ΔU).

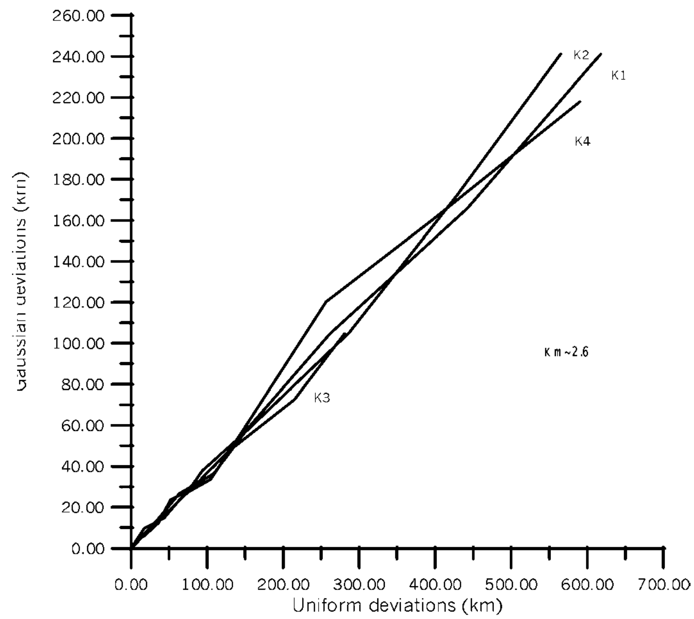

Figure 7.3. $\Delta G \times \Delta U$: Theoretical and Practical Cases, Systematic Errors.

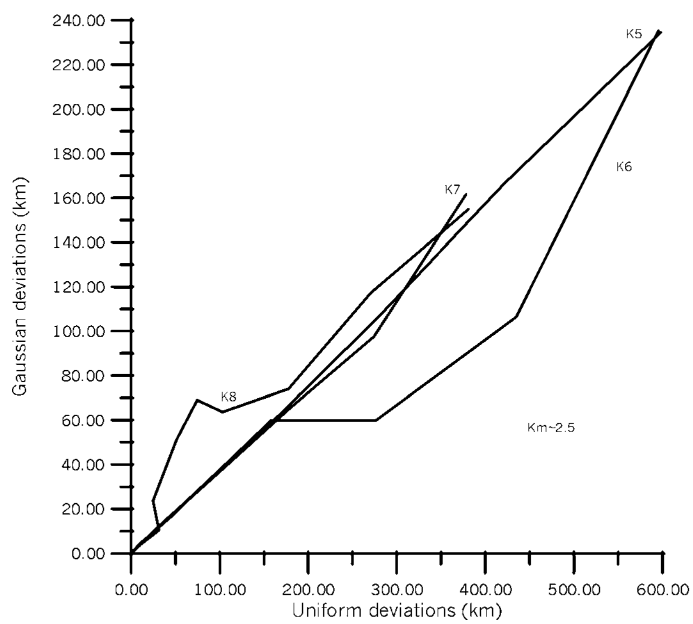

Figure 7.4. $\Delta G \times \Delta U$: Theoretical and Practical Cases, Operational Errors.

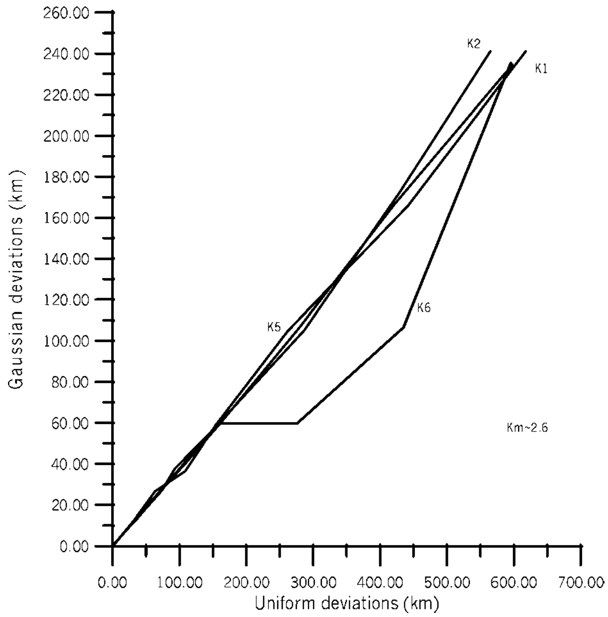

Figure 7.5. $\Delta G \times \Delta U$: Theoretical Case, Systematic and Operational Errors.

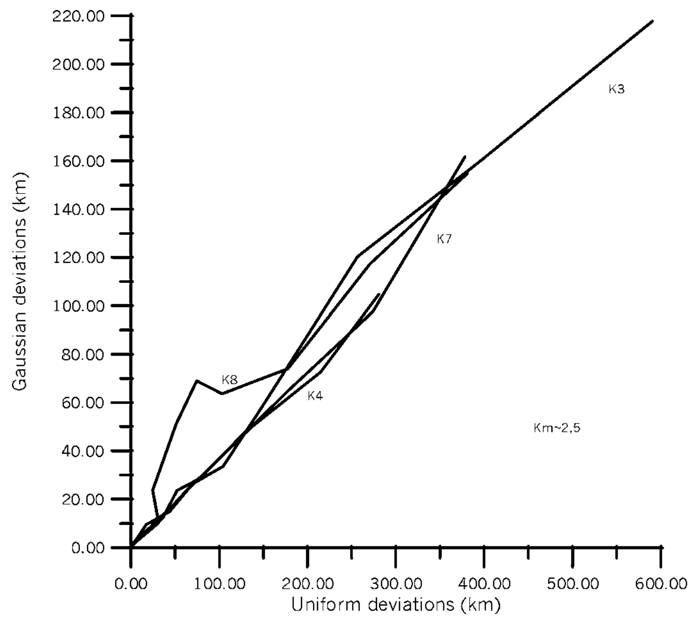

Figure 7.6. $\Delta G \times \Delta U$: Practical Case, Systematic and Operational Errors.

The mean linear coefficient between them is 2.6 in all cases: TUS, TUO, TGS, TGO and PUS, PUO, PGS, PGO. In theses graphics we introduced the numerical results of the out-plane angle deviations, that is, "yaw" angle deviations, DES3. The linear coefficients for these angle deviation are: k_2, k_4, k_6 and k_8, while for the "pitch" angle deviation,

k_1, k_3, k_5 and k_7. The algebraic results in equation (67) anticipated the value 3 (≈ 2.6 for numerical results). This shows consistency of our results.

8 Conclusions

In the algebraic developments, we obtained expression for $\mathcal{E}\{\Delta a(t_2)\}$ as series of even powers of $\sigma_{\Delta\alpha}$ dominated by the $(\sigma_{\Delta\alpha})^2$ term, to explain the near parabolic relations and others found, independent of the: 1) transfer orbit ("theoretical" or "practical"); 2) ensemble distribution (uniform or Gaussian); 3) time correlation/dependence (random-bias or white-noise). These results suggest and partially characterizes the progressive deformation of the trajectory distribution along the propulsive arc, turning 3-sigma ellipsoids into "banana" shaped volumes curved to the center of attraction (we call them "bananoids") due to the loss of optimality of the actual (with errors) trajectories with respect to the nominal (no errors) trajectory. A similar deformation but due to: a) the mean drag was studied by Carlton-Wippern [15]; b) initial condition Gaussian errors was shown by Junkins [14]. As his plots also suggest, such deformations can not be anticipated by covariance analysis ([9, 10, 12]) on linearized models with zero mean errors which propagate ellipsoids into ellipsoids always centered in the nominal (no errors) trajectory. Those results also characterize how close/far are Monte-Carlo analysis and covariance analysis for those examples. Other details about our numerical results can be found in Jesus [23].

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Stability of the Stationary Solutions of the Differential Equations of Restricted Newtonian Problem with Incomplete Symmetry

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Received: July 10, 2002; Revised: December 23, 2002

Abstract: We investigate the Lyapunov stability of the stationary solutions of the differential equations of restricted six-body problem with the gravitational centre. The gravitational field is created by bodies P_0, P_1, P_2, P_3 and P_4 with masses m_0, m_1, m_2, m_3 and m_4, respectively. In this gravitational field the movement of a body P with zero mass $(m = 0)$ is investigated. The bodies P_1, P_2, P_3 and P_4 form a rhombus, rotating uniformly around the centre of gravity P_0. In the article we have formulated necessary and sufficient conditions of Lyapunov stability and instability of equilibrium point of this model. All necessary analytical calculations are executed in the system of symbolical calculations (SSC) "Mathematica".

Keywords: *Hamiltonian systems; stability.*

Mathematics Subject Classification (2000): 37J25, 37J40.

1 Introduction

It is known, that the restricted Newtonian many-body problem is very important for a wide class of applications, from theoretical physics to celestial mechanics and astrodynamics [1, 6]. It is well known [4, 5], that the differential equations of this problem are in general not integrable, therefore Poincaré considered the first problem should be the search for the exact particular solutions and the research of their stability [1]. The latter problem is the most difficult in the qualitative theory of the differential equations and can be solved within the framework of the Kolmogorov-Arnold-Mozer () theory [12, 13].
With occurrence of the systems of symbolic calculations, for example, Mathematica [10], possibilities of performance of symbolic calculations have essentially increased. Such calculations are necessary for correct application of the well known Arnold-Mozer theorem [13, 15]. Let's consider the following restricted 6-body problem in Grebenikov-Elmabsout

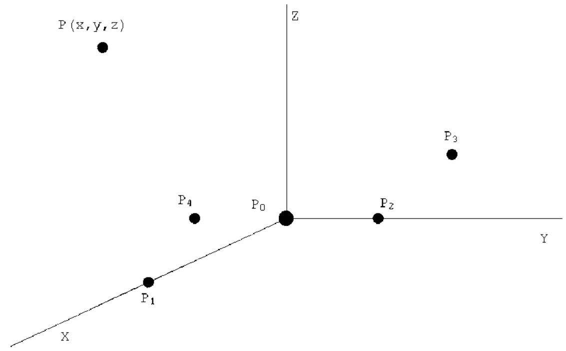

Figure 1.1

model [3, 7]. In the non-inertial Euclidean space P_0xyz there are six bodies P_0, P_1, P_2, P_3, P_4 and P with masses m_0, m_1, m_2, m_3, m_4 and m. It is shown, that in this model the bodies P_1, P_2, P_3 and P_4 move in one plane and form a rhombus, rotating uniformly around a body P_0 [2]. In this gravitational newtonian field, produced by mutual gravitation of five bodies, we investigate the motion of a body P with zero mass $m = 0$ (Figure 1.1).

The purpose of our work is the definition of the stationary solutions (states of equilibrium) of differential equations, describing this model, and the research of their Lyapunov stability by methods of computer algebra. It has been proved, that exact rhombus-like solutions do exist in this physical model, if the following conditions are executed [8]: a) the masses, located in the opposite vertices of a rhombus, are equal among themselves:

$$m_1 = m_3; \qquad m_2 = m_4; \tag{1}$$

b) relations of diagonals ρ_1, ρ_2, and masses m_1, m_2 of a rhombus are correlated as:

$$\lambda = \frac{\rho^3\left[8 - (1+\rho^2)^{3/2}\right]}{8\rho^3 - (1+\rho^2)^{3/2}}, \tag{2}$$

where $\dfrac{\rho_1}{\rho_2} = \rho$, $\dfrac{m_1}{m_2} = \lambda$.

2 Definition of Equilibrium State

Without loss of generality, it is possible to assume, that the gravitational rhombus rotates always in a plane P_0XY around an axis Z with a constant angular velocity ω.

It is obvious, that the sizes of a rhombus can be arbitrary, therefore we shall define coordinates of a rhombus as follows: $P_1(\alpha, 0)$, $P_2(0, 1)$, $P_3(-\alpha, 0)$, $P_4(0, -1)$.

In [9] it is shown that

$$m_2 = \frac{4m_0(1+\alpha^2)^{3/2}(\alpha^3 - 1) + m_1(8\alpha^3 - (1+\alpha^2)^{3/2})}{\alpha^3(8 - (1+\alpha^2)^{3/2})}, \tag{3}$$

therefore from conditions $\alpha > 0,\ m_0 > 0,\ m_1 > 0,\ m_2 > 0$ we receive admissible values of the parameter α: $1/\sqrt{3} < \alpha < \sqrt{3}$. For $\alpha \geq 1$ the masses can take any values and in the range $1/\sqrt{3} < \alpha < 1$ the relation

$$m_1 > \frac{4(1+\alpha^2)^{3/2}(1-\alpha^3)}{8\alpha^3 - (1+\alpha^2)^{3/2}}\, m_0 \tag{4}$$

should be satisfied. The angular velocity of rotation of the rhombus $P_1P_2P_3P_4$ is defined by the formula [2]

$$\omega = \sqrt{\frac{4fm_0(1+\alpha^2)^{3/2}(8\alpha^3 - (1+\alpha^2)^{3/2}) + fm_1(64\alpha^3 - (1+\alpha^2)^{3/2})}{4\alpha^3(1+\alpha^2)^{3/2}(8-(1+\alpha^2)^{3/2})}}, \tag{5}$$

where f is a gravitation constant.

Further for convenience we shall consider, that $f = 1$ and $m_0 = 0$.

The differential equations of motion of passive gravitating point P $(m = 0)$ in uniformly rotating Cartesian frame P_0XYZ are [4]:

$$\begin{aligned}
\frac{d^2X}{dt^2} &= \omega^2 X + 2\omega\,\frac{dY}{dt} - \frac{X}{r^3} + \frac{\partial R}{\partial X},\\
\frac{d^2Y}{dt^2} &= \omega^2 Y 2\omega\,\frac{dX}{r^3} - \frac{Y}{r^3} + \frac{\partial R}{\partial Y},\\
\frac{d^2Z}{dt^2} &= -\frac{Z}{r^3} + \frac{\partial R}{\partial Z},
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
R &= \sum_{j=1}^{4} m_j\left(\frac{1}{\Delta_j} - \frac{XX_j + YY_j + ZZ_j}{r_j^3}\right),\\
\Delta_j^2 &= (X-X_j)^2 + (Y-Y_j)^2 + (Z-Z_j)^2,\\
r^2 &= X^2 + Y^2 + Z^2, \qquad r_j^2 = X_j^2 + Y_j^2 + Z_j^2,
\end{aligned} \tag{7}$$

$X,\ Y,\ Z$ are the coordinates of the zero mass (point P), $X_j,\ Y_j,\ Z_j$ are the given coordinates of points P_j, ω is the angular velocity of rotation of the rhombus $P_1P_2P_3P_4$ around P_0. System (6) is not integrable in a general form, therefore we shall search for partial solutions, such as "equilibrium state". For this purpose we shall introduce a 6-dimensional phase space $x = X,\ y = Y,\ z = Z,\ u = \dfrac{dX}{dt},\ v = \dfrac{dY}{dt},\ w = \dfrac{dZ}{dt}$. Then the system (6) becomes

$$\begin{aligned}
&\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w,\\
&\frac{du}{dt} = \omega^2 x + 2\omega v - \frac{x}{r^3} + \frac{\partial R}{\partial x},\\
&\frac{dv}{dt} = \omega^2 y - 2\omega u - \frac{y}{r^3} + \frac{\partial R}{\partial y},\\
&\frac{dw}{dt} = -\frac{z}{r^3} + \frac{\partial R}{\partial z}.
\end{aligned} \tag{8}$$

Finding of equilibrium state of system (8) is reduced to the solution of the system of equations

$$
\begin{gathered}
u=0, \quad v=0, \quad ,w=0, \\
\omega^2 x-\frac{x}{(x^2+y^2+z^2)^{3/2}}-\left(\frac{m_1(x-\alpha)}{((x-\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_2 x}{(x^2+(y-1)^2+z^2)^{3/2}}\right. \\
\left.+\frac{m_3(x+\alpha)}{((x+\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_4 x}{(x^2+(y+1)^2+z^2)^{3/2}}\right)=0, \\
\omega^2 y-\frac{y}{(x^2+y^2+z^2)^{3/2}}-\left(\frac{m_1 y}{((x-\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_2(y-1)}{(x^2+(y-1)^2+z^2)^{3/2}}\right. \\
\left.+\frac{m_3 y}{((x+\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_4(y+1)}{(x^2+(y+1)^2+z^2)^{3/2}}\right)=0, \\
\frac{z}{(x^2+y^2+z^2)^{3/2}}-\left(\frac{m_1 z}{((x-\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_2 z}{(x^2+(y-1)^2+z^2)^{3/2}}\right. \\
\left.+\frac{m_3 z}{((x+\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_4 z}{(x^2+(y+1)^2+z^2)^{3/2}}\right)=0.
\end{gathered} \tag{9}
$$

From the last equation it follows that $z=0$, that is all stationary solutions lay in the plane P_0xy, and the solution of the system (9) is reduced to the solution of the following system

$$
\begin{gathered}
\omega^2 x-\frac{x}{(x^2+y^2+z^2)^{3/2}}-\left(\frac{m_1(x-\alpha)}{((x-\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_2 x}{(x^2+(y-1)^2+z^2)^{3/2}}\right. \\
\left.+\frac{m_3(x+\alpha)}{((x+\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_4 x}{(x^2+(y+1)^2+z^2)^{3/2}}\right)=0, \\
\omega^2 y-\frac{y}{(x^2+y^2+z^2)^{3/2}}-\left(\frac{m_1 y}{((x-\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_2(y-1)}{(x^2+(y-1)^2+z^2)^{3/2}}\right. \\
\left.+\frac{m_3 y}{((x+\alpha)^2+y^2+z^2)^{3/2}}+\frac{m_4(y+1)}{(x^2+(y+1)^2+z^2)^{3/2}}\right)=0.
\end{gathered} \tag{10}
$$

The following theorem takes place.

Theorem 2.1 *Necessary and sufficient condition of existence of the stationary solutions of the restricted six-body problem is decidability of system (10) with respect to the unknown x and y.*

The equations (10) are nonlinear, therefore the question on their decidability can be studied by graphic and iteration techniques. In terms of the “Mathematica” system a graphic solution of system (10) is constructed. For example, for $m_1=0.5$ and $\alpha=0.95$ the two curves are shown on Figure 2.1.

On this figure the bold points denote points P_0, P_1, P_2, P_3, P_4. Cross-points of the curves, laying on axes of coordinates, are denoted by N_i, other cross-points – by S_i. The points N_i and S_i are the equilibrium solutions of system (10). The calculations show, that the quantity of equilibrium states essentially depends both on the gravitational

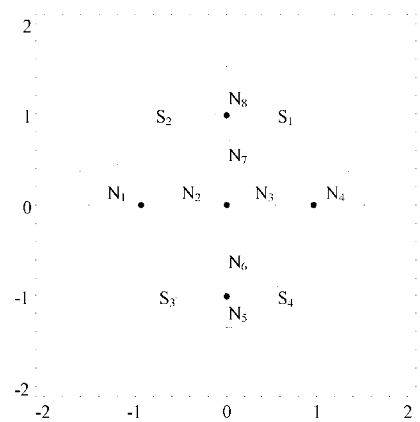

Figure 2.1

parameter m_1 and on the size of the diagonal α. Using Newton iteration method, we determine the coordinates of the equilibrium state for various values of the parameters m_1 and α.

3 Research of the Linear Stability of Equilibrium State

To investigate the linear stability of the equilibrium solutions of system of the differential equations (8) it is necessary to construct a linearized system of the differential equations in the neighborhood of points N_i and S_i, (Figure 2.1) with coordinates x_i^*, y_i^*, $z_i^* = 0$, and to study properties of eigenvalues of a matrix of this system. Denoting by x the phase vector $x = (u - u_i^*, v - v_i^*, w - w_i^*, x - x_i^*, y - y_i^*, z - z_i^*)$ and executing the procedure of linearization of the right parts of system (8) in the neighborhood of a phase point $x = 0$ in SSC "Mathematica", we shall get the system of linear differential equations

$$\frac{dx}{dt} = Ax. \tag{11}$$

Six-dimensional matrix A is

$$A = \begin{bmatrix} [0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a & b & 0 & 0 & 2\omega & 0 \\ b & c & 0 & -2\omega & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

The elements a, b, c, d of matrix A depend on the values x_i^*, y_i^*, m_1, α, whose expressions

m	α	N_1		S_1	
		λ_1, λ_2	λ_3, λ_4	λ_1, λ_2	λ_3, λ_4
0.001	0.99985	±2.35155	±1.97710i	±0.994867i	±0.099199i
	0.9999	±2.35146	±1.97703i	±0.990894i	±0.130508i
	1.0031	±2.34591	±1.97251i	±0.744772i	±0.651662i
	1.0032	±2.34574	±1.97237i	±0.34841 + 0.70044i	±0.34841 - 0.70044i
0.003	0.99954	±2.28897	±1.94015i	±0.98424i	±0.173624i
	0.9999	±2.28842	±1.93971i	±0.95927i	±0.273309i
	1.0022	±2.28494	±1.9369i	±0.736147i	±0.661479i
	1.0023	±2.28479	±1.93678i	±0.04689 + 0.70116i	±0.04689 - 0.70116i
0.009	0.99863	±2.20639	±1.89254i	±0.948869i	±0.310446i
	0.999	±2.20593	±1.89218i	±0.906597i	±0.411469i
	1.0001	±2.20458	±1.89111i	±0.73989i	±0.658941i
	1.0002	±2.20446	±1.89101i	±0.051454 + 0.7022i	±0.051454 - 0.7022i
0.01	0.9985	±2.19734	±1.88743i	±0.937409i	±0.341927i
	0.9998	±2.19578	±1.88619i	±0.738788i	±0.660504i
	0.9999	±2.19566	±1.8861i	±0.0537 + 0.7025i	±0.0537 - 0.7025i
0.03	0.99546	±2.09331	±1.831071i	±0.14454 + 0.71804i	±0.14454 - 0.71804i
	0.9955	±2.09328	±1.83104i	±0.16925 + 0.7227i	±0.16925 - 0.7227i
0.09	0.986537	±1.98254	±1.78116i	±0.47268 + 0.84128i	±0.47268 - 0.84128i
	0.987	±1.98234	±1.78102i	±0.50501 + 0.85608i	±0.50501 - 0.85608i
0.1	0.98507	±1.97277	±1.77794i	±0.49967 + 0.85574i	±0.49967 - 0.85574i
	0.99	±1.97088	±1.77667i	±0.62882 + 0.91929i	±0.62882 - 0.91929i
0.9	0.8874	±2.02623	±2.00395i	±1.1724 + 1.3235i	±1.1724 - 1.3235i
	0.89	±2.02549	±2.00323i	±1.1821 + 1.3243i	±1.1821 - 1.3243i

Table 3.1. Eigenvalues of matrix A.

are quite cumbersome, therefore we shall present the expressions for a and d:

$$\begin{aligned}
a = \omega^2 &+ \frac{3x_i^{*2}}{(x_i^{*2}+y_i^{*2})^{5/2}} - \frac{1}{(x_i^{*2}+y_i^{*2})^{3/2}} + \frac{3x_i^{*2}m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} \\
&- \frac{6x_i^*\alpha m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} + \frac{3\alpha m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{5/2}} - \frac{m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{3/2}} \\
&+ \frac{3x_i^{*2}m_2}{(x_i^{*2}+(y_i^*-1)^2)^{5/2}} - \frac{m_2}{(x_i^{*2}+(y_i^*-1)^2)^{5/2}} + \frac{3x_i^{*2}m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} \\
&- \frac{m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{3/2}} + \frac{6x_i^{*2}\alpha m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} + \frac{3\alpha^2 m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{5/2}} \\
&+ \frac{3x_i^{*2}m_4}{(x_i^{*2}+(y_i^*+1)^2)^{5/2}} - \frac{m_4}{(x_i^{*2}+(y_i^*+1)^2)^{3/2}}
\end{aligned} \tag{13}$$

$$\begin{aligned}
d = &- \frac{1}{(x_i^{*2}+y_i^{*2})^{3/2}} - \frac{m_1}{((x_i^*-\alpha)^2+y_i^{*2})^{3/2}} - \frac{m_2}{(x_i^{*2}+(y_i^*-1)^2)^{3/2}} \\
&- \frac{m_3}{((x_i^*+\alpha)^2+y_i^{*2})^{3/2}} - \frac{m_4}{(x_i^{*2}+(y_i^*+1)^2)^{3/2}}.
\end{aligned} \tag{14}$$

From the formula (14) it is clear, that $d < 0$. The eigenvalues of a matrix A are defined from the characteristic equation

$$\det(A-\lambda E) = (\lambda^2 - d)(\lambda^4 + (4\omega^2 - a - c)\lambda^2 + ac - b^2) = 0. \tag{15}$$

First multiplier of the equation (15) gives two pure imaginary eigenvalues, for example, λ_5 and λ_6. Using the instruction "Eigenvalues" of SSC "Mathematica" for calculation of eigenvalues, we have received other eigenvalues λ_1, λ_2, λ_3, λ_4 of matrix A at points N_1 and S_1 for various values of m_1 and α. Some of them are given in Table 3.1.

From Table 3.1 it is clear, that at point N_1 for any values of the parameters m_1 and α the eigenvalues of matrix A are not pure imaginary. The similar result is received for other points N_i.

At point S_1 for small enough values of m_1 and α, close to unit, the eigenvalues of matrix A are pure imaginary, that is the equilibrium solutions S_1 are stable in the first approximation. By the iterative method we calculate an interval of stability for m_1

$$0 < m_1 < m_1^{**} = 0.0250344906\dots. \tag{16}$$

The interval of stability for α depends on m_1, for each values of m_1 there is an interval of variation of α: (α^*, α^{**}). The calculated values of α^* and α^{**} for different m_1 are given in the following table:

m_1	α^*	α^{**}		m_1	α^*	α^{**}
0.001	0.9998476686	1.0031639276		0.002	0.9996953824	1.0026710278
0.003	0.9995431685	1.0022385906		0.004	0.9993910147	1.0018413555
0.009	0.9986311638	1.0001378981		0.01	0.9984793776	0.9998343232
0.02	0.9969648966	0.9972354376		0.025	0.9962099616	0.9962207882

Table 3.2.

The calculations executed for other points S_i, give the similar result. The carried out analysis allows to formulate the statements, following from the classical Lyapunov theorem on stability in the first approximation.

Theorem 3.1 *The stationary solutions of the differential equations of the restricted six-body problem, located on rotating axes of coordinates, are unstable for any values of mass m_1 and for any values of the relations of rhombus diagonals α.*

Theorem 3.2 *The stationary solutions of the differential equations of the restricted six-body problem, not located on the axes of coordinates, are stable in the first approximation for any value of parameter m_1 from the interval (15) and any value of parameter α from the interval $\alpha^* < \alpha < \alpha^{**}$.*

4 Research of Lyapunov Stability

The restricted 6-body problem is typically Hamiltonian, and, hence, differential equations, describing dynamics of our model, can be written a canonical form. Hence it follows, in particular, that the problem of stability of the stationary solutions S_1, S_2, S_3, S_4 in the sense of Lyapunov [5] can be solved only in the framework of KAM-theory [6, 15] on the basis of the well known Arnold-Mozer theorem [12, 13]. Now we formulate this theorem [6].

Theorem 4.1 *Let a Hamiltonian system*

$$\begin{aligned} \frac{dq_1}{dt} &= \frac{\partial K}{\partial p_1}, & \frac{dp_1}{dt} &= -\frac{\partial K}{\partial q_1}, \\ \frac{dq_2}{dt} &= \frac{\partial K}{\partial p_2}, & \frac{dp_2}{dt} &= -\frac{\partial K}{\partial q_2} \end{aligned} \tag{17}$$

by given with the Hamiltonian

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + \dots,$$

and let the origin be a singular point, such as the equilibrium state of system (17). Besides, let a canonical transformation

$$(q_1, q_2, p_1, p_2) \to (\psi_1, \psi_2, T_1, T_2)$$

exist, which yields

$$K(q_1, q_2, p_1, p_2) \equiv W(\psi_1, \psi_2, T_1, T_2),$$

where

$$\begin{gathered} W(\psi_1, \psi_2, T_1, T_2) = W_2(T_1, T_2) + W_4(T_1, T_2) + W_5(\psi_1, \psi_2, T_1, T_2) + \dots, \\ W_2 = \sigma_1 T_1 + \sigma_2 T_2, \qquad W_4 = c_{20} T_1^2 + c_{11} T_1 T_2 + c_{02} T_2^2. \end{gathered} \tag{18}$$

If:

(1) *eigenvalues of a matrix of linearized system (17) are the imaginary numbers* $\pm i\sigma_1$, $\pm i\sigma_2$*;*
(2) $n_1\sigma_1 + n_2\sigma_2 \neq 0$ *for* $|n_1| + |n_2| \leq 4$*;*
(3) $c_{20}\sigma_2^2 + c_{11}\sigma_1\sigma_2 + c_{02}\sigma_1^2 \neq 0$*,*

then the equilibrium

$$T_1 = T_2 = \psi_1 = \psi_2 = 0$$

of the Hamiltonian system

$$\begin{aligned} \frac{d\psi_1}{dt} &= \frac{\partial W}{\partial T_1}, & \frac{dT_1}{dt} &= -\frac{\partial W}{\partial \psi_1}, \\ \frac{d\psi_2}{dt} &= \frac{\partial W}{\partial T_2}, & \frac{dT_2}{dt} &= -\frac{\partial W}{\partial \psi_2} \end{aligned}$$

with the Hamiltonian (18) is Lyapunov stable.

Now we turn to a four-dimensional-phase space of Lagrangian coordinates and impulses (x, y, p_x, p_y). We shall get the Hamiltonian system of the 4-th order, equivalent to system (8):

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial h}{\partial p_x}, & \frac{dy}{dt} &= \frac{\partial h}{\partial p_y}, \\ \frac{dp_x}{dt} &= -\frac{\partial h}{\partial x}, & \frac{dp_y}{dt} &= -\frac{\partial h}{\partial y}, \end{aligned} \tag{19}$$

where the Hamiltonian h is expressed by the formula (see [9]):

$$
\begin{aligned}
h &= \omega(yp_x - xp_y) + \frac{1}{2}(p_x^2 + p_y^2) - (x^2+y^2)^{-1/2} \\
&\quad - m_1((x-\alpha)^2 + y^2)^{-1/2} - m_2((x^2 + (y-1)^2)^{-1/2} \\
&\quad - m_3((x+\alpha)^2 + y^2)^{-1/2} - m_4((x^2 + (y+1)^2)^{-1/2}.
\end{aligned} \tag{20}
$$

The elementary transformation makes any point S_i with coordinates x^*, y^* the beginning of coordinates: $X = x - x^*$, $Y = y - y^*$, $P_x = p_x - p_{x^*}$, $P_y = p_y - p_{y^*}$. For the Hamiltonian we get the expression:

$$
\begin{aligned}
H &= \omega((Y + y_*)(P_X + p_{x^*}) - (X + x^*)(P_Y + p_{y^*})) \\
&\quad + \frac{1}{2}((P_X + p_*)^2 + (P_Y + p_{y^*})^2) - ((X+x^*)^2 + (Y+y^*)^2)^{-1/2} \\
&\quad - m_1((X + x^* - \alpha)^2 + (Y+y^*)^2)^{-1/2} - m_2((X+x^*)^2 \\
&\quad + (Y + y^* - 1)^2)^{-1/2} - m_3((X + x^* + \alpha)^2 + (Y+y^*)^2)^{-1/2} \\
&\quad - m_4((X+x^*)^2 + (Y + y^* + 1)^2)^{-1/2}.
\end{aligned} \tag{21}
$$

In the new variables the Hamiltonian differential equations of motion have the form

$$
\begin{aligned}
\frac{dX}{dt} &= \frac{\partial H}{\partial P_X}, \qquad & \frac{dY}{dt} &= \frac{\partial H}{\partial P_Y}, \\
\frac{dP_X}{dt} &= -\frac{\partial H}{\partial X}, \qquad & \frac{dP_Y}{dt} &= -\frac{\partial H}{\partial Y}.
\end{aligned} \tag{22}
$$

The formulated Arnold-Mozer theorem is in applicable to system (22), as the Hamiltonian (22) is not a positively definite function of the variable (X, Y, P_X, P_Y) [5]. It is necessary to execute its further transformations. For this purpose it is necessary to construct Birkhoff normalization. This normalization will be executed for a certain equilibrium position. For example, we shall consider the point S_1, stable in the first approximation, with coordinates

$$x^* = 0.37355, \quad y^* = 0.971439,$$

calculated for $m_1 = 0.001$ and $\alpha = 0.99985$.

We build a sequence of Hamiltonian transformations, necessary for fulfilment of conditions of the Arnold-Mozer theorem.

4.1 Transformation 1

In a sufficiently small neighborhood of the point S_1 the analytical Hamiltonian (21) is presented in the form of a convergent power series:

$$H = H_2(X, Y, P_X, P_Y) + H_3(X, Y) + H_4(X, Y) + \dots,$$

where $H_k (k = 2, 3, \dots)$ is a homogeneous form of k-th degree, in our case

$$\begin{aligned} H_2 &= 0.414231X^2 - 0.915163Y^2 + 0.5(P_X^2 + P_Y^2) - 0.690873XY \\ &\quad + \omega(YP_X - XP_Y)), \\ H_3 &= -0.317928X^3 + 0.835341Y^3 - 1.049161X^2Y + 1.316579XY^2, \\ H_4 &= -0.19571X^4 - 0.73835Y^4 + 1.52426X^3Y + 1.65464X^2Y^2 \\ &\quad - 2.09451XY^3. \end{aligned} \tag{23}$$

Expression (23) indicates, that the quadratic form $H_2(X, Y, P_X, P_Y)$ contains the term $\omega(YP_X - XP_Y)$, which is the first obstacle on the way of investigation of the Lyapunov stability.

4.2 Transformation 2

Let's execute the linear transformation

$$[X, Y, P_X, P_Y] = B[q_1, q_2, p_1, p_2], \tag{24}$$

where symplectic matrix [10] B is defined so, that in the new transformed Hamiltonian

$$K(q_1, q_2, p_1, p_2) = K_2(q_1, q_2, p_1, p_2) + K_3(q_1, q_2, p_1, p_2) + K_4(q_1, q_2, p_1, p_2) + \dots$$

the quadratic form has a normal Birkhoff form [6, 14]

$$K_2 = \frac{1}{2}\,\sigma_1(q_1^2 + p_1^2) - \frac{1}{2}\,\sigma_2(q_2^2 + p_2^2),$$

where frequencies σ_1, σ_2: $\sigma_1 = |\lambda_1| = |\lambda_2|$, $\sigma_2 = |\lambda_3| = |\lambda_4|$, λ_1, λ_2, λ_3, λ_4 are the eigenvalues of linearized system for system (8) at point S_1.

Finding of elements of a matrix B is reduced to the solution of system of linear algebraic equations of the 16-th order. For the examined point S_1 $\sigma_1 = 0.994537$, $\sigma_2 = 0.102242$, and the matrix B has the form

$$B = \begin{bmatrix} 0 & 0 & 1.98114 & 5.328999 \\ -1.03191 & 0.38363 & -0.35829 & -1.29568 \\ -0.93804 & 0.16108 & 0.35842 & 1.29614 \\ 0.356334 & -0.13247 & 0.95557 & 5.29166 \end{bmatrix}. \tag{25}$$

The application of canonical transformation (24) with matrix (25) to Hamiltonian H gives the following expressions for the forms K_2, K_3 and K_4:

$$\begin{aligned} K_2 &= 0.49727(p_1^2 + q_1^2) - 0.05112(p_2^2 + q_2^2), \\ K_3 &= 0.700344p_1^3 - 3.771015p_1^2p_2 - 4.896871p_1p_2^2 + 0.452007p_2^3 \\ &\quad + 5.846024p_1^2q_1 + 32.621649p_1p_2q_1 + 45.164737p_2^2q_1 + 1.821347p_1q_1^2 \\ &\quad + 4.013408p_2q_1^2 - 0.917884q_1^3 - 2.173359p_1^2q_2 - 12.127657p_1p_2q_2 \end{aligned}$$

$$
\begin{aligned}
&- 16.790077p_2^2q_2 - 1.354234p_1q_1q_2 - 2.984107p_2q_1q_2 + 1.023717q_1^2q_2\\
&+ 0.251729p_1q_2^2 + 0.554696p_2q_2^2 - 0.380585q_1q_2^2 + 0.047163q_2^3,\\
K_4 = &- 6.249106p_1^4 - 69.140864p_1^3p_2 - 285.506731p_1^2p_2^2 - 521.541002p_1p_2^3\\
&- 355.623129p_2^4 - 5.919479p_1^3q_1 - 40.652926p_1^2p_2q_1 - 89.158571p_1p_1^2q_1\\
&- 61.002839p_2^3q_1 + 11.059209p_1^2q_1^2 + 62.773252p_1p_2q_1^2 + 88.314296p_2^2q_1^2\\
&+ 3.396809p_1q_1^3 + 8.059740p_2q_1^3 - 0.837196q_1^4 + 2.200668p_1^3q_2\\
&+ 15.113422p_1^2p_2q_2 + 33.146227p_1p_2^2q_2 + 22.678852p_2^3q_2 - 8.222901p_1^2q_1q_2\\
&- 46.674065p_1p_2q_1q_2 - 65.664708p_2^2q_1q_2 - 3.788466p_1q_1^2q_2 - 8.98904p_2q_1^2q_2\\
&+ 1.244968q_1^3q_2 + 1.528502p_1^2q_2^2 + 8.675942p_1p_2q_2^2 + 12.205991p_2^2q_2^2\\
&+ 1.408427p_1q_1q_2^2 + 3.341829p_2q_1q_2^2 - 0.694257q_1^2q_2^2 - 0.174536p_1q_2^3\\
&- 0.414127p_2q_2^3 + 0.172068q_1q_2^3 - 0.015992q_2^4.
\end{aligned}
$$

4.3 Transformation 3

Let's pass from the canonical variables (q_1, q_2, p_1, p_2) to the new canonical variables according to the Birkhoff formulas [14]

$$
\begin{aligned}
q_1 &= \sqrt{2\tau_1}\sin\theta_1, \qquad q_2 = \sqrt{2\tau_2}\sin\theta_2,\\
p_1 &= \sqrt{2\tau_1}\cos\theta_1, \qquad p_2 = \sqrt{2\tau_2}\cos\theta_2.
\end{aligned}
\tag{26}
$$

Transformation (26) eliminates expressions with the coordinates θ_1, θ_2 from the quadratic part of the new Hamiltonian F and leaves expressions, dependent only on the new variables τ_1, τ_2. If we present new Hamiltonian F in the form

$$F(\theta_1, \theta_2, \tau_1, \tau_2) = F_2(\tau_1, \tau_2) + F_3(\theta_1, \theta_2, \tau_1, \tau_2) + F_4(\theta_1, \theta_2, \tau_1, \tau_2) + \dots,$$

then after necessary symbolical transformations we shall receive

$$
\begin{aligned}
F_2 &= \sigma_1\tau_1 - \sigma_2\tau_2 = 0.994537\tau_1 - 0.102242\tau_2,\\
F_3 &= (0.197768\cos\theta_1 - 1.7831\cos 3\theta_1 + 2.18664\sin\theta_1 + 4.78281\sin 3\theta_1)\tau^{3/2}\\
&\quad + (6.46201\cos(2\theta_1 - \theta_2) + 0.342795\cos\theta_2 - 4.54683\cos(2\theta_1 + \theta_2)\\
&\quad + 25.3277\sin(2\theta_1 - \theta_2) - 1.62584\sin\theta_2 + 20.806\sin(2\theta_1 + \theta_2))\tau_1\sqrt{\tau_2}\\
&\quad - (6.5692\cos\theta_1 + 5.7507\cos(\theta_1 - 2\theta_2) + 1.53053\cos(\theta_1 + 2\theta_2)\\
&\quad - 63.3344\sin\theta_1 - 40.781\sin(\theta_1 - 2\theta_2) - 23.6299\sin(\theta_1 + 2\theta_2))\sqrt{\tau_1}\tau_2\\
&\quad + (1.35108\cos\theta_2 - 0.072612\cos 2\theta_2 - 11.7728\sin\theta_2 - 11.9062\sin 3\theta_2)\tau_1^{3/2},\\
F_4 &= (-5.09985 - 10.8238\cos 2\theta_1 - 9.07276\cos 4\theta_1 - 2.52267\sin 2\theta_1\\
&\quad - 4.65814\sin 4\theta_1)\tau_1^2 - (74.5687\cos(\theta_1 - \theta_2) + 70.691\cos(3\theta_1 - \theta_2)
\end{aligned}
$$

$$+ 70.081\cos(\theta_1+\theta_2) + 61.223\cos(3\theta_1+\theta_2) + 9.64362\sin(\theta_1-\theta_2)$$
$$+ 27.3509\sin(3\theta_1-\theta_2) + 6.8308\sin(\theta_1+\theta_2) + 21.362\sin(3\theta_1+\theta_2))\tau^{3/2}\sqrt{\tau}_2$$
$$- (196.358\tau_1\tau_2 + 371.598\cos 2\theta_1 + 211.36\cos(2\theta_1-2\theta_2) + 198.027\cos 2\theta_2$$
$$+ 164.69\cos(2\theta_1+2\theta_2) + 87.7501\sin 2\theta_1 + 57.3347\sin(2\theta_1-2\theta_2)$$
$$- 6.1244\sin 2\theta_2 + 33.2323(2\theta_1+2\theta_2))\tau_1\tau_2 - (298.027\cos(\theta_1-3\theta_2)$$
$$+ 810.55\cos(\theta_1-\theta_2) + 745.399\cos(\theta_1+\theta_2) + 232.19\cos(\theta_1+3\theta_2)$$
$$+ 48.8327\sin(\theta_1-3\theta_2) + 106.145\sin(\theta_1-\theta_2) + 73.522\sin(\theta_1+\theta_2)$$
$$+ 15.512\sin(\theta_1+3\theta_2))\sqrt{\tau}_1\tau_2^{3/2} - (527.356 + 711.214\cos 2\theta_2$$
$$+ 183.923\cos 4\theta_2 - 22.2647\sin 2\theta_2 - 11.5465\sin 4\theta_2)\tau_2^2.$$

4.4 Transformation 4

Let's construct the final canonical transformation

$$(\theta_1, \theta_2, \tau_1, \tau_2) \to (\psi_1, \psi_2, T_1, T_2) \tag{27}$$

which sets to zero the form of order of $3/2\, F_3(\theta_1, \theta_2, \tau_1, \tau_2)$, and excludes phase angles from the second-order form $F_4(\theta_1, \theta_2, \tau_1, \tau_2)$. Besides, the quadratic form $F_2(\tau_1, \tau_2)$ does not change. So, the transformed Hamiltonian should be

$$W(\psi_1, \psi_2, T_1, T_2) = W_2(T_1, T_2) + W_4(T_1, T_2) + W_5(\psi_1, \psi_2, T_1, T_2) + \dots. \tag{28}$$

We shall search the given transformation as

$$\begin{aligned}
\theta_1 &= \psi_1 + V_{13}(\psi_1, \psi_2, T_1, T_2) + V_{14}(\psi_1, \psi_2, T_1, T_2),\\
\theta_2 &= \psi_2 + V_{23}(\psi_1, \psi_2, T_1, T_2) + V_{24}(\psi_1, \psi_2, T_1, T_2),\\
\tau_1 &= T_1 + U_{13}(\psi_1, \psi_2, T_1, T_2) + U_{14}(\psi_1, \psi_2, T_1, T_2),\\
\tau_2 &= T_2 + U_{23}(\psi_1, \psi_2, T_1, T_2) + U_{24}(\psi_1, \psi_2, T_1, T_2),
\end{aligned} \tag{29}$$

where V_{13}, V_{14}, V_{23}, V_{24}, U_{13}, U_{14}, U_{23}, U_{24} are determined from some linear partial differential equations. For example, the equation for the unknown function U_{13} has the form

$$\frac{\partial U_{13}}{\partial \psi_1}\sigma_1 + \frac{\partial U_{13}}{\partial \psi_2}\sigma_2 = A_{13}(\psi_1, \psi_2, T_1, T_2),$$

where A_{13} is expressed by partial derivative of forms $F_3(\theta_1, \theta_2, \tau_1, \tau_2)$ and $F_4(\theta_1, \theta_2, \tau_1, \tau_2)$, in which the replacement (27) is executed. The solution, which guarantees the form (28) for the new Hamiltonian $W(\psi_1, \psi_2, T_1, T_2)$, is to be found by the method of characteristics [10] and has the form

$$\begin{aligned}
U_{13} = {} & (0.198854\cos\psi_1 + 1.7929\cos 3\psi_1 - 2.19865\sin\psi_1 - 4.809\sin 3\psi_1)T_1^{3/2}\\
& + (6.179\cos(2\psi_1-\psi_2) + 4.819\cos(2\psi_1+\psi_2) - 24.222\sin(2\psi_1-\psi_2)\\
& - 22.054\sin(2\psi_1+\psi_2))T_1\sqrt{T}_2 + (6.605\cos\psi_1 + 4.79616\cos(\psi_1-2\psi_2)\\
& - 63.6823\sin\psi_1 + 1.937\cos(\psi_1+2\psi_2) - 34.012\sin(\psi_1-2\psi_2)\\
& - 29.909\sin(\psi_1+2\psi_2))\sqrt{T}_1 T_2.
\end{aligned}$$

Carrying out transformation (29) with the found functions V_{13}, V_{14}, V_{23}, V_{24}, U_{13}, U_{14}, U_{23}, U_{24}, we receive for transformed Hamiltonian the final form (28), where

$$W_2 = \sigma_1 T_1 - \sigma_2 T_2 = 0.99453 T_1 - 0.102242 T_2,$$
$$W_4 = -197.657 T_1^2 - 5539.05 T_1 T_2 + 2591.95 T_2^2.$$

As a result of the executed transformations it is possible to assert the following.

1. The intervals for m_1: $(0, m_1^{**})$ and α: (α^*, α^{**}) are found. At each point of these intervals the linear system is stable.
2. The resonant curves are determined

$$\begin{cases} f_{1,-2}(m_1,\alpha) = \sigma_1(m_1,\alpha) - 2\sigma_2(m_1,\alpha), \\ f_{1,-2}(m_1,\alpha) = 0, \end{cases}$$
$$\begin{cases} f_{1,-3}(m_1,\alpha) = \sigma_1(m_1,\alpha) - 3\sigma_2(m_1,\alpha), \\ f_{1,-3}(m_1,\alpha) = 0, \end{cases} \tag{30}$$

   which should be excluded from the set of intervals of stability.
3. Third condition of the theorem is also executed, thus for the point S_1 the value of function $W_4(\sigma_1, \sigma_2)$ is equal to 2018.72.

The executed calculations for points S_2, S_3, S_4 give similar results.

Thus, the following statement is valid.

Theorem 4.1 *The equilibrium points, not lying on coordinate axes, are Lyapunov stable for any values of parameters m_1 from interval of stability $0 < m_1 < m_1^{**} = 0.0250344906\dots$ and any values of α from interval $\alpha^* < \alpha < \alpha^{**}$, except for the points, belonging to two resonant curves (30).*

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NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 3 Number 1 2003

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 ISSN 1562-8353 Printed in Ukraine
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