Volume 19, Number 1, 2019 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 19 Number 1 2019

CONTENTS

 ISSN 1562-8353 Printed in Ukraine
To receive contents and abstracts by e-mail, visit our Website at: **http://www.e-ndst.kiev.ua**

NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 19, No. 1, 2019

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

InforMath Publishing Group
http://www.e-ndst.kiev.ua

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. Nonlinear Dynamics and Systems Theory (ND&ST) is an international journal devoted to publishing peer-refereed, high quality, original papers, brief notes and review articles focusing on nonlinear dynamics and systems theory and their practical applications in engineering, physical and life sciences. Submission of a manuscript is a representation that the submission has been approved by all of the authors and by the institution where the work was carried out. It also represents that the manuscript has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that the authors have agreed that the copyright in the article shall be assigned exclusively to InforMath Publishing Group by signing a transfer of copyright form. Before submission, the authors should visit the website:

http://www.e-ndst.kiev.ua

for information on the preparation of accepted manuscripts. Please download the archive Sample_NDST.zip containing example of article file (you can edit only the file Samplefilename.tex).

(2) Manuscript and Correspondence. Manuscripts should be in English and must meet common standards of usage and grammar. To submit a paper, send by e-mail a file in PDF format directly to

Professor A.A. Martynyuk, Institute of Mechanics,
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine
e-mail: journalndst@gmail.com

or to one of the Regional Editors or to a member of the Editorial Board. Final version of the manuscript must typeset using LaTex program which is prepared in accordance with the style file of the Journal. Manuscript texts should contain the title of the article, name(s) of the author(s) and complete affiliations. Each article requires an abstract not exceeding 150 words. Formulas and citations should not be included in the abstract. AMS subject classifications and key words must be included in all accepted papers. Each article requires a running head (abbreviated form of the title) of no more than 30 characters. The sizes for regular papers, survey articles, brief notes, letters to editors and book reviews are: (i) 10-14 pages for regular papers, (ii) up to 24 pages for survey articles, and (iii) 2-3 pages for brief notes, letters to the editor and book reviews.

(3) Tables, Graphs and Illustrations. Each figure must be of a quality suitable for direct reproduction and must include a caption. Drawings should include all relevant details and should be drawn professionally in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach the electronic file of the artwork (preferably in PCX format).

(4) References. Each entry must be cited in the text by author(s) and number or by number alone. All references should be listed in their alphabetic order. Use please the following style:

Journal: [1] H. Poincare, Title of the article. *Title of the Journal* **volume** (issue) (year) pages. [Language]

Book: [2] A.M. Lyapunov, *Title of the Book.* Name of the Publishers, Town, year.

Proceeding: [3] R. Bellman, Title of the article. In: *Title of the Book.* (Eds.). Name of the Publishers, Town, year, pages. [Language]

(5) Proofs and Sample Copy. Proofs sent to authors should be returned to the Editorial Office with corrections within three days after receipt. The corresponding author will receive a sample copy of the issue of the Journal for which his/her paper is published.

(6) Editorial Policy. Every submission will undergo a stringent peer review process. An editor will be assigned to handle the review process of the paper. He/she will secure at least two reviewers' reports. The decision on acceptance, rejection or acceptance subject to revision will be made based on these reviewers' reports and the editor's own reading of the paper.

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 19 Number 1 2019

CONTENTS

Founded by A.A. Martynyuk in 2001.

Registered in Ukraine Number: KB 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2017: SNIP – 0.707, SJR – 0.316

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

AIMS AND SCOPE

Nonlinear Dynamics and Systems Theory is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed. This journal and the individual contributions published therein are protected under the copyright by International InforMath Publishing Group.

PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 4 issues in 2019, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (`mailto:journals@ebsco.com`), or website of the Journal: `http://e-ndst.kiev.ua`. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 1–9

Existence of Solutions for a Biological Model Using Topological Degree Theory

C. H. D. Alliera *

Departamento de Matemáticas, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires.

Received: January 11, 2018; Revised: December 21, 2018

Abstract: Topological degree theory is a useful tool for studying systems of differential equations. In this work, a biological model is considered. Specifically, we prove the existence of positive T-periodic solutions of a system of delay differential equations for a model with feedback arising on circadian oscillations in the drosophila period gene protein.

Keywords: *differential equations with delay; periodic solutions; models with feedback; topological degree; drosophila; circadian cycle.*

Mathematics Subject Classification (2010): 34K13, 92B05.

1 Introduction

The study of cellular control has been developed in many papers on mathematical analysis to determine the existence of stable oscillations in mRNA regulatory processes, see [5] and to understand circadian cycles and, in particular, of the cellular machinery that produces them, see [7].

In all cases, search for conditions on the parameters of the proposed systems has been carried out with the purpose of determining conditions for the existence of stable cycles and the cycles when the system solution may be even chaotic.

Let us consider a model proposed by Goldbeter [3], who showed the variation on PER: the period of *messenger of Ribo-Nucleic Acid (mRNA)* in Drosophila (often called "fruit flies") related to circadian rhythms. Our model does not consider temperature variation as shown in [6]. Here, a nonautonomous version of the model is considered with the aim of proving the existence of periodic solutions by means of a powerful topological tool: the Leray-Schauder degree (see [1] and [2]). In the original model, the existence of a positive steady state can be shown, under appropriate conditions, by the use of the Brouwer degree. As we shall see, when the parameters are replaced by periodic functions, essentially the same conditions yield the existence of positive periodic solutions.

* Corresponding author: mailto:calliera@dm.uba.ar

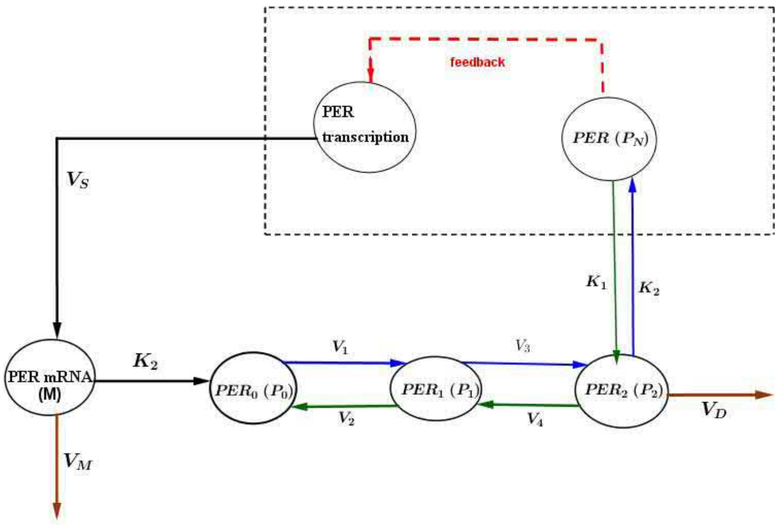

Figure 1: Model for the circadian variation in PER.

2 The Model

The following simplified model was proposed in [3]. Some more complex alternative models have been studied with light interaction and timeless (TIM) proteins (see [4]).

2.1 General features

1. This negative feedback will be described by an equation of Hill type in which n denotes the degree of cooperativity, and $K(t)$ is the threshold repression function.
2. To simplify the model, we consider that P_N behaves directly as a repressor.
3. The constants K_s, K_i and V_j denote the maximum rate and Michaelis constant of the kinase(s) and the phosphatase(s) involved in the reversible phosphorylation of P_0 into P_1, and of P_1 into P_2 are not negative.
4. Maximum accumulation rate of cytosol is denoted by V_s.
5. Cytosol is degraded enzymically, in a Michaelian manner, at a maximum rate V_m.
6. Functions of this system are:
   (a) Cytosolic concentration is denoted by M.
   (b) We consider only three states of the protein: unphosphorylated (P_0), monophosphorylated (P_1) and bisphosphorylated (P_2).
   (c) Fully phosphorylated form of PER (P_2) is degraded in a Michaelian manner, at a maximum rate V_d, and also transported into the nucleus, at a rate characterized by the apparent first-order rate constant k_1.

7. The rate of synthesis of PER, proportional to M, is characterized by an apparent first-order rate constant K_s.

8. Transport of the nuclear, bisphosphorylated form of PER (P_N) into the cytosol is characterized by the apparent first-order rate constant k_2.

9. The model could be readily extended to include a larger number of phosphorylated residues.

With this in mind, our non-autonomous version of Goldbeter's system reads:

$$\begin{aligned}
\frac{dM}{dt} &= \frac{V_S(t)K_1(t)^n}{K_1^n(t)+P_N(t)^n} - \frac{V_m(t)M(t)}{K_{m_1}(t)+M(t)},\\
\frac{dP_0}{dt} &= K_s(t)M(t) + \frac{V_2(t)P_1(t)}{K_2(t)+P_1(t)} - \frac{V_1(t)P_0(t)}{K_1(t)+P_0(t)},\\
\frac{dP_1}{dt} &= \frac{V_1(t)P_0(t)}{K_1(t)+P_0(t)} + \frac{V_4(t)P_2(t)}{K_4(t)+P_2(t)} - P_1(t)\left(\frac{V_2(t)}{K_2(t)+P_1(t)} + \frac{V_3(t)}{K_3(t)+P_1(t)}\right),\\
\frac{dP_2}{dt} &= \frac{V_3(t)P_1(t)}{K_3(t)+P_1(t)} + k_2(t)P_N(t) - P_2(t)\left(k_1(t) + \frac{V_4(t)}{K_4(t)+P_2(t)} + \frac{V_d(t)}{K_d(t)+P_2(t)}\right),\\
\frac{dP_N}{dt} &= k_1(t)P_2(t) - k_2(t)P_N(t),
\end{aligned} \tag{1}$$

where $K_i,\ i=1,2,3,4,d,m_1,s,\ k_1,\ k_2$ and $V_j,\ j=1,2,3,4,S,m,d$ are strictly positive, continuous T-periodic functions. We shall prove that, under accurate assumptions to be specified below, the system admits at least one positive T-periodic solution.

3 Existence of Positive Periodic Solutions

In order to apply the topological degree method to problem (1), let us consider the space of continuous T-periodic vector functions

$$C_T := \{u \in C(\mathbb{R},\mathbb{R}^5) : u(t) = u(t+T) \text{ for all } t\},$$

equipped with the standard uniform norm, and the positive cone

$$\mathcal{K} := \{u \in C_T : u_j \geq 0, j = 1,\dots,5\}.$$

Thus, the original problem can be written as $Lu = Nu$, where $L : C^1 \cap C_T \to C$ is given by $Lu := u'$ and the nonlinear operator $N : \mathcal{K} \to C_T$ is defined as the right-hand side of system (1). For convenience, the average of a function u shall be denoted by $\overline{u}$, namely $\overline{u} := \frac{1}{T}\int_0^T u(t)\,dt$. Also, identifying $\mathbb{R}^5$ with the subset of constant functions of C_T, we may define the function $\phi : [0,+\infty)^5 \to \mathbb{R}^5$ given by $\phi(x) := \overline{Nx}$.

For the reader's convenience, let us summarize the basic properties of the Leray-Schauder degree which, roughly speaking, can be regarded as an algebraic count of the zeros of a mapping $F : \overline{\Omega} \to E$, where E is a Banach space and $\Omega \subset E$ is open and bounded. In more precise terms, assume that $F = I - K$, where K is compact and $F \neq 0$ on $\partial\Omega$. The degree $deg_{LS}(F,\Omega,0)$ is defined as the Brouwer degree deg_B of its restriction $F|_V : \Omega \cap V \to V$, where V is an accurate finite-dimensional subspace of E. In particular, if the range of K is finite dimensional, then one may take V as the subspace spanned by $\mathrm{Im}(K)$. If $deg_{LS}(F, \Omega, 0)$ is different from 0, then F vanishes in Ω; moreover, the degree is invariant over a continuous homotopy $F_\lambda := I - K_\lambda$ with K_λ being compact and $F_\lambda \neq 0$ over $\partial\Omega$. Finally, we recall that if $\Delta : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $0 \in \Delta(A)$ for some open bounded $A \subset \mathbb{R}^n$, then $deg_B(\Delta, A, 0)$ is just the sign of the Jacobian determinant of Δ at the (unique) pre-image of 0. The following continuation theorem is a direct consequence of the standard topological degree methods (see e.g. [1]).

Theorem 3.1 *Assume there exists $\Omega \subset \mathcal{K}^\circ$ being open and bounded such that:*

a) The problem $Lu = \lambda Nu$ has no solutions on $\partial\Omega$ for $0 < \lambda < 1$.

b) $\phi(u) \neq 0$ for all $u \in \partial\Omega \cap \mathbb{R}^5$.

c) $deg_B(\phi, \Omega \cap \mathbb{R}^5, 0) \neq 0$.

Then (1) has at least one solution in $\overline{\Omega}$.

3.1 A priori bounds

Firstly, we shall find appropriate bounds for the solution of the problem $Lu = \lambda Nu$ with $\lambda \in (0, 1)$. For convenience, let us fix the following notation for the minima and maxima of all the functions involved in the model, namely

$$0 < v_i \leq V_i(t) \leq \mathcal{V}_i, \; 0 < \kappa_j \leq K_j(t) \leq \mathcal{K}_j, \; 0 < \hat{k}_l \leq k_l(t) \leq \Bbbk_l, \; \forall \; i, \; j, \; l.$$

Now assume that $u \in \mathcal{K}^\circ$ satisfies $Lu = \lambda Nu$ for some $0 < \lambda < 1$. Let us firstly consider a value t^* at which M achieves an absolute maximum, then $M'(t^*) = 0$ and hence

$$\frac{V_S(t^*)K_1(t^*)^n}{K_1^n(t^*) + P_N(t^*)^n} = \frac{V_m(t^*)M(t^*)}{K_{m_1}(t^*) + M(t^*)} \geq \frac{v_m M(t^*)}{\mathcal{K}_{m_1} + M(t^*)} := b_M(M(t^*)),$$

where the increasing function

$$b_M(x) := \frac{v_m x}{\mathcal{K}_{m_1} + x}$$

has inverse such that

$$b_M^{-1}(y) := \frac{\mathcal{K}_{m_1} y}{v_m - y}.$$

If

Hypothesis 3.1

$$v_m > \mathcal{V}_S,$$

then

$$M(t^*) = b_M^{-1}\left(\frac{V_S(t^*)K_1(t^*)^n}{K_1^n(t^*) + P_N(t^*)^n}\right) < b_M^{-1}\left(V_S(t^*)\right) \leq \frac{\mathcal{V}_S \mathcal{K}_{m_1}}{v_M - \mathcal{V}_S} := \mathcal{M}.$$

Next, suppose that P_0 achieves its absolute maximum at some point, denoted again t^*, then

$$K_s(t^*)M(t^*) + \frac{V_2(t^*)P_1(t^*)}{K_2(t^*) + P_1(t^*)} = \frac{V_1(t^*)P_0(t^*)}{K_1(t^*) + P_0(t^*)} \geq \frac{v_1 P_0(t^*)}{\mathcal{K}_1 + P_0(t^*)} := b_0(P_0(t^*)).$$

Again, we define an increasing and invertible function:

$$b_0(x) := \frac{v_1 x}{\mathcal{K}_1 + x} \to b_0^{-1}(y) := \frac{\mathcal{K}_1 y}{v_1 - y}.$$

Thus, under the condition

Hypothesis 3.2

$$\mathcal{K}_S \mathcal{M} + \mathcal{V}_2 < v_1$$

we deduce that

$$P_0(t^*) = b_0^{-1}\left(K_s(t^*)M(t^*) + \frac{V_2(t^*)P_1(t^*)}{K_2(t^*) + P_1(t^*)}\right) < \frac{\mathcal{K}_S\mathcal{M} + \mathcal{V}_2}{v_1 - (\mathcal{K}_S\mathcal{M} + \mathcal{V}_2)}\mathcal{K}_1 := \mathcal{P}_0.$$

Next, an upper bound $\mathcal{P}_1$ for P_1 is readily obtained in the following way. Let us denote again by t^* a value at which P_1 achieves its absolute maximum, then

$$\frac{V_1(t^*)P_0(t^*)}{K_1(t^*) + P_0(t^*)} + \frac{V_4(t^*)P_2(t^*)}{K_4(t^*) + P_2(t^*)} = P_1(t^*)\left(\frac{V_2(t^*)}{K_2(t^*) + P_1(t^*)} + \frac{V_3(t^*)}{K_3(t^*) + P_1(t^*)}\right).$$

When $P_1(t^*) \gg 0$, the right-hand side gets close to $V_2(t^*) + V_3(t^*)$, while the left-hand side is always less than or equal to $\frac{\mathcal{V}_1\mathcal{P}_0}{\kappa_1 + \mathcal{P}_0} + \mathcal{V}_4$ because $\frac{P_2}{K_4(t^*) + P_4} \leq 1$ and $\frac{x}{\kappa_1 + x}$ increase when $x = \mathcal{P}_0$.

Thus, the existence of $\mathcal{P}_1$ is guaranteed by the condition

Hypothesis 3.3

$$\frac{\mathcal{V}_1\mathcal{P}_0}{\kappa_1 + \mathcal{P}_0} + \mathcal{V}_4 < \min_{t\in\mathbb{R}}\{V_2(t) + V_3(t)\}.$$

The remaining upper bounds are obtained as follows. In the first place, define a new variable $Q := P_N + P_2$ which satisfies the equation:

$$\frac{dQ}{dt} = \frac{V_3(t)P_1(t)}{K_3(t) + P_1(t)} - P_2(t)\left(\frac{V_4(t)}{K_4(t) + P_2(t)} + \frac{V_d(t)}{K_d(t) + P_2(t)}\right).$$

If Q achieves its absolute maximum at t^*, then

$$\frac{\mathcal{V}_3\mathcal{P}_1}{\kappa_3 + \mathcal{P}_1} \geq \frac{V_3(t^*)P_1(t^*)}{K_3(t^*) + P_1(t^*)} - P_2(t^*)\left(\frac{V_4(t^*)}{K_4(t^*) + P_2(t^*)} + \frac{V_d(t^*)}{K_d(t^*) + P_2(t^*)}\right).$$

As before, if the condition

Hypothesis 3.4

$$\frac{\mathcal{V}_3\mathcal{P}_1}{\kappa_3 + \mathcal{P}_1} < \min_{t\in\mathbb{R}}(V_4(t) + V_d(t))$$

is assumed, then $P_2(t^*) \leq \tilde{P}$ for some $\tilde{P}$. Moreover, from the fourth equation of the system we deduce the existence of a constant C such that $\dfrac{dP_2}{dt} \geq -CP_2(t)$. Hence we obtain, for all t, that $P_2(t) \leq e^{CT}\tilde{P} := \mathcal{P}_2$. Then $Q'(t)$ is bounded. Besides, there exist $\hat{t}$ critical point of $\mathcal{P}_N$, in consequence

$$k_1(\hat{t})P_2(\hat{t}) = k_2(\hat{t})P_N(\hat{t}),$$

then $P_N(\hat{t})$ verifies:

$$P_N(\hat{t}) \leq \frac{k_1^*}{k_{2*}}\mathcal{P}_2,$$

thus

$$Q(\hat{t}) = P_N(\hat{t}) + P_2(\hat{t}) \leq \mathfrak{Q}_0 := \left(\frac{k_1^*}{k_{2*}} + 1\right)\mathcal{P}_2.$$

In this way, knowing that $Q' \leq \mathfrak{Q}_1$, by integrating up to a certain t in the interval $\mathcal{J} := [\hat{t}, \hat{t} + T]$ follows:

$$Q(t) = Q(\hat{t}) + \int_{\hat{t}}^{t} Q'(t) \leq \mathfrak{Q}_0 + \mathfrak{Q}_1 \underbrace{(t - \hat{t})}_{\leq T},\ t \in \mathcal{J}$$

in this way, there is also a $\mathcal{P}_N$ of $P_N(t)$, then

$$P_N(t) \leq Q(t) \leq \mathfrak{Q}_0 + \mathfrak{Q}_1 T := \mathcal{P}_N.$$

After upper bounds are established, we proceed with the lower bounds as follows. Assume that M achieves its absolute minimum at some t_*, then we use again the fact that $M'(t_*) = 0$ to obtain:

$$\frac{V_m(t_*)M(t_*)}{K_{m_1}(t_*) + M(t_*)} = \frac{V_S(t_*)K_1(t_*)^n}{K_1^n(t_*) + P_N(t_*)^n} \geq \frac{v_S\kappa_1^n}{\kappa_1^n + \mathcal{P}_N^n}.$$

As we did before:

$$\frac{V_m(t_*)M(t_*)}{K_{m_1}(t_*) + M(t_*)} \geq \frac{v_m M(t_*)}{\kappa_{m_1} + M(t_*)}$$

lets define the increasing and bijective function

$$\hat{b}_M(x) := \frac{v_m x}{\kappa_{m_1} + x},\ \hat{b}_M^{-1}(y) := \frac{\kappa_{m_1} y}{v_m - y}$$

this inverse is increasing too, thus:

$$M(t_*) \geq \hat{b}_M^{-1}\left(\frac{v_S\kappa_1^n}{\mathcal{K}_1^n + \mathcal{P}_N^n}\right) := \mathfrak{m}.$$

This shows that $M_1(t) \geq \mathfrak{m}$ for some positive constant $\mathfrak{m}$. In the same way, we find a lower bound $\mathfrak{p}_0$ for P_0 using the fact that

$$\frac{V_1(t_*)P_0(t_*)}{K_1(t_*) + P_0(t_*)} = K_s(t_*)M(t_*) + \frac{V_2(t_*)P_1(t_*)}{K_2(t_*) + P_1(t_*)} \geq \kappa_s\mathfrak{m}.$$

We know that

$$\frac{V_1(t_*)P_0(t_*)}{K_1(t_*) + P_0(t_*)} \geq \frac{v_1 P_0(t_*)}{\kappa_1 + P_0(t_*)},$$

this function is increasing and its inverse is also increasing:

$$\hat{b}_1(x) := \frac{v_1 x}{\kappa_1 + x} \to \hat{b}_1^{-1}(y) := \frac{\kappa_1 y}{v_1 - y},$$

therefore, it is defined $\mathfrak{p}_0 := \hat{b}_1^{-1}(\kappa_s\mathfrak{m})$.

Next, suppose that P_1 achieves its absolute minimum at t_*, then

$$P_1(t_*)\left(\frac{V_2(t_*)}{K_2(t_*)+P_1(t_*)}+\frac{V_3(t_*)}{K_3(t_*)+P_1(t_*)}\right)>\frac{V_1(t_*)P_0(t^*)}{K_1(t_*)+P_0(t_*)}\geq\frac{v_1\mathfrak{p}_0}{\mathcal{K}_1+\mathfrak{p}_0}>0$$

which yields the existence of a positive lower bound $\mathfrak{p}_1 := \frac{v_1\mathfrak{p}_0}{\mathcal{K}_1+\mathfrak{p}_0}$. Finally, positive lower bounds for P_2 and P_N are obtained by means of the function $Q = P_2 + P_N$. Indeed, if Q achieves its absolute minimum at some t_*, then

$$P_2(t_*)\left(\frac{V_4(t_*)}{K_4(t_*)+P_2(t_*)}+\frac{V_d(t_*)}{K_d(t_*)+P_2(t_*)}\right)\geq\frac{v_3\mathfrak{p}_1}{\mathcal{K}_3+\mathfrak{p}_1}$$

and we deduce that $P_2(t_*)$ cannot be arbitrarily small. As before, using the fact that $P_2' \geq -CP_2$ it is seen that $P_2(t) \geq e^{-CT}P_2(t_*)$ and the conclusion follows. This, in turn, yields a lower bound $\mathfrak{p}_N > 0$ for P_N.

4 Main Theorem

We are already in conditions of defining the open set $\Omega \subset \mathcal{K}^\circ$ as

$$\Omega := \{(M, P_0, P_1, P_2, P_N) \in C_T : \mathfrak{m} < M(t) < \mathcal{M}, \mathfrak{p}_0 < P_0(t) < \mathcal{P}_0,$$
$$\mathfrak{p}_1 < P_1(t) < \mathcal{P}_1, \mathfrak{p}_2 < P_2(t) < \mathcal{P}_2, \mathfrak{p}_N < P_N(t) < \mathcal{P}_N\}.$$

Theorem 4.1 *Assume that the previous conditions (3.1), (3.2), (3.3) and (3.4) hold. Then problem (1) has at least one positive $T-$periodic solution.*

Proof. In the previous section, the first condition of the continuation theorem was verified. It remains to prove that *b*) and *c*) are fulfilled as well. With this aim, set $\mathcal{Q} := \Omega\cap\mathbb{R}^5$ and recall that the function $\phi : \overline{\mathcal{Q}} \to \mathbb{R}^5$ is defined by $\phi(x) = \overline{Nx}$. We claim that each coordinate ϕ_j has different signs at the corresponding opposite faces of $\mathcal{Q}$.

Indeed, compute for example $\phi_1(\mathcal{M}, P_0, P_1, P_2, P_N)$ and $\phi_1(\mathfrak{m}, P_0, P_1, P_2, P_N)$ for $\mathfrak{p}_j \leq P_j \leq \mathcal{P}_j$:

$$\phi_1(\mathcal{M}, P_0, P_1, P_2, P_N) = \frac{1}{T}\int_0^T\left(\frac{V_S(t)K_1(t)^n}{K_1^n(t)+P_N}-\frac{V_m(t)\mathcal{M}}{K_{m_1}(t)+\mathcal{M}}\right)dt$$
$$< \mathcal{V}_S - \frac{v_m\mathcal{M}}{\mathcal{K}_{m_1}+\mathcal{M}} = 0,$$
$$\phi_1(\mathfrak{m}, P_0, P_1, P_2, P_N) = \frac{1}{T}\int_0^T\left(\frac{V_S(t)K_1(t)^n}{K_1^n(t)+P_N}-\frac{V_m(t)\mathfrak{m}}{K_{m_1}(t)+\mathfrak{m}}\right)dt$$
$$> \frac{v_S\kappa_1^n}{\mathcal{K}_1^n+\mathcal{P}_N^n}-\frac{\mathcal{V}_m\mathfrak{m}}{\kappa_{m_1}+\mathfrak{m}}\geq 0$$

provided that $\mathfrak{m}$ is small enough. In the same way, making the lower bounds smaller if necessary, we deduce that

$$\phi_2(M, \mathcal{P}_0, P_1, P_2, P_N) < 0 < \phi_2(M, \mathfrak{p}_0, P_1, P_2, P_N),$$

$$\phi_3(M, P_0, \mathcal{P}_1, P_2, P_N) < 0 < \phi_3(M, p_0, \mathfrak{p}_1, P_2, P_N),$$

$$\phi_4(M, P_0, P_1, \mathcal{P}_2, P_N) < 0 < \phi_4(M, p_0, p_1, \mathfrak{p}_2, P_N),$$
$$\phi_5(M, P_0, P_1, P_2, \mathcal{P}_N) < 0 < \phi_5(M, p_0, p_1, p_2, \mathfrak{p}_N).$$

Thus, condition $b)$ of the continuation theorem is verified. Moreover, we may define a homotopy as follows. Consider the center of $\mathcal{Q}$ given by

$$\wp := \left(\frac{\mathcal{M}+\mathfrak{m}}{2}, \frac{\mathcal{P}_0+\mathfrak{p}_0}{2}, \frac{\mathcal{P}_1+\mathfrak{p}_1}{2}, \frac{\mathcal{P}_2+\mathfrak{p}_2}{2}, \frac{\mathcal{P}_N+\mathfrak{p}_N}{2}\right)$$

and the function $\mathcal{H} : \overline{\mathcal{Q}} \times [0;1] \to \mathbb{R}^5$ given by

$$\mathcal{H}(x, \lambda) = (1-\lambda)(\wp - x) + \lambda\phi.$$

We need to verify that $\mathcal{H}$ does not vanish at $\partial\mathcal{Q}$. To this end, suppose, for example, that $\mathcal{H}(\mathcal{M}, P_0, P_1, P_2, P_N) = 0$ for some $\hat{\lambda} \in [0;1]$, then

$$0 = \mathcal{H}_1(\mathcal{M}, \hat{\lambda}) = (1-\hat{\lambda})\underbrace{\left(\frac{\mathcal{M}+\mathfrak{m}}{2} - \mathcal{M}\right)}_{<0} + \hat{\lambda}\underbrace{\phi_1(\mathcal{M}, P_0, P_1, P_2, P_N)}_{<0} < 0,$$

which is a contradiction. All the remaining cases follow in an analogous way. By the homotopy invariance of the Brouwer degree, it follows that

$$deg_B(\phi, \mathcal{Q}, 0) = deg_B(\wp - I, \mathcal{Q}, 0) = (-1)^5 \neq 0.$$

This proves the third condition of the continuation theorem and, therefore, the existence of a T-periodic solution is deduced.

5 Conclusion

Topological degree was used for proving existence of stable equilibrium in a generic model of circadian cycle. This theory allowed to demonstrate the existence of positive periodic solutions when parameters are replaced by fixed periodic functions. The relevance of finding periodic solutions in biological models relies mainly on the fact that periodic functions represent natural cycles, such as hormonal processes.

We show that topological degree can be successfully applied to find positive periodic orbits for some of these models in the non-autonomous case. It is worthy mentioning that, for diverse biological cycles, the behaviour is characterized by models with periodic parameters; thus, the present paper provides a useful mathematical tool to understand such models.

For future work, it would be interesting to consider a more general situation, in which the parameters are not periodic but almost-periodic functions, which attracted the attention of many researchers in the last decades. Here, the topological degree cannot be used anymore because of the lack of compactness of the associated operator; thus, a different approach is required, such as the use of fixed points in cones under monotonicity conditions that avoid the compactness assumption.

Acknowledgment

This work was partially supported by project UBACyT 20020120100029BA. I am grateful to Dr. Pablo Amster for his helpful comments.

References

[1] Amster, P. *Topological Methods in the Study of Boundary Value Problems.* New York: Springer, 2014.

[2] Amster, P. and Idels, L. Existence theorems for some abstract nonlinear non-autonomous systems with delays. *Commun. Nonlinear Sci. Numer. Simulat.* **19** (2014) 2974–2982.

[3] Goldbeter, A. A model for circadian oscillations in Drosophila Period Protein (PER). *Biological Sciences* **261** (1362) (1995) 319–324.

[4] Gonze, D., Leloup, J.C. and Goldbeter, A. Theoretical models for circadian rhythms in Neurospora and Drosophila. *C.R. Acad. Sci.* Paris, Sciences de la vie / Life Sciences **323** (2000) 57–67.

[5] Griffith, J. S. Mathematics of Cellular Control Processes I. Negative Feedback to One Gene *J. Theoret. Biol.* **20** (1968) 202–208.

[6] Majercak, J., Sidote, D., Hardin, P.E. and Edery, I. How a Circadian Clock Adapts to Seasonal Decreases in Temperature and Day Length *Neuron* **24** (1999) 219–230.

[7] Paetkau, V., Edwards, R. and Illner, R. A Model for generating circadian rithm by coupling ultradian oscillators. *Theoretical Biology and Medical Modelling* **3** (12) (2006) 1–10.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 10–20

Solution of 2D Fractional Order Integral Equations by Bernstein Polynomials Operational Matrices

M. Asgari [1], R. Ezzati [1,*] and H. Jafari [2,3]

[1] *Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*
[2] *Department of Mathematical Sciences, University of South Africa, UNISA003, South Africa*
[3] *Department of Mathematics, University of Mazandaran, Babolsar, Iran*

Received: October 11, 2017; Revised: December 12, 2018

Abstract: In this paper, we construct a new two-dimensional Bernstein polynomials operational matrix for solving 2-dimensional fractional order Volterra integral equations (2DFOVIE). By using this operational matrix, we reduce the original problem to a linear or nonlinear system of algebraic equations. We present some numerical examples to show the efficiency of the proposed method.

Keywords: *two-dimensional fractional integral equations; two-dimensional Bernstein polynomials; block pulse operational matrix; operational matrix of integration.*

Mathematics Subject Classification (2010): 26A33, 45G05.

1 Introduction

In the last few decades, various engineering and scientific problems involving fractional calculus were discussed. For example, electrochemical process [1, 2], earthquakes [3], economics [4], bioengineering [5], orthogonal splin collocation [6] and fractional optimal control problems [7, 8]. There are several analytical and numerical methods for solving one-dimensional and two-dimensional differential and integral equations of fractional order such as the Adomian decomposition [9], Variational iteration method [10, 11], Transform method [12], Homotopy perturbation method [13], and the methods of Harr and Chebyshev wavelet [14, 15] and Bernstein polynomials [16, 17].

The Bernstein polynomials play a conspicuous role in several areas of mathematics. These polynomials have been commonly used in the solution of differential equations,

* Corresponding author: mailto:ezati@kiau.ac.ir

integral equations, fractional optimal control problems and approximation theory [7, 8, 17–23]. In this work, we consider the following type of 2DVIEFO

$$u(x,y) - I_0^q u^p(x,y) = g(x,y), \quad q = (\alpha, \beta) \in (0,\infty) \times (0,\infty), \tag{1}$$

where $g(x,y)$ is a known function and $I_0^q u(x,y)$ is the left-sided mixed Riemann-Liouville integral of order q which is defined as [24]

$$(I_0^q u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1} u(\xi,\tau)\, d\tau\, d\xi. \tag{2}$$

Note: For $\alpha > 0$, the Riemann-Liouville integral (I^α) on the Lebesgue space $L^1[a,b]$ is defined as

$$(I_0^\alpha u)(t) = (I^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau. \tag{3}$$

In particular, for (2), we have

1. $(I_0^0 u)(x,y) = u(x,y)$,
2. $(I_0^\sigma u)(x,y) = \int_0^x \int_0^y u(\xi,\tau)\, d\tau\, d\xi, \quad (x,y) \in J,\ \sigma = (1,1)$,
3. $(I_0^r u)(x,0) = (I_0^r u)(0,y) = 0, \qquad x \in [0,a], y \in [0,b]$,
4. $I_0^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+\alpha)\Gamma(1+\omega+\beta)} x^{\lambda+\alpha} y^{\omega+\beta}, \qquad (x,y) \in J, \quad \lambda, \omega \in (-1,\infty)$.

We are looking for $u \in L^1(J)$, $J := [0,a] \times [0,b]$. The existence and uniqueness of (1) is investigated in [25].

We want to obtain the numerical solution of (1) by using two-dimensional Bernstein polynomials and block pulse functions. The rest of this paper is organized as follows. First, we briefly review some general concepts concerning one-dimensional and two-dimensional Bernstein polynomials, block pulse functions and derive the Bernstein polynomials operational matrix of two-dimensional integration of fractional order. In Section 3, the method is applied to solve linear or nonlinear 2DVIEFO. Section 4 exhibits an error estimation for the presented method. Section 5 illustrates several numerical examples to show the convergence and accuracy of the proposed method.

2 Bernstein Polynomials and Block Pulse Functions

2.1 One dimensional Bernstein polynomials (1D-BPs)

The nth degree Bernstein polynomials (BPs) on the interval $[0,1]$ are defined as

$$B_{i,n}(\tau) = \binom{n}{i} \tau^i (1-\tau)^{n-i}, \quad 0 \le i \le n. \tag{4}$$

The BPs on $[0,1]$ have the following properties [7]:

1. $B_{i,n}(\tau) \ge 0, i = 0, 1, \ldots, n, \qquad \tau \in [0,1]$,
2. $\sum_{i=0}^{n} B_{i,n}(t) = 1$,

3. $B_{i,n}(\tau) = (1-\tau)B_{i,n-1}(\tau) + \tau B_{i-1,n-1}(\tau), \quad i = 0, 1, \dots, n,$
4. $B_{i,n}(\tau) = \sum_{k=0}^{n-i}(-1)^k \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ k \end{pmatrix} \tau^{i+k}, \quad i = 0, 1, \dots, n.$

Theorem 2.1 *[26] Suppose that $H = L^2[0,1]$ is a Hilbert space with the inner product and $X = Span\{B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)\}$ is a closed subspace with finite dimensions, therefore X is a complete subspace of H. So, if $u \in H$ is an arbitrary element, it has a unique best approximation out of X such as x_0, that is*

$$\exists \ x_0 \in Y \quad s.t. \ \forall x \in X, \ \parallel u - x_0 \parallel_2 \leq \parallel u - x \parallel_2, \tag{5}$$

where $\parallel u \parallel_2 = \sqrt{< u, u >}$, $< u, v > = \int_0^1 u(\tau)v(\tau)\, d\tau$.

Thus, there exist unique coefficients $c_0, c_1, \dots, c_n$ such that

$$u(t) \simeq x_0 = \sum_{i=0}^{n} c_i B_{i,n}(t) = c^T \varphi(t), \tag{6}$$

where $c^T = [c_0, c_1, \dots, c_n]$, $\varphi(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \dots, B_{n,n}(\tau)]^T$.

Lemma 2.1 *If $\varphi_n(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \dots, B_{n,n}(\tau)]^T$ is a complete basis, then $\varphi_n(t) = AT_n(t)$, where A is an $(n+1) \times (n+1)$ upper triangular matrix with*

$$a_{i+1,j+1} = \begin{cases} (-1)^{j-i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ j-i \end{pmatrix}, & i \leqslant j, \\ 0, & i > j, \end{cases} \tag{7}$$

for $i, j = 0, 1, \dots, n$ and $T_n(\tau) = [1, \tau, \tau^2, \dots, \tau^n]^T$.

2.2 BPF and operational matrix

A set of BPF on $[0, 1)$ is defined as follows:

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \leq t < \frac{i+1}{m}, \quad i, j = 0, 1, \dots, m-1, \\ 0, & otherwise. \end{cases} \tag{8}$$

The above functions are orthogonal and disjoint, i.e.
$b_i(t)b_j(t) = \begin{cases} b_i(t) & i = j, \\ 0 & i \neq j, \end{cases}$ and $\int_0^1 b_i(t)b_j(t)\, dt = \frac{1}{m}\delta_{ij}$, where δ_{ij} is the Kronecker delta.

If $B_m(\tau) = [b_0(\tau), b_1(\tau), \dots, b_{m-1}(\tau)]^T$, the block pulse operational matrix of the fractional order integration F^α is [27]

$$I^\alpha B_m(\tau) = F^\alpha B_m(\tau),$$

where

$$F^\alpha = \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

with $\xi_s = (s+1)^{\alpha+1} - 2s^{\alpha+1} + (s-1)^{\alpha+1}$.

2.3 Operational matrix for fractional integral equation(1D)

If

$$\varphi(\tau) = \varphi_n(\tau) = [B_{0,n}(\tau), B_{1,n}(\tau), \dots, B_{n,n}(\tau)]^T,$$

then for the fractional integral equation (3), we have

$$I^\alpha \varphi_n(\tau) = P^\alpha \varphi_n(\tau), \tag{10}$$

with $n = m - 1$, the Bernstein polynomial might be expanded into an m-term BPF as

$$\varphi_m(\tau) = \phi_{m\times m} B_m(\tau), \tag{11}$$

now

$$I^\alpha \varphi_m(\tau) = I^\alpha \phi_{m\times m} B_m(\tau) = \phi_{m\times m} I^\alpha B_m(\tau) = \phi_{m\times m} F^\alpha B_m(\tau). \tag{12}$$

From equations (11) and (12), we have

$$I^\alpha \varphi_m(\tau) = \phi_{m\times m} F^\alpha B_m(\tau) = \phi_{m\times m} F^\alpha \phi^{-1}_{m\times m} \varphi_m(\tau). \tag{13}$$

Therefore,

$$P^\alpha_{m\times m} = \phi_{m\times m} F^\alpha \phi^{-1}_{m\times m}. \tag{14}$$

P^α is called an operational matrix for fractional integration based on the Bernstein polynomials [28].

2.4 Two-dimensional Bernstein polynomials (2D-BPs)

The Bernstein polynomials of degree mn on the interval $[0,1]\times[0,1]$ are defined by

$$B_{(i,m)(j,n)}(\mu,\nu) = \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \mu^i (1-\mu)^{m-i} \nu^j (1-\nu)^{n-j} \tag{15}$$

for $i = 0, 1, \dots, m,\ \ j = 0, 1, \dots, n$.

Similar to the 1D case, we have [19]:

1. $B_{(i,m)(j,n)}(\mu,\nu) \geq 0$,
2. $B_{(i,m)(j,n)}(\mu,\nu) = B_{(i,m)}(\mu) B_{(j,n)}(\nu)$,
3. $B_{(i,m)(j,n)}(\mu,\nu) = \sum_{k=0}^{m-i} \sum_{t=0}^{n-j} (-1)^{r+t} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} m-i \\ k \end{pmatrix} \begin{pmatrix} n-j \\ t \end{pmatrix} \mu^{i+k} \nu^{j+t}$,
4. $Q = < B_{(i,m)(j,n)}(\mu,\nu), B_{(k,m)(t,n)}(\mu,\nu) >$

$$= \int_0^1 \int_0^1 B_{(i,m)(j,n)}(\mu,\nu) B_{(k,m)(t,n)}(\mu,\nu)\, d\mu d\nu = \frac{\begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} \begin{pmatrix} n \\ t \end{pmatrix}}{(2m+1)(2n+1) \begin{pmatrix} 2m \\ i+k \end{pmatrix} \begin{pmatrix} 2n \\ j+t \end{pmatrix}},$$

for $i, k = 0, 1, \dots, m,\ \ j, t = 0, 1, \dots, n$.

Now, if we define $(m+1)\times(n+1)$-vector

$$\begin{aligned} \varphi_{mn}(\mu,\nu) = [&B_{(0,m)(0,n)}(\mu,\nu), \dots, B_{(0,m)(n,n)}(\mu,\nu), \\ &\dots, B_{(m,m)(0,n)}(\mu,\nu), \dots, B_{(m,m)(n,n)}(\mu,\nu)]^T, \end{aligned} \tag{16}$$

where $(\mu,\nu) \in [0,1]\times[0,1]$, then $\varphi_{mn}(\mu,\nu)$ is a complete basis.

2.5 Function expansion with 2D-BPs

We expand $u(\mu,\nu) \in L^2([0,1]\times[0,1])$ by 2D-BPs as

$$u(\mu,\nu) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij}\varphi_{ij}(\mu,\nu) \simeq \sum_{i=0}^{m}\sum_{j=0}^{n} u_{ij}\varphi_{ij}(\mu,\nu) = U^T\varphi(\mu,\nu) = \varphi^T(\mu,\nu)U, \quad (17)$$

where $\varphi(\mu,\nu)$ and U are $(m+1)(n+1)$ vectors. Components u_{ij} of U are obtained as

$$u_{ij} =< u(\mu,\nu), \varphi(\mu,\nu) >= \int_0^1\int_0^1 u(\mu,\nu)B_{(i,m)(j,n)}(\mu,\nu)\,d\mu\,d\nu. \quad (18)$$

Similarly, let $k(\mu,\nu,s,t)$ be defined on $[0,1]\times[0,1]\times[0,1]\times[0,1]$. It can be expanded with respect to 2D-BPs as

$$k(\mu,\nu,s,t) \simeq \varphi^T(\mu,\nu)K\psi(s,t), \quad (19)$$

where $\varphi(\mu,\nu)$ and $\psi(s,t)$ are 2D-BPs vectors of dimension $(m_1+1)(n_1+1)$ and $(m_2+1)(n_2+1)$, respectively, and K is the $(m_1+1)(n_1+1)\times(m_2+1)(n_2+1)$ two-dimensional Bernstein polynomials coefficient matrix.

2.6 Operational matrix for fractional integral equation(2D)

Suppose $B_{(i,m)}(\mu) = A_1T_m(\mu)$ and $B_{(j,n)}(\nu) = A_2T_n(\nu)$. Then

$$\varphi_{mn}(\mu,\nu) = MT_{mn}(\mu,\nu),$$

where

$$T_{mn}(\mu,\nu) = [1,\nu,\nu^2,\ldots,\nu^n,\mu,\mu\nu,\ldots,\mu\nu^n,\ldots,\mu^m,\mu^m\nu,\ldots,\mu^m\nu^n]^T,$$

and $M = A_1\otimes A_2$ and $\otimes$ denotes the Kronecker product.
Now, we present two-dimensional Bernstein polynomials operational matrices of fractional mode. Let $\varphi_{mn}(\mu,\nu)$ be defined as in (16). The fractional integration of the $\varphi_{mn}(\mu,\nu)$ can be approximately obtained as

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}\varphi_{mn}(\xi,\tau)\,d\xi\,d\tau \simeq P^r\varphi_{mn}(\mu,\nu), \quad (20)$$

where P^r is a $(m+1)(n+1)\times(m+1)(n+1)$ matrix and is called an operational matrix. Let operational matrices P^α and P^β satisfy (14), i.e.

$$\begin{aligned} I^\alpha\varphi_m(\mu) &= P^\alpha\varphi_m(\mu) = \phi_{m\times m}F^\alpha\phi^{-1}_{m\times m}\varphi_m(\mu),\\ I^\beta\varphi_n(\nu) &= P^\beta\varphi_n(\nu) = \phi_{n\times n}F^\beta\phi^{-1}_{n\times n}\varphi_n(\nu). \end{aligned} \quad (21)$$

From the disjointness property of two-dimensional Bernstein polynomials, we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}\varphi_{mn}(\xi,\tau)\,d\xi\,d\tau = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}\varphi_m(\xi)\,d\xi \times\frac{1}{\Gamma(\beta)}\int_0^y (y-\tau)^{\beta-1}\varphi_n(\tau)\,d\tau.$$

By using (21), we have

$$P^r = P^\alpha\otimes P^\beta. \quad (22)$$

2.7 Product operational matrix

In view of (1), we have $u^p(x,y)$. So, we need to evaluate the product of $\varphi_(x,y)$ and $\varphi^T(x,y)$, which is called the product matrix.

Lemma 2.2 *Suppose that $C_{(m+1)(n+1)}$ is an arbitrary vector. The operational matrix of product $\hat{C}_{(m+1)(n+1)\times(m+1)(n+1)}$ using BPs can be given as follows [29] :*

$$\varphi(x,y)\varphi^T(x,y)C \simeq \hat{C}^T\varphi(x,y). \tag{23}$$

Corollary 2.1 *Suppose $u(x,y) = U^T\varphi(x,y) = \varphi^T(x,y)U$ and $\hat{U}$ is the operational matrix of product. Then*

$$(u(x,y))^k = \varphi^T(x,y)\overline{U}_k, \tag{24}$$

where $k \in N$ and $\overline{U}_k = \hat{U}^{k-1}U$.

Proof. By using Lemma 2.2, for $k = 2$, we get

$$(u(x,y))^2 = U^T\varphi(x,y)\varphi^T(x,y)U = \varphi^T(x,y)\hat{U}U = \varphi^T(x,y)\overline{U}_2.$$

Also, if $k = 3$,

$$(u(x,y))^3 = U^T\varphi(x,y)\varphi^T(x,y)\hat{U}U = \varphi^T(x,y)\hat{U}^2U = \varphi^T(x,y)\overline{U}_3.$$

So, by induction we have

$$(u(x,y))^k = U^T\varphi(x,y)\varphi^T(x,y)\hat{U}^{k-2}U = \varphi^T(x,y)\hat{U}^{k-1}U = \varphi^T(x,y)\overline{U}_k.$$

3 Solving 2DFOVIE

In this section, two-dimensional Bernstein polynomials are applied to solve equation(1). Using the procedures mentioned in Section 2, we approximate functions $(u(x,y))^p$, $k(x,y,s,t)$ and $f(x,y)$ as follows:

$$\begin{aligned} (u(x,y))^p &= \varphi^T(x,y)\overline{U}_p = \overline{U}_p^T\varphi(x,y), \\ f(x,y) &= \varphi^T(x,y)F = F^T\varphi(x,y), \\ k(x,y,s,t) &= \varphi^T(x,y)K\varphi(x,y), \end{aligned} \tag{25}$$

where the $(m+1)(n+1)\times 1$ vectors $\overline{U}_p$, F and $(m+1)(n+1)\times(m+1)(n+1)$ matrix K are 2D-BPs coefficients of $(u(x,y))^p$, $f(x,y)$ and $k(x,y,s,t)$ respectively. Substituting equations(25) in equation(1), we have:

$$\varphi^T(x,y)U - \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}\varphi^T(x,y)K\varphi(s,t)\varphi^T(s,t)\overline{U}_p\,dt\,ds = \varphi^T(x,y)F.$$

By using (23), we get

$$\varphi^T(x,y)U - \frac{\varphi^T(x,y)K\,\hat{\overline{U}}_p^T}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1}\varphi(s,t)\,dt\,ds = \varphi^T(x,y)F.$$

From equation(20) and the above equation, we obtain

$$\varphi^T(x,y)U - \varphi^T(x,y)K\,\hat{\bar{U}}_p^T P^r\varphi(x,y) = \varphi^T(x,y)F,$$

or

$$U - K\,\hat{\bar{U}}_p^T P^r\varphi(x,y) = F. \tag{26}$$

Now, we collocate equation(26) in $(m+1)(n+1)$ Newton-Cotes nodes as

$$x_i = \frac{2i-1}{2(m+1)}\ ,\qquad y_j = \frac{2j-1}{2(n+1)}\ ,\qquad i = 1,2,\ldots,m+1,\qquad j = 1,2,\ldots,n+1.$$

So, we have a linear($p=1$) or nonlinear($p \geq 1$) algebraic system

$$U - B\psi = F, \tag{27}$$

where $B = K\,\hat{\bar{U}}_p^T P^r$, and

$$\psi = [\varphi(x_1,y_1),\varphi(x_1,y_2),\ldots,\varphi(x_1,y_{n+1}),\ldots\varphi(x_{m+1},y_1),\ldots,\varphi(x_{m+1},y_{n+1})]^T.$$

4 Error analysis

Theorem 4.1 *Suppose $u(x,y)$ is an exact solution of the equation (1) and $\hat{u}(x,y)$ shows its approximate solution by Bernstein polynomials, and*

1. $|\,(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)\,| < C,$

2. $(u(x,y))^p$ *is a Lipschitz continuous function, i.e.*

$$|(u(x,y))^p - (\hat{u}(x,y))^p| \leq L|u(x,y) - \hat{u}(x,y)|,$$

where L is a Lipschitz constant

3. $m_1 = m_2 = m$.

Then $\hat{u}(x,y)$ converges to $u(x,y)$, if $0 < \dfrac{LC}{\Gamma(\alpha)\Gamma(\beta)} < 1.$

Proof.

$$\|\, u(x,y) - \hat{u}(x,y)\,\|_\infty = \max_{0\leq x,y\leq 1} |u(x,y) - \hat{u}(x,y)|$$

$$= \max_{0\leq x,y\leq 1} \left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)((u(\xi,\tau))^p - (\hat{u}(\xi,\tau))^p)d\xi d\tau\right|$$

$$\leq \max_{0\leq x,y\leq 1} \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |(x-\xi)^{\alpha-1}(y-\tau)^{\beta-1}k(x,y,\xi,\tau)||(u(\xi,\tau))^p - (\hat{u}(\xi,\tau))^p|d\xi d\tau$$

$$\leq \max_{0\leq x,y\leq 1} \frac{CL}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |u(\xi,\tau) - \hat{u}(\xi,\tau)|\,d\xi d\tau$$

$$\leq \frac{CLxy}{\Gamma(\alpha)\Gamma(\beta)}\,\|\, u(\xi,\tau) - \hat{u}(\xi,\tau)\,\|_\infty \leq \frac{CL}{\Gamma(\alpha)\Gamma(\beta)}\,\|\, u(\xi,\tau) - \hat{u}(\xi,\tau)\,\|_\infty .$$

Therefore we get

$$\| u(x,y) - \hat{u}(x,y) \|_\infty \leq \frac{CL}{\Gamma(\alpha)\Gamma(\beta)} \| u(\xi,\tau) - \hat{u}(\xi,\tau) \|_\infty . \tag{28}$$

Equation (28) shows that if $0 < \frac{LC}{\Gamma(\alpha)\Gamma(\beta)} < 1$, then $\| u(\xi,\tau) - \hat{u}(\xi,\tau) \|_\infty \longrightarrow 0$.

5 Numerical Examples

To demonstrate the validity and applicability of this scheme, we use the present method for the following four examples. In view of (2), we rewrite (1) in the following form of 2DFOVIE:

$$u(x,y) - \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \int_0^y (x-\xi)^{\alpha-1}(y-\tau)^{\beta-1} k(x,y,\xi,\tau)\, u^p(\xi,\tau)\, d\xi\, d\tau = g(x,y) \tag{29}$$

Now, for different values of α, β, $k(x,y,\xi,\tau)$, p and $g(x,y)$, we solve (29).

Example 5.1 Let $\alpha = \frac{5}{3}$, $\beta = \frac{7}{3}$, $k(x,y,\xi,\tau) = \xi\tau\sqrt{xy}$, $p = 1$ and $g(x,y) = x^3(y^2 - y) - \frac{x^{\frac{17}{3}} y^{\frac{13}{3}} \sqrt{xy}(9y-16)}{5000}$ The exact solution is $u(x,y) = x^3(y^2 - y)$. We applied the proposed method to solve this example for various values of m and n. Also, we compare the numerical results with the exact solution. The results are tabulated in Table 1.

$x = y$	$m = n = 1$	$m = n = 2$	$m = n = 3$
0.0	6.292×10^{-6}	3.091×10^{-6}	7.394×10^{-6}
0.1	7.702×10^{-5}	3.942×10^{-4}	4.401×10^{-5}
0.2	1.261×10^{-3}	2.814×10^{-3}	9.460×10^{-5}
0.3	5.645×10^{-3}	4.417×10^{-3}	1.492×10^{-4}
0.4	1.533×10^{-2}	3.212×10^{-3}	2.022×10^{-4}
0.5	3.121×10^{-2}	1.926×10^{-4}	2.515×10^{-4}
0.6	5.180×10^{-2}	3.579×10^{-3}	2.919×10^{-4}
0.7	7.198×10^{-2}	4.819×10^{-3}	3.020×10^{-4}
0.8	8.187×10^{-2}	2.975×10^{-3}	2.257×10^{-4}
0.9	6.556×10^{-2}	5.010×10^{-4}	5.289×10^{-5}

Table 1: The maximum absolute errors in Example 5.1.

Example 5.2 Let $\alpha = \beta = \frac{5}{2}$, $k(x,y,\xi,\tau) = \sqrt{xy\xi}$, $p = 2$ and $f(x,y) = x\sqrt{y} - \dfrac{1}{420} x^{\frac{11}{2}} y^4$ with the exact solution $u(x,y) = x\sqrt{y}$. The maximum absolute errors are shown in Table 2.

Example 5.3 Let $\alpha = \frac{5}{2}$, $\beta = \frac{7}{2}$, $k(x,y,\xi,\tau) = (y+\xi)e^{-2\tau}$, $p = 2$ and $f(x,y) = xe^y - \dfrac{1024 x^{\frac{9}{2}} y^{\frac{7}{2}} (6x + 11y)}{1091475\pi}$ with the exact solution $u(x,y) = xe^y$. The maximum absolute errors are shown in Table 3.

$x=y$	$m=n=1$	$m=n=2$	$m=n=3$
0.0	5.603×10^{-5}	2.592×10^{-6}	1.327×10^{-5}
0.1	3.064×10^{-3}	1.273×10^{-3}	1.611×10^{-3}
0.2	4.032×10^{-3}	4.748×10^{-3}	1.497×10^{-3}
0.3	1.220×10^{-2}	4.694×10^{-3}	1.253×10^{-3}
0.4	1.818×10^{-2}	1.276×10^{-3}	3.797×10^{-3}
0.5	2.009×10^{-2}	3.948×10^{-3}	4.052×10^{-3}
0.6	1.666×10^{-2}	8.789×10^{-3}	1.451×10^{-3}
0.7	6.937×10^{-3}	1.070×10^{-2}	2.806×10^{-3}
0.8	9.786×10^{-3}	6.921×10^{-3}	5.652×10^{-3}
0.9	3.410×10^{-2}	5.490×10^{-3}	2.132×10^{-3}

Table 2: The maximum absolute errors in Example 5.2.

$x=y$	$m=n=1$	$m=n=2$	$m=n=4$
0.0	9.890×10^{-5}	3.578×10^{-4}	7.921×10^{-4}
0.1	6.034×10^{-3}	6.324×10^{-4}	9.468×10^{-4}
0.2	1.666×10^{-3}	3.307×10^{-4}	1.104×10^{-3}
0.3	9.537×10^{-3}	1.114×10^{-3}	1.266×10^{-3}
0.4	2.339×10^{-2}	8.532×10^{-4}	1.424×10^{-3}
0.5	3.514×10^{-2}	6.995×10^{-4}	1.566×10^{-3}
0.6	3.935×10^{-2}	3.095×10^{-3}	1.673×10^{-3}
0.7	2.987×10^{-2}	5.105×10^{-3}	1.675×10^{-3}
0.8	3.150×10^{-4}	4.622×10^{-3}	1.370×10^{-3}
0.9	5.916×10^{-2}	1.445×10^{-3}	2.511×10^{-4}

Table 3: The maximum absolute errors in Example 5.3.

Example 5.4 As the last example, let $\alpha=\frac{3}{2}$, $\beta=\frac{5}{2}$, $k(x,y,\xi,\tau)=\sqrt{xy\tau}$, $p=2$ and $f(x,y)=\sqrt{y}(\frac{-1}{180}x^3y^{\frac{7}{2}}+\sqrt{\frac{x}{3}})$ The exact solution of this example is $u(x,y)=\frac{\sqrt{3xy}}{3}$. The maximum absolute errors are shown in Table 4. Also, the obtained numerical results are compared with the method of block pulse operational matrix (BPOM) proposed in [23, 30].

6 Conclusion

A new approach to obtain numerical solution of 2DFOVIE based on the operational matrices of Bernstein polynomials has been presented. With the help of the operational matrix of fractional integration P^r and the collocation method, the given 2DFOVIE is reduced to a linear or nonlinear system of algebraic equations. Illustrative examples show that the proposed method can be a suitable method for solving these equations. All of computations are done by *Mathematica* 9.

$x = y$	$m = n = 1$	$m = n = 2$	$m = n = 3$	$\frac{m_1=m_2=32}{BPOM}$
0.0	4.091×10^{-2}	1.701×10^{-2}	9.354×10^{-3}	9.386×10^{-3}
0.1	1.171×10^{-2}	4.572×10^{-3}	5.766×10^{-3}	1.561×10^{-2}
0.2	1.017×10^{-2}	1.183×10^{-2}	3.740×10^{-3}	8.812×10^{-3}
0.3	2.472×10^{-2}	9.513×10^{-3}	2.911×10^{-3}	1.630×10^{-2}
0.4	3.196×10^{-2}	1.934×10^{-3}	7.428×10^{-3}	8.239×10^{-3}
0.5	3.186×10^{-2}	7.003×10^{-3}	7.270×10^{-3}	1.410×10^{-2}
0.6	2.444×10^{-2}	1.382×10^{-2}	2.893×10^{-3}	7.665×10^{-3}
0.7	9.702×10^{-3}	1.545×10^{-2}	3.149×10^{-3}	1.430×10^{-2}
0.8	1.236×10^{-2}	9.258×10^{-3}	6.781×10^{-3}	7.091×10^{-3}
0.9	4.176×10^{-2}	6.980×10^{-3}	2.666×10^{-3}	1.260×10^{-2}

Table 4: The maximum absolute errors in Example 5.4.

References

[1] Ichise, M., Nagayanagi, Y. and Kojima, T. An analog simulation of non-integer order transfer functions for analysis of electrode processes. *J. Electronical. Chem. Interfacial Electrochem* **33** (1971) 253–265.

[2] Sun, H. H., Onaral, B. and Tsao, Y. Application of positive reality principle to metal electrode linear polarization phenomena. *IEEE Trans. Biomed. Eng.* BME-**31** (10) (1984) 664–674.

[3] He, J. H. Nonlinear oscillation with fractional derivative and its applications. In: *International Conference on Vibrating Engineering 98*, China, 1998, 288–291.

[4] Baillie, R. T. Long memory processes and fractional integration in econometrics. *J. Econometrics* **73** (1996) 50–59.

[5] Magin, R. L. Fractional calculus in bioengineering. *Crit. Rev. Biomed. Eng.* **32** (1) (2004) 91–104.

[6] Pandy, R. K., Singh, O. P. and Singh, V. K. Efficient algorithms to solve singular integral equations of Abel type. *Computers and Mathematics with Applications* **57** (4) (2009) 664–676.

[7] Jafari, H. and Tajadodi, H. Fractional order optimal control problems via the operational matrices of Bernstein polynomials. *U.P.B Sci. Bull.*, Series A **76** (3) (2014) 115–128.

[8] Alipour, M. and Rostamy, D. BPs operational matrices for solving time varying fractional optimal control problems. *The Journal of Mathematics and Computer Science* **6** (2013) 292–304.

[9] Kharrat, M. Closed-form solution of European option under fractional Heston model. *Nonlinear Dynamics and Systems Theory* **18** (2) (2018) 191–195.

[10] Mohammed, M.A. and Fadhel, F.S. Solution of two-dimensional fractional order Volterra integro-differential equations. *Joyrnal of AL-Nahrain university* **12** (4) (2009) 185–189.

[11] Das, S. Analytical solution of a fractional diffusion equation by variational iteration method. *Comput. Math. Appl.* **57** (2009) 483–487.

[12] Erturk, V. S., Momani, S. and Odibat, Z. Application of generalized differential transform method to multi-order fractional differential equations. *Comm. Nonlinear Sci. Numer. Simmulat.* **13** (2008) 1642–1654.

[13] Zhang, X., Liu, J., Zhao, J. and Tang, B. Homotopy perturbation method for two-dimensional time-fractional wave equation. *Appl. Math. Modelling* **38** (23) (2014) 5545–5552.

[14] Saeedi, H., Mollahasani, N., Mohseni moghadam, M. and Chuev, G. N. An operational Harr wavelet method for solving fractional Volterra integral equations. *Int. J. Appl. Math. Sci.* **21** (3) (2011) 535–547.

[15] Zhu, L. and Fan, Q. Solving fractional nonlinear Fredholm integro-differential by the second kind Chebyshev wavelet. *Comm. Nonlinear Sci. Numer. Simmulat.* **17** (2012) 2333–2341.

[16] Doha, E.H., Bhrawy, A.H. and Saker, M.A. Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations. *Appl. Math. Lett.* **24** (2011) 559–565.

[17] Maleknejad, K., Hashemizadeh, E. and Basirat, B. Computational method based on Bernstein operational matrices for nonlinear VolterraFredholmHammerstein integral equations. *Commun. Nonlinear Sci. Numer. Simul.* **17** (1) (2012) 52–61.

[18] Hosseini Shekarabi, F., Maleknejad, K. and Ezzati, R. Application of two-dimensional Bernstein polynomials for solving mixed VolterraFredholm integral equations. *Afr. Mat.* **26** (2015) 1237–1251.

[19] Maleknejad, K., Hashemizadeh, E. and Basirat, B. A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm integro-differential equations. *Math. Comput. Model.* **55** (2012) 1363–1372.

[20] Maleknejad, K., Hashemizadeh, E. and Ezzati, R. New approach to the numerical solution of Volterra integral equations by using Bernsteins approximation. *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 647–655.

[21] Mandal, B. N. and Bhattacharya, S. Numerical solution of some classes of integral equations using Bernstein polynomials. *Appl. Math. Comput.* **190** (2007) 707–716.

[22] Yousefi, S. A. and Behroozifar, M. Operational matrices of Bernstein polynomials and their applications. *Int.J. Syst. Sci.* **41** (6) (2010) 709–716.

[23] Jabari Sabeg, D., Ezzati, R. and Maleknejad, K. Solving Two-Dimensional Integeral Equations of Fractional Order by Using Operational Mtrix of Two-Dimentional Shifted Legendre Polynomials. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 297–306.

[24] Vityuk, A. N. and Golushkov, A. V. Existence of solutions of systems of partial differential equations of fractional order. *Nonlinear Oscil.* **7** (2004) 318–325.

[25] Abbas S., and Benchohra, M. Fractional order integral equations of two independent variables. *Appl. Math. Comput.* **227** (2014) 755–761.

[26] Kreyszig, E. *Introductory Functional Analysis with Applications.* New York: John Wiley and Sons, 1978.

[27] Kilicman, A. and Al Zhour, Z.A.A. Kronecker operational matrices for fractional calculus and some applications. *Appl. Math. Comput.* **187** (1) (2007) 250–265.

[28] Pandey, R.K., Bhardwaj, A. and Syam, M.I. An efficient method for solving fractional differential equations using Bernstein polynomials. *Journal of Fractional Calculus and Applications* **5** (1) (2014) 129–145.

[29] Asgari, M. and Ezzati, R. Using operational matrix of two-dimensional Bernstein polynomials for solving two-dimensional integral equations of fractional order. *Appl. Math. Comput.* **307** (2017) 290–298.

[30] Najafalizadeh, S. and Ezzati, R. Numerical methods for solving two-dimensional nonlinear integral equations of fractional order by using two-dimensional block pulse operational matrix. *Appl. Math. Comput.* **280** (20) (2016) 46-56.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 21–35

Singular Analysis of Reduced ODEs of Rotating Stratified Boussinesq Equations Through the Mirror Transformations

B.S. Desale [1*] and K.D. Patil [2]

[1] *Department of Mathematics, University of Mumbai, Kalina, Santacruz (East), Mumbai 400 098, India.*

[2] *Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon 425 001, India.*

Received: March 11, 2018; Revised: December 13, 2018

Abstract: In this paper we have considered the system of six coupled non-linear ordinary differential equations (ODEs), which arose in the reduction of uniformly stratified fluid contained in a rotating rectangular box of dimension $L \times L \times H$ which is completely integrable if the Rayleigh number $Ra = 0$. In our investigations, we have shown that there exits a regular mirror system near movable singularities of these integrable ODEs. Moreover, we have used the mirror system to prove the convergence of Laurent series solutions obtained by the Painlevé method.

Keywords: *mirror transformation; mirror system; Painlevé test.*

Mathematics Subject Classification (2010): 37K10, 34M55.

1 Introduction

In general, we believed that the differential system is integrable due to some sort of underlying linear structure(s). But, when it comes to this concept, it is never clear what does it mean. On the other hand the integrability of nonlinear system is quite ambiguous. In this connection many mathematicians started to work over the investigation of integrability of nonlinear system. In 1889, Sophie Kowalevski [12] proved the complete integrability of the system of ordinary differential equations (ODEs) governing the motion of a spinning top moving under the influence of gravity. In her study, she was seeking

* Corresponding author: mailto:bhausaheb.desale@mathematics.mu.ac.in

analytic solutions whose singularities are movable poles. This was done by substituting a Frobenius series into the system of ODEs. Then, few years later, that is in 1897, Paul Painlevé [6] classified first and second order algebraic differential equations whose solutions exist in the complex domain and are devoid of movable essential singularities or movable branch points. ODEs possessing this property are said to be of the Painlevé type. Painlevé test in view of partial differential equations is generally known as WTC (Weiss-Tabor-Carnevale, [7]) test, which is further modified by S. Kichenassamy and G. K. Srinivasan [3]. So far various properties are considered as indicator of integrability: solitons, the Lax pair, the Bäcklund transformations, the underlying Hamiltonian formulations, Hirota's bilinear representation, etc. The relation between these properties has yet to be understood.

In 1999–2000, Hu J. and Yan M. [8, 9] introduced the mirror transformation, which is a new tool used in the singularity analysis of ODEs. With the help of this method we constructed the mirror system of given PDEs or ODEs successfully; we could focus commonly at the singularity structure and symplectic structure of the Hamiltonian system for each principle balance in the Painlevé test. Further to this study, Hu et al [11] proved that the mirror transformation is canonical for finite-dimensional Hamiltonian systems. Furthermore, in 2001 Yee [13] showed that linearization of the mirror systems near movable poles provides the possibility to construct the associated Backlund transformations. In continuous development of mirror transformations in 2011, Tat-Leung Yee [14] extended the mirror method with perturbations which was utilized for finer analysis of certain nonlinear equations possessing negative Fuchsian indices.

In connection with the basin scale dynamics, Maas [5] has considered the flow of fluid contained in a rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moment of mass and heat. The container is assumed to be steady, uniform rotation of an f-plane. With this assumption Maas [5] reduces the rotating stratified Boussinesq equation to a beautiful six coupled system of ODEs. Srinivasan et al. [4] extended this work and gave a detail mathematical analysis of the reduced system of six coupled ODEs. Furthermore, Desale and Patil [2] tested the system of six coupled ODEs (5) for complete integrability using the Painlevé test. Also, they investigated the case of non-integrability for $Ra \neq 0$ and thereby they have obtained weak solutions (in the form of logarithmic psi-series) in the different branches of leading order.

In this paper we have successfully implemented the mirror transformations and constructed the mirror system of (5) for $Ra = 0$ which is regular near movable singularity. Further, with the help of mirror transformation, we have proved that the Laurent series obtained by using the Painlevé test are convergent. In the following section we imploy the mirror transformation to find the mirror system of ideal rotating stratified Boussinesq equations.

2 Mirror System of Six Coupled Non-Linear ODEs

Consider the rotating stratified Boussinesq equations (see Majda [1], p. 1)

$$\begin{aligned}
\frac{D\vec{\mathbf{v}}}{Dt} + f(\hat{\mathbf{e_3}} \times \vec{\mathbf{v}}) &= -\nabla p + \nu(\Delta\vec{\mathbf{v}}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\
\operatorname{div}\vec{\mathbf{v}} &= 0, \\
\frac{D\tilde{\rho}}{Dt} &= \kappa\Delta\tilde{\rho},
\end{aligned} \tag{1}$$

where $\vec{\mathbf{v}}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturbation density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{\mathbf{v}} \cdot \nabla)$ is a convective derivative. For more about the rotating stratified Boussinesq equations one may see Majda [1]. Maas [5] reduces the system of equations (1) to the following system of six coupled ODEs:

$$\begin{aligned} Pr^{-1}\frac{d\vec{\mathbf{w}}}{dt} + f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} &= \hat{\mathbf{e_3}} \times \vec{\mathbf{b}} - (w_1,\, w_2,\, rw_3) + \hat{T}\vec{\mathbf{T}}, \\ \frac{d\vec{\mathbf{b}}}{dt} + \vec{\mathbf{b}} \times \vec{\mathbf{w}} &= -(b_1,\, b_2,\, \mu b_3) + Ra\vec{\mathbf{F}}. \end{aligned} \tag{2}$$

In these equations, $\vec{\mathbf{b}} = (b_1, b_2, b_3)$ is the center of mass, $\vec{\mathbf{w}} = (w_1, w_2, w_3)$ is the basin averaged angular momentum vector, $\vec{\mathbf{T}}$ is the differential momentum, $\vec{\mathbf{F}}$ are buoyancy fluxes, $f' = f/2r_h$ is the earth rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [5] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction. $\vec{\mathbf{F}} = (0, 1, 0)$, the wind effect is neglected, i.e. $\vec{\mathbf{T}} = 0$. For the Prandtl number Pr, equal to one, the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$\begin{aligned} \frac{d\vec{\mathbf{w}}}{dt} &= -f'\hat{\mathbf{e_3}} \times \vec{\mathbf{w}} + \hat{\mathbf{e_3}} \times \vec{\mathbf{b}}, \\ \frac{d\vec{\mathbf{b}}}{dt} &= -\vec{\mathbf{b}} \times \vec{\mathbf{w}} + Ra\vec{\mathbf{F}}. \end{aligned} \tag{3}$$

The system of ODEs (3) can be written component wise as

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \quad \dot{w_3} = 0, \\ &\dot{b_1} = w_2 b_3 - w_3 b_2, \quad \dot{b_2} = w_3 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \end{aligned} \tag{4}$$

Since $\dot{w_3} = 0$, this gives $w_3 = \text{constant} = k_1$. Consequently, we have the following system of ODEs:

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \\ &\dot{b_1} = w_2 b_3 - k_1 b_2, \quad \dot{b_2} = k_1 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \end{aligned} \tag{5}$$

In our earlier study [2], we have shown that the system of ODEs (5) is completely integrable provided that $Ra = 0$ and we have determined the solutions in the form of Laurent series with the help of the Painlevé method. Now our aim is to determine the mirror system of (5) and its solutions in the following form:

$$\begin{aligned} &w_1(t) = \theta^{-m_1}, \quad \theta' = l_0 + l_1\theta + l_2\theta^2 + l_3\theta^3 + l_4\theta^4 + \cdots, \\ &w_2(t) = \theta^{-m_2}\Big(w_{20} + w_{21}\theta + w_{22}\theta^2 + w_{23}\theta^3 + w_{24}\theta^4 + \cdots\Big), \\ &b_1(t) = \theta^{-m_3}\Big(b_{10} + b_{11}\theta + b_{12}\theta^2 + b_{13}\theta^3 + b_{14}\theta^4 + \cdots\Big), \\ &b_2(t) = \theta^{-m_4}\Big(b_{20} + b_{21}\theta + b_{22}\theta^2 + b_{23}\theta^3 + b_{24}\theta^4 + \cdots\Big), \\ &b_3(t) = \theta^{-m_5}\Big(b_{30} + b_{31}\theta + b_{32}\theta^2 + b_{33}\theta^3 + b_{34}\theta^4 + \cdots\Big), \end{aligned} \tag{6}$$

where $\theta = t - t_0$ and t_0 is an arbitrary position of singularity. We found that there were several possible cases of dominant balance of the system (5) similar to those in the Painlevé test. Among the several possible cases of principle dominant balance we have obtained the singular solution only in the following case of principle dominant balance:

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1, \tag{7}$$

and the exponent with this principle dominant balance are as follows:

$$m_1 = m_2 = -1, \quad m_3 = m_4 = m_5 = -2. \tag{8}$$

Since w_1, w_2 are of order 1 near the movable singularity, we can introduce the indicial normalization $w_1(t) = \theta^{-1}$ and try to calculate the formal $\theta-$ series of (6) with $m_2 = -1$, $m_3 = m_4 = m_5 = -2$. Since the system (5) is autonomous, the coefficients appearing in the series given by (5) are to be constant. Substituting the values of exponents from (8) into the equations (6) and then substituting these series into the system (5) and hence equating the like powers of θ on both sides, we obtain the following equations in leading order coefficients:

$$\begin{aligned} &l_0 = b_{20}, \quad -w_{20} l_0 = b_{10}, \quad -2b_{10} l_0 = w_{20} b_{30}, \\ &2b_{20} l_0 = b_{30}, \quad -2b_{30} l_0 = b_{20} - w_{20} b_{10}. \end{aligned} \tag{9}$$

Solving equations (9), we find two possible branches of leading order coefficients which are as follows:

$$l_0 = r_1', \; w_{20} = \pm\sqrt{-1 - 4r_1'^2}, \; b_{10} = \mp r_1' \sqrt{-1 - 4r_1'^2}, \; b_{20} = r_1', \; b_{30} = 2r_1'^2, \tag{10}$$

where r_1' is an arbitrary constant.

Definition 2.1 The leading exponents m_1, m_2, m_3, m_4, m_5 for system of ODEs (5) are Fuchsian, if the m_*-weighted degree of the right-hand side of (5) is $\leq m_i + 1$.

The m_*-weighted degree of polynomial in w_1, w_2, b_1, b_2, b_3 is found by taking the degree of $w_i's$, $i = 1, 2$, $b_i's$, $i = 1, 2, 3$ to be m_i, $i = 1, 2, 3, 4, 5$. And we verified that the exponents m_i's, $i = 1, 2, 3, 4, 5$ are Fuchsian for the system (5).

Remark 2.1 Since all leading order coefficients given by (10) are nonzero, the selection of leading exponents is natural and these exponents satisfy the Fuchsian condition.

So far in the employment of mirror transformations we have completed the two steps of algorithm, that is, we have determined leading order coefficients in principle dominant balance and exponents. Now, in the following section we will implement the third step of the algorithm and determine the resonances in the following way.

2.1 Resonances

Now we substitute the assumed θ-series (6) with the values of exponents given by (8) into the system of ODEs (5) and after doing some algebraic calculations we specify the following recursive relations to determine the coefficients w_{1j}, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 1, 2, 3, \ldots$ which are valid for $j \geq 2$:

$$M(j) \begin{pmatrix} l_j \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned}
&A_j = f' w_{2(j-1)}, \quad B_j = -\sum_{k=1}^{j-1} l_k w_{2(j-k)},\\
&C_j = -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k} b_{3(j-k)} - \sum_{k=1}^{j-1} l_k b_{1(j-k)},\\
&D_j = k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} l_k b_{2(j-k)},\\
&E_j = -\sum_{k=1}^{j-1} w_{2k} b_{1(j-k)} - \sum_{k=1}^{j-1} l_k b_{3(j-k)} \ ,
\end{aligned} \tag{12}$$

and matrix $M(j)$ is

$$M(j) = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ -w_{20} & (j-1)l_0 & -1 & 0 & 0 \\ -2b_{10} & -b_{30} & (j-2)l_0 & 0 & -w_{20} \\ -2b_{20} & 0 & 0 & (j-2)l_0 & 1 \\ -b_{30} & b_{10} & w_{20} & -1 & (j-2)l_0 \end{pmatrix}. \tag{13}$$

The above recursive relations $(11, 12)$ determine the unknown expansion coefficients uniquely unless the determinant of matrix $M(j)$ is zero. Those values of j at which the determinant of matrix $M(j)$ vanishes are called the *resonances*. Here, we observe that for both possible branches of leading order coefficients given in equations (10), the resonances are $j = 0,\ 2,\ 3,\ 4$. Since $j = 0$ is the resonance, one of the variable in (10) appears to be a resonance parameter, say $l_0 = r_1'$, and we should replace it by $\bar{r_1}$ (where $\bar{r_1} = \sqrt{-4-k_2^2}$, the arbitrary constant k_2 is the resonance parameter in the Painlevé test [2]), which satisfies the condition $\bar{r_1}^{-m_1} = r_1'$, that is, $\bar{r_1}^{-1} = r_1'$. Let us denote by $k_2 = r_1$ the resonance parameter, and hence we have $\bar{r_1} = \sqrt{-4-r_1^2}$. Now, we refresh the leading order coefficients given by (10) as follows:

$$\begin{aligned}
&l_0 = (\sqrt{-4-r_1^2})^{-1},\ w_{20} = \pm\frac{r_1}{\sqrt{-4-r_1^2}},\ b_{10} = \mp\frac{r_1}{(\sqrt{-4-r_1^2})^2},\\
&b_{20} = (\sqrt{-4-r_1^2})^{-1},\ b_{30} = \frac{2}{(\sqrt{-4-r_1^2})^2}.
\end{aligned} \tag{14}$$

2.2 Compatibility conditions

Further, we need to check the compatibility conditions for each resonance $j = 2,\ 3,\ 4$. We will do this for the first branch.

Case I: Consider the leading order coefficients

$$\begin{aligned}
&l_0 = (\sqrt{-4-r_1^2})^{-1},\ w_{20} = \frac{r_1}{\sqrt{-4-r_1^2}},\ b_{10} = -\frac{r_1}{(\sqrt{-4-r_1^2})^2},\\
&b_{20} = (\sqrt{-4-r_1^2})^{-1},\ b_{30} = \frac{2}{(\sqrt{-4-r_1^2})^2}.
\end{aligned} \tag{15}$$

• **Compatibility condition at** $j = 1$.

As $j = 1$ is not resonance, we get the unique solution. Since the recursion relations (11, 12) remain valid when $j \geq 2$, we directly substitute the equations (15) into the equations (6) and then into (5). After that, equating the like powers of θ on both sides of the resulting expansion, we obtain the system of linear equations which determine the coefficients l_1, w_{21}, b_{11}, b_{21} and b_{31} uniquely as

$$\begin{aligned} &l_1 = \frac{(-f' + k_1)r_1}{\sqrt{-4 - r_1^2}}, \quad w_{21} = \frac{2(f' - k_1)}{\sqrt{-4 - r_1^2}}, \\ &b_{11} = \frac{4f' + k_1 r_1^2}{r_1^2 + 4}, \quad b_{21} = \frac{k_1 r_1}{\sqrt{-4 - r_1^2}}, \quad b_{31} = \frac{2(f' - k_1)r_1}{4 + r_1^2}. \end{aligned} \tag{16}$$

• **Compatibility condition at the resonance** $j = 2$.
Now $j = 2$ is a resonance so that one of the coefficients in the computation of the system (11) at this level is independent. Let b_{32} be independent and let $b_{32} = r_2$ (the arbitrary coefficient), where r_2 is the second resonance parameter so that the values of coefficients are given in terms of r_2, which are as follows:

$$\begin{aligned} &l_2 = \frac{(r_2 - f'k_1)}{2}\sqrt{-4 - r_1^2}, \quad w_{22} = 0, \\ &b_{12} = \frac{r_1}{2}(f'k_1 - r_2), \quad b_{22} = \frac{1}{2}\left[r_2\sqrt{-4 - r_1^2} + \frac{f'(4f' + k_1 r_1^2)}{\sqrt{-4 - r_1^2}}\right], \quad b_{32} = r_2. \end{aligned} \tag{17}$$

• **Compatibility condition at the resonance** $j = 3$.
To check the compatibility condition at $j = 3$, we substitute the equations (15, 16, 17) into the system of ODEs (5), then we obtain a system of linear equations. While solving that linear system, we found the variable b_{23} to be independent. Now assign the arbitrary value to b_{23}, say $b_{23} = r_3$, and solving the corresponding system we obtain the following solution. At this level of resonance, we have the third resonance parameter r_3:

$$\begin{aligned} &l_3 = r_3,\ w_{23} = \frac{-r_3}{r_1},\ b_{13} = -\frac{1}{\sqrt{-4 - r_1^2}}\left(r_1 r_3 + \frac{2r_3}{r_1}\right), \\ &b_{23} = r_3,\ b_{33} = \frac{r_3}{\sqrt{-4 - r_1^2}}. \end{aligned} \tag{18}$$

• **Compatibility condition at the resonance** $j = 4$.
Now $j = 4$ is the fourth resonance and solving the system (11) for $j = 4$ involves the resonance parameter, say r_4. Solving the system (11) for this value of j, we obtain the following solution with b_{24} as an arbitrary constant with value r_4:

$$\begin{aligned} &l_4 = r_4 + \frac{f' r_3}{r_1}, \quad w_{24} = (k_1 - f')r_3, \\ &b_{14} = \frac{1}{\sqrt{-4 - r_1^2}}\left[-r_1 r_4 + (-2f' + k_1)r_3\right], \quad b_{24} = r_4, \\ &b_{34} = \frac{(f' - k_1)(2 + r_1^2)r_3}{r_1\sqrt{-4 - r_1^2}}. \end{aligned} \tag{19}$$

Substituting all the values of coefficients l_j, w_{2j}, b_{1j}, b_{2j} and b_{3j} for $j = 0, 1, 2, 3, 4\ldots$ into the equations (6), we get

$$
\begin{aligned}
\theta' &= \frac{1}{\sqrt{-4-r_1^2}} + \frac{(-f'+k_1)r_1}{\sqrt{-4-r_1^2}}\theta + \frac{1}{2}\sqrt{-4-r_1^2}(r_2 - f'k_1)\theta^2 + r_3\theta^3 \\
&+ (r_4 + \frac{f'r_3}{r_1})\theta^4 + \cdots, \\
w_2(t) &= \theta^{-1}\Big[\frac{r_1}{\sqrt{-4-r_1^2}} + \frac{2(f'-k_1)}{\sqrt{-4-r_1^2}}\theta - \frac{r_3}{r_1}\theta^3 + (k_1 - f')r_3\theta^4 + \cdots\Big], \\
b_1(t) &= \theta^{-2}\Big[-\frac{r_1}{(\sqrt{-4-r_1^2})^2} + (\frac{4f'+k_1r_1^2}{r_1^2+4})\theta + \frac{r_1}{2}(f'k_1 - r_2)\theta^2 - \frac{1}{\sqrt{-4-r_1^2}} \\
&\quad \left(r_1r_3 + \frac{2r_3}{r_1}\right)\theta^3 + \frac{1}{\sqrt{-4-r_1^2}}\left(-r_1r_4 + (-2f'+k_1)r_3\right)\theta^4 + \cdots\Big], \\
b_2(t) &= \theta^{-2}\Big[\frac{1}{\sqrt{-4-r_1^2}} + (\frac{k_1r_1}{\sqrt{-4-r_1^2}})\theta + \frac{1}{2}\left(r_2\sqrt{-4-r_1^2} + \frac{f'(4f'+k_1r_1^2)}{\sqrt{-4-r_1^2}}\right)\theta^2 \\
&+ r_3\theta^3 + r_4\theta^4 + \cdots\Big], \\
b_3(t) &= \theta^{-2}\Big[\frac{2}{(\sqrt{-4-r_1^2})^2} - (\frac{2(f'-k_1)r_1}{4+r_1^2})\theta + r_2\theta^2 + \frac{r_3}{\sqrt{-4-r_1^2}}\theta^3 \\
&+ \frac{(f'-k_1)(2+r_1^2)r_3}{r_1\sqrt{-4-r_1^2}}\theta^4 + \cdots\Big].
\end{aligned}
\tag{20}
$$

We have just finished the primary calculations of the system (11) and we have determined the resonance parameters, say r_1, r_2, r_3 and r_4. In the following subsection we obtain the mirror transformations and consequently, we determine the mirror system of (5).

2.3 Mirror system

In this subsection we will develop the mirror transformations by which we transform the system (5) to its mirror system. Thereby, we discuss the regularity of it.

Now the important step towards determining the mirror system is to introduce a new variable in which we develop the mirror system. Let us introduce the new variables ξ_1, ξ_2, ξ_3 and ξ_4 in the Laurent θ-series of w_2, b_1, b_2 and b_3 by successively truncating the expansion at the free parameters (resonance parameters) r_1, r_2, r_3 and r_4. Now we begin to truncate the θ-series of w_2 at the first resonance parameter r_1 by introducing the variable ξ_1 as

$$w_2(t) = \theta^{-1}\xi_1, \tag{21}$$

where

$$\xi_1 = \frac{r_1}{\sqrt{-4-r_1^2}} + \frac{2(f'-k_1)}{\sqrt{-4-r_1^2}}\theta - \frac{r_3}{r_1}\theta^3 + (k_1 - f')r_3\theta^4 + \cdots. \tag{22}$$

We convert this into

$$r_1 = \xi_1\bar{r_1} - 2(f'-k_1)\theta + \frac{r_3}{\xi_1}\theta^3 - r_3(f'-k_1)(\frac{2}{\xi_1^2\bar{r_1}} + \bar{r_1})\theta^4 + \cdots. \tag{23}$$

Upon substituting the value of r_1 in b_1, we get

$$\begin{aligned} b_1(t) &= -\frac{\xi_1}{\bar{r_1}}\theta^{-2} + \Big(\frac{-2f' - 2k_1}{\bar{r_1}^2} - k_1{\xi_1}^2\Big)\theta^{-1} + \Big[\frac{1}{2}(f'k_1 - r_2)\xi_1\bar{r_1} \\ &+ \frac{4k_1\xi_1(f'-k_1)}{\bar{r_1}}\Big] + \Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} - \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} - (f'k_1 - r_2)(f'-k_1) \\ &- \xi_1 r_3\Big]\theta + \Big[\frac{-2r_3(f'-k_1)}{{\xi_1}^2\bar{r_1}^3} - \xi_1 r_4 + \frac{(f'r_3 - 4k_1r_3)}{\bar{r_1}}\Big]\theta^2. \end{aligned} \tag{24}$$

Next we proceed to cut the θ-series of b_1 at r_2 by introducing the second variable, say ξ_2:

$$b_1(t) = -\frac{\xi_1}{\bar{r_1}}\theta^{-2} + \Big(\frac{-2f' - 2k_1}{\bar{r_1}^2} - k_1{\xi_1}^2\Big)\theta^{-1} + \xi_2, \tag{25}$$

where

$$\begin{aligned} \xi_2 &= \Big[\frac{1}{2}(f'k_1 - r_2)\xi_1\bar{r_1} + \frac{4k_1\xi_1(f'-k_1)}{\bar{r_1}}\Big] + \Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} - \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} \\ &- (f'k_1 - r_2)(f'-k_1) - \xi_1 r_3\Big]\theta + \Big[\frac{-2r_3(f'-k_1)}{{\xi_1}^2\bar{r_1}^3} - \xi_1 r_4 \\ &+ \frac{(f'r_3 - 4k_1r_3)}{\bar{r_1}}\Big]\theta^2 + \cdots . \end{aligned} \tag{26}$$

From the θ-series of ξ_2, we have

$$\begin{aligned} r_2 &= f'k_1 - \frac{2\xi_2}{\xi_1\bar{r_1}} + \frac{8k_1(f'-k_1)}{\bar{r_1}^2} + \frac{2}{\xi_1\bar{r_1}}\Big[\frac{-3r_3}{\xi_1\bar{r_1}^2} + \frac{4k_1(f'-k_1)^2}{\bar{r_1}^2} - \frac{2\xi_2(f'-k_1)}{\xi_1\bar{r_1}} \\ &- \xi_1 r_3\Big]\theta - \frac{2}{\xi_1\bar{r_1}}\Big[\frac{8r_3}{\xi_1^2\bar{r_1}^3}(f'-k_1) - \frac{8k_1(f'-k_1)^3}{\xi_1\bar{r_1}^3} + \frac{4\xi_2(f'-k_1)^2}{\xi_1^2\bar{r_1}^2} \\ &+ \frac{(f'+2k_1)r_3}{\bar{r_1}} + \xi_1 r_4\Big]\theta^2 + \cdots. \end{aligned} \tag{27}$$

Now, we substitute the value of r_2 into θ-series of b_2 and consequently, we update it. And then after cutting this series at the third resonance parameter r_3, we obtain the θ-series of b_2 as follows:

$$\begin{aligned} b_2(t) &= \frac{1}{\bar{r_1}}\theta^{-2} + k_1\xi_1\theta^{-1} + \Big[\frac{2k_1(f'-k_1)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1} + \frac{1}{2}f'k_1\xi_1^2\bar{r_1} + \frac{2f'^2}{\bar{r_1}} \\ &- \frac{\xi_2}{\xi_1}\Big] + \xi_3\theta, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \xi_3 &= \Big[\frac{-3r_3}{\xi_1^2\bar{r_1}^2} + \frac{4k_1(f'-k_1)^2}{\xi_1\bar{r_1}^2} - \frac{2\xi_2(f'-k_1)}{\xi_1^2\bar{r_1}} - 2f'k_1\xi_1(f'-k_1)\Big] + \Big[\frac{-8r_3(f'-k_1)}{\xi_1^3\bar{r_1}^3} \\ &+ \frac{8k_1(f'-k_1)^3}{\xi_1^3\bar{r_1}^3} - \frac{4\xi_2(f'-k_1^2)}{\xi_1^3\bar{r_1}^2} - \frac{(f'+k_1)r_3}{\xi_1\bar{r_1}} + \frac{2f'k_1(f'-k_1^2)}{\bar{r_1}}\Big]\theta + \cdots. \end{aligned} \tag{29}$$

From the θ-series of ξ_3 we have

$$\begin{aligned}
r_3 &= -\frac{\xi_1^2\bar{r_1}^2\xi_3}{3} + \frac{4}{3}k_1(f'-k_1)^2\xi_1 - \frac{2}{3}\xi_2\bar{r_1}(f'-k_1) - \frac{2}{3}f'k_1\xi_1^3\bar{r_1}^2(f'-k_1)\\
&- \Big[-\frac{8}{9}(f'-k_1)\xi_1\xi_3\bar{r_1} + \frac{8k_1(f'-k_1)^3}{9\bar{r_1}} - \frac{4\xi_2(f'-k_1)^2}{9\xi_1} - \frac{22}{9}(f'-k_1)^2f'k_1\xi_1^2\bar{r_1}\\
&- \frac{1}{9}(f'+k_1)\xi_1^3\xi_3\bar{r_1}^3 + \frac{4}{9}(f'+k_1)(f'-k_1)^2k_1\xi_1^2\bar{r_1}\\
&- \frac{2}{9}(f'+k_1)(f'-k_1)\xi_1\xi_2\bar{r_1}^2 - \frac{2}{9}f'k_1(f'+k_1)(f'-k_1)\xi_1^4\bar{r_1}^3\Big]\theta + \cdots.
\end{aligned} \tag{30}$$

Similarly, we truncate the θ series of b_3 at the resonance parameter r_4 and we obtain the following θ-series:

$$\begin{aligned}
b_3(t) &= \frac{2}{\bar{r_1}^2}\theta^{-2} - \frac{2(f'-k_1)\xi_1}{\bar{r_1}}\theta^{-1} + \Big[\frac{4(f'-k_1)(f'+k_1)}{\bar{r_1}^2} - \frac{2\xi_2}{\xi_1\bar{r_1}} + f'k_1\Big]\\
&+ \Big[\frac{2}{\bar{r_1}}\xi_3 + \frac{4f'k_1\xi_1(f'-k_1)}{\bar{r_1}} + \frac{\xi_1^2\xi_3\bar{r_1}}{3} - \frac{4k_1\xi_1(f'-k_1)^2}{3\bar{r_1}} + \frac{2}{3}\xi_2(f'-k_1)\\
&+ \frac{2}{3}f'k_1\xi_1^3\bar{r_1}(f'-k_1)\Big]\theta + \xi_4\theta^2.
\end{aligned} \tag{31}$$

Hence, we have

$$\begin{aligned}
\xi_4 &= \frac{2}{9}\xi_1\xi_3(7k_1-4f') - \frac{4k_1}{9\bar{r_1}^2}(f'-k_1)^2(7f'+8k_1) + \frac{4}{9\xi_1\bar{r_1}}(f'-k_1)\\
&\quad (4k_1-f')\xi_2 + \frac{4}{9}(f'-k_1)^2(-10f+k_1)k_1\xi_1^2 - \frac{2}{9}(2f'-k_1)\xi_1^3\xi_3\bar{r_1}^2\\
&- \frac{2}{9}(f'-k_1)(2f'-k_1)\xi_1\xi_2\bar{r_1} - \frac{4}{9}f'k_1\xi_1^4\bar{r_1}^2(f'-k_1)(2f'-k_1) - \frac{2}{\bar{r_1}}r_4\\
&+ \frac{4}{3}k_1\xi_1^2(f'-k_1)(f'^2-2f'k_1+k_1^2) + \cdots.
\end{aligned} \tag{32}$$

Using (21), (25), (28) and (31) with $w_1 = \theta^{-1}$, we get the change of variables $(w_1, w_2, b_1, b_2, b_3) \longleftrightarrow (\theta, \xi_1, \xi_2, \xi_3, \xi_4)$. The following is the conversion of given system into the mirror system in terms of the new variables θ, ξ_1, ξ_2, ξ_3 and ξ_4:

$$\begin{aligned}
\theta' &= \frac{1}{\bar{r_1}} + (k_1-f')\xi_1\theta + \Big[\frac{2(k_1f'-k_1^2+f'^2)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1}(1+\xi_1^2) - \frac{\xi_2}{\xi_1}\Big]\theta^2\\
&+ \xi_3\theta^3,\\
\xi_1' &= \Big[-(1+\xi_1^2)f' - \frac{2(f'+k_1)}{\bar{r_1}^2}\Big] + \Big[\frac{2(k_1f'-k_1^2+f'^2)}{\bar{r_1}} + \frac{1}{2}f'k_1\bar{r_1}\xi_1(1+\xi_1^2)\Big]\theta\\
&+ \xi_1\xi_3\theta^2,\\
\xi_2' &= \Big[\frac{(-1-\xi_1^2)(f'+k_1)}{\bar{r_1}} - \frac{4(f'+k_1)}{\bar{r_1}^3}\Big]\theta^{-2} + \Big[\frac{2\xi_1}{\bar{r_1}^2}(2f'^2-4k_1^2-3k_1f')\\
&- \frac{f'k_1\xi_1}{2}(3+5\xi_1^2) - k_1^2\xi_1 - (k_1-f')k_1\xi_1^3\Big]\theta^{-1} + \Big[-\frac{4k_1\xi_1^2(f'-k_1)^2}{3\bar{r_1}} - \frac{f'k_1^2\xi_1^4\bar{r_1}}{6}\\
&+ \frac{\xi_1\xi_3 + 5f'^2\xi_1^2k_1 - 3f'k_1^2\xi_1^2 - 3k_1^2f' - 3f'^2k_1 + 2k_1^3(1-\xi_1^2)}{\bar{r_1}} - \frac{f'k_1^2\bar{r_1}}{2} + \frac{k_1\xi_2}{\xi_1}\\
&+ \frac{1}{3}(\xi_1^3\xi_3\bar{r_1} + \xi_1\xi_2(2f'+k_1) + 2f'^2k_1\xi_1^4\bar{r_1}) - \frac{4(f'+k_1)(k_1f'-k_1^2+f'^2)}{\bar{r_1}^3}\\
&+ \frac{2\xi_2(f'+k_1)}{\xi_1\bar{r_1}^2}\Big] + \Big[\xi_1\xi_4 + k_1\xi_3(\xi_1^2-1) - \frac{2\xi_3(f'+k_1)}{\bar{r_1}^2}\Big]\theta,
\end{aligned} \tag{33}$$

$$
\begin{aligned}
{\xi_3}' &= \Big[\frac{(-1-\xi_1^2)(f'+k_1)}{\xi_1\bar{r_1}} - \frac{4(f'+k_1)}{\xi_1\bar{r_1}^3}\Big]\theta^{-3} + \Big[\frac{-2k_1f' - 8k_1^2 + 4f'^2}{\bar{r_1}^2} - k_1^2(\xi_1^2+1)\\
&- \frac{1}{2}f'k_1(\xi_1^2+1)\Big]\theta^{-2} + \Big[\frac{\xi_3 + 3f'^2k_1\xi_1 + 3f'k_1^2\xi_1}{\bar{r_1}} + k_1\xi_2 + \frac{1}{2}f'k_1^2\xi_1^3\bar{r_1} - \xi_3\\
&+ f'^2k_1\xi_1\bar{r_1}(\xi_1^2+1) - \frac{3k_1^2f' + 3f'^2k_1 - 2k_1^3(1-\xi_1^2)}{\xi_1\bar{r_1}} - \frac{f'k_1^2\bar{r_1}}{2\xi_1} + \frac{k_1\xi_2 + (1+\xi_1^2)f'\xi_2}{\xi_1^2}\\
&- \frac{4(f'+k_1)(k_1f'-k_1^2+f'^2)}{\xi_1\bar{r_1}^3} + \frac{4\xi_2(f'+k_1)}{\xi_1^2\bar{r_1}^2}\Big]\theta^{-1} + \Big[\frac{-2f'k_1\xi_1^2\bar{r_1}(k_1f'-k_1^2+f'^2)}{\bar{r_1}}\\
&- \frac{1}{2}f'^2k_1^2\bar{r_1}^2\xi_1^2(1+\xi_1^2) + \frac{k_1\xi_3(\xi_1^2-1)}{\xi_1} - \frac{2\xi_3(f'+k_1)}{\xi_1\bar{r_1}^2} - \frac{2\xi_2}{\xi_1\bar{r_1}}(k_1f'-k_1^2+f'^2)\\
&- \frac{f'k_1\bar{r_1}\xi_2(1+\xi_1^2)}{2\xi_1}\Big] + \Big[-f'k_1\xi_1^2\xi_3\bar{r_1} - \frac{\xi_2\xi_3}{\bar{r_1}}\Big]\theta,\\
{\xi_4}' &= \Big[\frac{\xi_1^2+1}{\bar{r_1}} + \frac{4}{\bar{r_1}^3}\Big]\theta^{-5} + \Big[k_1\xi_1(1+\xi_1^2) + \frac{4\xi_1}{\bar{r_1}^2}(-f'+2k_1) + \frac{4\xi_1}{3\bar{r_1}^2}(f'+k_1)\\
&+ \frac{\xi_1(\xi_1^2+1)(f'+k_1)}{3}\Big]\theta^{-4} + \Big[\frac{f'k_1\bar{r_1}}{2} + \frac{2f'k_1\xi_1^2\bar{r_1}}{3} - \frac{\xi_2}{\xi_1} + \frac{4}{\bar{r_1}^3}(f'^2-k_1^2)\\
&+ \frac{1}{\bar{r_1}}(4f'k_1 - 2k_1^2 + 2k_1^2\xi_1^2 - 2f'k_1\xi_1^2) - \frac{4\xi_2}{\xi_1\bar{r_1}^2} + \frac{1}{6}f'k_1\xi_1^4\bar{r_1} + \frac{8(f'^2-k_1^2)}{3\bar{r_1}^3}\\
&+ \frac{1}{3\bar{r_1}}(2k_1f'\xi_1^2 + 6k_1^2\xi_1^2 - 2f'^2\xi_1^2 - 2(k_1^2-f'^2)) + \frac{\xi_1^2\bar{r_1}k_1^2}{3}(\xi_1^2+1)\Big]\theta^{-3}\\
&+ \Big[\frac{2}{3}f'k_1^2\xi_1(1+\xi_1^2\bar{r_1} - 4\xi_1^2) - \frac{1}{6}f'k_1^2\xi_1\bar{r_1}^2(\xi_1^4-1) - \frac{2}{3}f'^2k_1\xi_1^3(1+\bar{r_1})\\
&- \frac{1}{3}f'^2k_1\xi_1^3\bar{r_1}^2(1+\xi_1^2) - \frac{1}{3}f'\xi_2(2+\bar{r_1}) + \frac{1}{3}k_1\xi_2(2-\bar{r_1}) - \frac{1}{3}\xi_1^2\xi_2\bar{r_1}(f'+k_1)\\
&+ \frac{1}{\bar{r_1}^2}(4\xi_3 + 4f'^2k_1\xi_1 - 12f'k_1^2\xi_1) + \frac{1}{\bar{r_1}}(2\xi_2(f'-2k_1) - 4f'k_1\xi_1(f'-k_1))\\
&+ \frac{1}{3\bar{r_1}^2}(20f'k_1\xi_1(f'+k_1) - 28k_1^3\xi_1 - 4\xi_1f'^3) + \frac{1}{3}(\xi_1^2\xi_3 - 4k_1^3\xi_1)\\
&+ \frac{1}{3\bar{r_1}}(-2k_1\xi_2 + 4k_1\xi_1(f'-k_1)^2)\Big]\theta^{-2} + \Big[2f'^3\xi_1^2k_1\bar{r_1}(1+\xi_1^2)\\
&- f'^2k_1^2\xi_1^2\bar{r_1}(1+\frac{1}{9}\xi_1^2) + \frac{1}{\bar{r_1}}(-8f'k_1^3\xi_1^2 + 12f'^3k_1\xi_1^2 + 4f'^3k_1 - 4f'^2k_1^2)\\
&- \frac{4\xi_2(f'^2-k_1^2)}{3\xi_1\bar{r_1}^2} - \frac{2(f'-k_1)k_1\xi_2}{3\xi_1} + \frac{8f'k_1(f'^2-k_1^2)}{\bar{r_1}^3} + \frac{1}{9}f'k_1\xi_1^4\bar{r_1}(-7k_1^2+4f'^2)\\
&+ \frac{1}{6}f'^2k_1^2\xi_1^4\bar{r_1}^3(1+\xi_1^2) + \frac{1}{3}\xi_1^3\xi_3\bar{r_1}(f' - \frac{1}{3}k_1) + \xi_1\xi_3\bar{r_1}(2+\frac{1}{3}k_1)\\
&+ \frac{4}{3}k_1\xi_1\xi_2(2f'-k_1) + \frac{2}{3}f'^2\xi_1\xi_2 + \frac{1}{2}f'k_1\bar{r_1}^2\xi_1(\xi_2+\xi_1^2\xi_3) + 2\xi_4\xi_3\bar{r_1}\\
&+ f'k_1^2\bar{r_1}(f'-k_1) + \frac{1}{3\bar{r_1}}(4\xi_1\xi_3(f'+2k_1) + 2f'^3k_1(1-\frac{17}{3}\xi_1^2) + 2f'k_1^3(-7+\xi_1^2)\\
&+ 8f'^2k_1^2(1+2\xi_1^2) + 4k_1^4(1-\frac{5}{3}\xi_1^2) + \frac{8f'^2(f'^2-k_1^2)}{3\bar{r_1}^3})\Big]\theta^{-1} + \Big[\frac{1}{\bar{r_1}^2}(2\xi_2\xi_3\\
&- 8f'k_1\xi_1(f'-k_1)(k_1f'-k_1^2+f'^2)) + \frac{1}{3\bar{r_1}^2}(8k_1(k_1f'-k_1^2+f'^2)(f'-k_1)^2\xi_1\\
&+ 4\xi_3(f'^{-}k_1^2)) + \frac{1}{3}(-f'k_1\xi_1^2\xi_3\bar{r_1}^2 + 2k_1^2f'\xi_1(f'-k_1)^2(1+\xi_1^2) - \xi_3(\xi_1^2-1)\\
&- 2(f'-k_1)\xi_1\xi_4) - \frac{\xi_2\xi_3}{\xi_1} + 2\xi_1^2\xi_3(f'k_1+\xi_2) - f'^2k_1^2\xi_1(f'-k_1)(1+\xi_1^2)\\
&\quad (2+\xi_1^2\bar{r_1}^2) - 4\xi_1^2(k_1f'-k_1^2+f'^2)(\xi_3 + f'k_1\xi_1(f'-k_1)) - 2\xi_1\xi_4(f'-k_1)\Big]\\
&+ \Big[-\frac{2\xi_1^2\xi_3\bar{r_1}}{3} + \frac{4k_1\xi_1\xi_3(f'-k_1)(-2f'-k_1)}{3\bar{r_1}} - \frac{4\xi_4(k_1f'-k_1^2+f'^2)}{\bar{r_1}}\\
&- 2f'k_1\xi_1^3\xi_3\bar{r_1} - f'k_1\xi_4\bar{r_1}(1+\xi_1^2) + \frac{2\xi_2\xi_4}{\xi_1}\Big]\theta - 2\xi_3\xi_4\theta^2.
\end{aligned}
\tag{34}
$$

By similar calculations, we can find the mirror system for the following branch of leading order coefficients:

$$\begin{aligned} &l_0 = (\sqrt{-4-r_1^2})^{-1}, \quad w_{20} = -\frac{r_1}{\sqrt{-4-r_1^2}}, \\ &b_{10} = \frac{r_1}{(\sqrt{-4-r_1^2})^2}, \quad b_{20} = (\sqrt{-4-r_1^2})^{-1}, \quad b_{30} = \frac{2}{(\sqrt{-4-r_1^2})^2}. \end{aligned} \tag{35}$$

The mirror system obtained so far for the present case of leading order coefficient is regular if and only if the following condition are satisfied:

$$\xi_1 = \frac{\sqrt{-4-\bar{r_1}^2}}{\bar{r_1}},\ \xi_2 = -\frac{26k_1^2\xi_1}{9\bar{r_1}},\ f' = k_1,\ \xi_3 = 0. \tag{36}$$

The most prominent thing for the singularity analysis is that the system is regular near $\theta = 0$, which corresponds to movable singularity of the system of six coupled ODEs (5).

3 Alternative Approach of the Convergence of Laurent Series in Painlevé Test

The convergence of Laurent series solution obtained by the Painlevé test is guaranteed by Kichenassamy and Littman [4]. But here we are going to present an alternative approach of the convergence of these series by making use of the mirror system and the Cauchy-Kowalevski theorem.

An ideal rotating, uniformly stratified system of six coupled ODEs (5) is completely integrable for the Rayleigh number $Ra = 0$. For $Ra = 0$, the Painlevé test produces the following formal solution of ODEs (5) for the first case of leading order coefficients:

$$\begin{aligned} w_1(t) &= \sqrt{-4-k_2^2}\tau^{-1} + \frac{(f'-k_1)k_2}{2} + \frac{\sqrt{-4-k_2^2}}{2}(-k_3 + f'k_1)\tau \\ &+ \Big[-\frac{k_4}{2} + \frac{f'k_2}{4}(-k_3+f'k_1)\Big]\tau^2 \\ &+ \{-\frac{k_5}{3} + \frac{f'\sqrt{-4-k_2^2}}{12k_2}\Big[f'k_2(k_3-f'k_1)+2k_4\Big]\}\tau^3 \\ &+ \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\ w_2(t) &= k_2\tau^{-1} + \Big[\frac{\sqrt{-4-k_2^2}}{2}\Big(-f'+k_1\Big)\Big] + \frac{(-k_3k_2+f'k_2k_1)}{2}\tau \\ &+ \sqrt{-4-k_2^2}\Big[\frac{k_4}{2k_2} + \frac{f'}{4}\Big(k_3 - f'k_1\Big)\Big]\tau^2 \\ &+ \Big[\frac{-k_5k_2}{3\sqrt{-4-k_2^2}} + \frac{f'}{12}\Big(f'k_2k_3 + 2k_4 - f'^2k_2k_1\Big)\Big]\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\ b_1(t) &= -k_2\tau^{-2} + f'\sqrt{-4-k_2^2}\tau^{-1} + \frac{(-k_2k_3+f'^2k_2)}{2} + \frac{k_4\sqrt{-4-k_2^2}}{k_2}\tau \\ &- \frac{k_5k_2}{\sqrt{-4-k_2^2}}\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \end{aligned} \tag{37}$$

$$
\begin{aligned}
b_2(t) &= \sqrt{-4-k_2^2}\tau^{-2} + f'k_2\tau^{-1} + \Big[\frac{\sqrt{-4-k_2^2}}{2}(k_3 - f'^2)\Big] + k_4\tau + k_5\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 - \frac{4k_5}{3\sqrt{-4-k_2^2}}\tau^2 \\
&- \frac{1}{6k_2}\Big(f'^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - f'^3k_2k_1 + 3f'k_2k_3k_1 - 6k_4k_1\Big)\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned} \tag{38}
$$

The above Laurent series contains the five arbitrary constant $w_{30} = k_1$, k_2, k_3, k_4 and k_5. Here $\tau = t - t_0$ and t_0 is an arbitrary position of singularity in complex domain. As we see, the above Laurent series has a movable pole type singularity, and using the Painlevé method we conclude that the above Laurent series (37) and (38) are convergent for small τ; and this convergence is guaranteed by Kichenassamy and Littman [4]. But for an alternative approach, we convert these series into an initial value problem for the mirror system (33) and (34). For this purpose we substitute the formal Laurent series (37) and (38) into the mirror transformation $w_1 = \theta^{-1}$, (21), (25), (28) and (31). After simplification, we obtain the following formal power series for θ, ξ_1, ξ_2, ξ_3 and ξ_4:

$$
\begin{aligned}
\theta &= (\sqrt{-4-k_2^2})^{-1}\tau - \frac{(f'-k_1)k_2}{(\sqrt{-4-k_2^2})^2}\tau^2 + \frac{1}{4(\sqrt{-4-k_2^2})^3} \\
&\quad (-8k_3 - 2k_2^2k_3 + 8f'k_1 + f'^2k_2^2 + k_1^2k_2^2)\tau^3 \\
&+ \Big[\frac{1}{2(\sqrt{-4-k_2^2})^2}\Big(k_4 + (-k_3 + f'k_1)(\frac{f'k_2}{2} - k_1k_2)\Big) - \frac{(f'-k_1)^3k_2^3}{8(\sqrt{-4-k_2^2})^4}\Big]\tau^4 \\
&+ \Big[\frac{k_5}{3(\sqrt{-4-k_2^2})^2} + \frac{1}{\sqrt{-4-k_2^2}}(-\frac{f'^2}{12}(k_3 - f'k_1) - \frac{f'k_4}{6k_2} + \frac{(-k_3+f'k_1)^2}{4}) \\
&+ \frac{1}{2(\sqrt{-4-k_2^2})^3}(-k_4(f'-k_1)k_2 + \frac{(f'-k_1)k_2^2}{4}(-k_3+f'k_1)(3k_1 - f')) \\
&+ \frac{(f'-k_1)^4k_2^4}{(16\sqrt{-4-k_2^2})^5}\Big]\tau^5 + \ldots, \\
\xi_1 &= \frac{k_2}{\sqrt{-4-k_2^2}} - \frac{2(f'-k_1)}{4+k_2^2}\tau - \frac{k_2(f'-k_1)^2}{(-4-k_2^2)^{\frac{3}{2}}}\tau^2 + \Big[\frac{1}{(-4-k_2^2)}\Big(\frac{k_2k_4}{2\sqrt{-4-k_2^2}} \\
&- \frac{1}{4}(f'k_1 - k_3)k_1k_2^2 + \frac{1}{8}(-f'+k_1)(8f'k_1 + f'^2k_2^2 + k_1^2k_2^2)\Big) \\
&+ \frac{k_4}{2k_2} - f'^2k_1 + \frac{k_1k_3}{4} - \frac{(f'-k_1)^3k_2^4}{8(-4-k_2^2)^2}\Big]\tau^3 + \ldots, \\
\xi_2 &= \frac{k_2}{2}(f'^2 - 3k_3 + 2f'k_1) + \frac{1}{(-4-k_2^2)}(f'-k_1)k_2(\frac{1}{4}k_2^2(f'+k_1) + 2(f'+2k_1)) \\
&+ \Big[\frac{1}{\sqrt{-4-k_2^2}}\Big(-\frac{k_2k_4}{2} + \frac{1}{4}(f'k_1 - k_3)(4f'-k_1)k_2^2 + 2f'^2k_1 - 3k_3(f'-k_1) \\
&- \frac{1}{4}f'k_1k_2^2(f'-k_1)\Big) + \sqrt{-4-k_2^2}(\frac{k_4}{2k_2} - f'^2k_1 + \frac{k_1k_3}{4}) + \frac{4k_1(f'-k_1)^2}{(-4-k_2^2)^{\frac{3}{2}}}\Big]\tau + \ldots,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
\xi_3 &= \sqrt{-4-k_2^2}\Big(\frac{3}{2}k_4 - \frac{2k_4}{k_2} - \frac{k_4k_2}{2} + \frac{5f'^2k_1}{k_2} - \frac{k_1k_3}{2} + \frac{1}{2}f'^2k_1k_2 + \frac{1}{4}f'k_1^2k_2 - \frac{f'^3}{k_2}\\
&+ \frac{2f'k_1^2}{k_2}\Big) + \frac{(f'-k_1)}{\sqrt{-4-k_2^2}}\Big[-2f'k_1k_2 + \frac{1}{2}k_2(k_1^2 - f'^2) - \frac{3}{8}f'k_1k_2^3 + \frac{1}{4}k_1^2k_2^3\\
&- \frac{4f'}{k_2}(f'-k_1)\Big] + \{-\frac{3}{8}k_1^2k_2^2k_3 + \frac{1}{2}f'k_2k_4 - \frac{3}{8}f'^2k_2^2k_3 + \frac{5}{8}f'^3k_1k_2^2 - \frac{15}{8}k_1k_2k_4\\
&- \frac{1}{2}f'^2k_1^2 - 2f'k_1^3 + \frac{3}{2}f'^3k_1 + \frac{7}{2}f'k_1k_3 - \frac{3}{2}k_1^2k_3 - f'^2k_3 - \frac{5}{8}f'^2k_1^2k_2^2\\
&+ f'k_1k_2^2k_3 - f'k_1(f'-k_1) + \frac{1}{4}k_1^3k_2^2(f' - \frac{1}{2}k_1^2) + \frac{1}{k_2}\Big[2f'^4 - 6f'k_3(f'-2k_1)\\
&- 6k_1^2(f'^2 + k_3 + 4f'k_1^3 + f'k_4 + \frac{2f'^3k_1}{k_2} + \frac{6k_3(f'-k_1)^2}{k_2} - k_1k_4 - \frac{2f'^2k_1^2}{k_2})\Big]\\
&+ (-4-k_2^2)\Big[-\frac{5}{12}f'^2k_3 + \frac{f'k_4}{6k_2} - \frac{13}{12}f'^3k_1 - \frac{1}{4}f'k_1k_3 + \frac{1}{2}(k_3^2 + f'^3k_1) - \frac{k_1k_4}{k_2}\\
&+ 2f'^2k_1^2 - \frac{k_1^2}{2}(4f'^2 - k_3) - \frac{1}{k_2^3}k_4(k_1 - 2f') + \frac{2f'^2k_1}{k_2^2}(f'-k_1) + \frac{k_1^2k_3}{2k_2^2}\Big]\\
&+ \sqrt{-4-k_2^2}\Big[\frac{5}{3}k_5 + \frac{1}{k_2}(6f'^2k_1 + 2k_1k_3 - 3f'k_3) - k_4 + \frac{7}{4}f'^2k_1k_2 - \frac{2k_4}{k_2^2}\\
&- f'k_2k_3\Big] + \frac{1}{(-4-k_2^2)}\Big[k_1(f'-k_1)^3(4 - \frac{8}{k_2^2} + \frac{k_2^4}{8} + \frac{1}{2}k_2^2) + \frac{4(f'-k_1)^3}{k_2}\\
&\quad (\frac{1}{4}k_2^2(f'+k_1) + 2(f'+2k_1))\Big]\}\tau + \ldots,\\
\xi_4 &= \frac{5}{18}k_5\sqrt{-4-k_2^2} + \frac{2}{3}(f'-k_1)^2(2f'+k_1)k_1 - 12f'k_1(f'-k_1)^2k_2^2\\
&- \frac{1}{\sqrt{-4-k_2^2}}(f'-k_1)^3(f'+k_1)k_2^2 + \frac{1}{3}(f'-k_1)^2\sqrt{-4-k_2^2} + \frac{(f'+k_1)^2k_2^2}{3(4+k_2^2)}\\
&\quad [12f'k_1 + f'k_1k_2^2 - 3(f'-k_1)^2k_2^2] + \frac{1}{3}k_2(f'-k_1)^2(\frac{8}{k_2^3} - \frac{2}{k_2} + \frac{1}{2}k_2\\
&- \frac{2(4+k_2^2)(f'k_1-k_3)}{k_2(f'-k_1)^2})\Big[\frac{f'^2+2f'k_1-(f'-k_1)(f'(8+k_2^2)+k_1(16+k_2^2))}{2(4+k_2^2)} - 3k_3\Big]\\
&+ 2(f'k_1-k_3)[3(f'-k_1)^2 - 2(f'^2-k_1^2)\sqrt{-4-k_2^2}] + \frac{7}{2}(4+k_2^2)\\
&\quad (f'k_1-k_3)^2 + \frac{1}{4}(f'-k_1)^2k_2^2k_3 - \frac{5}{2}(f'-k_1)k_2(f'^2k_1k_2 - f'k_2k_3 - 2k_4)\\
&+ \frac{72(f'-k_1)k_2(f'^2k_1k_2 - f'k_2k_3 - 2k_4)}{\sqrt{-4-k_2^2}} + 12(f'-k_1)(-4-k_2^2)\Big[-2f'^2k_1\\
&+ \frac{1}{2}k_1k_3 + \frac{k_4}{k_2} - \frac{k_2^4(f'-k_1)^4}{(4+k_2^2)^2} + \frac{k_2^2(-f'^2 - f'k_1^2 + k_1^3 + k_1(f'^2+2k_3)) + 4k_2k_4}{\sqrt{-4-k_2^2}}\Big]\\
&+ \ldots,
\end{aligned}
\tag{40}
$$

Thus, we have the mirror system (33) and (34) with the following initial data

$$
\begin{aligned}
\theta(0) &= 0, \quad \xi_1(0) = \frac{k_2}{\sqrt{-4-k_2^2}}, \\
\xi_2(0) &= \frac{k_2}{2}(-3k_3 + f'^2 + 2f'k_1) + \frac{1}{(-4-k_2^2)}(f'-k_1)k_2\Big[\frac{1}{4}k_2^2(f'+k_1) \\
&+ 2(f'+2k_1)\Big], \\
\xi_3(0) &= \sqrt{-4-k_2^2}\Big(\frac{3}{2}k_4 - \frac{2k_4}{k_2} - \frac{k_4k_2}{2} + \frac{5f'^2k_1}{k_2} - \frac{k_1k_3}{2} + \frac{1}{2}f'^2k_1k_2 + \frac{1}{4}f'k_1^2k_2 \\
&- \frac{f'^3}{k_2} + \frac{2f'k_1^2}{k_2}\Big) + \frac{(f'-k_1)}{\sqrt{-4-k_2^2}}\Big[(-2f'k_1k_2 + \frac{1}{2}k_2(k_1^2 - f'^2) - \frac{3}{8}f'k_1k_2^3 \\
&+ \frac{1}{4}k_1^2k_2^3 - \frac{4f'}{k_2}(f'-k_1)\Big], \\
\xi_4(0) &= \frac{5}{18}k_5\sqrt{-4-k_2^2} + \frac{2}{3}(f'-k_1)^2(2f'+k_1)k_1 - 12f'k_1(f'-k_1)^2k_2^2 + \ldots.
\end{aligned}
\tag{41}
$$

Now we are ready to show the convergence of (37)and (38) by using the Cauchy theorem [10, p.150-151]. From the differential equations (33) and (34) and the initial conditions (41) we see that the coefficients of variable in (33), (34) and initial value conditions (41) are analytic functions provided that $r_1 = k_2 \neq \pm 2i$. Thus, the initial value problem (33) and (34) with initial conditions (41) has unique analytic solutions which are convergent in the neighbourhood of $\theta = 0$.

Substituting the series (39) and (40) back into $w_1 = \theta^{-1}$, (21), (25), (28) and (31), we obtain the convergent power series for w_1, w_2, b_1, b_2 and b_3 which was not just formal. Furthermore, with some computation we see that these series are exactly (37) and (38). Therefore, we come to the conclusion that the Laurent series (37) and (38) are convergent. Thus, we summarise these results in terms of the following theorem.

Theorem 3.1 *For the principal Laurent series solution of the ideal rotating, uniformly stratified system of six coupled ODEs (3), there is a change of variables of the form (6) such that the system of ODEs (3) is transformed into a regular system of ODEs (33) and (34) for the new variables* $(\theta, \xi_1, \xi_2, \xi_3, \xi_4)$. *Further, the Laurent series (37) and (38) in the principle dominant balance are converted into the power series (39) and (40) with initial data (41) which are the analytic functions in terms of new variables and thus, the series solutions (39)with (40) are convergent in the neighbourhood of* $\theta = 0$.

4 Conclusion

The reduced system of ODEs (3) which arose in the reduction of uniformly stratified fluid contained in the rotating box of dimension $L \times L \times H$ is completely integrable if the Rayleigh number $Ra = 0$. By taking $Ra = 0$, we have obtained the mirror system for both possible branches of leading ordered coefficients of system (3). The main feature in the singularity analysis is that the mirror system is regular near $\theta = 0$, which corresponds to the movable singularity of the system (3) provided (36) holds. Also, we have shown that the formal Laurent series solutions arising from successful application of the Painlevé test to the system of ODEs (3) are convergent.

References

[1] Majada, A.J. and Shefter, M.G. Elementary stratified flows with instability at large Richardson number. *J. Fluid Mechanics* **376** (1998) 319–350.

[2] Desale, B.S. and Patil, K.D. Painlevé test to a reduced system of six coupled nonlinear ODEs. *Nonlinear Dynamics and System Theory* **10(4)** (2010) 349–361.

[3] Kichenassamy, S. and Srinivasan, G. The structure of WTC expansions and applications. *Journal of Phys. A: Math. Gen.* **28** (1995) 1977–2004.

[4] Kichenassamy, S. and Littman, W. Blow-up surfaces for nonlinear wave equations I. *Commun. in PDE* **18** (1993) 431–452.

[5] Maas, L.R.M. Theory Of basin scale dynamics of a stratified rotating fluid. *Surveys in Geophysics.* **25** (2004) 249–279.

[6] Paul, P. Sur les équations différentielles du Second Ordre et dórdre supérieur dont l'integrale générale est uniform. *Acta Math.* **25** (1902) 1–86.

[7] Weiss, J., Tabor, M. and Carnevale, G. The Painlevé property for partial differential equations. *Journal of Math. Phys.* **24** (1983) 522–526.

[8] Hu, J. and Yan, M. Singularity analysis for integrable systems by their mirrors. *Nonlinearity* **12** (1999) 1531–1543.

[9] Hu, J. and Yan, M. The mirror systems of integrable equations. *Studies in App. Maths.* **104** (2000) 67–90.

[10] Hu, J. and Yan, M. Local analyticity of solutions in the Painlevé test. In: *Proc. of the workshop on nonlinearity, integrability and all that: twenty years after needs' 79* (2000) 146–152.

[11] Hu, J., Yan, M. and and Yee, T. Mirror transformations of hamiltonian system. *Physica D.* **152** (2001) 110–123.

[12] Kowalevski, S. Sur le problè me de la rotation dún corps solide autour dún point fixe. *Acta Math.* **12** (1889) 177–232.

[13] Yee, T. L. Linearization of mirror transform. *Journal of Nonlinear Mathematical Physics* **9** (2001) 234–242.

[14] Yee, T. L. A new perturbative approach in nonlinear singularity analysis. *Journal of Mathematics and Statistics* **7(3)** (2011) 249–254.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 36–54

Real Time Analysis of Signed Marked Random Measures with Applications to Finance and Insurance

Jewgeni H. Dshalalow * and Kizza M. Nandyose

Department of Mathematics, Florida Institute of Technology
Melbourne, FL 32901, USA

Received: June 17, 2018;   Revised: December 9, 2018

Abstract: This paper deals with stationary and independent increments processes in real time initiated in [14] embellishing it to a two-dimensional signed random measure with position dependent marking. The real-valued component of the associated marked point process is non-monotone presenting an analytical challenge. We manage to investigate various characteristics of that component, including the nth drop or a sharp surge that find applications to finance (like option trading) and risk theory. The need for time sensitive feature of our study (i.e., an analytical association with real time parameter t) allows stochastic control implementation in sharp contrast with time insensitive analysis in the present literature. We proceed with the classical approach of fluctuation analysis of a particle running through a random grid of a convex set that the particle is trying to escape. We find the distribution of the first passage time and its location in space.

Keywords: *random walk; independent and stationary increments processes; fluctuations of stochastic processes; marked point processes; first passage time; signed marked random measures; time sensitive analysis.*

Mathematics Subject Classification (2010): 60G50, 60G51, 60G52, 60G55, 60G57, 60K05, 60K35, 60K40, 60G25, 90B18, 90B10, 90B15, 90B25.

1 Introduction

In many scientific, financial, and game theoretic processes, timing is of at most importance and a main strategic issue. Several studies have been done on the first passage time in fluctuation theory and their applications to queuing, stochastic games, seismology, and finance (cf. [1,2,8-10,11,12,13,15,16,19,22-24,27,30]). Fluctuation theory pertains to the behavior of an underlying process around a critical threshold and more generally, when a process escapes from a fixed manifold. The time when that passage takes place is referred

* Corresponding author: mailto:eugene@fit.edu

to as the first passage time. Another critical value of that situation is the new location of the process upon its escape. Besides the original topics mentioned above, fluctuation theory has become a stand-alone subject in numerous articles appeared through the decades of intense research, cf. [3-7,17,20,21,29,31].

In our most recent paper [15], we worked with time sensitive functionals of the same entities but under real time observation of a monotone process. We dealt with non-negative random measures and increment processes. In this paper we study a class of signed marked random measures $(\mathcal{A}, \mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty} (X_n, \pi_n)\, \varepsilon_{t_n}$ with position dependent marking, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Marks X_n's are non-negative, while marks π_n's are real-valued, with the support counting random measure $\sum_{n=0}^{\infty} \varepsilon_{t_n}$. This is a significant upgrade from [15], because not only is yet another component added, but it is non-monotone. Studies of non-monotone components are very few in the literature on fluctuations. Most prominent of them was by Lajos Takács [30]. However, the results in [30] were not tractable.

As in the theory of fluctuations, we focus on the behavior of $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ around a fixed threshold $M > 0$ with respect to its first component $\mathcal{A}$, referred to as an active component. With

$$A_n = X_0 + X_1 + \ldots + X_n \tag{1.1}$$

we have $\{A_n\}$ monotone non-decreasing, whereas

$$P_n = \pi_0 + \pi_1 + \ldots + \pi_n \tag{1.2}$$

is non-monotone, as π_k's are real-valued marks. Our interest is in an extreme behavior of the marginal process $(\mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}$ that is difficult to analyze due to the non-monotone nature of its marks. For that reason we introduce active mark X_n being nonnegative and integer-valued that is to oversee π_n. For instance, we might be curious when the process $(\mathit{\Pi}, \mathcal{T})$ changes its monotonicity or when it experiences its first extreme drop or a surge. For example, we set $X_0 = X_1 = \ldots = X_{n-1} = 0, X_n = 1$, if $\pi_0 > a, \pi_1 > a, \ldots, \pi_{n-1} > a$, and $\pi_n \le a$. In the general case, the increments X_i's need not be constant, but they can be random variables with particular marginal distributions. For a fixed positive integer M, we define the exit index as

$$\nu := inf\,\{n = 0, 1, \ldots : A_n \ge M\}\,. \tag{1.3}$$

Then, t_ν is called the first passage time of process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$. It is the first epoch when the crossing of M occurs. Obviously, t_ν is a stopping time relative to filtration $\mathcal{F}_t$. The respective excess values of A_ν and P_ν representing active and passive components, $\mathcal{A}$ and $\mathit{\Pi}$, respectively, are also of interest. We further assume that **A1** the increments $\{X_n, \pi_n, \Delta_n = t_n - t_{n-1}\}$ for $n = 0, 1, 2, \ldots$, $t_{-1} = 0$, of the process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ are independent (position dependent marking), that is, X_n and π_n are dependent only on Δ_n. **A2** for $n = 1, 2, \ldots$, $\{X_n, \pi_n, \Delta_n\}$ are identically distributed.

Associated with $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ is the "time sensitive counting" process

$$(N_t, \mathit{\Pi}_t) = (\mathcal{A}, \mathit{\Pi})\,[0, t] = \sum_{n=0}^{\infty} (X_n, \pi_n)\, \varepsilon_{t_n}\,[0, t]\,, t \ge 0. \tag{1.4}$$

We will be interested in the value of $(N_t, \mathit{\Pi}_t)$ of some t enclosed between $t_{\nu-1}$ and t_ν providing us with the information about $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ between two key reference points as well as $(N_t, \mathit{\Pi}_t)$ for $t \in [0, \tau_\nu)$ (that we will discuss later on, in Section 5).

So we target the joint Laplace- and Fourier-Stieltjes transform of the above r.v.'s:

$$\begin{aligned}\varPhi_\nu(t) = Ez^{N_t} e^{-i\eta\varPi_t} e^{-i\varphi P_{\nu-1}} u^{A_{\nu-1}} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t), \\ \|z\| \le 1, \|u\| \le 1, \|v\| \le 1, Re\vartheta_0 \ge 0, Re\vartheta \ge 0, \eta \in \mathbb{R}, \varphi \in \mathbb{R}, \phi \in \mathbb{R}.\end{aligned} \tag{1.5}$$

Note that because we manage to observe the process in real time, i.e., upon $t_0, t_1, t_2, \ldots$ (meaning that there are no changes between those epochs), it raises a question about a need in the continuous time interpolation. Indeed, in some past work (cf. Dshalalow and White [17]) when a process was observed over arbitrary time epochs (i.e., unrelated to $t_0, t_1, t_2, \ldots$), its continuous revival made perfect sense. In our case, however, it is more about associating the point process $t_0, t_1, t_2, \ldots$, especially the reference points $t_{\nu-1}, t_\nu$, with time t, than anything else. Its very obvious benefit is to know about the process over time related intervals like $[0, t]$ which was impossible with time insensitive versions. From a practical stand point, observing the process over arbitrary time epochs is more realistic than in real time. However, whenever it is possible to render, its second benefit lies in far more tractable results compared to delayed observations that additionally require the named point process to be Poisson or alike. Furthermore, we also obtain explicit characteristics of the continuous time parameter process in interval $[0, t_\nu)$ giving us a broad spectrum of information about process N_t. The associated functional will read

$$Ez^{N_t} e^{-i\eta\varPi_t} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0, t_\nu)}(t), \|z\| \le 1, \|v\| \le 1, Re\vartheta \ge 0, \eta \in \mathbb{R}, \phi \in \mathbb{R}. \tag{1.6}$$

Back to the random measure $(\mathcal{A}, \varPi, \mathcal{T})$, we recall that the passive component $\varPi$ is real-valued making this random measure signed. Studies related to signed random measures have previously been done in various topological and stochastic analysis contexts. In [19] Hellmund extended the idea of completely random measures to completely random signed measure and gave a characterization of this class of signed random measures. He demonstrated that the classes of Lévy random measures (utilized in Lévy adaptive regression kernel models) and Lévy bases (utilized in spatio-temporal modeling) are natural extensions of completely random signed measures and that independence is a fundamental concept in defining Lévy random measures and Lévy bases. Other concepts related to signed random measures are in the work by Smorodina and Faddeev [29] who studied symmetric stable signed measures and showed that they are limit measures of sums of independent random variables.

Various applications of fluctuation theory that we explore can also be found in stochastic signals such as time continuous readings for automated seizure detection and quantification using EEGs, heart attack activity monitoring through detection by EKGs, real time blood pressure monitoring, and the stock market. In this paper, we illustrate the applicability of our study by expounding on the case of stock prices. We are able to predict the time of the first drop of a stock (or first increase if we short it) at t_ν and thus, the highest price at $t_{\nu-1}$ at which we can sell it at that point in time.

Our model also applies to the classical risk problem originally posed by Filip Lundberg (see [27]). Assume that an insurance company starts at zero with the initial capital u and let the premium be a linear function with a constant premium rate c, so that the premium income of the company at time t is $u + ct$. Assume that the aggregate claims form a marked point process $\mathcal{Y} = \sum_{k=0}^{\infty} Y_k \varepsilon_{t_k}$, with t_k being the time of the kth claim and Y_k - the amount of claim. Now Lundberg postulated that $\mathcal{Y}$ was a marked Poisson process with position independent marking. We relax either condition by assuming that neither is $\mathcal{Y}$ Poisson, nor is it with position independent marking. If $\Delta_k = t_k - t_{k-1}$,we have $c\Delta_k$ premiums' increase from t_{k-1} to t_k. The mark $\pi_k = c\Delta_k - Y_k$ is the change of company's asset from t_{k-1} to t_k. Now,

$$\varPi = \sum_{k=0}^{\infty} \pi_k \varepsilon_{t_k}$$

is a purely signed marked random measure and

$$P_t = \varPi\, [0, t]$$

is the process describing the asset changes of the insurance company on interval $[0, t]$. Notice that P_t does not give us the true value of the company's asset at time t, because P_t is a piecewise constant interpolation of the true asset value process

$$R_t = u + ct - \sum_{k=0}^{\infty} Y_k \varepsilon_{t_k}\, [0, t]$$

known as the risk process. They coincide upon times $t_0, t_1, t_2, \ldots$ which is exactly what we need. Our process $(\mathcal{A}, \varPi, \mathcal{T})$ is defined through the active component

$$X_k = \begin{cases} 0, & \pi_k > 0, \\ 1, & \pi_k \le 0. \end{cases}$$

So we are interested in the moment when P_k becomes negative or zero for the first time (which would trigger $X_k = 1$). Thus, $\pi_0, \pi_1, \ldots, \pi_{\nu-1}$ are positive, while π_ν is negative or zero. $\{t_{\nu_n}\}$ is the embedded sequence of consecutive drops of P_t. Then obviously, the risk process R_t will become negative or zero only upon one of the epochs $\{t_{\nu_n}\}$, known as the ruin time of R_t.

Let $\mathcal{F}_t$ be the natural filtration with respect to the risk process R_t. Then, $\{t_{\nu_n}\}$ is a sequence of stopping times relative $\mathcal{F}_t$ that are also locally strong Markov points, that is either R_t and P_t have a locally strong Markov property at each point t_{ν_n}. Therefore, R_t and P_t conditionally regenerate upon these epochs. We can slightly modify P_t to make it semi-regenerative with respect to $\{t_{\nu_n}\}$.

While a further discussion on the risk process and its study as a semi-regenerative process is beyond the scope of this paper, the time of the first or the second or the nth drop of the risk process is of interest for statistics purposes and it is often raised by insurance companies.

We continue this paper in Section 2 through a further formalism of our model and introduce basics of discrete operational calculus earlier developed by Dshalalow [6,7] and Dshalalow and Iwezulu [13]. In Section 3, we use the method of stochastic decomposition previously developed in Dshalalow and Nandyose [15] and Dshalalow and White [17,18], only now embellished for non-monotone components. We establish a key formula for the functional $\varPhi_\nu(t)$ of (1.5) that we claim is analytically tractable. This claim is justified throughout Section 4 in a number of examples and special cases. We conclude our paper in Section 5 with time sensitive analysis where time t runs interval $[0, \tau_\nu)$ and find the joint transform of N_t, P_t, N_ν,and the first passage time t_ν in a fully closed form.

2 Formalism and Notation

We now return to the functional $\varPhi_\nu$. Note that we do not know the distribution of the random vector $(A_\nu - A_{\nu-1}, t_\nu - t_{\nu-1})$ nor is the latter independent of $(A_{\nu-1}, t_{\nu-1})$. The remedy for this predicament is the use of stochastic expansion that will include several steps. In the first step, we introduce the auxiliary sequence $\{\nu(p)\}$ of exit indices relative to the sequence $\{0, 1, \ldots\}$ of thresholds to be crossed by A_n, of which $\nu = \nu(M-1)$ was introduced in (1.3). Namely, let

$$\nu(p) = inf\{n = 0, 1, \ldots : A_n > p\}, p = 0, 1, \ldots \tag{2.1}$$

With p fixed, we have the sequence of functionals

$$\varPhi_{\nu(p)}(t) = Ez^{N_t} e^{-i\eta\varPi_t} e^{-i\varphi P_{\nu(p)-1}} u^{A_{\nu(p)-1}} e^{-i\phi P_{\nu(p)}} v^{A_{\nu(p)}} e^{-\vartheta_0 t_{\nu(p)-1} - \vartheta t_{\nu(p)}} \mathbf{1}_{[t_{\nu(p)-1}, t_{\nu(p)})}(t). \tag{2.2}$$

In our second step, we apply to $\varPhi_{\nu(p)}$ of (2.2) the transformation D_p defined as

$$D_p\{f(p)\}(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), ||x|| < 1, \tag{2.3}$$

where f is a real-valued function with the domain $\mathbb{N}_0 = \{0, 1, \ldots\}$. The inverse of D_p is the so-called $\mathcal{D}$-operator previously introduced in Dshalalow [6,7]:

$$\mathcal{D}_x^k \varphi(x, y) = \begin{cases} \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x} \varphi(x, y) \right], & k \geq 0 \\ 0, & k < 0. \end{cases} \tag{2.4}$$

From $\varPhi_{\nu(p)}(t) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}(t) \mathbf{1}_{\{v(p)=n\}}$, we have

$$\varPhi(t, x) := D_p\left[\varPhi_{\nu(p)}(t)\right](x) = \sum_{n=0}^{\infty} \varPhi_{\nu(p)}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x)$$

$$= \sum_{n=0}^{\infty} \varPhi_{\nu(p)=n}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x),$$

with

$$\varPhi_{\nu(p)=n}(t) = Ez^{N_t} u^{A_{n-1}} e^{-i\eta\varPi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \leq t < t_n\}} = F_n(t). \tag{2.5}$$

From $\mathbf{1}_{\{v(p)=n\}} = \mathbf{1}_{\{A_{n-1} \leq p\}} \mathbf{1}_{\{A_n > p\}}$,

$$D_p \mathbf{1}_{\{v(p)=n\}}(x) = (1-x) \sum_{p=0}^{\infty} x^p \mathbf{1}_{\{A_{n-1} \leq p\}} \mathbf{1}_{\{A_n > p\}}$$

$$= (1-x) \sum_{p=A_{n-1}}^{A_n - 1} x^p$$

$$= (1-x)\left(\sum_{p=0}^{A_n-1} x^p - \sum_{p=0}^{A_{n-1}-1} x^p \right) = (1-x)\left(\frac{1-x^{A_n}}{1-x} - \frac{1-x^{A_{n-1}}}{1-x} \right) = x^{A_{n-1}} - x^{A_n}$$

that yields

$$\Phi\left(t,x\right)=\sum_{n=0}^{\infty}F_{n}\left(t\right)\left(x^{A_{n-1}}-x^{A_{n}}\right)$$
$$=\sum_{n=0}^{\infty}\left[F_{n}\left(ux,v,z,\vartheta_{0},\vartheta,t\right)-F_{n}\left(u,vx,z,\vartheta_{0},\vartheta,t\right)\right],whereA_{-1}=0. \tag{2.6}$$

Finally, applying the Laplace transform to $\Phi\left(t,x\right)$ of (2.6) we have

$$\Phi^{*}\left(\theta,x\right)=\int_{t=0}^{\infty}e^{-\theta t}\Phi\left(t,x\right)dt=\sum_{n=0}^{\infty}\left[F_{n}^{*}\left(ux,v,z,\vartheta_{0},\vartheta,t\right)-F_{n}^{*}\left(u,vx,z,\vartheta_{0},\vartheta,t\right)\right]. \tag{2.7}$$

Now functionals F_n and their transforms F_n^* are subject to our scrutiny in Section 3.

3 Analysis of F_n

With $n=1,2,\dots$, we work on

$$F_{n}\left(t\right)=Ez^{N_{t}}u^{A_{n-1}}e^{-i\eta\Pi_{t}}e^{-i\varphi P_{n-1}}v^{A_{n}}e^{-i\phi P_{n}}e^{-\vartheta_{0}t_{n-1}-\vartheta t_{n}}\mathbf{1}_{\{t_{n-1}\leq t<t_{n}\}}, \tag{3.1}$$

(defined in (2.5)). (3.1) can be brought to the expression

$$F_{n}\left(t\right)=E[(zuv)^{A_{n-1}}e^{-i(\eta+\varphi+\phi)P_{n-1}}e^{-(\vartheta_{0}+\vartheta)t_{n-1}}v^{A_{n}-A_{n-1}}$$
$$\times e^{-i\phi(P_{n}-P_{n-1})}e^{-\vartheta(t_{n}-t_{n-1})}\mathbf{1}_{\{t_{n-1}\leq t<t_{n}\}}]$$
$$=E(zuv)^{A_{n-1}}e^{-i(\eta+\varphi+\phi)P_{n-1}}e^{-(\vartheta_{0}+\vartheta)t_{n-1}}v^{X_{n}}e^{-i\phi\pi_{n}}e^{-\vartheta\Delta_{n}}\mathbf{1}_{\{t_{n-1}\leq t<t_{n}\}},n=1,2\dots \tag{3.2}$$

The Laplace transform of F_n with the expectation unfolded reads

$$F_{n}^{*}\left(\theta\right)=\int_{t=0}^{\infty}e^{-\theta t}F_{n}\left(t\right)dt$$
$$=\sum_{k=0}^{\infty}(zuv)^{k}\sum_{j=0}^{\infty}v^{j}\int_{p=-\infty}^{\infty}e^{-i(\eta+\varphi+\phi)p}\int_{s\geq0}e^{-(\vartheta_{0}+\vartheta)s}e^{-\theta s}$$
$$\times\int_{q=-\infty}^{\infty}e^{-i\phi q}\int_{\delta\geq0}e^{-\vartheta\delta}\int_{t-s=0}^{\delta}e^{-\theta(t-s)}dt$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_{n}\otimes\pi_{n}\otimes\Delta_{n}}\left(k,j,dp,ds,dq,d\delta\right)$$
$$=\frac{1}{\theta}\sum_{k=0}^{\infty}(zuv)^{k}\sum_{j=0}^{\infty}v^{j}\int_{p=-\infty}^{\infty}e^{-i(\eta+\varphi+\phi)p}\int_{s\geq0}e^{-(\vartheta_{0}+\vartheta+\theta)s}\int_{q=-\infty}^{\infty}e^{-i\phi q}\int_{\delta\geq0}e^{-\vartheta\delta}$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_{n}\otimes\pi_{n}\otimes\Delta_{n}}\left(k,j,dp,ds,dq,d\delta\right)$$
$$-\frac{1}{\theta}\sum_{k=0}^{\infty}(zuv)^{k}\sum_{j=0}^{\infty}v^{j}\int_{p=-\infty}^{\infty}e^{-i(\eta+\varphi+\phi)p}\int_{s\geq0}e^{-(\vartheta_{0}+\vartheta+\theta)s}\int_{q=-\infty}^{\infty}e^{-i\phi q}\int_{\delta\geq0}e^{-(\vartheta+\theta)\delta}$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_{n}\otimes\pi_{n}\otimes\Delta_{n}}\left(k,j,dp,ds,dq,d\delta\right)$$

due to independence of $A_{n-1}\otimes P_{n-1}\otimes t_{n-1}$ and $X_n\otimes\pi_n\otimes\Delta_n$

$$=\frac{1}{\theta}E[(zuv)^{A_{n-1}}\,e^{-i(\eta+\varphi+\phi)P_{n-1}}e^{-(\vartheta_{0}+\vartheta+\theta)t_{n-1}}]$$

$$\times\left[Ev^{X_n}e^{-i\phi\pi_n}e^{-\vartheta\Delta_n}-Ev^{X_n}e^{-i\phi\pi_n}e^{-(\vartheta+\theta)\Delta_n}\right]$$
$$=\frac{1}{\theta}\mathit{\Gamma}_{n-1}\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right], \quad (3.3)$$

where

$$\mathit{\Gamma}_{n-1}(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)$$
$$=\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\gamma^{n-1}\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\text{ for } n\geq 1 \quad (3.4)$$

and

$$\gamma_0\left(u,\varphi,\vartheta\right)=Eu^{X_0}e^{-i\varphi\pi_0}e^{-\vartheta t_0},\ \gamma\left(u,\varphi,\vartheta\right)=Eu^{X_k}e^{-i\varphi\pi_k}e^{-\vartheta\Delta_k}, k=1,2,\ldots. \quad (3.5)$$

Summing up F_n for all $n=1,2,\ldots$, with (3.3-3.4) in mind, we formally arrive at the expression

$$\sum_{n=1}^{\infty}F_n^*\left(\theta\right)=\frac{1}{\theta}\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)$$
$$\times\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)}. \quad (3.6)$$

To warrant the convergence of the geometric series $\sum_{n=1}^{\infty}F_n^*\left(\theta\right)$, in the proposition below, we show that the norm $\left\|\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\right\|<1$.

Proposition 3.1 *The series*

$$\sum_{n=1}^{\infty}F^*\left(\theta\right)=\sum_{n=1}^{\infty}\int_{t=0}^{\infty}e^{-\theta t}E[(zuv)^{A_{n-1}}e^{-i(\eta+\varphi+\phi)P_{n-1}}$$
$$\times e^{-(\vartheta_0+\vartheta)t_{n-1}}v^{X_n}e^{-i\phi\pi_n}e^{-\vartheta\Delta_n}\mathbf{1}_{\{t_{n-1}\leq t<t_n\}}]dt$$

converges to

$$\sum_{n=1}^{\infty}F_n^*\left(\theta\right)=\frac{1}{\theta}\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)$$
$$\times\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)},$$

with

$$\left\|\gamma(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\right\|<1,$$

provided one of the following conditions is met:

$$Re\vartheta_0>0, or Re\vartheta>0, or Re\,\theta>0 or \left\|u\right\|<1, or \left\|v\right\|<1, or \left\|z\right\|<1.$$

Proof. The first part of the proposition is due to the above steps that formally ended in formula (3.6). Inequality (3.7) holds due to the following arguments:

$$\left\|\gamma(uvz,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\right\|\leq E\left\|(uvz)^{X_1}e^{-i(\eta+\varphi+\phi)\pi_n}e^{-(\vartheta_0+\vartheta+\theta)\Delta_1}\right\|$$
$$=\sum_{k=0}^{\infty}\left\|uvz\right\|^k\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(k,dt\right)$$
$$=\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(0,dt\right)+\sum_{k=1}^{\infty}\left\|uvz\right\|^k\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(k,dt\right)$$

$$= \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(0,dt) + \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(0,dt)$$

$$+ \sum_{k=1}^{\infty} \left\|uvz\right\|^k \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(k,dt)$$

$$+ \sum_{k=1}^{\infty} \left\|uvz\right\|^k \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(k,dt)$$

$$\le \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(0,dt) + e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(0,dt)$$

$$+\left\|uvz\right\| \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(k,dt) + \left\|uvz\right\| \sum_{k=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(k,dt),$$

since $\left\|uvz\right\| \ge \left\|uvz\right\|^k$ for $\left\|uvz\right\| \le 1$ and $k > 1$. Let

$$a := \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(0,dt),\ b := \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(0,dt)$$

$$c := \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(k,dt),\ d := \sum_{k=1}^{\infty} \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(k,dt).$$

Then clearly, $a+b+c+d = 1$ and thus,

$$a + e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} b + \left\|uvz\right\| c + \left\|uvz\right\| e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} d < 1$$

whenever $\left\|uvz\right\| < 1$ or $\mathrm{Re}(\vartheta_0+\vartheta+\theta) > 0$ and we are done with the proof. □

We continue with F_n for $n = 0$. F_0 is the functional of the underlying process on interval $[0, t_0)$. With $N_t = \Pi_t = A_{-1} = P_{-1} = t_{-1} = 0$ we have

$$F_0(t) = Ez^{N_t} u^{A_{n-1}} e^{-i\eta\Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{0\le t<t_0\}}$$

$$= Ev^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0)}(t).$$

The following is easy to prove.

Proposition 3.2 *Let* $F_0(t) = Ev^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0)}(t)$. *Then*

$$F_0^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right]. \tag{3.7}$$

With Proposition 3.2, we can augment the series $\sum_{n=1}^{\infty} F_n^*$ of formula (3.6) to include F_0^*:

$$\sum_{n=0}^{\infty} F_n^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right] + \frac{1}{\theta}\gamma_0(zuv, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)$$
$$\times \left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta)\right] \frac{1}{1-\gamma(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)}. \tag{3.8}$$

From (2.7) and (3.8) we arrive at

$$\Phi^*(\theta, x) = \sum_{n=0}^{\infty} \left[F_n^*(ux, v, z, \vartheta_0, \vartheta, t) - F_n^*(u, vx, z, \vartheta_0, \vartheta, t)\right]$$

$$= \frac{1}{\theta}\left[\gamma_0(v, \phi, \vartheta) - \gamma_0(v, \phi, \vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx, \phi, \vartheta) - \gamma_0(vx, \phi, \vartheta+\theta)\right]$$

$$+\frac{1}{\theta}\gamma_0(zuvx, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)\frac{1}{1-\gamma(zuvx, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)}$$

$$\times\left[\gamma(v, \phi, \vartheta) - \gamma(vx, \phi, \vartheta) + \gamma(vx, \phi, \vartheta+\theta) - \gamma(v, \phi, \vartheta+\theta)\right]. \qquad (3.9)$$

The Laplace transform $\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}\Phi_\nu(t)\,dt$ of the functional

$$\Phi_\nu(t) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)$$

can be extracted from $\Phi^*(\theta, x)$ of (4.9) using the $\mathcal{D}$-operator.

The entire effort in this section can be reduced to the following.

Theorem 3.1 *Let $\Phi_\nu(\theta)$ denote the Laplace transform of the functional*

$$\Phi_\nu(t) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) \qquad (3.10)$$

$$\|z\| \le 1, \|u\| \le 1, \|v\| \le 1, Re\vartheta_0 \ge 0, Re\vartheta \ge 0, \eta, \varphi, \phi \in \mathbb{R},$$

$$Then, with\, \|u\| < 1, or\;\; \|v\| < 1, or\;\; \|z\| < 1,\;\; or\;\; Re\vartheta_0 > 0, or\;\; Re\vartheta > 0,\;\; or\;\; Re\,\theta > 0, \qquad (3.11)$$

$$\Phi_\nu^*(\theta)$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\theta}\left[\gamma_0(v, \phi, \vartheta) - \gamma_0(v, \phi, \vartheta+\theta)\right] - \frac{1}{\theta}\left[\gamma_0(vx, \phi, \vartheta) - \gamma_0(vx, \phi, \vartheta+\theta)\right]\right.$$

$$+\frac{1}{\theta}\gamma_0(zuvx, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)\frac{1}{1-\gamma(zuvx, \eta+\varphi+\phi, \vartheta_0+\vartheta+\theta)}$$

$$\left.\times\left[\gamma(v, \phi, \vartheta) - \gamma(vx, \phi, \vartheta) + \gamma(vx, \phi, \vartheta+\theta) - \gamma(v, \phi, \vartheta+\theta)\right]\right\}. \qquad (3.12)$$

4 Applications to Option Trading

For an illustration, consider the following special case. Suppose that we observe a constantly fluctuating stock price of some company over the times $t_0 = 0, t_1, t_2, \ldots$ that starts off at time zero with a price π_0.

Case 1. Observation of process P_i upon the first drop.

1a. Suppose we are interested in the characteristics of the process around the period when the stock price drops for the first time. Because the stock prices cannot be modeled by a monotone process, we have the observed prices upon t's as the passive component, and introduce the active component

$$X_n = \begin{cases} 0, & \pi_n \geq 0, \\ 1, & \pi_n < 0. \end{cases} \tag{4.1}$$

Suppose π_0 is a nonnegative r.v. with some specified distribution and let $X_0 = \tau_0 = 0$.
So, $\gamma_0(z, \phi, \theta) = Ee^{-i\phi\pi_0}$ $(innotation) = \gamma_0(\phi)$.

Next, with $M = 1$ according to our assumption about the first drop, formula (3.12) further reduces to

$$\begin{aligned} \theta\Phi^*_\nu(\theta) = \gamma_0(\eta + \varphi + \phi) \frac{1}{1 - \gamma(0, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta)} \\ \times [\gamma(v, \phi, \vartheta) - \gamma(0, \phi, \vartheta) + \gamma(0, \phi, \vartheta + \theta) - \gamma(v, \phi, \vartheta + \theta)]. \end{aligned} \tag{4.2}$$

Because the active component is merely auxiliary, we are less interested in any information about $N_t, A_{\nu-1}, A_\nu$, as well as $P_{\nu-1}, t_{\nu-1}$, so we set $z = u = v = 1$ and $\varphi = \vartheta_0 = 0$ restricting the Laplace transform of Φ_ν to the marginal transform

$$\begin{aligned} &\int_{t=0}^{\infty} e^{-\theta t} E e^{-i\eta\Pi_t} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t)\, dt \\ &= \frac{1}{\theta}\gamma_0(\eta + \phi) \frac{1}{1 - \gamma(0, \eta + \phi, \vartheta + \theta)} \\ &\times [\gamma(1, \phi, \vartheta) - \gamma(0, \phi, \vartheta) + \gamma(0, \phi, \vartheta + \theta) - \gamma(1, \phi, \vartheta + \theta)], \end{aligned} \tag{4.3}$$

where

$$\gamma(z, \phi, \theta) = Ez^{X_1} e^{-i\phi\pi_1} e^{-\Delta_1\theta} \; and \gamma(0, \phi, \theta) = Ez^{X_1} e^{-i\phi\pi_1} e^{-\Delta_1\theta}\big|_{z=0}.$$

From

$$Ez^{X_1}\big|_{z=0} = P\{X_1 = 0\} + zP\{X_1 = 1\}\big|_{z=0} = P\{X_1 = 0\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1 \geq 0\}}$$

we have

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} e^{-\Delta_1\theta}.$$

Suppose now that Δ's and π's are independent, that is, the observation epochs and stock price changes are independent. This may not always apply, but it would simplify establishing of γ. Then

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} E e^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} \frac{\gamma}{\gamma + \theta},$$

if the observation epochs occur according to a Poisson point process of intensity γ. Our next assumption is that the marginal distribution of π_1 is Laplace with parameter μ and zero shift. That being said, the PDF of π_1 is

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x|}, x \in \mathbb{R}. \tag{4.4}$$

Then

$$\gamma(0,\phi,0)=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}=\int_{x=0}^{\infty}e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx=\frac{1}{2}\frac{\mu}{\mu+i\phi}.$$

Because $Ee^{-i\phi\pi_1}=Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq 0\}}+\mathbf{1}_{\{\pi_1<0\}}\right)$,we have

$$Ee^{-i\phi\pi_1}=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\int_{x=-\infty}^{0}e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx$$

$$=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\frac{1}{2}\frac{\mu}{\mu-i\phi}=\frac{1}{2}\mu\frac{2\mu}{\mu^2+\phi^2}=\frac{\mu^2}{\mu^2+\phi^2}.$$

Thus,

$$\gamma(1,\phi,\theta)=Ee^{-i\phi\pi_1}Ee^{-\Delta_1\theta}=Ee^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta}=\frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}.$$

Next the following two further marginals are of interest.

(i) With $\eta=\phi=0$ in (4.3),the functional

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\frac{1}{\theta}\frac{1}{1-\gamma(0,0,\vartheta+\theta)}\left[\gamma(1,0,\vartheta)-\gamma(0,0,\vartheta)+\gamma(0,0,\vartheta+\theta)-\gamma(1,0,\vartheta+\theta)\right]\tag{4.5}$$

represents the Laplace transform of the first passage time t_ν's marginal functional at the first drop with the time t falling between the pre-first passage time $t_{\nu-1}$ and t_ν. Here

$$\gamma(1,0,\vartheta+\theta)=\frac{\gamma}{\gamma+\vartheta+\theta}$$

$$\gamma(0,\phi,\theta)=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}Ee^{-\Delta_1\theta}=\frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta}$$

$$\gamma(0,0,\vartheta)=\frac{1}{2}\frac{\gamma}{\gamma+\vartheta}.$$

Therefore,

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\left(1+\frac{\gamma}{\gamma+2(\vartheta+\theta)}\right)\frac{\gamma}{2}\frac{1}{(\gamma+\vartheta)(\gamma+\vartheta+\theta)}\tag{4.6}$$

implying that the inverse of the Laplace transform is

$$Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)=\mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\}=\frac{\gamma}{2(\gamma+\vartheta)}e^{-\frac{t}{2}(\gamma+2\vartheta)}.\tag{4.7}$$

(ii) With $\eta=\vartheta=0$ in (4.3), we have the Laplace transform of the P_ν's marginal functional upon the first passage time t_ν jointly with the time t running between $t_{\nu-1}$ and t_ν.

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\frac{1}{\theta}\gamma_0(\phi)\frac{1}{1-\gamma(0,\phi,\theta)}\left[\gamma(1,\phi,0)-\gamma(0,\phi,0)+\gamma(0,\phi,\theta)-\gamma(1,\phi,\theta)\right].\tag{4.8}$$

Because

$$\gamma_0(\phi) = e^{-i\phi p_0}$$

(assuming the initial price $\pi_0 = p_0$ a.s. where p_0 is a constant)
and

$$\gamma(1,\phi,\theta) = \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta},$$

$$\gamma(0,\phi,\theta) = E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = \frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta},$$

$$1-\gamma(0,\phi,\theta) = \frac{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)},$$

and

$$\frac{1}{1-\gamma(0,0,\vartheta+\theta)} = 1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma},$$

we have that

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-\theta t}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\\
&= \frac{1}{\theta}e^{-i\phi p_0}\left(1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right)\\
&\times\left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}+\frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta}-\frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}\right]\\
&= e^{-i\phi p_0}\left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}\right]\left(1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right)\frac{1}{\gamma+\theta}.
\end{aligned} \tag{4.9}$$

Thus,

$$\begin{aligned}
Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) &= \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\right\}\\
&= \left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}\right]e^{-\left(\frac{\gamma t}{2}\left(\frac{\mu+2i\phi}{\mu+i\phi}\right)+i\phi p_0\right)}
\end{aligned} \tag{4.10}$$

and

$$EP_\nu\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = i\lim_{\phi\to 0}\frac{\partial}{\partial\phi}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = \frac{1}{2\mu}\left(\frac{\gamma t}{2}+\mu p_0-1\right)e^{-\frac{\gamma t}{2}} \tag{4.11}$$

$$\begin{aligned}
EP_\nu^2\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) &= -\lim_{\phi\to 0}\frac{\partial^2}{\partial\phi^2}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\\
&= \frac{1}{2\mu^2}\left(2+\left(\frac{\gamma t}{2}\right)^2+(\mu p_0)^2+2\mu p_0\frac{\gamma t}{2}-2\mu p_0\right)e^{-\frac{\gamma t}{2}}.
\end{aligned} \tag{4.12}$$

So

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\{t_{\nu-1}\leq t<t_\nu\} = \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\} = \frac{e^{-\frac{\gamma}{2}t}}{2}. \tag{4.13}$$

1b. One could be interested in when the passive component drops lower than R, for some $R<0$. Thus the active component reads now

$$X_n = \begin{cases}0, & \pi_n\geq R,\\ 1, & \pi_n<R.\end{cases} \tag{4.14}$$

With $M=1$ assumed and because

$$Ez^{X_1}\big|_{z=0} = P\{X_1=0\} + zP\{X_1=1\}\big|_{z=0} = P\{X_1=0\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1\geq R\}},$$

we have

$$\gamma(0,\phi,\theta) = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1}e^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta},$$

$$\gamma(0,\phi,0) = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1} = \int_{x=R}^{0} e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx + \int_{x=0}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx$$

$$= \frac{1}{2}\frac{\mu}{\mu-i\phi}\left[1-e^{(\mu-i\phi)R}\right] + \frac{1}{2}\frac{\mu}{\mu+i\phi}.$$

Since

$$Ee^{-i\phi\pi_1} = Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq R\}} + \mathbf{1}_{\{\pi_1<R\}}\right),$$

we have

$$Ee^{-i\phi\pi_1} = \frac{1}{2}\frac{\mu}{\mu-i\phi}\left[1-e^{(\mu-i\phi)R}\right] + \frac{1}{2}\frac{\mu}{\mu+i\phi} + \int_{x=-\infty}^{R} e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx = \frac{\mu^2}{\mu^2+\phi^2}.$$

Thus,

$$\gamma(1,\phi,\theta) = Ee^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = Ee^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta} = \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}$$

and with $\eta=\phi=0=\vartheta$ in (4.3), the functional

$$\int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \frac{1}{\theta}\frac{1}{1-\gamma(0,0,\theta)}\left[\gamma(1,0,0)-\gamma(0,0,0)+\gamma(0,0,\theta)-\gamma(1,0,\theta)\right] = e^{\mu R}\frac{1}{2\theta+\gamma e^{\mu R}}$$

and

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\{t_{\nu-1}\leq t<t_\nu\}$$
$$= \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\} = \frac{1}{2}e^{-\left(\frac{\gamma t e^{\mu R}}{2}-\mu R\right)} \tag{4.15}$$

which reduces to (4.13) when $R=0$.

Case 2. Observation of process P_i upon general Mth drop.

2a. For the general threshold level M (when the stock price drops Mth times), since the active process increments X_n are Bernoulli with $p=0.5$ due to the symmetric Laplace PDF of π_n defined in (4.4) above with zero shift and with

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = \Phi_\nu(t)\big|_{z,v,u,\vartheta=1,\eta,\varphi,\phi,\vartheta_0,\vartheta=0},$$

$$\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0(0)\frac{1}{1-\gamma(x,0,\theta)}\times\left[\gamma(1,0,0)-\gamma(x,0,0)+\gamma(x,0,\theta)-\gamma(1,0,\theta)\right], \tag{4.16}$$

where

$$\gamma(1,0,\theta) = \frac{\gamma}{\gamma+\theta}, \quad \gamma(x,0,0) = \frac{1+x}{2},\ \gamma(x,0,\theta) = (\frac{1+x}{2})\frac{\gamma}{\gamma+\theta}.$$

Therefore,

$$\Phi_\nu^*(\theta) = \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[1-\frac{1+x}{2}+\frac{1+x}{2}\frac{\gamma}{\gamma+\theta}-\frac{\gamma}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[\frac{1-x}{2}\frac{\theta}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\} - \mathcal{D}_x^{M-2}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\} = \frac{1}{(\gamma+2\theta)}\left(\frac{\gamma}{\gamma+2\theta}\right)^{M-1} = \frac{\gamma^{M-1}}{(\gamma+2\theta)^M}.$$

So

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = P\{t_{\nu-1}\le t < t_\nu\} = \mathcal{L}_\theta^{-1}\{\Phi_\nu(\theta)\} = \frac{1}{2}\frac{\left(\frac{\gamma t}{2}\right)^{M-1}}{(M-1)!}e^{-\frac{\gamma}{2}t}\ . \tag{4.17}$$

2b. Next we obtain the result for $E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ for general M and general shift parameter a in our model such that

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x-a|}, x\in\mathbb{R}.$$

After some algebra we have

$$\gamma(0,\phi,0) = E\mathbf{1}_{\{\pi_1\ge 0\}}e^{-i\phi\pi_1} = \int_{x=0}^{a} e^{-i\phi x}\frac{1}{2}\mu e^{\mu(x-a)}dx + \int_{x=a}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu(x-a)}dx$$

$$= \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2},$$

$$\gamma(0,\phi,\theta) = \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}$$

and

$$\gamma(0,\phi,0) = \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2},$$

$$Ee^{-i\phi\pi_1} = \frac{2\mu^2e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]$$

$$\gamma(1,\phi,\theta) = \left[\frac{2\mu^2e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}$$

$$\gamma(x,\phi,\theta) = (p+qx)\left[\frac{2\mu^2e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}.$$

Hence

$$\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\,dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0(0)\frac{1}{1-\gamma(x,0,\theta)} \times [\gamma(1,0,0) - \gamma(x,0,0) + \gamma(x,0,\theta) - \gamma(1,0,\theta)]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\theta}\frac{(\gamma+\theta)}{\gamma+\theta-p\gamma(2-\cosh(\mu a)) - q\gamma(2-\cosh(\mu a))x}\right.$$

$$\times\left[(2-\cosh(\mu a))-(p+qx)\,(2-\cosh\mu a))+(p+qx)\,(2-\cosh(\mu a))\,\frac{\gamma}{\gamma+\theta}\right.$$

$$\left.\left.-\,(2-\cosh(\mu a))\,\frac{\gamma}{\gamma+\theta}\right]\right\}$$

$$=\frac{\gamma^{M-1}(q\,(2-\cosh(\mu a)))^{M}}{(\gamma+\theta-p\gamma\,(2-\cosh(\mu a)))^{M}} \tag{4.18}$$

by the $\mathcal{D}$-operator inversion formulas from [12].

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}\,(t)=P\left\{t_{\nu-1}\le t<t_\nu\right\}=\mathcal{L}_\theta^{-1}\left\{\Phi_\nu^*\,(\theta)\right\}$$

$$=\gamma^{M-1}(q\,(2-\cosh(\mu a)))^{M}\,\frac{t^{M-1}}{(M-1)!}e^{-(\gamma-p\gamma(2-\cosh(\mu a)))t}$$

$$=q\,(2-\cosh(\mu a))\,\frac{(\gamma q\,(2-\cosh(\mu a))\,t)^{M-1}}{(M-1)!}e^{-(\gamma-p\gamma(2-\cosh(\mu a)))t}. \tag{4.19}$$

Notice that when $a=0$ (in the symmetric case), (4.19) reduces to (4.17) and the value of μ is irrelevant given it is finite.

5 Continuous Time Parameter Process on Interval $[0,t_\nu)$

Now consider the functional of passive process P being observed over the period $[0,t_\nu)$, jointly with the active process A_ν, the first passage time t_ν, and the counting processes N_t and Π_t. The functional satisfies the formula:

$$\overset{\wedge}{\Phi}_\nu\,(t)=Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\phi P_\nu}\upsilon^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t)$$

$$=\sum_{k=0}^{\infty}Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\phi P_\nu}\upsilon^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t)\mathbf{1}_{\{\nu=k\}}.$$

Since $\sum_{j=0}^{\nu}E\mathbf{1}_{[t_{\nu-j-1},t_{\nu-j})}(t)=E\mathbf{1}_{[0,t_\nu)}(t)$,

$$\overset{\wedge}{\Phi}_\nu\,(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}E[z^{A_{k-j-1}}\upsilon^{A_{k-j-1}}\upsilon^{\sum_{i=k-j}^{k}X_i}e^{-i\eta\Pi_{k-j-1}-i\phi P_{k-j-1}}$$
$$\times e^{-i\phi\sum_{i=k-j}^{k}\pi_i}e^{-\vartheta t_{k-j-1}}e^{-\vartheta\sum_{i=k-j}^{k}\Delta_i}\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)],$$

and applying the transformation D_p to $\overset{\wedge}{\Phi}_\nu\,(t)$ we have:

$$D_p\left[\overset{\wedge}{\Phi}_\nu\,(t)\right](x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}F_{jk}\,(t)\,x^{X_{k-j+1}+\cdots+X_{k-1}}$$

$$\times E\,(vx)^{X_{k-j+1}+\cdots+X_{k-1}}\,e^{-i\phi(\pi_{k-j+1}+\cdots+\pi_{k-1})}e^{-\vartheta(\Delta_{k-j+1}+\cdots+\Delta_{k-1})}$$
$$\times E\left(1-x^{X_k}\right)e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}\upsilon^{X_k},$$

where

$$F_{jk}\,(t)=$$
$$E\,(zvx)^{A_{k-j-1}}\,e^{-i(\eta+\phi)P_{k-j-1}}e^{-\vartheta t_{k-j-1}}\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,(vx)^{X_{k-j}}\,e^{-i\phi(\pi_{k-j})}e^{-\vartheta(\Delta_{k-j})},$$

$$\widetilde{F}(t) = Ez^A e^{-i\eta P} e^{-\vartheta T} \mathbf{1}_{[T,T+\Delta)}(t)\, v^X e^{-i\phi\pi} e^{-\vartheta\Delta}$$

under the assumptions that random vectors $A \otimes P \otimes T$ and $X \otimes \pi \otimes \Delta$ are independent. Then

$$\widetilde{F}^*(\theta) = \sum_r z^r \sum_m v^m \int_p e^{-i\eta p} \int_w e^{-i\phi w} \int_{s\ge 0} e^{-\vartheta s} e^{-\theta s}.$$

$$\times \frac{1}{\theta}\int_\delta \left(e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta}\right) P_{A\otimes P\otimes T\otimes X\otimes\pi\otimes\Delta}(r, m, dp, ds, dw, d\delta)$$

and because $A \otimes P \otimes T$ and $X \otimes \pi \otimes \Delta$ are independent,

$$= \frac{1}{\theta} E\left[z^A e^{-i\eta P} e^{-(\vartheta+\theta)T}\right]\left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta)\right].$$

Thus

$$F^*_{jk}(\theta) = \frac{1}{\theta}\gamma_0 {\delta^1}^{j-1}\left[\delta - \delta^1\right]\gamma^{k-j-1}\left[\delta^1 - \delta^{13}\right], \tag{5.1}$$

$$(i) \qquad \sum_{k>0}\sum_{j=1}^{k-1} F^*_{jk}(\theta) = \frac{1}{\theta}\gamma_0 \Psi\delta \sum_{k>0}\gamma^{k-2}\sum_{j=1}^{k-1}\left(\frac{\delta^1}{\gamma}\right)^{j-1} = \frac{1}{\theta}\gamma_0 \frac{\Psi\delta}{(1-\gamma)(1-\delta^1)}, \tag{5.2}$$

with notation $\gamma := \gamma(zvx, \eta+\phi, \vartheta)$ and $\gamma_0 := \gamma_0(zvx, \eta+\phi, \vartheta)$, and further

$$\delta^1 = \gamma(vx,\phi,\vartheta), \delta^1_0 = \gamma_0(vx,\phi,\vartheta), \delta = \gamma(v,\phi,\vartheta),$$

$$\delta^3 = \gamma(v,\phi,\vartheta+\theta), \delta^{13} = \gamma(vx,\phi,\vartheta+\theta), \quad \delta_0 = \gamma_0(v,\phi,\vartheta), \delta^{13}_0 = \gamma_0(vx,\phi,\vartheta+\theta),$$

$$\Gamma\delta = \delta - \delta^3 - \delta^1 + \delta^{13},\ \Lambda\delta = \frac{\Psi\delta}{1-\delta^1} + \Gamma\delta, \Psi\delta = \left(\delta - \delta^1\right)\left(\delta^1 - \delta^{13}\right).$$

(ii) Consider $j = k = 0$. $A_{-1} = t_{-1} = P_{-1} = 0$ for $t \in [0, t_0)$ and $N_t = A_{-1} = \Pi_t = 0$.

$$F_{00}(t) = E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_0} v^{A_0} e^{-i\phi P_0}\left(1 - x^{A_0}\right).$$

$$F^*_{00}(\theta) = \sum_r v^r \int_p e^{-i\phi p} \int_s e^{-\vartheta s} \int_{t=0}^s e^{-\theta t} dt P_{A_0\otimes P_0\otimes t_0}(r, dp, ds)$$

$$= \sum_r v^r \int_p e^{-i\phi p}\int_s e^{-\vartheta s}\frac{1}{\theta}\left[e^{-\vartheta s} - e^{-(\vartheta+\theta)s}\right] P_{A_0\otimes P_0\otimes t_0}(r, dp, ds) = \frac{1}{\theta}\Gamma\delta_0. \tag{5.3}$$

(iii) Consider $j = 0, k > 0$.

$$F_{0k}(t) = Ez^{N_t} v^{A_k} e^{-i\eta P_{k-1}} e^{-i\phi P_k} e^{-\vartheta t_k} \mathbf{1}_{[t_{k-1},t_k)}(t)\left(x^{A_{k-1}} - x^{A_k}\right)$$

$$= E(zvx)^{A_{k-1}} e^{-i(\eta+\phi)P_{k-1}} e^{-\vartheta t_{k-1}} \mathbf{1}_{[t_{k-1},t_k)}(t)\, v^{X_k}\left(1 - x^{X_k}\right) e^{-i\phi\pi_k} e^{-\vartheta\Delta_k}.$$

$$F^*_{0k}(\theta) = \int_t e^{-\theta t}\sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p}\sum_m v^m \int_q e^{-i\phi q}\int_s e^{-\vartheta s}$$

$$\times \int_\delta e^{-\vartheta\delta}\mathbf{1}_{[s,s+\delta)}(t)\, dt P_{A_{k-1}\otimes P_{k-1}\otimes T_{k-1}\otimes X_k\otimes\pi_k\otimes\Delta_k}(r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p}\sum_m v^m \int_q e^{-i\phi q}\int_s e^{-\vartheta s} e^{-\theta s}$$

$$\times \int_{t-s=0}^{\delta} e^{-\theta(t-s)} dt P_{A_{k-1}\otimes P_{k-1}\otimes T_{k-1}\otimes X_k\otimes \pi_k \otimes \Delta_k}(r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p} \sum_m v^m \int_q e^{-i\phi q} \int_\delta \left[e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta} \right]$$

$$\times P_{A_{k-1}\otimes P_{k-1}\otimes X_k\otimes \pi_k \otimes \Delta_k}(r, dp, m, dq, d\delta)$$

$$= \frac{1}{\theta}\gamma^{k-1}(zvx, \eta+\phi, \vartheta)\, \gamma_0 (zvx, \eta+\phi, \vartheta)$$

$$\times [\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta)]$$

and

$$\sum_{k>0} F_{0k}^{*}(\theta) = \frac{1}{\theta}\gamma_0 (zvx, \eta+\phi, \vartheta) \frac{1}{1-\gamma(zvx,\eta+\phi,\vartheta)}$$
$$\times [\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta)] = \frac{1}{\theta}\frac{\gamma_0}{1-\gamma}\mathit{\Gamma}\delta. \tag{5.4}$$

(iv) Consider $j = k > 0$. $F_{kk}(t) = E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} v^{A_k} e^{-i\phi P_k}\left(x^{A_{k-1}} - x^{A_k}\right)$

$$= E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} (vx)^{X_1+\cdots+X_{k-1}} e^{-i\phi(\pi_1+\cdots+\pi_{k-1})} e^{-\vartheta(\Delta_1+\cdots+\Delta_{k-1})}$$

$$\times \left[v^{X_k} - (vx)^{X_k} \right] e^{-i\phi\pi_k} e^{-\vartheta\Delta_k}$$

$$= E\mathbf{1}_{[0,t_0)}(t)\, e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \gamma^{k-1}(vx,\phi,\vartheta) [\gamma(v,\phi,\vartheta) - \gamma(vx,\phi,\vartheta)].$$

So,

$$\sum_{k>0} F_{kk}^{*}(\theta) = \frac{1}{\theta}\frac{\mathit{\Psi}\delta_0}{1-\delta^1}. \tag{5.5}$$

Altogether, from (i) through (iv) we have

$$\overset{\wedge}{\mathit{\Phi}}{}_{\nu}^{*}(\theta) = \int_{t=0}^{\infty} e^{-\theta t} \mathit{\Phi}_\nu(t)\, dt = \mathcal{D}_x^{M-1}\Big\{ \sum_{k>0}\sum_{j=1}^{k-1} F_{jk}^{*}(\theta) + F_{00}^{*}(t) + \sum_{k>0} F_{0k}^{*}(\theta) + \sum_{k>0} F_{kk}^{*}(\theta) \Big\}$$

$$= \mathcal{D}_x^{M-1}\Big\{ \frac{1}{\theta}\Big(\Lambda\delta_0 + \frac{\gamma_0}{1-\gamma}\Lambda\delta \Big) \Big\} \tag{5.6}$$

where $\Lambda\alpha = \mathit{\Gamma}\alpha + \frac{\mathit{\Psi}\alpha}{1-\delta^1}$ and $\alpha = \delta$ or δ_0. The Laplace inverse of (5.6) will permit the recovery of $\overset{\wedge}{\mathit{\Phi}}_\nu(t)$.

6 Conclusion

In this paper we study a class of signed marked random measures $(\mathcal{A}, \mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty}(X_n, \pi_n)\,\varepsilon_{t_n}$ with position dependent marking, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We target the critical behavior of the underlying stochastic process about a fixed threshold in the context of time sensitivity. The latter means that all related characteristics, such as first passage time and the location of the process upon crossing the threshold relate to deterministic time $t \geq 0$. The major benefit of this study is to utilize stochastic control over the process that must traditionally be considered on time interval $[0,t]\,, t \geq 0$. Using and further embellishing fluctuation theory, we find explicitly the functionals

$$\Phi_\nu\left(t\right) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}\left(t\right)$$

and

$$\overset{\wedge}{\Phi}_\nu\left(t\right) = Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\phi P_\nu}v^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t)$$

with respect to time $t \in [\tau_{\nu-1}, \tau_\nu)$ and $t \in [0, \tau_\nu)$, respectively. These functionals describe the status of underlying processes $N_t = \sum_{n=0}^{\infty} X_n \varepsilon_{t_n}[0,t]$ and $\Pi_t = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}[0,t]\,,$ along with other characteristics like the values of these processes upon the crossing as well as just prior to crossing the threshold.

We discuss various applications to the finance (stock option trading) and risk theory. A number of special cases and examples demonstrate analytic tractability of the results obtained.

References

[1] Al-Matar, N. and Dshalalow, J.H. A game-theoretic approach in single-server queues with maintenance. Time sensitive analysis. *Commun. in Appl. Nonlin. Analysis* **17** (1) (2010) 65–92.

[2] Al-Matar, N. and Dshalalow, J.H. Time sensitive functionals in classes of queues with sequential maintenance. *Stochastic Models* **27** (2011) 687–704.

[3] Bingham, N.H. Random walk and fluctuation theory. In: *Handbook of Statistics* (Eds. D.N. Shanbhag and C.R. Rao), Vol. 19, 2001, Elsevier Science, 171–213.

[4] Borokov, A.A. On the first passage time for one class of processes with independent increments. *Theor. Probab. Appl.* **10** (1964) 331–334.

[5] Coutin, L. and Dorobantu, D. First passage time law for some Levy processes with compound Poisson: Existence of a density. *Bernoulli* **17** (4) (2011) 1127–1135

[6] Dshalalow, J.H., First excess level of vector processes. *J. Appl. Math. Stoch. Anal.* **7** (3) (1994) 457–464.

[7] Dshalalow, J.H. On the level crossing of multidimensional delayed renewal processes. In: Special Issue *Stochastic Systems*, *Journ. Appl. Math. Stoch. Anal.* **10** (4) (1997) 355–361.

[8] Dshalalow, J.H. Fluctuations of recurrent processes and their application to the stock market. *Stoch. Anal. Appl.* **22** (1) (2004) 67–79.

[9] Dshalalow, J.H. On exit times of a multivariate random walk with some applications to finance. *Nonlinear Analysis* **63** (2005) 569–577.

[10] Dshalalow, J.H. Random walk analysis in antagonistic stochastic games. *Stochastic Analysis and Applications* (26) (2008) 738–783.

[11] Dshalalow, J.H. On multivariate antagonistic marked point processes. *Math. and Comp. Modeling* **49** (2009) 432–452.

[12] Dshalalow, J.H. *Stochastic Processes.* Lecture Notes, FIT, Melbourne, FL, 2015.

[13] Dshalalow, J.H. and Iwezulu, K. Discrete versus continuous operational calculus in antagonistic stochastic games. S*ão Paulo Journal of Math. Sci.*, Springer, NY, **11** (2017) 471–489.

[14] Dshalalow, J.H., Iwezulu, K. and White, R.T. Discrete operational calculus in delayed stochastic games. *Neural, Parallel, and Scientific Computations* **24** (2016) 55–64.

[15] Dshalalow, J.H. and Nandyose, K.M. Continuous time interpolation of monotone marked random measures and their applications. *Neural, Parallel, and Scientific Computations* **24** (2018) 119–141.

[16] Dshalalow, J.H. and White, R.T. On reliability of stochastic networks. *Neural, Parallel, and Scientific Computations* **21** (2013) 141–160.

[17] Dshalalow, J.H. and White, R.T. On strategic defense in stochastic networks. *Stochastic Analysis and Applications* **32** (2014) 365–396.

[18] Dshalalow, J.H. and White, R.T. Time sensitive analysis of independent and stationary increment processes. J*ournal of Mathematical Analysis and Applications* **443** (2016) 817–833.

[19] Hellmund, G. Completely random signed measures. *Statisitcs and Probability Letters* **79** (2009) 894–898.

[20] Hida, T. (Editor), Mathematical Approach to Fluctuations: Astronomy, Biology and Quantum Dynamics: Proceedings of the Iias Workshop: Kyoto, Japan, May 18-21, 1992, World Scientific Publishers, 1995.

[21] Kadankova, T.V. On the distribution of the moment of the first exit time from an interval and value of overjump through borders interval for the processes with independent increments and random walk. *Random Operators and Stochastic Equations* **13** (3) (2005) 219–244.

[22] Kadankova, T.V. Exit, passage, and crossing times and overshoots for a Poisson compound process with an exponential component. *Theor. Probability and Math. Statist.* **75** (2007) 23–29.

[23] Kyprianou, A.E. and Pistorius, M.R. Perpetual options and Canadization through fluctuation theory. *Ann. Appl. Prob.* **13** (3) (2003) 1077–1098.

[24] Mellander, E., Vredin, A, and Warne, A. Stochastic trends and economic fluctuations in a small open economy. *J. Applied Econom.* **7** (4) (1992) 369-394.

[25] Muzy, J., Delour1, J., and Bacry, E. Modelling fluctuations of financial time series: from cascade process to stochastic volatility model. *Eur. Phys. J. B* **17** (2000) 537–548.

[26] Redner, S. *A Guide to First-Passage Processes.* Cambridge University Press, Cambridge, 2001.

[27] Schmidli, H. *Risk Theory.* Springer Actuarial, Cham, Switzerland, 2017.

[28] Shinozuka, M. and Wu, W-F. On the first passage problem and its application to earthquake engineering. *Proceedings of Ninth World Conference on Earthquake Engineering*, August 2-9, (VIII) 1988, Tokyo-Kyoto, Japan.

[29] Smorodina, N. and Faddeev M., The Lvy-Khinchin representation of the one class of signed stable measures and some of its applications. *Acta Appl. Math.* **110** (2010) 1289–1308.

[30] Takács, L. On fluctuations of sums of random variables, in Studies in Probability and Ergodic Theory. In: *Advances in Mathematics; Supplementary Studies*, Vol. 2 (G.-C. Rota, Ed.) (1978) 45–93.

[31] Yin, C., Wen, Y., Zong, Z., and Shen, Y. The first passage time problem for mixed-exponential jump processes with applications in insurance and finance. *Abstract and Applied Analysis* (2014), 9 pages.

[32] Zolotarev, V.M. The first passage time of a level and the behavior at infnity for a class of processes with independent increments. *Theor. Probab. Appl.* **9** (1964) 653–664.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 55–67

Decentralized Stabilization for a Class of Nonlinear Interconnected Systems Using SDRE Optimal Control Approach

A. Feydi, S. Elloumi and N. Benhadj Braiek *

Advanced System Laboratory (Laboratoire des Systèmes Avancés – LSA), Tunisia Polytechnic School – EPT, University of Carthage. BP 743, 2078, La Marsa, Tunisia.

Received: September 9, 2018; Revised: December 10, 2018

Abstract: This paper presents a new approach to assure the decentralized optimal control of interconnected nonlinear systems based on the decentralized state-dependent riccati equation (SDRE). To remedy the problem of persistent stability in other works, we based our approach on the foundations of the Lyapunov theory. It allows developing a new sufficient condition to guarantee the global asymptotic stability of the systems under study. We conducted a simulation of this new control method on a numerical example. It demonstrated its efficiency and the sufficiency of the new stability conditions.

Keywords: *decentralized optimal control; state-dependent Riccati equation (SDRE); interconnected nonlinear systems; Lyapunov theory; Kronecker product.*

Mathematics Subject Classification (2010): 93D15, 34D23, 93A14.

1 Introduction

In recent years, the modern dynamical systems are getting more complex, highly interconnected, and mutually interdependent. This change is caused either by physical attributes, and/or a multitude of information and communication network constraints [1–3]. The important dimension and complexity of these large-scale systems often require a hierarchical decentralized architecture to analyze and control these systems [4–10]. Since these complex dynamic systems can be characterized by an interconnection between many subsystems, possible control strategies are generally based on a decentralized approach. The

* Corresponding author: mailto:naceur.benhadj@ept.rnu.tn

advantage of such method is to reduce the complexity and therefore make the implementation of the control law more feasible.

In fact, the decentralized control refers to a control design with local decisions. These decisions are based only on local information of the subsystems. This method is given considerable interest because it brings up significant solutions for the traditional control approach limitations such as the implementation constraints, cost and reliability considerations especially for large-scale systems.

Optimal control of nonlinear systems is one of the most challenging subjects in control theory. Indeed, the classical problems of optimal control are based on the solution of the Hamilton-Jacobi equation (HJE) [11, 12]. The solution to the HJE is a function of the state of the nonlinear system which makes it possible to characterize the quadratic optimal law of control sought under some hypotheses. However, in most cases it is impossible to solve it analytically, and despite recent progress, unsolved problems still exist and researchers often complain about the very limited applicability of contemporary theories because of conditions imposed on the system. This has led to numerous methods proposed in the literature for obtaining a suboptimal state feedback control law for the general case of nonlinear dynamic systems [13, 14].

The SDRE approach is one of the methods applied in the determination of a suboptimal quadratic control based on the solution of a state-dependent Riccati equation. This strategy provides an efficient algorithm for nonlinear state feedback control synthesis while retaining the nonlinearities of the complex dynamic system, thanks to the flexibility of the state-dependent weighting matrices [15, 16]. This approach, proposed by Pearson [17] and later extended by Wernli and Cook [18], was studied independently by Mracek and Cloutier [19]. It should be pointed out that, although it is a relatively simplified and practical technique for controlling nonlinear systems, the SDRE approach involves problems that deserve to be treated with great attention, in particular the stability problem of the system controller [20,21]. Elloumi and Benhadj Braiek [22,23] have developed a sufficient condition for the stability of nonlinear system with optimal control based on SDRE approach. In this paper, we extend this work to the case of large scale interconnected systems. In this direction we carried out the synthesis of decentralized optimal control law based on the SDRE technique. This approach aims to minimize a performance criterion in order to compute decentralized optimal control gains when some sufficient conditions developed using the Lyapunov theory are verified.

The rest of the paper is organized as follows: the second section is devoted to the description of the systems under study and the formulation of the problem. In the third section, we present the decentralized optimal control law based on the SDRE approach. The fourth section treats the stability of the system in question using the quadratic Lyapunov function. The simulation results are set out in the fifth section to illustrate the applicability of the developed approach. Finally, conclusions are drawn and future scope of study is outlined.

2 Description of the System Under Study and Problem Formulation

A nonlinear system can be described by the interconnection of subsystems as follows:

$$\begin{cases} \dot{x}_i = f_i\left(\left(x_i, x_j\right), u_i\left(t\right), t\right), \quad i \neq j, \\ y_i = h_i\left(x_i\right), \quad i = 1, ..., n, \ j = 1, ..., n, \end{cases} \tag{1}$$

where $x_i\,(t) \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are, respectively, the state, the control and the output of the i^{th} subsystem.

$f_i\,(x_i, x_j)$ and $h_i\,(x_i)$ are nonlinear functions of the state. Through the state-dependent coefficient (SDC) factorization, system designers can represent the nonlinear equations of the system under consideration as linear structures with state-dependent coefficients. Thus, the following procedure is similar to the optimal linear control (LQR) method, except that all matrices may depend on the states. Based on this concept, the state space equation for the nonlinear interconnected subsystem can be expressed as a linear-like state-space equation using direct SDC factorization as:

$$\begin{cases} \dot{x}_i\,(t) = A_i\,(x_i)\,x_i\,(t) + B_i\,(x_i)\,u_i\,(t) + \sum\limits_{j=1, j\neq i}^{n} H_{ij}\,(x_i, x_j) x_j\,(t)\,, \\ y_i\,(t) = C_i\,(x_i)\,x_i\,(t)\,, \quad i = 1, \ldots, n, \end{cases} \tag{2}$$

where $A_i\,(x_i)$ is the characteristic matrix that depends on the state of the i^{th} subsystem, $B_i\,(x_i)$ is the control vector of the i^{th} subsystem, $C_i\,(x_i)$ is the state-dependent observation matrix of the i^{th} subsystem and $H_{ij}\,(x_i, x_j)$ is the state -dependent interconnection matrix between the i^{th} and the j^{th} subsystem.

The global interconnected system can be defined by the following compact form:

$$\begin{cases} \dot{x} = A\,(x)\,x + B\,(x)\,u + H\,(x)\,x, \\ y = C\,(x)\,x, \end{cases} \tag{3}$$

with
$x^T = \left[x_1^T,\ x_2^T,\ \ldots\ ,\ x_n^T\right]$ being the state vector of the overall system; $x \in \mathbb{R}^n,\ n = \sum\limits_{i=1}^{n} n_i$;
$u^T = \left[u_1^T,\ u_2^T,\ \ldots\ ,\ u_n^T\right]$ being the control vector of the overall system ,
$A\,(x) = diag\,[A_i\,(x_i)]\,, B\,(x) = diag\,[B_i\,(x_i)]$ and $C\,(x) = diag\,[C_i\,(x_i)]$.
$H\,(x)$ is the global interconnection matrix given as follows:

$$H\,(x) = \begin{pmatrix} 0 & H_{12}\,(x) & \cdots & H_{1n}\,(x) \\ H_{21}\,(x) & 0 & \cdots & H_{2n}\,(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1}\,(x) & \cdots & \cdots & 0 \end{pmatrix}. \tag{4}$$

Our contribution consists in the application of a decentralized optimal control via the SDRE approach to nonlinear interconnected systems. We based on solving the decentralized state-dependent Riccati equations to obtain the local control gains. The synthesis of a decentralized control for the system in question is detailed in the following section.

3 Decentralized State-Dependent Riccati Regulation Theory

The decentralized state-dependent Riccati equation technique is a nonlinear control design method for the direct construction of nonlinear sub-optimal feedback controllers. The determination of such decentralized control is based on considering the decoupled subsystem, expressed as follows:

$$\begin{cases} \dot{x}_i = A_i\,(x_i)\,x_i + B_i\,(x_i)\,u_i\ , \quad i = 1, \ldots, n \\ y_i = C_i\,(x_i)\,x_i. \end{cases} \tag{5}$$

Note that $A_i(x_i)$ is not a unique matrix because there could be many possible choices in the direct (SDC) factorization. For this subsystem, the SDRE technique finds an input $u_i(t)$ that approximately minimizes the following performance criterion:

$$J_i = \frac{1}{2}\int_0^{\infty} \left(x_i^T Q_i(x_i)\, x_i + u_i^T R_i(x_i)\, u_i\right) dt, \tag{6}$$

where $Q_i(x_i) \in \mathbb{R}^{(n_i \times n_i)}$ and $R_i(x_i) \in \mathbb{R}^{(m_i \times m_i)}$ are symmetric, positive definite matrices. $x_i^T Q_i(x_i)\, x_i$ is a measure of the control accuracy and $u_i^T R_i(x_i)\, u_i$ is a measure of the control effort.

3.1 Existence of a control solution

The SDRE feedback control provides a similar approach as the algebraic Riccati equation (ARE) for LQR problems to the nonlinear regulation problem for the decoupled nonlinear subsystem (5) with cost functional (6). Indeed, once a SDC form has been found, the SDRE approach is reduced to solving a LQR problem at each sampling instant.

To guarantee the existence of such controller, the conditions in the following definitions must be satisfied [19].

- **Definition 3.1**: $A_i(x_i)$ is a controllable (stabilizable) parametrization of the nonlinear subsystem for a given region if $[A_i(x_i), B_i(x_i)]$ are pointwise controllable (stabilizable) in the linear sense for all x_i in that region.

- **Definition 3.2**: $A_i(x_i)$ is an observable (detectable) parametrization of the nonlinear subsystem for a given region if $[C_i(x_i), A_i(x_i)]$ are pointwise observable (detectable) in the linear sense for all x_i in that region.

Given these standing assumption, the state feedback decentralized controller is obtained in the following form:

$$u_i(x_i) = -K_i(x_i)\, x_i \tag{7}$$

and the state feedback decentralized gain for minimizing (6) is

$$K_i(x_i) = R_i^{-1}(x_i)\, B_i^T(x_i)\, P_i(x_i), \tag{8}$$

where $P_i(x_i)$ is the unique symmetric positive-definite solution of the decentralized state dependent Riccati equation (SDRE)

$$\begin{aligned} &A_i^T(x_i)\, P_i(x_i) + P_i(x_i)\, A_i(x_i) \\ &\qquad -P_i(x_i)\, B_i(x_i)\, R_i^{-1}(x_i)\, B_i^T(x_i)\, P_i(x_i) + C_i^T(x_i)\, Q_i(x_i)\, C_i(x_i) = 0. \end{aligned} \tag{9}$$

Remark 3.1: It is important to note that the existence of the decentralized optimal control for a particular parametrization of the subsystem is not guaranteed. Furthermore, there may be an infinite number of parametrizations of the subsystem, therefore the choice of parametrization is very important. The other factor which may determine the existence of a solution is the $Q_i(x_i)$ and $R_i(x_i)$ weighting matrices in the state dependent Riccati equation (9).

Remark 3.2. The greatest advantage of the state-dependent Riccati equation approach is that physical intuition is always present and the designer can directly control the performance by tuning the weighting matrices $Q_i(x_i)$ and $R_i(x_i)$. In other words, via the SDRE, the design flexibility of LQR formulation is directly translated to control the nonlinear interconnected systems. Moreover, $Q_i(x_i)$ and $R_i(x_i)$ are not only allowed to be constant, but can also vary as functions of states. In this way, different modes of behavior can be imposed in different regions of the state-space [21].

3.2 Optimality of the SDRE regulation

As $x_i \to 0$, $A_i(x_i) \to \partial f_i(0)/\partial x_i$ which implies that $P_i(x_i)$ approaches the linear ARE at the origin. Furthermore, the SDRE control solution asymptotically approaches the optimal control as $x_i \to 0$ and away from the origin the SDRE control is arbitrarily close to the optimal feedback. Hence the SDRE approach yields an asymptotically optimal feedback solution.

Let the Hamiltonian be defined by the following expression:

$$H_i(x_i, u_i, \lambda_i) = \tfrac{1}{2}\left[x_i^T Q_i(x_i)\, x_i + u_i^T R_i(x_i)\, u_i\right] + \lambda_i^T\left[A_i(x_i)\, x_i + B_i(x_i)\, u_i\right]. \tag{10}$$

Mracek and Cloutier developed the necessary conditions for the optimality of a general nonlinear regulator, that is the regulator governed by (5) and (6), and then extend these results to determine the optimality of the SDRE approach [19].

Theorem 1. *For the general multivariable nonlinear SDRE control case (i.e., $n > 1$), the SDRE nonlinear feedback solution and its associated state satisfy the first necessary condition for optimality $\partial H_i/\partial u_i = 0$ of the nonlinear optimal regulator problem defined by (5) and (6). Additionally, the second necessary condition for optimality $\dot{\lambda}_i = -\partial H_i \big/ \partial x_i$ is asymptotically satisfied at a quadratic rate.*

Proof. Pontryagin'S maximum principle states that necessary conditions for optimality are

$$\frac{\partial H_i}{\partial u_i} = 0, \quad \dot{\lambda}_i = -\frac{\partial H_i}{\partial x_i}, \quad \dot{x}_i = \frac{\partial H_i}{\partial \lambda_i}, \tag{11}$$

where H_i is the Hamiltonian. Using (7) yields

$$\frac{\partial H_i}{\partial u_i} = B_i^T(x_i)\left[\lambda_i - P_i(x_i)\, x_i\right] \tag{12}$$

and λ_i, the adjoint vector for the system, satisfies

$$\lambda_i = P_i(x_i)\, x_i, \tag{13}$$

and the first optimality condition (12) is satisfied identically for the nonlinear regulator problem. With the Hamiltonian defined in (10), the second necessary condition becomes

$$\begin{aligned} \dot{\lambda}_i = -x_i^T\left(\frac{\partial A_i(x_i)}{\partial x_i}\right)^T \lambda_i - u_i^T\left(\frac{\partial B_i(x_i)}{\partial x_i}\right)^T \lambda_i - Q_i(x_i)\, x_i \\ -\frac{1}{2} x_i^T \frac{\partial Q_i(x_i)}{\partial x_i} x_i - \frac{1}{2} u_i^T \frac{\partial R_i(x_i)}{\partial x_i} u_i. \end{aligned} \tag{14}$$

Taking the time derivative of (13) yields

$$\dot{\lambda}_i = \dot{P}_i(x_i)\,x_i + P_i(x_i)\,\dot{x}_i. \tag{15}$$

Substituting this result, along with (5), (7) and (14) into (9) leads to the SDRE necessary condition for optimality

$$\begin{aligned}
&\dot{P}_i(x_i)\,x_i + \frac{1}{2}x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,\frac{\partial R_i(x_i)}{\partial x_i}R_i^{-1}(x_i)\,B_i^T(x_i)\,P_i(x_i)\,x_i \\
&\qquad + x_i^T\left(\tfrac{\partial A_i(x_i)}{\partial x_i}\right)^T P_i(x_i)\,x_i + \frac{1}{2}x_i^T\frac{\partial Q_i(x_i)}{\partial x_i}x_i \\
&\qquad - x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\left(\frac{\partial B_i(x_i)}{\partial x_i}\right)^T P_i(x_i)\,x_i = 0.
\end{aligned} \tag{16}$$

Hence, whenever (16) is satisfied, the closed-loop SDRE solution satisfies all the first-order necessary conditions for an extremum of the cost functional.

4 Stability Study

In this section, we study the asymptotic stability of interconnected system based on the Lyapunov theory [10]. We begin with the stability study of each subsystem, thereafter we deal with the development of a sufficient condition to assure the asymptotic stability of the overall interconnected nonlinear system.

4.1 Stability of a decoupled nonlinear subsystem

Stability of SDRE systems is still an open problem. Local stability results are presented by Cloutier, D'souza and Mracek in the case when the closed-loop coefficient matrix is assumed to have a special structure.

The authors in [22,23] presented the optimal control solution for nonlinear subsystem using the SDRE method. The asymptotic stability of decoupled subsystem (5) with SDRE feedback control is guaranteed provided that

$$\begin{aligned}
&M_i(x_i) = -C_i^T(x_i)\,Q_i(x_i)\,C_i(x_i) - P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,B_i^T(x_i)\,P_i(x_i) \\
&- \left(I_n \otimes x_i^T P_i(x_i)\,B_i(x_i)\,R_i^{-1}(x_i)\,B_i^T(x_i)\right)\frac{\partial P_i(x_i)}{\partial x_i} + \left(I_n \otimes (x_i^T A_i^T(x_i)\right)\frac{\partial P_i(x_i)}{\partial x_i}
\end{aligned} \tag{17}$$

is negative definite for all $x_i \in \mathbb{R}^{n_i}$.

Now, to guarantee the asymptotic stability of the overall interconnected system (3), we carry out a stability study of interconnected system (2) with the decentralized control (7) as depicted in the following subsection.

4.2 Stability of a global interconnected system

In this paragraph, we present our contribution which consists in developing a sufficient condition to assure the asymptotic stability of the overall interconnected nonlinear system (3) with the decentralized control law (7). This study is based on the quadratic Lyapunov function

$$V(x) = x^T P(x)\,x, \tag{18}$$

where $P(x) = diag\left[P_i(x_i)\right]$.

The global asymptotic stability of the equilibrium state ($x = 0$) of system (3) is ensured when the time derivative $\dot{V}(x)$ of $V(x)$ is negative define for all $x \in \mathbb{R}^n$,

$$\dot{V}(x) = \dot{x}^T P(x)\, x + x^T \frac{dP(x)}{dt} x + x^T P(x)\, \dot{x}. \tag{19}$$

The use of expression (19) and the following equality:

$$\frac{dP(x)}{dt} = \left(I_n \otimes \dot{x}^T\right) \frac{\partial P(x)}{\partial x} \tag{20}$$

yields

$$\begin{aligned}\dot{V}(x) &= x^T\left[A^T(x)\,P(x) + P(x)\,A(x)\right]x + x^T\left[H^T(x)\,P(x) + P(x)\,H(x)\right]x \\ &-2x^T\left[P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x)\right]x + x^T\left(I_n \otimes \dot{x}^T\right)\frac{\partial P(x)}{\partial x}x,\end{aligned} \tag{21}$$

then

$$\begin{aligned}\dot{V}(x) &= x^T\left[A^T(x)\,P(x) + H^T(x)\,P(x)\right. \\ &\left.+P(x)\,A(x) + P(x)\,H(x) - 2P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x)\right]x \\ &+x^T\left[\left(I_n \otimes \left(x^T A^T(x) + x^T H^T(x) - x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right],\end{aligned} \tag{22}$$

where $\otimes$ is the Kronecker product notation whose definition and properties are detailed in the appendix. Using the state-dependent Riccati equation (9), expression (22) can be simplified as follows:

$$\begin{aligned}\dot{V}(x) &= x^T\left[-C^T(x)\,Q(x)\,C(x) - P(x)\,L(x)\,P(x)\right]x \\ &+x^T\left[H^T(x)\,P(x) + P(x)\,H(x)\right]x + x^T\left[\left(I_n \otimes \left(x^T A^T(x) + x^T H^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right]x \\ &-x^T\left[\left(I_n \otimes \left(x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\right)\frac{\partial P(x)}{\partial x}\right]x,\end{aligned} \tag{23}$$

where $L(x) = B(x)\,R^{-1}(x)\,B^T(x)\,, \forall\, x \in \mathbb{R}^n$.

To ensure the asymptotic stability of the overall systems (3) with the decentralized optimal control law (7), $\dot{V}(x)$ should be negative definite, which is equivalent to $M(x)$ being negative definite, with

$$\begin{aligned}M(x) &= -C^T(x)\,Q(x)\,C(x) - P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\,P(x) \\ &\quad - \left(I_n \otimes x^T P(x)\,B(x)\,R^{-1}(x)\,B^T(x)\right)\frac{\partial P(x)}{\partial x} \\ &\quad + \left(I_n \otimes x^T A^T(x) + x^T H^T(x)\right)\frac{\partial P(x)}{\partial x} + P(x)\,H(x) + H^T(x)\,P(x)\,.\end{aligned} \tag{24}$$

We need to simplify the manipulation of matrix $M(x)$ by expressing $\partial P(x)/\partial x$ in terms of $P(x) > 0,\ \forall\, x \in \mathbb{R}^n$. When deriving the SDRE (9) with respect to the state vector $x \in \mathbb{R}^n$, we get the following expression:

$$\begin{aligned}&\frac{\partial P(x)}{\partial x}A(x) + (I_n \otimes P(x))\frac{\partial A(x)}{\partial x} + \frac{\partial A^T(x)}{\partial x}P(x) + \left(I_n \otimes A^T(x)\right)\frac{\partial P(x)}{\partial x} + \frac{\partial \Phi(x)}{\partial x}\\ &-\frac{\partial P(x)}{\partial x}L(x)P(x) - (I_n \otimes P(x)L(x))\frac{\partial P(x)}{\partial x} - (I_n \otimes P(x))\frac{\partial L(x)}{\partial x}P(x) = 0\end{aligned} \tag{25}$$

with $\Phi(x) = C^T(x)Q(x)C(x)$, which gives

$$\left[I_n \otimes A^T(x) - I_n \otimes (P(x)L(x))\right]\frac{\partial P(x)}{\partial x} + \frac{\partial P(x)}{\partial x}\left[A(x) - L(x)P(x)\right] = W(x) \tag{26}$$

with

$$W(x) = (I_n \otimes P(x))\frac{\partial L(x)}{\partial x}P(x) - (I_n \otimes P(x))\frac{\partial A(x)}{\partial x} - \frac{\partial A^T(x)}{\partial x}P(x) - \frac{\partial \Phi(x)}{\partial x}. \tag{27}$$

To simplify the partial derivative expression $\partial P(x)/\partial x$ we use the functions Vec and mat and their properties defined in this paper appendix; so (26) becomes

$$\begin{aligned}Vec\left(\frac{\partial P(x)}{\partial x}\right) &= \left[I_n \otimes \left(I_n \otimes A^T(x) + I_n \otimes P(x)L(x)\right)\right.\\ &\quad \left.+ (A(x) - L(x)P(x)) \otimes I_n\right]^{-1} Vec(W(x))\end{aligned} \tag{28}$$

which leads to

$$\begin{aligned}\frac{\partial P(x)}{\partial x} = &\, mat_{(n^2,n)}\Big[\big(I_n \otimes \big[I_n \otimes A(x) + I_n \otimes A^T(x)\\ &-2\,(I_n \otimes L(x)P(x))\big]\big)^{-1} Vec(W(x))\Big].\end{aligned} \tag{29}$$

Therefore, we can state the following result.

Theorem 2. *The overall system (3) is globally asymptotically stabilizable by the optimal decentralized control law (7), with the cost function (6) if the matrix $M(x)$ defined by (24) is negative definite for all $x \in \mathbb{R}^n$.*

5 Simulation Results

In this section we will illustrate the performance of the decentralized SDRE approach, discussed in the previous paragraph, by a numerical example. We consider a nonlinear interconnected system defined by the following two subsystems of state equations:

$$\begin{cases} \sum 1: \begin{cases} \dot{x}_{11} = -2x_{11} + x_{11}x_{12}, \\ \dot{x}_{12} = x_{13} + x_{12}x_{11} + x_{22}^2 x_{21}, \\ \dot{x}_{13} = u_1 + x_{13}^2\left(x_{12}x_{11} + x_{11}^2\right) + x_{22}x_{21}, \end{cases} \\ \\ \sum 2: \begin{cases} \dot{x}_{21} = -x_{21} + x_{22}^2, \\ \dot{x}_{22} = x_6 + \left(x_{12}^2 x_{11} + x_{22}^2 x_{21}\right), \\ \dot{x}_{23} = u_2 + x_{23}^2\left(x_{12}x_{11}^2 + x_{22}x_{21}^2\right) + x_{23}^2 x_{21}^2, \end{cases} \end{cases} \tag{30}$$

with

- $x_1 = [x_{11}\ x_{12}\ x_{13}]^T, x_2 = [x_{21}\ x_{22}\ x_{23}]^T$ being the state vectors of subsystems $\sum 1$ and $\sum 2$,
- $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ being the inputs of the interconnected nonlinear system.

We solve equation (9) with

$$Q_1(x_1) = Q_2(x_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{31}$$

$$R_1(x_1) = R_2(x_2) = 0.1. \tag{32}$$

For interconnected nonlinear systems (30), we choose the following (SDC) parametrization:

$$A_1(x_1) = \begin{pmatrix} -2 & x_{11} & 0 \\ x_{12} & 0 & 1 \\ x_{13}^2 x_{12} & 0 & x_{13}x_{11}^2 \end{pmatrix}, A_2(x_2) = \begin{pmatrix} -1 & x_{22} & 0 \\ 0 & x_{22}x_{21} & 1 \\ x_{23}^2 x_{22}x_{21} + x_{23}^2 x_{21}^2 & 0 & 0 \end{pmatrix}.$$

The control matrices are given as follows:

$$B_1(x_1) = B_2(x_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The interconnection matrices between subsystem 1 and subsystem 2 are expressed as follows:

$$H_{12}(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ x_{22}^2 & 0 & 0 \\ x_{22} & 0 & 0 \end{pmatrix}, \quad H_{21}(x_1, x_2) = \begin{pmatrix} 0 & 0 & 0 \\ x_{12}^2 & 0 & 0 \\ x_{23}^2 x_{12} & 0 & 0 \end{pmatrix}.$$

The controllability matrices, respectively, for subsystem 1 and subsystem 2 are given as follows:

$$\begin{aligned} \zeta_1(x_1) &= \left[B_1(x_1)\ \ A_1(x_1)\,B_1(x_1)\ \ A_1^2(x_1)\,B_1(x_1)\right] \\ &= \begin{pmatrix} 0 & 0 & x_{11} \\ 0 & 1 & x_{13}x_{11}^2 \\ 1 & x_{13}x_{11}^2 & x_{13}^2 x_{11}^4 \end{pmatrix}, \end{aligned} \tag{33}$$

$$\begin{aligned} \zeta_2(x_2) &= \left[B_2(x_2)\ \ A_2(x_2)\,B_2(x_2)\ \ A_2^2(x_2)\,B_2(x_2)\right] \\ &= \begin{pmatrix} 0 & 0 & x_{22} \\ 0 & 1 & x_{22}x_{21} \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

$\zeta_1(x_1), \zeta_2(x_2)$ have a full order rank for all x_i, which can justify the good choice of (SDC) parametrization. Now, we referring to equation (9), we can write the following decentralized state-dependent Riccati equations:

$$\begin{cases} P_1(x_1)\,A_1(x_1) + A_1^T(x_1)\,P_1(x_1) + Q_1(x_1) \\ -P_1(x_1)\,B_1(x_1)\,R_1^{-1}(x_1)\,B_1^T(x_1)\,P_1(x_1) = 0, \\ \\ P_2(x_2)\,A_2(x_2) + A_2^T(x_2)\,P_2(x_2) + Q_2(x_2) \\ -P_2(x_2)\,B_2(x_2)\,R_2^{-1}(x_2)\,B_2^T(x_2)\,P_2(x_2) = 0. \end{cases} \tag{35}$$

The decentralized optimal control are expressed as follows:

$$u_1(x_1) = -0.1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_1(x_1) \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} \tag{36}$$

and

$$u_2(x_2) = -0.1 \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_2(x_2) \begin{pmatrix} x_{21} \\ x_{22} \\ x_{23} \end{pmatrix}. \tag{37}$$

• **Numerical simulation:**

Figure 1 (respectively Figure 2) shows the behavior of the first states variables x_{11}, x_{12} and x_{13}, (respectively, the second states variables x_{21}, x_{21} and x_{23} of interconnected system (30) controlled by the decentralized control laws illustrated in Figure 3. Initial conditions were taken as follows: $x_{11}(0) = x_{13}(0) = x_{21}(0) = x_{22}(0) = 0.1$, $x_{12}(0) = x_{23}(0) = 0$.

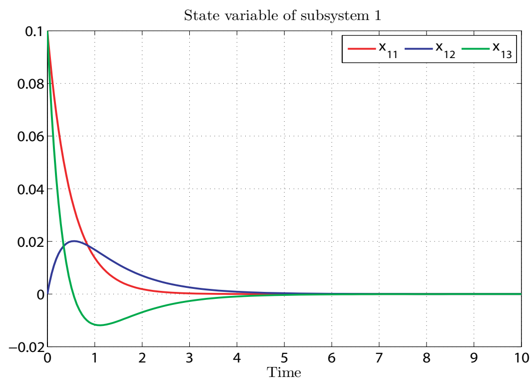

Figure 1: Closed loop reponses of x_1.

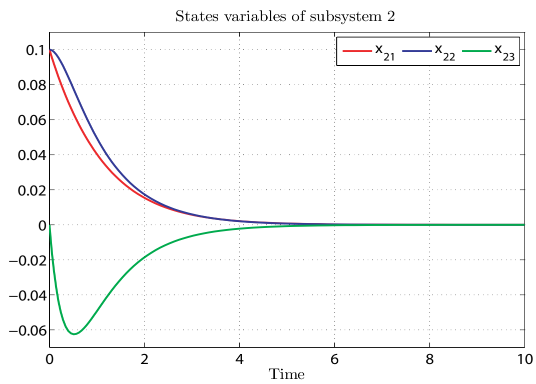

Figure 2: Closed loop reponses of x_2.

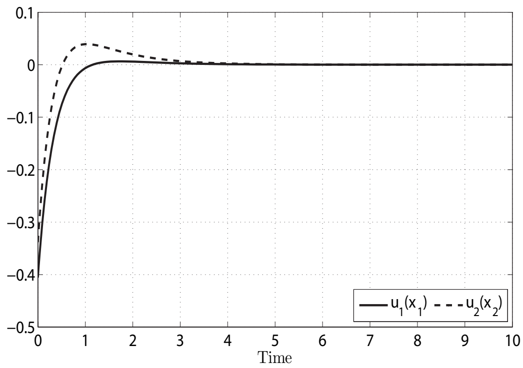

Figure 3: Decentralized control signals evolution.

We can note a satisfactory stabilization of state variables which converge into the origin point confirming the asymptotic stability of the controlled interconnected system using the decentralized SDRE approach.

6 Conclusion

In this paper, we have considered the method for feedback control of nonlinear interconnected systems using the decentralized state-dependent Riccati equation. This decentralized optimal approach is based on the solution of algebraic Riccati equation. Our first result was to determine and prove sufficient conditions that guarantee the global asymptotic stability of the overall interconnected system. We have then run some numerical simulations on a third order system. As expected, these simulations have shown the aptitude of the SDRE approach to be implemented easily and to give satisfactory result in terms of performance for a wide class of nonlinear interconnected systems. One of the possible perspectives that we can consider as a continuity of this research would be to investigate an optimal control for interconnected nonlinear systems via approximate methods.

Appendix

We recall hereafter the useful mathematical notations and properties concerning the Kronecker tensor product used in this paper.

A.1. Kronecker product:

The Kronecker product of $A\,(p \times q)$ and $B\,(r \times s)$ denoted by $A \otimes B$ is the $(pr \times qs)$ matrix defined by $[24, 25]$

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{pmatrix}. \tag{38}$$

A.2. Vec-function:

Vec-function is a linear algebra tool which is important in the multidimensional regression matrix representation. This operator is defined as follows [24, 25] :

$$A = (A_1\ A_2\ \dots\ A_n)\,; \quad Vec\,(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}, \tag{39}$$

where $\forall\, i \in \{1, \dots, n\}$, A_i is a vector of $\mathbb{R}^m$.

We recall the following useful rule of this function, given as follows:

$$Vec(E.A.C) = (C^T \otimes E)Vec(A). \tag{40}$$

A.3. Mat function :

An important matrix-valued linear function of a vector, denoted by $mat_{(n,m)}\,(\,.\,)$, was defined in [24, 25] as follows: if V is a vector of dimension $p = n.m$, then $M = mat_{(n,m)}\,(V)$ is the $(n \times m)$ matrix verifying $V = Vec\,(M)$.

References

[1] Yucelen, T., Medagam, P.V. and Pourboghrat, F. Nonlinear quadratic optimal control for cascaded multilevel static compensators. In: *39th North American Power Symposium*, Las Cruces, NM, USA, (2007) 523–527.

[2] Bahuguna, D. and Muslim, M. A study of nonlocal history-valued retarded differential equations using analytic semigroups. *Nonlinear Dynamics and Systems Theory* **6** (1) (2006) 63–75.

[3] Gruyitch, L.T. Consistent Lyapunov Methodology: Non-Differentiable Non-Linear Systems. *Nonlinear Dynamics and Systems Theory* **1**(1) (2001) 1–22.

[4] Elloumi, S., Abidi, B. and Benhadj Braiek, N. Decentralized optimal controller design for multimachine power systems using successive approximation approach. *Journal of the Franklin Institute* **350** (2013) 2994–3010.

[5] Guang-Hongand, Y. and Si-Ying, Z. Decentralized robust control interconnected systems with time-varying uncertainties. *Automatica* **32** (1996) 1603–1608.

[6] Hou, H., Nian, X., Xu, S., Sun, M. and Xiong, H. Robust decentralized control for large-scale web-winding systems: A linear matrix inequality approach. *Transactions of the Institute of Measurement and Control* **39** (7)(2017) 953–964.

[7] Li, Y. and Tong, S. Adaptive neural networks decentralized FTC design for nonstrict-feedback nonlinear interconnected large-scale systems against actuator faults. *IEEE transactions on neural networks and learning systems* **45** (2016) 138–149.

[8] Elloumi, S. and Benhadj Braiek, N. A decentralized stabilization approach of a class of nonlinear polynomial interconnected systems-application for a large scale power system. *Nonlinear Dynamics and Systems Theory* **12**(2) (2012) 159–172.

[9] Elloumi, S. and Benhadj Braiek, N. Robust decentralized control for multimachine power systems-the LMI approach. In: *IEEE International Conference on Systems, Man and Cybernetics* **6** IEEE, (2002).

[10] Chibani, A., Chadli, M. and Benhadj Braiek, N. A sum of squares approach for polynomial fuzzy observer design for polynomial fuzzy systems with unknown inputs. *International Journal of Control, Automation and Systems* **14**(1) (2016) 323–330.

[11] Huang, Y. and Lu, W.M. Nonlinear optimal control: Alternatives to Hamilton-Jacobi equation. In: *Proceedings of the 35th IEEE Conference on Decision and Control*, Kobe, Japan, Japan (1996) 3942–3947.

[12] Crandall, M.G., Evans, L.C. and Lions, P.L. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society* **282** (1984) 487–502.

[13] Won, C.H. and Biswas, S. Optimal control using an algebraic method for control affine nonlinear systems. *International Journal of Control* **80** (2007) 1491–1502.

[14] Wernli, A. and Cook, G. Suboptimal control for the nonlinear quadratic regulator problem. *Automatica* **11** (1975) 75–84.

[15] Primbs, J.A., Nevistić, V. and Doyle, J.C. Nonlinear optimal control: A control Lyapunov function and receding horizon perspective. *Asian Journal of Control* **1** (1) (1999) 14–24.

[16] Kamocki, R. Necessary and sufficient optimality conditions for fractional nonhomogeneous Roesser model. *Optimal Control Applications and Methods* **37** (4) (2016) 574–589.

[17] Pearson, J.D. Approximation methods in optimal control. *Journal of Electronics and Control* **13** (1962) 453-469.

[18] Wernli, A. and Cook, G. Suboptimal control for the nonlinear quadratic regulator problem. *Automatica* **11**(1) (1975) 75–84.

[19] Mracek, C.P. and Cloutier, J.R. Control designs for the nonlinear benchmark problem via the state-dependent Riccati equation method. *International Journal of Robust and Nonlinear Control* **8** (4-5) (1998) 401–433.

[20] Cimen, T. State-dependent Riccati equation (SDRE) control: A survey. *IFAC Proceedings* **41**(2) (2008) 3761–3775.

[21] Banks, H.T., Kwon, H.D., Toivanen, J.A. and Tran, H.T. A state-dependent Riccati equation-based estimator approach for HIV feedback control. *Optimal Control Applications and Methods* **27** (2006) 93-121.

[22] Elloumi, S. and Benhadj Braiek, N. On Feedback Control Techniques of Nonlinear Analytic Systems. *Journal of Applied Research and Technology* **12**(2014) 500-513.

[23] Elloumi, S., Sansa, I. and Benhadj Braiek, N. On the stability of optimal controlled systems with SDRE approach. In: *9th International Multi-Conference on Systems, Signals and Devices (SSD)*, Chemnitz, Germany (2012) 1–5.

[24] Bouafoura, M.K., Moussi, O. and Benhadj Braiek, N. A fractional state space realization method with block pulse basis. *Signal Processing* **91**(3) (2011) 492–497.

[25] Brewer, J.W. Kroneker products and matrix calculus in system theory. *IEEE Transaction on Circuits and Systems (CAS)* **25** (1978) 772–781.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 68–78

Analysis and Adaptive Control Synchronization of a Novel 3-D Chaotic System

F. Hannachi *

Department of Management Sciences, University of Tebessa, (12002), Algeria

Received: August 28, 2018; Revised: December 8, 2018

Abstract: In this paper, a new 3D chaotic system is introduced. Basic dynamical characteristics and properties of this new chaotic system are studied, namely the equilibrium points and their stability, the Lyapunov exponent, Lyapunov exponent spectrum and the Kaplan-Yorke dimension. Also, we derive new control results via the adaptive control method based on Lyapunov stability theory and the adaptive control theory of this new chaotic system with unknown parameters. The results are validated by numerical simulation using Matlab.

Keywords: *chaotic system; strange attractor; Lyapunov exponent; Lyapunov stability theory; adaptive control; synchronization.*

Mathematics Subject Classification (2010): 37B55, 34C28, 34D08, 37B25, 37D45, 93C40, 93D05.

1 Introduction

In mathematics and physics, chaos theory deals with the behavior of certain nonlinear dynamical systems that under certain conditions exhibit a phenomenon known as chaos, which is characterised by a sensitivity to initial conditions [1]. Chaos as an important nonlinear phenomenon has been studied in mathematics, engineering and many other disciplines. Since Lorenz discovered a three-dimensional autonomous chaotic system [2], many other systems have been introduced and analysed, we mention the Chen, Rössler and Lü systems [3,4,5]. After that hyperchaotic systems were constructed using many different methods. The synchronization of two chaotic systems was introduced in the work of Pecora and Carroll [6], then many different methodologies have been developed for synchronization of chaotic systems such as the OGY method [7], active contol method [8], sliding mode control [9], backstepping control [10], function projective method [11], adaptive control [12-14], etc.

In this work, a new chaotic system is introduced and we derive new control results via the adaptive control method based on Lyapunov stability theory and the adaptive control theory for this new chaotic system with unknown parameters. The results are validated by numerical simulation using Matlab.

* Corresponding author: mailto:farehhannachi@yahoo.com

1.1 Description of the novel chaotic system

In this research work, we propose a new 3D chaotic system with two quadratic nonlinearities, which is given in the system form as

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1), \\ \frac{dx_2}{dt} = cx_1 + x_1x_3, \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3), \end{cases} \tag{1}$$

where a, b, c are positive reals parameters. In the first part of this paper, we shall show that the system (1) is chaotic when the system parameters a, b and c take the values:

$$a = 13, b = 2.5, c = 50. \tag{2}$$

1.2 Basic properties

In this section, some basic properties of system (1) are given. We start with the equilibrum points of the system and check their stability at the initial values of the parameters a, b, c.

1.3 Equilibrum points

Putting equations of system (1) equal to zero, i.e.

$$a(x_2 - x_1) = 0, \quad cx_1 + x_1x_3 = 0, \quad -x_1x_2 + b(x_1 - x_3) = 0, \tag{3}$$

gives the three equilibrium points

$$p_0 = (0,0,0)\,, \quad p_{1,2} = \left(\frac{1}{2}b \mp \frac{1}{2}\sqrt{4bc + b^2}, \frac{1}{2}b \mp \frac{1}{2}\sqrt{4bc + b^2}, -c\right). \tag{4}$$

1.4 Stability

In order to check the stability of the equilibrum points we derive the Jacobian matrix at a point $p\,(x, y, z)$ of the system (1)

$$J(p) = \begin{pmatrix} -a & a & 0 \\ c+z & 0 & x \\ b-y & -x & -b \end{pmatrix}. \tag{5}$$

For p_0, we obtain $J(p_0) = \begin{pmatrix} -a & a & 0 \\ c & 0 & 0 \\ b & 0 & -b \end{pmatrix}$, with the characteristic polynomial equation $\lambda^3 + (a+b)\,\lambda^2 + (ab - ac)\,\lambda - abc = 0$, which has three eigenvalues

$$\lambda_1 = 19.811, \lambda_2 = -2.5, \lambda_3 = -32.811. \tag{6}$$

Since all the eigenvalues are real, the Hartma-Grobman theorem implies that p_0 is a saddle point which is unstable according to the Lyapunov theorem of stability.

By the same method, the eigenvalues of the Jacobian at p_1 are:

$$\lambda_1 = 0.993\,85 - 12.895i, \lambda_2 = 0.993\,85 + 12.895i, \lambda_3 = -17.488. \tag{7}$$

The eigenvalues of the Jacobian at p_2 are:

$$\lambda_1 = 0.763\,22 - 14.\,634i, \lambda_2 = 0.763\,22 + 14.\,634i, \lambda_3 = -17.\,026. \tag{8}$$

Then p_1 and p_2 are two unstable saddle-foci because none of the eigenvalues have zero real part and λ_1, λ_2 are complex.

1.5 Dissipativity

A dissipative dynamical system satisfies the condition

$$\nabla.V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} < 0. \tag{9}$$

In the case of the system (1), we have

$$\nabla.V = -(a + b). \tag{10}$$

For $a = 13, b = 2.5, c = 50$ we obtain $\nabla.V = -15.\,5 < 0$, and threfore dissipativity condition holds for this system. Also,

$$\frac{dV}{dt} = e^{-(a+b)} = 1.\,855\,4 \times 10^{-7}. \tag{11}$$

Then the volume of the attractor decreases by a factor of 0.00000018554.

2 Lyapunov Exponents and Kaplan-Yorke Dimension

Lyapunov exponents are used to measure the exponential rates of divergence and convergence of nearby trajectoiries, which is an important characterstic to judge whether the system is chaotic or not. The existence of at least one positive Lyapunov exponent implies that the system is chaotic.

For the chosen parameter values (2), the Lyapunov exponents of the novel chaotic system (1) are obtained using Matlab as:

$$L_1 = 1.4375, L_2 = -0.000166417, L_3 = -16.9373. \tag{12}$$

The Lyapunov exponents spectrum is shown in Fig. 1.

Since the spectrum of Lyapunov exponents (13) has a positive term L_1, it follows that the novel 3-D chaotic system (1) is chaotic. The Kaplan-Yorke dimension of system (1) is calculated as

$$D_{KL} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0849. \tag{13}$$

3 Adaptive Control of the Novel 3-D Chaotic System

This section describes an adaptive design of a globally stabilizing feedback controller for the chaotic system (1) with unknown parameters. The design is carried out using the adaptive control theory and Lyapunov stability theory.

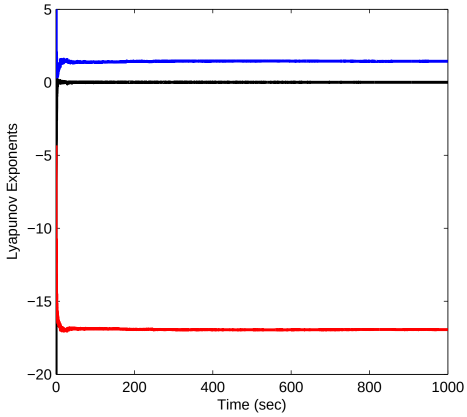

Figure 1: Lyapunov exponents spectrum.

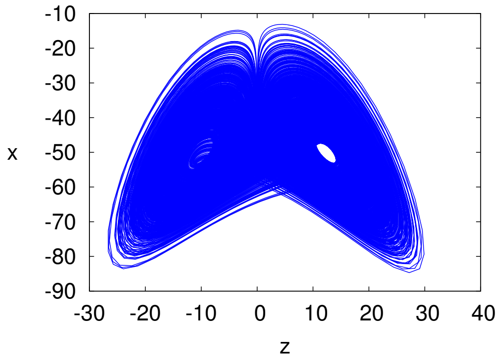

Figure 2: Projection of the strange attractor of the system (1) into the (z; x)-plane.

A controlled chaotic system of (1) is given by

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1) + u_1, \\ \frac{dx_2}{dt} = cx_1 + x_1x_3 + u_2, \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3) + u_3, \end{cases} \tag{14}$$

where a, b, c are unknown constant parameters, and u_1, u_2, u_3 are adaptive controllers to be found using the states x_1, x_2, x_3 and estimates $a_1(t), b_1(t), c_1(t)$ of the unknown parameters a, b, c, respectively.

We take the adaptive control law defined by

$$\begin{cases} u_1 = -a_1(t)(x_2 - x_1) - k_1x_1, \\ u_2 = -c_1(t)x_1 - x_1x_3 - k_1x_2, \\ u_3 = x_1x_2 - b_1(t)(x_1 - x_3) - k_3x_3, \end{cases} \tag{15}$$

where k_1, k_2, k_3 are positive gain constants.

Substituting (15) into (14), we obtain the closed-loop control system as

$$\begin{cases} \frac{dx_1}{dt} = (a - a_1(t))(x_2 - x_1) - k_1 x_1, \\ \frac{dx_2}{dt} = (c - c_1(t))x_1 - k_2 x_2, \\ \frac{dx_3}{dt} = (b - b_1(t))(x_1 - x_3) - k_3 x_3. \end{cases} \tag{16}$$

We define the parameter estimation errors as

$$e_a(t) = a - a_1(t), \quad e_c(t) = c - c_1(t), \quad e_b(t) = b - b_1(t). \tag{17}$$

By using (17), we rewrite the closed-loop system (16) as

$$\begin{cases} \frac{dx_1}{dt} = e_a(t)(x_2 - x_1) - k_1 x_1, \\ \frac{dx_2}{dt} = e_c(t)x_1 - k_2 x_2, \\ \frac{dx_3}{dt} = e_b(t)(x_1 - x_3) - k_3 x_3. \end{cases} \tag{18}$$

Differentiating (17) with respect to t, we obtain

$$\begin{cases} \frac{de_a(t)}{dt} = -\frac{da_1(t)}{dt}, \\ \frac{de_c(t)}{dt} = -\frac{dc_1(t)}{dt}, \\ \frac{de_b(t)}{dt} = -\frac{db_1(t)}{dt}. \end{cases} \tag{19}$$

To find an update law for the parameter estimates, we shall use the Lyapunov stability theory. We consider the quadratic Lyapunov function given by

$$V(x_1, x_2, x_3, e_a, e_b, e_c) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2\right). \tag{20}$$

which is a positive definite function on $\mathbb{R}^6$.

Differentiating V along the trajectories of the systems (18) and (19), we obtain the following:

$$\dot{V} = -\sum_{i=1}^{3} k_i x_i^2 + e_a\left(x_1x_2 - x_1^2 - \frac{da_1(t)}{dt}\right) + e_b\left(x_1x_3 - x_3^2 - \frac{db_1(t)}{dt}\right) + e_c\left(x_1x_2 - \frac{dc_1(t)}{dt}\right). \tag{21}$$

In view of (21), we take the parameter update law as follows

$$\begin{cases} \frac{da_1(t)}{dt} = x_1x_2 - x_1^2, \\ \frac{db_1(t)}{dt} = x_1x_3 - x_3^2, \\ \frac{dc_1(t)}{dt} = x_1x_2. \end{cases} \tag{22}$$

Theorem 3.1 *The 3-D novel chaotic system (14) with unknown parameters is globally and exponentially stabilized by the adaptive feedback control law (15) and the parameter update law (22), where k_1, k_2, k_3 are positive constants* 3.1.

Proof. Substituting the parameter update law (21) into (20), we obtain the time derivative of V as:

$$\dot{V} = -k_1x_1^2 - k_2x_2^2 - k_3x_3^2, \tag{23}$$

which is a negative definite function on $\mathbb{R}^6$. By the direct method of Lyapunov [15], it follows that $x_1, x_2, x_3, e_a, e_b, e_c$ are globally exponentially stable. □

3.1 Numerical simulations

We used the classical fourth-order Runge-Kutta method with the step size $h = 10^{-8}$ to solve the system of differential equations (14) and (22), when the adaptive control law (15) is applied.

The parameter values of the novel 3-D chaotic system (14) are chosen as in the chaotic case (2). The positive gain constants are taken as $k_i = 3$, for $i = 1, 2, 3$.

The initial conditions of the novel chaotic system (14) are chosen as $x_1(0) = 6.4, x_2(0) = -4.7, x_3(0) = 2.5$. Furthermore, as initial conditions of the parameter estimates of the unknown parameters, we have chosen: $a_1(0) = 2.5, b_1(0) = 5.3, c_1(0) = 4.8$.

In Figs. 3-4, the exponential convergence of the controlled states $x_1(t), x_2(t), x_3(t)$ and the time-history of the parameter estimates $a_1(t); b_1(t); c_1(t)$ are depicted, when the adaptive control law (15) and parameter update law (22) are implemented.

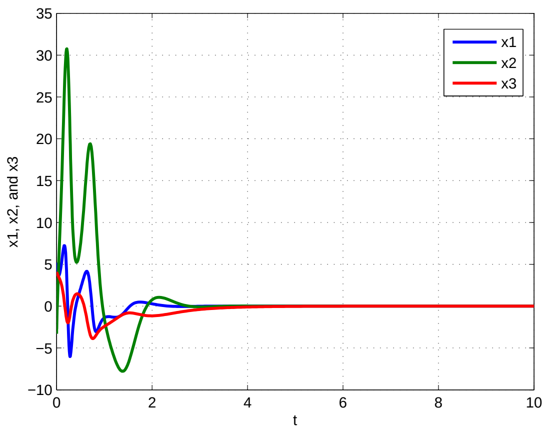

Figure 3: Exponential convergence of the controlled states $x_1(t); x_2(t); x_3(t)$.

4 Adaptive Synchronization of the Identical Novel 3-D Chaotic Systems

In this section, we derive an adaptive control law for globally and exponentially synchronizing the identical novel 3-D chaotic systems with unknown system parameters. Thus, the master system is given by the novel chaotic system dynamics

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1), \\ \frac{dx_2}{dt} = cx_1 + x_1 x_3, \\ \frac{dx_3}{dt} = -x_1 x_2 + b(x_1 - x_3). \end{cases} \tag{24}$$

Also, the slave system is given by the novel chaotic system dynamics

$$\begin{cases} \frac{dy_1}{dt} = a(y_2 - y_1) + u_1, \\ \frac{dy_2}{dt} = cy_1 + y_1 y_3 + u_2, \\ \frac{dy_3}{dt} = -y_1 y_2 + b(y_1 - y_3) + u_3. \end{cases} \tag{25}$$

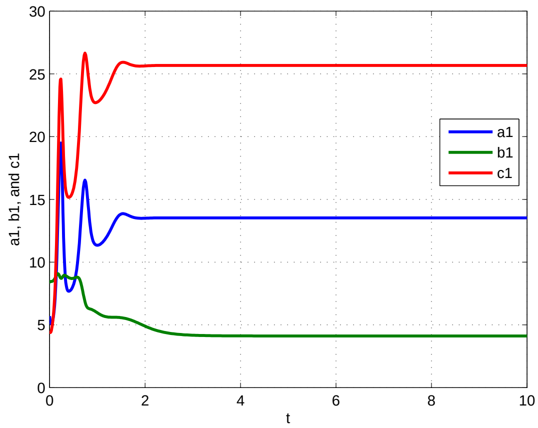

Figure 4: Time-history of the parameter estimates $a_1(t); b_1(t); c_1(t)$.

In (24) and (25), the system parameters a, b, c are unknown and the design goal is to find an adaptive feedback controls u_1, u_2, u_3 using the states x_1, x_2, x_3 and estimates $a_1(t), b_1(t), c_1(t)$ of the unknown parameters a, b, c, respectively. The synchronization error between the novel chaotic systems (24) and (25) is defined as

$$e_1 = y_1 - x_1, \quad e_2 = y_2 - x_2, \quad e_3 = y_3 - x_3. \tag{26}$$

Then (26) implies

$$\begin{cases} \dot{e}_1 = \dot{y}_1 - \dot{x}_1, \\ \dot{e}_2 = \dot{y}_2 - \dot{x}_2, \\ \dot{e}_3 = \dot{y}_3 - \dot{x}_3. \end{cases} \tag{27}$$

Thus, the synchronization error dynamics is obtained as

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1, \\ \dot{e}_2 = ce_1 + y_1 y_3 - x_1 x_3 + u_2, \\ \dot{e}_3 = b(e_1 - e_3) - y_1 y_2 + x_1 x_2 + u_3. \end{cases} \tag{28}$$

We take the adaptive control law defined by

$$\begin{cases} u_1 = -a_1(e_2 - e_1) - k_1 e_1, \\ u_2 = -c_1 e_1 - y_1 y_3 + x_1 x_3 - k_2 e_2, \\ u_3 = -b_1(e_1 - e_3) + y_1 y_2 - x_1 x_2 - k_3 e_3. \end{cases} \tag{29}$$

where k_1, k_2, k_3 are positive gain constants.

Substituting (29) into (28), we obtain the closed-loop error dynamics as

$$\begin{cases} \dot{e}_1 = (a - a_1)(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = (c - c_1)e_1 - k_2 e_2, \\ \dot{e}_3 = (b - b_1)(e_1 - e_3) - k_3 e_3. \end{cases} \tag{30}$$

The parameter estimation errors are defined as

$$e_a(t) = a - a_1(t), \quad e_c(t) = c - c_1(t), \quad e_b(t) = b - b_1(t). \tag{31}$$

Differentiating (31) with respect to t, we obtain

$$\begin{cases} \frac{de_a(t)}{dt} = -\frac{da_1(t)}{dt}, \\ \frac{de_c(t)}{dt} = -\frac{dc_1(t)}{dt}, \\ \frac{de_b(t)}{dt} = -\frac{db_1(t)}{dt}. \end{cases} \tag{32}$$

By using (31), we rewrite the closed-loop system (30) as

$$\begin{cases} \dot{e}_1 = e_a(e_2 - e_1) - k_1 e_1, \\ \dot{e}_2 = e_c e_1 - k_2 e_2, \\ \dot{e}_3 = e_b(e_1 - e_3) - k_3 e_3. \end{cases} \tag{33}$$

We consider the quadratic Lyapunov function given by

$$V(x_1, x_2, x_3, e_a, e_b, e_c) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + e_a^2 + e_b^2 + e_c^2\right). \tag{34}$$

which is a positive definite function on $\mathbb{R}^6$.

Differentiating V along the trajectories of the systems (33) and (32), we obtain the following:

$$\dot{V} = -\sum_{i=1}^{3} k_i e_i^2 + e_a\left(e_1 e_2 - e_1^2 - \frac{da_1(t)}{dt}\right) + e_b\left(e_1 e_3 - e_3^2 - \frac{db_1(t)}{dt}\right) + e_c\left(e_1 e_2 - \frac{dc_1(t)}{dt}\right). \tag{35}$$

In view of (35), we take the parameter update law as follows:

$$\begin{cases} \frac{da_1(t)}{dt} = e_1 e_2 - e_1^2, \\ \frac{db_1(t)}{dt} = e_1 e_3 - e_3^2, \\ \frac{dc_1(t)}{dt} = e_1 e_2. \end{cases} \tag{36}$$

Substituting (36) into (35), we get

$$\dot{V} = -\sum_{i=1}^{3} k_i e_i^2, \tag{37}$$

which is a negative definite function on $\mathbb{R}^3$. Hence, by the Lyapunov stability theory [15], it follows that $e_i(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for $i = 1, 2, 3$. Hence, we have proved the following theorem.

Theorem 4.1 *The novel 3-D chaotic systems (24) and (25) with unknown parameters are globally and exponentially synchronized for all initial conditions by the adaptive feedback control law (29) and the parameter update law (36), where k_1, k_2, k_3 are positive constants* 4.1.

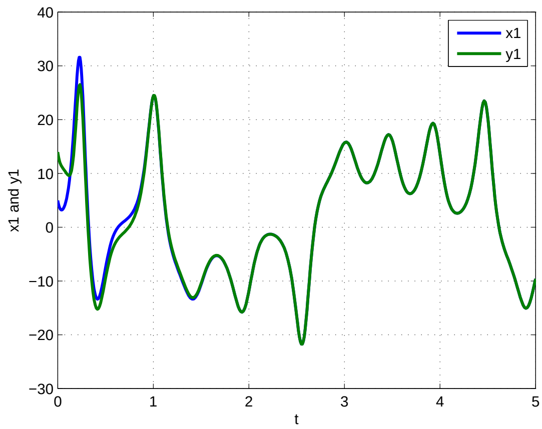

Figure 5: Synchronization of the states $x_1(t)$ and $y_1(t)$.

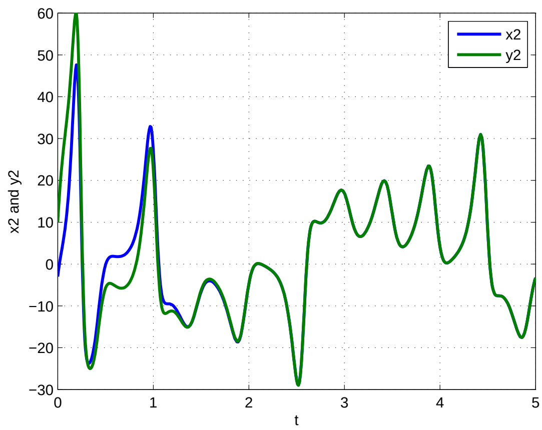

Figure 6: Synchronization of the states $x_2(t)$ and $y_2(t)$.

4.1 Numerical simulations

We used the classical fourth-order Runge-Kutta method with the step size $h = 10^{-8}$ to solve the system of differential equations (24), (25) and (36), when the adaptive control law (29) is applied.

The parameter values of the novel 3-D chaotic system (24) are chosen as in the chaotic case (2). The positive gain constants are taken as $k_i = 4$, for $i = 1, 2, 3$.

The initial conditions for the master system (24) are chosen as $x_1(0) = 5, x_2(0) =$

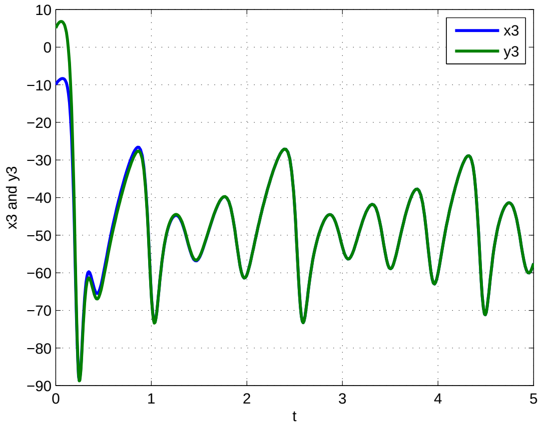

Figure 7: Synchronization of the states $x_3(t)$ and $y_3(t)$.

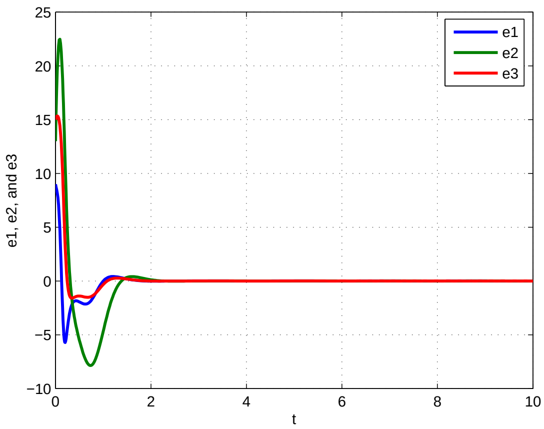

Figure 8: Time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$.

$-3, x_3(0) = -10$ and those for the slave system (25) are chosen as $y_1(0) = 14, y_2(0) = 10, y_3(0) = 5$. Furthermore, as initial conditions of the parameter estimates of the unknown parameters, we have chosen $a_1(0) = 10, b_1(0) = 15, c_1(0) = 20$. In Figs. 5-7, the synchronization of the states of the master system (24) and slave system (25) is depicted, when the adaptive control law (29) and parameter update law (36) are implemented. In Fig. 8, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ is depicted.

5 Conclusion

In this paper, a new chaotic system is introduced. Basic properties of this system are studied, namely, the equilibrum points and their stability, the Lyapunov exponent and the Kaplan-Yorke dimension. Moreover, adaptive control schemes have been proposed to stabilize and synchronize such two new chaotic systems. Numerical simulations using MATLAB have been made to illustrate our results for the new chaotic system with unknown parameters.

Acknowledgment

The author would like to thank the editor in chief and the referees for their valuable suggestions and comments.

References

[1] Ott, E. *Chaos in Dynamical Systems.* Cambridge University Press, Cambridge, 2002.

[2] Lorenz, E.N. Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences* **20** (5) (1963) 130–141.

[3] Chen, G. and Ueta, T. Yet another chaotic attractor. *International Journal of Bifurcation and Chaos* **9** (7) (1999) 1465–1466.

[4] Lü, J. and Chen, G. A new chaotic attractor coined. *International Journal of Bifurcation and Chaos* **12** (3) (2002) 659–661.

[5] Rössler, O. An equation for continuous chaos. *Physics Letters A* **57** (5) (1976) 397–398.

[6] Pecora, L. and Carroll, T. Synchronization in chaotic systems. *Physical Review Letters* **64** (8) (1990) 821–824.

[7] Grebogi, C. and Lai, Y. C. Controlling chaotic dynamical systems. *Systems and control letters* **31** (5) (1997) 307–312.

[8] Ho, M. and Hung, Y. Synchronization of two different chaotic systems using generalized active control. *Physics Letters A* **301** (5) (2002) 424–428.

[9] Sun, J., Shen, Y., Wang, X. et al. Finite time combination-combination synchronization of four different chaotic systems with unknown parameters via sliding mode control. *Nonlinear Dynamics* **76** (1) (2014) 383–397.

[10] Xiao-Qun, W. and Jun-An, L. Parameter identification and backstepping control of uncertain Lü system. *Chaos, Solitons and Fractals* **18** (1) (2003) 721–729.

[11] Othman, A. A., Noorani, M. S. M. and Al-Sawalha, M. M. Function projective dual synchronization of chaotic systems with uncertain parameters. *Nonlinear Dynamics and Systems Theory* **17** (2) (1963) 193–204.

[12] Adloo, H. and Roopaei, M. Review article on adaptive synchronization of chaotic systems with unknown parameters. *Nonlinear Dynamics* **65** (1) (2011) 141–159.

[13] Vaidyanathan, S., Vollos, C., Pham, V . and Madhavan, K. Analysis, adaptive control and synchronization of a novel 4-D hyperchaotic hyperjerk system and its SPICE implementation. *Archives of Control Sciences* **25** (1) (2003) 135–158.

[14] Vaidyanathan, S. and Vollos, C. Analysis and adaptive control of a novel 3-D conservative noequilibrium chaotic system. *Archives of Control Sciences* **25** (3) (2015) 333–353.

[15] Hahn, W. *The Stability of Motion.* Springer, Berlin, New York, 1967.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 79–87

Mathematical Model of C_d for Circular Cylinder Using Two Passive Controls at Re = 5000

C. Imron [1*] C.J. Kumalasari [1] B. Widodo [2] and T.Y. Yuwono [2]

[1] *Department of Mathematics, Faculty of Mathematics, Computation and Data Science, Institute of Technology Sepuluh Nopember, Surabaya, Indonesia*
[2] *Department of Mechanical Engineering, Faculty of Industrial Technology, Institute of Technology Sepuluh Nopember, Surabaya, Indonesia*

Received: December 8, 2017; Revised: December 10, 2018

Abstract: This study focuses on two passive controls. Passive control is the addition of a small object to an object to reduce the drag force of the object. In this case, two passive controls are placed in front of and in the rear of the main object. The distance between the main object and the two passive controls varies and the Reynolds number used is 5000. The main object is a circular cylinder, and its passive control in front is a cylinder of type-I at the distance S / D = 0.6, 1.2 ; 1.8; 2.4; 3.0 and in the rear is an elliptical or circular cylinder at the distance T / D = 0,6; 0,9; 1,2; 1,5; 1,8 and 2,1. In this study, we want to find an effective distance of the main object to two passive controls so that the drag coefficient of the main object is minimal compared to that with non-passive control or with one passive control in front. In addition, a mathematical model of the drag coefficient of circular cylinders with two passive controls at Re = 5000 will be obtained.

Keywords: *passive control; drag coefficient; cylinder.*

Mathematics Subject Classification (2010): 58D30, 65C20.

* Corresponding author: `mailto:imron-its@matematika.its.ac.id`

1 Introduction

Today, so many people are racing to create new technologies. Technological advancement is growing rapidly. Technology is actually a way and effort to improve the quality of human life [1]. New technologies can be created by conducting ongoing research, where the new technology is expected to change the behavior of users of these new technologies. Research related to fluid flow can be done by experiment or simulation. Study of the flow of fluids through objects with the aim of reducing the drag force of most objects is a paramount concern of the researchers.

Some researchers used one passive control placed in front of various shapes, such as cylindrical cylinders, type-I cylinders, type-D cylinders etc. Circular cylinders, elliptical cylinders or other shapes are commonly used objects for designing industrial chimneys, offshore and flyover structures and others. In this case, the design process should allow for the geometrical shape of the object because it affects the value of the drag coeffcient, so that for different geometric shapes the drag coeffcient values are also different. At the interaction between the fluid flow and the object the resulting fluid flow across a single object or multiple grouped objects will produce different flow characteristics.

In this study, we consider a boundary layer because it is seen that the liquid that flows through the surface of the object comes with the flow of particles around it. Basically, the boundary layer is an increase in shear stress which will affect the flow velocity in each layer [14]. The surface of the object will move slowly due to the friction force, so that the particle flow velocity around the object will be zero. While the other particles will interact,the velocity of the flow away from the object will be faster. This is due to increased shear stress.

There are some studies that use boundary layer concept, and the concept of the boundary layer can help to find the answer to the effect of shear stress having a very important role in flow characteristics around the object [2]. The research, among others, has been conducted on the flow of fluids through an object, such as a single cylindrical circular object [3], or a modified cylinder such as a cylinder of type-I or a cylinder of type-D [4,5] and a study has been conducted on a fluid stream through more than one object, i.e. fluid flow through more than one cylinder of various sizes and configurations, fluid flow through a circular cylinder with tandem configuration [6–9] and eliptical cylinders with their side configurations [10,11].

The existence of a drag force occurs when an object is bypassed by a fluid. In this case, the drag force is influenced by several parameters, one of which is the drag coefficient. One way to reduce the drag force on the objects bypassed is to add a smaller object in front of the main object called the passive control. The addition of passive control is carried out to reduce the coefficient by 48% [6], also one can find a mathematical model for a circular cylinder with two passive controls with the Reynolds number 5000. The cylinder of type-I is a circular cylinder obtained by cutting the left and right ends at a certain angle, so that the cylinder is shaped like I. The best cutting edge is 53^o, this is because the wake occured is wider than that at the other angle, forming also a wider and more annoying strong flow on the object wall.

In this study, we will get a mathematical model for a circular cylinder with two passive controls with the Reynolds number when these two passive controls effectively decrease the drag coefficient. The Reynolds number used is Re = 5000. Two passive controls will be used, the passive control in front is the cylinder of type-I and the passive control of type-I is placed perpendicular to the flow, while the passive control in the rear is landscape. The distance between the passive control in front and the circular cylinder is varying, as well as the distance between the passive control in the rear and the circular cylinder.

2 Numerical Method

The previously described problem can be solved by using the unstable incompressible fluid equation and the Navier-Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{v}\mathbf{v} = -\nabla P + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

where Re is the Reynolds number, $\mathbf{v}$ is the velocity, and P is the pressure. The Navier-Stokes equation can be solved by using SIMPLE algorithms and numerical methods. The first thing to do is to give the initial value for each variable. By ignoring the pressure components, we will find the velocity component of the momentum equation, so equation (1) becomes

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \cdot \mathbf{v}\mathbf{v} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v} \tag{3}$$

by using the finite difference method, we have

$$(f_x)_i = \frac{2f_{i+1} + 3f_i - 6f_{i-1} + f_{i-2}}{6\,\mathrm{dx}} \quad \text{and} \quad (f_y)_j = \frac{2f_{j+1} + 3f_j - 6f_{j-1} + f_{j-2}}{6\,\mathrm{dx}},$$

$$(f_{xx})_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{\mathrm{dx}^2} \quad \text{and} \quad (f_y y)_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{\mathrm{dx}^2},$$

and afterwards

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\mathbf{v}^{**} - \mathbf{v}^{*}}{\Delta t} = -\nabla P \tag{4}$$

because of equation (2), then equation (4) becomes

$$\frac{\nabla \cdot \mathbf{v}^{*}}{\Delta t} = -\Delta P \tag{5}$$

by using *SOR* (*Successive Over Relaxation*)

$$(P_n)_{i,j} = (1 - \epsilon)(P_{n-1})_{i,j} + \epsilon (P_n)_{i,j}. \tag{6}$$

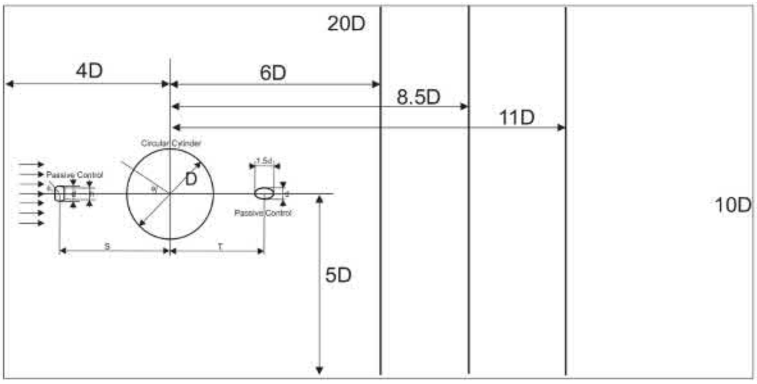

Figure 1: Design of the research system.

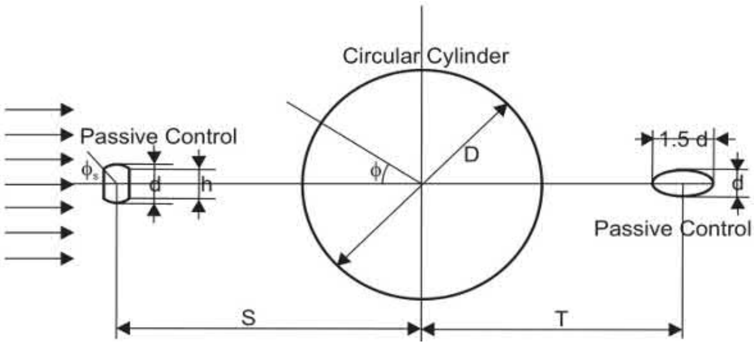

Figure 2: Schematic of two passive controls and a circular cylinder.

3 Main Result

Our research system is 10D 20D, where D is the diameter of the circular cylinder, placed at the distance of 4D from the front of the system and in the center of the system, as shown in Figure 1.

In this study, we used two passive controls. The first passive control is a cylinder of type-*I* placed in front of a circular cylinder at varying distance, i.e. S / D = 0.6,1,2,1,8,2,4 and 3.0. The second passive control are circular cylindrical and elliptical cylinders. The second passive control is placed in the rear of the circular cylinder at varying distance, i.e. T / D = 0.6,0,9,1,2,1,5,1,8 and 2.1 as shown in Figure 2.

3.1 Drag coefficient

The drag coeffcient of a single circular cylinder has been obtained by using the simulation program, the results are compared with experimental results and other simulation programs. We calculated that the drag coefficient of a single cylinder with Re = 100 is 1.356, while other researchers, with the same Reynolds number, have obtained: Zulhidayat has 1.4 and Five has 1.39 [12]. In this paper we will simulate a circular cylinder with two passive controls, and the Reynolds number used is 5000. The drag coefficient for a circular cylinder with Re = 5000 is 1.51.

S/D	0.6	1.2	1.8	2.4	3.0
C_D 5000	1.455	1.273	1.221	1.224	1.216

Table 1: Cd of a circular cylinder for Re=5000 with difference S/D.

Table 1 presents data on the drag coeffcient of a circular cylinder with a passive control, the cylinder of type-I, located at the front at varying distance. From the table it is clear that for the Reynold number Re = 5000, the best distance to get the minimum drag coeffcient is S/D = 1.8 or S/D = 3.0 with a drag coeffcient of 1.221 or 1.216. The value of the drag coeffcient is still smaller than the drag coeffcient without passive control.

C_{DO}	S/D				
T/D	0.6	1.2	1.8	2.4	3.0
0.6	1.116	1.012	0.973	0.992	0.987
0.9	1.205	1.043	1.015	1.007	1.008
1.2	1.169	1.014	0.977	0.990	0.986
1.5	1.412	1.277	0.916	1.265	1.245
1.8	1.557	1.284	1.225	1.222	1.195
2.1	1.401	1.384	1.220	1.209	1.191

Table 2: C_D of a circular cylinder for Re=5000 with difference S/D.

The drag coefficient of a circular cylinder with two passive controls at the front and in the rear. Passive control in front of the circular cylinder is the cylinder of type-I, while the passive control behind the circular cylinder is a small circular cylinder. The data on the drag coefficient with the Reynolds number Re = 5000 and the configuration as above, can be seen in Table 2. It appears that the passive control behind has a significant effect on the drag coeffcient, since the drag coeffcient is still smaller than that without passive control. The minimum drag coefficient of the configuration is 0.916, this occurs at S/D = 1.8 and T/D = 1.5.

3.2 Mathematical Model

In this case, the simulation result of the drag coefficient with two passive controls is interpolated to obtain the mathematical model. By using the bilinear interpolation approach one can make a mathematical model of the drag coefficient. Bilinear interpolation is the development of linear interpolation of two variables [13]. In this study we use the 2nd order bilinear interpolation. In this case, the variables used are (x, y) = (T / D, S / D). By taking the nine points of drag data that have been obtained from the simulation results in Table 2 we can get the interpolation formulation. Therefore, nine polynomial equations and nine unknown coefficients can be obtained. The polynomial interpolation function can be written as follows:

$$f(x,y) = a_{00} + a_{01}y + a_{02}y^2 + a_{10}x + a_{11}xy + a_{12}xy^2 + a_{20}x^2 + a_{21}x^2y + a_{22}x^2y^2. \quad (7)$$

Taking data from Table 3 and substituting (x, y)=(T/D, S/D) into f(x,y) we find the unknown coefficients. Therefore, we can obtain the mathematical model of the drag coefficient as follows:

$$\begin{aligned} E(x,y) \quad = \quad & 0.0275x^2y^2 - 0.0240x^2y + 0.0590x^2 - 0.1713xy^2 + 0.1792xy - 0.2958x \\ & +5.5913y^2 - 0.848y + 1.5542. \end{aligned} \quad (8)$$

$f(T/D, S/D)$	S/D		
T/D	0.6	1.8	3.0
0.6	1.116	0.973	0.987
1.2	1.169	0.977	0.986
1.8	1.557	1.225	1.195

Table 3: Nine drag data results for a circular cylinder

The error in the above mathematical model is calculated using an absolute error as follows:

$$e(x,y) \quad = \quad |E(x,y) - f(x,y)|. \quad (9)$$

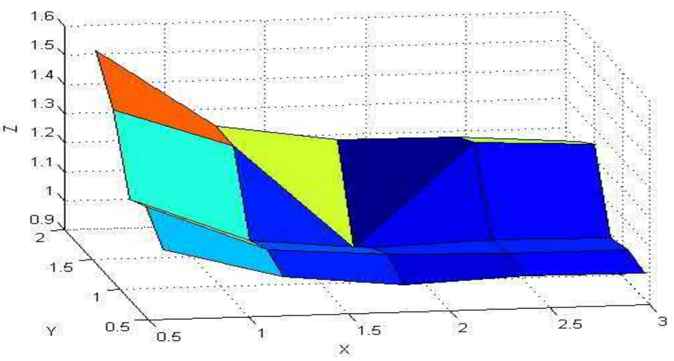

Figure 3: Graphic plot of Table 2 in Matlab.

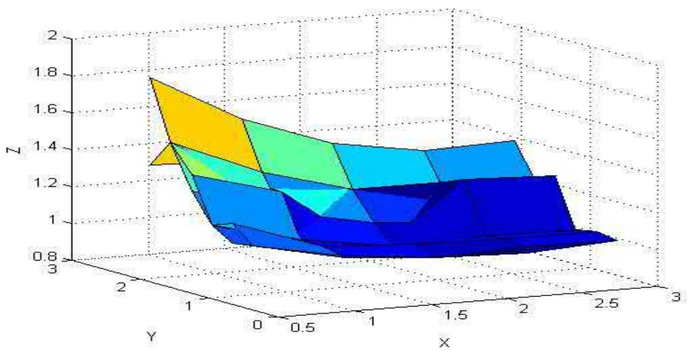

Figure 4: Comparison of the graphic plot of Table 2 and the graphic plot of Table 3.

If we simulate the original data in Table 2 as shown in Figure 3 and compare it with the simulation result by using bilinear interpolation shown in Figure 4 then it appears that the smallest absolute error is S / D = 0.6,1.8,3.0 , T / D = 0.6,1.2,1.8 with the value of Cd = 1.116, 1.169, 1.557, 0.973, 0.977, 1.225, 0.987, 0.986, 1.195 and also the obtained largest absolute error is in S / D =2.4 , T / D = 1.5 , with the value of Cd = 1.265. In other words, the error will not exceed the point of 0.2282.

3.3 Wake

In this study, the velocity data at the distance 6D, 8.5D and 11D from the center of the circular cylinder or at the distance 10D, 12.5D and 15D from the front of the system, are shown in Figure 1 . In both passive controls with Re = 5000 there is a wake. Also, it can be seen for the drag coefficient of the main circular cylinder that there is a significant decrease. In this case there is a decrease in the drag coefficient which affects the magnitude of the average velocity behind the circular cylinder.

It appears that Table 4 shows that a decrease in flow velocity behind the circular cylinder correspons to the decrease in the drag coefficient. In addition, the flow velocity near the circular cylinder (i.e., 6D from the center of the circular cylinder) will increase as the distance moves farther away from the center of the circular cylinder and will return equally to the speed without passive control.

Re	Single	1 PC	%	2 PC	%
5000	1.51	1.216	19.47	0.916	39.34

Table 4: C_D for $R_e = 5000$.

4 Conclusion

By using the bilinear interpolation approach one can make a mathematical model of the drag coefficient. Bilinear interpolation is the development of linear interpolation of two variables. In this study we use the 2nd order bilinear interpolation. Thus, a mathematical model can be formed for C_d of a circular cylinder using two passive controls at Re = 5000. The mathematical model can be written as follows:

$$\begin{aligned} E(x,y) &= 0.0275x^2y^2 - 0.0240x^2y + 0.0590x^2 - 0.1713xy^2 + 0.1792xy - 0.2958x \\ &\quad +5.5913y^2 - 0.848y + 1.5542. \end{aligned} \tag{10}$$

In addition, a reduction of the drag coefficient in a circular cylinder can be done by adding passive control. Passive control can be placed in front and/or behind. The drag coefficient can be reduced by up to 40% if using passive control in the form of the cylinder of type-I and an ellipse-shaped cylindrical back control rather than a passive control drag coefficient. This is reinforced by the decreasing flow velocity behind the circular cylinder.

Acknowledgment

We appreciate the support of the head of the Research and Community Service of the Institute of Technology Sepuluh Nopember (LPPM ITS) Surabaya, Indonesia, on behalf of the Ministry of Technology and Higher Education (Kemenristekdikti), who has given us a research grant under the contract No. 010/SP2H/LT/DRPM/IV/2017, April 20th, 2017, and the opportunity to disseminate our research by this paper.

References

[1] [online] Available : http://ariantiyoulie.blogspot.co.id/2014/01/01. December, 2017.

[2] Widodo, B. The Influence of Hydrodynamics on Pollutant Dispersion in the River. *International Journal of Contemporary Mathematical Sciences (IJCMS) ISSN 1312-7586*. HIKARI Ltd Journals and Books Publisher, Bulgaria, 2012, 2229–2234.

[3] Ladjedel, A.O. Yahiaoui, B.T. Adjlout, C.L. and Imine, D.O. Experimental and Numerical Studies of Drag Reduction on Circular Cylinder. *World Academy of Sciences*. Engineering and Technology, Bulgaria, 2011, 357–361.

[4] Igarashi, T. and Shiba, Y. Drag Reduction for D-Shape and I-Shape Cylinders (Aerodynamics Mechanism of ReductionDrag). *JSME International Journal* (Series B). 2006, 1036–1042.

[5] Triyogi, Y. and Wawan, A.W. Flow Characteristics Around a D-Type Cylinder Near a Plane Wall. In: *Regional Conferences on Mechanical and Aerospace Technology.* Bali, 2010.

[6] Bouak, F. and Lemay, J. Passive Control of the Aerodynamic Forces Acting on a Circular Cylinder. *Experimental Thermal and Fluid Science* (1998) 112–121.

[7] Tsutsui, T. and Igarashi, T. Drag Reduction of a Circular Sylinder in an Air-Stream. *Journal of Wind Engineering and Industrial Aerodynamics* (2002) 527–541.

[8] Triyogi Y. and Nuh, M. Using of a Bluff Body Cut from a Circular Cylinder as Passive Control to Reduce Aerodynamics Forces on a Circular Cylinder. In: *The International Conference on Fluid and Thermal energy Conversion.* Bali, Indonesia, 2003.

[9] Lee, Sang-Joon. Lee, Sang-Ik. and Park, Cheol-Woo. Reducing the Drag on a Circular Cylinder by Upstream Installation of a Small Control Rod. *Fluid Dynamic Research* (2004) 233–250.

[10] Imron, C. and Yunus, M. Mathematical Modeling of Pressure on Cylindrical Ellipse Using Side-by-Side Configuration. *International Journal of Computing Science and Applied Mathematics* **1** (1) (2015) 10–11.

[11] Imron, C. and Apriliani, E. Effect of Major Axis Length to the Pressure on Ellips. *International Journal of Computing Science and Applied Mathematics* **2** (3) (2016) 41–44.

[12] Widodo, B. Yuwono, T. and Imron, C. The influence of distance between passive control and circular cylinder on wake. *Journal of Physics Conference Series* **890** (1) (2017) 012053.

[13] The Wikipedia website [Online]. Available: https://en.wikipedia.org/wiki/$Bilinear_interpolation$. 29 October, 2017.

[14] Widodo, B. Yuwono, T. and Imron, C. The effectiveness of passive control to reduce the drag coefficient. *Journal of Physics Conference Series* **890** (1) (2017) 012044.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 88–96

A Variety of New Solitary-Solutions for the Two-mode Modified Korteweg-de Vries Equation

A. Jaradat [1], M.S.M. Noorani [1], M. Alquran [2*], and H.M. Jaradat [3]

[1] *School of Mathematical Sciences, University of Kebangsaan Malaysia, Bangi, Malaysia*
[2] *Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box (3030), Irbid (22110), Jordan*
[3] *Department of Mathematics, Al al-Bayt University, Mafraq, Jordan*

Received: October 13, 2018; Revised: December 12, 2018

Abstract: In this paper, we studied the nonlinear two-mode modified Korteweg-de Vries (TMmKdV) equation. We derived multiple singular soliton solutions to this new version of KdV equation by using the simplified form of Hirota's direct method. Also, kink and periodic solutions are extracted by using the tanh-expansion and the sine-cosine function methods. Finally, graphical analysis is conducted to show some physical features regarding TMmKdV equation.

Keywords: *two-mode mKdV; Hirota bilinear method; sine-cosine function method; multiple singular solutions; kink and periodic solutions.*

Mathematics Subject Classification (2010): 35C08, 74J35.

1 Introduction

Sergei V. Korsunsky [1] was the first who established the nonlinear two-mode Korteweg-de Vries (TMKdV) equation which reads

$$w_{tt} + (a_1 + a_2)w_{xt} + a_1 a_2 w_{xx} + ((\lambda_1 + \lambda_2)\frac{\partial}{\partial t} + (\lambda_1 a_2 + \lambda_2 a_1)\frac{\partial}{\partial x})ww_x \qquad (1)$$
$$+ ((\mu_1 + \mu_2)\frac{\partial}{\partial t} + (\mu_1 a_2 + \mu_2 a_1)\frac{\partial}{\partial x})w_{xxx},$$

where $w(x,t)$ is a field function representing the height of the free water surface above a flat bottom, a_1 and a_2 are the phase velocities, μ_1 and μ_2 are the dispersion parameters,

* Corresponding author: mailto:marwan04@just.edu.jo

λ_1 and λ_2 are the parameters of nonlinearity.

The modified Korteweg-de Vries (mKdV) equation for the one-dimensional propagation of solitary waves in a fluid is given by

$$w_t + \alpha w_{xxx} + \beta w^2 w_x = 0, \tag{2}$$

which is a generalized model in ocean dynamics, nonlinear lattice and plasma physics. In this paper we reconstruct and study the two-mode modified Korteweg-de Vries equation which describes the propagation of two wave modes of the same orientation. Now, the two-mode modified Korteweg-de Vries (TMmKdV) equation in a scaled-form reads

$$\begin{aligned} &w_{tt} + (a_1 + a_2)w_{xt} + a_1 a_2 w_{xx} + (\beta(\lambda_1 + \lambda_2)\frac{\partial}{\partial t} + \beta(\lambda_1 a_2 + \lambda_2 a_1)\frac{\partial}{\partial x})w^2 w_x \\ &+ (\alpha(\mu_1 + \mu_2)\frac{\partial}{\partial t} + \alpha(\mu_1 a_2 + \mu_2 a_1)\frac{\partial}{\partial x})w_{xxx}, \end{aligned} \tag{3}$$

where $a_1, a_2, \lambda_1, \lambda_2, \mu_1, \mu_2$ are some real numbers, $w(x,t)$ is a field function, a_1 and a_2 are the phase velocities, μ_1 and μ_2 are the dispersion parameters, λ_1 and λ_2 are the parameters of nonlinearity. Note that a_1, a_2 are considered to be distinct and $x, t \in (-\infty, \infty)$. Now we suggest the changes of variable by using the transformations [1–5]:

$$\begin{aligned} T &= (\mu_1 + \mu_2)^{\frac{-1}{2}} t, \\ X &= (\mu_1 + \mu_2)^{\frac{-1}{2}} (x - a_0 t), \\ a_0 &= \frac{a_1 + a_2}{2}, \\ W &= (\lambda_1 + \lambda_2)^{\frac{1}{2}} w. \end{aligned}$$

Therefore, equation (3) reduces to TMmKdV equation in a scaled form as

$$W_{TT} - a^2 W_{XX} + (\beta\frac{\partial}{\partial T} - \beta\lambda a\frac{\partial}{\partial X})W^2 W_x + (\alpha\frac{\partial}{\partial T} - \alpha\mu a\frac{\partial}{\partial X})W_{XXX}, \tag{4}$$

where

$$\begin{aligned} a &= \frac{a_1 - a_2}{2}, \\ \lambda &= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1}, |\lambda| \le 1, \\ \mu &= \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}, |\mu| \le 1, \end{aligned}$$

where $|\lambda| \le 1, |\mu| \le 1$ and a is defined above. Note that when $a = 0$, by integrating with respect to t, the two-mode modified Korteweg-de Vries equation (4) is reduced to the standard modified Korteweg-de Vries equation (2).

Finally, for more details about generating two-mode equations and physical features such models possess, we recommend for the readers the following references [6–14].

2 Multiple Soliton Solutions

In this section, we apply the simplified bilinear method [15–20], to find single soliton solutions and multiple soliton solutions for TMmKdV equation. First, we substitute

$$W(X,T) = e^{\varepsilon_i}, \qquad \varepsilon_i(X,T) = h_i X - \omega_i T$$

into the linear terms of (4) and solve the resulting equation to obtain the dispersion relation

$$\omega_i = \frac{\alpha h_i^3 \pm h_i\sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2}. \tag{5}$$

As a result ε_i becomes

$$\varepsilon_i(X,T) = h_i X - \frac{\alpha h_i^3 \pm h_i\sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2}T, \qquad i = 1, 2, \ldots. \tag{6}$$

Second, we propose the solutions of (4) in the form

$$W(X,T) = R\left(\arctan\left(\frac{h(X,T)}{k(X,T)}\right)\right)_X = R\frac{h_X k - k_X h}{h^2 + k^2}. \tag{7}$$

The auxiliary functions $h(X,T)$ and $k(X,T)$ for single-soliton solution are given by

$$\left\{ \begin{array}{c} h(X,T) = e^{\varepsilon_1(X,T)} = e^{h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T}, \\ k(X,T) = 1. \end{array} \right. \tag{8}$$

Substituting (7) and (8) into (4) and solving for R, we get

$$R = \pm 2\sqrt{\frac{6\alpha}{\beta}}. \tag{9}$$

Under the constraint condition $\lambda = \mu$, the single soliton solution is given by

$$\begin{aligned} W(X,T) &= 2h_1\sqrt{\frac{6\alpha}{\beta}}\frac{e^{h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T}}{1 + e^{2h_1 X - (\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2})T}} \\ &= h_1\sqrt{\frac{6\alpha}{\beta}}\text{sech}\left(\varepsilon_1(X,T)\right), \end{aligned} \tag{10}$$

where

$$\varepsilon_1(X,T) = h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T.$$

To find the two-soliton solution, we assume

$$\begin{aligned} h(X,T) &= e^{\varepsilon_1} + e^{\varepsilon_2} \\ &= e^{h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T} + e^{h_2 X - \frac{\alpha h_2^3 \pm h_2\sqrt{\alpha^2 h_2^4 + 4\alpha\mu a h_2^2 + 4a^2}}{2}T}, \\ k(X,T) &= 1 - c_{12}e^{h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T + h_2 X - \frac{\alpha h_2^3 \pm h_2\sqrt{\alpha^2 h_2^4 + 4\alpha\mu a h_2^2 + 4a^2}}{2}T}. \end{aligned} \tag{11}$$

Substituting (7) and (11) into (4) and solving for c_{12}, we see that the constraint condition of two soliton solutions exists only if $\lambda = \mu = \pm 1$ and the phase shift c_{12} is obtained by

$$c_{12} = \frac{(h_1 - h_2)^2}{(h_1 + h_2)^2} \tag{12}$$

and this can be generalized as

$$c_{ij} = \frac{(h_i - h_j)^2}{(h_i + h_j)^2}, \qquad 1 \le i < j \le 3. \tag{13}$$

To get the two-soliton solutions for (4), we substitute (11) and (12) into (7) and use $\lambda = \mu = 1$. As a result, we get

$$\begin{aligned} U(X,T) &= \frac{h_1 e^{h_1 X - r_1 T}\left(1 + \frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{2h_2 X - (\alpha h_2^3 \pm (\alpha h_2^3 + 2a))T}\right)\sqrt{\frac{6\alpha}{\beta}}}{\left(\frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{h_1 X - r_1 T + h_2 X - r_2 T} - 1\right)^2 + \left(e^{h_1 X - r_1 T} + e^{h_2 X - r_2 T}\right)^2} \\ &+ \frac{h_2 e^{h_2 X - r_2 T}\left(1 + \frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{2h_1 X - (\alpha h_1^3 \pm (\alpha h_1^3 + 2a))T}\right)\sqrt{\frac{6\alpha}{\beta}}}{\left(\frac{(h_1-h_2)^2}{(h_1+h_2)^2} e^{h_1 X - r_1 T + h_2 X - r_2 T} - 1\right)^2 + \left(e^{h_1 X - r_1 T} + e^{h_2 X - r_2 T}\right)^2}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} r_1 &= \frac{\alpha h_1^3 \pm (\alpha h_1^3 + 2a)}{2}, \\ r_2 &= \frac{\alpha h_2^3 \pm (\alpha h_2^3 + 2a)}{2}. \end{aligned}$$

For the three-soliton solutions, we use

$$\begin{cases} h(X,T) = e^{\varepsilon_1} + e^{\varepsilon_2} + e^{\varepsilon_3} + c_{123} e^{\epsilon_1 + \epsilon_2 + \varepsilon_3}, \\ k(X,T) = 1 - c_{12} e^{\epsilon_1 + \epsilon_2} - c_{13} e^{\epsilon_1 + \varepsilon_3} - c_{23} e^{\epsilon_2 + \varepsilon_3}, \end{cases} \tag{15}$$

where c_{ij} are given in (13). Substituting (7) and (15) into (4) and solving for c_{123} under the constraint condition $\lambda = \mu = \pm 1$, we find

$$c_{123} = c_{12} c_{13} c_{23}.$$

Finally, we reach to the fact that TMmKdV equation given in (4) has N-soliton solutions under the constraint condition $\lambda = \mu = \pm 1$ which can be obtained for finite N, where $N \ge 3$.

3 Singular Soliton Solutions

In this section we construct a multiple singular-soliton solution for (4) where the solution is assumed to be of the form

$$W(X,T) = R \ln \left(\frac{h(X,T)}{k(X,T)}\right)_X = R \frac{k h_X - h k_X}{kh}. \tag{16}$$

The dispersion relation as in the previous section is given by

$$\omega_i = \frac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2},$$

and hence $\varepsilon_i(X,T) = h_i X - \dfrac{\alpha h_i^3 \pm h_i \sqrt{\alpha^2 h_i^4 + 4\alpha\mu a h_i^2 + 4a^2}}{2} T$, $i = 1, 2, \ldots$.

For the singular one-soliton solution, we consider

$$h(X,T) = 1 + e^{\varepsilon_1(X,T)}, \quad k(X,T) = 1 - e^{\varepsilon_1(X,T)}. \tag{17}$$

Substituting (17) and (16) into (4) and solving for R, we get

$$R = \pm\sqrt{-6\alpha/\beta}. \tag{18}$$

Under the constraint condition $\lambda = \mu$, the single-soliton solution is given by

$$\begin{aligned} W(X,T) &= -2h_1\sqrt{\frac{-6\alpha}{\beta}}\frac{e^{h_1 X - \frac{\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2}}{2}T}}{e^{2h_1 X - (\alpha h_1^3 \pm h_1\sqrt{\alpha^2 h_1^4 + 4\alpha\mu a h_1^2 + 4a^2})T} - 1} \\ &= -h_1\sqrt{\frac{-6\alpha}{\beta}}\text{csch}\,(\varepsilon_1(X,T)). \end{aligned} \tag{19}$$

To obtain singular two-soliton solution, we set

$$\begin{cases} h(X,T) = 1 + e^{\varepsilon_1(X,T)} + e^{\varepsilon_2(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)}, \\ k(x,t) = 1 - e^{\varepsilon_1(X,T)} - e^{\varepsilon_2(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)}. \end{cases} \tag{20}$$

Substituting (18), (20) and (16) into (4) and solving for c_{12} lead to the two soliton solutions only if $\lambda = \mu = \pm 1$ and the same phase shift c_{12} obtained in (12) and hence c_{ij} given by (13).

To construct the singular three-soliton solution, we set

$$\begin{aligned} h(X,T) &= 1 + e^{\varepsilon_1(X,T)} + e^{\varepsilon_2(X,T)} + e^{\epsilon_3(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_1(X,T)} + c_{23}e^{\varepsilon_2(X,T)+\varepsilon_3(X,T)} \\ &+ c_{13}e^{\varepsilon_1(X,T)+\varepsilon_3(X,T)} + c_{123}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)+\varepsilon_3(X,T)}, \\ k(X,T) &= 1 - e^{\varepsilon_1(X,T)} - e^{\varepsilon_2(X,T)} - e^{\epsilon_3(X,T)} + c_{12}e^{\varepsilon_1(X,T)+\varepsilon_1(X,T)} + c_{23}e^{\varepsilon_2(X,T)+\varepsilon_3(X,T)} \\ &+ c_{13}e^{\varepsilon_1(X,T)+\varepsilon_3(X,T)} - c_{123}e^{\varepsilon_1(X,T)+\varepsilon_2(X,T)+\varepsilon_3(X,T)}. \end{aligned} \tag{21}$$

Repeating the same previous steps, we reach to the same fact that the three single-soliton solutions exists only under the constraint condition $\lambda = \mu = \pm 1$.

4 Solitary Ansatze Methods

In this part, we introduce in brief two methods, the tanh-technique and the sine-cosine function method to solve the problem (4).

4.1 The tanh method

The tanh technique [21–26] suggests the following solution

$$W(\zeta) = S(Y) = \sum_{i=0}^{M} b_i Y^i, \tag{22}$$

where $Y = \tanh(\delta\zeta)$. The index M can be determined by a balance procedure. Once we have M, we collect all coefficients of powers of Y in the resulting equation and set them to zero. Finally, we solve the obtained algebraic system to retrieve the values of the required coefficients b_i.

Now, we consider a new variable $\zeta = X - \gamma T$ to reduce (4) into the following differential equation

$$(\gamma^2 - a^2)W - \frac{\beta}{3}(\gamma + \lambda a)W^3 - \alpha(\gamma + \mu a)W'' = 0, \tag{23}$$

where $W = W(\zeta)$ and the prime denotes the ordinary derivative. By a blanching procedure for equation (23), the value of the parameter M is equal to 1 and thus $W(\zeta) = A + B \tanh(\delta\zeta)$. Substituting this proposed solution in (23) yields the following algebraic system:

$$\begin{aligned}
0 &= -Aa^2 - \frac{1}{3}\lambda A^3 a\beta - \frac{1}{3}A^3\beta\gamma + A\gamma^2,\\
0 &= -Ba^2 - \lambda A^2 B a\beta - A^2 B\beta\gamma + B\gamma^2 + 2\mu B a\alpha\delta^2 + 2B\alpha\gamma\delta^2,\\
0 &= -\lambda A B^2 a\beta - AB^2\beta\gamma,\\
0 &= -\frac{1}{3}\lambda B^3 a\beta - \frac{1}{3}B^3\beta\gamma - 2\mu B a\alpha\delta^2 - 2B\alpha\gamma\delta^2.
\end{aligned} \tag{24}$$

Solving the above system produces the following two-wave solution

$$W(X,T) = \pm\, \frac{\sqrt{-6\alpha\delta^2\left((-1+\lambda\mu)a + (\lambda-\mu)\gamma\right)}}{\sqrt{\beta\left((-1+\lambda^2)a + 2(-\lambda+\mu)\alpha\delta^2\right)}}\, \tanh\left(\delta\left(X - \gamma T\right)\right), \tag{25}$$

with $\gamma = (-\alpha\delta^2 \pm \sqrt{a^2 - 2\mu a\alpha\delta^2 + \alpha^2\delta^4})$. If the tanh-function is replaced by coth-function in (25), a new solution will be obtained.

4.2 The sine-cosine method

The sine-cosine technique [24, 25, 27–31] assumes the solution of (23) in the form of

$$W(\zeta) = A\sin^B(\delta\zeta), \tag{26}$$

or

$$W(\zeta) = A\cos^B(\delta\zeta), \tag{27}$$

To determine the values of $A,\ B,\ \gamma$ and δ, we substitute (26) in (23) to get

$$\begin{aligned}
0 &= \left(A\mu B a\alpha\delta^2 - A\mu B^2 a\alpha\delta^2 + AB\alpha\gamma\delta^2 - AB^2\alpha\gamma\delta^2\right)\sin^{B-2}(\delta z)\\
&\quad - \left(Aa^2 + A\gamma^2 + A\mu B^2 a\alpha\delta^2 + AB^2\alpha\gamma\delta^2\right)\sin^B(\delta z) - \left(\frac{1}{3}\lambda A^3 a\beta - \frac{1}{3}A^3\beta\gamma\right)\sin^{3B}(\delta z).
\end{aligned} \tag{28}$$

Now, equating the exponents $B-2$ and $3B$ in (28) and setting the coefficients of same power to zero, produce the following two-wave solution

$$W(X,T) = \frac{\sqrt{-6\alpha\delta^2\left((-1+\lambda\mu)a + (\lambda-\mu)\gamma\right)}}{\sqrt{\beta\left((-1+\lambda^2)a + (-\lambda+\mu)\alpha\delta^2\right)}}\, \csc\left(\delta\left(X - \gamma T\right)\right), \tag{29}$$

with $\gamma = \frac{1}{2}(-\alpha\delta^2 \pm \sqrt{4a^2 - 4\mu a\alpha\delta^2 + \alpha^2\delta^4})$.

Finally, by using the cosine-function method (27), another two-wave solution will be obtained being the same as given in (29) but with csc replaced by sec.

5 Numerical Example

In this section, we study some physical features of the solution of TMmKdV equation given in (25). In Figure 1, increasing the phase velocity a leads to a gradual increase in the space between the two-waves of TMmKdV equation. In Figure 2, decreasing of the nonlinearity parameter λ leads to interaction of the two-waves of the TMmKdV equation.

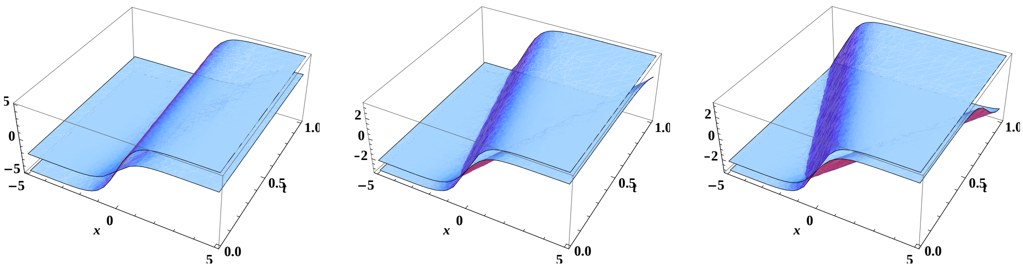

Figure 1: Behaviors of two-waves in (25) at the increasing phase velocity: $a = 1,\ 3,\ 5$ respectively. The assigned values for the other parameters are $\delta = \gamma = 1$, $\lambda = \frac{1}{2}$, $\mu = \frac{1}{4}$, $\alpha = -1$, $\beta = 1$.

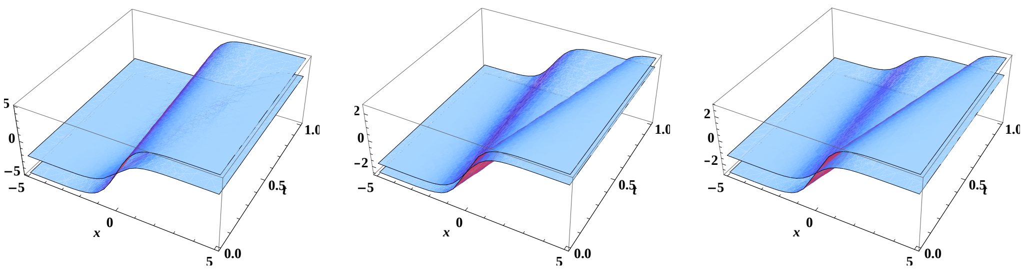

Figure 2: Behaviors of two-waves in (25) at the decreasing nonlinearity parameter: $\lambda = -\frac{1}{2},\ 0,\ \frac{1}{2}$ respectively. The assigned values for the other parameters are $\delta = \gamma = 1$, $s = 1$, $\mu = \frac{1}{4}$, $\alpha = -1$, $\beta = 1$.

6 Conclusion

In this paper we studied the solutions of the scaled TMmKdV equation which reads

$$W_{TT} - a^2 W_{XX} + (\beta\frac{\partial}{\partial T} - \beta\lambda a\frac{\partial}{\partial X})W^2 W_x + (\alpha\frac{\partial}{\partial T} - \alpha\mu a\frac{\partial}{\partial X})W_{XXX}.$$

We used three different methods, the simplified bilinear method, the tanh-technique and the sine-cosine function method. The following findings are observed in this work.

• When $\lambda = \mu = \pm 1$, TMmKdV equation admits multiple-soliton solutions by means of the simplified bilinear method.

• For arbitrary λ and μ, periodic solutions are obtained for TMmKdV equation by using the sine-cosine method.

• For arbitrary λ and μ, kink solutions are obtained for TMmKdV equation by using the tanh-expansion method.

Acknowledgment

Authors would like to thank the editor and the anonymous referees for their in-depth reading and insightful comments on an earlier version of this paper.

References

[1] Korsunsky, S.V. Soliton solutions for a second-order KdV equation. *Phys. Lett. A.* **185** (1994) 174–176.

[2] Lee, C.T. and Liu, J.L. A Hamiltonian model and soliton phenomenon for a two-mode KdV equation. *Rocky Mountain J. Math.* **41** (4) (2011) 1273–1289.

[3] Lee, C.C., Lee, C.T., Liu, J.L. and Huang, W.Y. Quasi-solitons of the two-mode Korteweg-de Vries equation. *Eur. Phys. J. Appl. Phys.* **52** (2010) 11301.

[4] Wazwaz, A.M. Multiple soliton solutions and other exact solutions for a two-mode KdV equation. *Math. Methods Appl. Sci.* **40** (6) (2017) 1277–1283.

[5] Wazwaz, A.M. A Two-Mode Burgers Equation of Weak Shock Waves in a Fluid: Multiple Kink Solutions and Other Exact Solutions. *Int. J. Appl. Comput. Math.* (2016), DOI 10.1007/s40819-016-0302-4.

[6] Wazwaz, A.M. Two-mode Sharma-Tasso-Olver equation and two-mode fourth-order Burgers equation: Multiple kink solutions. *Alexandria Eng. J.* (2017), https://doi.org/10.1016/j.aej.2017.04.003.

[7] Alquran, M. and Jarrah, A. Jacobi elliptic function solutions for a two-mode KdV equation. *Journal of King Saud University-Science* (2017), https://doi.org/10.1016/j.jksus.2017.06.010.

[8] Syam, M., Jaradat, H.M. and Alquran, M. A study on the two-mode coupled modified Korteweg-de Vries using the simplified bilinear and the trigonometric-function methods. *Nonlinear Dynamics* **90** (2) (2017) 1363–1371.

[9] Jaradat, H.M., Syam, M. and Alquran, M. A two-mode coupled Korteweg-de Vries: multiple-soliton solutions and other exact solutions. *Nonlinear Dynamics* **90** (1) (2017) 371–377.

[10] Alquran, M. Jaradat, H.M. and Syam, M. A modified approach for a reliable study of new nonlinear equation: two-mode Korteweg-de Vries-Burgers equation. *Nonlinear Dynamics* **91** (3) (2018) 1619–1626.

[11] Jaradat, I., Alquran, M. and Ali, M. A numerical study on weak-dissipative two-mode perturbed Burgers and Ostrovsky models: right-left moving waves. *Eur. Phys. J. Plus.* **133**: 164 (2018).

[12] Jaradat, I., Alquran, M., Momani, S. and Biswas, A. Dark and singular optical solutions with dual-mode nonlinear Schrodinger's equation and Kerr-law nonlinearity. *Optik* **172** (2018) 822–825.

[13] Alquran, M. and Yassin, O. Dynamism of two-modes parameters on the field function for third-order dispersive Fisher: application for fibre optics. *Optical and Quantum Electronics* **50** (9) (2018): 354.

[14] Yassin, O. and Alquran, M. Constructing new Solutions for some types of two-mode nonlinear equations. *Appl. Math. Inf. Sci.* **12** (2) (2018) 361–367.

[15] Wazwaz, A.M. Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities. *Nonlinear Dynamics* **83** (2016) 591–596.

[16] Wazwaz, A.M. Multiple kink solutions for two coupled integrable $(2+1)$-dimensional systems. *Applied Mathematics Letters* **58** (2016) 1-6.

[17] Hirota, R. Exact solutions of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27** (1971) 1192–1194.

[18] Hirota R., Exact N-soliton solutions of a nonlinear wave equation. *J. Math. Phys.* **14** (1973) 805–809.

[19] Jaradat, H.M, Awawdeh, F., Al-Shara', S., Alquran, M. and Momani, S. Controllable dynamical behaviors and the analysis of fractal burgers hierarchy with the full effects of inhomogeneities of media. *Romanian Journal of Physics* **60** (3-4) (2015) 324–343.

[20] Alquran, M., Jaradat, H.M., Al-Shara', S. and Awawdeh, F. A New Simplified Bilinear Method for the N-Soliton Solutions for a Generalized FmKdV Equation with Time-Dependent Variable Coefficients. *International Journal of Nonlinear Sciences and Numerical Simulation* **16** (6) (2015) 259–269.

[21] Wazwaz, A.M. A variety of distinct kinds of multiple soliton solutions for a (3+1)-dimensional nonlinear evolution equation. *Math. Meth. Appl. Sci.* **36** (3) (2013) 349–357.

[22] Maliet, W. and Heremanm, W. The tanh method: I. Exact solutions of nonlinear evolution and wave equations. *Physica Scripta* **54** (1996) 563–568.

[23] Alquran, M. and Al-Khaled, K. Mathematical methods for a reliable treatment of the (2+1)-dimensional Zoomeron equation. *Mathematical Sciences* **6** (12) (2012).

[24] Alquran, M. and Al-Khaled, K. The tanh and sine-cosine methods for higher order equations of Korteweg-de Vries type. *Physica Scripta* **84**(2011) 025010 (4pp).

[25] Alquran, M. and Al-Khaled, K. Sinc and solitary wave solutions to the generalized Benjamin-Bona-Mahony-Burgers equations. *Physica Scripta* **83**(2011) 065010 (6pp).

[26] Wazwaz, A.M. Peakons and soliton solutions of newly developed Benjamin-Bona-Mahony-like equations. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 209–220.

[27] Alquran, M., Jarrah, A. and Krishnan, E.V. Solitary wave solutions of the phi-four equation and the breaking soliton system by means of Jacobi elliptic sine-cosine expansion method. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 233–240.

[28] Alquran, M. Solitons and periodic solutions to nonlinear partial differential equations by the sine-cosine method. *Appl. Math. Inf. Sci.* **6** (1) (2012) 85–88.

[29] Ali, A.H.A., Soliman, A.A. and Raslan, K.R. Soliton solution for nonlinear partial differential equations by cosine-function method. *Phys. Lett. A.* **368** (2007) 299–304.

[30] Alquran, M. and Qawasmeh, A. Classifications of solutions to some generalized nonlinear evolution equations and systems by the sine-cosine method. *Nonlinear Studies* **20** (2) (2013) 261–270.

[31] Alquran, M., Al-Omary, R. and Katatbeh, Q. New explicit solutions for homogeneous KdV equations of third order by trigonometric and hyperbolic function methods. *Applications and Applied Mathematics AAM* **7** (1) (2012) 211–225.

Nonlinear Dynamics and Systems Theory, **19** (1) (2019) 97–110

Existence of Renormalized Solutions for Some Strongly Parabolic Problems in Musielak-Orlicz-Sobolev Spaces

A. Talha *, A. Benkirane and M.S.B. Elemine Vall

Laboratory LAMA, Faculty of Sciences Dhar El Mahraz
University Sidi Mohamed Ben Abdellah, P.O. Box 1796, Atlas Fez, Morocco

Received: December 19, 2016; Revised: December 13, 2018

Abstract: In this work, we prove an existence result of renormalized solutions in Musielak-Orlicz-Sobolev spaces for a class of nonlinear parabolic equations with two lower order terms and L^1-data.

Keywords: *parabolic problems, Musielak-Orlicz space, renormalized solutions.*

Mathematics Subject Classification (2010): 46E35, 35K15, 35K20, 35K60.

1 Introduction

We consider the following nonlinear parabolic problem:

$$(\mathcal{P})\begin{cases}\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(x,t,u,\nabla u) = f \quad \text{in } Q,\\ u = 0 \quad \text{on } \partial Q = \partial\Omega \times [0,T],\\ u(x,0) = u_0 \quad \text{on} \quad \Omega,\end{cases}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is an operator of Leray-Lions type, the lower order term $\Phi \in C^0(\mathbb{R},\mathbb{R}^N)$, g is a nonlinearity term which satisfies the growth and the sign condition and the data f belong to $L^1(Q)$. Under these assumptions the term $\operatorname{div}(\Phi(u))$ may not exist in the distributions sense, since the function $\Phi(u)$ does not belong to $(L^1_{\text{loc}}(Q))^N$.

In the setting of classical Sobolev spaces, the existence of a weak solution for the problem $(\mathcal{P})$ has been proved in [10] in the case of $\Phi \equiv g \equiv 0$. It is well known that this weak solution is not unique in general (see [16] for a counter-example in the stationary case).

* Corresponding author: mailto:talha.abdous@gmail.com

In order to obtain well-posedness for this type of problems the notion of renormalized solution has been introduced by Lions and DiPerna [12] for the study of Boltzmann equation (see also Lions [13] for a few applications to fluid mechanics models). This notion was then adapted to the elliptic version by Boccardo et al. [11]. At the same time, the equivalent notion of entropy solutions has been developed independently by Bénilan et al. [5] for the study of nonlinear elliptic problems.

The existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [8] in the case where $a(x,t,s,\xi)$ is independent of s, with $\Phi \equiv 0$ and $g \equiv 0$, by D. Blanchard, F. Murat and H. Redwane [9] with the large monotonicity on a. For measure data, $u = b(x,u)$ and $\Phi \equiv 0$, the existence of renormalized solution for the problem $(\mathcal{P})$ has been proved by Y. Akdim et al.[3] in the framework of weighted Sobolev space, by L. Aharouch, J. Bennouna and A. Touzani [1], and by A. Benkirane and J. Bennouna [6] in the Orlicz spaces and degenerated spaces.

In the Musielak framework, the existence of a weak solution for the problem $(\mathcal{P})$ has been proved by M.L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [2] where $\Phi \equiv 0$, the existence of entropy solutions for the problem $(\mathcal{P})$ has been studied by A. Talha, A. Benkirane and M.S.B. Elemine Vall in [19].

As an example of equations to which the present result can be applied, we give

$$\frac{\partial u}{\partial t} - \operatorname{div}\Big(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u + u|u|^{\sigma}\Big) + \frac{\operatorname{sign}(u)}{1+u^2}\,\varphi(x,|\nabla u|) = f \in L^1(Q),$$

where m is the derivative of φ with respect to t.

2 Preliminaries

2.1 Musielak-Orlicz-Sobolev spaces.

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfying the following conditions:
a) $\varphi(x,\cdot)$ is an N-function,
b) $\varphi(\cdot,t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function. For a Musielak-orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is $\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$. The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$ and a non negative function h integrable in Ω, we have

$$\varphi(x,2t) \le k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \ge 0. \tag{1}$$

When (1) holds only for $t \ge t_0 > 0$; then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions. We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, if for every positive constant c we have

$$\lim_{t \longrightarrow 0}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big) = 0, \quad (\text{resp. } \lim_{t\longrightarrow\infty}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big) = 0).$$

We define the functional $\rho_{\varphi,\Omega}(u) = \displaystyle\int_\Omega \varphi(x,|u(x)|)dx$, where $u : \Omega \longrightarrow \mathbb{R}$ is a Lebesgue measurable function.

We define the Musielak-Orlicz space (the generalized Orlicz spaces) by

$$L_\varphi(\Omega) = \Big\{u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable} \Big/ \rho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big) < +\infty, \ \text{for some } \lambda > 0\Big\}.$$

For a Musielak-Orlicz function we put: $\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$. ψ is called the Musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [14]. The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that $\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0$.

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in L_\varphi(\Omega)\Big\}$$

and

$$W^m E_\varphi(\Omega) = \Big\{u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in E_\varphi(\Omega)\Big\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \textstyle\sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

For $u \in W^m L\varphi(\Omega)$ these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [14] :

$$\text{there exists a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c. \tag{2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified with a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed. We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$) the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω. Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W^m_0 E_\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \Big\{f \in \mathcal{D}'(\Omega); \ f = \textstyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\Big\}$$

and

$$W^{-m}E_\psi(\Omega)=\Big\{f\in\mathcal{D}'(\Omega);\ f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha \text{ with } f_\alpha\in E_\psi(\Omega)\Big\}.$$

We say that a sequence of functions $u_n\in W^mL_\varphi(\Omega)$ is modular convergent to $u\in W^mL_\varphi(\Omega)$ if there exists a constant $k>0$ such that $\lim_{n\to\infty}\overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n-u}{k}\Big)=0$.

The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x}L_\varphi(Q)=\Big\{u\in L_\varphi(Q):\forall|\alpha|\le 1\ D_x^\alpha u\in L_\varphi(Q)\Big\}$$

and

$$W^{1,x}E_\varphi(Q)=\Big\{u\in E_\varphi(Q):\forall|\alpha|\le 1\ D_x^\alpha u\in E_\varphi(Q)\Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\|=\sum_{|\alpha|\le m}\|D_x^\alpha u\|_{\varphi,Q}$. We have the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F\\ W_0^{1,x}E_\varphi(Q) & F_0\end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F=W^{-1,x}L_\psi(Q)$ and it is shown that

$$W^{-1,x}L_\psi(Q)=\Big\{f=\sum_{|\alpha|\le 1}D_x^\alpha f_\alpha : f_\alpha\in L_\psi(Q)\Big\}.$$

This space will be equipped with the usual quotient norm $\|f\|=\inf\sum_{|\alpha|\le 1}\|f_\alpha\|_{\psi,Q}$, where the inf is taken on all possible decompositions $f=\sum_{|\alpha|\le 1}D_x^\alpha f_\alpha,\quad f_\alpha\in L_\psi(Q)$.

The space F_0 is then given by

$$F_0=\Big\{f=\sum_{|\alpha|\le 1}D_x^\alpha f_\alpha : f_\alpha\in E_\psi(Q)\Big\}=W^{-1,x}E_\psi(Q).$$

Let us give the following lemma which will be needed later.

Lemma 2.1 *[7]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*
i) *There exists a constant $c>0$ such that $\inf_{x\in\Omega}\varphi(x,1)\ge c$,*
ii) *There exists a constant $A>0$ such that for all $x,y\in\Omega$ with $|x-y|\le\frac12$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)}\le t^{\left(\frac{A}{log\left(\frac{1}{|x-y|}\right)}\right)},\quad \forall t\ge 1. \tag{3}$$

iii)

$$\textit{If } D\subset\Omega \textit{ is a bounded measurable set, then } \int_D\varphi(x,1)dx<\infty. \tag{4}$$

iv) *There exists a constant $C>0$ such that $\psi(x,1)\le C$ a.e in Ω.*
Under these assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $<S,u>$.

Lemma 2.2 (Poincaré inequality) *[18] Let φ be a Musielak-Orlicz function which satisfies the assumptions of Lemma 2.1, suppose that $\varphi(x,t)$ decreases with respect to one* *of coordinates of x. Then, there exists a constant $c > 0$ depending only on Ω such that*

$$\int_\Omega \varphi(x, |u(x)|)\, dx \leq \int_\Omega \varphi(x, c|\nabla u(x)|)\, dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{5}$$

3 Assumptions and Main Result

Let Ω be a bounded open set on $\mathbb{R}^N$ satisfying the segment property and $T > 0$, we denote $Q = \Omega \times [0, T]$, and let φ and γ be two Musielak-Orlicz functions such that $\gamma \prec\prec \varphi$ and φ satisfies the conditions of Lemma 2.2. Let $A : D(A) \subset W_0^{1,x} L_\varphi(Q) \longrightarrow W^{-1,x} L_\psi(Q)$ be a mapping given by $A(u) = -\text{div}(a(x, t, u, \nabla u))$, where $a : a(x, t, s, \xi) : \Omega \times [0, t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $(x, t) \in Q$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$,

$$|a(x, t, s, \xi)| \leq \beta\Big(c(x, t) + \psi_x^{-1} \varphi(x, \nu|\xi|)\Big), \tag{6}$$

$$\Big(a(x, t, s, \xi) - a(x, t, s, \xi')\Big)(\xi - \xi') > 0, \tag{7}$$

$$a(x, t, s, \xi).\xi \geq \alpha\varphi(x, |\xi|), \tag{8}$$

where $c(x, t)$ is a positive function, $c(x, t) \in E_\psi(Q)$ and $\beta, \nu, \alpha \in \mathbb{R}_+^*$.
Let $g : \Omega \times [0, t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x, t) \in \Omega \times [0, t]$ and $\forall s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|g(x, t, s, \xi)| \leq b(|s|)(c_2(x, t) + \varphi(x, |\xi|)), \tag{9}$$

$$g(x, t, s, \xi)s \geq 0, \tag{10}$$

where $c_2(x, t) \in L^1(Q)$ and $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and nondecreasing function. Furthermore, let

$$\Phi \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{11}$$

$$f \in L^1(Q) \text{ and } u_0 \text{ is an element of } L^1(Q). \tag{12}$$

For $\ell > 0$ we define the truncation at height ℓ: $T_\ell : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$T_\ell(s) = \begin{cases} s & \text{if } |s| \leq \ell, \\ \ell \frac{s}{|s|} & \text{if } |s| > \ell. \end{cases} \tag{13}$$

The definition of a renormalized solution for problem $(\mathcal{P})$ can be stated as follows.

Definition 3.1 A measurable function u defined on Q is a renormalized solution of Problem $(\mathcal{P})$ if

$$T_\ell(u) \in W_0^{1,x} L_\varphi(Q), \tag{14}$$

$$\int_{\{(x,t)\in Q; m\leq|u(x,t)|\leq m+1\}} a(x, t, u, \nabla u) \cdot \nabla u \, dxdt \longrightarrow 0 \text{ as } m \longrightarrow \infty, \tag{15}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have

$$\begin{aligned} &\frac{\partial S(u)}{\partial t} - \text{div}\,(a(x, t, u, \nabla u)S'(u)) + S''(u)a(x, t, u, \nabla u) \cdot \nabla u \\ &- \text{div}\,(\Phi(u)S'(u)) + S''(u)\Phi(u) \cdot \nabla u + g(x, t, u, \nabla u)S'(u) = fS'(u) \quad \text{in } \mathcal{D}'(Q), \end{aligned}$$

$$S(u)(t = 0) = S(u_0) \text{ in } \Omega. \tag{16}$$

We will prove the following existence theorem.

Theorem 3.1 *Assume that (6) to (11) hold true. Then, there exists a renormalized solution u of problem ($\mathcal{P}$) in the sense of Definition 3.1.*

Proof. The proof of Theorem 3.1 is divided into five steps.

Step 1: Approximate problem. Let consider us the following approximate problem

$$(\mathcal{P}_n)\begin{cases}\frac{\partial u_n}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\Big) + g_n(x,t,u_n,\nabla u_n) = f_n \quad \text{in } \mathcal{D}'(Q),\\ u_n = 0 \quad \text{on} \quad \partial\Omega\times(0,T),\\ u_n(t=0) = u_{0n} \quad \text{on } \Omega,\end{cases}$$

where $(f_n) \in L^1(Q)$ is a sequence of smooth functions such that $f_n f_n \to f$ in $L^1(Q) f$ in $L^1(Q)$, $\Phi_n(s) = \Phi(T_n(s))$ and $g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))$. Note that $g_n(x,t,s,\xi)s \geq 0$, $|g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$. Since Φ is continuous, we have $\Phi(T_n(s)) \leq c_n$, then the problem $(\mathcal{P}_n)$ has at least one solution $u_n \in W_0^{1,x}L_\varphi(Q)$ (see e.g. [2]).

Step 2: A priori estimates. We take $T_\ell(u_n)\chi_{(0,\tau)}$ as a test function in $(\mathcal{P}_n)$, we get for every $\tau \in (0,T)$

$$\int_\Omega \widehat{T_\ell}(u_n(\tau))\,dx + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\,dxdt + \int_{Q_\tau}\Phi_n(u_n)\cdot\nabla T_\ell(u_n)dxdt$$

$$= \int_{Q_\tau} f_n T_\ell(u_n)\,dxdt - \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_\ell(u_n)\,dxdt + \int_\Omega \widehat{T_\ell}(u_{0n})\,dx, \tag{17}$$

where

$$\widehat{T_\ell}(s) = \int_0^s T_\ell(\sigma)d\sigma = \begin{cases} \frac{s^2}{2}, & \text{if } |s| \leq \ell,\\ \ell|s| - \frac{s^2}{2}, & \text{if } |s| > \ell.\end{cases} \tag{18}$$

The Lipshitz character of Φ_n and the Stokes formula together with the boundary condition $u_n = 0$ on $(0,T)\times\partial\Omega$ make it possible to obtain

$$\int_{Q_\tau} \Phi_n(u_n)\cdot\nabla T_\ell(u_n)\,dxdt = 0. \tag{19}$$

Due to the definition of $\widehat{T_\ell}$ and (12) we have

$$0 \leq \int_\Omega \widehat{T_\ell}(u_{0n})\,dx \leq \ell\int_\Omega |u_{0n}|\,dx \leq \ell||u_0||_{L^1(\Omega)}. \tag{20}$$

Using the same argument as in [15], we can see that

$$\int_Q g_n(x,t,u_n,\nabla u_n)\,dxdt \leq C_g. \tag{21}$$

Here and below C_i denotes positive constants not depending on n and ℓ. By using (12), (19), (20), (21) we can deduce from (17) that

$$\int_\Omega \widehat{T_\ell}(u_n(\tau))\,dx + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\,dxdt \leq \ell C_0. \tag{22}$$

By using (22), (7) and the fact that $\widehat{T}_\ell(u_n) \geq 0$, we deduce that

$$\int_{Q_\tau} \varphi(x, |\nabla T_\ell(u_n)|)\, dxdt \leq \frac{1}{\alpha} \int_{Q_\tau} a(x,t,T_\ell(u_n), \nabla T_\ell(u_n)) \cdot \nabla T_\ell(u_n)\, dxdt \leq \ell C_1, \quad (23)$$

we deduce from the above inequality (22) that

$$\int_\Omega \widehat{T}_\ell(u_n(\tau))\, dx \leq \ell C_0, \text{ for almost any } \tau \text{ in } (0,T). \quad (24)$$

On the other hand, thanks to Lemma 2.2, there exists a constant $\lambda > 0$ depending only on Ω such that

$$\int_{Q_\tau} \varphi(x, |v|)\, dxdt \leq \int_{Q_\tau} \varphi(x, \lambda|\nabla v|)\, dxdt, \quad \forall v \in W_0^1 L_\varphi(\Omega). \quad (25)$$

Taking $v = \frac{T_\ell(u_n)}{\lambda}$ in (25) and using (23), one has

$$\int_{Q_\tau} \varphi(x, \frac{|T_\ell(u_n)|}{\lambda})\, dxdt \leq \ell C_1. \quad (26)$$

Then we deduce by using (26), that

$$\begin{aligned} meas\{|u_n| > \ell\} &\leq \frac{1}{\inf\limits_{x\in\Omega} \varphi(x, \frac{\ell}{\lambda})} \int_{Q_\tau} \varphi(x, \frac{1}{\lambda}|T_\ell(u_n)|)\, dxdt \\ &\leq \frac{C_1 \ell}{\inf\limits_{x\in\Omega} \varphi(x, \frac{\ell}{\lambda})} \quad \forall n, \quad \forall \ell \geq 0. \end{aligned} \quad (27)$$

By using the definition of φ, we can deduce

$$\lim_{\ell \longrightarrow \infty} (\text{meas}\{(x,t) \in Q_\tau : |u_n| > \ell\}) = 0 \quad (28)$$

uniformly with respect to n. Moreover, we have from (26) that $T_\ell(u_n)$ is bounded in $W_0^{1,x} L_\varphi(Q)$ for every $\ell > 0$. Consider now in $C^2(\mathbb{R})$ a nondecreasing function $\zeta_\ell(s) = s$ for $|s| \leq \frac{\ell}{2}$ and $\zeta_\ell(s) = \ell$ sign (s). Multiplying the approximating equation by $\zeta'_\ell(u_n)$, we obtain

$$\begin{aligned} &\frac{\partial(\zeta_\ell(u_n))}{\partial t} = \text{div}\, \big(a(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n)\big) - \zeta''_\ell(u_n) a(x,t,u_n,\nabla u_n) \cdot \nabla u_n \\ &+ \text{div}\, \big(\Phi_n(u_n)\zeta'_\ell(u_n)\big) - \zeta''_\ell(u_n)\Phi_n(u_n) \cdot \nabla u_n - g_n(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n) + f_n \zeta'_\ell(u_n) \end{aligned}$$

in the sense of distributions. Thanks to (26) and the fact that ζ'_ℓ has a compact support, $\zeta'_\ell(u_n)$ is bounded in $W_0^{1,x} L_\varphi(Q)$ while its time derivative $\frac{\partial(\zeta_\ell(u_n))}{\partial t}$ is bounded in $W_0^{-1,x} L_\varphi(Q) + L^1(Q)$, hence Corollary 4.5 of [2] allows us to conclude that $\zeta_\ell(u_n)$ is compact in $L^1(Q)$. Due to the choice of ζ_ℓ, we conclude that for each ℓ, the sequence $T_\ell(u_n)$ converges almost everywhere in Q. Therefore, following [8,9,15], we can see that there exists a measurable function $u \in L^\infty(0,T; L^1(\Omega))$ such that for every $\ell > 0$ and a subsequence, not relabeled,

$$u_n \to u \text{ a. e. in } Q, \quad (29)$$

and

$$T_\ell(u_n) \rightharpoonup T_\ell(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \tag{30}$$
$$\text{strongly in } L^1(Q) \text{ and a. e. in } Q.$$

Now we shall to prove the boundness of $(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)))_n$ in $(L_\psi(Q))^N$.
Let $\phi \in (E_\varphi(Q))^N$ with $||\phi||_{\varphi,Q} = 1$. In view of the monotonicity of a one easily has,

$$\int_Q \big[a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\phi)\big]\big[\nabla T_\ell(u_n) - \phi\big]\, dxdt \geq 0, \tag{31}$$

which gives

$$\begin{aligned}\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\phi\, dxdt \leq &\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\nabla T_\ell(u_n)\, dxdt\\ &+\int_Q a(x,t,T_\ell(u_n),\phi)\cdot\Big[\nabla T_\ell(u_n)-\phi\Big]\, dxdt.\end{aligned} \tag{32}$$

Using (6) and (23), we easily see that

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\cdot\phi\, dxdt \leq C_3. \tag{33}$$

And so, we conclude that $(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)))_n$ is a bounded sequence in $(L_\psi(Q))^N$.
Now, we prove that

$$\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}\int_{\left\{m\leq|u_n|\leq m+1\right\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt = 0. \tag{34}$$

Using in $(\mathcal{P}_n)$ the test function $v = T_1(u_n - T_m(u_n))$, we obtain

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t}, v\rangle + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt + \int_Q g_n(x,t,u_n,\nabla u_n)v\, dxdt\\ &+\int_Q \mathrm{div}\Big[\int_0^{u_n}\Phi_n(r)T_1'(u_n - T_m(u_n))dr\Big]\, dxdt = \int_Q f_n v\, dxdt.\end{aligned} \tag{35}$$

By using $\int_0^{u_n}\Phi_n(r)T_1'(u_n-T_m(u_n))dr \in W_0^{1,x}L_\varphi(Q)$ and the Stokes formula, we get

$$\begin{aligned}&\int_\Omega U_n^m(u_n(T))\, dx + \int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt\\ &\leq \int_Q (|f_n + g_n(x,t,u_n,\nabla u_n)|)|T_1(u_n - T_m(u_n))|\, dxdt + \int_\Omega U_n^m(x,u_{0n})\, dx,\end{aligned} \tag{36}$$

where $U_n^m(r) = \int_0^{u_n}\frac{\partial u_n}{\partial t}T_1(s - T_m(s))ds$. In order to pass to the limit as n tends to $+\infty$ in (36), we use $U_n^m(u_n(T)) \geq 0$, (12) and (21), we obtain that

$$\begin{aligned}\lim_{n\longrightarrow\infty}&\int_{\{m\leq u_n\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\, dxdt\\ &\leq \int_{\{|u_n|>m\}}(|f| + C_g)\, dxdt + \int_{\{|u_0|>m\}}|u_0|\, dx.\end{aligned} \tag{37}$$

Finally, by(12) and (37) we obtain (34).

Step 3: Almost everywhere convergence of the gradients. Fix $\ell > 0$ and let $\phi(s) = s\exp(\delta s^2), \delta > 0$. It is well known that when $\delta \geq (\frac{b(\ell)}{2\alpha})^2$ one has

$$\phi'(s) - \frac{b(\ell)}{\alpha}|\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R}. \tag{38}$$

Let $v_j \in \mathcal{D}(Q)$ be a sequence which converges to u for the modular convergence in $W_0^{1,x}L_\varphi(Q)$ and let $\omega_i \in \mathcal{D}(Q)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$. Set $\omega_{i,j}^\mu = T_\ell(v_j)_\mu + \exp(-\mu t)T_\ell(w_i)$, where $T_\ell(v_j)_\mu$ is the mollification with respect to time of $T_\ell(v_j)$. Note that $\omega_{i,j}^\mu$ is a smooth function having the following properties:

$$\frac{\partial}{\partial t}(\omega_{i,j}^\mu) = \mu(T_\ell(v_j) - \omega_{i,j}^\mu), \omega_{i,j}^\mu(0) = T_\ell(\omega_i), |\omega_{i,j}^\mu| \leq \ell, \tag{39}$$

$$\omega_{i,j}^\mu \to T_\ell(u)_\mu + \exp(-\mu t)T_\ell(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{40}$$

for the modular convergence as $j \to \infty$,

$$T_\ell(u)_\mu + \exp(-\mu t)T_\ell(w_i) \to T_\ell(u) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{41}$$

for the modular convergence as $\mu \to \infty$. Let now the function ρ_m on $\mathbb{R}$ with $m \geq \ell$ be defined by

$$\rho_m(s) = \begin{cases} 1, & \text{if } |s| \leq m, \\ m+1-|s|, & \text{if } m \leq |s| \leq m+1, \\ 0, & \text{if } |s| \geq m+1. \end{cases} \tag{42}$$

We set $\theta_{i,j}^{\mu,n} = T_\ell(u_n) - \omega_{i,j}^\mu$. Using the admissible test function $Z_{i,j,n}^{\mu,m} = \phi(\theta_{i,j}^{\mu,n})\rho_m(u_n)$ as test function in $(\mathcal{P}_n)$ and since $g_n(x,t,u_n,\nabla u_n)\phi(\theta_{i,j}^{\mu,n})\rho_m(u_n) \geq 0$ on $\{|u_n| > \ell\}$, we arrive at

$$\begin{aligned} &\langle \frac{\partial u_n}{\partial t}, Z_{i,j,n}^{\mu,m}\rangle + \int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_\ell(u_n) - \nabla\omega_{i,j}^\mu)\phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n)\,dxdt \\ &+ \int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n)\,dxdt \\ &+ \int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n)\,dxdt \\ &+ \int_Q \Phi_n(u_n)\cdot(\nabla T_\ell(u_n) - \nabla\omega_{i,j}^\mu)\phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n)\,dxdt \\ &+ \int_{\{|u_n|\leq\ell\}} g_n(x,t,u_n,\nabla u_n)\phi(\theta_{i,j}^{\mu,n})\rho_m(u_n)\,dxdt \leq \int_Q f_n Z_{i,j,n}^{\mu,m}\,dxdt. \end{aligned} \tag{43}$$

Denote by $\epsilon(n,j,\mu,i)$ any quantity such that $\lim\limits_{i\to\infty}\lim\limits_{\mu\to\infty}\lim\limits_{j\to\infty}\lim\limits_{n\to\infty}\epsilon(n,j,\mu,i) = 0$.

The very definition of the sequence $\omega_{i,j}^\mu$ makes it possible to establish the following lemma.

Lemma 3.1 *(cf.[2]) Let $Z_{i,j,n}^{\mu,m} = \phi(\theta_{i,j}^{\mu,n})\rho_m(u_n)$, we have for any $\ell \geq 0$*

$$\langle\frac{\partial u_n}{\partial t}, Z_{i,j,n}^{\mu,m}\rangle \geq \epsilon(n,j,i). \tag{44}$$

Concerning the right-hand of (43), by the almost everywhere convergence of u_n, we have $\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n) \rightharpoonup \phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u)$ weakly-* in $L^\infty(Q)$ as $n\to\infty$, and then

$$\int_Q f_n\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\,dxdt \to \int_Q f\phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(n)\,dxdt,$$

so that $\phi(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u) \rightharpoonup \phi(T_\ell(u)-T_\ell(u)_\mu-\exp(-\mu t)T_\ell(w_i))\rho_m(u)$ weakly star in $L^\infty(Q)$ as $j\to\infty$, and finally,

$$\phi(T_\ell(u)-T_\ell(u)_\mu-\exp(-\mu t)T_\ell(w_i))\rho_m(u) \rightharpoonup 0 \text{ weakly star as } \mu\to\infty.$$

Then, we deduce that

$$\langle f_n, \phi(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\rangle = \epsilon(n,j,\mu). \tag{45}$$

Similarly, Lebesgue's convergence theorem shows that

$$\Phi_n(u_n)\rho_m(u_n) \to \Phi(u)\rho_m(u) \text{ strongly in } (E_\psi(Q)^N) \text{ as } n\to\infty,$$

and

$$\Phi_n(u_n)\chi_{\{m\le|u_n|\le m+1\}}\phi'(T_\ell(u_n)-\omega_{i,j}^\mu) \to \Phi(u)\chi_{\{m\le u\le m+1\}}\phi'(T_\ell(u)-\omega_{i,j}^\mu)$$

strongly in $(E_\psi(Q)^N)$. Then by virtue of $\nabla T_\ell(u_n) \rightharpoonup \nabla T_\ell(u)$ weakly star in $(L_\varphi(Q)^N)$, and $\nabla u_n\chi_{\{m\le|u_n|\le m+1\}} = \nabla T_{m+1}(u_n)\chi_{\{m\le|u_n|\le m+1\}}$ a. e. in Q, one has

$$\begin{aligned}&\int_Q \Phi_n(u_n)\cdot(\nabla T_\ell(u_n)-\nabla\omega_{i,j}^\mu)\phi'(T_\ell(u_n)-\omega_{i,j}^\mu)\rho_m(u_n)\,dxdt\\ &\to \int_Q \Phi(u)\nabla(\nabla T_\ell(u)-\nabla\omega_{i,j}^\mu)\phi'(T_\ell(u)-\omega_{i,j}^\mu)\rho_m(u)\,dxdt\end{aligned}$$

as $n\to\infty$, and

$$\begin{aligned}&\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\nabla u_n\rho_m'(u_n)\,dxdt\\ &\to \int_{\{m\le|u_n|\le m+1\}} \Phi(u)\phi(T_\ell(u_n)-\omega_{i,j}^\mu)\nabla u\rho_m'(u)\,dxdt\end{aligned}$$

as $n\to+\infty$. Thus, by using the modular convergence of $\omega_{i,j}^\mu$ as $j\to+\infty$ and letting μ tend to infinity, we get

$$\int_Q \Phi_n(u_n)\cdot(\nabla T_\ell(u_n)-\nabla\omega_{i,j}^\mu)\phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n)\,dxdt = \epsilon(n,j,\mu) \tag{46}$$

and

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n)\,dxdt = \epsilon(n,j,\mu). \tag{47}$$

Concerning the third term of the right-hand side of (43) we obtain that

$$\begin{aligned}&\left|\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta_{n,j}^{\mu,i})\rho_m'(u_n)\,dxdt\right|\\ &\le \phi(2k)\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dxdt.\end{aligned}$$

Then by (34) we deduce that

$$|\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta^{\mu,i}_{n,j})\rho'_m(u_n)\ dxdt|\leq\epsilon(n,\mu,m). \tag{48}$$

Using the same technics as in the proof of Proposition 5.6 in [4], we obtain

$$\begin{aligned}&\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi^s\Big)\ dxdt\leq\varepsilon(n,j,\mu,i,s,m).\end{aligned} \tag{49}$$

To pass to the limit in (49) as n j, m, s tend to infinity, we obtain

$$\begin{aligned}&\lim_{s\to\infty}\lim_{n\to\infty}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n)-\nabla T_k(u)\chi^s\Big)\ dxdt=0.\end{aligned} \tag{50}$$

And thus, as in the elliptic case (see [18]), there exists a subsequence also denoted by u_n such that

$$\nabla u_n\to\nabla u \text{ a.e. in } Q. \tag{51}$$

Then, for all $k>0$, one has

$$\begin{aligned}&a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup a(x,t,T_k(u),\nabla T_k(u))\\&\qquad\text{weakly star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi).\end{aligned} \tag{52}$$

Step 4: In this step we prove that u ***satisfies (15).*** According to (50), one can pass to the limit as n tends to $+\infty$ for fixed $m\geq 0$ to obtain

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\int_{\{m\leq|u_n|\leq m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\ dxdt\\&=\int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)\ dxdt\\&\quad-\int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)\ dxdt\\&\quad=\int_{\{m\leq|u|\leq m+1\}}a(x,t,u,\nabla u)\cdot\nabla u\ dxdt.\end{aligned} \tag{53}$$

Taking the limit as $m\to+\infty$ in (53) and using the estimate (34) show that u satisfies (15). Following the same technique as that used in [2], and by using (29), (50) and Vitali's theorem, we have

$$g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u) \text{ strongly in } L^1(Q). \tag{54}$$

Step 5 : Passing to the limit. Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\mathrm{supp}(S') \subset [-K, K]$. Pointwise multiplication of the approximate equation $(\mathcal{P}_n)$ by $S'(u_n)$ leads to

$$\begin{aligned}\frac{\partial S(u_n)}{\partial t} &- \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)S'(u_n)\Big) + S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \\ &- \operatorname{div}\Big(S'(u_n)\Phi(u_n)\Big) + S''(u_n)\Phi(u_n)\cdot\nabla u_n \\ &+ g_n(x,t,u_n,\nabla u_n)S'(u_n) \\ &= f_nS'(u_n). \end{aligned} \tag{55}$$

In what follows we pass to the limit as n tends to $+\infty$ in each term of (55).
• Since S is bounded and continuous, then the fact that $u_n \longrightarrow u$ a.e. in Q, implies that $S(u_n)$ converges to $S(u)$ a.e. in Q and L^∞ weakly-$*$. Consequently,

$$\frac{\partial S(u_n)}{\partial t} \longrightarrow \frac{\partial S(u)}{\partial t} \quad \text{in } \mathcal{D}'(Q) \text{ as } n \text{ tends to } +\infty.$$

• Since $\mathrm{supp}(S') \subset [-K, K]$, we have for $n \geq K$,

$$a(x,t,u_n,\nabla u_n)S'(u_n) = a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \quad \text{a.e. in } Q.$$

The pointwise convergence of u_n to u and (52) as n tends to ∞ and the bounded character of S' permit us to conclude that

$$a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \longrightarrow a(x,t,T_K(u),\nabla T_K(u))S'(u) \text{ weakly star in } (L_\psi(Q))^N \tag{56}$$

as n tends to infinity.
• Regarding the 'energy' term, we have for $n \geq K$

$$S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n = S''(u_n)a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot\nabla T_K(u_n) \text{ a.e. in } Q.$$

The pointwise convergence of $S'(u_n) \longrightarrow S'(u)$ and (52) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \rightharpoonup S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) \text{ weakly star in } L^1(Q). \tag{57}$$

Recall that $S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) = S''(u)a(x,t,u,\nabla u)\cdot\nabla u$ a.e. in Q.
• Since $\mathrm{supp}(S') \subset [-K, K]$, we have

$$S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n)) \quad \text{a.e. in } Q. \tag{58}$$

As a consequence of (11) and (29), it follows that

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_K(u)) \quad \text{a.e. in } (E_\varphi(Q))^N, \tag{59}$$

we have $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $(L_\varphi(Q))^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a. e. in Q to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup S''(u)\Phi(u)\nabla u \quad \text{weakly in } L_\varphi(Q), \tag{60}$$

• Since $suppS' \subset [-K, K]$ and from (54), we have

$$S'(u_n)g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u)S'(u) \quad \text{strongly in } L^1(Q). \tag{61}$$

• Due to $f_n \longrightarrow f$ in $L^1(Q)$ and the fact that $u_n \longrightarrow u$ a.e. in Q, we have

$$S'(u_n)f_n \longrightarrow S'(u)f \quad \text{strongly in } L^1(Q). \tag{62}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in equation (55) and to conclude that

$$\begin{aligned}\frac{\partial S(u)}{\partial t} \;&-\; \operatorname{div}\Big(a(x,t,u,\nabla u)S'(u)\Big) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u\\ &-\; \operatorname{div}\Big(S'(u)\Phi(u)\Big) + S''(u)\Phi(u)\cdot\nabla u\\ &+\; g(x,t,u,\nabla u)S'(u)\\ &=\; fS'(u).\end{aligned} \tag{63}$$

It remains to show that $S(u)$ satisfies the initial condition.

To this end, firstly note that, S being bounded, $S(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (55) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. As a consequence, an Aubin's type lemma (see, e.g, [17]) implies that $S(u_n)$ lies in a compact set of $C^0([0,T], L^1(\Omega))$. It follows that, on the one hand, $S(u_n)(t=0) = S(u_{0n})$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$.

On the other hand, the smoothness of S implies that

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega.$$

As a conclusion of step 1 to step 6, the proof of Theorem 3.1 is complete.

Example 3.1 Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and $T > 0$, we denote by $Q = \Omega \times [0,T]$, and let φ and ψ be two complementary Musielak functions. Moreover, we assume that $\varphi(x,t)$ decreases with respect to one of coordinates of x (for example, $\varphi(x,t) = |t|^{p(x)}\log(1+t^3)$, $p(x) = e^{(-x_1^2+x_2^2+\cdots+x_N^2)}$. We set

$$a(x,t,s,\zeta) = (3+\cos^2(\varphi(x,s)))\psi_x^{-1}(\varphi(x,|\zeta|))\frac{\zeta}{|\zeta|},$$

$$g(x,t,s,\zeta) = \tfrac{\varphi(x,|\zeta|)}{1+s^2},\; \Phi(s) = (|s|^{r_1-1}s, ..., |s|^{r_N-1}s), \quad 1 \le r_1, ..., R_N < \infty.$$

It is easy to show that $a(x,t,s,\zeta)$ is the Caratheodory function satisfying the growth condition (6), the coercivity (8) and the monotonicity condition, while the Caratheodory function $g(x,t,s,\zeta)$ satisfies the condition (9) and (10), Finally, the hypotheses of Theorem 3.1 are satisfied. Therefore, the following problem

$$\begin{cases}\lim\limits_{m\longrightarrow\infty}\displaystyle\int_{\{(x,t)\in Q; m\le|u(x,t)|\le m+1\}} a(x,t,u,\nabla u)\cdot\nabla u \quad dxdt = 0,\\ \frac{\partial S(u)}{\partial t} - \operatorname{div}\Big(a(x,t,u,\nabla u)S'(u)\Big) + S''(u)a(x,t,u,\nabla u)\cdot\nabla u\\ -\operatorname{div}\Big(S'(u)\Phi(u)\Big) + S''(u)\Phi(u)\cdot\nabla u + g(x,t,u,\nabla u)S'(u) = fS'(u),\\ S(u)(t=0) = S(u_0) \text{ in } \Omega,\\ \text{for every function } S \text{ in } W^{2,\infty}(\mathbb{R}) \text{ and such that } S' \text{ has a compact support in } \mathbb{R}\end{cases} \tag{64}$$

has at least one renormalised solution for any $f \in L^1(Q)$.

References

[1] Aharouch, L., Bennouna, J. and Touzani, A. Existence of renormalized solutions of some elliptic problems in Orlicz spaces. *Rev. Mat. Complut.* **22** (1) (2009) 91–110.

[2] Ahmed Oubeid, M. L. , Benkirane, A. and Sidi El Vally, M. Strongly nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces. *Bol. Soc. Paran. Mat.* **33** (1) (2015) 191–223.

[3] Akdim, Y., Bennouna, J., Mekkour, M. and Redwane, H., Parabolic Equations with Measure Data and Three Unbounded Nonlinearities in Weighted Sobolev Spaces. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 107–126.

[4] Azroul, E., Redwane, H. and Rhoudaf, M. Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces. *Port. Math.* **66** (1) (2009) 29–63.

[5] Bénilan, P., Boccardo, L., Gallouet, T., Gariepy, R., Pierre, M. and Vazquez, J.L. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Sc. Norm. Super. Pisa* **22** (1995) 241–273.

[6] Benkirane, A. and Bennouna, J. Existence of solution for nonlinear elliptic degenrate equations. *Nonlinear Analysis* **54** (2003) 9–37.

[7] Benkirane, A., and Sidi El Vally, M. (Ould Mohamedhen Val). Some approximation properties in Musielak-Orlicz-Sobolev spaces. *Thai. J. Math.* **10** (2012) 371–381.

[8] Blanchard, D., and Murat, F. Renormalized solutions of nonlinear parabolic with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect, A* **127** (1997) 1137–1152.

[9] Blanchard, D., Murat, F., and Redwane, H. Existence and uniqueness of renormalized solution for fairly general class of non linear parabolic problems. *J. Differ. Equ.* **177** (2001) 331–374.

[10] Boccardo, L. and Gallouet, T. On some nonlinear elliptic equations with right-hand side measures. *Commun. Partial Differ. Equ.* **17** (1992) 641–655.

[11] Boccardo, L., Giachetti, D., Diaz, J.I. and Murat, F. Existence and regularity of renormalized solutions for some elliptic problems involving derivation of nonlinear terms. *J. Differ. Equ.* **106** (1993) 215–237.

[12] DiPerna, R.J. and Lions, P.L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. Math.* **130** (1989) 321–366.

[13] Lions, P.L. *Mathematical Topics in Fluid Mechanics, Incompressible Models*. Oxford Univ. Press, 1996.

[14] Musielak, J. Modular spaces and Orlicz spaces. *Lecture Notes in Math.* (1983) 10–34.

[15] Porretta, A. Existence results for strongly nonlinear parabolic equations via strong convergence of truncations. *Ann. Mat. Pura Appl.* **IV** (177) (1999) 143–172.

[16] Serrin, J. Pathological solution of elliptic differential equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **18** (1964) 385-387.

[17] Simon, J. Compact sets in the space $L^p(0, T, B)$, *Ann. Mat. Pura. Appl.* **146** (1987) 65–96.

[18] Talha. A., Benkirane, A. Strongly nonlinear elliptic boundary value problems in Musielak-Orlicz spaces, *Monatsh. Math.* **184** (2017) 1–32.

[19] Talha, A., Benkirane, A. and Elemine Vall, M.S.B. Entropy solutions for nonlinear parabolic inequalities involving measure data in Musielak-Orlicz-Sobolev spaces. *Bol. Soc. Paran. Mat.* **36** (2) (2018) 199–230.