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CONTENTS

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Nonlinear Dynamics and Systems Theory, **2**(2) (2002) 117–124

On a New Approach to Some Problems of Classical Calculus of Variations

N. Azbelev, E. Bravyi and S. Gusarenko†

Perm State Technical University, Research Center "Functional Differential Equations", 614600, Perm, Komsomol'skii pr. 29a, Russia

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Abstract: Employing the contemporary theory of functional differential equations, we propose an effective test on the existence of a minimum for a wide class of functionals in various Banach spaces.

Keywords: *Variational problem; boundary value problem; functional differential equation; sufficient conditions.*

Mathematics Subject Classification (2000): 49K25.

1 Introduction

The classical calculus of variations assumes that the functional is defined on a specific set and has a very characteristic form. Thus, some problems of minimization prove to be unsolvable in the frame of the classical calculus of variations. In the case of such an "unsolvable" situation D.Hilbert proposed to define the functional on a suitable set such that the functional under consideration obtains a point of a minimum on the set [4]. But what should be done if the "proper" set does not comply with the requirement of the known methods? The main idea of the contemporary theory of functional differential equations is that "any problem needs its proper space of functions" [5]. Using some given below elements of the mentioned theory, we are able to propose a new approach to certain problems of minimization.

2 Preliminaries

Let R^n be the space of vectors $\alpha = \mathrm{col}\{\alpha^1, \dots, \alpha^n\}$ with real components α^i, L_2 be the Banach space of square integrable functions $z\colon [a,b] \to R^1$ under the norm $\|z\|_{L_2} =$

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$\left(\int\limits_a^b z^2(s)\,ds\right)^{\frac{1}{2}}$, and D be a linear space of functions $x\colon [a,b] \to R^1$. Denote by $r = \operatorname{col}\{r^1,\dots,r^n\}\colon D \to R^n$ a system of linearly independent linear functionals on D. Let further $\mathcal{L}\colon D \to L_2$ be a linear operator and the system

$$\begin{aligned} \mathcal{L}x &= z, \\ rx &= \alpha \end{aligned} \tag{1}$$

have a unique solution $x \in D$ for each pair $\{z,\alpha\} \in L_2 \times R^n$. We define the norm $\|x\|_D = \|\mathcal{L}x\|_{L_2} + \|rx\|_{R^n}$. Then D becomes a Banach space. The solution x of (1) has the representation

$$x = Gz + Y\alpha. \tag{2}$$

Here the "Green operator" $G\colon L_2 \to \{x \in D\colon rx = 0\}$ is an integral one:

$$(Gz)(t) = \int\limits_a^b G(t,s)z(s)\,ds$$

whenever the space D is continuously embedded into the space C of continuous functions $x\colon [a,b] \to R^1$ under the norm $\|x\|_C = \max\limits_{t\in[a,b]} |x(t)|$; the finite-dimensional $Y\colon R^n \to D$ is defined by

$$(Y\alpha)(t) = \sum_{k=1}^n \alpha^k y_k(t),$$

where y_i, $i = 1,\dots,n$, are the solutions of the semi-homogeneous problems

$$\mathcal{L}x = 0, \qquad r^k x = \begin{cases} 1 & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i, \end{cases} \qquad k = 1,\dots,n.$$

According to (2) any pair $\{z,\alpha\} \in L_2 \times R^n$ defines an element $x \in D$ as well as any $x \in D$ defines a pair $\{z,\alpha\} \in L_2 \times R^n$ with $z = \mathcal{L}x$, $\alpha = rx$. Thus there exists an isomorphism $\mathcal{J} = \{G, Y\}\colon L_2 \times R^n \to D$ ($\mathcal{J}^{-1} = [\mathcal{L}, r]\colon D \to L_2 \times R^n$) between D and the direct product $L_2 \times R^n$. We will denote the fact by $D \simeq L_2 \times R^n$. Below we consider functionals on $D \simeq L_2 \times R^n$. The first example of $D \simeq L_2 \times R^n$ is the space of continuous functions $x\colon [a,b] \to R^1$ such that $x^{(i)}$, $i = 0,1,\dots,n-1$, are absolutely continuous and $x^{(n)} \in L_2$. In this case

$$\begin{gathered} (Gz)(t) = \int\limits_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,ds, \\ Y(t) = \left(1, t-a, \dots, \frac{(t-a)^{n-1}}{(n-1)!}\right), \\ \mathcal{L}x = x^{(n)}, \quad rx = \{x(a), \dot{x}(a), \dots, x^{(n-1)}(a)\}. \end{gathered}$$

In the capacity of a more complicated example consider the space of continuous functions $x\colon [0,1] \to R^1$ such that $\dot{x}$ is absolutely continuous on any $[c,d] \subset (0,1)$ and $t(1-t)\ddot{x}$ is square integrable on $[0,1]$. The space of such functions is isomorphic to $L_2 \times R^2$,

$$(Gz)(t) = \int\limits_0^1 G(t,s)z(s)\,ds, \tag{3}$$

$$G(t,s) = \begin{cases} -\dfrac{1-t}{1-s} & \text{if} \quad 0 \le s \le t \le 1, \\ -\dfrac{t}{s} & \text{if} \quad 0 \le t < s \le 1, \end{cases} \tag{4}$$

$$(Y\alpha)(t) = \alpha^1(1-t) + \alpha^2 t. \tag{5}$$

Some other examples can be found in $[3,5]$.

3 Main Assertions

Let $D \simeq L_2 \times R^n$, $\mathcal{J} = \{G,Y\}$, $\mathcal{J}^{-1} = [\mathcal{L}, r]$, the linear operator $T\colon D \to L_2$ be bounded, $D_\alpha \overset{def}{=} \{x \in D\colon rx = \alpha\}$. Consider the functional

$$J(x) = \int\limits_a^b \left(\frac{1}{2}(\mathcal{L}x)^2(t) - f(t,(Tx)(t))\right)dt$$

defined on an open set $\Omega \subset D_\alpha$. We will say that a point $x_0 \in D_\alpha$ such that $J(x) \ge J(x_0)$ for any x from a neighbourhood of x_0 is a point of a local minimum of J. The problem of the existence of such a point is denoted by

$$J(x) \to \min, \quad rx = \alpha. \tag{6}$$

In what follows $Q = TG$, $Q^*\colon L_2 \to L_2$ is the adjoint to $Q\colon L_2 \to L_2$, $\varphi(t,\theta) = \frac{\partial}{\partial\theta} f(t,\theta)$, $(F(y))(t) = \varphi(t,y(t))$, $\Psi(x) = GQ^*F(Tx) + Y\alpha$.

Theorem 3.1 *Let $\Psi\colon \Omega \to L_2$ be continuous and bounded. Then any point $x_0 \in \Omega$ of a local minimum of the functional $J(x)$ satisfies the equation*

$$x = \Psi(x).$$

Proof Using the substitution $x = Gz + Y\alpha$, we get the auxiliary functional $J_1(z)$ on L_2:

$$J_1(z) = J(Gz + Y\alpha) = \int\limits_a^b \left(\frac{1}{2}(\mathcal{L}(Gz+Y\alpha))^2(t) - f(t, T(Gz+Y\alpha)(t))\right)dt$$

$$= \int\limits_a^b \left(\frac{1}{2}z^2(t) - f(t,(Qz)(t) + (TY)(t)\alpha)\right)dt.$$

An element $x_0 = Gz_0 + Y\alpha$ is a point of a local minimum of $J(x)$ if z_0 is a point of a local minimum of $J_1(z)$. By virtue of the generalized Fermat theorem, the point z_0 such that $J_1(z) \geq J_1(z_0)$ for each z from a neighbourhood of z_0 satisfies the equality $\delta J_1(z_0, \xi) = 0$, where

$$J_1'(z_0)\xi = \delta J_1(z_0, \xi) \overset{def}{=} \lim_{\tau \to 0} \frac{J_1(z_0 + \tau\xi) - J_1(z_0)}{\tau}.$$

The differential $J_1'(z)\xi$ at the point z by the increment ξ has the form

$$J_1(z)\xi = \int_a^b \Big(z(t)\xi(t) - (F(Qz + TY\alpha))(t)(Q\xi)(t) \Big)\, dt.$$

Using the definition

$$\int_a^b y_1(t)(By_2)(t)\, ds = \int_a^b (B^* y_1)(t) y_2(t)\, dt,$$

we obtain

$$J_1(z)\xi = \int_a^b \Big(z(t) - (Q^* F(Qz + TY\alpha))(t) \Big) \xi(t)\, dt.$$

Therefore the point $z_0 \in L_2$ of a minimum of $J_1(z)$ satisfies the equation

$$z_0 = Q^* F(Qz_0 + TY\alpha).$$

Thus, the solution $x_0 \in D$ of problem (6) satisfies the equation

$$x = \Psi(x).$$

Theorem 3.2 *Let $M \subseteq \Omega$ be a nonempty closed convex set and the operator $\Psi\colon M \to M$ be contractive. Then there exists a unique point $x_0 \in M$ such that $J(x) \geq J(x_0)$ for any $x \in M_\alpha \overset{def}{=} \{x \in M : rx = \alpha\}$.*

To prove Theorem 3.2 we use the following well-known result [6, p.376].

Lemma 3.1 *Let functional ω be differentiable on a convex set M and, besides,*

$$\|\omega'(x_1) - \omega'(x_2)\| \leq q\|x_1 - x_2\|$$

on M. Then

$$|\omega(x_1) - \omega(x_2) - \omega'(x_2)(x_2 - x_1)| \leq \frac{q}{2}\|x_1 - x_2\|^2. \tag{7}$$

Proof of Theorem 3.2 By the conditions, the operator Ψ maps the set M_α into the set M_α. By the Banach principle, there exists a unique solution $x_0 \in M_\alpha$ to the equation $x_0 = \Psi(x_0)$. The Frechét differential of

$$\varphi(z) = \int_a^b f(t, (Qz)(t) + A(t)\alpha)\, dt,$$

where $A = TY$, is defined by

$$\varphi'(z)\xi = \int_a^b f_1'(t,(Qz)(t) + A(t)\alpha)(Q\xi)(t)\,dt = \int_a^b (Q^* f_1'(\cdot, Qz + A\alpha))(t)\xi(t)\,dt.$$

Take arbitrary points $x_1, x_2 \in M_\alpha$, $x_1 = \Lambda z_1 + Y\alpha$, $x_2 = \Lambda z_2 + Y\alpha$. We have

$$\begin{aligned}\|\varphi'(z_1) - \varphi'(z_2)\|_{L_2} &= \|Q^* f_1'(\cdot, Tx_1) - Q^* f_1'(\cdot, Tx_2)\|_{L_2} = \|\Psi(x_1) - \Psi(x_2)\|_D \\ &= \|\Psi(\Lambda z_1 + Y\alpha) - \Psi(\Lambda z_2 + Y\alpha)\|_D \le \|\Lambda(z_1 - z_2)\|_D = q\|z_1 - z_2\|_{L_2},\end{aligned}$$

where $q \in (0,1)$ is the contraction constant of Ψ on the set M_α. So, the operator φ' is a contraction on the set $S = \{z \in L_2 \colon \Lambda z + Y\alpha \in M_\alpha\}$ with the constant q. Thus estimate (7) is valid for the functional φ on S.

Let $z_0 = \delta x_0$. Note that the equality $x_0 = \Psi(x_0)$ implies $z_0 = Q^* F(T(\Lambda z_0 + Y\alpha)) = \varphi'(z_0)$. The equality

$$\begin{aligned} J(x) - J(x_0) &= J(\Lambda z + Y\alpha) - J(\Lambda z_0 + Y\alpha) \\ = \frac{1}{2}\int_a^b (z^2(t) - z_0^2(t))\,dt - \varphi(z) + \varphi(z_0) &= \frac{1}{2}(\|z\|^2 - \|z_0\|^2) - \varphi(z) + \varphi(z_0)\end{aligned}$$

is fulfilled for all $x \in M_\alpha$. Using Lemma 3.1, we get

$$\varphi(z) - \varphi(z_0) \le \varphi'(z_0)(z - z_0) + \frac{q}{2}\,\|z - z_0\|^2.$$

Then

$$\begin{aligned} J(x) - J(x_0) &\ge \frac{1}{2}\,(\|z\|^2 - \|z_0\|^2) - \varphi'(z_0)(z - z_0) - \frac{1}{2}\,q\|z - z_0\|^2 \\ &\ge \frac{1}{2}\,(\|z\|^2 - \|z_0\|^2) - \int_a^b z_0(t)z(t)\,dt + \|z_0\|^2 - \frac{1}{2}\,q\|z - z_0\|^2 \\ &\ge \frac{1}{2}\Big(\|z\|^2 - 2\int_a^b z_0(t)z(t)\,dt + \|z_0\|^2 - q\|z - z_0\|^2\Big) \ge \frac{1}{2}\,(1-q)\|z - z_0\|^2 \ge 0.\end{aligned}$$

Hence the functional J takes on its minimal value on the set M_α at the point x_0.

This completes the proof.

4 Examples

Example 4.1 The problem

$$\int_0^b \left(\frac{1}{2}((\mathcal{L}x)^2(t) - p(t)f((Tx)(t))\right) dt \longrightarrow \min, \tag{8}$$
$$x(b) - x(0) = 0,$$

where $\mathcal{L}x = \dot{x} + x(b)$ and $f(x) = \frac{1}{2}x^2$, was investigated in [1] for $Tx = x$ and in [5] for an arbitrary linear T. Consider the case with

$$(Tx)(t) = \begin{cases} x(h(t)) & \text{if} \quad h(t) \in [0,b], \\ 0 & \text{if} \quad h(t) \notin [0,b], \end{cases} \tag{9}$$

where the real function h is measurable.

Here

$$(Gz)(t) = \int_0^b G(t,s)z(s)\,ds,$$

$$G(t,s) = \begin{cases} \dfrac{1+b-t}{b} & \text{if} \quad 0 \le s \le t \le b, \\ \dfrac{1-t}{b} & \text{if} \quad 0 \le t < s \le b. \end{cases}$$

Then

$$(\Psi x)(t) = \int_0^b K(t,s)f'(x(h(s)))\,ds,$$

where $x(\xi) = 0$ for $\xi \notin [0,b]$;

$$K(t,s) = p(s) \begin{cases} \dfrac{1+(b-t)h(s)}{b} & \text{if} \quad 0 \le h(s) \le t \le b, \\ \dfrac{1+(b-h(s))t}{b} & \text{if} \quad 0 \le t < h(s) \le b, \\ 0 & \text{otherwise}. \end{cases}$$

Let

$$\mu(r) = \sup_{|x| \le r} |f'(x)|, \qquad q(r) = \sup_{|x_1| \le r,\, |x_2| \le r,\, x_1 \ne x_2} \frac{|f'(x_1) - f'(x_2)|}{|x_1 - x_2|},$$

$$(Kx)(t) = \int_0^b K(t,s)x(s)\,ds.$$

Then the operator Ψ maps the set $M(r) = \{x \in D\colon \|x\|_C \le r\}$ into itself if $\mu(r)\|K\|_{C \to C} \le r$, and Ψ is a contractive operator on $M(r)$ if $q(r)\|K\|_{L_2 \to L_2} < 1$. So, problem (8) has a solution under the condition

$$\int_0^b (1 + (b-s)s)^2 p^2(s)\,ds < \frac{b}{\left(q(r) + \frac{|f'(0)|}{r}\right)^2}.$$

Example 4.2 Consider the singular problem (see the quadratic case with a linear ordinary differential operator T in [2] and with a functional differential T in [5])

$$\begin{gathered}\int_0^1 \left(\frac{t^2(1-t)^2}{2}\,\ddot{x}^2(t) - p(t)f((Tx)(t))\right) ds \longrightarrow \min, \\ x(0) = \alpha^1, \quad x(1) = \alpha^2,\end{gathered} \tag{10}$$

where T is defined by (9) for $b = 1$. Let $(\mathcal{L}x)(t) = t(1-t)\ddot{x}(t)$, $rx = \{x(0), x(1)\}$. Define the elements of the space D by equalities (2)–(5).

Then

$$(\Psi x)(t) = \int_0^1 \int_0^1 G(t,s)G(h(\tau),s)p(\tau)f'(x(h(\tau)))\,ds\,d\tau + \alpha^1(1-t) + \alpha^2 t,$$

where $x(\xi) = 0$ for $\xi \notin [0,1]$.

Assume that there exists a non-decreasing function $\tilde{f}$ such that

$$|f'(x) - f'(y)| \le |x-y|\tilde{f}(\max\{|x|,|y|\}), \quad |f'(x)| \le \gamma|x|\tilde{f}(|x|).$$

Let $M = \{x \in D \colon rx = \alpha,\ |t(1-t)\ddot{x}(t)| \le r\}$ and denote $v(t) = (t-1)\ln(1-t) - t\ln t$, $u(t) = |\alpha^1|(1-t) + |\alpha^2|t$.

Then the operator Ψ maps the set M into itself if

$$\gamma \int_0^1 |p(t)|(v(t)r + u(t))\tilde{f}(v(t)r + u(t))\,dt \le r,$$

and Ψ is a contractive operator on M if

$$\sqrt{2}\int_0^1 |p(t)|\tilde{f}(v(t)r + u(t))\,dt < 1.$$

Hence problem (10) is solvable if $\gamma < \sqrt{2}$,

$$\int_0^1 |p(t)|\tilde{f}\left(\frac{\max\{|\alpha^1|,|\alpha^2|\}\gamma}{\sqrt{2}-\gamma}v(t) + u(t)\right) dt < \frac{1}{\sqrt{2}}$$

or

$$\int_0^1 |p(t)|\,dt < \frac{1}{\sqrt{2}\tilde{f}\left(\frac{\sqrt{2}\max\{|\alpha^1|,|\alpha^2|\}}{\sqrt{2}-\gamma}\right)}.$$

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Nonlinear Dynamics and Systems Theory, **2**(2) (2002) 125–144

The Relationship between Pullback, Forward and Global Attractors of Nonautonomous Dynamical Systems

D.N. Cheban[1], P.E. Kloeden[2] and B. Schmalfuß[3†]

[1] *Departement of Mathematics and Informatics, State University of Moldova, A. Mateevich Street 60, MD–2009 Chişinău, Moldova*
[2] *Fachbereich Mathematik, Johann Wolfgang Goethe Universität, D–60054 Frankfurt am Main, Germany*
[3] *Department of Sciences, University of Applied Sciences, D–06217 Merseburg, Germany*

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Abstract: Various types of attractors are considered and compared for nonautonomous dynamical systems involving a cocycle state space mapping that is driven by an autonomous dynamical system on a compact metric space. In particular, conditions are given for a uniform pullback attractor of the cocycle mapping to form a global attractor of the associated autonomous skew-product semi-dynamical system. The results are illustrated by several examples that are generated by differential equations on a Banach space with a uniformly dissipative structure induced by a monotone operator.

Keywords: *Nonautonomous dynamical system; skew-product flow; pullback attractor; global attractor; asymptotical stability; nonautonomous Navier-Stokes equation.*

Mathematics Subject Classification (2000): 34D20, 34D40, 34D45, 58F10, 58F12, 58F39, 35B35, 35B40.

1 Introduction

Nonautonomous dynamical systems can often be formulated in terms of a cocycle mapping for the dynamics in the state space that is driven by an autonomous dynamical system in what is called a parameter or base space. Traditionally the driving system

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is topological and the resulting cartesian product system forms an autonomous semi-dynamical system that is known as a skew-product flow. Results on global attractors for autonomous semi-dynamical systems can thus be adapted to such nonautonomous dynamical systems via the associated skew-product flow [5, 7, 8, 11, 13, 22, 34, 38].

A new type of attractor, called a pullback attractor, was proposed and investigated for nonautonomous deterministic dynamical systems and for random dynamical systems [12, 15, 19, 29, 31]. Essentially, it consists of a parametrized family of nonempty compact subsets of the state space that are mapped onto each other by the cocycle mapping as the parameter is changed by the underlying driving system. Pullback attraction describing such attractors to a component subset for a fixed parameter value is achieved by starting progressively earlier in time, that is, at parameter values that are carried forward to the fixed value. A deeper reason for this procedure is that a cocycle can be interpreted as a mapping between the fibers of a fiber bundle. For the pullback convergence the *image* fiber is fixed. (The kernels of a global attractor of the skew-product flows considered in [13] are very similar). This differs from the more conventional forward convergence where the parameter value of the limiting object also evolves with time, in which case the parametrized family could be called a forward attractor.

Pullback attractors and forward attractors can, of course, be defined for nonautonomous dynamical systems with a topological driving system [25 − 27]. In fact, when the driving system is the shift operator on the real line, forward attraction to a time varying solution, say, is the same as the attraction in Lyapunov asymptotic stability. The situation of a compact parameter space is dynamically more interesting as the associated skew-product flow may then have a global attractor. The relationship between the global attractor of the skew-product system and the pullback and forward attractor of the cocycle system is investigated in this paper. It will be seen that forward attractors are stronger than global attractors when a *compact* set of nonautonomous perturbations is considered. In addition, an example will be presented in which the cartesian product of the component subsets of a pullback attractor is not a global attractor of the skew-product flow. This set is, however, a maximal compact invariant subset of the skew-product flow. By a generalization of some stability results of Zubov [39] it is asymptotically stable. Thus a pullback attractor always generates a local attractor of the skew-product system, but this need not be a global attractor. If, however, the pullback attractor generates a global attractor in the skew-product flow and if, in addition, its component subsets depend lower continuously on the parameter, then the pullback attractor is also a forward attractor.

Several examples illustrating these results are presented in the final section.

In concluding this introduction, we note that although our assumption of the compactness of the parameter space P is a restriction, it nevertheless occurs in many important and interesting applications such as for nonautonomous differential equations with temporally almost periodic vector fields, where P is a compact subset of a function space defined by the hull of the vector field; see the example following Definition 2.2 in the next section. More generally, the vector fields could be almost automorphic in time [35] or be generated by an affine control system [14], in which case P is the space of measurable control taking values in a compact convex set, or the driving system could itself be an autonomous differential equation on a compact manifold P.

2 Nonautonomous Dynamical Systems and Their Attractors

A general nonautonomous dynamical system is defined here in terms of a cocycle mapping ϕ on a state space U that is driven by an autonomous dynamical system σ acting on a base space P, which will be called the parameter space. It is based on definitions in [4, 24]. In particular, let (U, d_U) be a complete metric space, let (P, d_P) be a compact metric space and let $\mathbb{T}$, the time set, be either $\mathbb{R}$ or $\mathbb{Z}$.

An autonomous dynamical system $(P, \mathbb{T}, \sigma)$ on P consists of a continuous mapping $\sigma\colon \mathbb{T} \times P \to P$ for which the $\sigma_t = \sigma(t, \cdot)\colon P \to P$, $t \in \mathbb{T}$, form a group of homeomorphisms on P under composition over $\mathbb{T}$, that is, satisfy

$$\sigma_0 = \mathrm{id}_P, \qquad \sigma_{t+\tau} = \sigma_t \circ \sigma_\tau$$

for all $t, \tau \in \mathbb{T}$. In addition, a continuous mapping $\phi\colon \mathbb{T}^+ \times U \times P \to U$ is called a cocycle with respect to an autonomous dynamical system $(P, \mathbb{T}, \sigma)$ if it satisfies

$$\phi(0, u, p) = u, \qquad \phi(t + \tau, u, p) = \phi(t, \phi(\tau, u, p), \sigma_\tau p)$$

for all $t, \tau \in \mathbb{T}^+$ and $(u, p) \in U \times P$.

Definition 2.1 The *triple* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is called a *nonautonomous dynamical system on the state space* U.

Let (X, d_X) be the cartesian product of (U, d_U) and (P, d_P). Then the mapping $\pi\colon \mathbb{T}^+ \times X \to X$ defined by

$$\pi(t, (u, p)) := (\phi(t, u, p), \sigma_t p)$$

forms a semi-group on X over $\mathbb{T}^+$ [33].

Definition 2.2 The *autonomous semi-dynamical system* $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$ is called the *skew-product dynamical system associated with the cocycle dynamical system* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$.

For example, let U be a Banach space and let the space $C = C(\mathbb{R} \times U, U)$ of continuous functions $f\colon \mathbb{R} \times U \to U$ be equipped with the compact open topology. Consider the autonomous dynamical system $(C, \mathbb{R}, \sigma)$, where σ is the shift operator on C defined by $\sigma_t f(\cdot, \cdot) := f(\cdot + t, \cdot)$ for all $t \in \mathbb{T}$. Let P be the hull $H(f)$ of a given functions $f \in C$, that is,

$$P = H(f) := \overline{\bigcup_{t \in \mathbb{R}} \{f(\cdot + t, \cdot)\}},$$

and denote the restriction of $(C, \mathbb{R}, \sigma)$ to P by $(P, \mathbb{R}, \sigma)$. Let $F\colon P \times U \to U$ be the continuous mapping defined by $F(p, u) := p(0, u)$ for $p \in P$ and $u \in U$. Then, under appropriate restrictions on the given function $f \in C$ (see Sell [33]) defining P, the differential equation

$$u' = p(t, u) = F(\sigma_t p, u) \tag{1}$$

generates a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$, where $\phi(t, p, u)$ is the solution of (1) with the initial value u at time $t = 0$.

Let dist_Y denote the Hausdorff distance (semi-metric) between two nonempty sets of a metric space (Y, d_Y), that is,

$$\mathrm{dist}_Y(A, B) = \sup_{a \in A} \inf_{b \in B} d_Y(A, B),$$

and let $\mathcal{D}(U)$ be either $\mathcal{D}_c(U)$ or $\mathcal{D}_b(U)$, classes of sets containing either the compact subsets or the bounded subsets of the metric space (U, d_U).

The definition of a global attractor for an autonomous semi-dynamical system $(X, \mathbb{T}^+, \pi)$ is well known. Specifically, a nonempty compact subset $\mathcal{A}$ of X which is π-invariant, that is, satisfies

$$\pi(t, \mathcal{A}) = \mathcal{A} \quad \text{for all} \quad t \in \mathbb{T}^+, \tag{2}$$

is called a *global attractor* for $(X, \mathbb{T}^+, \pi)$ with respect to $\mathcal{D}(X)$ if

$$\lim_{t \to \infty} \mathrm{dist}_X(\pi(t, D), \mathcal{A}) = 0 \tag{3}$$

for every $D \in \mathcal{D}(X)$. Conditions for the existence of such global attractors and examples can be found in [3, 8, 21, 37, 38]. Of course, semi-dynamical systems need not be a skew-product systems. When they are and when P is compact (in which case it suffices to consider the convergence (3) just for sets in $\mathcal{D}(U) \times \{P\}$, that is of the form $D \times P \subset X$, where $D \in \mathcal{D}(U)$), then the following definition will be used.

Definition 2.3 The *global attractor* $\mathcal{A}$ with respect to $\mathcal{D}(U) \times \{P\}$ of the skew-product dynamical system $(X, \mathbb{T}^+, \pi) = (U \times P, \mathbb{T}^+, (\phi, \sigma))$ with P compact will be called the *global attractor with respect to* $\mathcal{D}(U)$ *of the associated nonautonomous dynamical system* $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$.

Other types of attractors, in particular pullback attractors, that consist of a family of nonempty compact subsets of the state space of the cocycle mapping have been proposed for nonautonomous or random dynamical systems [15, 16, 27, 31, 32]. The main objective of this paper is to investigate the relationships between these different types of attractors.

Definition 2.4 Let $\widehat{A} = \{A(p)\}_{p \in P}$ be a family of nonempty compact sets of U for which $\bigcup_{p \in P} A(p)$ is pre-compact and let $\widehat{A}$ be ϕ-invariant with respect to a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$, that is, satisfies

$$\phi(t, A(p), p) = A(\sigma_t p) \quad \text{for all} \quad t \in \mathbb{T}^+, \quad p \in P. \tag{4}$$

The *family* $\widehat{A}$ is called a *pullback attractor* of $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with respect to $\mathcal{D}(U)$ if

$$\lim_{t \to \infty} \mathrm{dist}_U(\phi(t, D, \sigma_{-t} p), A(p)) = 0 \tag{5}$$

for any $D \in \mathcal{D}(U)$ and $p \in P$, or a *uniform pullback attractor* if the convergence (5) is uniform in $p \in P$, that is, if

$$\lim_{t \to \infty} \sup_{p \in P} \mathrm{dist}_U(\phi(t, D, \sigma_{-t} p), A(p)) = 0.$$

The *family* $\widehat{A}$ is called a *forward attractor* if the forward convergence

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,p), A(\sigma_t p)) = 0$$

holds instead of the pullback convergence (5), or a *uniform forward attractor* if this forward convergence is uniform in $p \in P$, that is, if

$$\lim_{t\to\infty} \sup_{p\in P} \mathrm{dist}_U(\phi(t,D,p), A(\sigma_t p)) = 0.$$

It follows directly from the definition that a pullback attractor is unique. Obviously, any uniform pullback attractor is also a uniform forward attractor, and vice versa.

If $\widehat{A}$ is a forward attractor for the nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$, then ([8], Lemma 4.2) the subset

$$\mathcal{A} = \bigcup_{p\in P} \big(A(p)\times\{p\}\big) \tag{6}$$

of X is the global attractor for the skew-product dynamical system $(X,\mathbb{T}^+,\pi)$; since the global attractor is unique so is the forward attractor. A weaker result holds when $\widehat{A}$ is a pullback attractor, but the inverse property is not true in general.

Although we could formulate our results with weaker assumptions we restrict our attention here to the case, which arises in certain important applications, where $\bigcup_{p\in P} A(p)$ is pre-compact and $\mathcal{D}(U)$ consists of compact or bounded sets. A further generalization which we will not consider here involves pullback attractors with a general domain of attraction $\mathcal{D}$ consisting of family of sets $D = \{D(p)\}_{p\in P}$ such that $\bigcup_{p\in P} D(p)$ is pre-compact or bounded in U, see [32].

The following existence result for pullback attractors is adapted from [16, 23].

Theorem 2.1 *Let $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$, with P compact be a nonautonomous dynamical system and suppose that there exists a family of nonempty sets $C = \{C(p)\}_{p\in P}$, $\bigcup_{p\in P} C(p)$ pre-compact such that*

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,\sigma_{-t}p), C(p)) = 0$$

for any bounded subset D of U and any $p\in P$. Then there exists a pullback attractor.

A related result is given by Theorem 4.3.4 in [8]: if the skew-product system $(X,\mathbb{T}^+,\pi)$ has a global attractor $\mathcal{A}$, then the nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ has a pullback attractor. The proof is based on the fact that the identical sets $C(p) \equiv \mathrm{pr}_U\,\mathcal{A}$ satisfy the assumptions of the previous theorem.

Alternatively, conditions can be given on the nonautonomous dynamical system to ensure the existence of a global attractor of the associated skew-product system. The following theorem is from [8].

Theorem 2.2 *Let $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ be a nonautonomous dynamical system with P* `compact` *for which*

(i) *ϕ is asymptotically compact, that is, for every bounded positive invariant set D and $p\in P$, there exists a compact set C such that*

$$\lim_{t\to\infty} \mathrm{dist}_U(\phi(t,D,p), C) = 0,$$

(ii) *there exists a bounded set B_0 that absorbs bounded subsets, that is, for every $p \in P$ and $D \in \mathcal{D}_b(U)$ there exists a $T_{p,D} \geq 0$ such that*

$$\phi(t, D, p) \subset B_0 \quad \textit{for all} \quad t \geq T_{p,D}.$$

Then the skew-product system $(X, \mathbb{T}^+, \pi)$ has a unique global attractor that attracts sets from $\mathcal{D}_b(U)$.

We now continue to derive properties of pullback attractors and the associated skew-product dynamical systems.

Lemma 2.1 *If $\widehat{A}$ is a pullback attractor of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$, where P is compact, then the subset $\mathcal{A}$ of X defined by (6) is the maximal π-invariant compact set of the associated skew-product dynamical system $(X, \mathbb{T}^+, \pi)$.*

Proof The π-invariance follows from the ϕ-invariance of $\widehat{A}$ via

$$\pi(t, \mathcal{A}) = \bigcup_{p \in P} (\phi(t, A(p), p), \sigma_t p) = \bigcup_{p \in P} (A(\sigma_t p), \sigma_t p) = \mathcal{A}.$$

Now $\mathcal{A} \subset \bigcup_{p \in P} A(p) \times P$, where P is compact and $\bigcup_{p \in P} A(p)$ is pre-compact, so $\mathcal{A}$ is pre-compact. Hence $\mathcal{B} := \bar{\mathcal{A}}$ is compact, from which it follows that

$$B(p) := \{u \colon (u, p) \in \mathcal{B}\}$$

is a compact set in U for each $p \in P$ and that the set

$$\bigcup_{p \in P} B(p) \subset \mathrm{pr}_1\, \mathcal{B}$$

is pre-compact. On the other hand, $\mathcal{B}$ is π-invariant since

$$\pi(t, \mathcal{B}) = \pi(t, \bar{\mathcal{A}}) = \overline{\pi(t, \mathcal{A})} = \bar{\mathcal{A}} = \mathcal{B}$$

for the continuous mapping $\pi(t, \cdot)$. In addition, $\phi(t, B(p), p) = B(\sigma_t p)$ holds, that is, the $B(p)$ are ϕ-invariant, since

$$\pi(t, \mathcal{B}) = \bigcup_{p \in P} (\phi(t, B(p), p), \sigma_t p) = \mathcal{B} = \bigcup_{p \in P} (B(\sigma_t p), \sigma_t p)$$

and $\sigma_t p = \sigma_t \hat{p}$ implies that $p = \hat{p}$ for the homeomorphism σ_t. The construction shows $B(p) \supset A(p)$. By the ϕ-invariance of the $B(p)$ and the pullback attraction property it follows then that $B(p) = A(p)$ such that $\mathcal{A} = \mathcal{B}$. Hence $\mathcal{A}$ is compact.

To prove that the compact invariant set $\mathcal{A}$ is maximal, let $\mathcal{A}'$ be any other compact invariant set the of skew-product dynamical system $(X, \mathbb{T}^+, \pi)$. Then $\widehat{A}' = \{A'(p)\}_{p \in P}$ is a family of compact ϕ-invariant subsets of U and by pullback attraction

$$\begin{aligned} \mathrm{dist}_U(A'(p), A(p)) &= \mathrm{dist}_U(\phi(t, A'(\sigma_{-t} p), \sigma_{-t} p), A(p)) \\ &\leq \mathrm{dist}_U(\phi(t, K, \sigma_{-t} p), A(p)) \to 0 \end{aligned}$$

as $t \to +\infty$, where $K = \overline{\bigcup_{p\in P} A'(p)}$ is compact. Hence $A'(p) \subseteq A(p)$ for every $p \in P$, i.e. $\widehat{A}' \subseteq \widehat{A}$, which means $\mathcal{A}$ is maximal for $(X, \mathbb{T}^+, \pi)$.

A set valued mapping M with $p \to M(p) \subset U$ for each $p \in P$ is called is *upper semi-continuous* if

$$\lim_{p\to p_0} d_U(M(p), M(p_0)) = 0 \quad \text{for any} \quad p_0 \in P,$$

lower semi-continuous if

$$\lim_{p\to p_0} d_U(M(p_0), M(p)) = 0 \quad \text{for any} \quad p_0 \in P,$$

and *continuous* if it is both upper and lower semi-continuous. Note that M is upper semi-continuous if and only if the its graph in $P \times U$ is closed in $P \times U$, see ([2, Proposition 1.4.8]). Then it follows straightforwardly from Lemma 2.1 that

Corollary 2.1 *The set valued mapping $p \to A(p)$ formed with the components sets of a pullback attractor $\widehat{A} = \{A(p)\}_{p\in P}$ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with P compact is upper semi-continuous.*

The following example shows that, in general, a pullback attractor need not also be a forward attractor nor form a global attractor of the associated skew-product dynamical system.

Example 2.1 Let f be the function on $\mathbb{R}$ defined by

$$f(t) = -\left(\frac{1+t}{1+t^2}\right)^2, \qquad t \in \mathbb{R},$$

and let $(P, \mathbb{R}, \sigma)$ be the autonomous dynamical system $P = H(f)$, the hull of f in $C(\mathbb{R}, \mathbb{R})$, with the shift operator σ. Note that

$$P = H(f) = \bigcup_{h\in\mathbb{R}} \{f(\cdot + h)\} \cup \{0\}.$$

Finally, let E be the evaluation functional on $C(\mathbb{R}, \mathbb{R})$, that is $E(p) = p(0) \in \mathbb{R}$.

Lemma 2.2 *The functional*

$$\gamma(p) = -\int_0^\infty e^{-\tau} E(\sigma_\tau p)\, d\tau = -\int_0^\infty e^{-\tau} p(\tau)\, d\tau$$

is well defined and continuous on P, and the function of $t \in \mathbb{R}$ given by

$$\gamma(\sigma_t p) = -e^t \int_t^\infty e^{-\tau} p(\tau)\, d\tau = \begin{cases} \dfrac{1}{1+(t+h)^2} & : \quad p = \sigma_h f \\ 0 & : \quad p = 0 \end{cases}$$

is the unique solution of the differential equation

$$x' = x + E(\sigma_t p) = x + p(t)$$

that exists and is bounded for all $t \in \mathbb{R}$.

The proof is by straightforward calculation, so will be omitted.

Consider now the nonautonomous differential equation

$$u' = g(\sigma_t(p), u), \tag{7}$$

where

$$g(p,u) := \begin{cases} -u - E(p)u^2 & : \quad 0 \le u\gamma(p) \le 1,\ p \ne 0, \\ -\dfrac{1}{\gamma(p)}\left(1 + \dfrac{E(p)}{\gamma(p)}\right) & : \quad 1 < u\,\gamma(p), \quad p \ne 0, \\ -u & : \quad 0 \le u, \quad p = 0. \end{cases}$$

It is easily seen that this equation has a unique solution passing through any point $u \in U = \mathbb{R}^+$ at time $t = 0$ defined on $\mathbb{R}$. These solutions define a cocycle mapping

$$\phi(t, u_0, p) = \begin{cases} \dfrac{u_0}{e^t(1 - u_0\gamma(p)) + u_0\gamma(\sigma_t p)} & : \quad 0 \le u_0\gamma(p) \le 1,\ p \ne 0, \\ u_0 + \dfrac{1}{\gamma(\sigma_t p)} - \dfrac{1}{\gamma(p)} & : \quad 1 < u_0\gamma(p),\ p \ne 0, \\ e^{-t}u_0 & : \quad 0 \le u_0,\ p = 0. \end{cases} \tag{8}$$

According to the construction, the cocycle mapping ϕ admits as its only invariant sets $A(p) = \{0\}$ for $p \in P$. To see that the $A(p) = \{0\}$ form a pullback attractor, observe that

$$\phi(t, u_0, \sigma_{-t}p) = \begin{cases} \dfrac{u_0}{e^t(1 - u_0\gamma(\sigma_{-t}p)) + u_0\gamma(p)} & : \quad 0 \le u_0\gamma(\sigma_{-t}p) \le 1,\ p \ne 0, \\ u_0 + \dfrac{1}{\gamma(p)} - \dfrac{1}{\gamma(\sigma_{-t}p)} & : \quad 1 < u_0\gamma(\sigma_{-t}p),\ p \ne 0, \\ e^{-t}u_0 & : \quad 0 \le u_0,\ p = 0. \end{cases}$$

In particular, note that $t \to \gamma(\sigma_t p)^{-1}$ is a solution of the differential equation (7). Since $\gamma(\sigma_{-t}p)^{-1}$ tends to $+\infty$ subexponentially fast for $t \to \infty$, it follows that

$$\phi(t, u, \sigma_{-t}p) \le \frac{1}{2}\, L\, e^{-\frac{1}{2}t}$$

for any $u \in [0, L]$, for any $L \ge 0$ and $p \in P$ provided t is sufficiently large. Consequently $\widehat{A} = \{A(p)\}_{p \in P}$ with $A(p) = \{0\}$ for all $p \in P$ is a pullback attractor for ϕ. In view of (8), the stable set $W^s(\mathcal{A}) := \{x \in X \mid \lim_{t \to +\infty} \operatorname{dist}_X(\pi(t,x), \mathcal{A}) = 0\}$ of $\mathcal{A}$, that is, the set of all points in X that are attracted to $\mathcal{A}$ by π, is given by

$$W^s(\mathcal{A}) = \{(u,p) \colon p \in P,\ u \ge 0,\ u\gamma(p) < 1\} \ne X.$$

Hence the cocycle mapping ϕ in this example admits a pullback attractor that is neither a forward attractor for ϕ nor a global attractor of the associated skew-product flow.

Other examples for different kinds of attractors are given by Scheutzow [30] for random dynamical systems generated by one dimensional stochastic differential equations. However, these considerations are based on the theory of Markov processes.

3 Asymptotic Stability in α-condensing Semi-dynamical Systems

To continue our investigation of the general relations between pullback attractors and skew-product flows we first need to derive some results from general stability theory. We start with some definitions.

Let $(X, \mathbb{T}^+, \pi)$ be a semi-dynamical system. The *ω-limit set* of a set M is defined to be

$$\omega(M) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \pi(t, M)}.$$

A set M is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\pi(t, \mathcal{U}_\delta(M)) \subset \mathcal{U}_\varepsilon(M)$ for $t \geq 0$. M is called a *local attractor* if there exists a neighborhood $\mathcal{U}(M)$ of M such that $\mathcal{U}(M) \subset W^s(M)$. A set M which is Lyapunov stable and a local attractor is said to be *asymptotically stable.* Note that any asymptotically stable compact set M also attracts compact sets contained in $\mathcal{U}(M)$.

Recall that a π-invariant compact set M is said to be *locally maximal* if there exists a number $\delta > 0$ such that any π-invariant compact set contained in the open δ-neighborhood $U_\delta(M)$ of M is in fact contained in M. In addition, a mapping $\gamma^x : \mathbb{T} \to X$ is called an *entire trajectory through* x of the semi-dynamical system $(X, \mathbb{T}^+, \pi)$ if

$$\pi(t, \gamma^x(\tau)) = \gamma^x(t + \tau) \quad \text{for all} \quad t \in \mathbb{T}^+, \ \tau \in \mathbb{T}, \ \gamma^x(0) = x.$$

Finally, the *alpha limit set* of an entire trajectory γ^x is defined by

$$\alpha_{\gamma^x} = \{y \in X : \exists\, \tau_n \to -\infty, \, \gamma^x(\tau_n) \to y\}.$$

Let α be a measure of noncompactness on the bounded subsets of a complete metric space (Y, d_Y) ([21], P.13 ff.). Then $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$ for all nonempty bounded subsets A and B with $\alpha(A) = 0$ whenever A is pre-compact. An example is the Kuratowski measure of noncompactness defined by

$$\alpha(A) = \inf\{d : A \quad \text{has a finite cover of diameter} \quad < d\}.$$

An autonomous semi-dynamical system $(X, \mathbb{T}^+, \pi)$ is called *α-condensing* if $\pi(t, B)$ is bounded and

$$\alpha(\pi(t, B)) < \alpha(B)$$

for all $t > 0$ for any bounded set B of X with $\alpha(B) > 0$.

Remark 3.1 Examples of α-condensing systems can be found in Hale [21].

Theorem 3.1 *Let M be a locally maximal compact set for an α-condensing semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then M is Lyapunov stable if and only if there exists a $\delta > 0$ such that*

$$\alpha_{\gamma^x} \cap M = \emptyset$$

for any entire trajectory γ^x through any $x \in U_\delta(M) \setminus M$.

Proof A proof of the necessity direction was given by Zubov in [39, Theorem 7] for a locally compact space X. This proof remains also true for a nonlocally compact space under consideration here. Indeed, let M be a compact invariant Lyapunov stable set for $(X, \mathbb{T}^+, \pi)$, but suppose that the assertion of the theorem is not true. Then there exist $x \notin M$, and a sequence $\tau_n \to -\infty$ such that $\rho(\gamma^x(\tau_n), M) \to 0$ for $n \to \infty$. Let $0 < \varepsilon < \rho(x, M)$ and $\delta(\varepsilon) > 0$ the corresponding positive number for the Lyapunov stability of set M, then for sufficiently large n we have $\rho(\gamma^x(\tau_n), M) < \delta(\varepsilon)$. Consequently, $\rho(\pi^t\gamma^x(\tau_n), M) < \varepsilon$ for all $t \geq 0$. In particular for $t = -\tau_n$ we have $\rho(x, M) = \rho(\pi^{-\tau_n}\gamma^x(\tau_n), M) < \varepsilon$. The obtained contradiction proves our assertion.

For the sufficiency direction, consider first the case $\mathbb{T}^+ = \mathbb{Z}^+$ and let $U_{\delta_0}(M)$ be a neighborhood such that M is locally maximal in $U_{\delta_0}(M)$. Suppose that M is not Lyapunov stable, but that the other condition of the theorem holds. Then there exist an $\varepsilon_0 > 0$ and sequences $\delta_n \to 0$, $x_n \in U_{\delta_n}(M)$, $k_n \to \infty$ such that $\pi(k, x_n) \in U_{\varepsilon_0}(M)$ for $0 \leq k \leq k_n - 1$ and $\pi(k_n, x_n) \notin U_{\varepsilon_0}(M)$. This ε_0 has to be chosen sufficiently small such that

$$\mathrm{dist}_X(\pi(1, U_{\varepsilon_0}(M)), M) < \frac{\delta_0}{2}.$$

Define $A = \{x_n\}$ and $B = \bigcup_{n\in\mathbb{N}} \{\pi(k, x_n) | 0 \leq k \leq k_n - 1\}$. Then $\alpha(A) = 0$ since A is pre-compact. In addition, $\pi(1, B) \subset U_{\delta_0}(M)$, so $\pi(1, B)$ is bounded. Suppose that B is not pre-compact, so $\alpha(B) > 0$. It follows by the properties of the measure of noncompactness for the non pre-compact set B that

$$\alpha(B) = \alpha(A \cup \pi(1, B) \cap B) \leq \max(\alpha(A), \pi(1, B)) = \alpha(\pi(1, B)) < \alpha(B)$$

which is a contradiction. This shows that B is pre-compact. Hence there exist subsequences (denoted with the same indices for convenience) such that

$$\pi(k_n - 1, x_n) \to \bar{x}, \quad \pi(k_n, x_n) \to \pi(1, \bar{x}) = \tilde{x} \in X \setminus U_{\varepsilon_0}(M) \quad \text{for} \quad n \to \infty,$$

the limit $\tilde{\gamma}^{\tilde{x}}(m) := \lim_{n' \to \infty} \gamma(n', m)$ exists for any $m \in \mathbb{Z}$ and some subsequence n' given by the diagonal procedure, where

$$\gamma(n, m) = \begin{cases} \pi(k_n + m, x_n) & : \quad -k_n \leq m < +\infty, \\ x_n & : \quad m < -k_n. \end{cases}$$

Note that $\tilde{\gamma}^{\tilde{x}}$ is an entire trajectory of the discrete-time semi-dynamical system above with $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x}$ and $\overline{\tilde{\gamma}^{\tilde{x}}(\mathbb{Z}^-)} \subset \bar{B}$. Thus the alpha limit set $\alpha_{\tilde{\gamma}^{\tilde{x}}}$ is nonempty, compact and invariant. In addition, $\alpha_{\tilde{\gamma}^{\tilde{x}}} \subset U_{\varepsilon_0}(M)$, hence $\alpha_{\tilde{\gamma}^{\tilde{x}}} \subset M$ because M is a locally maximal invariant compact set. On the other hand, $\tilde{\gamma}^{\tilde{x}}(0) = \tilde{x} \in U_{\varepsilon_0}(M) \setminus M$, so $\alpha_{\tilde{\gamma}^{\tilde{x}}} \cap M = \emptyset$ holds by the assumptions. This contradiction proves the sufficiency of the condition in the discrete-time case.

Now let $\mathbb{T}^+ = \mathbb{R}^+$ and suppose that $\alpha_{\gamma^x} \cap M = \emptyset$, where $x \notin M$ holds for the continuous-time system. Then it also holds for the restricted discrete-time system generated by $\pi_1 := \pi(1, \cdot)$ because any entire trajectory γ^x of the restricted discrete-time system can be extended to an entire trajectory of the continuous-time system via

$$\gamma^x(t) = \pi(\tau, \gamma^x(n)), \quad n \in \mathbb{Z}, \quad t = n + \tau, \quad 0 < \tau < 1.$$

Consequently, the set M is Lyapunov stable with respect to the restricted discrete-time dynamical system generated by π_1. Since M is compact, for every $\varepsilon > 0$ there exists a $\mu > 0$ such that

$$d_X(\pi(t, x), M) < \varepsilon \quad \text{for all} \quad t \in [0, 1], \;\; x \in U_\mu(M).$$

In view of the first part of the proof above, there is a $\delta > 0$ such that

$$d_X(\pi(n,x), M) < \min(\mu, \varepsilon) \quad \text{for} \quad x \in U_\delta(M) \quad \text{for} \quad n \in \mathbb{Z}^+.$$

The Lyapunov stability of M for the continuous dynamical system $(X, \mathbb{R}^+, \pi)$ then follows from the semi-group property of π.

The next lemma will be needed to formulate the second main theorem of this section.

Lemma 3.1 *Let M be a compact subset of X that is positively invariant for a semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then M is asymptotically stable if and only if $\omega(M)$ is locally maximal and asymptotically stable.*

Proof Suppose that M is asymptotically stable. Then there exists a closed positively invariant bounded neighborhood C of M contained in its stable set $W^s(M)$. The mapping π can be restricted to the complete metric space (C, d_X) to form a semi-dynamical system $(C, \mathbb{T}^+, \pi)$. Since M is a locally attracting set it attracts compact subsets of C. The assertion then follows by Theorems 2.4.2 and 3.4.2 in [21] because $\omega(M) = \bigcap_{t \in \mathbb{T}^+} \pi(t, M)$.

Suppose instead that $\omega(M)$ is asymptotically stable and locally maximal. Since M is compact, $\omega(M) = \bigcap_{t \geq 0} \pi(t, M)$. Hence there exist $\eta > 0$ and $\tau \in \mathbb{T}^+$ such that

$$\pi(\tau, M) \subset U_\eta(\omega(M)) \subset W^s(\omega(M)).$$

Now $\pi^{-1}(\tau, U_\eta(\omega(M)))$, where π^{-1} denotes the pre-image of $\pi(\tau, \cdot)$ for fixed τ, is an open neighborhood of M and $\pi(\tau, \pi^{-1}(\tau, U_\eta(\omega(M)))) \subset W^s(\omega(M))$. Hence for any $x \in \pi^{-1}(\tau, U_\eta(\omega(M))) \subset W^s(\omega(M))$ we have that $\pi(t, x)$ tends to $\omega(M)$ as $t \to \infty$, from which it follows that $\pi(t, x)$ also tends to M because $M \supset \omega(M)$.

Then, if M were not Lyapunov stable, there would exist $\varepsilon_0 > 0$, $\delta_n \to 0$, $x_n \in U_{\delta_n}(M)$ and $t_n \to \infty$ such that

$$\text{dist}_X(\pi(t_n, x_n), M) \geq \varepsilon_0. \tag{9}$$

For sufficiently large n_0, the set $\overline{\{x_n\}}_{n \geq n_0}$ would then be contained in the pre-image $\pi^{-1}(1, U_\eta(\omega(M)))$. Since M is compact, so is the set $\overline{\{x_n\}}_{n \geq n_0}$. This set would thus be attracted by $\omega(M) \subset M$, which contradicts (9).

Lemma 3.2 *Let M be a compact subset of X that is a positively invariant set for an asymptotically compact semi-dynamical system $(X, \mathbb{T}^+, \pi)$. Then the set M is asymptotically stable if and only if $\omega(M)$ is locally maximal and Lyapunov stable.*

Proof The necessity follows by Lemma 3.1. Suppose instead that $\omega(M)$ is locally maximal and Lyapunov stable; it is automatically π-invariant since it is an omega limit set. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\pi(t, U_\delta(\omega(M))) \subset U_\varepsilon(\omega(M)) \quad \text{for all} \quad t \geq 0.$$

By the assumption of asymptotical compactness, $\omega(U_\delta(\omega(M)))$ is nonempty and compact with

$$\lim_{t \to \infty} \text{dist}_X(\pi(t, U_\delta(\omega(M))), \omega(U_\delta(\omega(M)))) = 0$$

(see [21, Corollary 2.2.4]). Since $\omega(M)$ is locally maximal, $\omega(U_\delta(\omega(M))) \subset \omega(M)$ for sufficiently small $\delta > 0$, which means that $\omega(M)$ is asymptotically stable. The conclusion then follows by Lemma 3.1.

Corollary 3.1 *Let $(X, \mathbb{T}^+, \pi)$ be asymptotically compact and let M be a compact π-invariant set. Then M is asymptotically stable if and only if M is locally maximal and Lyapunov stable.*

Indeed, $M = \omega(M)$ here, so just apply Lemma 3.2.

The next theorem is a generalization to infinite dimensional spaces and α-condensing systems of Theorem 8 of Zubov [39] characterizing the asymptotic stability of a compact set.

Theorem 3.2 *Let $(X, \mathbb{T}^+, \pi)$ be an α-condensing semi-dynamical system and let $M \subset X$ be a compact invariant set. Then the set M is asymptotically stable if and only if*

(i) *M is locally maximal, and*
(ii) *there exists a $\delta > 0$ such that $\alpha_{\gamma^x} \cap M = \emptyset$ for any entire trajectory γ^x through any $x \in U_\delta(M) \setminus M$.*

Proof By Lemma 2.3.5 in [21] any α-condensing semi-dynamical system is asymptotically compact, so the assertion follows easily from Theorem 3.1 and Corollary 3.1.

A cocycle mapping ϕ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ will be called α-condensing if the set $\phi(t, B, P)$ is bounded and

$$\alpha(\phi(t, B, P)) < \alpha(B)$$

for all $t > 0$ for any bounded subset B of U with $\alpha(B) > 0$.

Lemma 3.3 *Suppose that the cocycle mapping ϕ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is α-condensing. Then the mapping π of the associated skew-product flow $(P, \mathbb{T}, \sigma)$ is also α-condensing.*

Proof Let $M = \bigcup_{p \in P} (M(p) \times \{p\})$ be a bounded set in X. Then M can be covered by finitely many balls $M_i \subset X$, $i = 1, \ldots, n$, of largest radius $\alpha(M) + \varepsilon$ for an arbitrary $\varepsilon > 0$. The sets $\operatorname{pr}_1 M_i \subset U$, $i = 1, \ldots, n$, cover $\operatorname{pr}_1 M$. The sets M_i are balls so $\alpha(\operatorname{pr}_1 M_i) = \alpha(M_i) < \alpha(M) + \varepsilon$ for $i = 1, \ldots, n$. It is easily seen that

$$\pi(t, M) = \bigcup_{p \in P} \{\pi(t, (M(p), p))\} = \bigcup_{p \in P} \{(\phi(t, M(p), p), \sigma_t p)\} \subset \phi(t, \operatorname{pr}_1 M, P) \times P.$$

Since ϕ is α-condensing, the set $\phi(t, \operatorname{pr}_1 M, P)$ is bounded. Hence

$$\begin{aligned} \alpha(\pi(t, M)) &\leq \alpha(\phi(t, \operatorname{pr}_1 M, P) \times P) \\ &\leq \alpha(\phi(t, \operatorname{pr}_1 M, P)) < \alpha(\operatorname{pr}_1 M) \leq \alpha(M) \quad \text{for each} \quad t > 0. \end{aligned} \tag{10}$$

The second inequality above is true by the compactness of P. Indeed, P can be covered by finitely many open balls P_i of arbitrarily small radius. Hence

$$\alpha(\phi(t, \operatorname{pr}_1 M, P) \times P) \leq \max_i \alpha(\phi(t, \operatorname{pr}_1 M, P) \times P_i) \leq \alpha(\phi(t, \operatorname{pr}_1 M, P)) + \varepsilon$$

for arbitrarily small $\varepsilon > 0$. The conclusion of the Lemma follows by (10).

4 Uniform Pullback Attractors and Global Attractors

It was seen earlier that the set $\bigcup_{p\in P}(A(p)\times\{p\})\subset X$ which was defined in terms of the pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ of a nonautonomous dynamical system $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ is the maximal π-invariant compact subset of the associated skew-product system $(X,\mathbb{T}^+,\pi)$, but need not be a global attractor. However, this set is always a *local* attractor under the additional assumption that the cocycle mapping ϕ is α-condensing.

Theorem 4.1 *Let $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ with P compact be an α-condensing dynamical system with a pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ and define $\mathcal{A}=\bigcup_{p\in P}(A(p)\times\{p\})$. Then*

(i) *The α-limit set α_{γ^x} of any entire trajectory γ^x passing through $x\in X\setminus\mathcal{A}$ is empty.*
(ii) *$\mathcal{A}$ is asymptotically stable with respect to π.*

Proof Suppose that there exists an entire trajectory γ^x through $x=(u,p)\in X\setminus\mathcal{A}$ such that $\alpha_{\gamma^x}\neq\emptyset$. Then there exists a subsequence $-\tau_n\to\infty$ such that $\gamma^x(\tau_n)$ converges to a point in α_{γ^x}. The set $K=\mathrm{pr}_1\overline{\bigcup_{n\in\mathbb{N}}\gamma^x(\tau_n)}$ is compact since $\overline{\bigcup_{n\in\mathbb{N}}\gamma^x(\tau_n)}$ is compact. Also $\widehat{A}=\{A(p)\}_{p\in P}$ is a pullback attractor, so

$$\lim_{n\to\infty}\mathrm{dist}_U(\phi(-\tau_n,K,\sigma_{\tau_n}p),A(p))=0$$

from which it follows that $u\in A(p)$. Hence $(u,p)\in\mathcal{A}$, which is a contradiction. This proves the first assertion.

By Lemma 3.3 $(X,\mathbb{T}^+,\pi)$ is α-condensing. According to Lemma 2.1 $\mathcal{A}$ is a maximal compact invariant set of $(X,\mathbb{T}^+,\pi)$ since $\widehat{A}$ is a pullback attractor of the cocycle ϕ. The second assertion then follows from Theorem 3.2 and from the first assertion of this theorem.

Remark 4.1 (i) The skew-product system in the example in Section 2 has only a *local* attractor associated with the pullback attractor.

(ii) If in addition to the assumptions of Theorem 4.1 the stable set $W^s(\mathcal{A})$ of $\mathcal{A}$ satisfies $W^s(\mathcal{A})=X$, then $\mathcal{A}$ is in fact a global attractor ([6, Lemma 7]).

Theorem 4.2 *Suppose that $\langle U,\phi,(P,\mathbb{T},\sigma)\rangle$ with P compact is a nonautonomous dynamical system with a pullback attractor $\widehat{A}=\{A(p)\}_{p\in P}$ and suppose that $W^s(\mathcal{A})=X$, where $\mathcal{A}=\bigcup_{p\in P}(A(p)\times\{p\})$.*

If the mapping $p\to A(p)$ is lower semi-continuous, then $\widehat{A}$ is a uniform pullback attractor and hence a uniform forward attractor.

Proof Suppose that the uniform convergence

$$\lim_{t\to\infty}\sup_{p\in P}\mathrm{dist}_U(\phi(t,D,p),A(\sigma_t p))=0$$

is not true for some $D\in\mathcal{D}_c$. Then there exist $\varepsilon_0>0$, a set $D_0\in\mathcal{D}_c$ and sequences $t_n\to\infty$, $p_n\in P$ and $u_n\in D_0$ such that:

$$\mathrm{dist}_U(\phi(t_n,u_n,p_n),A(\sigma_{t_n}p))\geq\varepsilon_0. \tag{11}$$

Now P is compact and $\mathcal{A}$ is a global attractor by Remark 4.1 (ii), so it can be supposed that the sequences $\{\phi(t_n, u_n, p_n)\}$ and $\{\sigma_{t_n} p\}$ are convergent. Let $\bar{u} = \lim_{n\to\infty} \phi(t_n, u_n, p_n)$ and $\bar{p} = \lim_{n\to\infty} \sigma_{t_n} p_n$. Then $\bar{u} \in A(\bar{p})$ because $\bar{x} = (\bar{u}, \bar{p}) \in \mathcal{A}$. On the other hand, by (11),

$$\begin{aligned}\varepsilon_0 &\le \operatorname{dist}_U(\phi(t_n, p_n, x_n), A(\sigma_{t_n} p_n)) \\ &\le \operatorname{dist}_U(\phi(t_n, p_n, x_n), A(\bar{p})) + \operatorname{dist}_U(A(\bar{p}), A(\sigma_{t_n} p_n)).\end{aligned}$$

By the lower semi-continuity of $p \to A(p)$ it follows then that $\bar{u} \notin A(\bar{p})$, which is a contradiction.

Remark 4.2 The example in Section 2 shows that Theorem 4.2 is in general not true without the assumption that $W^s(\mathcal{A}) = X$. In view of Corollary 2.1, the set valued mapping $p \to A(p)$ will, in fact, then be continuous here.

5 Examples of Uniform Pullback Attractors

Several examples illustrating the application of the above results, in particular of Theorem 4.2, are now presented. More complicated examples will be discussed in another paper.

5.1 Periodic driving systems

Consider a periodical dynamical system $(P, \mathbb{T}, \sigma)$, that is, for which there exists a minimal positive number T such that $\sigma_T p = p$ for any $p \in P$.

Theorem 5.1 *Suppose that a nonautonomous α-condensing dynamical system $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$ with a periodical dynamical system $(P, \mathbb{R}, \sigma)$ has a pullback attractor $\widehat{A} = \{A(p)\}_{p\in P}$. Then $\widehat{A}$ is a uniform forward attractor for $\langle U, \phi, (P, \mathbb{R}, \sigma)\rangle$.*

Proof Consider a sequence $p_n \to p$. By the periodicity of the driving system there exists a sequence $\tau_n \in [0, T]$ such that $p_n = \sigma_{\tau_n} p$. By compactness, there is a convergent subsequence (indexed here for convenience like the full one) $\tau_n \to \tau \in [0, T]$. Hence

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} \sigma_{\tau_n} p = \sigma_\tau p$$

which means $\tau = 0$ or T. Suppose that $\tau = T$. Then

$$\begin{aligned}\lim_{n\to\infty} \operatorname{dist}_U(A(p), A(p_n)) &= \lim_{n\to\infty} \operatorname{dist}_U(A(p), \phi(\tau_n, A(p), p)) \\ &= \operatorname{dist}_U(A(p), \phi(T, A(p), p)) \\ &= \operatorname{dist}_U(A(p), A(\sigma_T p)) = 0\end{aligned}$$

since ϕ is continuous and $A(p_n) = A(\sigma_{\tau_n} p) = \phi(\tau_n, A(p), p)$ by the ϕ-invariance of $\widehat{A}$. Hence the set valued $p \to A(p)$ is lower semi-continuous.

Now $\phi(nT, u_0, p) = \phi(nT, u_0, \sigma_{-nT} p)$ since $p = \sigma_{-nT} p$ by the periodicity of the driving system $(P, \mathbb{R}, \sigma)$. Hence from pullback convergence

$$\lim_{n\to\infty} \operatorname{dist}_U(\phi(nT, u_0, p), A(p)) = \lim_{n\to\infty} \operatorname{dist}_U(\phi(nT, u_0, \sigma_{-nT} p), A(p)) = 0$$

for any $(u_0, p) \in U \times P$. On the other hand

$$\sup_{s\in[0,T]} \mathrm{dist}_U(\phi(s+nT, u_0, p), A(\sigma_{s+nT}p))$$
$$= \sup_{s\in[0,T]} \mathrm{dist}_U(\phi(s, \phi(nT, u_0, p), \sigma_{nT}p), \phi(s, A(\sigma_{nT}p), \sigma_{nT}p)) = 0$$

by the cocycle property of ϕ and the ϕ-invariance of $\widehat{A}$. Hence

$$\lim_{t\to\infty} \mathrm{dist}_X((\phi(t, u_0, p), \sigma_t p), \mathcal{A}) = 0,$$

where $\mathcal{A} = \bigcup\limits_{p\in P} (A(p) \times \{p\})$. This shows that $W^s(\mathcal{A}) = X$. The result then follows by Theorem 4.2.

Consider the 2-dimensional Navier–Stokes equation in the operator form

$$\frac{du}{dt} + \nu Au + B(u) = f(t), \qquad u(0) = u_0 \in H, \tag{12}$$

which can be interpreted as an evolution equation on the rigged space $V \subset H \subset V'$, where V and H are certain Banach spaces. In particular, here $U = H$, which is in fact a Hilbert space, for the phase space. Then, from [36, Chapter 3],

Lemma 5.1 *The 2-dimensional Navier–Stokes equation (12) has a unique solution $u(\cdot, u_0, f)$ in $C(0, T; H)$ for each initial condition $u_0 \in H$ and forcing term $f \in C(0, T; H)$ for every $T > 0$. Moreover, $u(t, u_0, f)$ depends continuously on (t, u_0, f) as a mapping from $\mathbb{R}^+ \times H \times C(\mathbb{R}, H)$ to H.*

Now suppose that f is a periodic function in $C(\mathbb{R}, H)$ and define $\sigma_t f(\cdot) := f(\cdot + t)$. Then $P = \bigcup\limits_{t\in\mathbb{R}} \sigma_t f$ is a compact subset of $C(\mathbb{R}, H)$. By Lemma 5.1 the mapping $(t, u_0, p) \to \phi(t, u_0, p)$ from $\mathbb{R}^+ \times H \times C(\mathbb{R}, H) \to H$ defined by $\phi(t, u_0, p) = u(t, u_0, p)$ is continuous and forms a cocycle mapping with respect to σ on P. By [36, Theorem III.3.10] the mapping ϕ is completely continuous and hence α-condensing.

Lemma 5.2 *The nonautonomous dynamical system $\langle H, \phi, (P, \mathbb{R}, \sigma)\rangle$ generated by the Navier–Stokes equation (12) with periodic forcing term in $C(\mathbb{R}, H)$ has a pullback attractor.*

Proof The solution of the Navier–Stokes equation satisfies an energy inequality

$$\|u(t)\|_H^2 + \lambda_1 \nu \int_0^t \|u(\tau)\|_H^2 d\tau \le \|u_0\|_H^2 + \frac{1}{\nu} \int_0^t \|p(\tau)\|_{V'}^2 \, d\tau,$$

where λ_1 is the smallest eigenvalue of A. It follows that the balls $B(p)$ in H with center zero and square radii

$$R^2(p) = \frac{1}{\nu} \int_{-\infty}^0 e^{\nu\lambda_1\tau} \|p(\tau)\|_{V'}^2 \, d\tau$$

is a pullback attracting family of sets in the sense of Theorem 2.1. In particular, $C(p) := \phi(1, B(\sigma_{-1}p), \sigma_{-1}p)$ satisfies all of the required properties of Theorem 2.1 because $\phi(1, \cdot, p)$ is completely continuous.

This theorem and Theorem 5.1 give

Theorem 5.2 *The nonautonomous dynamical system $\langle H, \phi, (P, \mathbb{R}, \sigma)\rangle$ generated by the Navier–Stokes equation (12) with periodic forcing term in $C(\mathbb{R}, H)$ has a uniform pullback attractor which is also a uniform forward attractor.*

Remark 5.1 See [18] for a related result involving a different type of nonautonomous attractor.

5.2 Pullback attractors with singleton component sets

Now pullback attractors with singleton component sets, that is with

$$A(p) = \{a(p)\}, \quad a(p) \in U,$$

will be considered.

Lemma 5.3 *Let $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ a nonautonomous dynamical system and let $\widehat{A} = \{A(p)\}_{p\in P}$ be a pullback attractor with singleton component sets. Then the mapping $p \to A(p)$ is continuous, hence lower semi-continuous.*

Proof This follows from Corollary 2.1 since the upper semi-continuity of a set valued mapping $p \to A(p)$ reduces to continuity when the $A(p)$ are single point sets.

It follows straightforwardly from this lemma and Theorem 4.2 that

Theorem 5.3 *Suppose that $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ with compact P has at pullback attractor $\widehat{A} = \{A(p)\}_{p\in P}$ with singleton component sets which generates a global attractor $\mathcal{A} = \bigcup_{p\in P} A(p) \times \{p\}$. Then $\widehat{A}$ is a uniform pullback attractor and, hence, also a uniform forward attractor.*

The previous theorem can be applied to differential equations on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ of the form

$$u' = F(\sigma_t p, u), \tag{13}$$

where $F \in C(P \times H, H)$ is uniformly dissipative, that is, there exist $\nu \geq 2$, $\alpha > 0$

$$\langle F(p, u_1) - F(p, u_2), u_1 - u_2\rangle \leq -\alpha \|u_1 - u_2\|^\nu \tag{14}$$

for any u_1, $u_2 \in H$ and $p \in P$.

Theorem 5.4 [10] *The nonautonomous dynamical system $\langle H, \varphi, (P, \mathbb{T}, \sigma)\rangle$ generated by (13) has a uniform pullback attractor that consists of singleton component subsets.*

For example, the equation

$$u' = F(\sigma_t p, x) = -u|u| + g(\sigma_t p)$$

with $g \in C(P, \mathbb{R})$ satisfies

$$\langle u_1 - u_2, F(\sigma_t p, u_1) - F(\sigma_t p, u_2)\rangle \leq -\frac{1}{2}\,|u_1 - u_2|^2(|u_1| + |u_2|) \leq -\frac{1}{2}\,|u_1 - u_2|^3,$$

which is condition (14) with $\alpha = \frac{1}{2}$ and $\nu = 3$.

The above considerations apply also to nonlinear nonautonomous partial differential equations with a uniform dissipative structure, such as the dissipative quasi-geostrophic equations

$$\omega_t + J(\psi, \omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x, y, t), \tag{15}$$

with relative vorticity $\omega(x, y, t) = \Delta\psi(x, y, t)$, where $J(f, g) = f_x g_y - f_y g_x$ is the Jacobian operator. This equation can be supplemented by homogeneous Dirichlet boundary conditions for both ψ and ω

$$\psi(x, y, t) = 0, \quad \omega(x, y, t) = 0 \quad \text{on} \quad \partial D, \tag{16}$$

and an initial condition,

$$\omega(x, y, 0) = \omega_0(x, y) \quad \text{on} \quad D,$$

where D is an arbitrary bounded planar domain with area $|D|$ and piecewise smooth boundary. Let U be the Hilbert space $L_2(D)$ with norm $\|\cdot\|$.

Theorem 5.5 *Assume that*

$$\frac{r}{2} + \frac{\pi\nu}{|D|} > \frac{1}{2}\,\beta\left(\frac{|D|}{\pi} + 1\right)$$

and that the wind forcing $f(x, y, t)$ is temporally almost periodic with its $L^2(D)$-norm bounded uniformly in time $t \in \mathbb{R}$ by

$$||f(\cdot,\cdot,t)|| \le \sqrt{\frac{\pi r}{3|D|}}\left[\frac{r}{2} + \frac{\pi\nu}{|D|} - \frac{1}{2}\,\beta\left(\frac{|D|}{\pi} + 1\right)\right]^{\frac{3}{2}}.$$

Then the dissipative quasigeostrophic model (15) – (16) has a unique temporally almost periodic solution that exists for all time $t \in \mathbb{R}$.

The proof in [17] involves explicitly constructing a uniform pullback and forward absorbing ball in $L^2(D)$ for the vorticity ω, hence implying the existence of a uniform pullback attractor as well as a global attractor for the associate skew-product flow system for which the component sets are singleton sets. The parameter set P here is the hull of the forcing term f in $L^2(D)$ and a completely continuous cocycle mapping $\phi(t, u_0, p) = \omega(t, u_0, p)$ with respect to the shift operator σ on P that is continuous in all variables.

5.3 Distal dynamical systems

A function $\gamma^{(u,p)}\colon \mathbb{R} \to U$ represents an entire trajectory $\gamma^{(u,p)}$ of a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ if $\gamma^{(u,p)}(0) = u \in U$ and $\phi(t, \gamma^{(u,p)}(\tau), \sigma_\tau p) = \gamma^{(u,p)}(t+\tau)$ for $t \ge 0$ and $\tau \in \mathbb{R}$. A nonautonomous dynamical system is called *distal on* $\mathbb{T}^-$ if

$$\inf_{t\in\mathbb{T}^-} d_U\left(\gamma^{(u_1,p)}(t), \gamma^{(u_2,p)}(t)\right) > 0$$

for any entire trajectories $\gamma^{(u_1,p)}$ and $\gamma^{(u_2,p)}$ with $u_1 \ne u_2 \in U$ and any $p \in P$. A nonautonomous dynamical system is said to be *uniformly Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$d_U(\phi(t, u_1, p), \phi(t, u_2, p)) < \delta$$

for all u_1, $u_2 \in U$ with $d_U(u_1, u_2) < \varepsilon$, $p \in P$ and $t \ge 0$. Finally, an autonomous dynamical system $(P, \mathbb{T}, \sigma)$ is called *minimal* if P does not contain proper compact subsets which are σ-invariant.

The following lemma is due to Furstenberg [20] (see also [4, Chapter 3] or [28, Chapter 7, Proposition 4]).

Lemma 5.4 *Suppose that a nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is distal on $\mathbb{T}^-$ and that $(P, \mathbb{T}, \sigma)$ is minimal. In addition suppose that a compact subset $\mathcal{A}$ of X is π-invariant with respect to the skew-product system $(X, \mathbb{T}^+, \pi)$. Then the mapping $p \to A(p) := \{u \in U : (u, p) \in \mathcal{A}\}$ is continuous.*

The following theorem gives the existence of uniform forward attractors.

Theorem 5.6 *Suppose that the nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is uniformly Lyapunov stable and that the skew-product system $(X, \mathbb{T}^+, \pi)$ has a global attractor $\mathcal{A} = \bigcup_{p \in P} A(p) \times \{p\}$. Then $\widehat{A} = \{A(p)\}_{p \in P}$ is a uniform forward attractor for $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$.*

Proof Suppose that the nonautonomous dynamical system $\langle U, \phi, (P, \mathbb{T}, \sigma)\rangle$ is not distal. Then there is a $p_0 \in P$, a sequence $t_n \to \infty$ and entire trajectories $\gamma^{(u_1, p_0)}$, $\gamma^{(u_2, p_0)}$ with $u_1 \neq u_2$ such that

$$\lim_{n \to \infty} d_U\Big(\gamma^{(u_1, p_0)}(-t_n), \gamma^{(u_2, p_0)}(-t_n)\Big) = 0.$$

Let $\varepsilon = d_U(u_1, u_2) > 0$ and choose $\delta = \delta(\varepsilon) > 0$ from the uniformly Lyapunov stability property. Then

$$d_U\Big(\gamma^{(u_1, p_0)}(-t_n), \gamma^{(u_2, p_0)}(-t_n)\Big) < \delta$$

for sufficiently large n. Hence

$$d_U\Big(\phi(t, \gamma^{(u_1, p_0)}(-t_n), \sigma_{-t_n} p_0), \phi(t, \gamma^{(u_2, p_0)}(-t_n), \sigma_{-t_n} p_0)\Big) < \varepsilon$$

for $t \geq 0$ and, in particular,

$$\varepsilon = d_U(u_1, u_2) = d_U\Big(\phi(t_n, \gamma^{(u_1, p_0)}(-t_n), \sigma_{-t_n} p_0), \phi(t_n, \gamma^{(u_2, p_0)}(-t_n), \sigma_{-t_n} p_0)\Big) < \varepsilon$$

for $t = t_n$, which is a contradiction. The nonautonomous dynamical system is thus distal, so $p \to A(p)$ is continuous by Lemma 5.4. The result then follows from Theorem 4.2 since $\{A(p)\}_{p \in P}$ generates a pullback attractor.

This theorem will now be applied to the nonautonomous differential equation (13) on a Hilbert space H, which is assumed to generate a cocycle ϕ that is continuous on $\mathbb{T}^+ \times P \times H$ and asymptotically compact.

Theorem 5.7 *Suppose that $F \in C(H \times P, H)$ satisfies the dissipativity conditions*

$$\langle F(u_1, p) - F(u_2, p), u_1 - u_2 \rangle \leq 0, \tag{17}$$

$$\langle F(u, p), u \rangle \leq -\mu(|u|) \tag{18}$$

for u_1, u_2, $u \in H$ and $p \in P$, where $\mu \colon [R, \infty) \to \mathbb{R}^+ \setminus \{0\}$. Suppose also that (13) generates a cocycle ϕ that is continuous and asymptotically compact. Finally, suppose that $(P, \mathbb{T}, \sigma)$ is a minimal dynamical system.

Then the nonautonomous dynamical system $\langle H, \phi, (P, \mathbb{T}^+, \sigma)\rangle$ has a uniform pullback attractor.

Proof It follows by the chain rule applied to $\|u\|^2$ for a solution of (13) that

$$\|\phi(t, u, p)\| < \|u\| \quad \text{for} \quad |u| > R, \quad t > 0 \quad \text{and} \quad p \in P.$$

Hence the nonautonomous dynamical system $(X, \mathbb{T}^+, \pi)$ has a global attractor [9]. On the other hand, by (17),

$$\|\phi(t, u_1, p) - \phi(t, u_2, p)\| \le \|u_1 - u_2\|$$

for $t \ge 0$, $p \in P$ and $u_1, u_2 \in H$. Theorem 5.6 then gives the result.

The above theorem holds for a differential equation (13) on $H = \mathbb{R}$ with

$$F(p,u) = \begin{cases} -(u+1) + g(p) & : \quad u < -1, \\ g(p) & : \quad |u| \le 1, \\ -(u-1) + g(p) & : \quad u > 1, \end{cases}$$

where $g \in C(P, \mathbb{R})$.

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Stability of an Autonomous System with Quadratic Right-Hand Side in the Critical Case

J. Diblík[1†] and D. Khusainov[2‡]

[1] *Department of Mathematics, Faculty of Electrical Engineering and Computer Science, Brno University of Technology, Technická 8, 616 00 Brno, Czech Republic*
[2] *Department of Complex Systems Modeling, Faculty of Cybernetics, Kiev University, Vladimirskaja 64, Kiev 252033, Ukraine*

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Abstract: In this paper an autonomous system of differential equations with quadratic right-hand side is considered. In the case when the matrix of linear approximation has just one zero eigenvalue, the stability of trivial solution is investigated. System is written in the vectors-matrices form and under some additional conditions a Liapunov function of the quadratic form is constructed. A guaranteed zone of stability of trivial solution is given as well.

Keywords: *Zero eigenvalue; Lyapunov stability.*

Mathematics Subject Classification (2000): 34A34, 34D20, 93D30.

1 Introduction

Many problems of biological sciences, medicine sciences etc. lead to investigation of systems that are described by means of ordinary differential equations with quadratic right-hand sides (e.g. [3, 5]). Zero solution of the system with quadratic right-hand side in the case of presence of zero eigenvalue of matrix of corresponding linear part can be, in general, unstable. This effect occurs already in the scalar case. For instance, the trivial solution of simple scalar equation $\dot{x} = -x^2$ is unstable, since the solution of the initial

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problem $x(t_0) = x_0$, given by formula $x(t) = 1/(t - t_0 + x_0^{-1})$ has, in the case $x_0 < 0$, a limit $\lim\limits_{t \to t_0 - x_0^{-1} - 0} x(t) = -\infty$.

We shall refer to as *critical cases* such cases when between the eigenvalues of the matrix of corresponding linear approximation there is at least one zero eigenvalue and the remaining eigenvalues have negative real parts. Then stability or instability cannot be established by the linear approximation. In the present paper one of critical cases for autonomous system with quadratic right-hand side, which allows the stability, is considered. This system consists of $n+1$ equations and has the form

$$\begin{gathered}\dot{x}_i = \sum_{s=1}^{n} a_{is} x_s + \sum_{k,s=1}^{n} b^i_{ks} x_k x_s + 2 \sum_{k=1}^{n} b^i_{k,n+1} x_k z + b^i_{n+1,n+1} z^2, \qquad i = 1 \dots, n, \\ \dot{z} = \sum_{k,s=1}^{n} b^{n+1}_{ks} x_k x_s + 2 \sum_{k=1}^{n} b^{n+1}_{k,n+1} x_k z + b^{n+1}_{n+1,n+1} z^2, \end{gathered} \tag{1}$$

where the coefficients a_{is} and b^m_{kl} are constant (we suppose $b^m_{kl} = b^m_{lk}$ if both coefficients exist) and it is supposed that the matrix of linear approximation has just one zero eigenvalue.

For further investigation system (1) is written in the unified vectors-matrices form. As a tool of investigation, a Liapunov function of quadratic form is used. When the full derivative of the Liapunov function along the trajectories of system (1) is estimated, the coefficients of the resulting form are chosen in such a way that guarantees its nonpositivity in a neighbourhood of zero equilibrium state.

Moreover, as a consequence of the performed computations, in the case of stability a concrete neighbourhood of zero solution is found, where fulfilling of the definition of stability is guaranteed. This is possible due to the involved vectors-matrices method. For such kinds of neighbourhoods the term *guaranteed zone of stability* was involved previously (see e.g. [7]). To the best of our knowledge there is no result (for the discussed critical case) which is considered by means of the vectors-matrices method. The estimation of guaranteed zone of stability is new as well.

In the sequel the norms, used for vectors and matrices, are defined as

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

for the vector $x = (x_1, \dots, x_n)$ and

$$\|A\| = \left(\lambda_{\max}(A^T A) \right)^{1/2}$$

for any $m \times n$-matrix A. Here and in the sequel $\lambda_{\max}(\cdot)$ (or $\lambda_{\min}(\cdot)$) is maximal (or minimal) eigenvalue of the corresponding symmetric and positive definite matrix ([6]).

2 Preliminaries

Let us consider an autonomous system with quadratic right-hand sides

$$\dot{y}_i = \sum_{s=1}^{n+1} c^i_s y_s + \sum_{k,s=1}^{n+1} d^i_{ks} y_k y_s, \qquad i = 1, \dots, n+1, \tag{2}$$

where c_s^i and d_{ks}^i with $d_{ks}^i = d_{sk}^i$, $i, k, s = 1, \dots, n+1$ are constants.

If the matrix of linear approximation of system (2), i.e. the matrix of the system

$$\dot{y}_i = \sum_{s=1}^{n+1} c_s^i y_s, \qquad i = 1, \dots, n+1,$$

has just one zero eigenvalue, then there exists a linear regular transformation of the form

$$x_i = \sum_{s=1}^{n+1} l_s^i y_s, \qquad i = 1, \dots, n,$$

$$z = \sum_{s=1}^{n+1} l_s^{n+1} y_s,$$

(where l_s^i and l_s^{n+1}, $s = 1, \dots, n+1$, $i = 1, \dots, n$ are constants and x_i, $i = 1, \dots, n$; z are new dependent variables) which transform this system to the system (1) (see e.g. [9]). Therefore the investigation of the system (1) instead of the general case of system (2) is well grounded.

We begin with some necessary definitions of stability. Let us consider the general system of differential equations

$$\dot{y} = f(t, y), \quad y \in \mathbb{R}^{n+1} \tag{3}$$

with $f\colon [t^*, \infty) \times \Omega \to \mathbb{R}^{n+1}$, where Ω is a connected domain containing the origin of coordinates and $f(t, 0) \equiv 0$ for all $t \in [t^*, \infty)$. Besides it is supposed that through each point $(t_0, y_0) \in [t^*, \infty) \times \Omega$ just one solution $y(t) = y(t; t_0, y_0)$ passes. Maximal right-hand interval of existence of this solution we denote as $J^+(t_0, y_0)$. By definition, $y(t_0; t_0, y_0) = y(t_0)$.

Definition 2.1 [8, 11] *Solution* $y \equiv 0$ of the system (3) is called *stable* if for every $\varepsilon > 0$ and every $t_0 \in [t^*, \infty)$ there exists a $\delta > 0$ such that for any $y_0 \in \mathbb{R}^{n+1}$ with $\|y_0\| < \delta$ and for any $t \in J^+(t_0, y_0)$ it follows: $\|y(t, t_0, y_0)\| < \varepsilon$.

Definition 2.2 [10, 11] *Solution* $y \equiv 0$ of the system (3) is called *uniformly stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $t_0 \in [t^*, \infty)$, $y_0 \in \mathbb{R}^{n+1}$ with $\|y_0\| < \delta$ and any $t \in J^+(t_0, y_0)$ it follows: $\|y(t, t_0, y_0)\| < \varepsilon$.

Obviously, for the autonomous systems under consideration the notions of stability and uniform stability are equivalent. Stability of system (1) will be investigated by means of the direct (second) Liapunov method. For this the following general result is necessary.

Theorem 2.1 [8, 11] *If there exist a function* $V\colon [t^*, \infty) \times \Omega \to \mathbb{R}^+$, $V \in C^1$ *and an increasing continuous function* $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with* $a(0) = 0$ *such that for all* $(t, y) \in [t^*, \infty) \times \Omega$*:*

(1) $V(t, y) \ge a(\|y\|)$*;* $V(t, 0) = 0$*;*

(2) $\dot{V}(t, y) \le 0$*,*

where $\dot{V}$ *is the total derivative of the function* V *along the trajectories of system (3), then the solution* $y \equiv 0$ *of this system is stable.*

Note that function V is usually called the *Liapunov* function. In addition to the establishment of the fact of stability (or asymptotic stability), the second Liapunov method gives a possibility of estimation of the domain (guaranteed zone) of stability (or asymptotic stability), i.e. gives a possibility of estimation of a set of initial data $y_0 \in \mathbb{R}^{n+1}$ for which the corresponding definitions of stability hold. The guaranteed zone of stability can be defined with the aid of the Liapunov function as a set of $y \in \mathbb{R}^{n+1}$ such that

$$V(t, y) < \alpha,$$

where $\alpha = \text{const}$, in the situation when Theorem 2.1 is valid. If this set is equal to the space $\mathbb{R}^{n+1}$ (i.e. if α can be taken as any positive number), we say that the trivial solution is *globally stable*. In particular, for linear autonomous systems stability (asymptotic stability) is always global.

If the system (3) is linear and autonomous, i.e. has the form

$$\dot{y} = Ay,$$

where A is an $n \times n$-matrix, we can look for a Liapunov function of the quadratic form

$$V(y) = y^T H y.$$

Then

$$\dot{V}(y) = y^T (A^T H + HA) y.$$

Let, moreover, A be asymptotically stable (i.e. all its eigenvalues have negative real parts). Then always there exists a symmetric positive definite $n \times n$-matrix H such that the symmetric matrix

$$C = -A^T H - HA$$

is positive definite too (see $[1, 2, 4, 8]$). The set of the matrices H, satisfying this property, generates a convex cone (on the set of positive definite matrices) and the zero matrix serves as its vertex. The corresponding function V satisfies all conditions formulated above (in Theorem 2.1).

3 Matrix Forms of System (1)

Let us consider the system (1). For investigation of stability of its trivial solution it will be useful to rewrite system (1) in the vectors-matrices form. Further we will use: matrices $X_i,\ i = 1, \dots, n$ of the type $n \times n$ and $Z_i,\ i = 1, \dots, n$ of the type $n \times 1$ having the property that only the i-th row of which can be nonzero; symmetric matrices $B_l,\ l = 1, \dots, n+1$ of the type $n \times n$; vectors $b_l,\ l = 1, \dots, n+1$ of the type $n \times 1$; matrix A of the type $n \times n$; vector θ of the type $n \times 1$; matrix Θ of the type $n \times n$ and vector x of the type $n \times 1$. They are defined according to the following formulas:

$$X_i = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad Z_i = \begin{pmatrix} 0 \\ . \\ z \\ . \\ 0 \end{pmatrix}, \quad B_l = \begin{pmatrix} b_{11}^l & b_{12}^l & \dots & b_{1n}^l \\ b_{21}^l & b_{22}^l & \dots & b_{2n}^l \\ \dots & \dots & \dots & \dots \\ b_{n1}^l & b_{n2}^l & \dots & b_{nn}^l \end{pmatrix},$$

$$b_l = \begin{pmatrix} b^l_{1,n+1} \\ b^l_{2,n+1} \\ \dots\dots \\ b^l_{n,n+1} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \end{pmatrix},$$

$$\Theta = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

Then the system (1) can be rewritten in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A & \theta \\ \theta^T & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

$$+ \begin{pmatrix} X_1 & Z_1 & \dots & X_n & Z_n & \Theta & \theta \\ \theta^T & 0 & \dots & \theta^T & 0 & x^T & z \end{pmatrix} \cdot \begin{pmatrix} B_1 & b_1 \\ b_1^T & b^1_{n+1,n+1} \\ \dots & \dots \\ B_{n+1} & b_{n+1} \\ b_{n+1}^T & b^{n+1}_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

or in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A + r_{11}(x,z) & r_{12}(x,z) \\ r_{21}^T(x,z) & r_{22}(x,z) \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix} \tag{4}$$

with

$$\begin{gathered} r_{11} = \sum_{l=1}^{n}(X_l B_l + Z_l b_l^T), \qquad r_{12} = \sum_{l=1}^{n}(X_l b_l + Z_l b^l_{n+1,n+1}), \\ r_{21}^T = x^T B_{n+1} + z b_{n+1}^T, \quad r_{22}(x,z) = x^T b_{n+1} + z b^{n+1}_{n+1,n+1}. \end{gathered} \tag{5}$$

4 Main Result

Before formulation of the main result let us introduce necessary abbreviations:

$$\begin{gathered} \tilde b_{n+1} = \left(b^1_{n+1,n+1},\ b^2_{n+1,n+1},\ \dots,\ b^n_{n+1,n+1}\right)^T, \\ \tilde B_{n+1} = \begin{pmatrix} b^1_{1,n+1} & b^1_{2,n+1} & \dots & b^1_{n,n+1} \\ b^2_{1,n+1} & b^2_{2,n+1} & \dots & b^2_{n,n+1} \\ \dots & \dots & \dots & \dots \\ b^n_{1,n+1} & b^n_{2,n+1} & \dots & b^n_{n,n+1} \end{pmatrix}, \\ R(H) = 2(\tilde B^T_{n+1} H + H \tilde B_{n+1} + B_{n+1}), \end{gathered} \tag{6}$$

where H is an $n \times n$ matrix and

$$\bar B^T = \begin{pmatrix} b^1_{11} & \dots & b^1_{1n} & b^1_{21} & \dots & b^1_{2n} & \dots & b^1_{n1} & \dots & b^1_{nn} \\ b^2_{11} & \dots & b^2_{1n} & b^2_{21} & \dots & b^2_{2n} & \dots & b^2_{n1} & \dots & b^2_{nn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b^n_{11} & \dots & b^n_{1n} & b^n_{21} & \dots & b^n_{2n} & \dots & b^n_{n1} & \dots & b^n_{nn} \end{pmatrix}. \tag{7}$$

Theorem 4.1 *Let the matrix A be asymptotically stable and the coefficient $b^{n+1}_{n+1,n+1} = 0$. If there exists a symmetric positive definite $n \times n$ matrix H such that the matrix*

$$C = -A^T H - HA$$

is positive definite too and, moreover, the relation

$$H\tilde{b}_{n+1} + 2b_{n+1} = 0 \tag{8}$$

holds, then the trivial solution of system (1) is stable. A guaranteed zone of stability contains an ellipse

$$x^T Hx + z^2 \leq \alpha$$

with

$$\alpha = \frac{\lambda_{\max}(H) \cdot (\lambda_{\min}(C))^2}{\lambda_{\max}(H)\|R(H)\|^2 + 4\|H\bar{B}^T\|^2}.$$

Remark 4.1 With respect to the definition of stability we note, that if conditions of Theorem 4.1 hold, then (as it follows from the proof) for each solution (x, z) of system (1) defined by the initial data (t_0, x_0, z_0) with $x_0^T Hx_0 + z_0^2 \leq \alpha$ we have $J^*(t_0, x_0, z_0) = [t_0, \infty)$.

Proof of Theorem 4.1 Let us seek for a Liapunov function of the hypermatrix form

$$V(x, z) = (x^T, z) \cdot \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix},$$

where $h_{n+1,n+1}$ is a positive constant. Its total derivative along the trajectories of system (4) takes the form

$$\dot{V}(x, z) = (x^T, z) \cdot \left\{ \begin{pmatrix} A^T + r_{11}^T(x, z) & r_{21}(x, z) \\ r_{12}^T(x, z) & r_{22}(x, z) \end{pmatrix} \cdot \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} H & \theta \\ \theta^T & h_{n+1,n+1} \end{pmatrix} \cdot \begin{pmatrix} A + r_{11}(x, z) & r_{12}(x, z) \\ r_{21}^T(x, z) & r_{22}(x, z) \end{pmatrix} \right\} \cdot \begin{pmatrix} x \\ z \end{pmatrix}.$$

After computing we get

$$\dot{V}(x, z) = (x^T, z) \cdot \begin{pmatrix} g_{11}(x, z) & g_{12}(x, z) \\ g_{12}^T(x, z) & g_{22}(x, z) \end{pmatrix} \cdot \begin{pmatrix} x \\ z \end{pmatrix}$$

with

$$g_{11}(x, z) = (A + r_{11}(x, z))^T H + H(A + r_{11}(x, z)),$$
$$g_{12}(x, z) = r_{21}(x, z)h_{n+1,n+1} + Hr_{12}(x, z)$$

and

$$g_{22}(x, z) = 2h_{n+1,n+1}r_{22}(x, z).$$

With the aid of (5) we express

$$g_{11}(x,z) = (A^T H + HA) + \sum_{l=1}^{n} \left((X_l B_l)^T H + H(X_l B_l)\right) + \sum_{l=1}^{n} \left((Z_l b_l^T)^T H + H(Z_l b_l^T)\right),$$

$$g_{12}(x,z) = \left(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l)\right) + \left(h_{n+1,n+1} b_{n+1} z + H \sum_{l=1}^{n} (Z_l b^l_{n+1,n+1})\right)$$

and

$$g_{22}(x,z) = 2h_{n+1,n+1}(b_{n+1}^T x + b_{n+1,n+1}^{n+1} z).$$

Then the total derivative takes the form

$$\begin{aligned}
\dot V(x,z) &= x^T (A^T H + HA)x + x^T \left\{ \sum_{l=1}^{n} \left((X_l B_l)^T H + H(X_l B_l)\right) \right\} x \\
&+ x^T \left\{ \sum_{l=1}^{n} \left((Z_l b_l^T)^T H + H(Z_l b_l^T)\right) \right\} x \\
&+ 2x^T \left(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l) \right) z \\
&+ 2x^T \left(h_{n+1,n+1} b_{n+1} z + H \sum_{l=1}^{n} (Z_l b^l_{n+1,n+1}) \right) z \\
&+ 2h_{n+1,n+1}(b_{n+1}^T x + b_{n+1,n+1}^{n+1} z) z^2.
\end{aligned}$$

Let us consider some addends of this expression separately.

1. Symmetric matrix $C = -A^T H - HA$ is, in accordance with conditions of Theorem 4.1, positive definite.

2. Let us denote

$$X = \begin{pmatrix} X_1^T \\ X_2^T \\ \dots \\ X_n^T \end{pmatrix}.$$

Then, by using (7), we transform the expression in the second addend:

$$\sum_{l=1}^{n} \left((X_l B_l)^T H + H(X_l B_l)\right) = (\bar B^T X)^T H + H(\bar B^T X).$$

3. With the aid of (6) we transform the expression in the third addend:

$$\sum_{l=1}^{n} Z_l b_l^T = z \tilde B_{n+1}.$$

4. For the fourth term we get

$$2x^T \left(h_{n+1,n+1} B_{n+1} x + H \sum_{l=1}^{n} (X_l b_l) \right) z = 2z x^T \left(h_{n+1,n+1} B_{n+1} + H \tilde B_{n+1} \right) x.$$

5. The fifth addend turns into

$$2x^T\left(h_{n+1,n+1}b_{n+1}z + H\sum_{l=1}^{n}(Z_l b^l_{n+1,n+1})\right)z = 2x^T\big(h_{n+1,n+1}b_{n+1} + H\tilde{b}_{n+1}\big)z^2.$$

After above transformations the total derivative is simplified as

$$\begin{aligned}\dot{V}(x,z) = & -x^TCx + x^T\left((\bar{B}^TX)^TH + H(\bar{B}^TX)\right)x + zx^T\left(\tilde{B}^T_{n+1}H + H\tilde{B}_{n+1}\right)x\\ & + zx^T\big(2h_{n+1,n+1}B_{n+1} + (\tilde{B}^T_{n+1}H + H\tilde{B}_{n+1})\big)x\\ & + 2x^T\big(2h_{n+1,n+1}b_{n+1} + H\tilde{b}_{n+1}\big)z^2 + 2h_{n+1,n+1}b^{n+1}_{n+1,n+1}z^3\end{aligned}$$

and, finally, if we take into account (8) (since obviously it can be put $h_{n+1,n+1} = 1$) and condition $b^{n+1}_{n+1,n+1} = 0$, it becomes

$$\dot{V}(x,z) = -x^T\big\{C - \big((\bar{B}^TX)^TH + H(\bar{B}^TX)\big) - zR(H)\big\}x. \tag{9}$$

Estimation of (9) gives (we take into account the property $\|X\| = \|x\|$)

$$\begin{gathered}-x^T\left\{C - \big((\bar{B}^TX)^TH + H(\bar{B}^TX)\big) - zR(H)\right\}x\\ -x^TCx + x^T\left\{\big((\bar{B}^TX)^TH + H(\bar{B}^TX)\big) - zR(H)\right\}x\\ \le -\lambda_{\min}(C)\|x\|^2 + \lambda_{\max}\left\{\big((\bar{B}^TX)^TH + H(\bar{B}^TX)\big) - zR(H)\right\}\|x\|^2\\ \le \big(-\lambda_{\min}(C) + 2\|H\tilde{B}^T\|\cdot\|x\| + \|R(H)\|\cdot\|z\|\big)\|x\|^2.\end{gathered}$$

Then for stability of the system (1) it is sufficient that

$$\lambda_{\min}(C) - 2\|H\bar{B}^T\|\cdot\|x\| - \|R(H)\|\cdot\|z\| \ge 0. \tag{10}$$

Since

$$V(x,z) = x^THx + z^2 \le \lambda_{\max}(H)\|x\|^2 + \|z\|^2, \tag{11}$$

as it follows from (10) and (11), a guaranteed zone of stability has the form

$$x^THx + z^2 \le \alpha$$

with

$$\alpha = \frac{\lambda_{\max}(H)\cdot(\lambda_{\min}(C))^2}{\lambda_{\max}(H)\|R(H)\|^2 + 4\|H\bar{B}^T\|^2}.$$

The theorem is proved.

The assertion (formulated in Remark 4.1) concerning the maximal interval of existence follows, obviously, from the method of the proof which uses the Liapunov function.

Corollary 4.1 *From the proof of Theorem 4.1 and from representation (9) the following corollary follows: Let all assumptions of Theorem 4.1 be valid and, moreover,*

$$H\bar{B}^T = 0, \qquad R(H) = 0.$$

Then the trivial solution of the system (1) is globally stable. In this case for the global stability it is sufficient if the matrix $C = -A^T H - HA$ *is only positive semi-definite.*

5 Example

Let us consider system (1) for $n = 2$, i.e. the system of three equations

$$\begin{aligned} \dot{x} &= -x + ay + b_{11}^1 x^2 + b_{22}^1 y^2 + b_{33}^1 z^2 + 2b_{12}^1 xy + 2b_{13}^1 xz + 2b_{23}^1 yz, \\ \dot{y} &= \qquad\quad -y + b_{11}^2 x^2 + b_{22}^2 y^2 + b_{33}^2 z^2 + 2b_{12}^2 xy + 2b_{13}^2 xz + 2b_{23}^2 yz, \\ \dot{z} &= \qquad\qquad\qquad b_{11}^3 x^2 + b_{22}^3 y^2 + b_{33}^3 z^2 + 2b_{12}^3 xy + 2b_{13}^3 xz + 2b_{23}^3 yz, \end{aligned} \tag{12}$$

where a, b_{ij}^k, i, j, $k = 1, 2, 3$, $i \le j$ are constants, $a > 0$ and $b_{33}^2 \ne 0$. As it follows from Theorem 4.1, the stability of trivial solution of system (12) will be proved if there exists a symmetric positive definite matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

such that the matrix $C = -A^T H - HA$ is positive definite too and, except for

$$b_{33}^3 = 0, \quad H\tilde{b}_3 + 2b_3 = 0 \tag{13}$$

with $b_3^T = (b_{13}^3, b_{23}^3)$ and $\tilde{b}_3^T = (b_{33}^1, b_{33}^2)$. It is easy to see that

$$C = \begin{pmatrix} 2h_{11} & 2h_{12} - ah_{11} \\ 2h_{12} - ah_{11} & 2(h_{22} - ah_{12}) \end{pmatrix}.$$

Combining conditions for positive definiteness of matrix C and conditions (13) we get (note that from the existence of positive definite matrix C follows the existence of positive definite matrix H – see end of the Section 2):

$$\begin{cases} h_{11} > 0, \\ 4h_{11}(h_{22} - ah_{12}) - (2h_{12} - ah_{11})^2 > 0, \\ h_{11} b_{33}^1 + h_{12} b_{33}^2 = -2b_{13}^3, \\ h_{12} b_{33}^1 + h_{22} b_{33}^2 = -2b_{23}^3. \end{cases} \tag{14}$$

The last two equations yield

$$\begin{aligned} h_{12} &= -\frac{1}{b_{33}^2}\,(h_{11} b_{33}^1 + 2b_{13}^3), \\ h_{22} &= \frac{1}{(b_{33}^2)^2}\,[(b_{33}^1)^2 h_{11} + 2(b_{33}^1 b_{13}^3 - b_{23}^3 b_{33}^2)]. \end{aligned} \tag{15}$$

With the aid of (15) and the second inequality in (14) we obtain:

$$a^2h_{11}^2 + \frac{8}{(b_{33}^2)^2}\,(b_{33}^1b_{13}^3 + b_{23}^3b_{33}^2)h_{11} + 16\left(\frac{b_{13}^3}{b_{33}^2}\right)^2 < 0.$$

Then, for $h_{11} > 0$,

$$b_3^T\tilde{b}_3 = b_{13}^3b_{33}^1 + b_{23}^3b_{33}^2 < -a|b_{13}^3b_{33}^2|$$

is necessary and sufficient. Using the vectors b_3 and $\tilde{b}_3$ we see that h_{11} can vary within the interval

$$\begin{aligned} -\frac{4}{a^2}\cdot\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2} + \sqrt{\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2}\right]^2 - a^2\left(\frac{b_{13}^3}{b_{33}^2}\right)^2}\,\right] &< h_{11} \\ < -\frac{4}{a^2}\cdot\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2} - \sqrt{\left[\frac{b_3^T\tilde{b}_3}{(b_{33}^2)^2}\right]^2 - a^2\left(\frac{b_{13}^3}{b_{33}^2}\right)^2}\,\right]&. \end{aligned} \tag{16}$$

So, for stability of the trivial solution of system (12) the following conditions are sufficient

$$\begin{aligned} &b_{33}^3 = 0, \\ &b_3^T\tilde{b}_3 < -a|b_{13}^3b_{33}^2|. \end{aligned} \tag{17}$$

Remark 5.1 The first two inequalities in (14) define a convex cone in the space (h_{11}, h_{12}, h_{22}) with the vertex at the origin. Next two equations define a straight line in this space. Consequently, geometrically relations (14) express the conditions of an intersection of a straight line and a cone.

Let us consider a partial case of the system (12) when $B_1 = B_2 = B_3 = \tilde{B}_3 = 0$ and $b_{33}^3 = 0$. Then $H\bar{B}^T = 0$, $R(H) = 0$ and the system has the form

$$\begin{aligned} \dot{x} &= -x + ay + b_{33}^1z^2, \\ \dot{y} &= \quad\quad - y + b_{33}^2z^2, \\ \dot{z} &= \quad\quad\quad 2b_{13}^3xz + 2b_{23}^3yz. \end{aligned} \tag{18}$$

Suppose that b_{33}^1, b_{33}^2, b_{13}^3 and b_{23}^3 satisfy conditions (17). Let h_{11} be taken in accordance with inequalities (16) and compute h_{12} and h_{22} by formulas (15). Then the corresponding Liapunov function has the form

$$V(x, y, z) = h_{11}x^2 + 2h_{12}xy + h_{22}y^2 + z^2$$

and its derivative is

$$\dot{V}(x, y, z) = -2\left\{h_{11}x^2 + 2\left(h_{12} - \frac{1}{2}\,ah_{11}\right)xy + (h_{22} - ah_{12})y^2\right\}.$$

This derivative is negative semi-definite in $\mathbb{R}^3$, i.e. the trivial solution of (18) is globally stable. This is in accordance with Corollary 4.1.

Stationary points of (18) are solutions of the system

$$\begin{aligned} -x + ay + \qquad\qquad b^1_{33}z^2 &= 0,\\ -y + \qquad\qquad b^2_{33}z^2 &= 0,\\ 2b^3_{13}xz + 2b^3_{23}yz &= 0. \end{aligned}$$

Seeking for the stationary point of this system we see that, in view of the inequality in (17), the stationary point is the only one – namely, the origin of coordinates.

Let us consider some of possible *limiting* cases.

1. Suppose that instead of inequality the equality in (17) holds and, moreover,

$$b_3^T\tilde{b}_3 = b^3_{13}b^1_{33} + b^3_{23}b^2_{33} = -ab^3_{13}b^2_{33} \quad \text{and} \quad b^3_{13}b^2_{33} > 0. \tag{19}$$

Then there exists a set of stationary points which lies on the curve

$$x = (ab^2_{33} + b^1_{33})z^2, \quad y = b^2_{33}z^2, \quad -\infty < z < \infty.$$

In accordance with (16) we put (as a limiting case)

$$h_{11} = -\frac{4}{a^2}\cdot\frac{b_3^T\tilde{b}_3}{(b^2_{33})^2} = \frac{4}{a}\cdot\frac{b^3_{13}}{b^2_{33}}.$$

Then formulas (15) give

$$\begin{aligned} h_{12} &= -\frac{2b^3_{13}}{b^2_{33}}\left(\frac{2}{a}\cdot\frac{b^1_{33}}{b^2_{33}} + 1\right),\\ h_{22} &= \frac{2}{(b^2_{33})^2}\left(\frac{2}{a}\cdot\frac{b^3_{13}(b^1_{33})^2}{b^2_{33}} + b^1_{33}b^3_{13} - b^3_{23}b^2_{33}\right). \end{aligned} \tag{20}$$

Conditions of positivity definiteness of the Liapunov function have the form

$$h_{11} > 0, \quad h_{11}h_{22} - h_{12}^2 > 0. \tag{21}$$

If we take into account (20), these inequalities take the form

$$b^3_{13}b^2_{33} > 0, \quad \frac{2}{a}\cdot\frac{b^3_{13}}{(b^2_{33})^3}\cdot(b^3_{13}b^1_{33} + b^3_{23}b^2_{33}) + \frac{(b^3_{13})^2}{(b^2_{33})^2} < 0.$$

If (19) holds, then the second inequality turns into

$$-\frac{(b^3_{13})^2}{(b^2_{33})^2} < 0$$

and always holds. So, if (19) holds, the trivial solution of the system (18) is globally stable.

2. Suppose that instead of inequality the equality in (17) holds and, moreover,

$$b_3^T\tilde{b}_3 = b^3_{13}b^1_{33} + b^3_{23}b^2_{33} = ab^3_{13}b^2_{33} \quad \text{and} \quad b^3_{13}b^2_{33} < 0. \tag{22}$$

Then only the origin is a stationary point. Let us put (in accordance with (16) in a limiting case)

$$h_{11} = -\frac{4}{a^2} \cdot \frac{b_3^T \tilde{b}_3}{(b_{33}^2)^2} = -\frac{4}{a^2} \cdot \frac{b_{13}^3}{b_{33}^2}.$$

Then formulas (15) give

$$\begin{aligned} h_{12} &= -\frac{2b_{13}^3}{b_{33}^2}\left(-\frac{2}{a} \cdot \frac{b_{33}^1}{b_{33}^2} + 1 \right), \\ h_{22} &= \frac{2}{(b_{33}^2)^2}\left(-\frac{2}{a} \cdot \frac{b_{13}^3 (b_{33}^1)^2}{b_{33}^2} + b_{33}^1 b_{13}^3 - b_{23}^3 b_{33}^2 \right). \end{aligned} \tag{23}$$

Conditions (21) for positivity definiteness of the Liapunov function take (in view of (23)) the form

$$b_{13}^3 b_{33}^2 < 0, \qquad \frac{2}{a} \cdot \frac{b_{13}^3}{(b_{33}^2)^3} \cdot (b_{13}^3 b_{33}^1 + b_{23}^3 b_{33}^2) - \frac{(b_{13}^3)^2}{(b_{33}^2)^2} > 0.$$

If (22) holds, then the second inequality turns into

$$\frac{(b_{13}^3)^2}{(b_{33}^2)^2} > 0$$

and always holds too. So, if (22) holds, then the trivial solution of the system (18) is globally stable too.

Remark 5.2 In the above considered particular limiting cases **1** and **2** the condition (14) can be geometrically interpreted as a contact of a straight line with a convex cone.

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Statistical Analysis of Nonimpulsive Orbital Transfers under Thrust Errors, 1

A.D.C. Jesus[1], M.L.O. Souza[2] and A.F.B.A. Prado[2]

[1] *Departamento de Física, Universidade Estadual de Feira de Santana (UEFS), KM. 03 BR. 116 – Campus Universitário, Caixa Postal 252-294, CEP. 44.031-460, Feira de Santana, BA, Brazil*

[2] *Instituto Nacional de Pesquisas Espaciais (INPE), Avenida dos Astronautas 1758, Caixa Postal 515, 12227-010, São Paulo, SP, Brazil*

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Abstract: In this paper we present the first part of an extensive study of nonimpulsive orbital transfers under thrust errors. We emphasize the first part of the numerical implementation (Monte-Carlo) of the study but mention the first algebraic explanation for some of the numerical results. Its main results suggest and partially characterize the progressive deformation of the trajectory distribution along the propulsive arc, turning 3sigma ellipsoids into banana shaped volumes curved to the center of attraction (we call them "bananoids") due to the loss of optimality of the actual (with errors) trajectories with respect to the nominal (no errors) trajectory.

Keywords: *Orbits; transfer; nonimpulsive; thrust errors; Monte-Carlo analysis.*

Mathematics Subject Classification (2000): 70M20, 65C05, 62L70.

1 Introduction

Most space missions need orbit transfers to reach their goals. These trajectories/orbits are reached sequentially through transfers between them by changing their keplerian elements, by firing apogee motors or other sources of force. These thrusts have linear and/or angular misalignments that displace the vehicle with respect to its nominal directions. The mathematical treatment for these errors can be done by complementary approaches (deterministic, probabilistic, minimax, etc.). In the literature, already reviewed by Souza, *et al.* [1], we highlight.

In the deterministic approach: Rodrigues [2] analyzed the effects of the errors in nonimpulsive thrust on coplanar transfers of a nonpunctual model of a satellite. As such, it is the only work we got considering the attitude motion, the center of mass misalignments, and the reduction of thrust with use, etc. Santos-Paulo [3] analyzed the effects of errors in impulsive thrusts on coplanar or noncoplanar transfers of punctual model of a satellite. Other related papers are Schultz [4] and Rocco [5].

In the probabilistic approach: Porcelli and Vogel [6] presented an algorithm for the determination of the orbit insertion errors in biimpulsive noncoplanar orbital transfers (perigee and apogee), using the covariance matrices of the sources of errors. Adams and Melton [7] extended such algorithm to ascent transfers under a finite thrust, modeled as a sequence of impulsive burns. They developed an algorithm to compute the propagation of the navigation and direction errors among the nominal trajectory, with finite perigee burns. Rao [8] built a semi-analytic theory to extend covariance analysis to long-term errors on elliptical orbits. Howell and Gordon [9] also applied covariance analysis to the orbit determination errors and they develop a station-keeping strategy of Sun-Earth L1 libration point orbits. Junkins, *et al.* [10] and Junkins [11] discussed the precision of the error covariance matrix method through nonlinear transformations of coordinates. He also found a progressive deformation of the initial ellipsoid of trajectory distribution (due to gaussian initial condition errors), that was not anticipated by the covariance analysis of linearized models with zero mean errors. Its main results also characterize how close or how far are Monte-Carlo analysis and covariance analysis for those examples. Carlton-Wippern [12] proposed differential equations in polar coordinates for the growth of the mean position errors of satellites (due to errors in the initial conditions or in the drag), by using an approximation of Langevin's equation and a first order perturbation theory. Alfriend [13] studied the effects of drag uncertainty via covariance analysis.

In the minimax approach: see russian authors, mainly.

However, all these analyses are approximated. This motivated an exhaustive numerical but exact analysis (by Monte-Carlo), and a partial algebraic analysis done by Jesus [14] under the supervision of the two other authors, to highlight and to study effects not shown in those analyses.

In this work we present the 1^{st} part of the numerical implementation of that Monte-Carlo analysis of the nonimpulsive orbital transfers under thrust vector errors. The results were obtained for two transfers: the first, a low thrust transfer between high coplanar orbits, used by Biggs [15, 16] and Prado [17]; the second, a high thrust transfer between middle noncoplanar orbits (the first transfer of the EUTELSAT II-F2 satellite) implemented by Kuga, *et al.* [18].

The simulations were done for both transfers with minimum fuel consumption. The optimization method used by Biggs [15, 16] and Prado [17] was adapted to the case of transfers with thrust errors. The "pitch" and "yaw" angles were taken as control variables such that the overall minimum fuel consumption defines each burn of the thrusts.

The error sources that we considered were the magnitude errors, the "pitch" and "yaw" direction errors of the thrust vector, as causes of the deviations found in the keplerian elements of the final orbit. Each deviation was introduced separately along the orbital transfer trajectory. We studied two types of errors for each one of these causes: the systematic/constructional/assembly errors (modeled as random-bias) and the operational errors (modeled as white-noise). The random-bias errors are unknown but constants during all the transfer arc, while the white-noise errors change along the transfer arc. These error sources introduced in the orbital transfer dynamics cause effects in the keplerian elements of the final orbit at the final instant.

In this work we present an statistical analysis of the effects of these errors on the mean of the deviations of the keplerian elements of the final orbit with respect to the reference orbit (final orbit without errors in the thrust vector) for both transfers. The

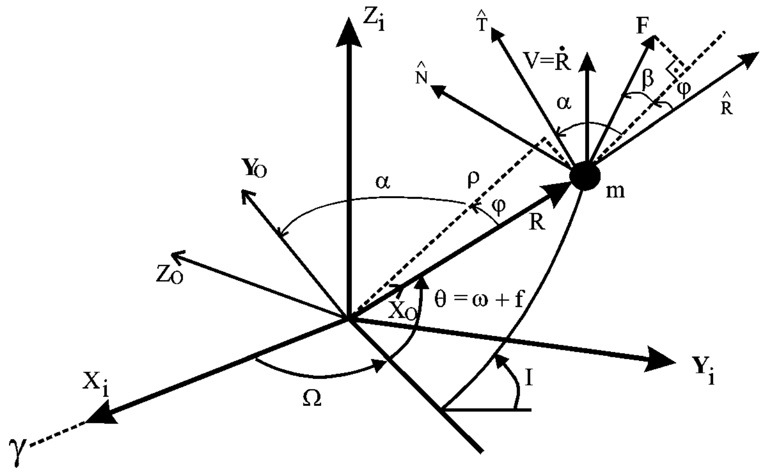

Figure 2.1. Reference system used in this work.

approach that we used in this work for the treatment of the errors was the probabilistic one, assuming these as having zero mean uniform probability density function.

2 Mathematical Formulation and Coordinates Systems

The orbital transfer problem studied can be formulated in the following way:

1) Globally minimize the performance index: $J = m(t_0) - m(t_f)$;
2) With respect to α: $[t_0, t_f] \to R$ ("pitch" angle) and β: $[t_0, t_f]$ ("yaw" angle) with $\alpha, \beta \in C^{-1}$ in $[t_0, t_f]$;
3) Subject to the dynamics in inertial coordinates X_i, Y_i, Z_i of Figure 2.1: $\forall\, t \in [t_0, t_f]$;

$$m\,\frac{d^2X}{dt^2} = -\mu m\,\frac{X}{R^3} + F_x, \tag{1}$$

$$m\,\frac{d^2Y}{dt^2} = -\mu m\,\frac{Y}{R^3} + F_x, \tag{2}$$

$$m\,\frac{d^2Z}{dt^2} = -\mu m\,\frac{Z}{R^3} + F_x, \tag{3}$$

$$\begin{gathered} F_x = F\left[\cos\beta\sin\alpha\left(\cos\Omega\cos\theta - \sin\Omega\cos I\sin\theta\right) + \sin\beta\sin\Omega\sin I \right. \\ \left. - \cos\beta\cos\alpha\left(\cos\Omega\sin\theta + \sin\Omega\cos I\cos\theta\right)\right], \end{gathered} \tag{4}$$

$$\begin{gathered} F_y = F\left[\cos\beta\sin\alpha\left(\sin\Omega\cos\theta + \cos\Omega\cos I\sin\theta\right) - \sin\beta\cos\Omega\sin I \right. \\ \left. - \cos\beta\cos\alpha\left(\sin\Omega\sin\theta - \cos\Omega\cos I\cos\theta\right)\right] \end{gathered} \tag{5}$$

$$F_z = F\left(\cos\beta\sin\alpha\sin I\sin\theta + \cos\beta\cos\alpha\sin I\cos\theta + \sin\beta\cos I\right). \tag{6}$$

Or in orbital coordinates (radial R, transversal T, and binormal N) of Figure 2.1:

$$ma_R(t) = F\cos\beta(t)\sin\alpha(t) - \frac{\mu m}{R^2(t)}, \tag{7}$$

$$ma_T(t) = F\cos\beta(t)\cos\alpha(t), \tag{8}$$

$$ma_N(t) = F\sin\beta(t), \tag{9}$$

$$a_R(t) = \dot{V}_R - \frac{V_T^2}{R} - \frac{V_N^2}{R}, \tag{10}$$

$$a_T(t) = \dot{V}_T + \frac{V_R V_T}{R} - V_N \dot{I}\cos\theta - V_N\dot{\Omega}\sin I\sin\theta, \tag{11}$$

$$a_N(t) = \dot{V}_N + \frac{V_R V_N}{R} + V_T \dot{I}\cos\theta + V_T\dot{\Omega}\sin I\sin\theta, \tag{12}$$

$$V_R = \dot{R}, \tag{13}$$

$$V_T = R(\dot{\Omega}\cos I + \dot{\theta}), \tag{14}$$

$$V_N = R(-\dot{\Omega}\sin I\cos\Omega + \dot{I}\sin\Omega), \tag{15}$$

$$\theta = \omega + f. \tag{16}$$

These equations were obtained by: 1) writing in coordinates of the dexterous rectangular reference system with inertial directions $OX_iY_iZ_i$ the Newton's laws for the motion of a satellite with mass m, with respect to this reference system, centered in the Earth's center of mass O with X_i axis toward the Vernal point, X_iY_i plane coincident with Earth's Equator, and Z_i axis toward the Polar Star approximately; 2) rewriting them in coordinates of the dexterous rectangular reference system with radial, transversal, binormal directions SRTN, centered in the satellite center of mass; helped by 3) a parallel system with $OX_oY_oZ_o$ directions, centered in the Earth's center of mass O, X_o axis toward the satellite, X_oY_o plane coincident with the plane established by the position R and velocity V vectors of the satellite, and Z_o axis perpendicular to this plane; and helped by 4) the instantaneous keplerian coordinates (Ω, I, ω, f, a, e). These equations were later rewritten and simulated by using 5) 9 state variables, defined and used by Biggs [15, 16] and Prado [17], as functions of an independent variable s shown in Figure 2.2.

The nonideal thrust vector, with magnitude and direction errors, is given by:

$$\vec{F_E} = \vec{F} + \Delta\vec{F}, \tag{17}$$

$$\vec{F_E} = \vec{F_R} + \vec{F_T} + \vec{F_N}, \tag{18}$$

$$|\vec{F_E}| = F_E, \qquad |\vec{F}| = F, \tag{19}$$

$$F_R = (F + \Delta F)\cos(\beta + \Delta\beta)\sin(\alpha + \Delta\alpha), \tag{20}$$

$$F_T = (F + \Delta F)\cos(\beta + \Delta\beta)\cos(\alpha + \Delta\alpha), \tag{21}$$

$$F_N = (F + \Delta F)\sin(\beta + \Delta\beta), \tag{22}$$

where: $\vec{F}$, $\vec{F_E}$ and $\Delta\vec{F}$ are: the thrust vector without errors, the thrust vector with errors, and the error in the thrust vector, respectively; $\Delta\alpha$ e $\Delta\beta$ are the errors in the

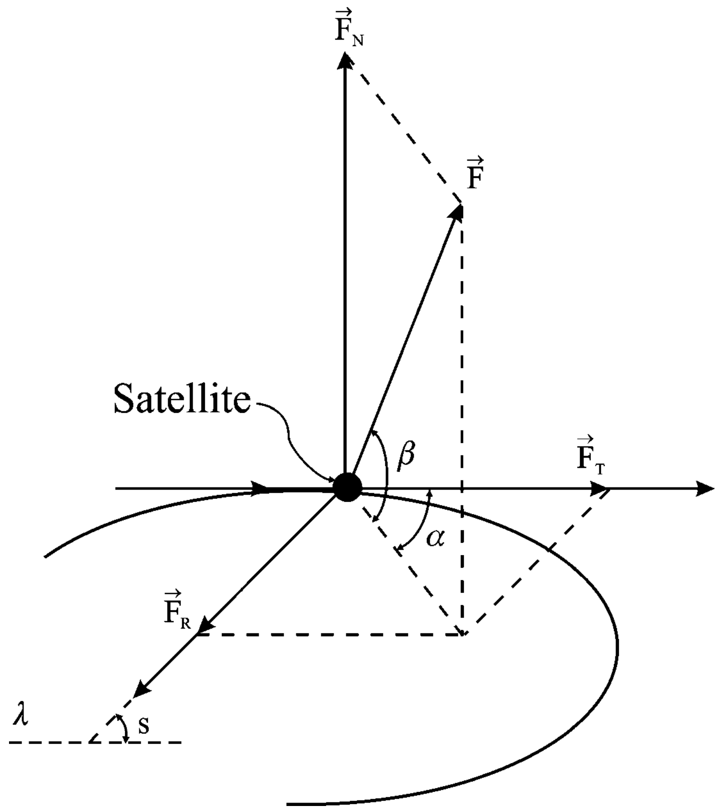

Figure 2.2. Thrust vector applied to the satellite and the s variable.

"pitch" and in the "yaw" angles, respectively; F_R, F_T e F_N are the components of the thrust vector with errors $\vec{F_E}$ in the radial, transversal and normal directions, respectively. The magnitude error, ΔF, was computed as a percentage of the nominal force, while the direction errors $\Delta\alpha$ and $\Delta\beta$ were computed in units of angle. They are varied inside given ranges, that is, $\pm DES1.F$ for ΔF, $\pm DES2$ for $\Delta\alpha$ and $\pm DES3$ for $\Delta\beta$. This variation will correspond to the implementation of the random numbers that satisfy a uniform probability distribution into those ranges. In this way, for each implementation of the orbital transfer arc, values of α and β are chosen, whose errors are inside the range, that produce the direction for the overall minimum fuel consumption.

3 Numerical Results

The simulations were performed with 1000 realizations for each transfer, that is, 1000 runs were done with random values for each DES1, DES2 and DES3, such that the results obtained for the final keplerian elements represent the arithmetic mean of 1000 realizations (mean over the ensemble). The value 1000 was chosen to represent the set of runs because the mean deviations in all final keplerian elements with respect to their references converge to their minimal for this number of runs.

Figures 3.1 and 3.2 show the mean deviations in the final semi-major axis and eccentricity versus the number of runs, respectively. These plots were done for systematic pitch direction with $DES2 = 1,0^o$.

The computation of the mean deviations of the final keplerian elements with respect to their references can be estimated by the arithmetic mean of them, for 1000 runs as representatives. So, we can estimate mean deviation of any final keplerian element, ΔK as

$$\overline{\Delta K} = E\{\Delta K\} = \int_{-\infty}^{\infty} \xi f_{\Delta K}(\xi)\, d\xi \cong \sum_{i=1}^{N=1000} \frac{\Delta K_i}{N}. \tag{23}$$

It is important to remark that equation (23) estimates a mean in the ensemble and not in the time. In this work we present only these estimates for the final semi-major axis and eccentricity with respect to their reference. Figures 3.3 to 3.14 present the behavior of them as functions of the maximum (random-bias and white noise) direction errors. For the random-bias errors we found the following results:

1) Uniform random-bias errors: semi-major axis(a), "pitch" errors.

We observe, in these plots (Figure 3.3 and Figure 3.4), behaviors very similar for both maneuvers, although they are very different from each other. We easily observe that the values of the mean semi-major axis present a region of decrease sufficiently defined according to the growth of the maximum "pith" error, DES2. These figures suggest a nonlinear law between these elements for both cases, that is, they suggest a cause vs. effect relation in the orbital transfer phenomenon, not depending of the maneuver studied.

2) Uniform random-bias errors: semi-major axis(a), "yaw" errors

Once more these plots (Figure 3.5 and Figure 3.6) show behaviors well defined and similar for the semi-major axis as function of the maximum "yaw" error, DES3, for both maneuvers studied. That is, there is a region of decrease well defined between the elements a and DES3.

3) Uniform random-bias errors: eccentricity(e), "pitch" and "yaw" errors.

These plots show (Figure 3.7 and Figure 3.8) the nonlinear behavior of the mean final eccentricity with the maximum "pitch" and "yaw" deviations. They were done only for the second maneuver because in the first one the change of the eccentricity is close to zero for the usual values of DES2 and DES3. They were plotted with precision of 10^{-3} for the eccentricity.

For the white-noise errors the results were very similar to the results obtained for the random-bias errors but, the curves for the "pitch" errors present a more defined pattern with respect to those for the "yaw" errors, where small fluctuations appear in its final form. It is possible to see that the influence of the out-of-plane ("yaw") errors is so strong in the definition of the orbital transfer trajectory.

4) Uniform white-noise errors: semi-major axis(a), "pitch" errors.

The Figures 3.9 and 3.10 show the results for white-noise "pitch" errors in the semi-major axis.

5) Uniform white-noise errors: semi-major axis(a), "yaw" errors.

The Figures 3.11 and 3.12 show the results for white-noise "yaw" errors in the semi-major axis.

Figures 3.9 to 3.12 show clearly the influence of the white-noise errors when the second maneuver is simulated with errors in "yaw". The region of decrease still exist, as well as the nonlinear relation, but there are fluctuations in the growth of the maximum "yaw" error.

6) Uniform white-noise errors: eccentricity(e), "pitch" and "yaw" errors.

These plots (Figures 3.13 and 3.14) show that the values of the eccentricity also fluctuate for practical maneuvers with the white-noise errors in "yaw", but keeping the region of growth similar to the one verified for the random-bias errors case. So, we can say that all these results suggest and partially characterize the progressive deformation of the trajectory distribution along the propulsive arc. It occurs due to the loss of optimality of the actual trajectories (with errors) with respect to the nominal trajectories (without errors).

The dependence of the final keplerian elements with the magnitude errors for any of the cases was practically null, specially for the mean deviation of the final semi-major axis, since the perturbations occurred in this element were probably due to its estimator and they were comparable to the numerical errors of the experiment, as shown in Figures 3.15 and 3.16. They show that the mean deviation in the final semi-major axis is much smaller than the cone $\pm 1\sigma$(standard deviation of the deviation in the final semi-major axis).

The values for DES1, DES2 and DES3 used in these plots range from usual values to nonusual values, with the aim to verify the general behaviors. Obviously, it is not usual to have a "pitch" error equal to $30,0^o$ or a magnitude error equal to $30,0\%$, for example.

Conclusions

This work presented results of the thrust vector errors implementation for nonimpulsive orbital transfer maneuvers. It was verified that, in any case, the mean deviation in the final semi-major axis presents a nonlinear (approximately parabolic) dependence with the maximum error in thrust direction. The same results were verified for the mean deviation in the final eccentricity, for the second transfer. The respective dependencies with the thrust magnitude errors were not verified. The general results suggest a progressive deformation of the trajectory distribution along the propulsive arc. This deformation may be associated to the loss of the optimality of the actual trajectories with respect to the nominal trajectory.

Acknowledgments

The partial financial support from CAPES (Brazilian agencie) is acknowledged.

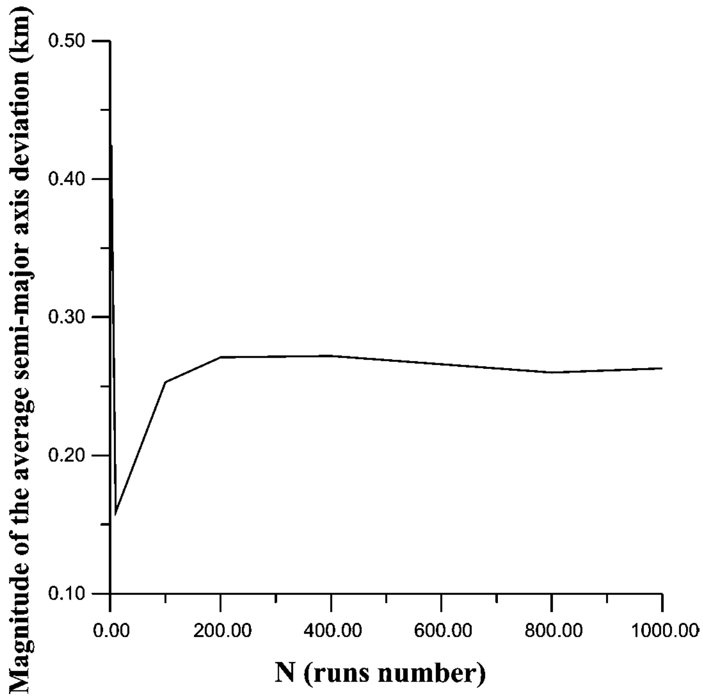

Figure 3.1. Magnitude of the average semi-major axis deviations vs. N.

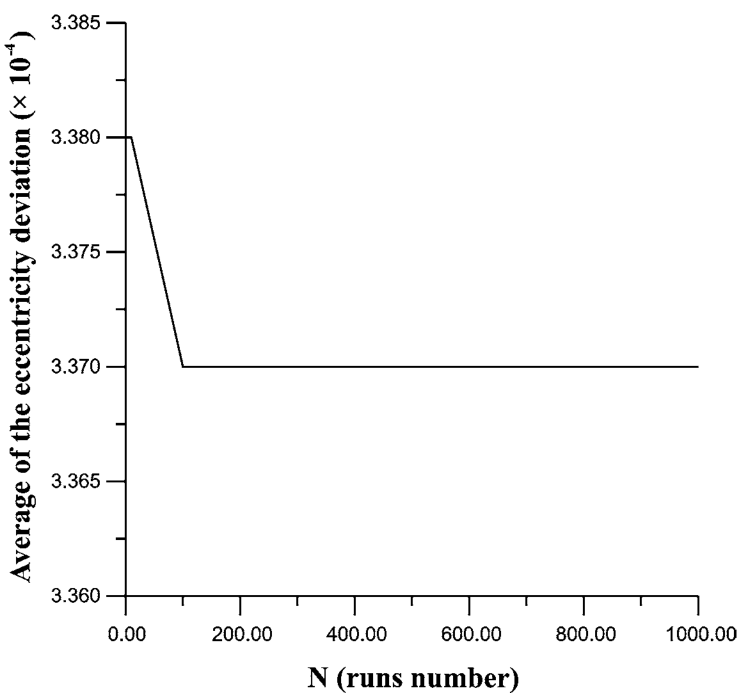

Figure 3.2. Average of the eccentricity $(\times 10^{-4})$ deviation vs. N.

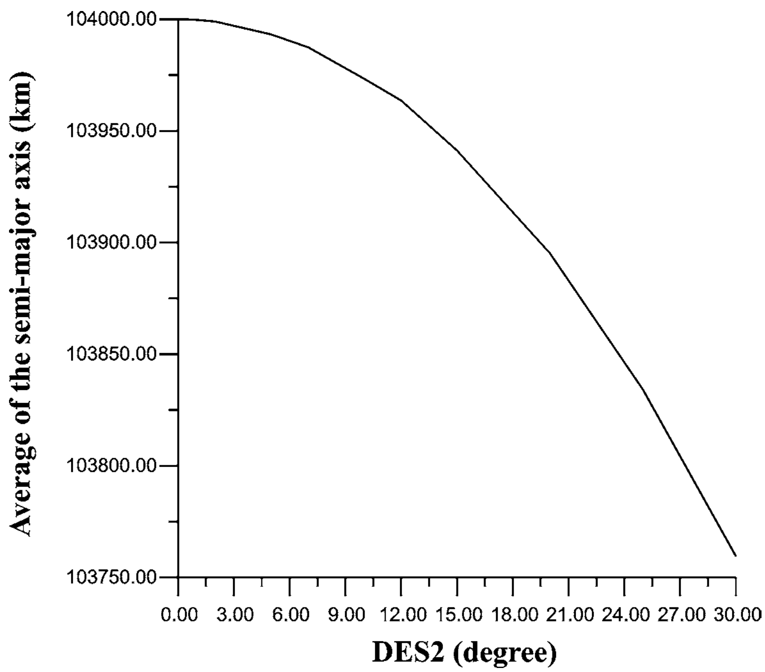

Figure 3.3. First Maneuver: $E\{a(t_f)\}$ vs. $DES2$.

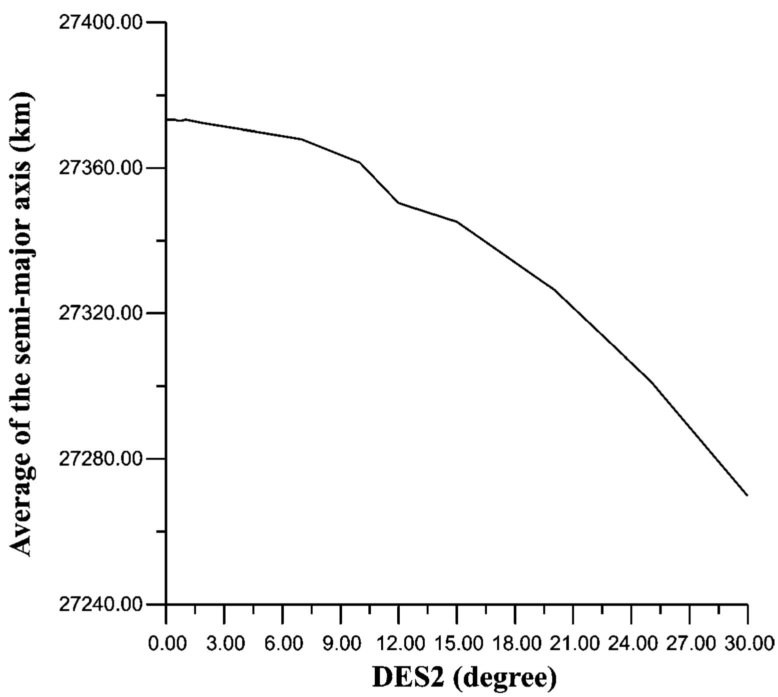

Figure 3.4. Second Maneuver: $E\{a(t_f)\}$ vs. $DES2$.

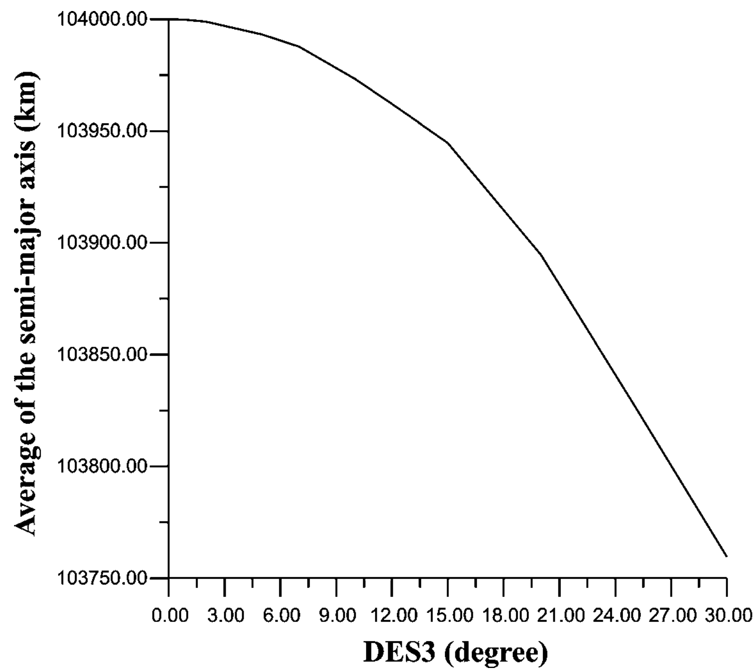

Figure 3.5. First Maneuver: $E\{a(t_f)\}$ vs. $DES3$.

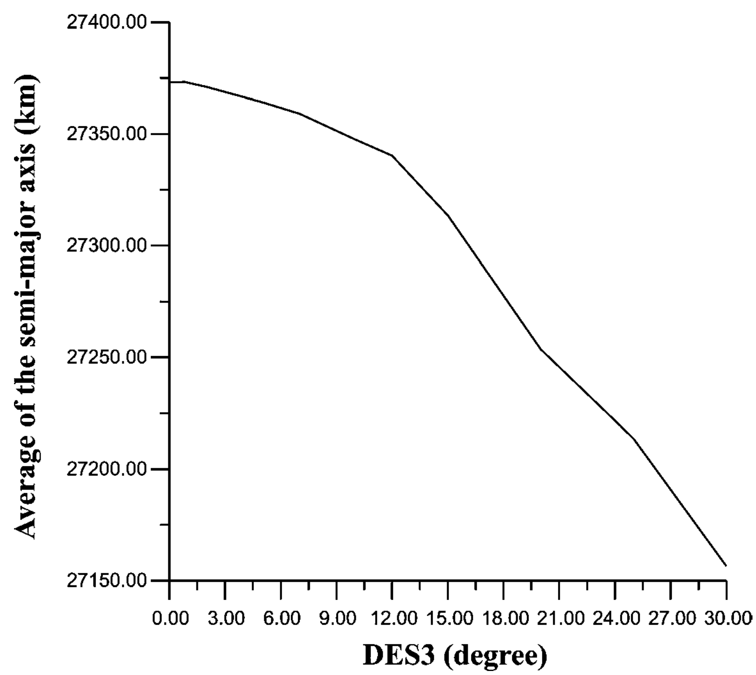

Figure 3.6. Second Maneuver: $E\{a(t_f)\}$ vs. $DES3$.

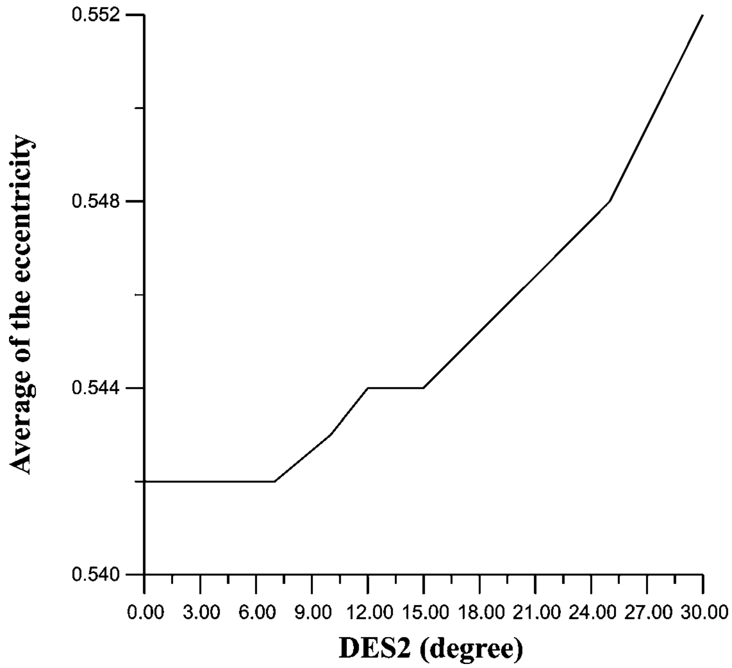

Figure 3.7. Second Maneuver: $E\{e(t_f)\}$ vs. $DES2$.

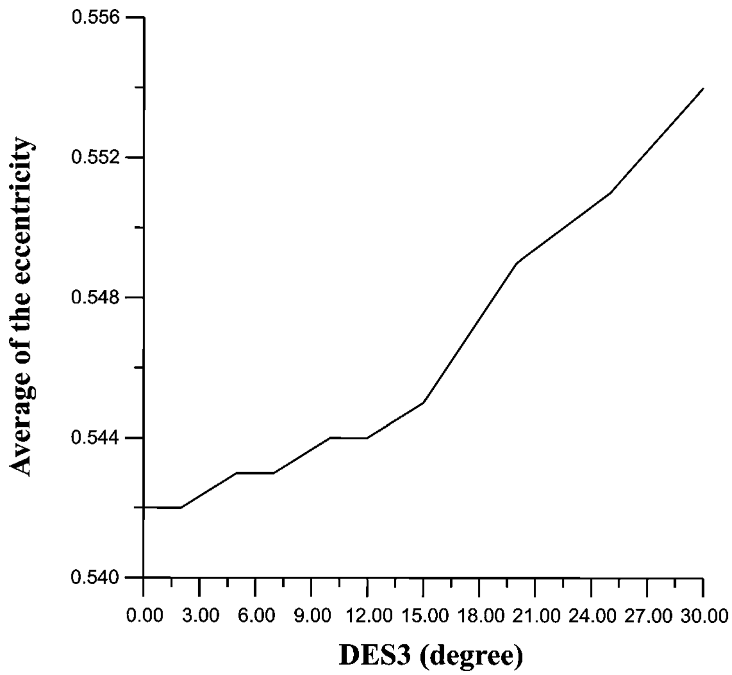

Figure 3.8. Second Maneuver: $E\{e(t_f)\}$ vs. $DES3$.

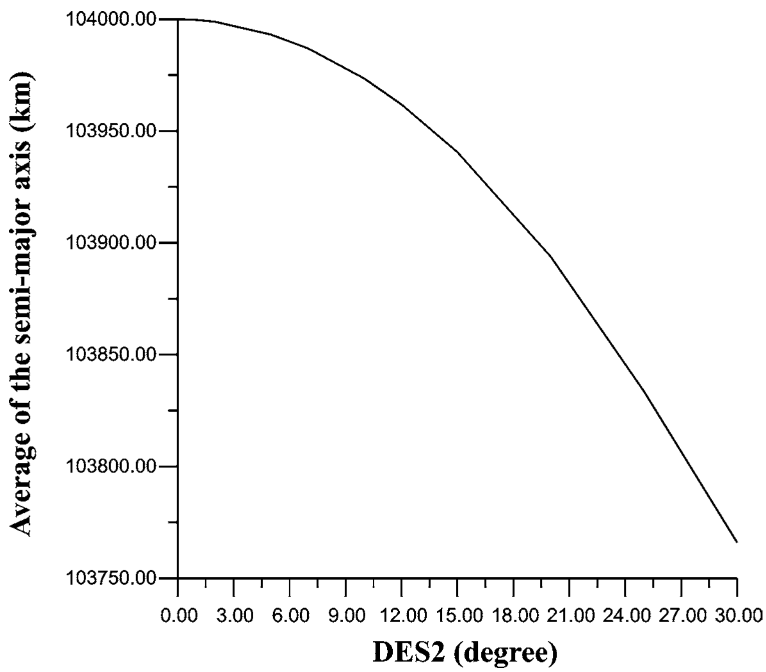

Figure 3.9. First Maneuver: $E\{a(t_f)\}$ vs. $DES2$.

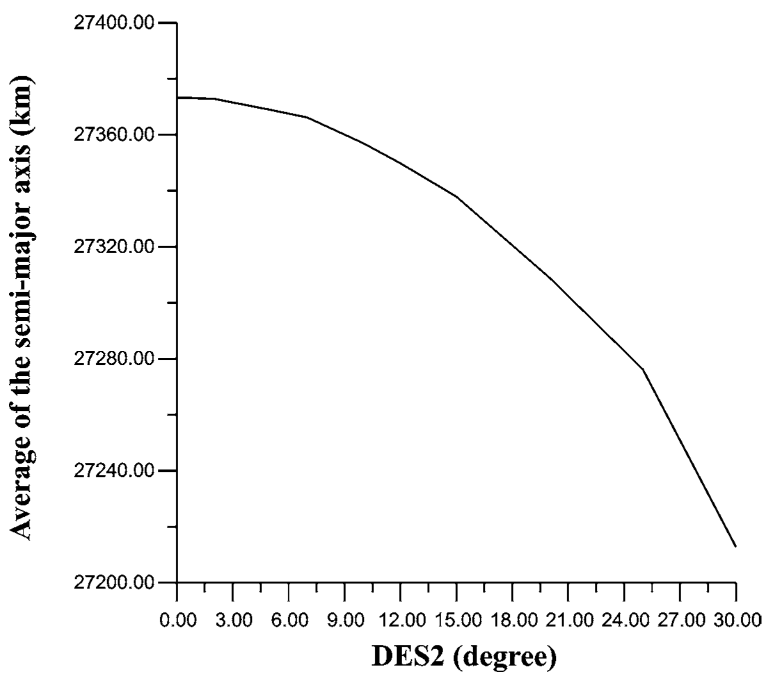

Figure 3.10. Second Maneuver: $E\{a(t_f)\}$ vs. $DES2$.

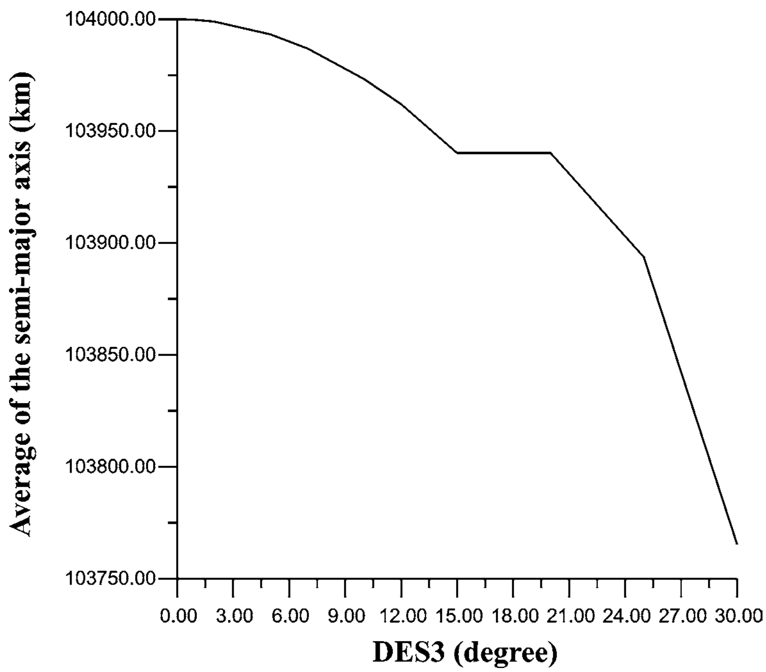

Figure 3.11. First Maneuver: $E\{a(t_f)\}$ vs. $DES3$.

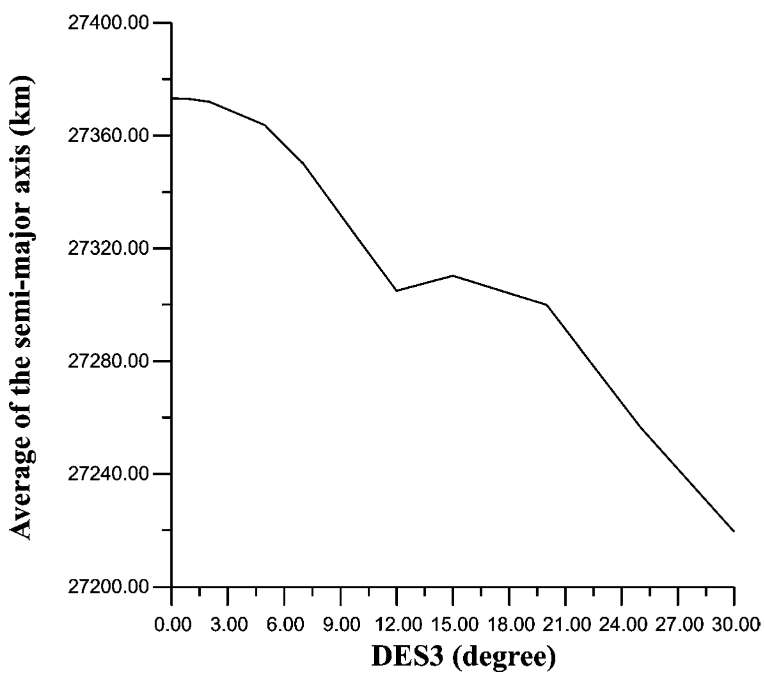

Figure 3.12. Second Maneuver: $E\{a(t_f)\}$ vs. $DES3$.

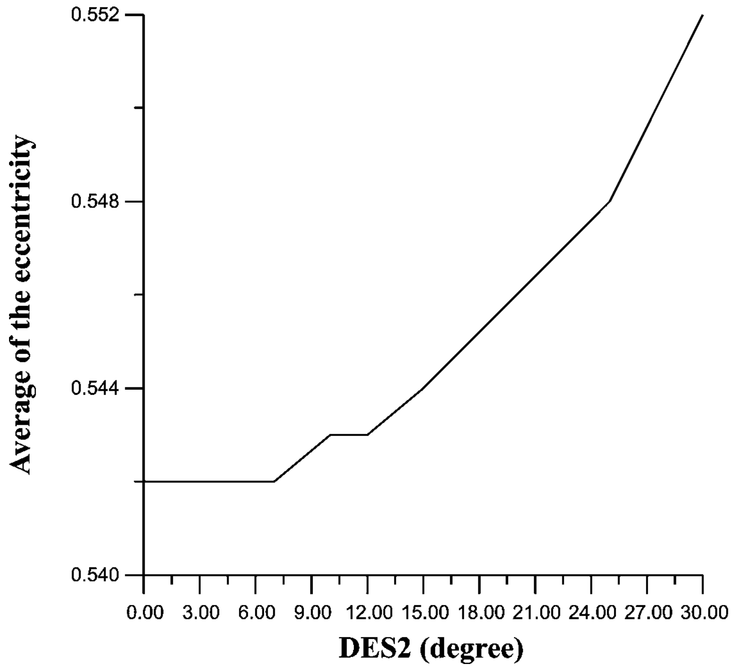

Figure 3.13. Second Maneuver: $E\{e(t_f)\}$ vs. $DES2$.

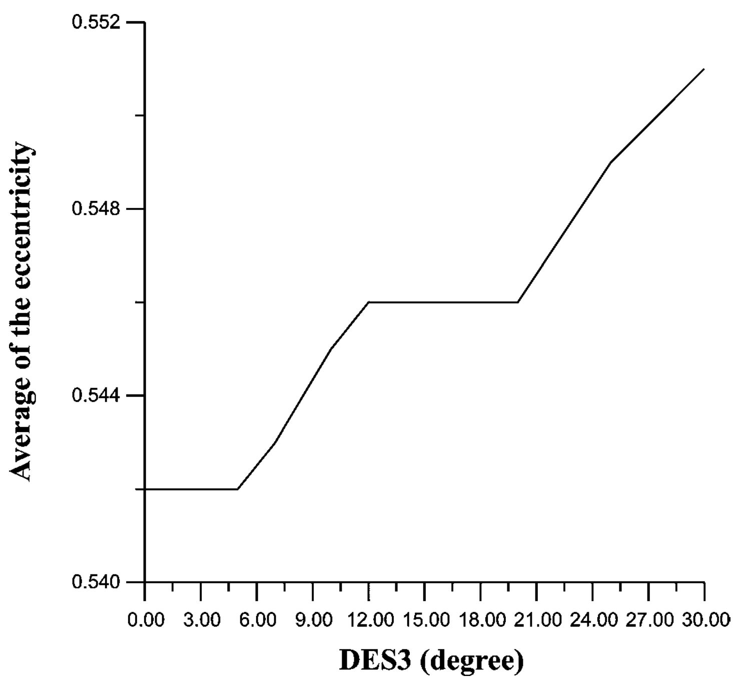

Figure 3.14. Second Maneuver: $E\{e(t_f)\}$ vs. $DES3$.

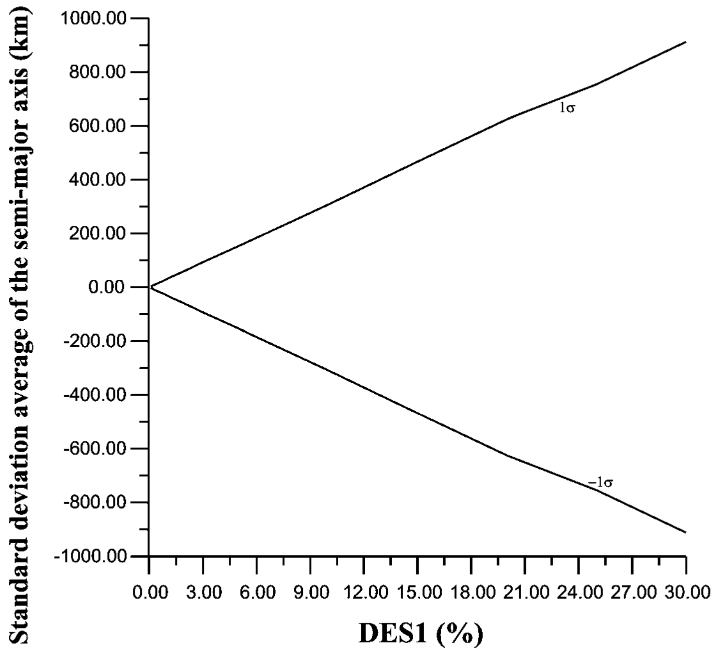

Figure 3.15. First Maneuver: $E\{\Delta a(t_f)\}$ vs. $DES1$ and its σ (standard deviation).

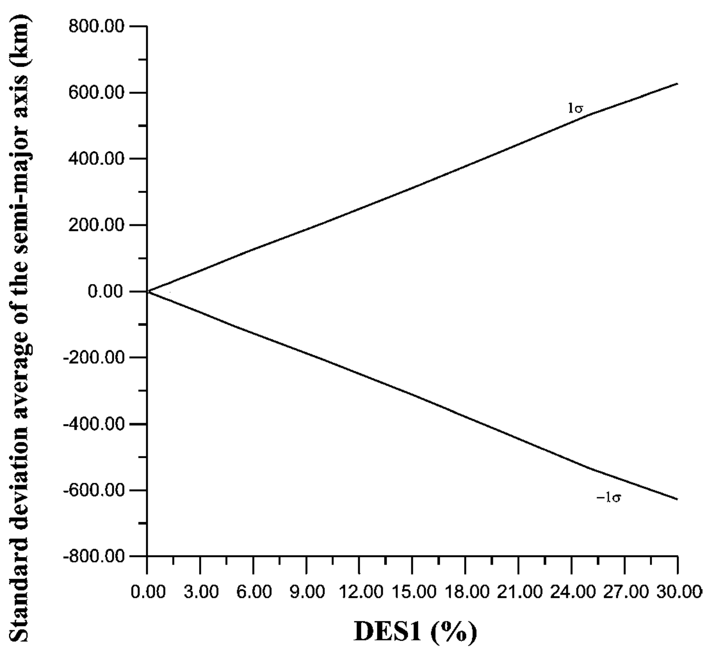

Figure 3.16. Second Maneuver: $E\{\Delta a(t_f)\}$ vs. $DES1$ and its σ (standard deviation).

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Impulsive Stabilization and Application to a Population Growth Model*

Xinzhi Liu[1] and Xuemin Shen[2]

[1] *Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.*

[2] *Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.*

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Abstract: This paper studies the problem of impulsive stabilization of a system of autonomous ordinary differential equations. Necessary and sufficient conditions are established for a given state, which need not be an equilibrium point of the system, to be impulsively stabilizable. These results are applied to a three-species population growth model. In the population growth model, it is shown that by impulsively regulating one species, the population of all three species can be maintained at a positive level, which otherwise would drop to a level of extinction for one of the species.

Keywords: *Impulsive stabilization; control; population growth model.*

Mathematics Subject Classification (2000): 34A37, 34d20, 93B05, 93C15.

1 Introduction

In this paper we shall investigate the problem of impulsively controlling a system of autonomous ordinary differential equations so as to keep solutions close to a given state, p, which need not be an equilibrium point of the system. Consider the following autonomous system

$$x' = f(x), \tag{1}$$

where $f \in C^1[D, \mathbb{R}^n]$, $D \subset \mathbb{R}^n$ is open. Let the space $X = \mathbb{R}^n$ be decomposed into the direct sum $X = Y \oplus Z$, where Y is an m-dimensional subspace of X, $1 \le m < n$, and $Z = Y^{\perp}$. We call Y and Z the impulsive and non-impulsive subspaces respectively. Any vector $x \in X$, or vector function $f(x)$, may be expressed uniquely as $x = y + z$,

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or $f(x) = f_y(x) + f_z(x)$, where y, $f_y(x) \in Y$ and $z, f_z(x) \in Z$. We shall utilize this decomposition throughout this chapter and remark that the subscripts y and z on any vector in x shall denote its unique portion in Y and Z respectively, while the vectors y and z shall represent the unique portions of x.

Let U be the set of admissible controls u, where $u = \{(t_k, \Delta y_k)\}_{k=1}^{\infty}$ and

(i) $0 \le t_1 < t_2 < \ldots < t_k < \ldots$, and $t_k \to \infty$ as $k \to \infty$,

(ii) $\Delta y_k \in Y$, $k = 1, 2, \ldots$, $|\Delta y_k| \le A$ for some positive constant A.

Consider the impulsive control system associated with system (1)

$$\begin{cases} y' = f_y(x), \\ z' = f_z(x), & t \neq t_k, \quad k = 1, 2, \ldots, \\ y(t_k^+) = y(t_k) + \Delta y_k, & k = 1, 2, \ldots, \\ x(0) = x_0. \end{cases} \tag{2}$$

For a comprehensive treatment of impulsive differential equations see Lakshmikantham, Bainov and Simeonov (1989). Let $\phi(t, x)$ be a solution of (1). Then for each $u \in U$, $x(t) = x(t, x_0, u)$ is a solution of (2) given by

$$x(t) = \phi(t - t_{k-1}, x_{k-1}^+), \qquad t \in (t_{k-1}, t_k], \quad k = 1, 2, \ldots, \tag{3}$$

where $t_0 = 0$, $x_0^+ = x_0$, and

$$x_k^+ = \phi(t_k - t_{k-1}, x_{k-1}^+) + \Delta y_k, \quad k = 1, 2, \ldots. \tag{4}$$

Note from (3) and (4) that since f is C^1, ϕ is a continuous function of t, and since $\Delta y_k \in Y$, it follows that $z(t)$ is continuous for all $t \ge 0$, while $y(t)$ is continuous on each interval $(t_{k-1}, t_k]$, where $y(t) + z(t)$ is the decomposition of $x(t)$.

2 Criteria for Stabilizability

We begin by stating the concept of impulsive stabilization of a point p. For $\alpha > 0$, let $B_\alpha(p) = \{x \in \mathbb{R}^n : |p - x| < \alpha\}$.

Definition 2.1 A point $p \in D$ is said to be

(S_1) *impulsively stabilizable* if for any given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for each $x_0 \in B_\delta(p)$ there exists a $u \in U$ such that $x(t) \in B_\epsilon(p)$, for all $t \ge 0$, where $x(t) = x(t, x_0, u)$ is any solution of (2);

(S_2) *asymptotically impulsively stabilizable* if for any given $\epsilon > 0$, there exists a $\sigma = \sigma(\epsilon) > 0$ such that for each $x_0 \in B_\sigma(p)$ there exists a $u \in U$ such that $x(t) \in B_\epsilon(p)$, for all $t \ge 0$, and $\lim_{t\to\infty} x(t) = p$;

(S_3) *impulsively unstabilizable* if (S_1) fails to hold.

It should be noted that the point p in the above definition is, in general, not an equilibrium point of the system, in contrast to those found in the standard control theory (see for example Sontag, 1990). The type of stability defined above may be considered a special case of stability in terms of two measures, a concept expounded by Lakshmikantham and Liu (1989), and Liu (1990).

2.1 Necessary conditions

It follows from the above definition that the vector field f must be tangent to the impulsive subspace at p for p to be impulsively stabilizable as indicated in the theorem below.

Theorem 2.1 *If $f(p) \notin Y$, then p is impulsively unstabilizable.*

Proof If $f(p) \notin Y$, then $f_z(p) = v \neq 0$. By continuity of f, there exists an $\epsilon > 0$ such that

$$|\mathrm{Proj}_v f_z(x)| > 0, \quad \forall\, x \in \overline{B_\epsilon(p)}, \tag{5}$$

where Proj_v denotes the orthogonal projection onto the one-dimensional subspace defined by $\mathrm{span}\,\{v\}$. By continuity of the projection function and by the compactness of $\overline{B_\epsilon(p)}$, inequality (5) implies that there exists $m > 0$ such that

$$|\mathrm{Proj}_v f_z(x)| \geq m > 0, \quad \forall\, x \in \overline{B_\epsilon(p)}; \tag{6}$$

physically, m is the minimum speed in the positive v direction for all points in $\overline{B_\epsilon(p)}$. For any δ, $0 < \delta < \epsilon$, choose $x_0 = y_0 + z_0$ in $B_\delta(p)$. Since $z(t) = z(t, x_0, u)$ is continuous in t, (6) implies

$$|\mathrm{Proj}_v(z(t) - z_0)| \geq mt, \quad \forall\, t \geq 0, \quad \text{provided} \quad x(t) \in \overline{B_\epsilon(p)}. \tag{7}$$

Since the projection is orthogonal, (7) implies

$$|z(t) - z_0| \geq mt, \quad \forall\, t \geq 0, \quad \text{provided} \quad x(t) \in \overline{B_\epsilon(p)}, \tag{8}$$

but $|z(t) - z_0| \leq |x(t) - x_0| \leq |x(t) - p| + |p - x_0|$, hence from (8) we have

$$|x(t) - p| \geq mt - |p - x_0| \geq mt - \delta, \quad \forall\, t \geq 0,$$

so that $|x(t) - p| > \epsilon$ for t sufficiently large; consequently p is impulsively unstabilizable.

2.2 Impulsively invariant sets

To motivate and help illustrate our subsequent theorem we shall embark on a short discussion of the problem of finding a control u that will create an invariant set of system (2).

Consider a system in $\mathbb{R}^3$ and a point p for which $f(p)$ is aligned with the x-axis. Let $Y = \mathrm{span}\{(1,0,0)^T\}$ and $Z = \mathrm{span}\{(0,1,0)^T, (0,0,1)^T\}$. Such a system meets the necessary condition of Theorem 2.1. Consider a closed curve C, lying in the plane through p parallel to Z such that p is in the interior of C. Generate a "cylinder", S, by constructing, through each point of C, a line segment of length 2ℓ parallel to Y such that its midpoint lies on C, (see Figure 2.1). The boundary of the cylinder is composed of the cylinder's wall (the line segments) and its two ends which are surfaces parallel to Z.

Our aim is to make the cylinder, S, invariant with the application of impulses in the x_1-direction. Consider a point x_0, starting within the cylinder. The trajectory $\phi(t, x_0)$ will either stay within S or will reach the cylinder wall or the ends. Suppose $\phi(t^*, x_0) = A$ for some time t^*, where A is a point on one of the cylinder ends. It is then easy to see that an impulse of strength less than 2ℓ in the positive or negative x_1-direction as appropriate, will send the trajectory back into the interior of S. If however $\phi(t^*, x_0) = B$, where B is a point on the cylinder wall, then an impulse in Y can only carry the trajectory to some other point on the cylinder wall along the line through B parallel to Y. If along this line segment there is a point Q where the vector field f is moving into the cylinder then an

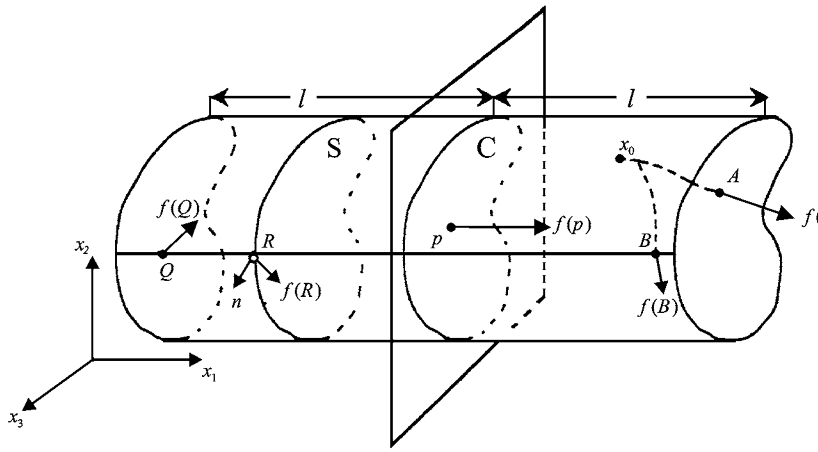

Figure 2.1. An invariant cylinder.

impulse to Q will keep $x(t)$ within S for at least a short time longer. If there is a point R along the line segment at which the velocity field is tangent to the cylinder wall and for which $\phi(t, R)$ lies either on or inside S for some positive time interval then an impulse to R would also keep $x(t)$ within S. If such points Q or R exist for each B on the cylinder wall then the cylinder S can be made invariant provided the sum of the time intervals between successive impulses is unbounded. We now formalize the preceding discussion on invariant sets.

Theorem 2.2 *Let $\Omega_z \subset Z$ be an open bounded region whose boundary is C^1, and let Ω_y be an open bounded region in Y. Define the "cylinder", Ω, and its wall, W, by $\Omega = \Omega_y \oplus \Omega_z$, and $W = \Omega_y \oplus \partial\Omega_z$. We assume $\Omega \subset D$. Let n be the unit outward normal to Ω defined on W, and define the set N as*

$$N = \{w \in W \colon f \cdot n\big|_w \leq 0\}^{O(W)},$$

where the superscript $O(W)$ denotes the interior of the set N with respect to W.

If $\mathrm{Proj}_z(N) = \partial\Omega_z$, *then for any $x_0 \in \overline{\Omega}$, there exists a $u \in U$ such that $x(t, x_0, u) \in \overline{\Omega}$, for all $t \geq 0$.*

Proof See Liu and Willms (1994).

2.3 Sufficient conditions

Sufficient conditions for impulsive stabilizability are given in the following theorem. Essentially, the conditions imposed assure that for any positive ϵ, there exists an impulsively invariant set contained within $B_\epsilon(p)$. The proof itself, although somewhat intuitive, is quite long and technical, for which reason the reader is referred to Liu and Willms (1994).

Theorem 2.3 *Let $p = p_y + p_z$ be a point in D and let $v \in C^1[Z, \mathbb{R}]$ be a positive* *definite function with respect to p_z. Define the sets*

$$
\begin{aligned}
I &= \{x \in D \mid \nabla v(z) \cdot f_z(x) \leq 0\}^O\,, \\
I_\alpha &= \mathrm{Proj}_z\,(I \cap (B^y_\alpha(p_y) \oplus Z)) \cup \{p_z\}, \\
J &= \{x \in D \mid \nabla v(z) \cdot f_z(x) < 0\}\,, \\
J_\alpha &= \mathrm{Proj}_z\,(J \cap (B^y_\alpha(p_y) \oplus Z)) \cup \{p_z\},
\end{aligned}
$$

where the superscript O denotes the interior of the set, Proj_z *denotes the orthogonal projection onto the Z subspace, and $B^y_\alpha(p_y)$ is the m-dimensional α-ball around p_y in the Y subspace.*

(a) *If I_α is a neighbourhood of p_z, for all $\alpha > 0$, then p is impulsively stabilizable.*
(b) *If J_α is a neighbourhood of p_z, for all $\alpha > 0$, then p is asymptotically impulsively stabilizable.*

We remark that the openness of the set I in the above theorem is an essential requirement without which the theorem does not hold.

3 Application

In this section, we shall consider a fish population growth model. Suppose the owner of a resort on a small northern lake wishes to attract fishermen by increasing the population of two particular species of game fish in his lake. Upon looking into the matter he discovers that the cost of stocking his lake with these species is excessive while the cost of stocking his lake with the main prey species of these game fish is comparatively economical. The owner therefore wishes to determine how high he can keep the game species population by stocking the lake with the feeder fish. A model for this situation may be presented as below,

$$
\begin{aligned}
\dot{N}_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3), \\
\dot{N}_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3), \\
\dot{N}_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3),
\end{aligned}
\tag{9}
$$

where N_1 is the feeder fish population, N_2 and N_3 are the game fish populations, all of the b_i, e_i, a_{ij} are constants, $b_i - a_{ii}N_i$ is the per capita birth rate of the population N_i, $-a_{12}N_2 - a_{13}N_3$ represents the effect of the predation, $a_{21}N_1$, $a_{31}N_1$ represent the prey's contribution to the predator's growth rate, $-a_{23}N_3$, $-a_{32}N_2$ represent the effect of the competition between the predators, and e_2, e_3 are the fishing efforts applied by the anglers. From Theorem 2.1, the candidate positive points, p, that could be stabilized are those satisfying

$$
\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} b_2 - e_2 + a_{21}p_1 \\ b_3 - e_3 + a_{31}p_1 \end{pmatrix}. \tag{10}
$$

In what follows, we shall assume that all constants are positive, and furthermore, equation (10) tells us that $b_2 - e_2 > 0$ and $b_3 - e_3 > 0$, that is to say that the fishing efforts of the game fish e_2 and e_3 cannot exceed their birth rates b_2 and b_3.

Selecting p such that (10) is satisfied, we choose the Lyapunov function

$$v = N_2 - p_2 - p_2 \ln \frac{N_2}{p_2} + N_3 - p_3 - p_3 \ln \frac{N_3}{p_3}.$$

Differentiating with respect to t, substituting for $b_2 - e_2$ and $b_3 - e_3$ from (10) and rearranging gives

$$\begin{aligned} \dot{v} &= (N_2 - p_2)[a_{21}(N_1 - p_1) - a_{22}(N_2 - p_2) - a_{23}(N_3 - p_3)] \\ &\quad + (N_3 - p_3)[a_{31}(N_1 - p_1) - a_{32}(N_2 - p_2) - a_{33}(N_3 - p_3)] \\ &= -(N_2 - p_2, N_3 - p_3) \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} N_2 - p_2 \\ N_3 - p_3 \end{pmatrix} \\ &\quad + (N_1 - p_1)[a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)]. \end{aligned} \tag{11}$$

Let

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

We note that if det $A > 0$ then $\dot{v} < 0$ if the second term of (11) is negative. Likewise, if det $A = 0$ then $\dot{v} < 0$ if the second term is strictly negative.

Based on this, we can determine the regions in the positive orthant that belong to the set $\mathcal{J} = \{\dot{v} < 0\}$ and hence give impulses in the other regions to bring them to points in $\mathcal{J}$.

Case 1: Consider the case det $A > 0$. Here the set $\mathcal{J}$ is given by points in the positive orthant which make the second term of (11) negative, that is,

$$\begin{aligned} \mathcal{J} &= \{N_1 \geq p_1, a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\} \\ &\quad \cup \{N_1 \leq p_1, a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\} \\ &\quad \cup \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0, \ N_2 \neq p_2, \ N_3 \neq p_3\}. \end{aligned} \tag{12}$$

Note that we must exclude the case $N_2 = p_2$ and $N_3 = p_3$ from $\mathcal{J}$, as otherwise, $\dot{v} = 0$.

Divide the positive orthant into the following regions:

$$\begin{aligned} \Omega_1 &= \{N_1 \geq p_1, \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\ \Omega_2 &= \{N_1 > p_1, \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\ \Omega_3 &= \{N_1 \leq p_1, \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\ \Omega_4 &= \{N_1 < p_1, \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\ \Omega_5 &= \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0, \ N_2 \neq p_2, \ N_3 \neq p_3\}, \\ \Omega_6 &= \{N_2 = p_2, \ N_3 = p_3\}. \end{aligned} \tag{13}$$

Clearly, $\mathcal{J}$ is made up of Ω_1, Ω_3 and Ω_5, so, at most, we need to specify impulses in the other three regions.

Impulses in Ω_2 and Ω_4 are as follows:

$$\Delta N_1 = \begin{cases} c - N_1, & \mathbf{N} \in \Omega_2, \; 0 < c \le p_1, \\ d - N_1, & \mathbf{N} \in \Omega_4, \; d \ge p_1. \end{cases} \tag{14}$$

This ensures that points in Ω_2 are moved to points in $\Omega_3 \subset \mathcal{J}$ and points in Ω_4 are moved to points in $\Omega_1 \subset \mathcal{J}$.

We will now show that impulses are not required in region Ω_6. Our system of ODE's in Ω_6 reduces to

$$\begin{aligned} \dot N_1 &= N_1(b_1 - a_{11}N_1 - a_{12}p_2 - a_{13}p_3), \\ \dot N_2 &= p_2(b_2 - e_2 + a_{21}N_1 - a_{22}p_2 - a_{23}p_3) \\ &= p_2 a_{21}(N_1 - p_1) \qquad \text{(using (10))}, \\ \dot N_3 &= p_3(b_3 - e_3 + a_{31}N_1 - a_{32}p_2 - a_{33}p_3) \\ &= p_3 a_{31}(N_1 - p_1) \qquad \text{(using (10))}. \end{aligned} \tag{15}$$

It is clear that points where $\dot N_2 \neq 0$ and/or $\dot N_3 \neq 0$ will cause N_2 and/or N_3 to increase or to decrease and points will therefore leave Ω_6. We are therefore concerned with points where both $\dot N_2 = 0$ and $\dot N_3 = 0$. We see from (15) that this occurs only if $N_1 = p_1$. Now because both $\dot N_2$ and $\dot N_3$ are equal to zero, we must have $\dot N_1 \neq 0$, or else we have a positive equilibrium point, which we have assumed does not exist. It follows that N_1 must then either increase or decrease, in which case we move to regions in Ω_6 where either $N_1 > p_1$, or $N_1 < p_1$. At this new point, we no longer have $\dot N_2 = 0$ and $\dot N_3 = 0$, so that we, again, leave Ω_6, by the same reasoning as before. It follows that solutions passing through Ω_6 naturally leave there and move to one of the other five regions, where impulses have already been specified.

It follows that the required set of impulses in the case of det $A > 0$ is as specified by equation (14).

Case 2: Now consider the case det $A = 0$. In this case we must ensure that the second term of (11) is strictly negative, in which case we obtain the set $\mathcal{J}$ given by

$$\begin{aligned} \mathcal{J} = &\{N_1 < p_1, \; a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\} \\ &\cup \{N_1 > p_1, \; a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)\}. \end{aligned} \tag{16}$$

Unless one further analyzes what happens on the plane $a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0$, one cannot specify impulses in N_1 that guarantee survival of all three fish. The reason for this is that one does not know the behaviour of solutions on this plane, and since this plane is parallel to the N_1 axis, an impulse in N_1 does not move points off of the plane. A problem will certainly occur if solutions move along this plane towards extinction in N_2, or N_3, which is quite possible.

With this in mind, we try to find regions on this plane where solutions leave the plane, and thus we can specify impulses in the remaining regions to move to these "good" regions of the plane. It turns out that the entire plane without the line $\{N_2 = p_2, \; N_3 = p_3\}$ may be added to the set $\mathcal{J}$, provided certain conditions on the a_{ij} hold, which guarantee that both terms of (11) are not equal to zero at the same time. Call this set $\mathcal{S}$, that is, let

$$\mathcal{S} = \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0, \; N_2 \neq p_2, \; N_3 \neq p_3\}.$$

We now derive these conditions on the a_{ij}, so that we may add the set $\mathcal{S}$ to $\mathcal{J}$.

The first term of (11) equal to zero gives

$$a_{22}(N_2 - p_2)^2 + (a_{23} + a_{32})(N_2 - p_2)(N_3 - p_3) + a_{33}(N_3 - p_3)^2 = 0.$$

Using the quadratic formula, and that $a_{22} > 0$ as given, we have

$$\begin{aligned} N_2 - p_2 &= \frac{-(a_{23} + a_{32}) \pm \sqrt{(a_{23} + a_{32})^2 - 4a_{22}a_{33}}}{2a_{22}}\,(N_3 - p_3) \\ &= \frac{-(a_{23} + a_{32}) \pm \sqrt{(a_{23} - a_{32})^2}}{2a_{22}}\,(N_3 - p_3) \quad \text{(using det } A = 0 \text{ and factoring)} \\ &= \begin{cases} -\dfrac{a_{32}}{a_{22}}\,(N_3 - p_3), \\ -\dfrac{a_{23}}{a_{22}}\,(N_3 - p_3). \end{cases} \end{aligned}$$

The second term of (11) equals zero for points inside the set $\mathcal{S}$ and they satisfy

$$N_2 - p_2 = -\frac{a_{31}}{a_{21}}\,(N_3 - p_3).$$

It follows that if we have

$$\frac{a_{31}}{a_{21}} \neq \frac{a_{32}}{a_{22}}, \quad \text{and} \quad \frac{a_{31}}{a_{21}} \neq \frac{a_{23}}{a_{22}} \tag{17}$$

both terms are cannot equal zero at the same time, or equivalently, the first term is non-zero in the set $\mathcal{S}$. This means that $\dot{v} < 0$ in this set, and we may, as a result, add it to $\mathcal{J}$.

As in the case det $A > 0$, we split the positive orthant into the regions as shown, in which the regions Ω_1, Ω_3, and Ω_5 belong to $\mathcal{J}$.

$$\begin{aligned} \Omega_1 &= \{N_1 > p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\ \Omega_2 &= \{N_1 \le p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\ \Omega_3 &= \{N_1 < p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) > 0\}, \\ \Omega_4 &= \{N_1 \ge p_1,\ \ a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) < 0\}, \\ \Omega_5 &= \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0,\ \ N_2 \neq p_2,\ \ N_3 \neq p_3\}, \\ \Omega_6 &= \{N_2 = p_2,\ \ N_3 = p_3\}. \end{aligned} \tag{18}$$

The only thing different here from (13) are the inequalities in the N_1, p_1 terms.

One can show in exactly the same manner as in the Case 1, that solutions in Ω_6 naturally tend to one of the other regions.

Further, our set of impulses is the same as in (14), with the inequalities in the constants c and d changed, that is, the required set of impulses in the case $\det A = 0$ is given by

$$\Delta N_1 = \begin{cases} c - N_1, & \mathbf{N} \in \Omega_2,\ \ 0 < c < p_1, \\ d - N_1, & \mathbf{N} \in \Omega_4,\ \ d > p_1. \end{cases} \tag{19}$$

Provided the conditions (17) are satisfied.

We now show that, even if (17) does not hold, one may still specify impulses in exactly the same manner as above.

Consider first the case $\dfrac{a_{31}}{a_{21}} = \dfrac{a_{23}}{a_{22}}$. Again, we try to add the set

$$\mathcal{S} = \{a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3) = 0, \;\; N_2 \neq p_2, \;\; N_3 \neq p_3\}$$

to $\mathcal{J}$. Again, we may do this provided points in $\mathcal{S}$ don't remain there, possibly leading to extinction in one of the species. Consider what happens in $\mathcal{S}$ by looking at the system (9). We have

$$\begin{aligned} \dot N_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3), \\ \dot N_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3) \\ &= N_2[a_{21}(N_1 - p_1) - \{a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)\}] \quad \text{(using (10))} \\ &= N_2 a_{21}(N_1 - p_1) \end{aligned} \tag{20}$$

<u>*Aside:*</u>

$$\begin{aligned} a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3) &= \frac{a_{21}a_{23}}{a_{31}}(N_2 - p_2) + a_{23}\,(N_3 - p_3) \\ &\qquad \Big(\text{using } \frac{a_{31}}{a_{21}} = \frac{a_{23}}{a_{22}}\Big) \\ &= \frac{a_{23}}{a_{31}}\,[a_{21}(N_2 - p_2) + a_{31}(N_3 - p_3)] \\ &= 0 \qquad \text{(as we are in } \mathcal{S}) \end{aligned}$$

and

$$\begin{aligned} \dot N_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3) \\ &= N_3[a_{31}(N_1 - p_1) - \{a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3)\}] \quad \text{(using (q0))} \\ &= N_3 a_{31}(N_1 - p_1) \end{aligned}$$

<u>*Aside:*</u>

$$\begin{aligned} a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3) &= \frac{a_{22}a_{33}}{a_{23}}(N_2 - p_2) + a_{33}(N_3 - p_3) \\ &\qquad (\text{using } \det A = 0) \\ &= \frac{a_{33}}{a_{23}}[a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)] \\ &= 0. \qquad \text{(from above)} \end{aligned}$$

It follows that if $N_1 = p_1$, then $\dot N_2 = 0$ and $\dot N_3 = 0$. Further $\dot N_1 \neq 0$ or else we have a positive equilibrium point. If $\dot N_1 > 0$, then N_1 increases, so that at the next point, we have $N_1 > p_1$. This in turn implies $\dot N_2 > 0$ and $\dot N_3 > 0$. Similarly, if $\dot N_1 < 0$, then N_1 decreases, so at the next point we have $\dot N_2 < 0$ and $\dot N_3 < 0$. Note, that no matter what happens, $\dot N_2$ and $\dot N_3$ have the same sign, so that $a_{21}\dot N_2 + a_{31}\dot N_3 \neq 0$, that is, solutions will leave $\mathcal{S}$. It follows that all solutions in $\mathcal{S}$ will leave $\mathcal{S}$ and hence we will not encounter the problem of extinction.

In the same manner, one analyzes what happens in the second case $\dfrac{a_{31}}{a_{21}} = \dfrac{a_{32}}{a_{22}}$. Here, the system (9) reduces to

$$\begin{aligned}
\dot{N}_1 &= N_1(b_1 - a_{11}N_1 - a_{12}N_2 - a_{13}N_3),\\
\dot{N}_2 &= N_2(b_2 - e_2 + a_{21}N_1 - a_{22}N_2 - a_{23}N_3)\\
&= N_2[a_{21}(N_1 - p_1) - \{a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)\}] \qquad (21)\\
&\qquad \text{(using (10))},\\
\dot{N}_3 &= N_3(b_3 - e_3 + a_{31}N_1 - a_{32}N_2 - a_{33}N_3)\\
&= N_3[a_{31}(N_1 - p_1) - \{a_{32}(N_2 - p_2) + a_{33}(N_3 - p_3)\}]\\
&\qquad \text{(using (10))}\\
&= N_3\left[a_{31}(N_1 - p_1) - \left\{\frac{a_{22}a_{33}}{a_{23}}(N_2 - p_2) + a_{33}(N_3 - p_3)\right\}\right]\\
&\qquad \text{(using } \det A = 0 \text{ to substitute for } a_{32})\\
&= N_3\left[a_{31}(N_1 - p_1) - \frac{a_{33}}{a_{23}}\ \{a_{22}(N_2 - p_2) + a_{33}(N_3 - p_3)\}\right]. \qquad (22)
\end{aligned}$$

Solving for $a_{22}(N_2 - p_2) + a_{23}(N_3 - p_3)$ in (21) and substituting into (22) gives

$$\begin{aligned}
\dot{N}_3 &= N_3\left[a_{31}(N_1 - p_1) - \frac{a_{33}}{a_{23}}\left\{a_{21}(N_1 - p_1) - \frac{\dot{N}_2}{N_2}\right\}\right]\\
&= N_3\left[\left\{a_{31} - \frac{a_{33}a_{21}}{a_{23}}\right\}(N_1 - p_1) + \frac{a_{33}}{a_{23}}\,\frac{\dot{N}_2}{N_2}\right]\\
&= N_3\left[\frac{a_{31}a_{23} - a_{33}a_{21}}{a_{23}}\,(N_1 - p_1) + \frac{a_{33}}{a_{23}}\,\frac{\dot{N}_2}{N_2}\right]\\
&= N_3\,\frac{a_{33}}{a_{23}}\,\frac{\dot{N}_2}{N_2},
\end{aligned}$$

since

$$\begin{aligned}
\frac{a_{31}}{a_{21}} = \frac{a_{32}}{a_{22}} &\Rightarrow a_{21}a_{32} - a_{22}a_{31} = 0\\
&\Rightarrow a_{21}\,\frac{a_{22}a_{33}}{a_{23}} - a_{22}a_{31} = 0\\
&\qquad \text{(using } \det A = 0)\\
&\Rightarrow a_{21}a_{33} - a_{23}a_{31} = 0\\
&\Rightarrow a_{31}a_{23} - a_{33}a_{21} = 0.
\end{aligned}$$

It follows that in the set $\mathcal{S}$ the system of ODE's satisfies

$$\dot{N}_3 = \frac{a_{33}}{a_{23}}\,\frac{N_3}{N_2}\,\dot{N}_2.$$

Since we are concerned with points satisfying $a_{21}\dot{N}_2 + a_{31}\dot{N}_3 = 0$, we may encounter problems if

$$\frac{a_{33}}{a_{23}}\,\frac{N_3}{N_2} = -\frac{a_{21}}{a_{31}} \quad \Leftrightarrow \quad N_3 = -\frac{a_{21}a_{23}}{a_{31}a_{33}}\,N_2.$$

But we are given that all constants are positive, so that the above equation implies N_2 and N_3 are of opposite sign. This however, will never occur, since we are dealing with the positive orthant.

It follows that all points in $\mathcal{S}$ will leave $\mathcal{S}$ in this case, and we have no possibility of extinction.

In conclusion, we have shown that the points satisfying (10) of the system (9) can be globally asymptotically stabilized by giving the set of impulses as specified by (14) in the case of $\det A > 0$, and as specified by (19) in the case of $detA = 0$.

4 Conclusion

In this paper, we have established, respectively, necessary and sufficient conditions for a point p to be impulsively stabilizable, i.e. Theorem 2.1 and Theorem 2.3. These results are applied to a three-species population growth model and an impulsive control program is obtained to stabilize the point p. A constructive approach for actually determining an appropriate feedback control law may be generated from the notion of impulsively invariant sets described in Section 2.2 as follows. A Lyapunov function, v, is constructed so that it satisfies the conditions of Theorem 2.3. Then for any positive ϵ, we can choose a constant c such that the level set $v = c$ is contained within an ϵ-neighbourhood of p_z. The set $\{z\colon \|z\| \le \epsilon,\ v(z) < c\}$ defines Ω_z, the non-impulsive portion of our invariant cylinder. We then choose an α such that $\Omega \subset B_\epsilon(p)$, where $\Omega = B^y_\alpha(p_y) \oplus \Omega_z$. Since Ω is an invariant cylinder, for all points $q \in \partial\Omega$ from which the trajectory induced by f leaves $\overline{\Omega}$ there is at least one impulse $\Delta y(q)$ such that $q + \Delta y(q)$ is either in the interior of Ω or at a point on $\partial\Omega$ from which the vector field f will keep the trajectory within $\overline{\Omega}$ for some positive time interval. For each such q we choose one of these $\Delta y(q)$ and define our feedback control law as $q \mapsto q + \Delta y(q)$. This law is implemented by observing the trajectory of an initial point within Ω and firing the appropriate impulse $\Delta y(q)$ each time the trajectory reaches a point q. In this manner the system is kept within a ϵ-neighbourhood of the desired point p. In the population growth model, a control design procedure for asymptotic impulsive stabilization is derived. It is shown that by impulsively regulating one species, the population of all three species can be maintained at a positive level, which otherwise would drop to a level of extinction for one of the species. This control program enables an resort owner to lower his cost of maintaining the game fish in his lake by stocking the lake with the feeder fish. Consequently, the resort owner's profit will be maximized. The results developed in this paper may be applied to other real world problems.

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Stability of Dynamic Systems on the Time Scales

S. Sivasundaram

Department of Computing and Mathematics, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114, USA

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Abstract: The paper dwells on the problems of stability of dynamical systems on a time scale. The paper is divided into the following sections: local existence and uniqueness, dynamic inequalities, existence of extremal solutions, comparison results, linear variation of parameters, nonlinear variation of parameters, global existence and stability, comparison theorems, stability criteria, etc.

Keywords: *Dynamical systems on a time scale; stability.*

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1 Introduction

In both natural and engineering systems the lowest level is usually characterized by continuous variable dynamics and the highest by a logical decision making mechanism. The interaction of these different levels, with their different types of information, leads to a hybrid system. Many complicated control systems today (e.g. those for flight control, manufacturing systems, and transportation) have vast amount of computer code at their highest level. More pervasively, programmable logic controllers are widely used industrial process control. Virtually all control systems today issue continuous variable controls and perform logical checks that determine the mode, and hence the control algorithms the continuous variable system is operating under at any given moment.

Hybrid control systems are control systems that involve both continuous systems that involve both continuous and discrete dynamics and continuous and discrete controls. The continuous dynamics of such a system is usually modeled by a controlled vector field or difference equation. Its hybrid nature is expressed by a dependence on some discrete phenomena, corresponding to discrete states, dynamics and controls. The prototypical hybrid systems are digital controllers, computers, and subsystems modeled as finite automata coupled with controllers and plants modeled by partial or ordinary differential equations or difference equations. Thus such systems arise whenever one mixes logical decision making with continuous control laws. More specifically, real world examples of hybrid systems include systems with relays, switches, and hysteresis; disk drivers, transmissions, step motors; constrained robots; automated transportation systems; and modern flexible manufacturing and flight control systems.

In control theory, there has certainly been a lot of related work in the past, including variable structure systems, jump linear systems, systems with impulse effect, impulse control, and piecewise deterministic processes.

The mathematical modeling of several important dynamic processes has been via difference equations or differential equations. Difference equations also appear in the study of discretization methods for differential equations. In recent years, however, the investigation of the theory of difference equations (discrete time dynamic systems) has assumed a greater importance as a well desired discipline. In spite of this tendency of independence, there is a striking similarity or even duality between the theories of continuous and discrete dynamic systems. Many results in the theory of difference equations have been obtained as more or less natural discrete analogs of corresponding results of differential equations. Nevertheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation exhibits the chaotic behavior which can only happen for higher order differential equations. Moreover, additional assumptions are often required in the discrete case in order to overcome the topological deficiency of lacking connectedness. From a modeling point of view, it is perhaps more realistic to model a phenomenon by a dynamic system which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals called time scale. In this survey paper we discuss the stability of dynamics systems on time scale.

2 Preliminaries

In this paper we use the calculus obtained in [1, 2] for unifying discrete and continuous dynamic systems.

Let $\mathbb{T}$ be a time scale (closed nonempty subset of R) with $t_0 \ge 0$ as a minimal element and no maximal element.

The points $\{t\}$ of $\mathbb{T}$ are classified as:

right-dense (rd), if $\sigma(t) = t$;

left-dense (ld), if $\rho(t) = t$;

right-scattered (rs), if $\sigma(t) > t$;

left-scattered (ls), if $\rho(t) < t$, where $\sigma(t)$, $\rho(t)$ are jump operators defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}.$$

Set $\mu^*(t) = \sigma(t) - t$ (*called graininess*), so that

$$\mathbb{T} \equiv R \Rightarrow \mu^*(t) = 0, \qquad \mathbb{T} \equiv Z \Rightarrow \mu^*(t) = 1.$$

Definition 2.1 The mapping $u\colon \mathbb{T} \to R$ is *rd-continuous* if it is continuous at each right-dense point and $\lim\limits_{s\to t^-} f(s) = f(t^-)$ exist at each left-dense.

Definition 2.2 A mapping $u\colon \mathbb{T} \to R$ is said to be *differentiable at* $t \in \mathbb{T}$, if there exists an $\alpha \in R$ such that for any $\epsilon > 0$ there exists a neighborhood U of t satisfying

$$|u(\sigma(t)) - u(s) - \alpha(\sigma(t) - s)| \le \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

Derivative of u is denoted by $u^{\triangle}(t)$.

Note: $\mathbb{T} = R \Rightarrow u^{\triangle} = \alpha = \frac{du(t)}{dt}$,

$\mathbb{T} = Z \Rightarrow u^{\triangle} = \alpha = u(t+1) - u(t)$.

If u is differentiable at t, then it is continuous at t.

If u is continuous at t and t is right-scattered, then u is differentiable and

$$u^{\triangle}(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.3 For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+u^{\triangle}(t)$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu^*(t,s)} < D^+u^{\triangle}(t) + \epsilon \quad \text{for} \quad s \in N_\epsilon, \quad s > t,$$

where $\mu(t,s) = \sigma(t) - s$.

In case t is rs and u is continuous at t, we have, as in the case of the derivative,

$$D^+u^{\triangle}(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.4 Let h be a mapping from $\mathbb{T}$ to R. The mapping $g\colon \mathbb{T} \to R$ is called the *antiderivative* of h on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\triangle}(t) = h(t)$ for $t \in \mathbb{T}$.

The following known properties of the antiderivative are useful.

(a) If $h\colon \mathbb{T} \to R$ is rd-continuous, then h has the antiderivative $g\colon t \to \int_s^t h(s)\,ds$, $s, t \in \mathbb{T}$.

(b) If the sequence $\{h_n\}_{n\in N}$ of rd-continuous functions $\mathbb{T} \to R$ converge uniformly on $[r,s]$ to rd-continuous function h then

$$\left(\int_r^s h_n(t)\,dt\right)_{n\in N} \to \int_r^s h(t)\,dt, \quad \text{in} \quad R.$$

A basic tool employed in the proofs is the following induction principle, well suited for time scales.

Suppose that for any $t \in \mathbb{T}$, there is a statement $A(t)$ such that the following conditions are verified:

(I) $A(t_0)$ is true;

(II) If t right-scattered and $A(t)$ is true, then $A(\sigma(t))$ is also true;

(III) For each right-dense t there exists a neighborhood U such that whenever $A(t)$ is true, $A(s)$ is also true for all $s \in U$, $s \ge t$;

(IV) For left-dense t, $A(s)$ is true for all $s \in [t_0, t)$ implies $A(t)$ is true.

Then the statement $A(t)$ is true for all $t \in \mathbb{T}$.

3 Local Existence and Uniqueness

In this section, we shall consider the initial value problem for dynamic systems on time scales and prove local existence and uniqueness results corresponding to Peano's and Perron's theorems. Let $\mathbb{T}^k$ represent the set of all nondegenerate points of the time scale $\mathbb{T}$.

Consider the initial value problem (IVP)

$$x^{\Delta} = f(t,x), \quad t \in \mathbb{T}^k, \quad x(t_0) = x_0, \tag{3.1}$$

where $f\colon \mathbb{T}^k \times R^n \to R^n$ and f is rd-continuous on $\mathbb{T}^k \times R^n$.

A map $x\colon \mathbb{T}^k \to R^n$ is a solution of IVP (3.1) if $x(t)$ is an antiderivative of $f(t,x(t))$ on $\mathbb{T}^k$ and satisfies $x(t_0) = x_0$.

Theorem 3.1 *Let $f \in C_{rd}[R_0, R^n]$, where $R_0 = [t_0, t_0 + a] \times B$, $[t_0, t_0 + a]$ is understood as $[t_0, t_0 + a] \cap \mathbb{T}^k$ and $B = \{x \in R^n \colon |x - x_0| \le b\}$. Then the IVP (3.1) has at least one solution $x(t)$ on $[t_0, t_0 + \alpha]$, where $\alpha = \min\left(a, \frac{b}{M}\right)$, M being the bound of $f(t,x)$ on R_0.*

Proof See [3] and cf. [1, 2].

Next we shall consider Perron type uniqueness result.

Theorem 3.2 *Assume that*

(i) *$g \in C_{rd}\big[[t_0, t_0 + a] \times [0, 2b], R_+\big]$ and for every t_1, $t_0 \le t_1 \le t_0 + a$, $u(t) \equiv 0$ is the only solution of*

$$u^{\Delta} = g(t,u), \quad u(t_1) = 0, \quad on \quad [t_1, t_0 + a];$$

(ii) *$f \in C_{rd}[R_0, R^n]$ and for each $t \in [t_0, t_0 + a]$, there exists a compact neighborhood U_t such that f^t in $U_t \times B$ satisfies*

$$|f(t,x) - f(t,y)| \le g(t, |x - y|), \quad (t,x),\ (t,y) \in U_t \times B.$$

Then the IVP (3.1) has a unique solution $x(t)$ on $[t_0, t_0 + a]$.

Proof See [3].

4 Dynamic Inequalities

In this section, we shall prove basic results on dynamic inequalities, that are needed to prove existence of extremal solutions. We shall first prove a result relative to a system of strict dynamic inequalities and then consider a similar result for nonstrict inequalities which is needed for later discussion. All inequalities between vectors are to be understood componentwise hereafter.

We need the following definition before we proceed further

Definition 4.1 A *function* $f \in C[R^n, R^n]$ is said to be *quasimonotone nondecreasing* if $x \le y$ and $x_i = y_i$ for some $1 \le i \le n$ implies $f_i(x) \le f_i(y)$.

Theorem 4.1 *Let $\mathbb{T}$ be the time scale with $t_0 \ge 0$ minimal element and no maximal element, $v, w\colon \mathbb{T} \to R^n$ be the rd-continuous mappings that are differentiable for each $t \in \mathbb{T}$ and satisfy*

$$v^\Delta(t) \le f(t, v(t)), \quad w^\Delta(t) \ge f(t, w(t)), \quad t \in \mathbb{T}, \tag{4.1}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $f(t,x)$ is quasimonotone nondecreasing in x, and, for $1 \le i \le n$, $f_i(t,x)\mu^*(t) + x_i$ is nondecreasing in x_i for each $t \in \mathbb{T}$.
Then $v(t_0) < w(t_0)$ implies $v(t) < w(t)$, for $t \in \mathbb{T}$.

Proof See [4].

The next result deals with nonstrict dynamic inequalities

Theorem 4.2 *Let $\mathbb{T}$ be the time scale as before, $v, w\colon \mathbb{T} \to R^n$ be the rd-continuous mappings that are differentiable for each $t \in \mathbb{T}$ and satisfy*

$$v^\Delta(t) \le f(t, v(t)), \quad w^\Delta(t) \ge f(t, w(t)), \quad t \in \mathbb{T}, \tag{4.2}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $f(t,x)$ is quasimonotone nondecreasing in x and for each i, $1 \le i \le n$, $f_i(t,x)\mu^*(t) + x_i$ is nondecreasing in x_i for $t \in \mathbb{T}$. Then $v(t_0) \le w(t_0)$ implies $v(t) \le w(t)$, $t \in \mathbb{T}$, provided f satisfies

$$f_i(t,x) - f_i(t,y) \le L \sum_{i=1}^{n} (x_i - y_i), \quad x \ge y. \tag{4.3}$$

Proof See [3].

5 Existence of Extremal Solutions

Using the result on strict dynamic inequalities proved in Section 4, we shall discuss, in this section, the existence of extremal solutions for dynamic systems. For that consider the IVP

$$u^\Delta = g(t,u), \quad u(t_0) = u_0 > 0, \tag{5.1}$$

where $g \in C_{rd}[R_0, R^n]$, $R_0 = \{([t_0, t_0 + a] \cap \mathbb{T}) \times B\}$,

$$B = \{u \in R^n : |u - u_0| \le b\}, \quad \text{and} \quad |g(t,u)| \le M \quad \text{on} \quad R_0.$$

Theorem 5.1 *Assume that*

(i) *$g(t,u)$ is quasimonotone nondecreasing in u,*
(ii) *for each i, $1 \le i \le n$, $g_i(t,u)\mu^*(t) + u_i$ is nondecreasing in u_i for each $t \in \mathbb{T}$.*

Then there exist minimal and maximal solutions of (5.1) on $I \equiv [t_0, t_0 + \eta] \cap \mathbb{T}$, where $\eta = \min\left(a, \frac{b}{2M+b}\right)$.

Proof See [4].

6 Comparison Results

Having the existence results and the theory of dynamic inequalities at our disposal, it is now easy to prove comparison results.

Theorem 6.1 *Let the assumptions of Theorem 5.1 hold and let* $m\colon I \equiv [t_0, t_0 + a) \cap \mathbb{T} \to R^n$ *be a mapping that is differentiable for each* $t \in I$ *and that satisfies*

$$m^{\Delta}(t) \leq g(t, m(t)), \qquad t \in I.$$

Then $m(t_0) \leq u_0$ *implies that* $m(t) \leq r(t)$, $t \in I$, *where* $r(t)$ *is the maximal solution of (5.1) existing on* I.

Proof See [4].

Definition 6.1 Let $x \in C^1_{rd}[\mathbb{T}, R^n]$. Given an $\epsilon > 0$, if there exists a neighbourhood N_ϵ of $t \in \mathbb{T}$ satisfying

$$\frac{1}{\mu(t,s)}\left[|x(\sigma(t))| - |x(\sigma(t)) - \mu(t,s)x^{\triangle}(t)\|\right] < \left[x, x^{\triangle}\right]_+ + \epsilon$$

for each $s \in N_\epsilon$ and $s > t$, where $\mu(t,s) = \sigma(t) - s$, then we say that $\left[x, x^{\triangle}\right]_+$ is the *generalised derivative* of $x(t)$. In case, $t \in \mathbb{T}$ is rs, then we have

$$\left[x, x^{\triangle}\right]_+ = \frac{1}{\mu^*(t)}\left[|x(\sigma(t))| - |x(t)|\right],$$

where $\mu^*(t) = \mu(t,t)$.

We can now prove the following comparison result.

Theorem 6.2 *Suppose that*

$$\left[x, x^{\triangle}\right]_+ \leq g(t, |x|) \quad on \quad \mathbb{T} \times R,$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R]$ *and* $g(t,u)\mu^*(t) + u$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$, *where* $x\colon \mathbb{T} \to R^n$ *is any rd-continuously differentiable function such that* $|x_0| \leq u_0$. *Then* $|x(t)| \leq r(t)$, $t \in \mathbb{T}$, *where* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of (5.1) existing on* $\mathbb{T}$.

Proof See [5, p.86] and [3].

7 Linear Variation of Parameters

Let $(\mathbb{T}, \mu, X)$ a dynamical triple, and $B(X)$ be a Banach algebra with unity of the continuous endomorphims on a Banach space X.

A *mapping* $A\colon \mathbb{T}^k \to B(X)$ is called *regressive*, if for each $t \in \mathbb{T}^k$ the mapping $A(t)\mu^*(t) + id\colon X \to X$ is invertible. This is the case e.g if $|A(t)\mu^*(t)| < 1$ for all $t \in \mathbb{T}$. Obviously in case $\mathbb{T} = R$ any A is regressive (since $\mu^* = 0$) and in case $\mathbb{T} = Z$, A is regressive if $|A(t)| < 1$ (since $\mu^* \equiv 1$).

Suppose $A\colon \mathbb{T}^k \to B(X)$ is rd-continuous and regressive and $F\colon \mathbb{T}^k \times X \to X$ is rd-continuous, then a mapping $x\colon \mathbb{T}^k \to X$ is called a *solution of the dynamic equation*

$$x^{\triangle} = A(t)x + F(t,x) \tag{7.1}$$

if $x^{\triangle}(t) = A(t)x(t) + F(t, x(t))$ for all $t \in \mathbb{T}^k$.

If a solution $x(\cdot)$ of (7.1) in addition satisfies the condition $x(\tau) = \eta$ for a pair $(\tau,\eta) \in \mathbb{T}^k \times X$, it is called a *solution of the initial value problem* (IVP)

$$x^{\triangle} = A(t)x + F(t,x), \qquad x(\tau) = \eta. \tag{7.2}$$

Consider the IVP, in the Banach algebra $B(X)$,

$$x^{\triangle} = A(t)x, \qquad x(\tau) = I, \tag{7.3}$$

where I is the unity of $B(X)$. By Theorem 3.2 , it admits exactly one solution $\Phi_A(\tau) := x(\cdot;\tau,I)$. We call it *principal solution.* The corresponding transition function is defined to be $\Phi_A(t,\tau) := \Phi_A(\tau)(t)$. In the particular case, when $A\colon \mathbb{T}^k \to B(X)$ is constant, we call the transition function, exponential function $(e_L(t,\tau))$. If $X = R$ and $C\colon \mathbb{T}^k \to R_+$, then $C(t)\mu^*(t) + 1 > 0$ satisfies the regressive property we can set $e_C(t,\tau) := \Phi_A(t,\tau)$.

Theorem 7.1 *We consider the IVP (7.2) with rd-continuous and regressive right-hand side. Then the solution of (7.2) is given by*

$$x(t) = \Phi_A(t,\tau)\eta + \int_\tau^t \Phi_A(t,\sigma(s))F(s,x(s)) \triangle s.$$

Proof See [1, 2].

8 Nonlinear Variation of Parameters

Theorem 8.1 *Let $\mathbb{T} = [\tau, s]$ be some compact measure chain. Assume that $f \in C_{rd}[\mathbb{T}^k \times R^n, R^n]$, and possesses rd-continuous partial derivatives f_x on $\mathbb{T}^k \times R^n$. Let L be a nonnegative constant with $L\mu(s,\tau) < 1$ and $|f_x(t,x)| \le L$ on $\mathbb{T}^k \times R^n$. Let the solution $x_0(t) = x(t,\tau,\eta)$ of*

$$x^{\triangle} = f(t,x), \quad x(\tau) = \eta, \quad \textit{exists for} \quad t \ge \tau. \tag{8.1}$$

Then

(i) $\Phi(t,\tau,\eta) = x_\eta(t,\tau,\eta)$ *exists and is the solution of*

$$y^{\triangle} = H(t,\tau,\eta)y, \tag{8.2}$$

where $H(t,\tau,\eta) = \lim\limits_{h\to 0} \int_0^1 f_x(t, px(t,\tau,\eta) - (1-p)x(t,\tau,\eta+h)) \triangle p$ *such that* $\Phi(\tau,\tau,\eta)$ *is the unit matrix;*

(ii) $\Psi(t,\tau,\eta) = x_\tau^{\triangle}(t,\tau,\eta)$ *exists, is the solution of*

$$z^{\triangle} = H(t,\sigma(\tau),\tau,\eta)z \tag{8.3}$$

such that $\Psi(\sigma(\tau),\tau,\eta) = -f(\tau,\eta)$, *where*

$$H(t,\sigma(\tau),\tau,\eta) = \int_0^1 f_x(t, px(t,\sigma(\tau),\eta) - (1-p)x(t,\tau,\eta)) \triangle p;$$

(iii) *the function* $\Phi(t,\tau,\eta)$, $\Psi(t,\tau,\eta)$ *satisfy the relation*

$$\Psi(t,\tau,\eta) = -\Phi(t,\sigma(\tau),\eta)f(\tau,\eta) + \int_{\sigma(\tau)}^{t} \Phi(t,\sigma(s),\eta)\big[H(s,\sigma(\tau),\tau,\eta) - H(s,\sigma(\tau),\tau,\eta)\big]\Psi(s,\tau,\eta) \triangle s.$$

Proof See [3].

Theorem 8.2 *Let* $\mathbb{T} = [\tau, s]$ *be some compact measure chain. Assume that* $f,\ F \in C_{rd}[\mathbb{T}^k \times R^n, R^n]$, *and* f_x *exists and be rd-continuous on* $\mathbb{T}^k \times R^n$.

Let L *be a nonnegative constant with* $L\mu(s,\tau) < 1$ *and* $|f_x(t,x)| \le L$ *on* $\mathbb{T}^k \times R^n$.

If $x(t,\tau,\eta)$ *is the solution* $x^{\triangle} = f(t,x)$, $x(\tau) = \eta$, *exists for* $t \ge \tau$, *any solution* $y(t,\tau,\eta)$ *of* $y^{\triangle} = f(t,y) + F(t,y)$, *with* $y(\tau) = \eta$, *satisfies the integral equation*

$$y(t,\tau,\eta) = x(t,\tau,\eta) + \int_{\tau}^{t} \Phi(t,\sigma(s),y(s))\,F(s,y(s)) \triangle s + \int_{\tau}^{t}\int_{\sigma(s)}^{t} \Phi(t,\sigma(p),y(s))\big[H(p,\sigma(s),s,y(s)) - H(p,\sigma(s),y(s))\big]\,\Psi(p,s,y(s)) \triangle p \triangle s.$$

Proof See [3].

Remark 8.1 It is easy to see from the definitions of $H(p,\sigma(s),s,y(s))$ and $H(p,\sigma(s), y(s))$ that they are identical if the measure chain is R, and consequently, the foregoing variation of parameter formula reduces to the usual Alekseev's formula (see [6]).

9 Global Existence and Stability

As an application of comparison Theorem 6.2, we shall prove, in this section, a global existence result and a simple stability result.

Theorem 9.1 *Assume that*

(i) $f \in C_{rd}[\mathbb{T}\times R^n, R^n]$, $g \in C_{rd}[\mathbb{T}\times R_+, R_+]$, $g(t,u)$ *is non-decreasing in* u *for each* $t \in \mathbb{T}$, *where* $\mathbb{T}$ *is the time scale with* $t_0 \ge 0$ *as the minimal element and has no maximal element and*

$$|f(t,x)| \le g(t,|x|) \quad \text{for} \quad (t,x) \in \mathbb{T}\times R^n;$$

(ii) *the maximal solution* $r(t)$ *of the scalar IVP*

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{9.1}$$

exists on $\mathbb{T}$.

Then the largest interval of existence of any solution $x(t)$ of

$$x^{\Delta} = f(t,x), \qquad x(t_0) = x_0, \tag{9.2}$$

with $|x_0| \le u_0$ *is* $\mathbb{T}$.

Proof See [3].

To prove a simple stability result, we need the following definition of stability.

Definition 9.1 The trivial *solution* of (9.2) is said to be

(i) *stable* if given an $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta > 0$ such that $|x_0| \le \delta$ implies $|x(t)| \le \epsilon$, $t \ge t_0$;
(ii) *asymptotically stable* if it is stable and $\lim\limits_{t\to\infty} |x(t)| = 0$.

We are now in a position to prove a typical result on stability in terms of comparison principle (cf. [7, p.13]).

Theorem 9.2 *Assume that*

(i) $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $f(t,0) \equiv 0$, $g(t,0) \equiv 0$ *and for* $(t,x) \in \mathbb{T} \times R^n$,

$$[x, f(t,x)]_+ \equiv \lim_{h\to 0^+} \frac{1}{h}\left[|x + hf(t,x)| - |x|\right] \le g(t,|x|); \tag{9.3}$$

(ii) $g(t,u)\mu^*(t)$ *is nondecreasing in* u *for each* $t \in \mathbb{T}$.

Then the stability properties of the trivial solution of the IVP

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{9.4}$$

imply the corresponding stability properties of the trivial solution of

$$x^{\Delta} = f(t,x), \quad x(t_0) = x_0, \quad t_0 \ge 0, \quad t \in \mathbb{T}. \tag{9.5}$$

Proof Let the trivial solution of (9.4) be stable. Then, given $\epsilon > 0$ and $t_0 \ge 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$u \le u_0 < \delta \quad \text{implies} \quad u(t) < \epsilon, \quad t \in \mathbb{T}. \tag{9.6}$$

It is easy to claim that with these ϵ and δ, the trivial solution of (9.5) is also stable. If this were false, there would exist a solution $x(t)$ of (9.5) with $|x_0| < \delta$ and a $t_1 \in \mathbb{T}$, $t_1 > t_0$ such that $\epsilon \le |x(t_1)|$ and $|x(t)| \le \epsilon$, $t \in [t_0,t_1)$.

For $t \in [t_0,t_1]$, using condition (9.3), we get

$$m^{\Delta}(t) \le g(t,m(t)), \quad t \in [t_0,t_1], \quad m(t_0) \le u_0,$$

where $m(t) = |x(t)|$. Consequently, Theorem 6.2 yields

$$|x(t)| \le r(t), \quad t \in [t_0,t_1].$$

At $t = t_1$, we arrive at the contradiction

$$\epsilon \le |x(t_1)| \le r(t_1) < \epsilon,$$

proving the claim. One can prove similarly other concepts of stability, and we omit the details.

10 Comparison Theorems

Consider the dynamic system

$$x^{\triangle} = f(t,x), \quad x(t_0) = x_0, \tag{10.1}$$

where $f \in C_{rd}[\mathbb{T} \times R^n, R^n]$, and $x^{\triangle}$ denotes the derivative of x with respect to $t \in \mathbb{T}$. We shall assume, for convenience, that the solutions $x(t) = x(t, t_0, x_0)$ of (10.1) exist and are unique for $t \ge t_0$.

Definition 10.1 Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$. Then we define the generalized derivative of $V(t,x)$ relative to (10.1) as follows: given $\epsilon > 0$, there exists a neighbourhood $N(\epsilon)$ of $t \in \mathbb{T}$ such that

$$\frac{1}{\mu(t,s)}\left[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s) f(t, x(t))\right] < D^+V^{\triangle}(t, x(t)) + \epsilon$$

for each $s \in N(\epsilon)$ and $s > t$, where $\mu(t,s) = \sigma(t) - s$ and $x(t)$ is any solution of (10.1).

In case, $t \in \mathbb{T}$ is right scattered and $V(t, x(t))$ is continuous at t, we have

$$D^+V^{\triangle}(t, x(t)) = \frac{1}{\mu^*(t)}\left[V(\sigma(t), x(\sigma(t))) - V(t, x(t))\right],$$

where $\mu^*(t) = \mu(t,t)$.

We are now in a position to prove the following comparison theorem in terms of Lyapunov function $V(t,x)$.

Theorem 10.1 *Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t,x)$ be locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, and let*

$$D^+V^{\triangle}(t, x(t)) \le g(t, V(t,x)),$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R_+]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$ and $r(t) = r(t, t_0, u_0)$ is the maximal solution of $u^{\triangle} = g(t,u)$, $u(t_0) = u_0 \ge 0$, existing on $\mathbb{T}$. Then, $V(t_0, x_0) \le u_0$ implies that $V(t, x(t)) \le r(t, t_0, u_0)$, $t \in \mathbb{T}$, $t \ge t_0$.

Proof See [3] and cf. [8].

Remark 10.1 If the inequalities between vectors is understood as componentwise, then Theorem 10.1 is valid for $V \in C_{rd}[\mathbb{T} \times R^n, R^N_+]$, $g \in C_{rd}[\mathbb{T} \times R^N_+, R^N_+]$, provided $g(t,u)$ is quasimonotone nondecreasing in u and $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$. The proof requires slight modification since we now need to use comparison result for differential systems relative to Vector Lyapunov functions. We therefore omit proving such a result.

We shall next discuss another comparison result which connects the solutions to dynamic systems which can be employed in perturbation theory.

Consider another dynamic system

$$x^{\Delta} = F(t,x), \qquad x(t_0) = x_0, \tag{10.2}$$

where $F \in C_{rd}[\mathbb{T} \times R^n, R^n]$. Relative to the system (10.1), let us assume that the following assumption (H) holds:

(H) the solutions $y(t,t_0,x_0)$ of (10.2) exist for all $t \geq t_0$, unique and rd-continuous with respect to the initial data and $|y(t,t_0,x_0)|$ is locally Lipschitzian in x_0.

For any $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$ and any fixed $t \in \mathbb{T}$, we define $D^+V^{\Delta}(s, y(t,s,x(s)))$ as follows: given $\epsilon > 0$, there exists a neighbourhood $N(\epsilon)$ of $s \in \mathbb{T}$, $t_0 \leq s \leq t$ such that

$$\frac{1}{\mu(s,r)}\big[V(\sigma(s), y(t,\sigma(s),x(\sigma(s)))) - V(s, y(t,s,x(\sigma(s))) - \mu(s,r)F(s,x(s))\big]$$
$$< D^+V(s,y(t,s,x(s))) + \epsilon$$

for each $r \in N(\epsilon)$ and $r > s$. As before, if $s \in \mathbb{T}$ is right-scattered and $V(s,y(t,s,x(s)))$ is continuous at s, then

$$D^+V(s,y(t,s,x(s))) = \frac{1}{\mu^*(s)}\big[V(\sigma(s), y(t,\sigma(s),x(\sigma(s))) - V(s, y(t,s,x(\sigma(s))\big],$$

with $\mu^*(s) = \mu(s,s)$.

We then have the following general comparison result which includes Theorem 10.1 as a special case.

Theorem 10.2 *Assume that the assumption (H) holds. Suppose that*

(i) $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, *$V(s,x)$ is locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd and for $t_0 < s \leq t$, $x \in R^n$*

$$D^+V^{\Delta}(s, y(t,s,x)) \leq g(s, V(s, y(t,s,x)));$$

(ii) $g \in C_{rd}[\mathbb{T}^k \times R_+, R]$, *$g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and the maximal solution $r(t,t_0,u_0)$ of*

$$u^{\Delta} = g(t,u), \quad u(t_0) = u_0 \geq 0, \quad \textit{exists for} \quad t \in \mathbb{T}.$$

Then, if $x(t) = x(t,t_0,x_0)$ is any solution of (10.1) we have

$$V(t, x(t,t_0,x_0)) \leq r(t,t_0,u_0), \qquad t \in \mathbb{T}, \tag{10.3}$$

provided $V(t_0, y(t,t_0,x_0)) \leq u_0$.

Proof See [3] and [8].

11 Stability Criteria

In this section, we shall consider some simple stability results. We list a few definitions concerning the stability of the trivial solution of (10.1) which we assume to exit (for details see [5]).

Definition 11.1 The *trivial solution* $x = 0$ of (10.1) is said to be

($S1$) *equi-stable*, if for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is rd-continuous in t_0 for each ϵ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| < \epsilon$ for $t \geq t_0$;

($S2$) *uniformly stable*, if the δ in ($S1$) is independent of t_0;

($S3$) *quasi-equi asymptotically stable*, if for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exist positive $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that $|x_0| < \delta_0$ implies $|x(t, t_0, x_0)| < \epsilon$ for $t \geq t_0 + T$;

($S4$) *quasi-uniformly asymptotically stable*, if δ_0 and T in ($S3$) are independent of t_0;

($S5$) *equi-asymptotically stable*, if ($S1$) and ($S3$) hold simultaneously;

($S6$) *uniformly asymptotically stable*, if ($S2$) and ($S4$) hold simultaneously.

Corresponding to the definitions ($S1$) to ($S6$), we can define the stability notions of the trivial solution $u = 0$ of (10.1) below. For example, the trivial solution $u = 0$ of (10.1) is equi-stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a function $\delta_0 = \delta_0(t_0, \epsilon)$ that is rd-continuous in t_0 for each ϵ, such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon$, $t \geq t_0$.

We are now in a position to prove a general result which provides sufficient conditions for stability criteria.

Theorem 11.1 *Assume that*

(i) $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, *$V(t,x)$ is locally Lipschitzian in x for each right dense $t \in \mathbb{T}$;*

(ii) *$b(|x|) \leq V(t,x) \leq a(|x|)$ for $(t,x) \in \mathbb{T} \times R^n$, where a, $b \in \mathcal{K} = \big[\sigma \in C[R_+, R_+]$: $\sigma(0) = 0$ and $\sigma(u)$ is increasing in $u\big]$;*

(iii) *$f(t,0) = 0$, $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and*

$$D^+V^\Delta(t,x) \leq g(t, V(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n.$$

Then the stability properties of the trivial solution of

$$u^\Delta = g(t,u), \qquad u(t_0) = u_0 \geq 0 \tag{11.1}$$

imply the corresponding stability properties of the trivial solution of (10.1).

Proof Let $\epsilon > 0$ and $t_0 \in \mathbb{T}$ be given. Suppose that the trivial solution of (11.1) is equi-stable. Then given $b(\epsilon) > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$u_0 < \delta_1 \;\; \Rightarrow \;\; u(t) < b(\epsilon), \quad t \in \mathbb{T} \tag{11.2}$$

where $u(t) = u(t, t_0, u_0)$ is any solution of (11.1). Choose $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(\delta) < \delta_1. \tag{11.3}$$

We claim that if $|x_0| < \delta$, then $|x(t)| < \epsilon$, $t \in \mathbb{T}$, where $x(t) = x(t, t_0, x_0)$ is any solution of (11.1). If this is not true, there would exist a $t_1 \in \mathbb{T}$, $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (11.1) satisfying

$$|x(t)| < \epsilon, \quad t_0 \leq t < t_1 \quad \text{and} \quad |x(t_1)| \geq \epsilon. \tag{11.4}$$

Setting $m(t) = V(t, x(t))$ for $t_0 \le t \le t_1$ and using condition (iii), we get by Theorem 11.1, the estimate

$$V(t, x(t)) \le r(t, t_0, u_0), \quad t_0 \le t \le t_1, \tag{11.5}$$

where $r(t, t_0, u_0)$ is the maximal solution of (11.1) with $V(t_0, x_0) \le u_0$. Now the relations (11.2), (11.4), (11.5) and the assumption (ii) yield

$$b(\epsilon) \le b(|x(t_1)|) \le V(t, x(t_1)) \le r(t_1, t_0, u_0) < b(\epsilon),$$

since $u_0 = V(t_0, x_0) \le a(|x_0|) < a(\delta) < \delta_1$ by (11.3). This contradiction proves the claim.

Other stability properties may be proved in a similar manner and hence the proof is complete.

We can obtain from Theorem 11.1, a result analogous to Lyapunov's first theorem for continuous and discrete cases immediately. This we state as a corollary.

Corollary 11.1 *The function $g(t, u) \equiv 0$ is admissible in Theorem 11.1 to yield uniform stability of the trivial solution of (11.1).*

A result analogous to Lyapunov's second theorem cannot be obtained directly from Theorem 11.1, since when we choose $g(t, u) = -c(u)$, $c \in \mathcal{K}$, $g(t, u)\mu^*(t) + u$ does not satisfy the monotone condition. However, a minor change in the argument proves such a result.

Corollary 11.2 *The choice of the function $g(t, u) = -c(u)$, $c \in \mathcal{K}$, in Theorem 11.1 implies uniform asymptotic stability of the trivial solution of (11.1).*

Proof See [3].

Usually, Lyapunov's second theorem has the assumption $D^+V^\Delta(t, x(t)) \le -c_0(|x|)$, $c_0 \in \mathcal{K}$. Since $V(t, x)$ has the upper estimate in (ii), which means $V(t, x)$ is decrescent, it is easy to show that $D^+V^\Delta(t, x(t)) \le -c(|x|)$, where $c(u) = c_0^{-1}(a(u))$. Of course, one can prove directly with the assumption $D^+V^\Delta(t, x(t)) \le -c_0(|x|)$ following the proof of Corollary 11.2.

When $V(t, x)$ is not assumed to be decrescent, that $V(t, 0) \equiv 0$, we have Marachkov's result for differential equations. We can extend such a result in the present set up.

Theorem 11.2 *Assume that $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t, x)$ is locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, $b(|x|) \le V(t, x)$ and $V(t, 0) \equiv 0$. Suppose further that $D^+V^\Delta(t, x(t)) \le -c(|x|)$, and $|f(t, x)| \le M$ for $(t, x) \in \mathbb{T} \times R^n$. Then the trivial solution is equi-asymptotically stable.*

Proof Let $\epsilon > 0$, and $t_0 \in \mathbb{T}$ be given. Since $V(t, x)$ is rd continuous and $V(t, 0) \equiv 0$, it is possible to find a $\delta = \delta(t_0, \epsilon) > 0$ satisfying $V(t_0, x_0) < b(\epsilon)$ iff $|x_0| < \delta$.

We claim that $|x_0| < \delta$ implies $|x(t)| < \epsilon$, $t \ge t_0$, the proof of which follows from Theorem 10.1 . Now set $\epsilon = \rho$, for some $\rho > 0$ and designate by $\delta_0 = \delta(t_0, \rho)$ so that we have $|x_0| < \delta_0$ implies $|x(t)| < \rho$, $t \ge t_0$. To prove the theorem, we need to show that $\liminf_{t\to\infty} |x(t)| \ne 0$, then there exists a $T > 0$ such that for a given $\eta > 0$, we have $|x(t)| > \eta$, $t \ge T$. As a result, we get arguing as in Corollary 11.2, the estimate

$V(t,x(t)) \le V(t_0,x_0) - \int_T^t c(|x(s)|)\Delta s, t \ge T$, which yields a contradiction, for large t, $0 \le V(t_0,x_0) - c(\eta)(t-T)$.

Hence $\liminf_{t\to\infty} |x(t)| = 0$. Suppose then $\limsup_{t\to\infty} |x(t)| \ne 0$. Then, given an $\epsilon > 0$, there exist a divergent sequence $\{t_k\}$ such that $|x(t_k)| > \epsilon$. Each $t_k \in \mathbb{T}$ may belong to one of the following cases:

(i) t_k is rs and ls;
(ii) t_k is rs and ld;
(iii) t_k is rd and ls;
(iv) t_k is rd and ld;

without loss of generality, we can assume that there is a divergent subsequence $\{t_i\}$ of $\{t_k\}$ such that all t_i belong to one of the above four cases. In case (i), we have

$$V(\sigma(t_i), x(\sigma(t_i))) \le V(t_i, x(t_i)) - \mu^*(t_i)c(|x(t_i)|),$$

which yields, by successive application,

$$\begin{aligned} 0 \le V(\sigma(t_i), x(\sigma(t_i))) &\le V(t_0,x_0) - \sum_{j=1}^{i} \mu^*(t_j)c(|x(t_j)|) \\ &\le V(t_0,x_0) - c(\epsilon)\eta i. \end{aligned}$$

This leads to a contradiction as $i \to \infty$ since $\mu^*(t_i)$ is constant for each i, say η.

In cases (ii) to (iv), we can find another divergent sequence $\{t_i^*\}$ such that $t_i < t_i^*$ or $t_i^* < t_{i+1}$ satisfying

$$\begin{array}{llll} |x(t_i)| = \epsilon, & |x(t_i^*)| = \frac{1}{2}\epsilon, & \frac{1}{2}\epsilon < |x(t)| < \epsilon, & t \in (t_i, t_i^*) \quad \textit{or} \\ |x(t_i^*)| = \frac{1}{2}\epsilon, & |x(t_{i+1})| = \epsilon, & \frac{1}{2}\epsilon < |x(t)| < \epsilon, & t \in (t_i^*, t_{i+1}). \end{array}$$

Since $|f(t,x)| \le M$, it is easy to find $t_i - t_i^* > \frac{\epsilon}{2M}$ and $t_{i+1} - t_i^* > \frac{\epsilon}{2M}$.

It therefore follows that

$$0 \le V(t_i^*, x(t_i^*)) \le V(t_0,x_0) - ic\big(\tfrac{1}{2}\epsilon\big)\,\tfrac{\epsilon}{2M} < 0, \quad \text{for large} \quad i.$$

This is a contradiction. Similarly, we get contradiction for other case. Hence we have $\lim_{t\to\infty} |x(t)| = 0$ and the proof is complete.

12 A Technique in Stability Theory

As an application of Theorem 10.2, we shall consider a typical result on stability and asymptotic behavior of solutions of (10.2).

Theorem 12.1 *Assume that (H) holds and (i) of Theorem 10.2 is verified. Suppose that $g \in C_{rd}[\mathbb{T}\times R_+, R]$, $g(t,u)\mu^*(t)+u$ is nondecreasing in u for each $t \in \mathbb{T}$, $g(t,0) \equiv 0$, $f(t,0) \equiv 0$, $F(t,0) \equiv 0$ and for $(t,x) \in \mathbb{T} \times R^n$,*

$$b(|x|) \le V(t,x) \le a(|x|), \qquad a,\, b \in \mathcal{K}. \tag{12.1}$$

Furthermore, suppose that the trivial solution of (10.2) is uniformly stable and $u = 0$ of

$$u^{\Delta} = g(t,u), \qquad u(t_0) = u_0 \ge 0 \tag{12.2}$$

is uniformly asymptotically stable. Then, the trivial solution of (10.2) is uniformly asymptotically stable.

Proof Let $\epsilon > 0$ and $t_0 \in \mathbb{T}$ be given. The uniform stability of $u = 0$ of (12.2) implies that given $b(\epsilon) > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(\epsilon) > 0$ such that if $u_0 \le \delta_1$, then

$$u(t,t_0,u_0) < b(\epsilon), \qquad t \ge t_0. \tag{12.3}$$

Let $\delta_2 = a^{-1}(\delta_1)$. Since $y = 0$ of (10.2) is uniformly stable, given $\delta_2 > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(\epsilon)$ such that

$$|y(t,t_0,x_0)| < \delta_2, \quad t \ge t_0 \quad \text{if} \quad |x_0| < \delta. \tag{12.4}$$

We claim that $|x_0| < \delta$ also implies that $|x(t,t_0,x_0)| < \epsilon$, $t \in \mathbb{T}$, where $x(t,t_0,x_0)$ is any solution of (10.2). If this is not true, there would exist a solution $x(t,t_0,x_0)$ of (10.2) with $|x_0| < \delta$ and a $t_1 > t_0$ such that $|x(t_1,t_0,x_0)| \ge \epsilon$, $t_0 \le t \le t_1$. Then, by Theorem 10.2, we have

$$V(t,x(t,t_0,x_0)) \le r(t,t_0,V(t_0,y(t,t_0,x_0))), \qquad t_0 \le t \le t_1.$$

Consequently, by (12.1), (12.3) and (12.4), we get

$$\begin{aligned} b(\epsilon) &\le V(t_1,x(t_1,t_0,x_0)) \le r(t_1,t_0,a(|y(t_1,t_0,x_0)|)) \\ &\le r(t_1,t_0,a(\delta_2)) \le r(t_1,t_0,\delta_1) < b(\epsilon). \end{aligned}$$

This contradiction proves that $x = 0$ of (10.2) is uniformly stable.

To show uniform asymptotic stability, we set $\delta = \delta_0$. Then, from the foregoing argument, we have

$$b(|x(t,t_0,x_0)|) \le V(t,x(t,t_0,x_0)) \le r(t,t_0,V(t_0,y(t,t_0,x_0)))$$

for all $t \in \mathbb{T}$, if $|x_0| \le \delta_0$. From this it follows that

$$b(|x(t,t_0,x_0)|) \le r(t,t_0,\delta_1), \qquad t \in \mathbb{T}$$

which implies the uniform asymptotic stability of $x = 0$ because of the assumption that $u = 0$ of (12.2) is uniformly asymptotically stable. Hence the proof is complete.

Setting $F(t,x) = f(t,x) + R(t,x)$ in Theorem 12.1, we see that although the unperturbed system (10.2) is only uniformly stable, the perturbed system (10.2) is uniformly asymptotically stable, an improvement caused by the perturbing term.

13 Strict Stability

Consider the initial value problem

$$x^{\Delta} = f(t,x), \quad t \in \mathbb{T}, \quad x(t_0) = x_0, \tag{13.1}$$

where $f\colon \mathbb{T} \times R^n \to R^n$ and f is rd-continuous on $\mathbb{T} \times R^n$.

Let $K = \big\{a \in C_{rd}[\mathbb{T}, R_+]\colon a(u)$ is strictly increasing in u, $a(0) = 0$ and $a(u) \to \infty$ as $u \to \infty\big\}$.

Definition 13.1 The *trivial solution* of (13.1) is said to be

(S1) *strictly stable*, if given $\epsilon_1 > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta_1 = \delta_1(t_0, \epsilon_1) > 0$ such that $|x_0| < \delta_1$ implies $|x(t)| < \epsilon_1,\ t \geq t_0$, and for every $0 < \delta_2 \leq \delta_1$, there exist an $0 < \epsilon_2 < \delta_2$ such that

$$\delta_2 < |x_0| \quad \text{implies} \quad \epsilon_2 < |x(t)|, \quad t \geq t_0; \tag{13.2}$$

(S2) *strictly uniformly stable*, if δ_1, δ_2 and ϵ_2 are independent of t_0;

(S3) *strictly attractive*, if given $\alpha_1 > 0,\ \epsilon_1 > 0$ and $t_0 \in \mathbb{T}$ for every $\alpha_2 \leq \alpha_1$ there exists $\epsilon_2 < \epsilon_1$ and $T_1 = T_1(t_0, \epsilon_1),\ T_2 = T_2(t_0, \epsilon_1)$ such that

$$\alpha_2 \leq |x_0| \leq \alpha_1 \quad \text{implies} \quad \epsilon_2 < |x(t)| < \epsilon_1, \quad \text{for} \quad t_0 + T_1 \leq t \leq t_0 + T_2;$$

(S4) *strictly uniformly attractive*, if T_1, T_2 in (S3) are independent of t_0;

(S5) strictly asymptotically stable if (S3) holds and the trivial solution is stable;

(S6) strictly uniformly asymptotically stable if (S4) holds and the trivial solution is uniformly stable;

Remark 13.1 It is important to note that (S1) and (S3), or (S2) and (S4) cannot hold at the same time. If in (S1) it is not possible to find an ϵ_2 satisfying (13.2), we shall say that the trivial solution is stable. This can happen when $|x(t)| \to 0$ as $t \to \infty$ or $\liminf |x(t)| = 0$ and $\limsup |x(t)| \neq 0$.

Theorem 13.1 *Assume that*

(H_1) *for each* $0 < \eta < \rho,\ V_\eta \in C_{rd}[\mathbb{T} \times S_\rho, R_+],\ V_\eta$ *is locally Lipschitzian in* x *and for* $(t, x) \in \mathbb{T} \times S_\rho$ *and* $|x| \geq \eta$,

$$b_1(|x|) \leq V_\eta(t, x) \leq a_1(|x|), \quad a_1, b_1 \in K, \quad \text{and} \quad D^+V_\eta^\Delta(t, x) \leq 0; \tag{13.3}$$

(H_2) *for each* $\sigma,\ 0 < \sigma < \rho,\ V_\sigma \in C_{rd}[\mathbb{T} \times S_\rho, R_+],\ V_\sigma$ *is locally Lipschitzian in* x *and for* $(t, x) \in \mathbb{T} \times S_\rho$ *and* $|x| \leq \sigma$,

$$b_2(|x|) \leq V_\eta(t, x) \leq a_2(|x|), \quad a_2, b_2 \in K, \quad \text{and} \quad D^+V_\eta^\Delta(t, x) \geq 0. \tag{13.4}$$

Then the trivial solution is strictly uniformly stable.

Proof See [15].

Theorem 13.2 *Let the assumptions of Theorem 13.1 hold except that the conditions (13.3) and (13.4) are replaced by*

$$D^+V_\eta^\Delta(t, x) \leq -c_1(|x|),$$

and

$$D^+V_\eta^\Delta(t, x) \geq -c_2(|x|),$$

where $c_1, c_2 \in K$.

Then the trivial solution of (13.1) is uniformly strictly asymptotically stable.

Proof See [15].

Before proving the general result in terms of the comparison principle, we need to consider the comparison differential system

$$u_1^{\Delta} = g_1(t, u_1), \qquad u_1(t_0) = u_0 \ge 0, \tag{13.5a}$$

$$u_2^{\Delta} = g_2(t, u_2), \qquad u_2(t_0) = u_0 \ge 0, \tag{13.5b}$$

where $g_1,\, g_2 \in C_{rd}[\mathbb{T} \times R_+, R]$.

We shall say that the comparison system (13.5) is strictly stable, if given $\epsilon_1 > 0$ and $t_0 \in \mathbb{T}$, there exist a $\delta_1 > 0$ such that $u_0 < \delta_1$ implies $u_1(t) < \epsilon_1,\ t \ge t_0$, and for every $\delta_2 \le \delta_1$, there exists an $\epsilon_2,\ 0 < \epsilon_2 < \delta_2$ such that $\delta_2 < u_0$ implies that $\epsilon_2 < u_2(t)$, $t \ge t_0$. Where, $u_1(t)$, $u_2(t)$ are any solutions of (13.5a) and (13.5b) respectively.

Based on these definition, we can formulate other strict stability notions. Next result is formulated in terms of comparison principles.

Theorem 13.3 *Let the assumptions of Theorem 13.3 hold except that the conditions (13.3) and (13.4) are replaced by*

$$D^+V_\eta^{\Delta}(t,x) \le g_1(t, V_\eta(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n.$$

and

$$D^+V_\sigma^{\Delta}(t,x) \ge g_2(t, V_\sigma(t,x)), \qquad (t,x) \in \mathbb{T} \times R^n,$$

where $g_2(t,u) \le g_1(t,u),\ g_1,\, g_2 \in C_{rd}[\mathbb{T} \times R_+, R],\ g_1(t,0) \equiv 0,\ g_2(t,0) \equiv 0.$

Then any strict stability concept of the comparison system implies the corresponding strict stability concept of the trivial solution of (13.1) respectively.

Proof See [15].

For several allied results, see [9 − 15].

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Analysis of Time-Controlled Switched Systems by Stability Preserving Mappings

Guisheng Zhai[1*], Bo Hu[2†], Ye Sun[2] and A.N. Michel[2]

[1] *Faculty of Systems Engineering, Wakayama University, 930 Sakaedani, Wakayama 640-8510, Japan*
[2] *Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, U.S.A*

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Abstract: In this paper, we study the stability of a wide class of switched systems using stability preserving mappings. By considering an existing result and extending it to a general class of switched systems, we show that stability preserving mappings constitute an important and practical tool in stability analysis and design of switched systems.

Keywords: *Switched systems; stability analysis; stability preserving mappings.*

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1 Introduction

We study the stability of a large class of switched systems using stability preserving mappings [1 − 4]. By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time subsystems and a rule orchestrating the switching between the subsystems. Recently, there has been increasing interest in the stability analysis and switching control design of such systems (for recent progress in this field, see the survey papers [5, 6] and the references cited therein). It is known that when considering the switching method among several given subsystems, there are two main approaches for stability analysis or design: in one the switching depends only on time while in the other, the switching depends on the state and/or output of the system. In this paper, we focus our attention on the case of switching among subsystems determined by time, and in this sense we use the term *time-controlled switched system*. For such switched

*Corresponding author. E-mail: zhai@sys.wakayama-u.ac.jp

†Presently with Avant! Corporation, 46571 Bayside Parkway, Fremont, CA 94538. Email: bhu1@avanticorp.com.

systems, there are several existing results. The paper [7] shows that when all subsystems are linear time-invariant and all subsystem matrices are Hurwitz stable, we can choose each subsystem's activation time interval (called *dwell time*) sufficiently large so that the switched system is exponentially stable. In [8], a dwell time scheme is analyzed for local asymptotic stability of nonlinear switched systems with the activation time being used as a dwell time. In [9, 10], Hespanha extends the concept of "*dwell time*" to "*average dwell time*", by showing that when the average time interval between consecutive switchings is sufficiently large, the switched system is exponentially stable. In the recent papers [11, 12], the authors extended the above stability results to the case where both Hurwitz stable and unstable subsystems exist, by showing that if the average dwell time is chosen sufficiently large and the total activation time of unstable subsystems is relatively small compared with that of Hurwitz stable subsystems, then global exponential stability of a desired decay rate is guaranteed.

In this paper, we aim to extend the above results for a wide class of switched systems. The switched system under consideration is composed of N subsystems and is described by

$$\begin{cases} \dot{x}(t) = f_{i_k}(t, x(t), x(\tau_k)), & \tau_k \le t < \tau_{k+1}, \\ x(t) = g_{i_{k+1}}(t, x(t^-), x(\tau_k)), & t = \tau_{k+1}, \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state, $\tau_0, \tau_1, \tau_2, \cdots, \tau_k, \cdots$ are the switching points, and $i_k \in I_N = \{1, 2, \cdots, N\}$ denotes the number of the subsystem that is activated during the time interval $\tau_k \le t < \tau_{k+1}$. For all $i \in I_N$, it is assumed that $f_i \in C^1[\Re^+ \times \Re^n \times \Re^n, \Re^n]$ and $f_i(t, 0, 0) = 0$, $g_i \in C[\Re^+ \times \Re^n \times \Re^n, \Re^n]$ and $g_i(t, 0, 0) = 0$. Clearly, the differential equation in (1) determines the dynamical behavior of the system over the indicated time intervals while the second equation specifies the amount of the state change when switching occurs.

It is well known that the switched system (1) can be regarded as a discontinuous dynamical system. There are many results within this framework (for example, [1−4, 8]). In these references, the notion of stability preserving mapping is very important and effective in analyzing discontinuous dynamical systems. In this paper, we first use stability preserving mappings to recall an existing result and then extend our considerations to more general switched systems. In contrast with the general results given in [1−4, 8], we will in this paper take advantage of specific properties of switched systems to obtain some practical results.

The remainder of the present paper is organized as follows. In Section 2, we review some of the stability preserving mapping theory for discontinuous/hybrid dynamical systems established in [1−4, 8]. In Section 3, stability results for various cases of the switched system (1) are established. Finally, in Section 4 we make some concluding remarks.

2 Preliminaries

In the interests of completeness and clarity, we summarize in this section some of the stability preserving mappings theory developed in [1−4, 8]. To do this, we need to recall the definition of dynamical system and discontinuous/hybrid dynamical system.

Dynamical systems are families of motions determined by evolutionary processes (see, e.g., [1]). The evolution of such processes takes place over time which we denote by T. Every motion of a dynamical system depends on *initial data* (t_0, a), where $t_0 \in T$ is called *initial time* and $a \in A \subset X$ is called *initial point*, where X, the *state space*, is a *metric space* with metric d (i.e., (X, d) is a metric space), and A is an appropriate subset of X. For a given (t_0, a), we denote a *motion*, if it exists, by $p(t, t_0, a)$, $t \in T_{t_0,a}$, where $T_{t_0,a} = [t_0, t_1) \cap T$, and where t_1 may be finite or infinite. Thus, a motion is a mapping $p(\cdot, t_0, a)\colon T_{t_0,a} \to X$ with $p(t_0, t_0, a) = a$, and the family of motions which makes up the dynamical system is obtained by varying the initial point a over the set A and the initial time t_0 over T_0, an appropriate subset of T (the set of initial times). If we denote such a family by S, then the dynamical system is signified by the quintuple $\{T, X, A, S, T_0\}$. When $T = T_0$, we simply write $\{T, X, A, S\}$, and when all is clear from context, we will simply speak of a dynamical system S (rather than a dynamical system $\{T, X, A, S, T_0\}$). If $T = R^+ = [0, \infty)$, we speak of a *continuous-time dynamical system* and when $T = N = \{0, 1, 2, \cdots\}$ we speak of a *discrete-time dynamical system*. If $T = R^+$ and all $p \in S$ are continuous with respect to t, we speak of a *continuous dynamical system*. If $T = R^+$ and the elements of S are not continuous with respect to t, we speak of a *discontinuous dynamical system* (DDS). Most frequently, the system motions are determined by means of the solutions of initial-value problems.

Hybrid dynamical systems are capable of exhibiting simultaneously several kinds of dynamic behavior in different parts of the system (e.g., continuous-time dynamics, discrete-time dynamics, logic commands, discrete events, jump phenomena, and the like). For such systems, a general model which appears to be suitable for the *qualitative analysis* of *general* hybrid dynamical systems was introduced in [1−4]. This model incorporates a concept of *generalized time*. If we generalize the dynamical systems considered in the above paragraph by replacing the usual concept of time with the general time space (T, ρ), we end up with a notion of *hybrid dynamical system* $\{T, X, A, S, T_0\}$ (HDS) which includes most of the specific classes of dynamical systems considered in the literature as special cases. Presently, T is a totally ordered space with relation "$\prec$" which is bounded from below by $t_{min} \in T$ and for the metric ρ, triangle inequality is replaced by "triangle equality".

For $\{T, X, A, S, T_0\}$, a set $M \subset A$ is said to be *invariant with respect to system S* if $a \in M$ implies that $p(t, a, t_0) \in M$ for all $t \in T_{a,t_0}$, all $t_0 \in T_0$ and all $p(\cdot, a, t_0) \in S$. We will state the above more compactly by saying that M *is an invariant set of* S, or (S, M) *is invariant*. If in particular, $M = \{x_0\}$, then x_0 is called an *equilibrium*.

In the following, d denotes the metric on X (i.e., (X, d) is a metric space).

Let $\{T, X, A, S, T_0\}$ be an HDS and let $M \subset A$ be an invariant set for S. We say that (S, M) is *stable* if for every $\epsilon > 0$ and $t_0 \in T_0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in T_{a,t_0}$ and for all $p(\cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$. We say that (S, M) is *uniformly stable* if $\delta = \delta(\epsilon)$. If (S, M) is stable and if for any $t_0 \in T_0$ there exists an $\eta = \eta(t_0) > 0$ such that $\lim\limits_{t\to\infty} d(p(t, a, t_0), M) = 0$ (i.e., for every $\epsilon > 0$, there exists a $t_\epsilon \in T$ such that $d(p(t, a, t_0), M) < \epsilon$ whenever $t \in T$ and $t_\epsilon \prec t$) for all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \eta$, then (S, M) is said to be *asymptotically stable*. We call (S, M) *uniformly asymptotically stable* if (S, M) is uniformly stable and if there exists a $\delta > 0$ and for every $\epsilon > 0$ there exists a $\tau = \tau(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon$ for all $t \in \{t \in T_{a,t_0}\colon \rho(t, t_0) \geq \tau\}$ and all $p(\cdot, a, t_0) \in S$ whenever $d(a, M) < \delta$. We call (S, M) *exponentially stable* if there exists $\alpha > 0$, and for every $\epsilon > 0$ and $t_0 \in T_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $d(p(t, a, t_0), M) < \epsilon e^{-\alpha\cdot\rho(t,t_0)}$ for all $t \in T_{a,t_0}$ and for all $p(\cdot, a, t_0) \in S$, whenever $d(a, M) < \delta$. The notions of *uniform asymptotic stability in the large*, and *global exponential stability* are defined similarly. Finally, we call (S, M) *unstable* if (S, M) is not stable. It has been shown (for example, [1−4]) that, *by using the isometric mapping* $e\colon T \to R^+$ *given by* $e(t) = \rho(t, t_{min})$, *the qualitative analysis of invariant sets of hybrid dynamical systems defined on abstract time space T can be reduced to the qualitative analysis of the same invariant sets of the corresponding discontinuous dynamical systems defined on* R^+ (t_{min} is the minimum element on T determined by the relation "$\prec$"). Thus, in the qualitative analysis of such hybrid systems, we can confine ourselves to the qualitative analysis of appropriate DDS. For further details, see [1].

We now introduce the concept of stability preserving mapping between two discontinuous dynamical systems $\{R^+, X_1, A_1, S_1\}$ (with invariant set M_1) and $\{R^+, X_2, A_2, S_2\}$ (with invariant set M_2). Such mappings will serve as a basis for developing a general comparison (stability) theory for discontinuous dynamical systems. For example, if a stability preserving mapping has been established between S_1 and S_2, and if the stability properties of (S_2, M_2) are well understood, then it will be possible to deduce the stability properties of (S_1, M_1) from those of (S_2, M_2).

Definition 2.1 Let $\{R^+, X_1, A_1, S_1\}$ and $\{R^+, X_2, A_2, S_2\}$ be two discontinuous dynamical systems with invariant sets $M_1 \subset A_1$ and $M_2 \subset A_2$, respectively. We say that $V\colon X_1 \times R^+ \to X_2$ is a *stability preserving mapping* from S_1 to S_2 (or more explicitly, from (S_1, M_1) to (S_2, M_2)) if V satisfies the following conditions:

(i) $S_2 = \mathcal{V}(S_1) \stackrel{\triangle}{=} \{q(\cdot, b, t_0)\colon q(t, b, t_0) = V(p(t, a, t_0), t)$, with $b = V(a, t_0)$ and $T_{b,t_0} = T_{a,t_0}$, $a \in A_1$, $t_0 \in R^+\}$;

(ii) $M_2 = V(M_1 \times R^+) \stackrel{\triangle}{=} \{x \in X_2\colon x = V(x_1, t')$ for some $x_1 \in M_1$ and $t' \in R^+\}$;

(iii) the invariance of (S_1, M_1) is *equivalent* to the invariance of (S_2, M_2), i.e., (S_1, M_1) is invariant if and only if (S_2, M_2) is invariant; and

(iv) the stability, uniform stability, asymptotic stability, uniform asymptotic stability, exponential stability, uniform asymptotic stability in the large, and exponential stability in the large of (S_1, M_1) and (S_2, M_2) are equivalent, respectively (i.e., (S_1, M_1) is stable if and only if (S_2, M_2) is stable; (S_1, M_1) is uniformly stable if and only if (S_2, M_2) is uniformly stable; and so forth.)

The above definition states that the function V from $X_1 \times R^+$ into X_2 induces a mapping $\mathcal{V}\colon S_1 \to S_2$ and that under $\mathcal{V}$ several stability properties of (S_1, M_1) and $(S_2, V(M_1 \times R^+))$ are preserved.

Lemma 2.1 [1] *Let $\{R^+, X_i, A_i, S_i\}$, $i = 1, 2$, be two discontinuous dynamical systems and let $M_i \subset A_i$, $i = 1, 2$, be closed sets. Assume there exists $V\colon X_1 \times R^+ \to X_2$ which satisfies*

(i) *$\mathcal{V}(S_1) \subset S_2$, where $\mathcal{V}(S_1)$ and M_2 are defined as in Definition 2.1;*

(ii) *there exist ψ_1, $\psi_2 \in K$ defined on R^+ such that*

$$\psi_1(d_1(x, M_1)) \leq d_2(V(x, t), M_2) \leq \psi_2(d_1(x, M_1)) \tag{2}$$

for all $x \in X_1$, and $t \in R^+$, where d_1, d_2 are the metrics defined on X_1 and X_2, respectively. ($\psi \in K$ means that $\psi \in C[R^+, R^+]$, $\psi(0) = 0$, and $\psi(r)$ is monotonically increasing in r.)

Then,

(a) *the invariance of (S_2, M_2) implies the invariance of (S_1, M_1);*

(b) *the stability, uniform stability, asymptotic stability, and uniform asymptotic stability of (S_2, M_2) imply the same corresponding types of stability for (S_1, M_1); and*

(c) *if in (2),* $\psi_1(r) = ar^b$, $a > 0$, $b > 0$, *then the exponential stability of* (S_2, M_2) *implies the exponential stability for* (S_1, M_1)*; and*

(d) *if in (2),* $\lim\limits_{r\to\infty} \psi_1(r) = \infty$ *and if* M_1 *and* M_2 *are bounded and closed, then the global uniform asymptotic stability of* (S_2, M_2) *implies the global uniform asymptotic stability for* (S_1, M_1)*; and*

(e) *if in (c),* M_1 *and* M_2 *are bounded and closed, then the global exponential stability of* (S_2, M_2) *implies the global exponential stability for* (S_1, M_1)*.*

There may be a temptation to view the notions of *stability preserving mapping* and *Lyapunov function* as being identical concepts. This, however, is not correct, as can be seen by considering, e.g., for $\{T, X, A, S, T_0\}$ with $X = R^n$ and $M = \{0\}$, the function $V(x) = x \in R^n$. This function is clearly a stability preserving mapping. However, by any standards, it hardly qualifies as being a Lyapunov function.

3 Stability Analysis of Several Classes of Switched Systems

In the present section, we apply the stability preserving mapping theory in the analysis of several classes of switched systems described by (1).

First, we consider the linear case of the system (1) given by

$$\begin{cases} \dot{x}(t) = A_{i_k} x(t), & \tau_k \le t < \tau_{k+1}, \\ x(t) = B_{i_{k+1}} x(t^-), & t = \tau_{k+1}, \quad k \in N, \end{cases} \tag{3}$$

where A_{i_k}, $B_{i_{k+1}} \in \Re^{n\times n}$, and $E \overset{\triangle}{=} \{\tau_0, \tau_1, \dots : \tau_0 < \tau_1 < \cdots\}$ is a fixed, unbounded, closed, discrete set. For this switched system, we obtain the following result according to [4].

Lemma 3.1 *Assume that*

(i) *there exists a constant* $\alpha > 0$ *such that for all* $i \in I_N$, $\|A_i\| < \alpha$, *where* $\|\cdot\|$ *denotes the matrix norm induced by the Euclidean vector norm;*

(ii) $\sup\limits_k \{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where* λ *is a constant;*

(iii) $\|B_{i_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau_k)}\| < q < 1$, *where* q *is a constant,* $\forall\, k \in N$.

Then, the equilibrium $x_e = 0$ *of the switched system (3) is uniformly asymptotically stable in the large.*

We now show how one could obtain the above result by the stability preserving mapping theory (Lemma 2.1). Let $S_1 = S_{(3)}$ ($S_{(3)}$ denotes the dynamical system determined by the solutions of (3)), $M_1 = \{0\}$, and choose $y(t) = V(x(t)) = (x^T x)^{1/2}$. Along the solutions of (3), we have for $\tau_k \le t < \tau_{k+1}$,

$$\begin{aligned} \dot{y}(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} (x(t)^T \dot{x}(t) + \dot{x}(t)^T x(t)) \\ &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} (x(t)^T (A_{i_k} + A_{i_k}^T) x(t)) \\ &\le \alpha y(t). \end{aligned} \tag{4}$$

Also, at $t = \tau_{i_{k+1}}$, we have

$$y(t) = |x(t)| \leq \|B_{i_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau_k)}\| |x(\tau_k)| \leq q y(\tau_k). \tag{5}$$

Now consider the discontinuous dynamical system described by the scalar-valued inequalities

$$\begin{cases} \dot{y}(t) \leq \alpha y(t), & \alpha > 0 - \text{constant}, \quad \tau_k \leq t < \tau_{k+1}, \\ y(\tau_{k+1}) \leq q|y(\tau_k)|, & 0 < q < 1, \quad \forall\, k \in N. \end{cases} \tag{6}$$

Let $S_2 = S_{(6)}$ denote the dynamical system determined by (6), $M_2 = \{0\}$, and let d_1, d_2 be the metrics determined by the Euclidean norms on $\Re^n$ and $\Re^1$. Then, the function $V(x)$ induces a mapping $\mathcal{V}$ from S_1 to S_2 (see Definition 2.1), and $\mathcal{V}(S_1) \subset S_2$. Since the equilibrium $y_e = 0$ of (6) is uniformly asymptotically stable in the large (as can be verified by solving (6) directly), and since all the conditions of Lemma 2.1 are satisfied, we conclude that the equilibrium $x_e = 0$ of (3) is uniformly asymptotically stable in the large.

From Lemma 3.1, we obtain the following result, which is an extension of the results that appeared in [7] and [8].

Theorem 3.1 *Assume that A_i is Hurwitz stable for all $i \in I_N$. Then, there exists a constant $T > 0$ such that if every subsystem is activated over a time interval larger than T, then the switched system (3) is exponentially stable.*

Proof Since every A_i is Hurwitz stable, there exist positive scalars K and η such that $\|e^{A_i t}\| \leq K e^{-\eta t}$. Also, we can always find a positive scalar β such that $\|B_i\| \leq \beta$ for all $i \in I_N$. For any positive scalar $q < 1$, we choose

$$T > \frac{1}{\eta} \ln\left(\frac{\beta K}{q}\right). \tag{7}$$

When $\tau_{k+1} - \tau_k > T$, we let $l_k = \left\lfloor \dfrac{\tau_{k+1} - \tau_k}{T} \right\rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, and we let

$$\tau_{k,i} = \begin{cases} \tau_k + (i-1)T & \text{for} \quad 1 \leq i \leq l_k, \\ \tau_{k+1} & \text{for} \quad i = l_k + 1. \end{cases} \tag{8}$$

Obviously, during the interval $[\tau_{k,i}, \tau_{k,i+1})$ $(i = 1, \cdots, l_k)$ the i_k-th subsystem is activated. Now, since $T \leq \tau_{k,i+1} - \tau_{k,i} < 2T$, and since

$$\|B_{i_{k,i+1}} e^{A_{i_k}(\tau_{k,i+1}-\tau_{k,i})}\| \leq \beta K e^{-\eta T} < q < 1, \tag{9}$$

it follows from Lemma 3.1 that the equilibrium $x_e = 0$ of the switched system (3) is uniformly asymptotically stable in the large, and thus exponentially stable in this case.

Next, we consider the more general class of switched systems

$$\begin{cases} \dot{x}(t) = A_{i_k} x(t) + M_{i_k} x(\tau_k), & \tau_k \leq t < \tau_{k+1}, \\ x(t) = B_{i_{k+1}} x(t^-) + N_{i_{k+1}} x(\tau_k), & t = \tau_{k+1}, \quad k \in N, \end{cases} \tag{10}$$

where A_{i_k}, M_{i_k}, $B_{i_{k+1}}$, $N_{i_{k+1}} \in \Re^{n\times n}$. For such systems, we now prove the following *new result.*

Theorem 3.2 *Assume that*

(i) *there exist two constants* $\alpha > 0$, $\gamma > 0$ *such that for all* $i \in I_N$, $\|A_i\| < \alpha$, $\|M_i\| < \gamma$;

(ii) $\sup\limits_k \{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where* λ *is a constant;*

(iii) $\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int\limits_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| < q < 1$, *where* q *is a constant,* $\forall\, k \in N$.

Then, the equilibrium $x_e = 0$ *of the switched system (10) is uniformly asymptotically stable in the large.*

Proof Let $S_1 = S_{(10)}$, $M_1 = \{0\}$, and choose $y(t) = V(x(t)) = (x^T x)^{1/2}$. Along the solutions of (10), we have for $\tau_k \le t < \tau_{k+1}$,

$$
\begin{aligned}
\dot{y}(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}}(x(t)^T \dot{x}(t) + \dot{x}(t)^T x(t)) \\
&= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}} \left(x(t)^T (A_{i_k} + A_{i_k}^T) x(t) + x(t)^T M_{i_k} x(\tau_k) + x(\tau_k)^T M_{i_k}^T x(t) \right) \\
&\le \alpha y(t) + \gamma y(\tau_k).
\end{aligned} \tag{11}
$$

Also, at $t = \tau_{i_{k+1}}$, we have according to the condition (iii)

$$
\begin{aligned}
& y(t) = |x(t)| \\
\le & \left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int\limits_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| |x(\tau_k)| \le q y(\tau_k).
\end{aligned} \tag{12}
$$

Similarly as in the proof of Lemma 3.1, the function $V(x)$ induces a mapping $\mathcal{V}$ from S_1 to the discontinuous dynamical system $S_2 = S_{(13)}$ determined by the scalar-valued inequalities

$$
\begin{cases}
\dot{y}(t) \le \alpha y(t) + \gamma y(\tau_k), & \alpha > 0, \quad \gamma > 0, \quad \tau_k \le t < \tau_{k+1}, \\
y(\tau_{k+1}) \le q|y(\tau_k)|, & 0 < q < 1, \quad \forall\, k \in N.
\end{cases} \tag{13}
$$

Since the equilibrium $y_e = 0$ of (13) is uniformly asymptotically stable in the large, and since all the conditions of Lemma 2.1 are satisfied, we conclude that the equilibrium $x_e = 0$ of (10) is uniformly asymptotically stable in the large.

From Theorem 3.2, we obtain the following interesting result.

Lemma 3.2 *Assume that* A_i*'s,* $i \in I_N$, *are Hurwitz stable, and thus there exist constants* $K > 0$, $\eta > 0$ *such that* $\|e^{A_i t}\| \le K e^{-\eta t}$ *for all* $t \ge 0$, *and assume that* $\|B_i\| < \beta$, $\|N_i\| < \mu < 1$, $\|M_i\| < \gamma$, *where* β, μ, γ *are positive constants. If* $\dfrac{\beta\gamma K}{\eta} + \mu = q_0 < 1$, *then there exist constants* $T_2 > T_1 > 0$ *such that when every subsystem is activated over a time interval of duration* T *satisfying* $T_1 < T < T_2$, *the entire switched system is exponentially stable.*

Proof The condition (iii) of Theorem 3.2 is calculated as

$$
\begin{aligned}
&\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| \\
&\qquad \le \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + \beta\gamma K \int_0^{\tau_{k+1}-\tau_k} e^{-\eta\tau}\, d\tau + \mu \qquad (14)\\
&\qquad \le \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + \frac{\beta\gamma K}{\eta} + \mu \\
&\qquad = \beta K e^{-\eta(\tau_{k+1}-\tau_k)} + q_0.
\end{aligned}
$$

Therefore, for any $0 < q < q_0$, we can always choose $T_1 > 0$ such that when $\inf_k(\tau_{k+1} - \tau_k) \ge T_1$, we have $\beta K e^{-\eta(\tau_{k+1}-\tau_k)} + q_0 \le q < 1$. Pick any $T_2 > T_1$. Then, if every subsystem is activated over a time interval of magnitude T satisfying $T_1 < T < T_2$, then by Theorem 3.2 we can conclude that the entire switched system is uniformly asymptotically stable in the large, and thus exponentially stable in this case.

Note that in Theorem 3.1, τ_k can be ∞ (because the system is autonomous), while in Lemma 3.2 this case must be excluded since the term $M_{i_k}x(\tau_k)$ in the system (10) depends specifically on $x(\tau_k)$. Therefore, an upper bound T_2 is required to avoid the case that only one subsystem is activated after some time instant.

Finally, we consider the nonlinear switched systems determined by equations of the form

$$
\begin{cases}
\dot{x}(t) = A_{i_k}x(t) + M_{i_k}x(\tau_k) + F_{i_k}(t, x(t), x(\tau_k)), & \tau_k \le t < \tau_{k+1}, \\
x(t) = B_{i_{k+1}}x(t^-) + N_{i_{k+1}}x(\tau_k) + G_{i_{k+1}}(t, x(t^-), x(\tau_k)), & t = \tau_{k+1},\ k \in N,
\end{cases} \tag{15}
$$

where $A_{i_k}, M_{i_k}, B_{i_{k+1}}, N_{i_{k+1}} \in \Re^{n\times n}$, $F_{i_k}, G_{i_{k+1}} \in C[\Re^+ \times \Re^n \times \Re^n, \Re^n]$, $F_{i_k}(t,0,0) = 0$, $G_{i_{k+1}}(t,0,0) = 0$ for all $t \in R^+$, and

$$
\lim_{x\to 0,\ z\to 0} \frac{F_{i_k}(t,x,z)}{\sqrt{\|x\|^2 + \|z\|^2}} = 0, \qquad \lim_{x\to 0,\ z\to 0} \frac{G_{i_{k+1}}(t,x,z)}{\sqrt{\|x\|^2 + \|z\|^2}} = 0 \tag{16}
$$

hold uniformly for all $t \in \Re^+$, $k \in N$. Obviously, the system (15) may be a consequence of a *linearization process* of the system (1) about the point $x_e = 0$. We now prove the following *new* result.

Theorem 3.3 *Assume that*

(i) *there exist three positive constants α, γ, β such that for all $i \in I_N$, $\|A_i\| < \alpha$, $\|M_i\| < \gamma$, $\|B_i\| < \beta$;*

(ii) $\sup_{k\in N}\{\tau_{k+1} - \tau_k\} \le \lambda < \infty$, *where λ is a constant;*

(iii) $\left\| B_{i_{k+1}} \left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\, d\tau \right] M_{i_k} \right) + N_{i_{k+1}} \right\| < q < 1$, *where q is a constant, for all $k \in N$.*

Then, the equilibrium $x_e = 0$ of the switched system (15) is uniformly asymptotically stable.

To prove Theorem 3.3, we need the following preliminary result.

Lemma 3.3 *For any given* $\epsilon > 0$, *there exists a* $\delta_1 = \delta_1(\epsilon) > 0$ *such that*

$$\begin{cases} \left\| \int\limits_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\, d\tau \right\| \le \epsilon \|x(\tau_k)\|, \\ \|G_{i_{k+1}}(\tau_{k+1}, x(\tau_{k+1}^-), x(\tau_k))\| \le \epsilon(\|x(\tau_{k+1}^-)\| + \|x(\tau_k)\|), \\ \|x(t)\| \le c_0 \|x(\tau_k)\|, \end{cases} \tag{17}$$

whenever $\|x(\tau_k)\| \le \delta_1$ *for* $k \in N$, $t \in [\tau_k, \tau_{k+1})$, *where* $c_0 = (1 + \lambda + \lambda\gamma)e^{(\alpha+1)\lambda}$.

Proof By continuity, there exists a $\delta_2 > 0$ such that $\|F_{i_k}(t, x(t), x(\tau_k))\| \le \|x(t)\| + \|x(\tau_k)\|$ whenever $\|x(t)\| \le \delta_2$, $\|x(\tau_k)\| \le \delta_2$. We show that there exists a $\delta_3 > 0$ $\left(\delta_3 < \dfrac{\delta_2}{c_0}\right)$ such that $\|x(t)\| \le \delta_2$ for all $t \in [\tau_k, \tau_{k+1})$ whenever $\|x(\tau_k)\| \le \delta_3$. Otherwise, since $\delta_3 < \delta_2$, there exists a $t_0 \in (\tau_k, \tau_{k+1})$ such that $\|x(t)\| \le \delta_2$ for all $t \in [\tau_k, t_0)$ and $\|x(t_0)\| = \delta_2$. From the first equation of (15), we have

$$x(t) = x(\tau_k) + M_{i_k} x(\tau_k)(t - \tau_k) + \int\limits_{\tau_k}^{t} (A_{i_k} x(\tau) + F_{i_k}(\tau, x(\tau), x(\tau_k)))\, d\tau. \tag{18}$$

Now for $t \in [\tau_k, t_0)$, we have

$$\|x(t)\| \le (1 + \lambda + \lambda\gamma)\|x(\tau_k)\| + \int\limits_{\tau_k}^{t} (\alpha + 1)\|x(\tau)\|\, d\tau. \tag{19}$$

By the Grownwall inequality, we obtain

$$\|x(t)\| \le (1 + \lambda + \lambda\gamma)e^{(\alpha+1)(t-\tau_k)}\|x(\tau_k)\|, \tag{20}$$

and hence

$$\|x(t_0)\| \le (1 + \lambda + \lambda\gamma)e^{(\alpha+1)\lambda}\delta_3 = c_0\delta_3 < \delta_2, \tag{21}$$

which is a contradiction. Thus, our conclusion follows. In addition, we know that whenever $\|x(\tau_k)\| \le \delta_3$, $\|x(t)\| \le c_0\|x(\tau_k)\|$ for $t \in [\tau_k, \tau_{k+1})$.

Now, for given $\epsilon > 0$, let $\epsilon = \epsilon_1 \lambda e^{\alpha\lambda}(c_0 + 1)$. There exists a $\delta_4 > 0$ such that

$$\begin{gathered} \|F_{i_k}(t, x(t), x(\tau_k))\| \le \epsilon_1(\|x(t)\| + \|x(\tau_k)\|), \\ \|G_{i_{k+1}}(t, x(t^-), x(\tau_k))\| \le \epsilon_1(\|x(t^-)\| + \|x(\tau_k)\|), \end{gathered} \tag{22}$$

whenever $\|x(t)\| + \|x(\tau_k)\| \le \delta_4$. Let $\delta_1 = \min\left\{\delta_3, \dfrac{\delta_4}{c_0 + 1}\right\}$. When $\|x(\tau_k)\| \le \delta_1$, we have for all $t \in [\tau_k, \tau_{k+1})$, $\|x(t)\| \le c_0\|x(\tau_k)\|$. Thus $\|x(t)\| + \|x(\tau_k)\| \le \delta_4$, and then (22) is true. Furthermore, we obtain by (22) that

$$\begin{aligned} \left\| \int\limits_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\, d\tau \right\| &\le \epsilon_1 e^{\alpha\lambda} \int\limits_{\tau_k}^{\tau_{k+1}} (\|x(t)\| + \|x(\tau_k)\|)\, d\tau \\ &\le \epsilon_1 e^{\alpha\lambda}\lambda(c_0 + 1)\|x(\tau_k)\| = \epsilon\|x(\tau_k)\|. \end{aligned} \tag{23}$$

Since $\epsilon_1 < \epsilon$, we get from the second inequality of (22) that

$$\|G_{i_{k+1}}(t_{k+1}, x(t_{k+1}^-), x(\tau_k))\| \leq \epsilon(\|x(t_{k+1}^-)\| + \|x(\tau_k)\|. \tag{24}$$

This completes the proof of the lemma.

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3 Let $S_1 = S_{(15)}$, and let $V(x) = (x^T x)^{1/2}$. For any solution $x(t)$ of (15), let $y(t) = V(x(t))$. Then for $t \in [\tau_k, \tau_{k+1})$,

$$\begin{aligned} \dot{y}(t) &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}}(x(t)^T \dot{x}(t) + \dot{x}(t)^T x(t)) \\ &= \frac{1}{2}\,(x(t)^T x(t))^{-\frac{1}{2}}(x(t)^T (A_{i_k} + A_{i_k}^T)x(t) + x(t)^T M_{i_k} x(\tau_k) + x(\tau_k)^T M_{i_k}^T x(t) \\ &\quad + x(t)^T F_{i_k}(t, x(t), x(\tau_k)) + F_{i_k}(t, x(t), x(\tau_k))^T x(t)). \end{aligned} \tag{25}$$

Assumption (16) implies that for $\epsilon > 0$ ($\epsilon < 1$, which will be specified later) there exists $\delta = \delta(\epsilon) > 0$ such that

$$\begin{aligned} \|F_{i_k}(t, x(t), x(\tau_k))\| &\leq \epsilon(\|x(t)\| + \|x(\tau_k)\|), \\ \|G_{i_{k+1}}(t, x(t^-), x(\tau_k))\| &\leq \epsilon(\|x(t^-)\| + \|x(\tau_k)\|) \end{aligned} \tag{26}$$

for all $x \in B(\delta) = \{x \in R^n \colon \|x\| < \delta\}$ and $t \in R^+,\ k \in N$. According to Lemma 3.3, for the given $\epsilon > 0$, there exists a $\delta_1 > 0$ such that (17) holds. Updating δ with $\min\{\delta, \delta_1\}$ and combining (25), (26), we obtain for $t \in [\tau_k, \tau_{k+1})$,

$$\begin{aligned} \dot{y}(t) &\leq \frac{1}{2}\,(x(t)^T x(x))^{-\frac{1}{2}}((2\alpha + 2\epsilon)\|x(t)\|^2 + (2\gamma + 2\epsilon)\|x(t)\|\|x(\tau_k)\|) \\ &\leq (\alpha + 1)y(t) + (\gamma + 1)y(\tau_k). \end{aligned} \tag{27}$$

We now apply Lemma 2.1. We let $X_1 = B(\delta)$ to derive local stability results. For $t = \tau_{k+1}$, we have

$$\begin{aligned} y(\tau_{k+1}) &= \|x(\tau_{k+1})\| = \|B_{i_{k+1}} x(\tau_{k+1}^-) + N_{i_{k+1}} x(\tau_k) + G_{i_{k+1}}(\tau_{k+1}, x(\tau_{k+1}^-), x(\tau_k))\| \\ &\leq \left\| B_{i_{k+1}}\left(e^{A_{i_k}(\tau_{k+1}-\tau_k)} + \left[\int_0^{\tau_{k+1}-\tau_k} e^{A_{i_k}\tau}\,d\tau\right] M_{i_k}\right) + N_{i_{k+1}}\right\| \|x(\tau_k)\| \\ &\quad + \|B_{i_{k+1}}\| \left\| \int_{\tau_k}^{\tau_{k+1}} e^{A_{i_k}(\tau_{k+1}-\tau)} F_{i_k}(t, x(t), x(\tau_k))\,d\tau \right\| + \epsilon(\|x(\tau_{k+1}^-)\| + \|x(\tau_k)\|) \\ &\leq (q + \epsilon(\beta + c_0 + 1))\|x(\tau_k)\|. \end{aligned} \tag{28}$$

Since $q < 1$, there exists an $\epsilon_0 > 0$ such that

$$q_0 = q + \epsilon_0(\beta + c_0 + 1) < 1. \tag{29}$$

Clearly $B(\delta(\epsilon_0))$ is included in the region of attraction.

Now consider the discontinuous system determined by

$$\begin{cases} \dot{y}(t) \le (\alpha+1)y(t) + (\gamma+1)y(\tau_k), & \tau_k \le t < \tau_{k+1}, \\ y(\tau_{k+1}) \le q_0|y(\tau_k)|, & \forall\, k \in N. \end{cases} \tag{30}$$

The function $V(x(t))$ induces a mapping $\mathcal{V}$ from $S_1 = S_{(15)}$ to $S_2 = S_{(30)}$ which satisfies $\mathcal{V}(S_1) \subset S_2$. Since the equilibrium $y_e = 0$ of (30) is uniformly asymptotically stable, it now follows from Lemma 2.1 that the equilibrium $x_e = 0$ of system (15) is uniformly asymptotically stable.

4 Concluding Remarks

In this paper, we have analyzed the stability properties of a large class of switched systems by using the stability preserving mapping theory. By first considering an existing result and then analyzing more general switched systems, we have shown that the stability preserving mapping theory is very practical in the stability analysis and design of switched systems. We suggest that the same idea applies also to logic-based switched systems or discrete event systems, provided that we can model the state change when switchings occur.

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Asymptotic Behavior and Stability of the Solutions of Functional Differential Equations in Hilbert Space

V.V. Vlasov†

Moscow Institute of Physics and Technology,
Dolgoprudny, Moscow Region, 141700, Russia
and
Department of Mathematics and Mechanics,
Moscow State University, Moscow, 117234, Russia

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Abstract: In the following article we present the results on the asymptotic behavior and stability of the strong solutions for functional differential equations (FDE). We also formulate several results on spectral properties (completeness and basisness) of exponential solutions of the above-mentioned equations. It is relevant to underline that our approach for researching FDE is based on the spectral analysis of the operator pencils which are the symbols (characteristic quasipolynomials) with operator coefficients. The article is divided into two parts. The first part is devoted to researching FDE in a Hilbert space, the second part to researching FDE in a finite-dimensional space.

Keywords: *Functional-differential equation; solvability; stability; operator-valued function; asymptotic behavior; basisness.*

Mathematics Subject Classification (2000): 34K25, 34K20, 34L99.

1 Introduction

In the first part of this article we present results on the unique solubility of initial-boundary-value problems for a certain class of linear difference-differential equation of neutral type with coefficients that are operator-valued functions taking values in a set of operators (in general, unbounded) in a Hilbert space. We consider the case of variable

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time-lag. Moreover, we establish results for asymptotic behavior and the stability of the solutions of the above-mentioned equations.

These results extend certain results obtained in $[1-3, 35-38, 40-43]$.

In the second part of this article we study the asymptotic behavior of the solutions of difference-differential equations (in finite-dimensional space $H = C^m$) in more complicated and delicate situations in which there are the chains of roots of characteristic quasipolynomials which are lying on or approaching the imaginary axis (so-called critical and supercritical cases). There are several results, devoted to the analysis of this situation (see for more details $[19-22]$).

Besides, it is relevant to underline that our approach is seriously different to those methods used in cited works. Our estimates of the solutions are based on Riesz basis property of the system of exponential solutions. In turn, this result is based on the researching of the resolvent of the generator of the C^0-semigroup of the shift operator naturally connected with the initial-value problem for difference-differential equation.

These results generalize certain results obtained in $[4, 35-37, 39]$.

It is relevant to note that at the end of Section 1 and of Section 2 we give references for and brief comments on a comparison of our results with the results of previously published works on the subject.

2 FDE in Infinite Dimensional Space

Let H be a separable Hilbert space, let A be a positive self-adjoint operator in H with a bounded inverse, and let I be the identity operator in H. We convert the domain $\mathrm{Dom}(A^\alpha)$ of operator A^α $(\alpha > 0)$ into a Hilbert space H_α by introducing the norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|$ on $\mathrm{Dom}(A^\alpha)$.

We denote by $W_2^1((a,b),A)$ $(-\infty < a < b \le +\infty)$ the space of functions with values in H such that $A^j v^{(1-j)}(t) \in L_2((a,b),H)$ $(j = 0,1)$, endowed with the norm

$$\|v\|_{W_2^1(a,b)} \equiv \left(\int_a^b \left(\|v^{(1)}(t)\|^2 + \|Av(t)\|^2\right) dt \right)^{\frac{1}{2}}.$$

Here and throughout $v^{(j)}(t) \equiv \frac{d^j}{dt^j} v(t)$, $j = 0, 1, \ldots$. See Chapter 1 in [5] for more detailed information and a description of the space $W_2^1((a,b),A)$.

Along with $W_2^1((a,b),A)$ we introduce the two spaces $L_{2,\gamma}((a,b),H)$ and $W_{2,\gamma}^1((a,b),A)$ of functions with values in H, with norms defined by the relations

$$\|v\|_{L_{2,\gamma}} \equiv \left(\int_a^b \exp(-2\gamma t)\|v(t)\|^2\, dt \right)^{\frac{1}{2}},$$

$$\|v\|_{W_{2,\gamma}^1(a,b)} \equiv \|\exp(-\gamma t) v(t)\|_{W_2^1(a,b)}, \quad \gamma \in \mathbb{R}.$$

We consider the following problem on the semiaxis $\mathbb{R}_+ = (0, +\infty)$

$$\begin{aligned} \mathcal{U}u \equiv \frac{du}{dt} + Au(t) + B_0(t)CAu(t) \\ + \sum_{j=1}^n \left(B_j(t) S_{g_j}(Au)(t) + D_j(t) S_{g_j}\left(\frac{du}{dt}\right)(t) \right) = f(t), \end{aligned} \tag{1}$$

$$u(+0) = \phi_0. \tag{2}$$

Here $B_0(t)$, $B_j(t)$ and $D_j(t)$ $(j = 1, 2, \dots, n)$ are strongly continuous (see [6]) operator-valued functions with values in the ring of bounded operators in the space H, C is a compact operator in H, and $\phi_0 \in H_{\frac{1}{2}}$.

We define the operators S_{g_j} as follows

$$\begin{aligned}
&(S_{g_j} v)(t) = v(g_j(t)), && g_j(t) \geq 0,\\
&(S_{g_j} v)(t) = 0, && g_j(t) < 0, \quad j = 0, 1, 2, \dots, n,
\end{aligned}$$

where $g_j(t)$ $(j = 1, 2, \dots, n)$ are real-valued functions with continuous derivatives on the semiaxis $\mathbb{R}_+$ such that $g_j(t) \leq t$; $\frac{d}{dt} g_j(t) > 0$ $(j = 1, 2, \dots, n)$, and $g_0(t) = t$, $t \in \mathbb{R}_+$. We shall denote by $g_j^{-1}(t)$ the inverse functions of $g_j(t)$, $h_j(t) = t - g_j(t)$.

Definition 2.1 We call a *vector-valued function* $u(t)$ a *strong solution of equation* (1) if this function is in the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ for some value of $\gamma \geq 0$ and satisfies (1) almost everywhere on $\mathbb{R}_+$.

We define the quantities

$$\begin{aligned}
r_1(\gamma) &= \sup_{\lambda\colon\, \Re\lambda > \gamma} \|A(\lambda I + A)^{-1}\|,\\
r_2(\gamma) &= \sup_{\lambda\colon\, \Re\lambda > \gamma} |\lambda|\, \|(\lambda I + A)^{-1}\|, \quad \gamma \geq 0.
\end{aligned}$$

Theorem 2.1 *Let us suppose that* $B_0(t) \equiv 0$ *and there exists* γ_0 *such that*

$$\sigma(\gamma_0) < 1, \tag{3}$$

where

$$\begin{aligned}
\sigma(\gamma) = r_1(\gamma) \sum_{j=1}^{n} \sup_{t \in [g_j^{-1}(0), +\infty)} &\left[\exp\left(-\gamma(t - g_j(t))\right) \|B_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right]\\
+ r_2(\gamma) \sum_{j=1}^{n} \sup_{t \in [g_j^{-1}(0), +\infty)} &\left[\exp\left(-\gamma(t - g_j(t))\right) \|D_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right].
\end{aligned}$$

Then for every $\gamma > \gamma_0$ *the operator* V_γ*, acting according to the rule* $V_\gamma u \equiv (\mathcal{U}u, u(+0))$*, takes the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ *onto* $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ *and has a bounded inverse.*

We now turn to the problem often called the initial-value problem:

$$\begin{aligned}
&\frac{du}{dt} + Au(t) + B_0(t) C A u(t)\\
&+ \sum_{j=1}^{n} (B_j(t) A u(g_j(t)) + D_j(t) u^{(1)}(g_j(t))) = f_0(t), \quad t \in \mathbb{R}_+,
\end{aligned} \tag{1°}$$

$$u^{(m)}(t) = y_m(t), \quad t \in \mathbb{R}_- = (-\infty, 0), \quad u(+0) = \phi_0, \quad m = 0, 1. \tag{2°}$$

It is known (see, for details Chapter 1 in [8]) that problem (1°), (2°) can be reduced to one of the form (1), (2). In this case the vector-valued function $f(t)$ is defined as follows:

$$f(t) = f_0(t) - \sum_{j=1}^{n} \left[B_j(t) T^{g_j}(A y_0)(t) + D_j(t) T^{g_j}(y_1)(t) \right], \tag{4}$$

where the operators T^{g_j} are defined in a following way

$$(T^{g_j} v)(t) = 0, \quad g_j(t) \geq 0, \qquad (T^{g_j} v)(t) = v(g_j(t)), \quad g_j(t) < 0.$$

Definition 2.2 We call a *vector-valued function* $u(t)$ belonging to the space $W^1_{2,\gamma}(\mathbb{R}_+, A)$ for some $\gamma \geq 0$ the *strong solution of the problem* (1°), (2°), if $u(t)$ satisfies equation (1) with the function $f(t)$ defined by the expression (4) and the condition (2) in the sense of convergence in the space $H_{\frac{1}{2}}$.

On the basis of Theorem 2.1 it is possible to obtain

Theorem 2.2 *Let us suppose that the conditions of Theorem 2.1 are satisfied and there exists* $\gamma_1 > 0$, *such that*

$$\sigma_1(\gamma_1) < +\infty, \tag{5}$$

$$\sigma_1(\gamma) = \sum_{j=1}^{n} \sup_{t\in[0,g_j^{-1}(0))} \left[\exp\left(-\gamma(t - g_j(t))\right) \|B_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right]$$
$$+ \sum_{j=1}^{n} \sup_{t\in[0,g_j^{-1}(0))} \left[\exp\left(-\gamma(t - g_j(t))\right) \|D_j(t)\| \left(\frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right].$$

Then for every $\gamma \geq \gamma_* = \max(\gamma_0, \gamma_1)$, *every vector-valued functions* $(Ay_0)(t)$, $y_1(t) \in L_{2,\gamma}(\mathbb{R}_-, H)$, $f(t) \in L_{2,\gamma}(\mathbb{R}_+, H)$ *and every vector* $\phi_0 \in H_{\frac{1}{2}}$ *there exists a unique solution* $u(t)$ *of the problem* (1°), (2°) *belonging to the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ *and satisfying the inequality*

$$\begin{aligned} \|u(t)\|_{W^1_{2,\gamma}(\mathbb{R}_+,A)} \leq d_1 \Big(&\|f_0(t)\|^2_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|Ay_0\|^2_{L_{2,\gamma}(\mathbb{R}_-,H)} \\ &+ \|y_1\|^2_{L_{2,\gamma}(\mathbb{R}_-,H)} + \|\phi_0\|^2_{\frac{1}{2}} \Big)^{\frac{1}{2}} \end{aligned} \tag{6}$$

with constant d *independent of* $(f_0(t), (Ay_0)(t), y_1(t), \phi_0)$.

In the following theorem we investigate the case $\gamma_0 = 0$ which is important in applications.

Theorem 2.3 *Let us suppose that* $B_0(t) \equiv 0$ *and the following inequality*

$$\sum_{j=1}^{n} \left[\overline{\lim_{t\to+\infty}} \left(\|B_j(t)\|^2 \frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} + \overline{\lim_{t\to+\infty}} \left(\|D_j(t)\|^2 \frac{1}{g_j^{(1)}(t)} \right)^{\frac{1}{2}} \right] < 1 \tag{7}$$

satisfies.

Then the conclusion of Theorem 2.1 holds with constant $\gamma_0 = 0$ *and for* $\gamma_0 = 0$ *and* $f(t) \in L_2(\mathbb{R}_+, H)$

$$\lim_{t\to+\infty} \|u(t)\|_{\frac{1}{2}} = 0.$$

The following statement is connected with the equation of retarded type $(D_j(t) \equiv 0$, $j = 1, 2, \ldots, n)$.

Theorem 2.4 *Let us suppose* $D_j(t) \equiv 0$, $j = 1, 2, \ldots, n$, *operator-valued functions* $B_j(t)$ *are represented by the expression* $B_j(t) = B_j^0(t) C_j$, *where* C_j *are the compact* *operators in the space* H, $B_j^0(t)$ *— are strongly continuous operator-valued functions taking values in the ring of bounded operators in* H *and such that*

$$\sup_{t\in\mathbb{R}_+}\|B_0(t)\| < +\infty, \qquad \sup_{t\in\mathbb{R}_+}\left(\|B_j^0(t)C_j\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}} < +\infty. \tag{8}$$

Then there exists $\gamma_0 \ge 0$ *such that for every* $\gamma \ge \gamma_0$ *the operator* V_γ *takes the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ *onto* $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ *and has a bounded inverse.*

Let us denote by α_0 the infimum of the operator A (see the definition in [6]).

Theorem 2.5 is devoted to the case of negative γ_0.

Theorem 2.5 *Let us suppose the conditions of Theorem 2.4 are satisfied,* $B_0(t) \equiv 0$, *the inequality*

$$\sum_{j=1}^{n}\left(\sup_{t\in[g_j^{-1}(0),+\infty)}\|B_j(t)C_j\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}} < 1 \tag{9}$$

holds and the delays $h_j(t)$ *are bounded:* $0 < \theta_1 \le h_j(t) \le \theta_2 < \infty$; $\theta_1, \theta_2 = \mathrm{const}$.

Then there exists $\delta > 0$ *such that for every* $\gamma > \max(-\delta, -\alpha)$ *the operator* V_γ *takes the space onto the* $L_{2,\gamma}(\mathbb{R}_+, H) \oplus H_{\frac{1}{2}}$ *and has a bounded inverse.*

We present the result which is the corollary of Theorem 2.5.

Theorem 2.6 *Let us suppose the conditions of Theorem 2.5 are satisfied,* $f_0(t) \equiv 0$, *and the inequality*

$$\omega_0 = \max_{j=\overline{1,\,n}}\ \sup_{t\in[0,g_j^{-1}(0))}|g_j(t)| < +\infty$$

holds.

Then there exists $\delta > 0$ *such that for every initial functions* $y_0(t)$, $y_1(t)$ *such that* $Ay_0(t)$, $y_1(t) \in L_2((-\omega_0, 0), H)$ *and every vector* $\phi_0 \in H_{\frac{1}{2}}$ *there exists the unique solution* $u(t)$ *of the problem* (1°), (2°) (*for* $f_0 \equiv 0$), *belonging to the space* $W^1_{2,\gamma}(\mathbb{R}_+, A)$ (*for* $\gamma > \max(-\delta, -\alpha_0)$) *and satisfying the inequality*

$$\|e^{-\gamma t}u(t)\|_{W_2^1(\mathbb{R}_+,A)} \le d_2\left(\|\phi_0\|^2_{\frac{1}{2}} + \|Ay_0\|^2_{L_2(-\omega_0,0)} + \|y_1\|^2_{L_2(-\omega_0,0)}\right)^{\frac{1}{2}} \tag{10}$$

with the constant d_2 *independent of* $(\phi_0, (Ay_0)(t), y_1(t))$.

Let us present several remarks connected with the conditions of the results that were formulated above.

Remark 2.1 Under the additional restriction $h_j(t) \ge \alpha > 0$, $t \ge 0$, $j = 1, 2, \dots, n$ the sufficient condition for the existence γ_0 in the inequality (3) is the following

$$\begin{aligned}\Delta \equiv \sum_{j=1}^{n}\Bigg[&\sup_{t\in[g_j^{-1}(0),+\infty)}\left(\|B_j(t)\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}}\\ &+ \sup_{t\in[g_j^{-1}(0),+\infty)}\left(\|D_j(t)\|^2\,\frac{1}{g_j^{(1)}(t)}\right)^{\frac{1}{2}}\Bigg] < +\infty.\end{aligned} \tag{11}$$

Remark 2.2 If the inequality (11) holds and the delays $h_j(t) \geq \alpha > 0$, then the inequality (3) holds for every γ_0 such that

$$\gamma_0 > \frac{1}{\alpha} \max(\ln \Delta, 0). \tag{12}$$

Remark 2.3 The condition (5) guarantees that the right-hand part of (4) belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$ for every vector-valued functions $(Ay_0)(t),\ y_1(t) \in L_{2,\gamma}(\mathbb{R}_-, H)$.

Moreover, if the functions $g_j(t)$ satisfy the condition

$$\omega_0 = \max_{j=\overline{1,\,n}} \sup_{t \in [0, g_j^{-1}(0))} |g_j(t)| < +\infty,$$

then the second part of (4) belongs to the space $L_{2,\gamma}(\mathbb{R}_+, H)$ for every $\gamma \in \mathbb{R}$ and every vector-valued functions $y_0(t)$, $y_1(t)$ such that $(Ay_0)(t),\ y_1(t) \in L_2((-\omega_0, 0), H)$.

Remark 2.4 The inequality (7) is essential. It may be demonstrated by the following example.

Example 2.1 Let us suppose $H \equiv \mathbf{C},\ n = 1,\ A = \text{const} > 0,\ B_1(t) \equiv -1,\ B_0(t) \equiv D_1(t) \equiv 0$. Let us consider the following problem

$$\frac{du}{dt} + Au(t) - S_{t-h}(Au)(t) = \chi(0,h)A, \quad t \in \mathbb{R}_+, \tag{E1}$$
$$u(+0) = \phi_0 = 1,$$

where $\chi(0,h)$ is a characteristic function of the interval $(0,h)$.

In this case the condition (7) is not satisfied (the left-hand side of (7) is equal to 1). Equation (E1) has a unique solution $u(t) \equiv 1$ which does not belong to the space $W_2^1(\mathbb{R}_+)$.

Remark 2.5 The condition (11) is also essential. It may be demonstrated by the following example.

Example 2.2 Let us suppose $H = C,\ n = 1,\ B_0(t) \equiv D_1(t) \equiv 0,\ g_1(t) = t - 1,\ A = \text{const} > 0,\ B_1(t) = -2t\exp(2t - (A+1))$. Let us consider the following problem

$$\frac{du}{dt} + Au + B_1(t)u(t-1) = 0, \quad t \in \mathbb{R}_+, \tag{E2}$$
$$u(t) = y(t) = \exp(t^2 - At), \quad t \in [-1, 0], \quad u(+0) = \phi_0 = 1.$$

In this case equation (E2) has the solution $u(t) = \exp(t^2 - At)$ which does not belong to the space $W_{2,\gamma}^1(\mathbb{R}_+)$ for any $\gamma \in \mathbb{R}$.

Owing to the fact that

$$\sup_{t \in [1, +\infty)} \|B_1(t)\| = +\infty,$$

the condition (11) is not satisfied.

Remark 2.6 The inequality (12) is essential. It may be demonstrated by the following example.

Example 2.3 Let us suppose $H = C$, $n = 1$, $B_0(t) \equiv B_1(t) \equiv 0$, $A = \text{const} > 0$, $D_1(t) \equiv D = \text{const}$, $h > 0$. Let us consider the following equation

$$\frac{du}{dt} + Au(t) + D\frac{du}{dt}(t-h) = 0, \quad t \in \mathbb{R}_+. \tag{E3}$$

In this case the roots of the associated quasipolynomial

$$l(\lambda) = \lambda + A + \lambda D e^{-\lambda h}$$

are asymptotically approaching the line $\Re\lambda = \frac{\ln|D|}{h} = \Delta$ (see, for example [7]), being on the left-hand side if

$$\Re\lambda_q < \frac{\ln|D|}{h}.$$

Thus it is impossible to change Δ by $\Delta - \varepsilon$ (for any $\varepsilon > 0$) in the inequality (12).

The papers [30−33] were devoted to the spectral problem, namely, to studying operator-valued functions, which are symbols of the considered equations in the autonomous case.

Now we are going to present certain results for the asymptotic behavior of the strong solutions of FDE in the autonomous case. These results are based on information on the symbols (characteristic quasipolynomials) of the above-mentioned equations. In turn, these symbols are operator-valued functions (operator pencils) taking values in a set of unbounded operators in a Hilbert space.

The papers [30−33] dealt with operator-valued functions of the form

$$\begin{aligned} Ł(\lambda) = \lambda I + A + B_0 C A + \sum_{j=1}^{n} (B_j A + \lambda D_j)\exp(-\lambda h_j) + \\ + \left(\int_0^\infty \exp(-\lambda t) K(t)\, dt\right) A + \lambda\left(\int_0^\infty \exp(-\lambda t) Q(t)\, dt\right). \end{aligned} \tag{13}$$

Here B_0, B_j and D_j are bounded operators in the space H, $0 = h_0 < h_1 < \dots < h_n = h$, the operator functions $e^{-\varkappa t}K(t)$ and $e^{-\varkappa t}Q(t)$ take values in the ring of bounded operators acting in the space H and such that the operator functions $e^{-\varkappa t}K(t)$ and $e^{-\varkappa t}Q(t)$ are Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$, and λ $(\lambda \in \mathbb{C})$ is a spectral parameter.

A number of papers were devoted to studying characteristic quasipolynomials, the distribution of its zeros, and its estimates in the case of a finite-dimensional space H. We only mention monographs [7, 9, 10] and papers [19, 29].

The operator-valued functions of the form (13) have been studied much less in the case of infinite-dimensional spaces and, in particular, a Hilbert space H. Moreover, we do not know any paper (except [17, 18]) specifically dedicated to studying operator functions of the form (13).

It is noteworthy that there are new unexpected phenomena in the case of infinite-dimensional spaces. Some illustrative examples were given in [30, 32].

Let us proceed by formulating the certain results from [30−33, 35].

Lemma 2.1 *Let B_0, B_j, and D_j $(j = 1, 2, \ldots, n)$ be bounded operators in the space H, let the operator functions $K(t)$ and $Q(t)$ take values in the ring of bounded operators, acting in the space H, let the operator functions $\exp(-\varkappa t)K(t)$ and $\exp(-\varkappa t)Q(t)$ be Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$. Then there exists $M_0 \geq \varkappa$ such that in the half-plane $\Pi(M_0) \equiv \{\lambda\colon \Re\lambda > M_0\}$ the operator function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies the inequality*

$$\|(L(\lambda)(\lambda I + A)^{-1})^{-1}\| \leq \mathrm{const}\,. \tag{E4}$$

The following lemma defines conditions for the meromorphity of the operator-valued function $Ł^{-1}(\lambda)$.

Lemma 2.2 *Let the assumptions of Lemma 2.1 be satisfied. In addition, suppose that B_j $(j = 1, 2, \ldots, n)$ are compact operators in the space H, the operator functions $K(t)$ and $Q(t)$ take values in the ring of compact operators on H and additionally satisfy the condition $K(t) = Q(t) = 0$ whereas $t > h \stackrel{\text{def}}{=} h_n$. Then the spectrum of $Ł^{-1}(\lambda)$ consists of isolated characteristic numbers of a finite algebraic multiplicity that are finite-dimensional poles of $Ł^{-1}(\lambda)$.*

The following two statements complement Lemma 2.1 in the case of delay equations, i.e. $D_j \equiv 0$, $j = 1, 2, \ldots, n$, $Q(t) \equiv 0$.

Lemma 2.3 *Let the assumptions of Lemma 2.2 be satisfied and let $D_j \equiv 0$, $j = 1, 2, \ldots, n$, and $Q(t) \equiv 0$. Then for any $a \geq 0$ there exists $b > 0$ such that in the domain*

$$Q(a,b) \equiv \mathbb{C}\backslash\big(\{\lambda\colon \Re\lambda \leq -a\} \cup \{\lambda\colon\, -a \leq \Re\lambda \leq M_0,\ |\Im\lambda| \leq b\}\big),$$

the operator-valued function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies inequality (E4).

Lemma 2.4 *Let the assumptions of Lemma 2.2 be satisfied, let $D_j = 0$, $j = 1, 2, \ldots, n$, let $Q(t) \equiv 0$, let the operators B_j can be represented in the form $B_j = C_j A^{-\theta_j}$ $(j = 1, 2, \ldots, n)$, where C_j are bounded operators in the space H, $\theta_j \in (0, 1]$, $K(s) = K_1(s)A^{-\theta_0}$, where $\theta_0 \in (0, 1]$, and the operator function $K_1(s)$ takes values in the ring of bounded operators in the space H and is Bochner integrable on the interval $(0, h)$. Then there exists a constant $N_0 > 0$ such that in the domain*

$$\Phi(N_0) \equiv \mathbb{C}\backslash\big(\{\lambda\colon |\lambda| \leq N_0\} \cup \{\lambda\colon \Re\lambda < 0,\, |\Im\lambda| \leq N_0 \exp(-q\Re\lambda)\}\big),$$

where

$$q = \max\left(\max_{j=1,n} \frac{h_j}{\theta_j},\ \frac{h}{\theta_0}\right),$$

the operator-valued function $L^{-1}(\lambda)$ exists, is holomorphic, and satisfies inequality (E4).

Here it is useful to note the following.

Remark 2.7 Under the assumptions of Lemma 2.3, the assertion of Lemma 2.1 is valid for any constant $M_0 > \max \lambda_q$, where by λ_q we denote characteristic numbers of the operator-valued function $Ł(\lambda)$.

The major parts of papers [3, 30, 33, 35] are devoted to studying a more specific case of problem (1), (2) related to autonomous equation (1) with $f_0(t) \equiv 0$.

We assume that $B_0(t) \equiv B_0$, $B_j(t) \equiv B_j$, and $D_j(t) \equiv D_j$ and the functions $h_j(t) \equiv h_j$ $(j = 1, 2, \dots, n)$ are independent of t, i.e. B_0, B_j, and D_j are bounded operators in the space H, the operator functions $K(t)$ and $Q(t)$ satisfy the assumptions of Lemma 2.1, and h_j are numbers such that $0 = h_0 < h_1 < \dots < h_n = h$.

For convenience we formulate the resultant problem:

$$\begin{aligned}\frac{du}{dt} + Au(t) + B_0CAu(t) + \sum_{j=1}^{n} \big(B_jAu(t-h_j) + D_ju^{(1)}(t-h_j)\big)\\ + \int_{-\infty}^{t} \big(K(t-s)Au(s) + Q(t-s)u^{(1)}(s)\big)\,ds = 0, \quad t \in \mathbb{R}_+,\end{aligned} \tag{$1^{\circ\circ}$}$$

$$\begin{gathered}u^{(m)}(t) = y_m(t), \quad t \in \mathbb{R}_- = (-\infty, 0), \quad m = 0, 1;\\ u(+0) = \varphi_0.\end{gathered} \tag{$2^{\circ\circ}$}$$

Following the lines of [24], we introduce the operators $\mathbf{F}_1$ and $\mathbf{F}_2$, acting in the space $L_2((-h, 0), H)$:

$$\begin{gathered}(\mathbf{F}_1v)(t) = -\sum_{j=1}^{n} \chi(-h_j, 0)(t)B_jv(-t-h_j) - \int_{-h}^{t} K(-s)v(s-t)\,ds,\\ (\mathbf{F}_2v)(t) = -\sum_{j=1}^{n} \chi(-h_j, 0)(t)D_jv(-t-h_j) - \int_{-h}^{t} Q(-s)v(s-t)\,ds,\\ t \in [-h, 0),\end{gathered}$$

where $\chi(-h_j, 0)(t)$ are characteristic functions of the intervals $(-h_j, 0)$.

The next statement is useful when studying spectral problems.

Assertion 2.1 *Let B_0, B_j, D_j $(j = 1, 2, \dots, n)$ be bounded operators in the space H, and let the operator functions $K(t)$ and $Q(t)$ satisfy the assumptions of Lemma 2.2. Then any strong solution $u(t)$ of problem $(1^{\circ\circ})$, $(2^{\circ\circ})$ satisfies the inequalities*

$$d_1\|u\|_{W_2^1(0,h)} \le \big(\|\varphi\|_{1/2}^2 + \|\mathbf{F}_1(Ay_0)(t) + \mathbf{F}_2(y_1)(t)\|_{L_2(-h,0)}^2\big)^{1/2} \le d_2\|u\|_{W_2^1(0,h)}$$

with constants d_1 and d_2 independent of $(\varphi_0, \mathbf{F}_1(Ay_0), \mathbf{F}_2(y_1))$.

By U_α we denote the set of strong solutions of equation $(1^{\circ\circ})$ such that $\exp(\alpha t)u(t) \in L_2(\mathbb{R}_+, H)$, $\alpha \in \mathbb{R}$.

On the base of the canonical system of eigenvectors and adjoint eigenvectors $x_{q,j,0}$, $x_{q,j,1}, \dots, x_{q,j,s}$ $(j = 1, 2, \dots, p_q,\ s = 0, 1, \dots, r_{pq})$ of the operator-valued function $\mathrm{L}(\lambda)$ we construct the system of elementary (exponential) solutions of equation $(1^{\circ\circ})$:

$$y_{q,j,s}(t) = \exp(\lambda_q t)\bigg(\frac{t^s}{s!}\,x_{q,j,0} + \frac{t^{s-1}}{(s-1)!}\,x_{q,j,1} + \dots + x_{q,j,s}\bigg).$$

Lemma 2.5 *Let $D_j = 0$ $(j = 1, 2, \ldots, n)$, let $Q(t) \equiv 0$, let B_j $(j = 1, 2, \ldots, n)$ be compact operators in the space H, let the operator function $K(t)$ take values in the ring of compact operators acting in the space H, and let $K(t) = 0$, $t > h$. Then for an arbitrary $\alpha \geq 0$ any strong solution $u(t)$ of problem $(1^{\circ\circ})$, $(2^{\circ\circ})$ can be expressed in the form*

$$u(t) = \sum_{\Re\lambda_q \geq -\alpha} \sum_{j=1}^{p_q} \sum_{s=0}^{r_{pq}} c_{q,j,s} y_{q,j,s}(t) + w_\alpha(t),$$

where the vector-valued function $w_\alpha(t)$ belongs to U_α, and the coefficients $c_{q,j,s}$ satisfy the inequalities

$$|c_{q,j,s}| \leq d_q \big(\|\varphi_0\|^2_{1/2} + \|\mathbf{F}_1(Ay_0)(t)\|^2_{L_2(-h,0)}\big)^{1/2}$$

with the constants d_q independent of $(\varphi_0, \mathbf{F}_1(Ay_0)(t))$.

Corollary 2.1 *Let the conditions of Lemma 2.5 be satisfied, and let the solution $u(t)$ belong to U_α. Then there exists $\delta > 0$ such that $u(t) \in U_{\alpha+\delta}$.*

Lemma 2.6 *Let B_0, B_j, and D_j $(j = 1, 2, \ldots, n)$ be bounded operators in the space H, let $K(t)$ and $Q(t)$ be operator functions taking values in the ring of bounded operators on the space H and such that $\exp(-\varkappa t)K(t)$ and $\exp(-\varkappa t)Q(t)$ are Bochner integrable on the semiaxis $\mathbb{R}_+$ for some $\varkappa \geq 0$. Then the assertion of Theorem 2.1 is valid for any constant $\gamma_0 = M_0$, where the constant M_0 is defined in Lemma 2.1.*

Lemma 2.7 *Let the assumptions of Lemma 2.5 with $\varkappa = 0$ be valid, and let the operator function $(Ł(\lambda)(\lambda I + A)^{-1})^{-1}$ be bounded and continuous in the operator norm on the imaginary axis and satisfy the inequality*

$$\sup_{\lambda:\ \Re\lambda \geq 0} \|Ł(\lambda)(\lambda I + A)^{-1} - I\| < 1.$$

Then the assertion of Theorem 2.1 is valid with the constant $\gamma_0 = 0$; moreover, any solution of the problem with $\gamma = \gamma_0 = 0$ and $f(t) \in L_2(\mathbb{R}_+, H)$ satisfies the relation

$$\lim_{t \to +\infty} \|u(t)\|_{1/2} = 0.$$

Remark 2.8 We point out that the proposed approach to definition and understanding the solution of the problem (1°), (2°) is by no means the only one possible.

To date there is an extensive literature (covering mainly the case of finite-dimensional space) where one can find various approaches to the interpretation of solutions and various methods of solution and analysis of initial-boundary-value problems for functional differential equations. Here we restrict ourselves to drawing attention to monographs and papers [7−12] devoted to this subject and papers [13−17] treating the case of equations in Banach and, in particular, in Hilbert spaces.

Our approach to the interpretation of the solutions of FDE in a Hilbert space is based on the approach presented in [8, 11], and it is its development to FDE in abstract spaces.

We remark that the results that were formulated (Theorems 2.1−2.6) may be obtained in the same way as Theorem 1 [1] and Theorem 1 [3]. Certain differences arise in estimates of the integral operators in the description of the integral equation which is equivalent in the sense of solvability of the problem (1), (2).

We point out that, although there are many papers devoted to the study of functional differential equations in a Banach space, in particular, in Hilbert space, they consider mainly delay-type equations. We know of far fewer papers considering abstract equations of neutral type. The papers closest to the subject of present work are $[14-15, 18]$.

In the papers that we know ($[14-15, 18]$) the restrictions on the coefficients $B_j(t)A$ and $D_j(t)$ $(j = 1, 2, \dots, n)$ of the delays are more severe. For example, in most papers (see, in particular, $[14-15]$) the authors assume that the coefficients of the delays ($B_j(t)A$ and $D_j(t)$) are bounded operators. The authors of $[13, 16-17]$ assume in the case of delay-type equations $(D_j(t) \equiv 0,\ j = 1, 2, \dots, n)$ that the coefficients $B_j(t)$ are independent of t.

It is relevant to underline that in articles $[42-46]$ we obtained results on Fredholm solubility and the properties of the strong solutions of FDE of n-th order of convolution type (including integro-differential equations) the symbols of which are operator-valued functions representable as operator bundle of n-th order perturbed by operator-valued functions of special type (bounded or decreasing at infinity). In $[44-48]$ we proved also the result about multiple minimality of the system root vectors and exponential solutions.

In turn in the papers $[30-33, 35, 40, 46, 47]$ we proved the results on asymptotic behavior of the strong solutions of FDE in a Hilbert space, and in particular, the results on the nonexistence of nontrivial solutions decreasing more rapidly than any exponent (the problem of so-called small solutions or the Phraghmen–Lindelöf principle).

3 FDE in Finite-Dimensional Space

We are going to study the asymptotic behavior of the solutions of the following equation

$$\sum_{j=0}^{n}\left(B_j u(t-h_j) + D_j \frac{du}{dt}(t-h_j)\right) + \int_0^h B(s)u(t-s)\,ds = 0, \qquad t \in \mathbb{R}_+. \tag{14}$$

Here B_j, D_j $(j = 0, 1, \dots, n)$ are $(m \times m)$ matrices with constant elements, the real numbers h_j satisfy the inequalities $0 = h_0 < h_1 < \dots < h_n = h$, the elements $B_{ij}(s)$ of the matrix $B(s)$ belong to the space $L_2((0, h), C)$.

Let us introduce the matrix-valued function

$$\mathcal{L}(\lambda) = \sum_{j=0}^{n}(B_j + \lambda D_j)\exp(-\lambda h_j) + \int_0^h \exp(-\lambda s)B(s)\,ds, \quad \lambda \in C, \tag{15}$$

and the complex-valued function $l(\lambda) = \det \mathcal{L}(\lambda)$ often called by the characteristic quasipolynomial of equation (14).

Let us denote by λ_q the zeroes of the function $l(\lambda)$ numbered in increasing order of its modulars (counting multiplicity).

The eigenvectors appearing in a canonical system of eigen and associated (root) vectors corresponding to λ_q we denote by $x_{q,j,0}$, and associated vector of order s by $x_{q,j,s}$ (the index j shows where is vector $x_{q,j,0}$ in a sequence of the vectors in specially chosen basis of subspace of solutions of the equation $\mathcal{L}(\lambda_q)x = 0$).

We introduce the system of exponential (elementary) solutions of equation (14)

$$y_{q,j,s}(t) = \exp(\lambda_q t)\left(\frac{t^s}{s!}x_{q,j,0} + \frac{t^{s-1}}{(s-1)!}x_{q,j,1} + \dots + x_{q,j,s}\right). \tag{16}$$

Let us denote by $W_2^1((a,b),C^m)$ $(-\infty < a < b \leq +\infty)$ the Sobolev space of functions with values in C^m endowed by the norm

$$\|v\|_{W_2^1(a,b)} \equiv \left(\int_a^b \left(\|v^{(1)}(t)\|_{C^m}^2 + \|v(t)\|_{C^m}^2 \right) dt \right)^{\frac{1}{2}}.$$

Along with $W_2^1((a,b),C^m)$ we introduce $W_{2,\gamma}^1((a,b),C^m)$ as the space of functions with values in C^m endowed by the norm

$$\|v\|_{W_{2,\gamma}^1(a,b)} \equiv \left(\int_a^b e^{-2\gamma t} \left(\|v^{(1)}(t)\|_{C^m}^2 + \|v(t)\|_{C^m}^2 \right) dt \right)^{\frac{1}{2}}, \qquad \gamma \in \mathbb{R}.$$

We state for equation (14) the following initial conditions

$$\begin{gathered} u(t) = y(t), \quad t \in [-h,0], \quad u(+0) = y(-0), \\ y(t) \in W_2^1((-h,0),C^m). \end{gathered} \tag{17}$$

Definition 3.1 We call a *vector-valued function* $u(t)$ belonging to the space $W_{2,\gamma}^1((-h,+\infty),C^m)$ for certain $\gamma \in \mathbb{R}$ a *strong solution of the problem* (14), (17), if $u(t)$ satisfies equation (14) almost everywhere on the semiaxis $\mathbb{R}_+$ and condition (17).

First of all we formulate an a priori estimate for the strong solutions of the problem (14), (17).

Lemma 3.1 *Let us suppose* $\det D_0 \neq 0$. *Then there exists* $\gamma_0 \geq 0$ *such, that for every* $\gamma \geq \gamma_0$ *the problem (14), (17) has a unique strong solution* $u(t) \in W_{2,\gamma}^1((-h,+\infty),$ $C^m)$ *for every initial function* $y(t) \in W_2^1((-h,0),C^m)$*, and this solution* $u(t)$ *satisfy the inequality*

$$\|u\|_{W_{2,\gamma}^1((-h,+\infty),C^m)} \leq d\|y\|_{W_2^1((-h,0),C^m)} \tag{18}$$

with constant d *independent of function* $y(t)$.

Keeping in mind Lemma 2.1 let us introduce (in a way similar to that in [7]) the semigroup U_t of bounded operators, acting in the space $W_2^1((-h,0),C^m)$ according to the rule

$$(U_t y)(s) = u(t+s), \quad -h \leq s \leq 0, \quad t \geq 0.$$

Here $u(t)$ is the solution of the problem (14), (17) corresponding to the initial function $y(s)$.

In the following theorem we present the description of the generator of C^0-semigroup U_t.

Theorem 3.1 *Let us suppose* $\det D_0 \neq 0$. *Then* U_t *is* C^0*-semigroup of the operators acting in the space* $W_2^1((-h,0),C^m)$ *with generator* $\mathcal{D}$ *such that*

$$(\mathcal{D}\phi)(s) = \frac{d\phi}{ds}(s), \qquad s \in (-h,0),$$

$$\begin{aligned} \operatorname{Dom}\mathcal{D} = \Bigg\{ \phi \in W_2^2((-h,0),C^m),\ & \sum_{j=0}^m \left(B_j \phi(-h_j) + D_j \phi^{(1)}(-h_j) \right) \\ & + \int_0^h B(s)\phi(-s)\,ds = 0 \Bigg\}. \end{aligned} \tag{19}$$

Proposition 3.1 *Let us suppose* $\det D_0 \neq 0$. *Then the spectrum of the operator* $\mathcal{D}$ *is the set* Λ *of the zeroes* λ_q *of the function* $l(\lambda)$ *and the exponential solutions* $y_{q,j,s}(t)$ *(see (16)) are its root vectors and form a minimal system in the space* $W_2^1((-h,0),C^m)$.

In the following theorem we present the result on completeness of the system of exponential solutions.

Theorem 3.2 *Let us suppose* $\det D_0 \neq 0$, $\det D_n \neq 0$. *Then the system of elementary solutions* $\{y_{q,j,s}(t)\}$ *is complete in the space* $W_2^1((-h,0),C^m)$.

In the following proposition we show the localization of the spectrum of the operator $\mathcal{D}$.

Proposition 2.2 *Let us suppose* $\det D_0 \neq 0$, $\det D_n \neq 0$. *Then there exist constants* α *and* β *such that the set* Λ *is lying in the strip* $\{\lambda\colon \alpha < \Re\lambda < \beta\}$.

Let us denote by V_{λ_q} the span of the elementary solutions $y_{q,j,s}(t)$, corresponding to λ_q, by ν_q the multiplicity of λ_q and by $\varkappa = \sup\limits_{\lambda_q\in\Lambda} \Re\lambda_q$, $N = \max\limits_{\lambda_q\in\Lambda} \nu_q$.

Now we present one of our main results on the behavior of the strong solutions of the problem (14), (17).

Theorem 3.3 *Let us suppose that* $\det D_0 \neq 0$, $\det D_n \neq 0$ *and the set* Λ *is separate:*

$$\inf_{\lambda_p\neq\lambda_q} (\operatorname{dist}(\lambda_p,\lambda_q)) > 0.$$

Then any strong solution $u(t)$ *of the problem (14), (17) satisfies the inequality*

$$\|u(t+\cdot)\|_{W_2^1(-h,0)} \leq d(t+1)^{N-1} \exp(\varkappa t)\|y\|_{W_2^1(-h,0)}, \quad t \geq 0, \tag{20}$$

with constant d *independent of* $y(t)$.

The theorem is based on the following result.

Theorem 3.4 *Let us suppose the conditions of Theorem 3.3 are satisfied.*
Then the system of subspaces V_{λ_q} $(\lambda_q \in \Lambda)$ *forms a Riesz basis of subspaces of the space* $W_2^1((-h,0),C^m)$.

Let $B_p(\lambda_q)$ be a disk with radius ρ and with a center at the point λ_q. We introduce the domain

$$G_\rho(\Lambda) \equiv C \setminus \bigcup_{\lambda_q\in\Lambda} B_\rho(\lambda_q).$$

Assertion 3.1 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$. *Then there exists a system of closed contours* $\Gamma_n = \{\lambda\colon \Re\lambda = \beta,\ c_n \leq \Im\lambda \leq c_{n+1}\} \cup \{\lambda\colon \Re\lambda = \alpha,\ c_n \leq \Im\lambda \leq c_{n+1}\} \cup l_{n+1}$, $n \in \mathbb{Z}$, *which entirely lies in the domain* G_ρ *for some sufficiently small* $\rho > 0$. *In addition, the following conditions are satisfied:*

(i) *The sequence of real numbers* $\{c_n\}$ $(n \in \mathbb{Z})$ *lying on the semiaxes* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *is such that* $0 < \delta \leq c_{n+1} - c_n \leq \Delta < +\infty$; *piecewise smooth curves* l_n, *joining the points* $(\Re\lambda = \beta, \Im\lambda = c_n)$ *and* $(\Re\lambda = \alpha, \Im\lambda = c_n)$ *do not intersect, and their lengths* $d(l_n)$ *are uniformly bounded with respect to* n *(here* δ *and* Δ *are positive constants).*

(ii) *The number* $N(\Gamma_n)$ *of zeros* λ_q *(with regard to their multiplicities) lying inside the contour* Γ_n *is uniformly bounded with respect to* n:

$$\max_{n\in\mathbb{Z}} N(\Gamma_n) \leq M < +\infty;$$

(iii) *There exists a constant* c *such that* $\sup\limits_{\lambda\in\Gamma_n} |\lambda|\, \|\mathcal{L}^{-1}(\lambda)\| \leq c$.

We introduce the set $\{\mathcal{P}_n\}$ of Riesz spectral projections, corresponding to the contours Γ_n:

$$(\mathcal{P}_n f) = -\frac{1}{2\pi i}\int\limits_{\Gamma_n} R(\lambda,\mathbb{D})f\,d\lambda, \qquad n\in\mathbb{Z},$$

in this case we assume that contours have the counterclockwise orientation.

The following Theorems 3.5 and 3.6 generalize the Theorems 3.3 and 3.4.

Theorem 3.5 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$. *Then there exists a system of contours* Γ_n $(n\in\mathbb{Z})$, *satisfying the conditions* (i) − (iii) *of Assertion 3.1, such that the corresponding system of subspaces* $\mathcal{W}_n = \mathcal{P}_n W_2^1((-h,0),C^m)$ *forms a Riesz basis of subspaces of the space* $W_2^1((-h,0),C^m)$.

On the basis of Theorem 3.5 can be obtained the following

Theorem 3.6 *Let* $\det D_0 \neq 0$ *and* $\det D_n \neq 0$. *Then any strong solution* $u(t)$ *of the problem (14), (17) satisfies the inequality*

$$\|u(t+\cdot)\|_{W_2^1((-h,0),C^m)} \le d_1(t+1)^{M-1}\exp(\varkappa t)\|y\|_{W_2^1((-h,0),C^m)}, \qquad t\ge 0, \tag{21}$$

with the constant M *defined in Assertion 3.1 and constant* d_1 *independent of* $y(t)$.

The following theorem generalizes Theorem 3.3 in the case $B(s)\equiv 0$.

Theorem 3.7 *Let* $D_0 \neq 0$, $\inf_{\lambda_p\neq\lambda_q}(\mathrm{dist}(\lambda_p,\lambda_q)) > 0$, $B(s)\equiv 0$.
Then any strong solution $u(t)$ *of the problem (14), (17) satisfies the inequality (20).*

It is relevant to underline that the estimate (20) is also valid for the well-known example of Gromova and Zverkin (see [20], and the remarks in the monograph [7]). In their example $m=1,\ n=1,\ D_0=-D_1=1,\ B_0=B_1=a=\mathrm{const}>0,\ N=1,\ \varkappa=0.$ Moreover, if we introduce the following norm:

$$\|u\|^*_{W_2^1(-h,0)} = \left(\int\limits_{-h}^{0}\left(|u^{(1)}(s)|^2 + a^2|u(s)|^2\right)ds + a\left(|u(0)|^2+|u(-h)|^2\right)\right)^{\frac{1}{2}}$$

which is equivalent to the traditional norm in the space $W_2^1((-h,0),C)$ the exponential solutions $e^{\lambda_q t}$ will be orthogonal in scalar product $\langle\cdot,\cdot\rangle^*_{W_2^1}$ associated with norm $\|\cdot\|^*_{W_2^1(-h,0)}$.

In addition to the Theorems 3.3, 3.4 and we present (formulate) results on the asymptotic behavior of the strong solutions of scalar difference-differential equation of the m-th order.

Let us denote by $W^m_{2,\gamma}((a,b),C)$ weighted Sobolev space of complex-valued functions with norm

$$\|u\|_{W^m_{2,\gamma}(a,b)} = \left[\int\limits_a^b \exp(-2\gamma t)\left(\sum_{j=0}^m |u^{(j)}(t)|^2\right)dt\right]^{\frac{1}{2}}, \qquad \gamma\in\mathbb{R}.$$

We study the following initial value problem:

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}u^{(j)}(t-h_k)+\int_0^h a(s)u(t-s)\,ds=0, \qquad t\in\mathbb{R}_+; \tag{22}$$

$$\begin{gathered} u(t)=y(t), \quad t\in[-h,0], \\ u^{(j)}(+0)=y^{(j)}(-0), \quad j=0,1,\dots,m-1. \end{gathered} \tag{23}$$

Here a_{kj} are the complex coefficients, real numbers h_j satisfy the inequalities $0 = h_0 < h_1 < \dots < h_n = h$, the function $a(s)\in L_2((0,h),C)$, the initial data $y(s)\in W_2^m((-h,0),C)$.

Definition 3.2 We call the *complex-valued function* $u(t)$ belonging to the space $W_{2,\gamma}^m((-h,+\infty),C)$ for some $\gamma\geq 0$ the *strong solution of the problem* (22), (23), if $u(t)$ satisfies equation (22) almost everywhere on the semiaxis $\mathbb{R}_+$ and the initial conditions (23).

Let us denote by ν_q the multiplicities of the zeroes λ_q of the function $l(\lambda)$

$$l(\lambda)=\sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}\lambda^j\exp(-\lambda h_k)+\int_0^h a(s)e^{-\lambda s}\,ds. \tag{24}$$

Theorem 3.8 *Let us suppose $a_{0m}\neq 0$, $a_{nm}\neq 0$, and the set Λ of all zeroes λ_q of the function $l(\lambda)$ is separate (that's $\inf\limits_{\lambda_p\neq\lambda_q}\operatorname{dist}(\lambda_q,\lambda_p)>0$).*

Then the strong solution $u(t)$ of the problem (22), (23) satisfies the inequality

$$\|u(t+\cdot)\|_{W_2^m(-h,0)}\leq d(t+1)^{N-1}\exp(\varkappa t)\|y\|_{W_2^m(-h,0)}, \qquad t\geq 0, \tag{25}$$

with constant d independent of $y(t)$. Here $N=\max\limits_{\lambda_q\in\Lambda}\nu_q$, $\varkappa=\sup\limits_{\lambda_q\in\Lambda}\Re\lambda_q$.

This theorem is based on the following result.

Theorem 3.9 *Let us suppose that the conditions of Theorem 3.7 are satisfied.*
Then the following system of functions

$$v_{q,m}=\frac{t^r\exp(\lambda_q t)}{(|\lambda_q|^m+1)}, \quad \lambda_q\in\Lambda, \quad r=0,1,\dots,\nu_q-1; \tag{26}$$

form a Riesz basis in the space $W_2^m(-h,0)$.

Remark 3.1 The inequality $a_{nm}\neq 0$ is essential for Riesz basisness. Indeed it is not difficult to verify that for the following difference-differential equation

$$\frac{du}{dt}+au(t)+bu(t-h)=0$$

the system of normed exponential solutions $y_q(t)=a_qe^{\lambda_q t}$ $(\|y_q(t)\|_{W_2^1(-h,0)}=1)$ is not uniformly minimal. This fact may be easily obtained by calculating the scalar product $\langle y_{q+1}(t),\overline{y}_q(t)\rangle_{W_2^1(-h,0)}$.

Using the well-known asymptotics of the zeroes λ_q of the quasipolynomial

$$l(\lambda) = \lambda + a + be^{-\lambda h}$$

one can verify that

$$\langle y_{q+1}(t), \overline{y}_q(t) \rangle_{W_2^1(-h,0)} \to 1 \quad (q \to +\infty). \tag{27}$$

The statement (27) is a contradiction of uniform minimality of the system $\{y_q(t)\}_{\lambda_q \in \Lambda}$.

Remark 3.2 It is relevant to underline that critical and supercritical cases are realized for quasipolynomials (24), when

$$|a_{0m}| = |a_{nm}|$$

(see [19, 20] for more details).

Remark 3.3 It is known that in the case $k(s) \equiv 0$ constant N satisfies the following inequality

$$N \leq m(n+1) - 1.$$

It is relevant to underline that one of the first results about geometrical properties of elementary solutions of an equation similar to [22] was obtained by Levinson and McCalla in 1974 in [23]. In [23] a result on the completeness and minimality of the system of exponential solutions for the equation of the retarded type $(a_{ni} = 0,\ i = 1, 2, \dots, n)$ was obtained.

The generalization of this result for retarded equations $D_j \equiv 0,\ j = 1, \dots, n$ in the space $\mathbb{R}^n$ was obtained by Delfour and Manitius in [24]. In turn, the strongest results on the completeness of the exponential autonomous FDE were obtained by Lunel [25−27]. It is important to underline that in [25−27] Lunel also considered the problem of so-called "small solutions" which is deeply connected with the problem of the completeness of the exponential solutions.

The problem of small solutions was also researched by Hale [7], Henry [28] and Kappel [29] in finite-dimensional space $H = \mathbb{R}^m(C^m)$.

Certain results about minimality of the elementary solutions and the problem of small solutions (Phraghmen–Lindelöf Principle) for FDE in a Hilbert space was obtained by author in [30−33, 40].

In cited papers [30−33, 35] one can also find results on the spectral properties of the operator-valued functions (operator pencils) that are the symbols (characteristic quasipolynomials) of the autonomous FDE with operator coefficients in a Hilbert space (see also references in [30−33]).

Recently results on Riesz basisness in the space $L_2((-h, 0), C^m)$ of the exponential solutions for FDE of neutral type with a different understanding of solvability and definition of solutions have been obtained by Lunel and Yakubovich in [34].

For a more complete description of our results on Riesz basisness and estimates of the strong solutions presented in this article see [2−4, 35−37, 39].

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