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Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

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SPECIAL ISSUE
Recent Trends in Theoretical Aspects and Computational Methods in Differential and Difference Equations

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Special Issue
Recent Trends in Theoretical Aspects and Computational Methods in Differential and Difference Equations

Preface

Firstly, in recognition of Professor I.P. Stavroulakis's significant contributions to nonlinear dynamics and systems theory, we include a Personage in Science to introduce his biographical sketch and scientific activities.

After that, the first paper, entitled "On stability of a second order integro-differential equation", obtains new stability condition for the second order integro-differential equation.

In the second paper, entitled "Application of extended Fan sub-equation method to generalized Zakharov equation", the extended Fan sub-equation method is applied to obtain exact analytical solutions of the generalized Zakharov equation.

The third paper, entitled "Lie group classification of a generalized coupled Lane-Emden-Klein-Gordon-Fock system with central symmetry", is concerned with the symmetry analysis of a generalized Lane-Emden-Klein-Fock system with central symmetry. Several cases for the non-equivalent forms of the arbitrary elements are obtained.

The fourth paper, entitled "Numerical solutions of fractional chemical kinetics system", studies the numerical solution of the fractional chemical kinetics model using the operational matrices of fractional integration and multiplication based on the Bernstein polynomials.

In the fifth paper, entitled "A recursive solution approach to linear systems with non-zero minors", a recursive algorithm is presented to solve linear system of differential equations which has advantage over other existing algorithms.

The sixth paper, entitled "Comparison of new iterative method and natural homotopy perturbation method for solving nonlinear time-fractional wave-like equations with variable coefficients", investigated a comparison between an iterative method which is presented by Dafterdar and Jafari and natural homotopy perturbation method (NHPM) for solving nonlinear time-fractional wave-like equations with variable coefficients.

In the seventh paper, entitled "Mathematical analysis of a differential equation modeling charged elements aggregating in a relativistic zero-magnetic field", the authors analyze, in spaces of distributions with finite higher moments, discrete mass and momentum dependent equations describing the movement of charged particles (electrons, ions) aggregating and moving in a relativistic zero-magnetic field. The model is a combination of two processes (kinetic and aggregation), each of which is proven to be separately conservative.

The eighth paper, entitled “Oscillation of second order nonlinear differential equations with several sub-linear neutral terms”, is concerned with some new sufficient conditions for oscillation of all solutions of a class of second order differential equations with several sub-linear neutral terms.

The ninth paper, entitled “Approximate analytical solutions for transient heat transfer in two-dimensional straight fins”, studies the numerical solution of the problem on heat transfer in two dimensional straight fins. The three-dimensional differential transform method (3D DTM) is used to construct the approximate analytical solutions.

In the tenth paper, entitled “Complete symmetry and μ-symmetry analysis of the Kawahara-KdV type equation”, the ordinary and μ-symmetries methods are used for the Kawahara-KdV type equation.

The eleventh paper, entitled “A phase change problem including space-dependent latent heat and periodic heat flux”, investigated a mathematical model related to a problem of phase-change process with periodic surface heat flux and space-dependent latent heat. The homotopy analysis method has been used to acquire the solution to the problem.

In the last paper entitled “Dual phase synchronization of chaotic systems using nonlinear observer based technique” the dual phase synchronization is achieved using nonlinear state observer technique and stability theory. The Qi and Newton-Leipnik systems are considered during demonstration of dual phase synchronization.

We would like to express our warmest thanks to authors who submitted their papers to be considered for publication in this Special Issue. We highly appreciate the contributions from the reviewers for their careful and critical evaluation of the manuscripts. It is our pleasure to thank Professor A.A. Martynyuk, Editor-in-Chief of ND&ST, for his support and encouragement during the process of editing this Special Issue.

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PERSONAGE IN SCIENCE

Professor Emeritus I.P. Stavroulakis

H. Jafari [1*], G. Ladas [2], and I. Gyori [3]

[1] *Department of Mathematical Sciences, University of South Africa, Florida Campus, 0003 South Africa*
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[3] *Department of Mathematics, University of Pannonia, Hungary*

Ioannis P. Stavroulakis (denoted by IPS throughout this paper) was born on January 2, 1949 (registered as on December 28, 1948) on the island of Crete, Greece. After six years at the primary school in his birthplace Episkopi-Rethymnis he continued Gymnasium (High school) in Rethymnon (the capital town of the prefecture).

Graduating from high school he took state entrance examinations and was accepted at the University of Ioannina, Department of Mathematics, Faculty of Sciences. In 1971 he graduated from the University of Ioannina and right after he was accepted for post-graduate studies at the City University of New York obtaining a Master's Degree in Mathematics in 1973. His doctoral work began in 1973 under the direction of the late Vassilios A. Staikos who was a student of the late Demetrios Kappos who in turn was a student of Constantine Caratheodory. His Ph.D. thesis defense at the University of Ioannina was in 1976. At this point it should be noted that IPS was (chronologically) the first from all the graduates of the Department of Mathematics, University of Ioannina who obtained a doctor's degree. It is to be also mentioned that during both his under-graduate and post-graduate studies he was holding scholarships: IKY (State Scholarships Foundation), Graduate University Scholarship (CUNY), Teaching Assistantship (Univ. of Ioannina) and Research Fellowship (The National Hellenic Research Foundation).

In the same year 1976 he accepted an academic position at the University of Ioannina, while in the following three academic years he taught at the University of Crete and participated in the organization of the **Departments of Mathematics and Physics** during the **first three academic years (1977–80)** of their establishment.

During 1981–84, while on Sabbatical, IPS had a position as Visiting Assistant Professor in the Division of Applied Mathematics, Brown University and also taught as an Assistant Professor in the Department of Mathematics, University of Rhode Island, USA. In 1985 he was elected Associate Professor, while in 1991 elected (full) Professor and worked at the University of Ioannina until 2015 where he was elected Professor Emeritus. He has also held the following positions:

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- Visiting Researcher, Ibaraki University, Japan, 06-07/1992, 06-08/1994, 06-08/1995
- Visiting Scholar, The Flinders University of South Australia, Adelaide, 10-11/1994
- Visiting Professor, Boston University, USA, 01-08/1995
- Visiting Professor, Ankara University, Turkey, 2016-17
- Visiting Professor, Dept. of Mathematical Sciences, Univ. of South Africa, 2017-18.

He has taught several courses on Mathematical Analysis and Ordinary, Difference and Partial Differential Equations at:

- **University of Ioannina**
- **University of Crete**
- **University of Rhode Island**
- **University of Tirana**
- **University of Gjirokastra**
- **Hellenic Open University**
- **Ankara University**

and has also supervised 7 post-doctoral researchers (from Slovakia, China, Georgia, Egypt, Turkey, Albania), 30 Ph. D. Theses and 4 Master Theses (Advisory Committee or Jury).

Research in various aspects of the Qualitative Theory of Ordinary, Functional, Difference, and Partial Differential Equations. In particular: *Study of the oscillatory and asymptotic behavior of delay, advanced, mixed, neutral differential and difference equations and of dynamic equations on time scales. First and higher order linear and non-linear equations with one or several monotone or non-monotone arguments.*

Upon his invitation more than 50 researchers from several foreign Universities (from Italy, Hungary, Japan, USA, China, Bulgaria, USSR, Czech Republic, Slovakia, Morocco, Albania, Jugoslavia, Israel, Ukraine, Skopje, Georgia, Poland, Egypt, Turkey) have visited the University of Ioannina and collaborated with him. He is the author of 3 books and more than 130 research papers most of them are of **high quality** and have been published in **superior** journals such as:

- **Proc. Roy. Soc. Edinburgh**
- **J. Nonlinear Analysis TMA**
- **SIAM J. Math. Anal.**
- **J. Math. Anal. Appl;**
- **J. Difference Equ. Appl.**
- **J. London Math. Soc.**
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- **Differentsialnye Uravn.**
- **Comput. Math. Appl.**
- **Appl. Math. Comp.**
- **Pacific J. Math.**
- **Proc. Amer. Math. Soc.**
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- **J. Differential Equations**

and also have been cited very frequently (**2500** citations) by (more than **500**) authors of (more than **1000**) books, monographs, theses and papers on the subject.

He is a referee and/or reviewer in more than 75 research journals and an editor of the following journals:

- Nonlinear Dynamics and Systems Theory (**Managing Editor**)
- Memoirs on Differential Equations and Mathematical Physics
- STUDIES of the University of Žilina
- Journal of Advanced Research in Differential Equations
- Journal of the Egyptian Mathematical Society
- Journal Mathematics and Natural Sciences
- Journal of Computational Analysis and Applications
- International Journal: Mathematical Manuscripts (IJMM)
- Bulletin of Mathematical Analysis and Applications
- Alexandria Journal of Mathematics
- Mathematical Sciences Letters (**Editor-in-Chief**)
- Communications.

It should be emphasized that IPS has been served as Managing Editor of this journal from the first year of its foundation in 2001 and has done a great job.

He has also been **contractor, coordinator and/or leader** in many research/scientific projects of the European Union (TEMPUS, ERASMUS/SOCRATES, TEMPUS PHARE, INTERREG II) ; Japan (”Ampre” Foundation, Canon Foundation); Australia (Visiting Research Fellowship, The Flinders University of South Australia); Ministry of National Economy, Ministry of Education and Ministry of Development; CINAMIL - Centro de Investigao, Desenvolvimento e Inovao da Academia Militar, Portugal, TUBITAK (B. 14.2.TBT.0.06.01.03.220.01, 07/10/2013) of the total amount of more than **1.000.000 (one million euro)**.

He has been a member of:

– **IKY** (State Scholarships Foundation, Athens, Greece) Examination Committee for Postgraduate and Postdoctoral Scholarships;

– **DIKATSA - DOATAP** (Inter-University Center for the recognition of foreign academic titles, Athens, Greece), **Mathematics Committee**;

– **President of the panel**: Mathematical Modelling for Social and Economic Sciences, Evaluation **Archimedes Prize 2001**, European Commission, Brussels, 8–11 October 2001;

– **Academic Expert Meeting**, TEMPUS JEP Selection 2004, European Training Foundation, Brussels, 7–11 February 2005.

He has been invited as a member of the **scientific/organizing** committee and/or as a **keynote/plenary/invited speaker** at many international conferences and universities and delivered **more than 130 lectures at 110 Universities in 30 countries** around the world.

He was awarded the following awards/distinctions:

- **Ampere Foundation Fellowship**
- **Canon Foundation in Europe Research Fellowship**
- **The Flinders University of South Australia Visiting Research Fellowship**
- **Ministry of National Economy Fellowship**
- **The National Hellenic Research Foundation Fellowship**
- **DOCTOR HONORIS CAUSA (Honorary Doctorate)**

It is to be pointed out that he is **the first scientist awarded Honorary Doctorate** from the University of Gjirokastra.

In addition to the administrative experience described above, it is to be mentioned that he has served at the University of Ioannina as a **Director** (5 years) of the Section of Mathematical Analysis; **Deputy Chairman** (2 years) and also as a **Chairman** (2 years) of the Department of Mathematics. It should be also emphasized that he served as the first **Acting Chairman** of the **Department of Informatics** during the **first two years** of its operation.

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On Stability of a Second Order Integro-Differential Equation

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Abstract: There exists a well-developed stability theory for integro-differential equations of the first order and only a few results on integro-differential equations of the second order. The aim of this paper is to fill up this gap. Explicit tests for uniform exponential stability of linear scalar delay integro-differential equations of the second order

$$\ddot{x}(t) + \int_{g(t)}^{t} G(t,s)\dot{x}(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = 0$$

are obtained.

Keywords: *exponential stability; second order delay integro-differential equations; a priory estimation; Bohl-Perron theorem.*

Mathematics Subject Classification (2010): 34K40, 34K20, 34K06.

1 Introduction

Beginning with the classical book of Volterra [1] integro-differential equations and, more generally, functional differential equations have many applications in biology, physics, mechanics (see, for example, [2, 4–7, 22, 26]). In particular, second order integro-differential equations appear in stability problems of viscoelastic shells [3]. There are many papers devoted to stability of the first order integro-differential equations [8–11, 18] and only few papers on stability for the second order equations [12–14]. Oscillation conditions for the first and the second order functional differential equations can be found in papers [15–17].

The aim of the present paper is to fill up this gap and obtain new explicit exponential stability conditions for the equation

$$\ddot{x}(t) + \int_{g(t)}^{t} G(t,s)\dot{x}(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = 0. \tag{1}$$

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Papers [12–14] are devoted to some asymptotic properties of partial cases of (1). In [12] an asymptotic behavior of solutions is studied using analysis of a generalized characteristic equation. In [14] the authors obtain stability results by an application of the Lyapunov functional method. In [13] the authors use a connection between asymptotic properties of (1) (for some special kernels $G(t,s), H(t,s)$) and a system of infinite number of ordinary differential equations.

To obtain new stability tests, we apply the method based on the Bohl-Perron theorem together with a priori estimations of solutions, integral inequalities for fundamental functions of linear delay equations and various transformations of a given equation. We consider equation (1) in more general assumptions than in the above mentioned papers: all kernels and delays are measurable functions, derivative of a solution is an absolutely continuous function.

2 Preliminaries

Denote

$$a(t) = \int_{g(t)}^{t} G(t,s)ds,\ b(t) = \int_{h(t)}^{t} H(t,s)ds,$$

$$a_1(t) = \int_{g(t)}^{t} G(t,s)(t-s)ds,\ b_1(t) = \int_{h(t)}^{t} H(t,s)(t-s)ds.$$

We consider scalar delay differential equation (1) under the following conditions:
(a1) $G(t,s) \ge 0, H(t,s) \ge 0$ are Lebesgue measurable on $t \ge s \ge 0$, h, g are measurable on $[0,\infty)$ functions, a, b, a_1, b_1 are essentially bounded on $[0,\infty)$ functions;
(a2) $0 < a_0 \le a(t) \le A_0, 0 < b_0 \le b(t) \le B_0$ for all $t \ge t_0 \ge 0$ and some fixed $t_0 \ge 0$;
(a3) $0 \le t - g(t) \le \sigma, 0 \le t - h(t) \le \tau$ for $t \ge t_0$ and some $\sigma > 0, \tau > 0$ and $t_0 \ge 0$.

Along with (1), we consider for each $t_0 \ge 0$ an initial value problem

$$\ddot{x}(t) + \int_{g(t)}^{t} G(t,s)\dot{x}(s)ds + \int_{h(t)}^{t} H(t,s)x(s)ds = f(t), \tag{2}$$

$$x(t) = \varphi(t),\ \dot{x}(t) = \psi(t),\ t \le t_0, \tag{3}$$

where $f : [t_0,\infty) \to R$ is a Lebesgue measurable locally essentially bounded function, $\varphi : (-\infty, t_0] \to R$, $\psi : (-\infty, t_0) \to R$ are Borel measurable bounded functions.

Further, we assume that the above conditions hold without mentioning it.

A function x with a locally absolutely continuous on $[t_0,\infty)$ derivative $x' : R \to R$ is called **a solution of problem** (2) if it satisfies the equation (2) for almost all $t \in [t_0,\infty)$ and the equalities in (3) for $t \le t_0$.

There exists a unique solution of problem (2)-(3), see [6, 21].

Equation (1) is **(uniformly) exponentially stable** if there exist positive numbers M and γ such that the solution of problem (3)with $f \equiv 0$ satisfies the estimate

$$\max\{|x(t)|, |\dot{x}(t)|\} \le Me^{-\gamma(t-t_0)} \sup_{t\in(-\infty,t_0]} \max\{|\psi(t)|, |\varphi(t)|\},\quad t \ge t_0, \tag{4}$$

where M and γ do not depend on $t_0 \ge 0$ and functions ψ, φ.

Next, we present the Bohl-Perron theorem [6, 19].

Lemma 2.1 *Assume that the solution x of the problem* (2) *with the initial conditions $x(t) = \dot{x}(t) = 0, t \leq t_0$, and its derivative $\dot{x}$ are bounded on $[t_0, +\infty)$ for any essentially bounded function f on $[t_0, +\infty)$. Then equation (1) is exponentially stable.*

Consider now an ordinary differential equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0 \tag{5}$$

and denote by $X(t, s)$ the fundamental function of (5).

Lemma 2.2 *[20] If $A_0 \geq a(t) \geq a_0 > 0, B_0 \geq b(t) \geq b_0 > 0, t \geq t_0$ and $a_0^2 \geq 4B_0$, then $X(t, s) \geq 0$, equation* (5) *is exponentially stable and*

$$\int_{t_0}^{t} X(t, s)b(s)ds < 1.$$

For a fixed bounded interval $I = [t_0, t_1]$, consider the space $L_\infty[t_0, t_1]$ of all essentially bounded on I functions with the norm $\|y\|_{[t_0,t_1]} = \text{esssup}_{t\in I}|y(t)|$, denote

$$\|f\|_{[t_0,+\infty)} = \text{esssup}_{t\geq t_0}|f(t)|$$

for an unbounded interval, E is the identity operator.

In the sequel, we use the concept of a non-singular M-matrix. For convenience, we recall this notion.

Definition 2.1 [[24]] An $m \times m$ matrix $A = (a_{ij})_{i,j=1}^m$ is called a non-singular M-matrix if $a_{ij} \leq 0,\ i, j = 1, \dots, m,\ i \neq j$ and one of the following equivalent conditions holds:

1. There exists a positive inverse matrix A^{-1}.
2. All the principal minors of matrix A are positive.

3 Explicit Stability Conditions

Theorem 3.1 *Assume that for some $t_0 \geq 0$ and $t \geq t_0$ $a_0^2 \geq 4B_0$ and the following condition holds*

$$\begin{aligned} &\|a\|_{[t_0,\infty)} \left\|\frac{a_1}{a}\right\|_{[t_0,\infty)} + \left\|\frac{b_1}{b}\right\|_{[t_0,\infty)} \left(\left\|\frac{b}{a}\right\|_{[t_0,\infty)} + \|b\|_{[t_0,\infty)} \left\|\frac{a_1}{a}\right\|_{[t_0,\infty)}\right) \\ &\qquad + \left\|\frac{a_1}{b}\right\|_{[t_0,\infty)} \left(\|b\|_{[t_0,\infty)} + \|a\|_{[t_0,\infty)} \left\|\frac{b}{a}\right\|_{[t_0,\infty)}\right) < 1. \end{aligned} \tag{6}$$

Then equation (1) *is exponentially stable.*

Proof. For simplicity we omit the index in the norm $\|\cdot\|_{[t_0,+\infty)}$ of functions.

Consider problem (2) with $\|f\| < \infty$, where $x(t) = \dot{x}(t) = 0,\ t \leq t_0$. We will prove that the solution x and its derivative are bounded functions on $[t_0, +\infty)$. First we have to obtain estimates for $x, \dot{x}, \ddot{x}, t \in I = [t_0, t_1], t_1 > t_0$. Rewrite equation (2)

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = \int_{g(t)}^{t} G(t, s)(\dot{x}(t) - \dot{x}(s))ds + \int_{h(t)}^{t} H(t, s)(x(t) - x(s))ds + f(t)$$

$$= \int_{g(t)}^{t} G(t,s) \int_{s}^{t} \ddot{x}(\tau) d\tau ds + \int_{h(t)}^{t} H(t,s) \int_{s}^{t} \dot{x}(\tau) d\tau ds + f(t).$$

Hence

$$x(t) = \int_{t_0}^{t} X(t,s) b(s) \left[\frac{1}{b(s)} \int_{g(s)}^{s} G(s,\xi) \int_{\xi}^{s} \ddot{x}(\tau) d\tau d\xi \right.$$

$$\left. + \frac{1}{b(s)} \int_{h(s)}^{s} H(s,\xi) \int_{\xi}^{s} \dot{x}(\tau) d\tau d\xi \right] ds + f_1(t),$$

where $X(t,s)$ is the fundamental function of equation (5) and $f_1(t) = \int_{t_0}^{t} X(t,s) f(s) ds$. Since $X(t,s)$ has an exponential estimate, f_1 is essentially bounded on $[t_0, \infty)$.

By Lemma 2.2 we have

$$\|x\|_{[t_0,t_1]} \leq \left\| \frac{a_1}{b} \right\| \|\ddot{x}\|_{[t_0,t_1]} + \left\| \frac{b_1}{b} \right\| \|\dot{x}\|_{[t_0,t_1]} + \|f_1\|. \tag{7}$$

Rewrite now (2) in another form:

$$\ddot{x}(t) + a(t)\dot{x}(t) = \int_{g(t)}^{t} G(t,s) \int_{s}^{t} \ddot{x}(\tau) d\tau ds - \int_{h(t)}^{t} H(t,s) x(s) ds + f(t).$$

Hence

$$\dot{x}(t) = \int_{t_0}^{t} e^{-\int_s^t a(\xi) d\xi} a(s) \left[\frac{1}{a(s)} \int_{g(s)}^{s} G(s,\xi) \int_{\xi}^{s} \ddot{x}(\tau) d\tau d\xi \right.$$

$$\left. - \frac{1}{a(s)} \int_{h(s)}^{s} H(s,\xi) x(\xi) d\xi \right] ds + f_2(t),$$

where $f_2(t) = \int_{t_0}^{t} e^{-\int_s^t a(\xi) d\xi} f(s) ds$ is an essential bounded on $[t_0, \infty)$ function.

Hence

$$\|\dot{x}\|_{[t_0,t_1]} \leq \left\| \frac{a_1}{a} \right\| \|\ddot{x}\|_{[t_0,t_1]} + \left\| \frac{b}{a} \right\| \|x\|_{[t_0,t_1]} + \|f_2\|. \tag{8}$$

From equation (2) we have

$$\|\ddot{x}\|_{[t_0,t_1]} \leq \|a\| \|\dot{x}\|_{[t_0,t_1]} + \|b\| \|x\|_{[t_0,t_1]} + \|f\|. \tag{9}$$

Denote $Y = \{\|x\|_{[t_0,t_1]}, \|\dot{x}\|_{[t_0,t_1]}, \|\ddot{x}\|_{[t_0,t_1]}\}^T, F = \{\|f_1\|, \|f_2\|, \|f\|, \}^T$. Inequalities (7)-(9) imply $Y \leq BY + F$, where

$$B = \begin{pmatrix} 0 & \|\frac{b_1}{b}\| & \|\frac{a_1}{b}\| \\ \|\frac{b_1}{b}\| & 0 & \|\frac{a_1}{a}\| \\ \|b\| & \|a\| & 0 \end{pmatrix}.$$

Hence $AY \leq F$, where $A = E - B$. Theorem conditions imply that A is an M-matrix then $Y \leq A^{-1}F$, where $A^{-1}F$ is a constant vector which does not depend on the interval I. Hence the solution of (2) with its derivative are bounded functions on $[t_0, \infty)$, therefore by Lemma 2.1 equation (1) is exponentially stable.

Corollary 3.1 *Assume that for some* $t_0 \geq 0$ *and* $t \geq t_0$, $a_0^2 \geq 4B_0$ *and the following condition holds*

$$\sigma\|a\|_{[t_0,\infty)}+\tau\Big(\left\|\frac{b}{a}\right\|_{[t_0,\infty)}+\sigma\|b\|_{[t_0,\infty)}\Big)+\sigma\left\|\frac{a}{b}\right\|_{[t_0,\infty)}\Big(\|a\|_{[t_0,\infty)}\left\|\frac{b}{a}\right\|_{[t_0,\infty)}+\|b\|_{[t_0,\infty)}\Big)<1. \tag{10}$$

Then equation (1) *is exponentially stable.*

Proof. For simplicity we omit the index in the norm on functions. We have $t-s \leq t-g(t) \leq \sigma$ for $g(t) \leq s \leq t$. Similarly, $t-s \leq t-h(t) \leq \tau$ for $h(t) \leq s \leq t$. Hence

$$a_1(t) = \int_{g(t)}^{t} G(t,s)(t-s)ds \leq \int_{g(t)}^{t} G(t,s)\sigma ds = \sigma a(t),$$

$$b_1(t) = \int_{h(t)}^{t} H(t,s)(t-s)ds \leq \int_{h(t)}^{t} H(t,s)\tau ds = \tau b(t).$$

Then

$$\|a\|\left\|\frac{a_1}{a}\right\|+\left\|\frac{b_1}{b}\right\|\left(\left\|\frac{b}{a}\right\|+\|b\|\left\|\frac{a_1}{a}\right\|\right)+\left\|\frac{a_1}{b}\right\|\left(\|b\|+\|a\|\left\|\frac{b}{a}\right\|\right)$$

$$\leq \sigma\|a\|+\tau\left(\left\|\frac{b}{a}\right\|+\sigma\|b\|\right)+\sigma\left\|\frac{a}{b}\right\|\left(\|a\|\left\|\frac{b}{a}\right\|+\|b\|\right)<1.$$

By Theorem 3.1 equation (1) is exponentially stable.

Corollary 3.2 *Assume there exist*

$$\lim_{t\to\infty} a(t) = a > 0, \lim_{t\to\infty} b(t) = b > 0, \lim_{t\to\infty} a_1(t) = a_1 > 0, \lim_{t\to\infty} b_1(t) = b_1 > 0.$$

If

$$a^2 \geq 4b,\ \ 3a_1 + \frac{b_1(1+a_1)}{a} < 1,$$

then the equation (1) is exponentially stable.

Limits in the corollary 3.2 exist, for example, for kernels of the form $M(t-s)^n e^{-\gamma(t-s)}$ where $n \geq 0$ is a natural number.

Example 3.1 Consider the following equation

$$\ddot{x}(t) + M_1\int_{t-\sigma}^{t} e^{-\alpha_1(t-s)}\dot{x}(s)ds + M_2\int_{t-\tau}^{t} e^{-\alpha_2(t-s)}x(s)ds = 0, \tag{11}$$

where $\alpha > 0, \beta > 0, \sigma > 0, \tau > 0$.

We have

$$a(t) = a = M_1\int_{t-\sigma}^{t} e^{-\alpha_1(t-s)}ds = \frac{M_1}{\alpha_1}\left(1-e^{-\alpha_1\sigma}\right),$$

$$b(t) = b = M_2\int_{t-\tau}^{t} e^{-\alpha_2(t-s)}ds = \frac{M_2}{\alpha_2}\left(1-e^{-\alpha_2\tau}\right),$$

$$a_1(t) = a_1 = M_1\int_{t-\sigma}^{t}(t-s)e^{-\alpha(t-s)}ds = \frac{M_1}{\alpha}\left(\frac{1}{\alpha}-e^{-\alpha\sigma}(\sigma+\frac{1}{\alpha})\right),$$

$$b_1(t) = b_1 = M_2\int_{t-\tau}^{t}(t-s)e^{-\alpha_2(t-s)}ds = \frac{M_2}{\alpha_2}\left(\frac{1}{\alpha_2}-e^{-\alpha_2\tau}(\tau+\frac{1}{\beta})\right).$$

Hence, if $a^2 \geq 4b, 3a_1 + \frac{b_1(1+a_1)}{a} < 1$, then equation (11) is exponentially stable.

Corollary 3.3 *Assume for $t \geq t_0$*

$$0 < a_0 \leq a(t) \leq A_0, 0 < b_0 \leq b(t) \leq B_0, a_0^2 \geq 4B_0,$$

$$0 < \sigma_0 \leq t - g(t) \leq \sigma, 0 < \tau_0 \leq t - h(t) \leq \tau$$

and

$$\frac{A_0\sigma^3}{2a_0\sigma_0} + \frac{B_0^2\tau^3}{2a_0b_0\tau_0\sigma_0}(1 + \frac{A_0\sigma^2}{2}) + \frac{A_0B_0\tau\sigma_2}{2b_0\tau_0}\left(1 + \frac{A_0\sigma}{a_0\sigma_0}\right) < 1.$$

Then the equation (1) is exponentially stable.

Proof. The proof follows from the inequalities

$$a_0\sigma_0 \leq a(t) \leq A_0\sigma, b_0\tau_0 \leq b(t) \leq B_0\tau, a_1(t) \leq A_0\frac{\sigma^2}{2}, b_1(t) \leq B_0\frac{\tau^2}{2}$$

and Theorem 3.1.

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Oscillation of Second Order Nonlinear Differential Equations with Several Sub-Linear Neutral Terms

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Abstract: Some new sufficient conditions for oscillation of all solutions of a class of second order differential equations with several sub-linear neutral terms are given. Our results generalize and extend those reported in the literature. Examples are included to illustrate the importance of the results obtained.

Keywords: *second order neutral differential equation; sub-linear neutral term; oscillation.*

Mathematics Subject Classification (2010): 34C10, 34K11.

1 Introduction

In this paper, we study the oscillatory behavior of second order differential equations with several sub-linear neutral terms of the form

$$(a(t)z'(t))' + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1}$$

where $m > 0$ is an integer, $z(t) = x(t) + \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t))$ and we assume that

(H_1) $0 \leq \alpha_i \leq 1$ for $i = 1, 2, ..., m$ and β are the ratios of odd positive integers;

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(H_2) $a, p_i, q : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions for $i = 1, 2, ..., m$ with

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty; \tag{2}$$

(H_3) $\tau_i, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $\tau_i(t) < t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$ and $\tau_i(t), \sigma(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, ..., m$.

By a solution of equation (1), we mean a function $x \in C([T_x, \infty), \mathbb{R})$, $T_x \geq t_0$, which has the property $az' \in C^1([T_x, \infty), \mathbb{R})$ and satisfies equation (1) on $[T_x, \infty)$. We consider only those solutions x of equation (1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and assume that the equation (1) possesses such solutions. As usual, a solution of equation (1) is called oscillatory if it has a zero on $[T, \infty)$ for all $T \geq T_x$; otherwise it is called nonoscillatory. If all solutions of a differential equation are oscillatory, then the equation itself is called oscillatory.

The problem of investigating the oscillatory behavior of solutions of particular functional differential equations received a great attention in the past decades, see, for example, [1] – [20] for recent references. However, there are few results dealing with the oscillation of second order differential equations with a sub-linear neutral term, see [3, 8, 19], even though, such equations arise in many applications, see [9]. In establishing some new criteria for the oscillation of solutions of such equations, we reduce the equation to an equation with linear neutral term, using some inequalities.

Thus, by using some elementary inequalities, we obtained in this paper some new oscillation results, which are new, extend and complement those established in [2–5, 14–17, 19, 20].

2 Oscillation Results

In what follows, all functional inequalities considered here are assumed to hold eventually, that is, they are satisfied for all t large enough. Due to the assumptions and the form of the equation (1), we can deal only with eventually positive solutions of equation (1).

We begin with the following lemma.

Lemma 2.1 *If a and b are nonnegative, then*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \quad \textit{for} \ \ 0 < \alpha \leq 1, \tag{3}$$

where equality holds if and only if $a = b$.

Proof. The proof of the lemma can be found in [9]. □

To simplify our notation, for any function $\rho : [t_0, \infty) \to \mathbb{R}^+$ which is positive, continuous decreasing to zero, we set

$$\begin{aligned} P(t) &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i) p_i(t)\right), \\ Q(t) &= q(t) P^{\beta}(\sigma(t)) \end{aligned}$$

and

$$R(t) = \int_{t_1}^{t} \frac{1}{a(s)} ds.$$

Remark 2.1 It follows from condition (2), that the lower bound t_1 is an absolutely unimportant constant in the intended oscillatory criteria.

Lemma 2.2 *Assume condition* (2) *and let x be a positive solution of equation* (1). *Then the corresponding function z satisfies*

$$z(t) > 0,\ z'(t) > 0,\ \textit{and}\ (a(t)z'(t))' < 0,\ t \geq t_1 \geq t_0, \tag{4}$$

$$z(t) \geq R(t)a(t)z'(t),\ t \geq t_1 \tag{5}$$

and

$$\frac{z(t)}{R(t)}\ \textit{is decreasing for}\ t \geq t_1. \tag{6}$$

Proof. Assume that x is a positive solution of (1). Then $(a(t)z'(t))' < 0$ for $t \geq t_1 \geq t_0$ which in view of (2) implies $z'(t) > 0$ for $t \geq t_1 \geq t_0$. Since $a(t)z'(t)$ is decreasing, we have

$$z(t) \geq \int_{t_1}^{t} a(s)z'(s)\frac{1}{a(s)}ds \geq a(t)z'(t)R(t).$$

Moreover, using the previous inequality, we have

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{a(t)z'(t)R(t) - z(t)}{a(t)R^2(t)} \leq 0.$$

We can conclude that $\frac{z(t)}{R(t)}$ is decreasing for $t \geq t_1$. $\square$

Theorem 2.1 *Let $\beta > 1$ and conditions $(H_1) - (H_3)$ and* (2) *hold. Let*

$$\int_{t_1}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s)P^{\beta}(\sigma(s))ds\ du = \infty. \tag{7}$$

Assume that there is a positive continuous decreasing function $\rho : [t_0, \infty) \to (0, \infty)$ tending to zero, such that $P(t)$ is positive for $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)Q(s) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty, \tag{8}$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, some $t_1 \geq t_0$ and for $i = 1, 2, ..., m$. It is easy to see that $z(t) > 0$ for $t \geq t_1$, and from Lemma 2.2 (4) holds.

Now from the definition of z, we have

$$\begin{aligned} x(t) &= z(t) - \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t)) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)z^{\alpha_i}(t) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)(\alpha_i z(t) + (1-\alpha_i)) \\ &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t)\right) z(t) - \sum_{i=1}^{m} (1-\alpha_i)p_i(t), \end{aligned} \tag{9}$$

where we have used inequality (3) with $b = 1$. Since $z(t)$ is positive and increasing and $\rho(t)$ is positive and decreasing to zero, there is a $t_2 \geq t_1$ such that

$$z(t) \geq \rho(t) \text{ for } t \geq t_2. \tag{10}$$

Using (10) in (9), we obtain

$$x(t) \quad \geq \quad \left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{\rho(t)} \sum_{i=1}^{m} (1-\alpha_i) p_i(t)\right) z(t) = P(t) z(t)$$

and substituting this in equation (1) yields

$$(a(t)z'(t))' + q(t) P^{\beta}(\sigma(t)) z^{\beta}(\sigma(t)) \leq 0,\ t \geq t_2. \tag{11}$$

From condition (7) it follows that $z(t) \to \infty$ as for $t \to \infty$ and for $\beta > 1$, inequality

$$z^{\beta}(\sigma(t)) > z(\sigma(t))$$

holds. Using this inequality in (11), we obtain

$$(a(t)z'(t))' + Q(t) z(\sigma(t)) \leq 0, t \geq t_2. \tag{12}$$

Define the function

$$w(t) = \mu(t) \frac{a(t)z'(t)}{z(\sigma(t))},\ t \geq t_2.$$

Then $w(t) > 0$ for $t \geq t_2$ and

$$w'(t) = \mu'(t) \frac{a(t)z'(t)}{z(\sigma(t))} + \mu(t) \frac{(a(t)z'(t))'}{z(\sigma(t))} - \frac{\mu(t)a(t)z'(t)}{z^2(\sigma(t))} z'(\sigma(t)).\sigma'(t). \tag{13}$$

Since $a(t)z'(t)$ is positive and nonincreasing, we obtain

$$a(t)z'(t) \leq a(\sigma(t)) z'(\sigma(t)). \tag{14}$$

Using (14) and (12) in (13), and completing the square, we see that

$$w'(t) \leq -\mu(t)Q(t) + \frac{a(\sigma(t))(\mu'(t))^2}{4\mu(t)\sigma'(t)}.$$

An integration of the last inequality from t_2 to t yields

$$\int_{t_2}^{t} \left[\mu(s)Q(s) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds \leq w(t_2),$$

and on taking lim sup as $t \to \infty$, we obtain a contradiction with (8). This completes the proof. □

Next, we present new oscillation results for equation (1) with $\beta > 1$.

Theorem 2.2 *Let $\beta > 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$ such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)q(s)P^{\beta}(\sigma(s))\rho^{\beta-1}(\sigma(s)) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] = \infty, \tag{15}$$

then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, some $t_1 \geq t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we see that (11) holds. Now using (10) in (11), we obtain

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))\rho^{\beta-1}(\sigma(t))z(\sigma(t)) \leq 0,\ t \geq t_2.$$

The rest of the proof is similar to that of Theorem 2.1 and hence it is omitted. □

If $\beta = 1$, then from Theorem 2.2 one can immediately obtain the following oscillation results for the equation (1).

Theorem 2.3 *Let $\beta = 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$, such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\mu(s)q(s)P(\sigma(s)) - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty, \tag{16}$$

then every solution of equation (1) is oscillatory.

Next, we obtain an oscillation result for the equation (1) in the case $0 < \beta < 1$.

Theorem 2.4 *Let $0 < \beta < 1$ and conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive continuous and decreasing function $\rho(t) : [t_0, \infty) \to \mathbb{R}^+$ tending to zero as $t \to \infty$, such that $P(t)$ is positive for all $t \geq t_0$. If there exists a positive function $\mu(t) \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^{t} \left[\frac{\mu(s)q(s)P^{\beta}(\sigma(s))R^{\beta-1}(\sigma(s))}{K^{1-\beta}} - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)}\right] ds = \infty \tag{17}$$

for every constant $K > 0$, then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0,\ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we obtain (11). Now (11) can be written as

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))\frac{z^{\beta-1}(\sigma(t))}{R^{\beta-1}(\sigma(t))}z(\sigma(t)) \leq 0 \tag{18}$$

for all $t \geq t_2 \geq t_1$. Since $\frac{z(t)}{R(t)}$ is decreasing, there is a constant $K > 0$ such that

$$\frac{z(t)}{R(t)} \leq K \text{ for } t \geq t_2. \tag{19}$$

Using (19) and $\beta < 1$, in (18), we have

$$(a(t)z'(t))' + q(t)\frac{P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))}{K^{1-\beta}}z(\sigma(t)) \leq 0,\ t \geq t_2.$$

We define function $w(t)$ as in proof of Theorem 2.1. Proceeding exactly as in the proof of Theorem 2.1, we get

$$w'(t) \leq -\mu(t)q(t)\frac{P^{\beta}(\sigma(t))R^{\beta-1}(\sigma(t))}{K^{1-\beta}} + \frac{a(\sigma(t))(\mu'(t))^2}{4\mu(t)\sigma'(t)}.$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_0}^{t} \left[\frac{\mu(s)q(s)P^{\beta}(\sigma(s))R^{\beta-1}(\sigma(s))}{K^{1-\beta}} - \frac{a(\sigma(s))(\mu'(s))^2}{4\mu(s)\sigma'(s)} \right] ds \le w(t_2),$$

and on taking limsup as $t \to \infty$, we have a contradiction with (17). □

Next, we use a comparison method to prove our results for the case $\beta \in (0, \infty)$.

Theorem 2.5 *Let conditions $(H_1) - (H_3)$ and (2) hold. Assume that there is a positive, continuous and decreasing function $\rho(t) : [t_0, \infty) \to \mathbb{R}^+$ tending to zero such that $P(t)$ is positive for all $t \ge t_0$. If the first order delay differential equation*

$$w'(t) + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))w^{\beta}(\sigma(t)) = 0, \ t \ge t_1 \tag{20}$$

is oscillatory, then every solution of equation (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0, \ x(\tau_i(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \ge t_1$, for some $t_1 \ge t_0$ and $i = 1, 2, ..., m$. Proceeding as in the proof of Theorem 2.1, we see that (11) holds. Using (5) in (11), we obtain

$$(a(t)z'(t))' + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))(a(\sigma(t))z'(\sigma(t)))^{\beta} \le 0, \ t \ge t_1. \tag{21}$$

Set $w(t) = a(t)z'(t)$. Thus $w(t) > 0$, and

$$w'(t) + q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))w^{\beta}(\sigma(t)) \le 0.$$

By Lemma 2.2 of [17], the equation (20) has a positive solution which is a contradiction. This completes the proof. □

Using the results of [8] and [18], one can easily obtain the following corollaries from Theorem 2.5.

Corollary 2.1 *Let all conditions of Theorem 2.5 hold with $\beta = 1$ for all $t \ge t_0$. If*

$$\liminf_{t \to \infty} \int_{\sigma(t)}^{t} q(s)P(\sigma(s))R(\sigma(s))ds > \frac{1}{e},$$

then every solution of equation (1) is oscillatory.

Corollary 2.2 *Let all conditions of Theorem 2.5 hold with $0 < \beta < 1$ for all $t \ge t_0$. If*

$$\int_{t_0}^{\infty} q(t)P^{\beta}(\sigma(t))R^{\beta}(\sigma(t))dt = \infty,$$

then every solution of equation (1) is oscillatory.

Corollary 2.3 *Let all conditions of Theorem 2.5 hold with $\beta > 1$ for all $t \ge t_0$. If $\sigma(t) = t - \delta, \ \delta > 0$, and*

$$\liminf_{t \to \infty} \beta^{-\frac{t}{\delta}} \log(q(t)P^{\beta}(t-\delta)R^{\beta}(t-\delta)) > 0,$$

then every solution of equation (1) is oscillatory.

3 Examples

In this section, we provide some examples to illustrate the main results.

Example 3.1 Consider the differential equation with sub-linear neutral terms

$$\left(t\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{1}{5}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x^3\left(\frac{t}{2}\right)=0,\ t\geq 8. \tag{22}$$

Here $a(t)=t,\ p_1(t)=\frac{1}{t},\ p_2(t)=\frac{1}{t^2},\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=\frac{t}{3},\ \sigma(t)=\frac{t}{2},\ q(t)=t^{\gamma}$, $\alpha_1=\frac{1}{3},\ \alpha_2=\frac{1}{5}$ and $\beta=3$. Let $\rho(t)=\frac{1}{t}$ then $\rho(t)\to 0$ as $t\to\infty$ and $\eta(t)=\frac{1}{t}$ and

$$\begin{aligned} P(t) &= \left(1-\frac{1}{3t}-\frac{1}{5t^2}-t\left(\frac{2}{3t}+\frac{4}{5t^2}\right)\right) \\ &= \left(\frac{1}{3}-\frac{1}{3t}-\frac{1}{5t^2}-\frac{4}{5t}\right)=\frac{5t^2-17t-3}{15t^2}>0 \ \text{ for } \ t\geq 8. \end{aligned}$$

By taking $\mu(t)=t$, we see that

$$\lim_{t\to\infty}\sup\int_8^t\left(\frac{3}{2}s^{\gamma-1}\left(\frac{5s^2-34s-12}{15s^2}\right)^3-\frac{1}{4}\right)ds=\infty$$

provides $\gamma>1$. So by Theorem 2.2, every solution of equation (22) is oscillatory.

Example 3.2 Consider the differential equation with sub-linear neutral terms

$$\left(t\left(x(t)+\frac{1}{t}x^{\frac{3}{5}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{1}{3}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x\left(\frac{t}{2}\right)=0. \tag{23}$$

Here $a(t)=t,\ p_1(t)=\frac{1}{t},\ p_2(t)=\frac{1}{t^2},\ \tau_1(t)=\frac{t}{2},\ \tau_2(t)=\frac{t}{3},\ \sigma(t)=\frac{t}{2},\ q(t)=t^{\gamma}$, $\alpha_1=\frac{3}{5},\ \alpha_2=\frac{1}{3}$ and $\beta=1$. Let $\rho(t)=\frac{1}{t}$ then $\rho(t)\to 0$ as $t\to\infty$ and

$$\begin{aligned} P(t) &= 1-\frac{3}{5t}-\frac{1}{3t^2}-t\left(\frac{2}{5t}+\frac{2}{3t^2}\right) \\ &= \left(1-\frac{3}{5t}-\frac{1}{3t^2}-\frac{2}{5}-\frac{2}{3t}\right)=\frac{3}{5}-\frac{19}{15t}-\frac{1}{3t^2} \\ &= \frac{1}{15t^2}(9t^2-19t-5), \\ P\left(\frac{t}{2}\right) &= \left(\frac{9t^2-38t-20}{15t^2}\right)>0 \text{ for } t\geq 8. \end{aligned}$$

By taking $\mu(t)=t$, we see that

$$\lim_{t\to\infty}\sup\int_8^t\left(s^{\gamma+1}\left(\frac{9s^2-38s-20}{15s^2}\right)-\frac{1}{4}\right)ds=\infty$$

provides $\gamma\geq -1$. By Theorem 2.3, every solution of equation (23) is oscillatory.

Example 3.3 Consider the differential equation with sub-linear neutral terms

$$\left(t^{\frac{1}{2}}\left(x(t)+\frac{1}{t}x^{\frac{1}{3}}\left(\frac{t}{2}\right)+\frac{1}{t^2}x^{\frac{5}{7}}\left(\frac{t}{3}\right)\right)'\right)'+t^{\gamma}x^{\frac{1}{3}}\left(\frac{t}{2}\right)=0. \tag{24}$$

Here $a(t)=t^{\frac{1}{2}}$, $p_1(t)=\frac{1}{t}$, $p_2(t)=\frac{1}{t^2}$, $\alpha_1=\frac{1}{3}$, $\alpha_2=\frac{5}{7}$, $\beta=\frac{1}{3}$, $q(t)=t^{\gamma}$, $\tau_1(t)=\frac{t}{2}$, $\tau_2(t)=\frac{t}{3}$ and $\sigma(t)=\frac{t}{2}$. Let $\rho(t)=\frac{1}{t}$, then $\rho(t)\to 0$ as $t\to\infty$ and

$$\begin{aligned}
P(t) &= 1-\frac{1}{3t}-\frac{5}{7t^2}-t\left(\frac{2}{3t}+\frac{2}{7t^2}\right)\\
&= 1-\frac{1}{3t}-\frac{5}{7t^2}-\frac{2}{3}-\frac{2}{7t}=\left(\frac{1}{3}-\frac{13}{21t}-\frac{5}{7t^2}\right),\\
P(\sigma(t)) &= \left(\frac{1}{3}-\frac{26}{21t}-\frac{20}{7t^2}\right)=\frac{(7t^2-26t-60)}{21t^2}>0,\ t\ge 8,\\
R(t) &= \int_8^t \frac{1}{s^{1/2}}ds=2\sqrt{t}-4\sqrt{2}.
\end{aligned}$$

By taking $\mu(t)=1$, we see that

$$\lim_{t\to\infty}\sup\int_8^t K^{1/3-1}s^{\gamma}\left(\frac{7s^2-26s-60}{21s^2}\right)^{\frac{1}{3}}\left(2s^{\frac{1}{2}}-4\sqrt{2}\right)^{-\frac{2}{3}}ds=\infty$$

provides $\gamma\ge\frac{1}{3}$. By Theorem 2.4, every solution of equation (22) is oscillatory.

4 Conclusion

The results presented in this paper are new and complement to those of [3, 17, 19, 20]. Further it would be of interest to use this method to study equation (1) with $\alpha_i>1$ for $i=1,2,...,m$, that is, equation (1) with several superlinear neutral terms. Also, the results established in [2–5, 14–17, 19, 20] cannot be applied to equations (22) to (24), since the neutral term contains more than one sub-linear neutral term. Thus the results obtained in this paper are applicable to several classes of neutral type differential equations.

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Approximate Analytical Solutions for Transient Heat Transfer in Two-Dimensional Straight Fins

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Abstract: In this paper we analyse the heat transfer in two-dimensional straight fins. Both heat transfer coefficient and thermal conductivity are temperature dependent. The resulting 2+1 dimension partial differential equation (PDE) is rendered nonlinear and difficult to solve exactly, particularly with prescribed initial and boundary conditions. The three-dimensional differential transform method (3D DTM) is used to construct the approximate analytical solutions. The effects of parameters, appearing in the boundary value problem (BVP), on temperature profile of the fin are studied.

Keywords: *3D DTM; approximate solutions; 2D staright fins, heat transfer.*

Mathematics Subject Classification (2010): 35K57, 35G30, 35K05, 74A15, 41A58.

1 Introduction

Fins are surfaces that extend from a primary body to a surrounding fluid. They are predominantly used to increase the heat transfer rate between the body and its surroundings. Fins are designed in such a way that they increase the surface area of an object and hence its contact with the environment. They come in various shapes, geometries and profiles that cater for a diverse range of problems and applications (the reader is referred to [1] for a detailed theory). Fins are widely used in devices that exchange heat, common examples would include vehicle engine radiators, refrigerators, air conditioning devices and compressors. Consequently, the study of heat transfer in fins continues to be of interest.

Two-dimensional fin problems have received much attention, however, it is assumed in most works that the thermal conductivity and the heat transfer coefficient are constants, and the internal heat generation is omitted. In [2], the authors provided the approximate

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solutions using homotopy analysis for the transient problem with constant thermal properties. Moitsheki and Rowjee [3] constructed exact solutions for a two-dimensional steady state problem with the temperature-dependent thermal conductivity, heat transfer coefficient and internal heat generation. Analysis of transient heat transfer in straight fins of various shapes and with constant heat flux was carried out in [4]. A two-dimensional rectangular fin with variable heat transfer coefficient was analysed using the Fourier series approach [5]. In [6], two-dimensional trapezoidal fins were analysed wherein heat loss through fins at various slopes were compared. Exact solutions for heat transfer in rectangular fins were constructed in [7].

In this paper, the two-dimension heat flow in straight fins is analysed using the 3D DTM. The DTM was introduced in [8] and an account for the higher dimension DTM may be found in [9]. In Section 2, a mathematical description of the problem in question is provided. A brief account of the DTM is provided in Section 3. In Section 4, approximate analytical solutions are constructed. Some discussions and conclusion are given in Section 5.

2 Mathematical Description

The fin is attached to a primary surface of temperature T_b. The coordinate system has the origin at the intersection of the fin surface and the fin tip, with the X-axis extending towards the fin base and the Y-axis extending towards the centre of the fin. The fin height is L and the length from the X-axis to the center of the fin is δ. The temperature of the surrounding fluid into which the fin extends is designated by T_s. The thermal conductivity and the heat transfer coefficient are dependent on temperature and are denoted by $K(T)$ and $H(T)$, respectively. For our problem under consideration we assume no internal heat generation. Therefore, in the dimensionless variables and parameters, the governing BVP is given by (see also [1])

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left[k(\theta)\frac{\partial\theta}{\partial x}\right] + E^2\frac{\partial}{\partial y}\left[k(\theta)\frac{\partial\theta}{\partial y}\right], \tag{1}$$

subject to the initial condition

$$\theta(0,x,y) = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{2}$$

and the boundary conditions

$$\theta(\tau,1,y) = 1, \quad 0 \le y \le 1, \quad \tau > 0, \tag{3}$$

$$\frac{\partial\theta}{\partial x} = 0, \quad x = 0, \quad 0 \le y \le 1, \quad \tau > 0, \tag{4}$$

$$k(\theta)\frac{\partial\theta}{\partial y} = -Bih(\theta)\theta, \quad y = 0, \quad 0 \le x \le 1, \quad \tau > 0, \tag{5}$$

$$\frac{\partial\theta}{\partial y} = 0, \quad y = 1, \quad 0 \le x \le 1, \quad \tau > 0, \tag{6}$$

where the dimensionless quantities are given by

$$t = \frac{L^2\rho c_p}{K_a}\tau, \quad X = Lx, \quad Y = \delta y, K = K_a k, \quad H = H_b h, \quad T = (T_b - T_s)\theta + T_s,$$

with τ, x, y, k, h and θ being the dimensionless variables. K_a and H_b are the ambient thermal conductivity and the fin base heat transfer coefficient, respectively, and $E = \frac{L}{\delta}$ and $Bi = \frac{\delta H_b}{K_a}$ are the fin extension factor and the Biot number, respectively. An account of studies of diffusion equations in higher dimensions may be found, for example, in [10].

For practicality purposes, two cases will be considered for the relation of the thermal conductivity and temperature [11], namely, the linear function relation and the power law. We also consider the heat transfer coefficient given by the power law. The two cases for the thermal conductivity (see e.g. [12,13]) are given by

Case (i) the power law

$$k(\theta) = \theta^n, \tag{7}$$

where n is a dimensionless constant and

Case (ii) the linear function

$$k(\theta) = 1 + \beta\theta, \tag{8}$$

where $\beta = \epsilon(T_b - T_s)$ is the thermal conductivity parameter and ϵ is the thermal conductivity gradient. For most engineering applications the heat transfer coefficient has a power law relation with temperature [1], that is,

$$h(\theta) = \theta^m. \tag{9}$$

Here m is a dimensionless constant, which in engineering applications takes values from -3 to 3.

3 A Brief Account of the p-Dimensional DTM

For an analytic multivariable function $f(x_1, x_2, ..., x_p)$, we have the p-dimensional transform given by

$$F(k_1, k_2, ..., k_p) = \frac{1}{k_1!k_2!...k_p!}\left[\frac{\partial^{k_1+k_2+...+k_p} f(x_1, x_2, ..., x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_p^{k_p}}\right]\Bigg|_{(x_1,x_2,...,x_p)=(0,0,...,0)}. \tag{10}$$

The upper and lower case letters stand for the transformed and the original functions, respectively. The transformed function is also referred to as the T-function, the differential inverse transform is given by

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_p=0}^{\infty} F(k_1, k_2, ..., k_p)\prod_{l=1}^{p} x_l^{k_l}. \tag{11}$$

It can be easily deduced that the substitution of equation (10) into equation (11) gives the Taylor series expansion of the function $f(x_1, x_2, ..., x_p)$ about the point $(x_1, x_2, ..., x_p) = (0, 0, ..., 0)$. This is given by

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_p=0}^{\infty} \frac{\prod_{l=1}^{p} x_l^{k_l}}{k_1!k_2!...k_p!}\left[\frac{\partial^{k_1+k_2+...+k_p} f(x_1, x_2, ..., x_p)}{\partial x_1^{k_1}\partial x_2^{k_2}...\partial x_p^{k_p}}\right]\Bigg|_{x_1=0,...,x_p=0}. \tag{12}$$

For real world applications the function $f(x_1, x_2, ..., x_p)$ is given in terms of a finite series for some $q, r, s \in \mathbb{Z}$. Then equation (11) becomes

$$f(x_1, x_2, ..., x_p) = \sum_{k_1=0}^{q}\sum_{k_2=0}^{r}...\sum_{k_p=0}^{s} F(k_1, k_2, ..., k_p)\prod_{l=1}^{p} x_l^{k_l}. \tag{13}$$

Original function $f(x_1, x_2, ..., x_p)$	T-function $F(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = \lambda g(x_1, x_2, ..., x_p)$	$F(k_1, k_2, ..., k_p) = \lambda G(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = g(x_1, x_2, ..., x_p) \pm p(x_1, x_2, ..., x_p)$	$F(k_1, k_2, ..., k_p) = G(k_1, k_2, ..., k_p)$ $\pm P(k_1, k_2, ..., k_p)$
$f(x_1, x_2, ..., x_p) = \frac{\partial^{r_1+r_2+...+r_p} g(x_1,x_2,...,x_p)}{\partial x_1^{r_1} \partial x_2^{r_2} ... \partial x_p^{r_p}}$	$F(k_1, k_2, ..., k_p) = \frac{(k_1+r_1)!...(k_p+r_p)!}{k_1!...k_p!}$ $(k_1 + r_1, ..., k_p + r_p)$
$f(x_1, x_2, ..., x_p) = \prod_{l=1}^{p} x_l^{e_l}$	$F(k_1, k_2, ..., k_p) = \delta(k_1 - e_1,$ $k_2 - e_2, ..., k_p - e_p)$

Table 1: Theorems and operations performed in the p-dimensional DTM.

We now give some important operations and theorems performed in the p-dimensional DTM in Table 1. Those have been derived using the definition in (10) together with previously obtained results [14].

In the table

$$\delta(k_1 - e_1, k_2 - e_2, ..., k_p - e_p) = \begin{cases} 1, & \text{if } k_i = e_i \text{ for } i = 1, 2, .., p. \\ 0, & \text{otherwise.} \end{cases}$$

4 Approximate Analytical Solutions

4.1 Constant and linear function thermal conductivity

The work presented in this section will cover two cases, namely, the linear model case with $\beta = 0$, and the nonlinear case with $\beta \neq 0$. Equation (1) may be given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[(1 + \beta\theta)\frac{\partial \theta}{\partial x}\right] + E^2 \frac{\partial}{\partial y}\left[(1 + \beta\theta)\frac{\partial \theta}{\partial y}\right], \tag{14}$$

subject to the conditions (2) - (6). We now apply the three-dimensional DTM to the governing equation (14) and the above mentioned conditions to obtain the approximate analytical solution

$$\begin{aligned} \theta(\tau, x, y) = \; & c\tau + c\tau y + c\tau y^2 + c\tau y^3 + c\tau y^4 + c\tau y^5 + c\tau y^6 + c\tau y^7 + ... \\ & + c\tau x^2 - \frac{Bic^{m+1}}{(1 + \beta c)}\tau y x^2 - \frac{5c}{E^2}\tau y^2 x^2 + \frac{5Bic^{m+1}}{3E^2(1 + \beta c)}\tau y^3 x^2 + ... \\ & + c\tau x^3 - \frac{Bic^{m+1}}{(1 + \beta c)}\tau y x^3 - \frac{9c}{E^2}\tau y^2 x^3 + \frac{3Bic^{m+1}}{E^2(1 + \beta c)}\tau y^3 x^3 + ... \\ & \vdots \end{aligned} \tag{15}$$

For this problem we will choose the boundary $x = 1$. Along this boundary c must satisfy the equation

$$c\tau + c\tau y + c\tau y^2 + ... + c\tau - \frac{Bic^{m+1}}{(1 + \beta c)}\tau y - \frac{5c}{2E^2}\tau y^2 ... + c\tau - \frac{Bic^{m+1}}{(1 + \beta c)}\tau y - \frac{9c}{E^2}\tau y^2 + ... = 1. \tag{16}$$

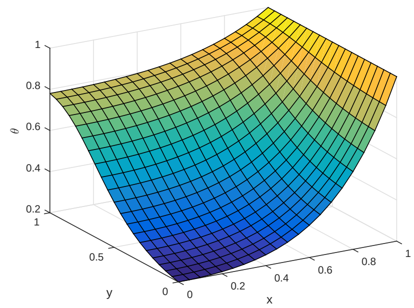

Figure 1: Approximate analytical solutions for a two-dimensional rectangular fin with a constant thermal conductivity ($\beta = 0$) for $\tau = 0.4$. The parameters are set such that $E = 2.8$, $Bi = 0.2$, and $m = 3$.

(y,x)	0	0.2	0.4	0.6	0.8	1
0	0.2000	0.2100	0.2520	0.3567	0.5779	1
0.2	0.2493	0.2590	0.2993	0.3986	0.6064	1
0.4	0.3465	0.3558	0.3933	0.4829	0.6646	1
0.6	0.5075	0.5157	0.5475	0.6194	0.7574	1
0.8	0.6883	0.6943	0.7168	0.7650	0.8528	1
1	0.7801	0.7837	0.7975	0.8286	0.8894	1

Table 2: Approximate analytical solutions for a two-dimensional rectangular fin with a constant thermal conductivity for $\tau = 0.4$.

Upon solution of (16) one obtains an expression for $\theta(\tau, x, y)$.The solution $\theta(\tau, x, y)$ will be discontinuous in the y direction. Taking the first six terms in every direction, that is, taking the first 216 terms of the series, we give the profile and plot for the case $\beta = 0$ over the (x, y) plane. The solution is depicted in Figures 1 and 2, and the numerical account is provided in Table 2.

4.2 Power law thermal conductivity

In this section we focus on the rectangular fin with a power law thermal conductivity. The problem is given by the equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^n \frac{\partial \theta}{\partial x}\right] + E^2 \frac{\partial}{\partial y}\left[\theta^n \frac{\partial \theta}{\partial y}\right], \tag{17}$$

which is subject to the conditions presented in (2)- (6). Applying the three-dimensional DTM to the governing equation (17) and the above mentioned conditions one obtains

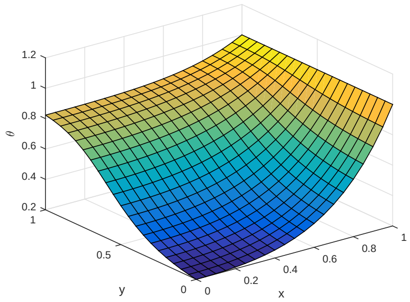

Figure 2: Approximate analytical solutions for a two-dimensional rectangular fin with a linear function thermal conductivity ($\beta = 2$) for $\tau = 0.4$. The parameters are set such that $E = 3.2$, $Bi = 0.2$, and $m = 3$.

the series solution

$$\begin{aligned}\theta(\tau, x, y) = \quad & c\tau - Bic^m\tau y + c\tau y^2 + c\tau y^3 + c\tau y^4 + c\tau y^5 + c\tau y^6 + c\tau y^7 + ... \\ & c\tau x^2 - Bic^m\tau y x^2 - \frac{18c + 2E^2c - 2BiE^2c^m - 3}{2E^2}\tau y^2 x^2 + ... \\ & c\tau x^3 - Bic^m\tau y x^3 - \frac{40c + 2E^2c - 2BiE^2c^m - 3}{2E^2}\tau y^2 x^3 + ... \\ & \vdots \end{aligned} \tag{18}$$

In order to find a value for c, we choose the boundary $x = 1$. This results in an equation in terms of τ and y given by

$$\begin{aligned} & c\tau - Bic^m\tau y + ... + c\tau - Bic^m\tau y + ... + c\tau - Bic^m\tau y + ... = 1. \\ & \vdots \end{aligned} \tag{19}$$

The obtained value of c can then be substituted back into (18) to get an expression for $\theta(\tau, x, y)$. The solution is depicted in Figure 3. It turns out that the 3D DTM works well only when $n = 1$, which is equivalent to rescaling of the linear thermal conductivity in equation (15). A question arises of whether this observation is the only case in this problem for which DTM is efficient. Figures 4 and 5 depict the temperature profiles for transient heat transfer.

5 Conclusion

As far as we know, the 3D DTM has never been applied to transient problems of heat transfer in 2D straight fins with temperature-dependent thermal properties. We have demonstrated that these methods are effective in providing approximate analytical solutions. Figures 1 to 3 provide the temperature profile of heat transfer in the 2D rectangular straight fins. One may notice that the transient solutions approach the steady state solution in Figures 4 and 5. Numerical results are provided in Table 2. The dependency of the thermal properties on temperature rendered the considered equation nonlinear. The effects of the Biot number and aspect ratio were studied in [3]. Similar results are

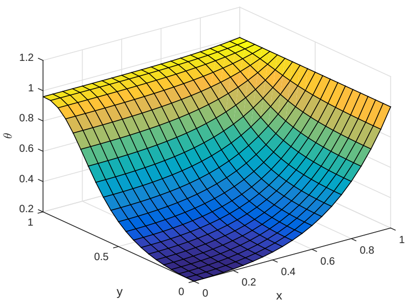

Figure 3: Approximate analytical solutions for a two-dimensional rectangular fin with a power law thermal conductivity for $\tau = 0.4$. The parameters are set such that $E = 25$, $Bi = 0.2$, and $m = 3$.

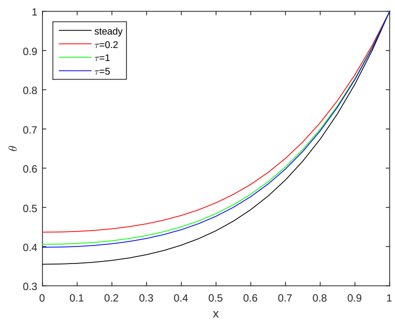

Figure 4: Plots of the transient profile for varying τ, against the steady state profile along $y = 0.5$. The parameters are set such that $E = 2.9$, $Bi = 0.2$, and $m = 3$.

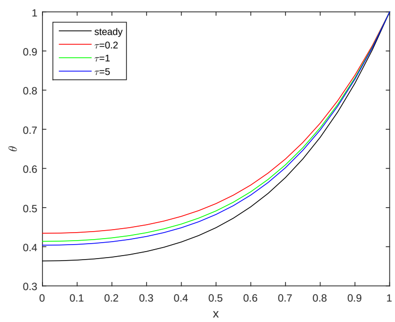

Figure 5: Plots of the transient profile for varying τ, against the steady state profile along $y = 0.5$. The parameters are set such that $E = 2.9$, $Bi = 0.1$, and $m = 2$.

obtained in this study, namely, that the fin performance decreases with the increased aspect ratio and the large Biot number yields a decreased fin efficiency.

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Mathematical Analysis of a Differential Equation Modeling Charged Elements Aggregating in a Relativistic Zero-Magnetic Field

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Abstract: We analyze, in spaces of distributions with finite higher moments, discrete mass and momentum dependent equations describing the movement of charged particles (electrons, ions) aggregating and moving in a relativistic zero-magnetic field. The model is a combination of two processes (kinetic and aggregation), each of which is proven to be separately conservative. Under specific hypothesis, notably on the relativistic work and aggregation rate, we prove existence results for the full model using the perturbation theory and the subordination principle. This result may have a great impact, especially in the full control of the total number of charged particles described by the model.

Keywords: *fractional differential model; magnetic field; perturbation; kinetic processes; subordination principle; aggregation; well-posedness.*

Mathematics Subject Classification (2010): 26A33, 12H20, 34D10, 46S20.

1 Introduction

It is well known [1] that magnetic fields can be produced by charged particles moving in the space. The particles such as electrons or ions, produce complicated but well known magnetic fields that depend on their charge, and their momentum. There are numerous applications and implications of the effects caused by the movements of charged particles in (zero) magnetic fields. The most common example, in consequence of the recent discoveries in the technology of ultrahigh intensity lasers and high current relativistic

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charged bunch sources, is the use of laser pulses together with charged bunches for excitation of strong waves (for example, plasma containing charged particles). The excited waves can be used, for example, for acceleration of charged particles and focusing of bunches [2,3]. Another example in optics is the production of pulses of light of extremely short duration using the mode-locking technique [3]. In biophysics it was proved [4] that the 250-fold screening of the geomagnetic field, which is a "zero" magnetic field with an induction, affects early embryogenesis and the capacity of some animals (a mouse, for instance) to reproduce.

On the other side, various types of pure aggregation equations have been comprehensively analyzed in numerous works (see, e.g., [5–12]). Conservative and nonconservative regimes for pure fragmentation equations have been thoroughly investigated, sometime leading to dishonesty in the process, that is, a process in which models are based on the principle of conservation of mass (individuals, or particles) but which generate solutions that are not conservative.

It is possible to combine the two processes described above into one unique model (the full model). However the analysis and the well posedness of this model are still hardly explored in the domain of mathematical and abstract analysis. Kinetic-type models with diffusion, growth or decay were globally investigated in [13–16], where the authors showed that the transport part does not affect the breach of the conservation laws.

At a macroscopic level, the discrete mass of charged particles (molar or relative molar mass) can be considered during the modeling. Thus, we obtain the following generalized model derived from the combination of Vlasov-Maxwell equations [17] and aggradation equation [18]:

$$\begin{aligned} D_t^\alpha g(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\ &\quad + \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\ g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t\in\mathbb{R}, \quad n=1,2,3\ldots, \end{aligned} \tag{1}$$

where D_t^α is defined as

$$D_t^\alpha g(t,x,p,n) = \frac{\partial^\alpha}{\partial t^\alpha} g(t,x,p,n) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\frac{\partial}{\partial r} g(r,x,p,n)dr, \tag{2}$$

with $0<\alpha\le 1$ and represents the fractional derivative of the function g in the sense of Caputo [19], where Γ is the gamma-function $\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt$. Moreover, the distribution function $g_n \equiv g(t,x,p,n)$ describes the density of groups of size n, that is, the number of particles (electrons or ions) having approximately the momentum p near the position x at time t. Here the independent variables (x,p,n) take values in a set $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{N}$ and γ is a Lorentz factor. We assume that the mass n of a cluster in motion is dependent on γ and the rest mass n_0, $n=\gamma n_0$. This implies that the relativistic momentum relation takes the same form as for the classical momentum, $p=\gamma p_0$. $a_n = a(x,p,n)\ge 0$ is the average aggregation rate, that is, the average number at which clusters of size n undergo splitting, $b_{n,m}=b(x,p,n,m)\ge 0$ is the average number of n-groups produced upon the splitting of m-groups. Equation (3) is really complex: the first member on its right-hand side represents the kinetic process due to the effect of charged particles in the relativistic zero-magnetic field E, while the second term represents the fission of groups of size n (the loss due to the fragmentation) and the third term is the fission to form groups of size n (the gain due to the fragmentation). The analysis of such a model required us to proceed step by step as we will see in the following sections. To analyse the generalized model (1) with $0 < \alpha \leq 1$, we need to start with the case $\alpha = 1$. We shall therefore fully study the well-posedness for the case $\alpha = 1$ and then extend the analysis to the general case $0 < \alpha \leq 1$ by exploiting the subordination principle [6, 20–22].

2 Existence Results: The Case $\alpha = 1$

2.1 Well-posedness of the full model

The case $\alpha = 1$ yields from (1) the following model

$$\begin{aligned}
\frac{\partial g}{\partial t}(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\
&\quad + \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\
g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t \in \mathbb{R}, \quad n = 1,2,3\ldots.
\end{aligned} \tag{3}$$

Throughout this work we assume that the following hypotheses are satisfied.

(**H1**): $b_{n,m} = 0$ for all $m \leq n$ (since a group of size $m \leq n$ cannot split to form a group of size n);

(**H2**): $a_1 = 0$ (a cluster of size one cannot split);

(**H3**): $\sum_{m=1}^{n-1} mb_{m,n} = n,\ (n = 2,3,...)$, (the sum of all individuals obtained by fragmentation of an n-group is equal to n);

The total number of particles, no matter the momentum in the space, is given by

$$U(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} ng(t,x,p,n)dxdp = \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(t,x,p,n)dxdp.$$

This number is normally not changed by interactions among groups, so we expect the following conservation law to be satisfied:

$$\frac{d}{dt}U(t) = 0. \tag{4}$$

Since $g_n = g(t,x,p,n)$ is the density of groups of size n with the momentum p near the position x at time t and the total number of particles is expected to be conserved, it is appropriate to work in the Banach space

$$\mathcal{X}_1 := \{\mathbf{h} = (h_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N} \ni (x,p,n) \to h_n(x,p),\ \|\mathbf{h}\|_1 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n|h_n(x,p)|dxdp < \infty\}. \tag{5}$$

We choose to restrict our analysis to a smaller class of functions, the class of distributions with finite higher moments

$$\{\ \mathcal{X}_r := \{\mathbf{h} = (h_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N} \ni (x,p,n) \to h_n(x,p),\ \|\mathbf{h}\|_r := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n^r|h_n(x,p)|dxdp < \infty\}, \tag{6}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x,p,n) \longrightarrow g(x,p,n) = g_n(x,p)$ is such that $\mathbf{g} = (g_n(x,p))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$ we can rewrite (3) in a more compact form

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{g} &= \mathbf{T}\mathbf{g} - \mathcal{A}\mathbf{g} + \mathfrak{B}\mathcal{A}\mathbf{g} := \mathbf{T}\mathbf{g} + \mathcal{F}\mathbf{g}, \\ \mathbf{g}_{|_{t=0}} &= \overset{\circ}{\mathbf{g}}. \end{aligned} \tag{7}$$

Here $\mathbf{g}$ is the vector $(g(t,x,p,n))_{n\in\mathbb{N}}$, $\mathcal{A}$ is the diagonal matrix $(a_n)_{n\in\mathbb{N}}$, $\mathfrak{B} = (b_{n,m})_{1\leq n\leq m-1, m\geq 2}$, $\mathbf{T}$ is the transport expression defined as $(g(t,x,p,n))_{n\in\mathbb{N}} \longrightarrow \left(\tilde{\mathcal{T}}_n[g(t,x,p,n)]\right)_{n=1}^{\infty}$ with

$$\tilde{\mathcal{T}}_n[g(t,x,p,n)] := -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p}. \tag{8}$$

$\overset{\circ}{\mathbf{g}}$ is the initial vector $(\overset{\circ}{g}_n(x,p))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_r$ and $\mathcal{F}$ is the fragmentation expression defined by

$$\mathcal{F}\mathbf{g} := \left(-a_n g(t,x,p,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m g(t,x,p,m)\right)_{n=1}^{\infty}. \tag{9}$$

Proposition 2.1 *The fragmentation model described by* (9) *is formally conservative.*

Proof. We aim to show that (4) is satisfied, that is,

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n g(t,x,p,n)dxdp = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\frac{\partial}{\partial t}g(t,x,p,n)dxdp = 0.$$

It suffices to show that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=1}^{\infty} a_m|g_m(x,p)|mdxdp = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\left(\sum_{m=n+1}^{\infty} b_{n,m}a_m|g_m(x,p)|\right)dxdp.$$

Making use of assumptions (**H1**)–(**H3**), we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\left(\sum_{m=n+1}^{\infty} b_{n,m}a_m|g_m(x,p)|\right)dxdp \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x,p)|\left(\sum_{n=1}^{\infty} nb_{n,m}\right)dxdp \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x,p)|\left(\sum_{n=1}^{m-1} nb_{n,m}\right)dxdp \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x,p)|mdxdp \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=1}^{\infty} a_m|g_m(x,p)|mdxdp, \end{aligned} \tag{10}$$

which ends the proof.

In this work, for any subspace $S \subseteq \mathcal{X}_r$, we will denote by S_+ the subset of S defined as $S_+ = \{\mathbf{h} = (h_n)_{n=1}^{\infty} \in S; h_n(x,p) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{h} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{h}) := \sum_{n=1}^{\infty} n^q h_n$$

of all orders $q \in [0, r]$. Imposing $r > 1$ ensures that a significant amount of mass after fragmentation is concentrated in small particles. This has the physical interpretation that surface effects are reduced, i.e. it is unlikely that a large cluster will fragment into large groups, therefore making more clusters with small sizes and concentrated at the origin. In $\mathcal{X}_r$, we define the operators $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}\mathbf{h} := (a_n h_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{h} \in \mathcal{X}_r : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |h_n(x,p)| dx dp < \infty\}; \tag{11}$$

$$\mathbf{B}\mathbf{h} := (B_n h_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m h_m \right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{12}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (H1)-(H3). Now let us prove that $\mathbf{B}$ is well defined on $D(\mathbf{A})$. Using the condition (H1)–(H3), we can prove that [5]

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{13}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. Using this inequality we have, for every $\mathbf{h} \in D(\mathbf{A})$,

$$\begin{aligned}
&\|\mathbf{B}\mathbf{h}\|_r \\
= \;& \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |h_m(x,p)| \right) dx dp \\
= \;& \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| \left(\sum_{n=1}^{\infty} n^r b_{n,m} \right) dx dp \\
= \;& \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| \left(\sum_{n=1}^{m-1} n^r b_{n,m} \right) dx dp = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| m^r dx dp \\
= \;& \|\mathbf{A}\mathbf{h}\|_r < \infty.
\end{aligned}$$

Then $\|\mathbf{B}\mathbf{h}\|_r \leq \|\mathbf{A}\mathbf{h}\|_r$, for all $\mathbf{h} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A} + \mathbf{B},\, D(\mathbf{A}))$ is well-defined.

3 Analysis of the Transport Operator in $\Lambda = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N}$

Our primary objective in this section is to analyze the solvability of the Cauchy problem for the transport equation

$$\frac{\partial}{\partial t} g(t,x,p,n) = -\frac{\gamma p}{n} \frac{\partial g(t,x,p,n)}{\partial x} + qE \frac{\partial g(t,x,p,n)}{\partial p}, \tag{14}$$

$$g(0,x,p,m) = \overset{o}{g}_n(x,p), \quad t \in \mathbb{R}, \quad n = 1,2,3\ldots$$

or its compact form

$$\frac{\partial}{\partial t}\mathbf{p} = \mathbf{T}\mathbf{p}, \qquad \mathbf{p}_{|_{t=0}} = \overset{\circ}{\mathbf{p}}. \tag{15}$$

in the space $\mathcal{X}_r$.

3.1 Setting

We note that the operators on the right-hand side of (7) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus we need to work with parameter-dependent operators that can be "glued"together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter. Let us consider the space $\mathcal{X} := L_g(S,X)$ where $1 \le p < \infty$, (S,dm) is a measure space and X is a Banach space. Let us suppose that we are given a family of operators $\{(A_s, D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae:

$$\mathcal{D}(\mathbb{A}) := \{h \in \mathcal{X}; h(s) \in D(A_s) \text{ for almost every } s \in S, \ \ \mathbb{A}h \in \mathcal{X}\}, \tag{16}$$

and, for $h \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}h)(s) := A_s h(s), \tag{17}$$

for every $s \in S$. We have the following proposition.

Proposition 3.1 *(see [5, 13, 14]). If for almost any $s \in S$ the operator A_s is m-dissipative in X, and the function $s \longrightarrow R(\lambda, A_s)h(s)$ is measurable for any $\lambda > 0$ and $h \in \mathcal{X}$, then the operator $\mathbb{A}$ is an m-dissipative operator in $\mathcal{X}$. If $(G_s(t))_{t\ge 0}$ and $(\mathcal{G}(t))_{t\ge 0}$ are the semigroups generated by A_s and $\mathbb{A}$, respectively, then for almost every $s \in S$, $t \ge 0$, and $h \in \mathcal{X}$ we have*

$$[\mathcal{G}(t)h](s) := G_s(t)h(s). \tag{18}$$

Using the above ideas, we introduce relevant operators in the present applications. In the transport part of (7), the variable n is the parameter and x is the main variable. We set

$$\mathbb{X} := L_1(\mathbb{R}^3 \times \mathbb{R}^3, dxdp) := \{\psi : \|\psi\| = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} |\psi(x,p)|dxdp < \infty\}$$

and define in $\mathbb{X}$ the operators $(\mathcal{T}_n, D(\mathcal{T}_n))$ as

$$\begin{aligned} &\mathcal{T}_n g_n = \tilde{\mathcal{T}}_n g_n, \quad \text{with } \tilde{\mathcal{T}}_n g_n \ \text{ represented by (8)} \\ &D(\mathcal{T}_n) := \{g_n \in \mathbb{X}, \ \ \mathcal{T}_n g_n \in \mathbb{X}\}, \quad n \in \mathbb{N}. \end{aligned} \tag{19}$$

Then we introduce the operator $\mathbf{T}$ in $\mathcal{X}_r$ defined by

$$\begin{aligned} &\mathbf{T}\mathbf{g} = (\mathcal{T}_n g_n)_{n\in\mathbb{N}}, \\ &D(\mathbf{T}) = \{\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \mathcal{X}_r, \ g_n \in D(\mathcal{T}_n) \text{ for almost every } n \in \mathbb{N}, \ \mathbf{T}\mathbf{g} \in \mathcal{X}_r\}. \end{aligned} \tag{20}$$

Making use of Proposition 3.1, we can take $\mathbb{A} = \mathbf{T}$, $\mathcal{X} = \mathcal{X}_r = L_1(\mathbb{N}, \mathbb{X}) = L_1(\Lambda, d\mu dm_r) = L_1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N}, d\mu dm_r)$, where $\mathbb{N}$ is equipped with the weighted counting measure dm_r with weight n^r and $d\mu = dxdp = d\mathbf{z}$ is the Lebesgue measure in $\mathbb{R}^6$. In the notation of the proposition, $(\mathbb{N}, dm_r) = (S, dm)$, $\mathbb{X} = X$ and $A_s = \mathcal{T}_n$, therefore $(\mathcal{T}_n, D(\mathcal{T}_n))_{n\in\mathbb{N}}$ is a family of operators in $\mathbb{X}$ and using (17), we have

$$(\mathbf{T}\mathbf{g})_n := \mathcal{T}_n g_n. \tag{21}$$

Here, $\mathcal{T}_n g_n$ is understood in the sense of distribution. Now we can properly study the transport operator $\mathbf{T}$. Let us fix $n \in \mathbb{N}$. We consider the function $\digamma_n : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\digamma_n(x,p) = (-\frac{\gamma p}{n},\ qE(x,p))$. For each $n \in \mathbb{N}$, we assume the following:

(**H4**): $\digamma_n$ is globally Lipschitz continuous;

(**H5**): $\digamma_n \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)$; and $\mathrm{div}\digamma_n \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$;

(**H6**): $\overset{o}{g}_n \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Let us set $\mathbf{z} = (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$, we rely on the following definition.

Definition 3.1 A function g_n is called a (weak) L^∞-solution to (14) if $g_n \in L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and moreover, for every test function $\Psi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\int\limits_{\mathbb{R}^6} \Psi(\mathbf{z}) g_n(t,\mathbf{z}) d\mathbf{z} = \int\limits_{\mathbb{R}^6} \Psi(\mathbf{z}) \overset{o}{g}_n(\mathbf{z}) d\mathbf{z} + \int\limits_0^t d\sigma \int\limits_{\mathbb{R}^6} g_n(\sigma,\mathbf{z})(\digamma_n(\sigma,\mathbf{z})\cdot\nabla\Psi(\mathbf{z}) + \Psi(\mathbf{z})\mathrm{div}\digamma_n(\sigma,\mathbf{z}))d\mathbf{z},$$

$t \in \mathbb{R}$.

Lemma 3.1 *In $\mathbb{X}$ the existence and uniqueness of L^∞-solutions to (14) hold if the above assumptions (**H4**)-(**H6**) are satisfied.*

We prove it by uniquely solving the characteristic ordinary differential equations

$$\begin{aligned} \dot{\gimel}_n(s) &= \digamma_n(\gimel_n(s)), \quad s \in \mathbb{R}, \\ \gimel_n(t) &= \mathbf{z}, \end{aligned} \tag{22}$$

with $\mathbf{z} \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}$, which have one and only one solution $\gimel_n(s)$ taking values in $\mathbb{R}^3 \times \mathbb{R}^3$. Thus we find the flow $(\phi^n_{t,s})$, $t,\ s \in \mathbb{R}$ generated by $\digamma_n$ with $\phi^n_{t,s} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$, that is,

1. $\phi^n_{t,s}(\mathbf{z}) = \gimel_n(s)$, where $\gimel_n(s)$ $s \in \mathbb{R}$, solves (22),
2. $\phi^n_{t,s}(\mathbf{z}) = \phi^n_{\tau,s}(\phi^n_{t,\tau}(\mathbf{z}))$, $t,\ s,\ \tau \in \mathbb{R}$,
3. The transformations $\phi^n_{t,s} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$ are Lipschitz-homeomorphism.

Note that the functions $\phi^n_{t,s}$ possess many more desirable properties as listed in [5, 23–25] that are relevant for studying the transport operator in $\mathcal{X}_r$. Then making use of $g_n(t, \phi^n_{0,t}(\mathbf{z})) = \overset{o}{g}_n(\mathbf{z})$, we obtain the unique solution to (14) given by

$$g_n(t,x,p) = \overset{o}{g}_n((\phi^n_{0,t})^{-1}(x,p)).$$

It is obvious that this solution belongs to $D(\mathcal{T}_n)$. Therefore the operator $(\mathcal{T}_n, D(\mathcal{T}_n))$ generates a semigroup given by

$$[G_{\mathcal{T}_n}(t) g_n](x,p) = g_n((\phi^n_{0,t})^{-1}(x,p)), \tag{23}$$

$g_n \in \mathbb{X}$. For existence and uniqueness in the full space $\mathcal{X}_r$, we state the following.

Proposition 3.2 *Under the conditions of Lemma 3.1, there is one and only one L^∞-solution to (15) holding in $\mathcal{X}_r$ and belonging to $D(\boldsymbol{T})$.*

Proof. The proof follows immediately from relation (21) and Lemma 3.1

4 Generalization: Existence Results for $0 < \alpha \leq 1$

Now, as we have fully analized the special case (3), proved its well-posedness and shown its existence results, we can come back to the general model (1):

$$\begin{aligned} D_t^\alpha g(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\ &\quad + \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\ g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t\in\mathbb{R}, \quad n = 1,2,3\ldots. \end{aligned} \tag{24}$$

This model can be written in the same way as the perturbed transport equation (7) above to read as

$$\begin{aligned} D_t^\alpha \mathbf{g} &= \mathbf{Tg} - \mathbf{Ag} + \mathbf{Bg}, \\ \mathbf{g}_{|t=0} &= \overset{o}{\mathbf{g}}. \end{aligned} \tag{25}$$

To process we need the following.

Definition 4.1 ([21, 26]) Consider an operator Q applied in the fractional model

$$D_t^\alpha(g(x,t)) = Qg(x,t), \quad 0<\alpha<1, \ x, \ t>0, \tag{26}$$

subject to the initial condition

$$g(x,0) = f(x), \qquad x>0 \tag{27}$$

and defined in a Banach space X_1. A family $(G_Q(t))_{t>0}$ of bounded operators on X_1 is called a solution operator of the fractional Cauchy problem (26)-(27) if

(i) : $G_Q(0) = I_{X_1}$;
(ii) : $G_Q(t)$ is strongly continuous for every $t \geq 0$;
(iii) : $QG_Q(t)f = G_Q(t)Qf$ for all $f \in D(Q)$;
(iv) : $G_Q(t)D(Q) \subset D(Q)$;
(v) : $G_Q(t)f$ is a (classical) solution of the model (26) – (27) for all $f \in D(Q)$, $t \geq 0$.

It is well known [5] that an operator $\widetilde{Q} \in \mathcal{G}(M,\omega)$ means $\widetilde{Q}$ generates a C_0-semigroup $(G_{\widetilde{Q}}(t))_{t>0}$ so that there exists $M>0$ and ω such that

$$\|G_{\widetilde{Q}}(t)\| \leq Me^{\omega t}. \tag{28}$$

Whence, by analogy, if the fractional Cauchy problem (26)-(27) has a solution operator $(G_Q(t))_{t>0}$ verifying (28), then we say that $Q \in \mathcal{G}^\alpha(M,\omega)$. The solution operator $(G_Q(t))_{t>0}$ is positive if

$$G_Q(t) \geq 0$$

and contractive if

$$\|G_Q(t)\|_{X_1} \leq 1, \tag{29}$$

and we say $Q \in \mathcal{G}^\alpha(1,0)$.

This leads to the following existence result.

Proposition 4.1 *Assume that the conditions of Lemma 3.1 hold, then for (25) there is an extension* $(\mathcal{K}_\alpha, D(\mathcal{K}_\alpha))$ *of* $(\boldsymbol{T} - \boldsymbol{A} + \boldsymbol{B},\ \ D(\boldsymbol{T}) \cap D(\boldsymbol{A}))$ *that generates a positive solution operator on* $\mathcal{X}_r$*, denoted by* $(G_{\mathcal{K}_\alpha}(t))_{t\geq 0}$.

Proof. The proof follows from the subordination principle [6, 20–22], by considering the existence result for (7) with $\alpha = 1$ and extending it to $0 < \alpha \leq 1$.

5 Results and Conclusion

We have analyzed, in the space $\mathcal{X}_r$ of distributions with finite higher moments, the generalized mass dependent discrete model (1), describing the movement of charged particles (electrons, ions) aggregating and moving in a relativistic zero-magnetic field. We showed existence of a solution g to (1) that is positive. Therefore, the evolution of the number of charged particles, given by this solution, is the same as the one predicted by the local law given in (4) which was used to construct the model. This is not always true since the analysis of certain models sometimes leads to the breach of the mass conservation law (called shattering) and that has been attributed to a phase transition creating a dust of "zero-size" particles with nonzero mass [9], which are beyond the model's resolution. Then we can use the full combination model (1) to study and control the dynamics of a number of charged particles moving in a relativistic zero-magnetic field. This work generalizes the preceding ones with the combination of the mass dependent relativistic kinetic and aggregation kernels which were not considered before. This work will therefore help addressing the problem of identifying and characterizing the full generator of our model which is still an unsolved issue.

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Application of Extended Fan Sub-Equation Method to Generalized Zakharov Equation

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Abstract: In this paper, the extended Fan sub-equation method is applied to obtain exact solutions of the generalized Zakharov equation. Applying this method, we obtain various solutions which are benefit to further understand the concepts of the complicated nonlinear physical phenomena. This method is straightforward, and it can be applied to many nonlinear equations. In this work, we use Mathematica for computations and programming.

Keywords: *extended Fan sub-equation method; generalized Zakharov equation; solitary wave solution.*

Mathematics Subject Classification (2010): 35-XX, 35Qxx.

1 Introduction

Nonlinear partial differential equations (PDEs) appear in many fields, such as fluid mechanics, solid state physics, plasma physics, chemical physics, nonlinear optics , and so on. Thus, nonlinear PDEs play an important role in the study of nonlinear science, especially in the study of nonlinear physical science. Exact solutions of nonlinear PDEs can provide much physical information to understand the mechanism that governs these physical models or provide better knowledge of the physical problems and possible applications [2]. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. Therefore, finding exact solutions of nonlinear PDEs has been of great significance. In the past decades, many researchers have paid more attention to various powerful methods for obtaining exact solutions to nonlinear PDEs. Some of the most

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important methods are the Jacobi elliptic method [4], Taylor-series expansion method [6], simplest equation method [9], the transformed rational function method [11], variational iteration method [12], tanh-sech method [14], sine-cosine method [1,15], $\frac{G'}{G}$-expansion method [17], exp function method [7], homotopy analysis method [8], and so on.

Yomba [16] demonstrated that the F-expansion method, the tanh and the extended tanh function method belonged to a class of methods called the sub-equation methods, because we can obtain exact solutions of the complicated nonlinear PDEs in use and study some simple nonlinear ordinary differential equations. These methods consist of solving the nonlinear PDEs under a suggestion that a polynomial in a variable satisfies an equation (named the sub-equation). Fan [5] recently developed a new algebraic method, called the Fan sub-equation method, for obtaining exact analytical solutions to nonlinear equations. These solutions include polynomial solutions, trigonometric periodic wave solutions, exponential solutions, rational solutions, hyperbolic and solitary wave solutions. The powerful Fan sub-equation method is widely applied by many scientists, see [3] and the references therein. In this paper, the extended Fan sub-equation method will be used to find exact solutions for the generalized Zakharov equation. We show the extended Fan sub-equation method is a very powerful mathematical technique for finding exact solutions of nonlinear differential equations. Here the exact solutions of the nonlinear PDEs can be expressed as a polynomial and the degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms in the considered equation. The aim of this paper is to find exact solutions of the generalized Zakharov equation by using the extended Fan sub-equation method as follows.

The form of the generalized Zakharov equation is [10]

$$\begin{cases} iu_t + u_{xx} - 2\alpha|u|^2u + 2uv = 0, \\ v_{tt} - v_{xx} + (|u|^2)_{xx} = 0. \end{cases} \tag{1}$$

Here the coefficient α is a real arbitrary constant. The nonlinear self-interaction in the high-frequency subsystem, such as a term corresponding to a self-focusing effect in plasma physics can be described via the third term of the first equaton in (1). The rest of this paper is organized as follows. In Section 2, we describe the extended Fan sub-equation method for solving nonlinear PDEs. In Section 3, we give an application of the proposed method to the generalized Zakharov equation. In Section 4, some conclusions are given.

2 Extended Fan Sub-Equation Method for Finding the Exact Solutions of Nonlinear PDEs

In this section, we illustrate the basic idea of the extended Fan sub-equation method for solving nonlinear differential equations. We consider a nonlinear PDE in two independent variables x, t and dependent variable u. Then by means of an appropriate transformation, it can be reduced to a nonlinear ordinary differential equation(ODE) as follows:

$$P(u, u', u'', u''', ...) = 0. \tag{2}$$

Here prime denotes the derivative with respect to ξ. Exact solution for this equation can be constructed as follows:

$$u(\xi) = \frac{A_{-n}}{\psi(\xi)^n} + ... + \frac{A_{-1}}{\psi(\xi)} + A_0 + A_1\psi(\xi) + ... + A_n\psi(\xi)^n; \quad A_n \neq 0. \tag{3}$$

Here A_i $(i = 0, 1, 2, \cdots, n)$ are constants to be determined later. Also, $\psi = \psi(\xi)$ satisfies the following ODE:

$$\psi'(\xi) = \epsilon\sqrt{\sum_{i=0}^{4} \omega_i \psi^i}, \tag{4}$$

where $\epsilon = \pm 1$ and ω_i are constants. Thus the derivatives with respect to ξ can be calculated with respect to the variable ψ as follows:

$$\frac{du}{d\xi} = \epsilon\sqrt{\sum_{i=0}^{4} \omega_i \psi^i}\,\frac{du}{d\psi}, \tag{5}$$

$$\frac{d^2u}{d\xi^2} = \frac{1}{2}\sum_{i=0}^{4} i\omega_i \psi^{i-1}\frac{du}{d\psi} + \sum_{i=0}^{4}\omega_i\psi^i\frac{d^2u}{d\psi^2}, \ldots. \tag{6}$$

The solutions of equation (4) are:

- Case 1. When $\omega_0 = \omega_1 = \omega_3 = 0$, we have the following solutions

$$\psi = \sqrt{-\frac{\omega_2}{\omega_4}}\, sech(\sqrt{\omega_2}\xi); \quad \omega_2 > 0,\ \omega_4 < 0, \tag{7}$$

$$\psi = \sqrt{-\frac{\omega_2}{\omega_4}}\, \sec(\sqrt{-\omega_2}\xi); \quad \omega_2 < 0,\ \omega_4 > 0, \tag{8}$$

$$\psi = -\frac{\epsilon}{\sqrt{\omega_4}\xi}; \quad \omega_2 = 0,\ \omega_4 > 0. \tag{9}$$

- Case 2. When $\omega_1 = \omega_3 = 0$, $\omega_0 = \frac{\omega_2^2}{4\omega_4}$, we have the following solutions

$$\psi = \epsilon\sqrt{-\frac{\omega_2}{2\omega_4}}\, \tanh(\sqrt{-\frac{\omega_2}{2}}\xi); \quad \omega_2 < 0,\ \omega_4 > 0, \tag{10}$$

$$\psi = \epsilon\sqrt{\frac{\omega_2}{2\omega_4}}\, \tan(\sqrt{\frac{\omega_2}{2}}\xi); \quad \omega_2 > 0,\ \omega_4 < 0. \tag{11}$$

- Case 3. When $\omega_1 = \omega_3 = 0$, we have the following solutions

$$\psi = \sqrt{-\frac{\omega_2 m^2}{\omega_4(2m^2-1)}}\, cn(\sqrt{\frac{\omega_2}{2m^2-1}}\xi, m);\ \omega_2 > 0, \omega_4 < 0, \omega_0 = \frac{1-m^2}{(2m^2-1)^2}, \tag{12}$$

$$\psi = \epsilon\sqrt{-\frac{\omega_2 m^2}{\omega_4(m^2+1)}}\, sn(\sqrt{-\frac{\omega_2}{m^2+1}}\xi, m);\ \omega_2 < 0, \omega_4 > 0, \omega_0 = \frac{\omega_2^2 m^2}{2\omega_4(m^2+1)}, \tag{13}$$

where m is the modulus. In limiting cases, the Jacobi elliptic function solutions can degenerate to hyperbolic function solutions and trigonometric function solutions, for example, $sn(\xi) \to tanh(\xi)$ as $m \to 1$, and $sn(\xi) \to sin(\xi)$ as $m \to 0$.

- Case 4. When $\omega_0 = \omega_1 = \omega_4 = 0$, we have the following solutions

$$\psi = -\frac{\omega_2}{\omega_3}\, sech^2(\frac{\sqrt{\omega_2}}{2}\xi); \quad \omega_2 > 0, \tag{14}$$

$$\psi = -\frac{\omega_2}{\omega_3}\, \sec^2(\frac{\sqrt{-\omega_2}}{2}\xi); \quad \omega_2 < 0, \tag{15}$$

$$\psi = \frac{1}{\omega_3 \xi^2}; \quad \omega_2 = 0. \tag{16}$$

Substituting (3)-(6) into equation (2) and collecting all terms with the same powers of ψ together, the left-hand side of equation (2) is converted into a polynomial. After setting each coefficients of this polynomial to zero, we obtain a set of algebraic equations in terms of A_n (n=0,1,2,...,n). Solving the system of algebraic equations and then substituting the results and the general solutions of (7)-(16) into equation (3), gives solutions of equation (2).

3 Application of the Extended Fan Sub-Equation Method

In this section, we apply the extended Fan sub-equation method for solving the generalized Zakharov equation as follows.

Example 3.1 We consider the generalized Zakharov equation in the form

$$iu_t + u_{xx} - 2\alpha|u|^2 u + 2uv = 0, \tag{17}$$

$$v_{tt} - v_{xx} + (|u|^2)_{xx} = 0. \tag{18}$$

For obtaining exact solutions of (17) and (18), we use

$$u(x,t) = \rho(x,t)\, e^{i(kx+\lambda t)}, \tag{19}$$

where k, λ are constants which should to be determined later. Substituting equation (19) into equations (17) and (18), we get

$$i(\rho_t + 2k\rho_x) + \rho_{xx} - (\lambda + k^2)\rho - 2\alpha\rho^3 + 2\rho v = 0, \tag{20}$$

$$v_{tt} - v_{xx} + \rho^2_{xx} = 0. \tag{21}$$

We take the traveling wave transformation

$$\rho = \rho(\xi), \quad v = v(\xi), \quad \xi = \omega(x - 2kt), \tag{22}$$

here ω is a constant which should be determined later. Then equations (20) and (21) are reduced into two nonlinear ODEs

$$\omega\rho'' - (\lambda + k^2)\rho - 2\alpha\rho^3 + 2\rho v = 0, \tag{23}$$

$$(4k^2 - 1)v'' + (\rho^2)'' = 0, \tag{24}$$

integrating equation (24) with respect to ξ, we have

$$v = \frac{\rho^2}{1 - 4k^2}. \tag{25}$$

Substituting equation (25) into equation (23) yields

$$\omega^2\rho'' - (\lambda + k^2)\rho - 2\alpha\rho^3 + \frac{2}{1-4k^2}\rho^3 = 0. \tag{26}$$

Balancing ρ'' with ρ^3 in (26) gives n=1. Thus the extended Fan sub-equation method admits the following solution

$$\rho(\xi) = \frac{A_{-1}}{\psi(\xi)} + A_0 + A_1\psi(\xi), \tag{27}$$

where A_{-1}, A_0, A_1 are constants to be determined and ψ satisfies equation (4).

By substituting equations (27) and (4) into equation (26), collecting the coefficients of ψ^i and setting them to be zero, a set of algebraic equations is obtained. Solving this set of algebraic equations using *Mathematica* [13], we get

- $$A_0 = 0,\ A_1 = \frac{\omega\sqrt{\omega_4\beta}}{\sqrt{1+\alpha\beta}},\ A_{-1} = \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\,\omega\sqrt{\omega_4(1+\alpha\beta)}},\ \beta = -1+4k^2, \tag{28}$$
  $\omega_0 = \omega_0,\ \omega_1 = \omega_3 = 0,\ \omega_2 = \omega_2,\ \omega_4 \neq 0.$

- $$A_0 = \frac{\sqrt{\beta\gamma}}{4\sqrt{3}},\ A_1 = \frac{\sqrt{3\beta\gamma}\omega^2\omega_3}{2[5\omega^2\omega_2 - 2(\lambda+k^2)]},\ A_{-1} = \frac{\sqrt{\beta\gamma}[-2(\lambda+k^2)-\omega^2\omega_2]}{24\sqrt{3}\omega^2\omega_3}, \tag{29}$$
  $\gamma = 10\omega^2\omega_2 - (1+4k),\ \omega_0 = \omega_0,\ \omega_1 = 0,\ \omega_2, \omega_3 \neq 0,\ \omega_4 = \omega_4.$

By using (28), (27) and cases (7)-(13) respectively, we get

$$\begin{aligned}\rho_1(x,t) = {} & \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}} cosh[\sqrt{\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}} \\ & sech[\sqrt{\omega_2}(\omega(x-2kt))],\end{aligned} \tag{30}$$

$$\begin{aligned}\rho_2(x,t) = {} & \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}} cos[\sqrt{-\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}} \\ & sec[\sqrt{\omega_2}(\omega(x-2kt))],\end{aligned} \tag{31}$$

$$\rho_3(x,t) = -\frac{\sqrt{\beta}}{\sqrt{1+\alpha\beta}}\left\{\frac{[(\lambda+k^2)-\omega^2\omega_2](\omega(x-2kt))}{6\,\epsilon\,\omega} + \frac{\epsilon\,\omega}{\omega(x-2kt)}\right\}, \tag{32}$$

$$\begin{aligned}\rho_4(x,t) = {} & \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}} coth[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}} \\ & tanh[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))],\end{aligned} \tag{33}$$

$$\begin{aligned}\rho_5(x,t) = {} & \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}} cot[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}} \\ & tan[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))],\end{aligned} \tag{34}$$

$$\rho_6(x,t) = \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(2m^2-1)}}{6\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m)},$$

$$\frac{\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(2m^2-1)(1+\alpha\beta)}}\,cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m), \tag{35}$$

$$\rho_7(x,t) = \frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(m^2+1)}}{6\,\epsilon\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m)},$$

$$\frac{\epsilon\,\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(m^2+1)(1+\alpha\beta)}}\,sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m). \tag{36}$$

Substituting (30)-(36) into (19) and (25) respectively, we have

$$u_1(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}}cosh[\sqrt{\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}}$$
$$sech[\sqrt{\omega_2}(\omega(x-2kt))]\}e^{i(kx+\lambda t)},$$
$$v_1(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}}cosh[\sqrt{\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}}$$
$$sech[\sqrt{\omega_2}(\omega(x-2kt))]\}^2,$$

$$u_2(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}}cos[\sqrt{-\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}}$$
$$sec[\sqrt{\omega_2}(\omega(x-2kt))]\}e^{i(kx+\lambda t)},$$
$$v_2(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta}}{6\omega\sqrt{-\omega_2(1+\alpha\beta)}}cos[\sqrt{-\omega_2}(\omega(x-2kt))] + \frac{\omega\sqrt{-\omega_2\beta}}{\sqrt{1+\alpha\beta}}$$
$$sec[\sqrt{\omega_2}(\omega(x-2kt))]\}^2,$$

$$u_3(x,t) = \{-\frac{\sqrt{\beta}}{\sqrt{1+\alpha\beta}}\left\{\frac{[(\lambda+k^2)-\omega^2\omega_2](\omega(x-2kt))}{6\,\epsilon\,\omega} + \frac{\epsilon\,\omega}{\omega(x-2kt)}\right\}\}e^{i(kx+\lambda t)},$$
$$v_3(x,t) = \frac{1}{1-4k^2}\{-\frac{\sqrt{\beta}}{\sqrt{1+\alpha\beta}}\left\{\frac{[(\lambda+k^2)-\omega^2\omega_2](\omega(x-2kt))}{6\,\epsilon\,\omega} + \frac{\epsilon\,\omega}{\omega(x-2kt)}\right\}\}^2,$$

$$u_4(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}}coth[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}}$$
$$tanh[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))]\}e^{i(kx+\lambda t)},$$
$$v_4(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}}coth[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}}$$
$$tanh[\sqrt{-\frac{\omega_2}{2}}(\omega(x-2kt))]\}^2.$$

$$u_5(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}}cot[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}}$$
$$tan[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))]\}e^{i(kx+\lambda t)},$$
$$v_5(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{2\beta}}{6\,\epsilon\,\omega\sqrt{-\omega_2(1+\alpha\beta)}}cot[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))] + \frac{\epsilon\,\omega\sqrt{-\omega_2\beta}}{\sqrt{2(1+\alpha\beta)}}$$
$$tan[\sqrt{\frac{\omega_2}{2}}(\omega(x-2kt))]\}^2,$$

$$u_6(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(2m^2-1)}}{6\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m)}$$
$$\frac{\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(2m^2-1)(1+\alpha\beta)}}\,cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m)\}e^{i(kx+\lambda t)},$$
$$v_6(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(2m^2-1)}}{6\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m)}$$
$$\frac{\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(2m^2-1)(1+\alpha\beta)}}\,cn(\sqrt{\frac{\omega_2}{2m^2-1}}\,\omega(x-2kt),m)\}^2,$$

$$u_7(x,t) = \{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(m^2+1)}}{6\,\epsilon\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m)}$$
$$\frac{\epsilon\,\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(m^2+1)(1+\alpha\beta)}}\,sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m)\}e^{i(kx+\lambda t)},$$
$$v_7(x,t) = \frac{1}{1-4k^2}\{\frac{[(\lambda+k^2)-\omega^2\omega_2]\sqrt{\beta(m^2+1)}}{6\,\epsilon\,\omega\sqrt{-\omega_2\,m^2(1+\alpha\beta)}}\frac{1}{sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m)}$$
$$\frac{\epsilon\,\omega\sqrt{-\omega_2 m^2\beta}}{\sqrt{(m^2+1)(1+\alpha\beta)}}\,sn(\sqrt{-\frac{\omega_2}{m^2+1}}\,\omega(x-2kt),m)\}^2.$$

By using (29), (27) and cases (14) and (15) respectively, we get

$$\rho_8(x,t) = \frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cosh^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)] + \frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sech^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)], \tag{37}$$
$$\rho_9(x,t) = \frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cos^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)] + \frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sec^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)]. \tag{38}$$

Substituting (37)-(38) into (19) and (25) respectively, we have

$$u_8(x,t)=\{\frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cosh^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)]+\frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sech^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)]\}e^{i(kx+\lambda t)},$$
$$v_8(x,t)=\frac{1}{1-4k^2}\{\frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cosh^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)]+\frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sech^2[\frac{\sqrt{\omega_2}}{2}\omega(x-2kt)]\}^2,$$

$$u_9(x,t)=\{\frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cos^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)]+\frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sec^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)]\}e^{i(kx+\lambda t)},$$
$$v_9(x,t)=\frac{1}{1-4k^2}\{\frac{[2(\lambda+k^2)+\omega^2\omega_2]\sqrt{\beta\gamma}}{24\sqrt{3}\,\omega^2\omega_2}cos^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)]+\frac{\sqrt{\beta\gamma}}{4\sqrt{3}}$$
$$-\frac{\sqrt{3\beta\gamma}\,\omega^2\omega_2}{2(5\omega^2\omega_2-2(\lambda+k^2))}sec^2[\frac{\sqrt{-\omega_2}}{2}\omega(x-2kt)]\}^2.$$

4 Conclusion

We have applied the extended Fan sub-equation method to solve nonlinear partial differential equations. As an application of the proposed method, some exact analytical solutions of the generalized Zakharov equation are successfully obtained. These solutions include hyperbolic function solutions, trigonometric function solutions and rational function solutions. Moreover, the proposed method is shown to be a simple, yet powerful algorithm for handling the systems of PDEs. *Mathematica* has been used for computations and programming in this paper.

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Comparison of New Iterative Method and Natural Homotopy Perturbation Method for Solving Nonlinear Time-Fractional Wave-Like Equations with Variable Coefficients

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Abstract: In this paper, we present a comparison between the new iterative method (NIM) and the natural homotopy perturbation method (NHPM) for solving nonlinear time-fractional wave-like equations with variable coefficients. The two methods introduced an efficient tool for solving this type of equations. The results show that the NIM has an advantage over the NHPM because it takes less time and uses only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of NHPM. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.

Keywords: *nonlinear time-fractional wave-like equations, Caputo fractional derivative, new iterative method, natural homotopy perturbation method.*

Mathematics Subject Classification (2010): Primary 35L05, 35R11; Secondary 35A35, 26A33.

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1 Introduction

The fractional calculus which deals with derivatives and integrals of arbitrary orders plays a vital role in many fields of applied science and engineering [4]. Recently, nonlinear fractional partial differential equations are successfully applied to many mathematical models in mathematical biology, aerodynamics, rheology, diffusion, electrostatics, electrodynamics, control theory, fluid mechanics, analytical chemistry and so on.

Several analytical and numerical methods have been proposed to solve nonlinear fractional partial differential equations. The most commonly used ones are: the adomian decomposition method (ADM) [8] variational iteration method (VIM) [10], fractional difference method (FDM) [4], homotopy perturbation method (HPM) [3].

In this paper, the main objective is to introduce a comparative study of nonlinear time-fractional wave-like equations with variable coefficients by using the new iterative method (NIM) which uses only the inverse operator and the natural homotopy perturbation method (NHPM) which is a coupling of the natural transform and the homotopy perturbation method (HPM) using He's polynomials.

Consider the following nonlinear time-fractional wave-like equations:

$$\begin{aligned} D_t^\alpha v &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \\ &+ \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) + S(X,t) \end{aligned} \tag{1}$$

with the initial conditions

$$v(X,0) = a_0(X), v_t(X,0) = a_1(X), \tag{2}$$

where D_t^α is the Caputo fractional derivative operator of order α, $1 < \alpha \leq 2$.

Here $X = (x_1, x_2, ..., x_n)$, F_{1ij}, G_{1i} are nonlinear functions of X, t and v, F_{2ij}, G_{2i} are nonlinear functions of derivatives of v with respect to x_i and x_j, respectively. Also H, S are nonlinear functions and k, m, p are integers.

In the classical case, these types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows [7].

2 Basic Definitions

In this section, we give some basic definitions and important properties of fractional calculus theory and natural transform, which will be used in this paper.

Definition 2.1 [4] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as follows:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, t > 0. \tag{3}$$

Definition 2.2 [4] The Caputo fractional derivative operator of order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ of a function $f \in C^n_{-1}$ is defined as follows:

$$D_t^\alpha f(t) = I_t^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}, t > 0. \tag{4}$$

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation:

$$I_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, t > 0. \tag{5}$$

Definition 2.3 [1] The natural transform is defined over the set of functions $A = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$ by the following integral:

$$\mathcal{N}^+ [f(t)] = R^+(s,u) = \frac{1}{u} \int_0^{+\infty} e^{-\frac{st}{u}} f(t) dt, s, u \in (0, \infty). \tag{6}$$

Definition 2.4 [6] The natural transform of the Caputo fractional derivative of order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ is defined as follows:

$$\mathcal{N}^+ [D_t^\alpha f(t)] = R_\alpha^+(s,u) = \frac{s^\alpha}{u^\alpha} R^+(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0^+). \tag{7}$$

3 The New Iterative Method (NIM)

In this section, we introduce the new iterative method for solving equations (1) and (2).

Applying the inverse operator I_t^α on both sides of equation (1) and using (5), we get

$$\begin{aligned} v(X,t) &= \sum_{k=0}^{n-1} v^{(k)}(X,0) \frac{t^k}{k!} + I_t^\alpha \left(\sum_{i,j=1}^{n} F_{1ij}(X,t,v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \right. \\ &\left. + \sum_{i=1}^{n} G_{1i}(X,t,v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) \right) + I_t^\alpha \left(S(X,t) \right). \end{aligned} \tag{8}$$

Let

$$\begin{aligned} g(X,t) &= \sum_{k=0}^{n-1} v^{(k)}(X,0) \frac{t^k}{k!} + I_t^\alpha \left(S(X,t) \right), \\ N(v(X,t)) &= I_t^\alpha \left(\sum_{i,j=1}^{n} F_{1ij}(X,t,v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \right. \\ &\left. + \sum_{i=1}^{n} G_{1i}(X,t,v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) \right). \end{aligned} \tag{9}$$

Thus, (8) can be written in the following form:

$$v(X,t) = g(X,t) + N(v(X,t)), \tag{10}$$

where g is a known function, N is a nonlinear operator of v.

The nonlinear operator N can be decomposed in the same way as in [2].

So, the solution of equation (10) can be written in the following series form:

$$v(X,t) = \sum_{i=0}^{\infty} v_i(X,t) = g(X,t) + N\left(\sum_{i=0}^{\infty} v_i(X,t)\right). \tag{11}$$

4 The Natural Homotopy Perturbation Method (NHPM)

In this section, we describe the application of the natural homotopy perturbation method (NHPM) for equations (1) and (2). First we define

$$\begin{aligned} Nv &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}), \\ Mv &= \sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}), Kv = H(X,t,v). \end{aligned} \tag{12}$$

Equation (1) is written in the form

$$D_t^{\alpha} v(X,t) = Nv(X,t) + Mv(X,t) + Kv(X,t) + S(X,t), t > 0. \tag{13}$$

Apply the natural transform on both sides of (13) and use (7), after that, we take the inverse natural transform, we obtain

$$v(X,t) = L(X,t) + \mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left[Nv(X,t) + Mv(X,t) + Kv(X,t)\right]\right), \tag{14}$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions.

Now we apply the homotopy perturbation method and the nonlinear terms can be decomposed in the same way as in [9], we get

$$\begin{aligned} \sum_{n=0}^{\infty} p^n v_n(X,t) &= L(X,t) + p\left[\mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left[\sum_{n=0}^{\infty} p^n H_n(v) + \sum_{n=0}^{\infty} p^n K_n(v)\right.\right.\right. \\ &\left.\left.\left. + \sum_{n=0}^{\infty} p^n J_n(v)\right]\right)\right], \end{aligned} \tag{15}$$

where $H_n(v)$, $K_n(v)$ and $J_n(v)$ are He's polynomials [5].

By using the coefficient of the like powers of p in equation (15), the following approximations are obtained:

$$\begin{aligned} p^0 &: v_0(X,t) = L(X,t), \\ p^1 &: v_1(X,t) = \mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left[H_0(v) + K_0(v) + J_0(v)\right]\right), \\ p^2 &: v_2(X,t) = \mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}\left[H_1(v) + K_1(v) + J_1(v)\right]\right) \\ &\quad \ldots \end{aligned} \tag{16}$$

Hence, the solution of equations (1) and (2) is given by

$$v(X,t)=\sum_{n=0}^{\infty} v_n(X,t). \tag{17}$$

5 Illustrative Examples and Numerical Results

Example 5.1 Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} v=\frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy})-\frac{\partial^2}{\partial x \partial y}(xyv_xv_y)-v, t>0, 1<\alpha \leq 2, \tag{18}$$

with the initial conditions

$$v(x,y,0)=e^{xy}, v_t(x,y,0)=e^{xy}, (x,y)\in \mathbb{R}^2. \tag{19}$$

5.1 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (18) and (19), we have

$$\begin{aligned} v_0 &= (1+t)\, e^{xy}, v_1=-\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\ v_2 &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}... \end{aligned} \tag{20}$$

So, the solution of equations (18) and (19) is

$$v(x,y,t)=\left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy}. \tag{21}$$

In the special case, $\alpha=2$, the series (21) has the closed form

$$v(x,y,t)=(\cos t+\sin t)\, e^{xy}. \tag{22}$$

5.2 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (18) and (19), we have

$$\begin{aligned} p^0 &: v_0(x,y,t)=(1+t)e^{xy}, p^1: v_1(x,y,t)=-\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\ p^2 &: v_2(x,y,t)=\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}... \end{aligned} \tag{23}$$

Therefore, the solution of equations (18) and (19) can be expressed by

$$v(x,y,t)=\left(1+t-\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-...\right)e^{xy}. \tag{24}$$

Taking $\alpha=2$ in equation (24), we obtain the exact solution as

$$v(x,y,t)=(\cos t+\sin t)\, e^{xy}. \tag{25}$$

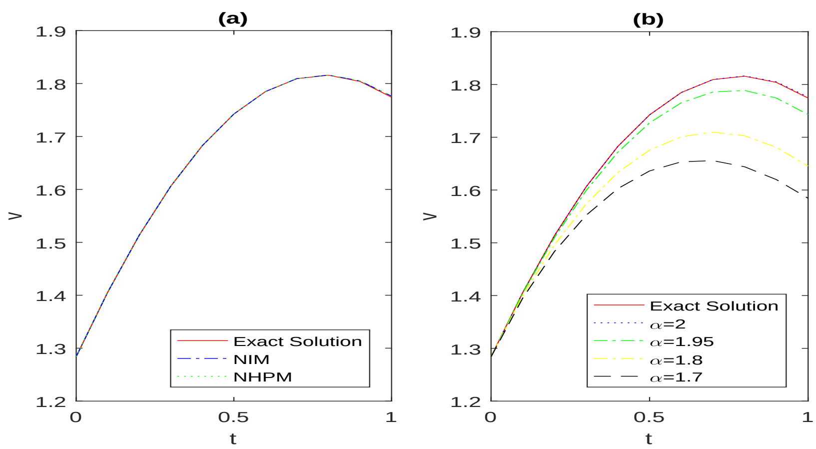

Figure 1: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha = 2$ and $x = y = 0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x = y = 0.5$.

	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$
$t/x, y$	0.5	0.5	0.7	0.7
0.1	1.8085×10^{-9}	1.8085×10^{-9}	2.2991×10^{-9}	2.2991×10^{-9}
0.3	1.3536×10^{-6}	1.3536×10^{-6}	1.7208×10^{-6}	1.7208×10^{-6}
0.5	2.9725×10^{-5}	2.9725×10^{-5}	3.7787×10^{-5}	3.7787×10^{-5}
0.7	2.2882×10^{-4}	2.2882×10^{-4}	2.9089×10^{-4}	2.9089×10^{-4}
0.9	1.0547×10^{-3}	1.0547×10^{-3}	1.3407×10^{-3}	1.3407×10^{-3}

Table 1: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.1, when $\alpha = 2$.

Example 5.2 Consider the following nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^{\alpha} v = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v, t > 0, 1 < \alpha \leq 2, \tag{26}$$

with the initial conditions

$$v(x, 0) = e^x, v_t(x, 0) = e^x, x \in \left]0, 1\right[. \tag{27}$$

5.3 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (26) and (27), we have

$$\begin{aligned} v_0 &= (1 + t) e^x, v_1 = \left(\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}\right) e^x, \\ v_2 &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 2)}\right) e^x ... \end{aligned} \tag{28}$$

So, the solution of equations (26) and (27) is

$$v(x,t)=\left(1+t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+...\right)e^{x}. \tag{29}$$

In the special case, $\alpha=2$, the series (29) has the closed form

$$v(x,t)=e^{x+t}. \tag{30}$$

5.4 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (26) and (27), we have

$$\begin{aligned} p^{0} &: v_0(x,t)=(1+t)e^{x}, p^{1}: v_1(x,t)=\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{x},\\ p^{2} &: v_2(x,t)=\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{x}... \end{aligned} \tag{31}$$

Therefore, the solution of equations (26) and(27) can be expressed by

$$v(x,t)=\left(1+t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+...\right)e^{x}. \tag{32}$$

Taking $\alpha=2$ in equation (32), we obtain the exact solution as

$$v(x,t)=e^{x+t}. \tag{33}$$

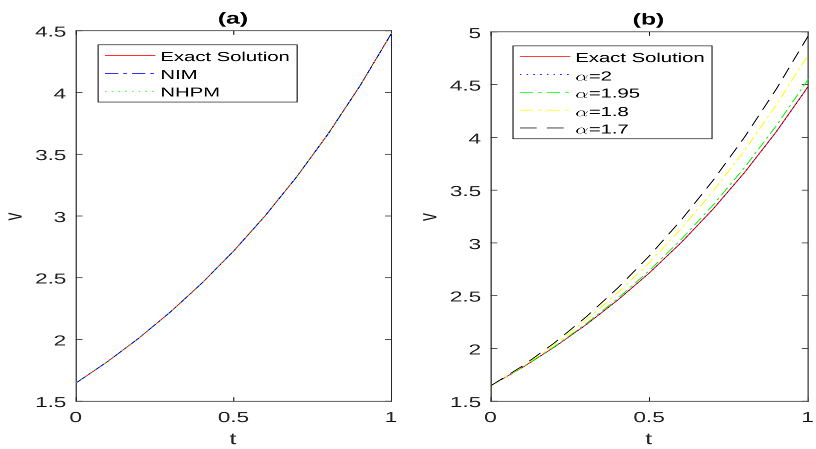

Figure 2: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha=2$ and $x=0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x=0.5$.

	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$
t/x	0.5	0.5	0.7	0.7
0.1	2.323×10^{-9}	2.323×10^{-9}	2.8373×10^{-9}	2.8373×10^{-9}
0.3	1.7436×10^{-6}	1.7436×10^{-6}	2.1297×10^{-6}	2.1297×10^{-6}
0.5	3.8504×10^{-5}	3.8504×10^{-5}	4.7029×10^{-5}	4.7029×10^{-5}
0.7	2.9890×10^{-4}	2.9890×10^{-4}	3.6507×10^{-4}	3.6507×10^{-4}
0.9	1.3929×10^{-3}	1.3929×10^{-3}	1.7013×10^{-3}	1.7013×10^{-3}

Table 2: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.2, when $\alpha = 2$.

Example 5.3 Consider the following one-dimensional nonlinear time-fractional wave-like equation with variable coefficients

$$D_t^\alpha v = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, t > 0, 1 < \alpha \leq 2, \tag{34}$$

with the initial conditions

$$v(x,0) = 0, v_t(x,0) = x^2, x \in \,]0,1[\,. \tag{35}$$

5.5 Application of the NIM

By applying the steps involved in NIM as presented in Section 3 to equations (34) and (35), we have

$$v_0 = tx^2, v_1 = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, v_2 = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 ... \tag{36}$$

So, the solution of equations (34) and (35) is

$$v(x,t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - ... \right). \tag{37}$$

In the special case, $\alpha = 2$, the series (37) has the closed form

$$v(x,t) = x^2 \sin t. \tag{38}$$

5.6 Application of the NHPM

By applying the steps involved in NHPM as presented in Section 4 to equations (34) and (35), we have

$$p^0 : v_0(x,t) = tx^2, p^1 : v_1(x,t) = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} x^2, p^2 : v_2(x,t) = \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} x^2 ... \tag{39}$$

Therefore, the solution of equations (34) and(35) can be expressed by

$$v(x,t) = x^2 \left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - ... \right). \tag{40}$$

Taking $\alpha = 2$ in equation (40), we obtain the exact solution as

$$v(x,t) = x^2 \sin t. \tag{41}$$

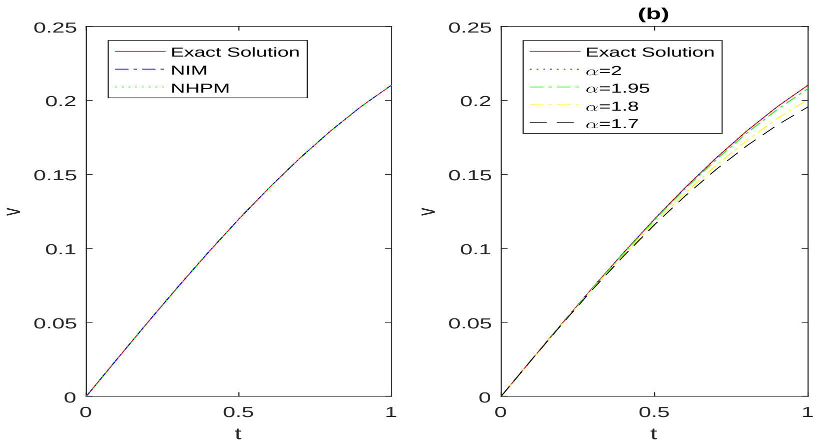

Figure 3: (a) The comparison of the 3−term approximate solution by NIM, NHPM and the exact solution, when $\alpha = 2$ and $x = 0.5$, (b) The behavior of the exact solution and the 3−term approximate solution by NIM and NHPM for different values of α when $x = 0.5$.

	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$	$\lvert v_{exact} - v_{NIM}\rvert$	$\lvert v_{exact} - v_{NHPM}\rvert$
t/x	0.5	0.5	0.7	0.7
0.1	4.9596×10^{-12}	4.9596×10^{-12}	9.7209×10^{-12}	9.7209×10^{-12}
0.3	1.0835×10^{-8}	1.0835×10^{-8}	2.1236×10^{-8}	2.1236×10^{-8}
0.5	3.8618×10^{-7}	3.8618×10^{-7}	7.5692×10^{-7}	7.5692×10^{-7}
0.7	4.0574×10^{-6}	4.0574×10^{-6}	7.9524×10^{-6}	7.9524×10^{-6}
0.9	2.346×10^{-5}	2.346×10^{-5}	4.5982×10^{-5}	4.5982×10^{-5}

Table 3: The absolute errors for differences between the exact solution and 3−term approximate solution by NIM and NHPM for Example 5.3, when $\alpha = 2$.

The numerical results (see Figures 1,2 and 3) affirm that when α approaches 2, our results approach the exact solutions. In Tables 1,2 and 3, the absolute errors obtained by NIM are the same as the results obtained by NHPM.

Remark 5.1 In general, the results obtained show that the method described by NIM is a very simple and easy method compared to the other methods and gives the approximate solution in the form of series, this series in closed form gives the corresponding exact solution of the given problem.

Remark 5.2 In this paper, we only apply three terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.

6 Conclusion

In this paper, we have compared between the new iterative method (NIM) and the natural homotopy perturbation method (NHPM) for solving nonlinear time-fractional wave-like equations with variable coefficients. The two methods are powerful and efficient methods and both give approximations of higher accuracy and closed form solutions, if any. The comparison gives similar results and supplies quantitatively reliable results. It is worth mentioning that the NIM has an advantage over the NHPM because it takes less time and uses only the inverse operator to solve the nonlinear problems and there is no need to use any other inverse transform as in the case of NHPM. The two methods are powerful mathematical tools for solving other nonlinear fractional differential equations.

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Complete Symmetry and μ-Symmetry Analysis of the Kawahara-KdV Type Equation

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Abstract: The goal of this paper is complete analysis of the Kawahara-KdV type equation using the ordinary symmetry and μ-symmetry methods. In other words, the Lie symmetry, symmetry reduction, differential invariant and conservation laws for the Kawahara-KdV type equation are provided. And in the second part the μ-symmetry, order reduced equations, Lagrangian and μ-conservation laws for the Kawahara-KdV type equation are presented.

Keywords: *Lie symmetry; μ-symmetry; Kawahara-KdV type equation; symmetry reduction; differential invariant; conservation law; order reduced equations; Lagrangian; variational problem; μ-conservation law.*

Mathematics Subject Classification (2010): 58D19, 58J70, 70G65, 76M60.

1 Introduction

The symmetry method is a powerful tool of differential geometry for accurate analysis of a mathematical model as a description of a system in many areas of applied mathematics and physics. Dispersive wave equations arise in many areas when the third order derivative in the KdV (Korteweg de Vries) equation approaches zero. It is necessary to take account of the higher order effect of dispersion in order to balance the nonlinear effect.

The Kawahara-KdV equation, modified Kawahara-KdV equation and Kawahara-KdV type equation, respectively, are given as:

$$u_t + uu_x + u_{xxx} - \gamma_1 u_{xxxxx} = 0, \quad u_t + 3u^2 u_x + u_{xxx} - \gamma_2 u_{xxxxx} = 0,$$
$$u_t + u_x + uu_x + u_{xxx} - \gamma u_{xxxxx} = 0, \tag{1}$$

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where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}^+$. When the cubic KdV type equation is weak, a lot of physical phenomena are described by the Kawahara-KdV type equations [6]. Especially, the Kawahara-KdV type equation as a specific form of the Benney-Lin equation describes the one-dimensional evolution problems. The λ-symmetries method is a special method for order reduction of ODEs. In 2004, Gaeta and Morando developed this method to a μ-symmetries method for PDEs, where $\mu = \lambda_i dx^i$ is a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and also μ is a compatible. The concepts of variational problem and conservation law and their relationship with λ-symmetries of ODEs were presented by Muriel, Romero and Olver (2006). More precisely, they have extended the formulation of Nother's theorem for λ-symmetry of ODEs. Continuing this trend, in 2007, Cicogna and Gaeta generalized the results obtained by Muriel, Romero and Olver in the case of λ-symmetries for ODEs to the case of μ-symmetries for PDEs.

The outline of this paper is as follows. Section 2 is devoted to the Lie symmetry analysis, reduction and differential invariant of equations (1). We will find all conservation laws for equations (1) in Section 3. In Section 4, we compute the μ-symmetry and order reduction of equations (1). Section 5 deals with the Lagrangian of equations (1) in potential form. Finally, in the last section, μ-conservation laws of equations (1) are obtained.

2 Lie Symmetry Analysis, Reduction and Differential Invariant of the Kawahara-KdV Type Equation

The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions [8].

First of all, we obtain the vector fields of equations (1) as follows: $\mathbf{v}_1 = \partial_x(space\,translation)$, $\mathbf{v}_2 = -\partial_t(time\,translation)$, $\mathbf{v}_3 = t\partial_x + \partial_u(Galilean\,boost)$. The commutation relations between vector fields is given by Table 2.

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$-\mathbf{v}_1$
$\mathbf{v}_3$	0	$\mathbf{v}_1$	0

Table 1: The commutator table of equations (1)

Note that the Lie algebra g is solvable , because $\mathrm{g}'' = [\mathrm{g}', \mathrm{g}'] = 0 \subset \mathrm{g}' = [\mathrm{g}, \mathrm{g}] =< \mathbf{v}_1 >\subset \mathrm{g}$. The one-parameter groups $G_1 : (x + \epsilon, t, u)$, $G_2 : (x, t - \epsilon, u)$ and $G_3 : (\epsilon t + x, t, u + \epsilon)$ are generated by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, respectively, so that the entries give the transformed point $exp(\epsilon \mathbf{v}_i)(x, t, u) = (\tilde{x}, \tilde{t}, \tilde{u})$. Since each group G_i is a symmetry group, this fact implies that if $u = f(x, t)$ is a solution of equations (1), so are the functions $u_1 = f(x - \epsilon, t)$, $u_2 = f(x, t + \epsilon)$ and $u_3 = f(x - \epsilon t, t) + \epsilon$.

For better cognition, we now try to classify the infinite set of solutions of equations (1). This is, in fact, the categorized orbits of the influence of groups. In general, for each $s-$parameter subgroup H of G, there is a family of group-invariant solutions ($s \leq p$) and it is not usually feasible to list all solutions via this method, because there are infinite number of $s-$parameter subgroups. Now we classify them according to the conjugacy property, and this is an effective method to find an optimal system of subgroup in terms of conjugacy in equivalent. This matter is equivalent to finding an optimal system of subalgebras, a list of subalgebras with the property that any other subalgebra is conjugate to one subalgebra in that list. Table 2 shows adjoint representation to compute.

$Ad(\exp(\varepsilon \mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + \varepsilon \mathbf{v}_1$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 - \varepsilon \mathbf{v}_1$	$\mathbf{v}_3$

Table 2: Adjoint representation table of equations (1)

Theorem 2.1 *An optimal system of one-dimensional Lie algebras of equations (1) is provided by* $a_2\mathbf{v}_2 + \mathbf{v}_3$ *and* $a_1\mathbf{v}_2$.

Proof. The adjoint representation was determined in Table 2, and the matrices M_i^ε of F_i^ε, $i = 1, 2, 3$, with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & -\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3^\varepsilon = \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we will make coefficients a_i as simple as possible, by acting these matrices on a vector field $\mathbf{V}$ alternatively. First, suppose that $a_3 \neq 0$, so we can assume that $a_3 = 1$, and by M_1^ε or M_2^ε, the coefficients of $\mathbf{v}_1$ vanish and $\mathbf{V}$ reduces to case 1. The second mode will be the same. □

Assume G acts projectably on M and Δ is a system of differential equation defined in it. By using the Lie-group method the number of independent variables can be reduced and the reduced system of differential equation is in quotient manifold M/G. If s denotes the dimension of the orbit of G, then there are precisely $(p-s)$ invariants which depend on x and play the role of independent variables $y = (y^1, ..., y^{p-s)})$ [7].

Now by integrating the characteristic equation, the invariants will be calculated. All results are coming in Table 2. In the following, differential invariants are computed. Let

operator	y	v	u	reduced equations
$\mathbf{v}_1$	t	u	$v(y)$	$v_y = 0$
$\alpha\,\mathbf{v}_2$	x	u	$v(y)$	$v_y + v\,v_y + v_{y^3} - \gamma\,v_{y^5} = 0$
$\mathbf{v}_3$	t	$x - t\,u$	$\frac{1}{t}(x - v(y))$	$1 - v_y = 0$
$\alpha\,\mathbf{v}_2 + \mathbf{v}_3$	$t^2 + 2\alpha\,x$	$t + \alpha\,u$	$\frac{1}{\alpha}(v(y) - t)$	$-1 + \alpha\,v_y + v\,v_y + \alpha^3 v_{y^3} - \alpha^5 v_{y^5} = 0$

Table 3: Reduction of equations (1).

us remind, if G is a symmetry group for a system with functionally differential invariants, then the system can be rewritten in terms of these invariants. Table 2 shows differential invariants of the equation (1) up to order 3.

vector field	up to the 3-rd order
$\mathbf{v}_1$	$t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$
$\mathbf{v}_2$	$x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$
$\mathbf{v}_3$	$t, \frac{-x}{t} + u, \frac{x}{t}u_x + u_t, \frac{x}{t}u_{xx} + u_{xt}, \frac{x^2}{t^2}u_{xx} + \frac{2x}{t^2}(xu_{xx} + tu_{xt}) + u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$

Table 4: Differential invariants of invariant(1).

3 Conservation Laws for the Kawahara-KdV Type Equation

Suppose that the Kawahara-KdV type equation is an isolated system, a particular measurable property of this system is called a conservation law which does not change as the system evolves over time. Consider $\Phi = (\Phi^1(x, u^{(n)}), ..., \Phi^p(x, u^{(n)}))$ is a p-tuple of smooth functions on $J^{(n)}M$. In characteristic form, a local conservation law is

$$Div\Phi = D_1\Phi^1(x, u^{(n)}) + ... + D_n\Phi^n(x, u^{(n)}) = Q.\Delta, \quad Q = (Q_1, ..., Q_L),$$

where $\Phi^i s$ and Q are the fluxes and characteristics of the conservation law. In this section, the conservation law is calculated by the multiplier method and also remind the Euler operator with respect to U^j is $E_{U^j} = \frac{\partial}{\partial U^j} - D_i\frac{\partial}{\partial U^j} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s}\frac{\partial}{\partial U^j_{i_1\cdots i_s}} + \cdots$.

The next theorem shows that the range of Div is a subset of the Euler operator's kernel.

Theorem 3.1 *The equations $E_{U^j}F(x, U, \partial_U, \cdots, \partial_U^s) \equiv 0$, $j = 1, \cdots, q$ hold for arbitrary $U(x)$ if and only if $F(x, U, \partial_U, \cdots, \partial_U^s)$ is in the range of Div [7, 8].*

Theorem 3.2 *The set of equations $E_{U^j}(\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\Delta_\nu(x, u^{(n)})) \equiv 0$, $j = 1, \cdots q$, holds for arbitrary functions $U(x)$, if and only if the set $\{\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\}_{\nu=1}^l$ yields a local conservation law for the system [7, 8].*

Now, to find all local conservation law multipliers of the form $\Lambda = \xi(x, t, u)$, we have

$$E_U[\xi(x, t, U)(U_t + U_x + UU_x + U_{xxx} - \gamma U_{xxxxx})] \equiv 0,$$

where $U(x, t)$ are arbitrary functions. The solution of the determining system is $\xi = 1$, U, $t + tU - x$. In other words, $D_t\Psi + D_x\Phi \equiv \xi(U_t + U_x + UU_x + U_{xxx} - \gamma U_{xxxxx})$, that is, (Ψ, Φ) determines a nontrivial local conservation law of the system. Further, (Ψ, Φ) are calculated by using the homotopy operator and all results are shown in Table 3.

4 μ-Symmetry and Order Reduction for Kawahara-KdV Type Equation

Let D_i be a total derivative up to x^i, $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$ and $\mu = \lambda_i dx^i$ be a horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ and compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$. Suppose $\Delta(x, u^{(k)}) = 0$ is a scalar PDE, involving p independent variables $x = (x^1, ..., x^p)$ and one dependent variable $u = u(x^1, ..., x^p)$ of order k.

Let $X = \sum_{i=1}^p \xi^i\partial_{x^i} + \varphi\partial_u$ be a vector field on M, $Y = X + \sum_{J=1}^k \Psi_J\,\partial_{u_J}$ be the μ-prolongation of X on jet space J^kM if $\Psi_{J,i} = (D_i + \lambda_i)\Psi_J - u_{J,m}(D_i + \lambda_i)\xi^m$, $(\Psi_0 = \varphi)$. Suppose $\mathcal{S}_\Delta \subset J^kM$ is a solution manifold for $\Delta = 0$ and $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, then X is said to be a μ-symmetry for Δ. Generally, in this thread if $\mu = 0$, ordinary prolongation

$\xi(x,t,u)$	Ψ	Φ
1	$\Psi = u + \frac{1}{2}u^2 + u_{x^2} - \gamma u_{x^4}$	$\Phi = u$
u	$\Psi = \frac{1}{2}u^2 + \frac{1}{3}u^3 + uu_{x^2} - \frac{1}{2}u_x u_x$ $+\gamma(-uu_{x^4} + u_x u_{x^3} + \frac{1}{2}u_{x^2}u_{x^3})$	$\Phi = \frac{1}{2}u^2$
$t + tu - x$	$\Psi = tu + tu^2 - xu - \frac{x}{2}u^2 + \frac{t}{3}u^3 + u_x$ $+tu_{x^2}(1+u) - xu_{x^2} - \gamma u_{x^4}(tu + t - x)$ $+\gamma tu_x u_{x^3} - \gamma\frac{t}{2}u_{x^2}u_{x^2} - \gamma u_{x^3} - \frac{t}{2}u_x u_x$	$\Phi = tu + \frac{t}{2}u^2 - xu$

Table 5: Conservation laws for equations (1).

and ordinary symmetry is going to happen. Suppose $\mu = \lambda_i dx^i$ is a horizontal 1-form and compatible on $\mathcal{S}_\Delta$ and X is a vector field on M, then the exponential vector field $V = \exp\big(\int \mu\big)X$ is a general symmetry for Δ if and only if X is a μ-symmetry for Δ.

Theorem 4.1 *Let Δ be a scalar PDE of order k for $u = u(x^1, ..., x^p)$, $X = \xi^i(\frac{\partial}{\partial x^i}) + \varphi(\frac{\partial}{\partial u})$ be a vector field on M, with characteristic $Q = \varphi - u_i\xi^i$ and Y be the μ-prolongation of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $\mid J \mid= 0, 1, ..., k-1$. [4]*

The process of calculating μ-symmetries of a given equation $\Delta = 0$ of order n is similar to that for the ordinary symmetries. Generally, if X is a generic vector field acting in M, then its μ-prolongation Y of order n for a generic $\mu = \lambda_i dx^i$, acting in $J^{(n)}M$ and applying Y to Δ and the obtained expression to $\mathcal{S}_\Delta \subset J^{(n)}M$, the result will be Δ_* up to ξ, τ, φ and λ_i. If we require λ_i to be functions on $J^{(k)}M$, all the dependences on u_J will be explicit, and one obtains a system of determining equations. This system should be complemented with the compatibility conditions between the λ_i. If we determine a priori the form μ, we are left with a system of linear equation for ξ, τ, φ; similarly, if we fix a vector field X and try to find the μ for which it is a μ-symmetry of the given equation Δ, we have a system of quasilinear equations for the λ_i [4].

To continue the μ-symmetry analysis of equations (1), let $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form and with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$ when $\Delta = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on M, in order to compute μ-prolongation of order 5 of X, we have $Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + ... + \Psi^{ttttt}\partial_{u_{ttttt}}$, where coefficients Y are as follows:

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi - u_t(D_x + \lambda_1)\tau\,, \\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi - u_t(D_t + \lambda_2)\tau\,, ...
\end{aligned}$$

By applying Y to equations (1) and substituting $(1/\gamma)(u_t + u_x + uu_x + u_{xxx})$ for u_{xxxxx}, we obtain the following system:

$$\gamma\tau_{uuuuu} = 0, \quad \gamma\xi_{uuuuu} = 0, \quad 5\gamma\tau_u = 0, \quad ... \quad , 10\gamma(3\tau_{xu} + \tau\lambda_{1u} + 3\tau_u\lambda_1) = 0. \qquad (2)$$

For any choice of the type $\lambda_1 = D_x[f(x,t)] + g(x)$, $\lambda_2 = D_t[f(x,t)] + h(t)$, where $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions and the functions λ_1 and λ_2 satisfy the compatibility condition. For instance, two cases studied to obtain the μ-symmetry and order reduction of equations (1) are as follows:

i) When $g(x) = 0$ and $h(t) = 0$, then by substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t)$ into the system of (2) and solving that system, we deduce $\xi = (c_1 t + c_2)F(x,t), \tau = F(x,t)$ and $\varphi = c_1 F(x,t)$, where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function and c_1 and c_2 are arbitrary constants. Then $X = \Big((c_1 t + c_2)\partial_x + \partial_t + c_1\partial_u\Big)F(x,t)$ is a μ-symmetry of equations (1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + D_t f(x,t)dt\Big)X$ of exponential type and order reduction of equations (1) is $Q = \varphi - \xi u_x - \tau u_t = \Big(c_1 - (c_1 t + c_2)u_x - u_t\Big)F(x,t)$.

ii) When $g(x) = 0$ and $h(t) = 1/(t + c_1)$, where c_1 is an arbitrary constant, then by substituting the functions $\lambda_1 = D_x f(x,t)$ and $\lambda_2 = D_t f(x,t) + 1/(t + c_1)$ into the system of (2) and solving them, we deduce $\xi = F(x,t), \tau = 0$, and $\varphi = 1/(t + c_1)\, F(x,t)$ where $f(x,t) = -\ln(F(x,t))$ and $F(x,t)$ is an arbitrary positive function. Then $X = \Big(\partial_x + 1/(t + c_1)\,\partial_u\Big)F(x,t)$ is a μ-symmetry of equations (1) and corresponds to an ordinary symmetry $V = \exp\Big(\int D_x f(x,t)dx + (D_t f(x,t) + 1/(t + c_1))dt\Big)X$ of exponential type. In this case reduction of equations (1) is $Q = \varphi - \xi u_x - \tau u_t = \Big(\frac{1}{t+c_1} - u_x\Big)F(x,t)$.

5 Lagrangian of the Kawahara-KdV Type Equation in Potential Form

In this section, we show that equations (1) do not admit a variational problem since they are of odd order, but equations (1) in potential form admit a variational problem.

Theorem 5.1 *Let $\Delta = 0$ be a system of differential equation. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$ if and only if the Frechet derivative D_Δ is self-adjoint: $D^*_\Delta = D_\Delta$ [8].*

In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula $L[u] = \int_0^1 u.\Delta[\lambda u]d\lambda$ and the Frechet derivative of $\Delta_{KK_u} : u_t + u_x + uu_x + u_{xxx} - \gamma u_{xxxxx} = 0$ is $D_{\Delta_{KK_u}} = D_t + (1 + u)D_x + D_x^3 - \gamma D_x^5 + u_x$. Obviously, it does not admit a variational problem since $D^*_{\Delta_{KK_u}} \neq D_{\Delta_{KK_u}}$. But the well-known differential substitution $u = v_x$ yields the related transformed Kawahara-KdV type equation as $\Delta_{KK_v} : v_{xt} + v_{xx} + v_x v_{xx} + v_{xxxx} - \gamma v_{xxxxxx} = 0$, that is called "the Kawahara-KdV type equation in potential form" and its Frechet derivative is $D_{\Delta_{KK_v}} = D_x D_t + v_{xx} D_x + (1 + v_x)D_x^2 + D_x^4 - \gamma D_x^6$, which is self-adjoint, i.e. $D^*_{\Delta_{KK_v}} = D_{\Delta_{KK_v}}$ and has a Lagrangian of the form

$$L[v] = \int_0^1 v.\Delta_{KK_v}[\lambda v]d\lambda = -\frac{1}{2}\Big(v_x v_t + v_x^2 + \frac{1}{3}v_x^3 - v_{xx}^2 + \gamma v_{xxx}^2\Big) + \text{Div}P.$$

Hence, the Lagrangian of the Kawahara-KdV type equation in potential form Δ_{KK_v}, up to Div-equivalence is

$$\mathcal{L}_{\Delta_{KK_v}}[v] = -\frac{1}{2}\Big(v_x v_t + v_x^2 + \frac{1}{3}v_x^3 - v_{xx}^2 + \gamma v_{xxx}^2\Big)\,. \qquad (3)$$

6 μ-Conservation Laws of the Kawahara-KdV Type Equation

A conservation law is a relation $\text{Div}\,\mathbf{P} := \sum_{i=1}^p D_i P^i = 0$, where $\mathbf{P} = (P^1, \cdots, P^p)$ is a $p-$dimensional vector. Let $\mu = \lambda_i dx^i$ be a horizontal one-form and $D_i\lambda_j = D_j\lambda_i$.

A μ-conservation law is a relation as $(D_i+\lambda_i)P^i=0,$ where P^i is a vector and the $M-$vector P^i is called a μ-conserved vector.

Theorem 6.1 *Consider the $n-th$ order Lagrangian $\mathcal{L}=\mathcal{L}(x,u^{(n)})$ and the vector field X, then X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}]=0$ if and only if there exists a $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i=0$.*

Suppose $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_t)$ is the first order Lagrangian and the vector field $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, then the $M-$vector $P^i:=\varphi\,(\partial\mathcal{L}/\partial u_i)$ is a μ-conserved vector. Also, suppose $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_{tt})$ is the second order Lagrangian and the vector field $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, then the $M-$vector $P^i:=\varphi(\partial\mathcal{L}/\partial u_i)+[(D_j+\lambda_j)\varphi](\partial\mathcal{L}/\partial u_{ij})-\varphi D_j(\partial\mathcal{L}/\partial u_{ij})$ is a μ-conserved vector. The $M-$vector P^i is obtained for the third order Lagrangian in the following theorem.

Theorem 6.2 *Consider the $3-$rd order Lagrangian $\mathcal{L}=\mathcal{L}(x,t,u,u_x,...,u_{ttt})$ and the vector field X, then $X=\varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}]=0$ if and only if the $M-$vector $P^i:=\varphi\frac{\partial\mathcal{L}}{\partial u_i}+[(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}-(D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big)$ satisfies the μ-conservation law $(D_i+\lambda_i)P^i=0$.*

Proof. Let $X=\varphi\,(\partial/\partial u)$ be a μ-symmetry for $\mathcal{L}$, its $3-$rd order μ-prolongation is $Y=\varphi\frac{\partial}{\partial u}+[(D_x+\lambda_1)\varphi]\frac{\partial}{\partial u_x}+[(D_t+\lambda_2)\varphi]\frac{\partial}{\partial u_t}+...+[(D_t+\lambda_2)^3\varphi]\frac{\partial}{\partial u_{ttt}}$, then by applying Y on the Lagrangian $\mathcal{L}$, we have

$$
\begin{aligned}
&Y[\mathcal{L}]=\varphi\frac{\partial\mathcal{L}}{\partial u}+[(D_x+\lambda_1)\varphi]\frac{\partial\mathcal{L}}{\partial u_x}+[(D_t+\lambda_2)\varphi]\frac{\partial\mathcal{L}}{\partial u_t}+...+[(D_t+\lambda_2)^3\varphi]\frac{\partial\mathcal{L}}{\partial u_{ttt}}=\varphi\\
&\big(\frac{\partial\mathcal{L}}{\partial u}-D_x\varphi\frac{\partial\mathcal{L}}{\partial u_x}-D_t\varphi\frac{\partial\mathcal{L}}{\partial u_t}+D_x^2\varphi\frac{\partial\mathcal{L}}{\partial u_{xx}}+...-D_t^3\varphi\frac{\partial\mathcal{L}}{\partial u_{ttt}}\big)+(D_x+\lambda_1)\Big[\varphi\frac{\partial\mathcal{L}}{\partial u_x}+[(D_j\\
&+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{xj}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{xj}}-(D_k+\lambda_k)\big([(D_j+\lambda_j)\varphi].\frac{\partial\mathcal{L}}{\partial u_{jkx}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkx}}\big)\Big]+(D_t+\lambda_2)\\
&\big[\varphi\frac{\partial\mathcal{L}}{\partial u_t}+[(D_j|+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{tj}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{tj}}-(D_k+\lambda_k)\big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jkt}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkt}}\big)\big].
\end{aligned}
$$

We put $P^i:=\varphi\frac{\partial\mathcal{L}}{\partial u_i}+[(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}-(D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big)$. Then $Y[\mathcal{L}]=\varphi E(\mathcal{L})+(D_i+\lambda_i)P^i$, where E is the Euler-Lagrange operator. The Euler-Lagrange equations $E(\mathcal{L})$ vanishes, hence this reduces to $Y[\mathcal{L}]=(D_i+\lambda_i)P^i$. This shows that $Y[\mathcal{L}]=0$ implies $(D_i+\lambda_i)P^i=0$. □

We consider the $3-$rd order Lagrangian (3) for the Kawahara-KdV type equation in potential form $\Delta_{KK_v}=v_{xt}+v_{xx}+v_xv_{xx}+v_{xxxx}-\gamma v_{xxxxxx}=E(\mathcal{L}_{\Delta_{KK_v}})$. Suppose $X=\varphi\partial_v$ is a vector field for $\mathcal{L}_{\Delta_{KK_v}}[v]$. Let $\mu=\lambda_1dx+\lambda_2dt$ be a horizontal one-form with the compatibility condition $D_t\lambda_1=D_x\lambda_2$ when $\Delta_{KK_v}=0$. In order to compute μ-prolongation of order 3 of X, we have $Y=\varphi\partial_v+\Psi^x\partial_{v_x}+\Psi^t\partial_{v_t}+\Psi^{xx}\partial_{v_{xx}}+...+\Psi^{ttt}\partial_{v_{ttt}}$, where coefficients Y are as follows:

$$\Psi^x=(D_x+\lambda_1)\varphi,\ \Psi^t=(D_t+\lambda_2)\varphi,\ \Psi^{xx}=(D_x+\lambda_1)\Psi^x,\ ...,\ \Psi^{ttt}=(D_t+\lambda_2)\Psi^{tt}.$$

Thus, the μ-prolongation Y acts on the $\mathcal{L}_{\Delta_{KKv}}[v]$, and substituting $\Big(v_x^2+\frac{1}{3}v_x^3-v_{xx}^2+\gamma v_{xxx}^2\Big)/-v_x$ for v_t, we obtain the system as follows:

$$\varphi_{vv}=0\,,\quad(-1/6)\varphi_v=0\,,\quad...,\quad\gamma(\varphi\lambda_{1v}+3\lambda_1\varphi_v+3\varphi_{xv})\ =0. \tag{4}$$

Suppose $\varphi = F(x,t)$, where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{KKv}}[v] = 0$, then a special solution of the system (4) is given by $\lambda_1 = -F_x(x,t)/F(x,t), \lambda_2 = -F_t(x,t)/F(x,t)$, where $D_t\lambda_1 = D_x\lambda_2$. Hence $X = F(x,t)\partial_v$ is a μ-symmetry for $\mathcal{L}_{\Delta_{KKv}}[v]$, then by Theorem 6.1, there exists a $M-$vector P^i satisfying the μ-conservation law $(D_i + \lambda_i)P^i = 0$. Then by Theorem 6.2, the $M-$vector P^i is

$$P^1 = -\frac{1}{2}F(x,t)\big(v_t + 2v_x + v_x^2 + 2v_{xxx} - 2\gamma v_{xxxxx}\big), P^2 = -\frac{1}{2}F(x,t)v_x, \tag{5}$$

and $(D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2 = 0$ is a μ-conservation law for the 3-rd order Lagrangian $\mathcal{L}_{\Delta_{KK_v}}[v]$. Therefore, the μ-conservation law for equations (1) in potential form $\Delta_{KKv} = E(\mathcal{L}_{\Delta_{KK_v}})$ is $D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0$, where P^1 and P^2 are the $M-$vectors P^i of (5).

Remark 6.1 The μ-conservation law for equations (1) in potential form Δ_{KK_v}, satisfies Noether's theorem for μ-symmetry, i.e. $(D_i + \lambda_i)P^i = QE(\mathcal{L}_{\Delta_{KK_v}})$.

We consider the Kawahara-KdV type equation in potential form $\Delta_{KK_v} = v_{xt} + v_{xx} + v_xv_{xx} + v_{xxxx} - \gamma v_{xxxxxx} = 0$, or equivalently, $D_x(v_t + v_x + \frac{1}{2}v_x^2 + v_{xxx} - \gamma v_{xxxxx}) = 0$, or $v_t + v_x + \frac{1}{2}v_x^2 + v_{xxx} - \gamma v_{xxxxx} = f(t)$, where $f(t)$ is an arbitrary function. If we substitute $f(t) - v_x - \frac{1}{2}v_x^2 - v_{xxx} + \gamma v_{xxxxx}$ by v_t and substitute u by v_x in the $M-$vector P^i of (5), then we obtain the $M-$vectors

$$P^1 = -\frac{1}{2}F(x,t)\big(f(t) + u + \frac{1}{2}u^2 + u_{xx} - \gamma u_{xxxx}\big), P^2 = -\frac{1}{2}F(x,t)u. \tag{6}$$

Also, the μ-conservation law for equations (1) is $D_xP^1 + D_tP^2 + \lambda_1P^1 + \lambda_2P^2 = 0$, where P^1 and P^2 are the $M-$vectors P^i of (6).

Remark 6.2 Equations (1) satisfy the characteristic form, i.e. $(D_i+\lambda_i)P^i = (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 = Q\Delta_{KK_u}$.

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A Phase Change Problem including Space-Dependent Latent Heat and Periodic Heat Flux

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Abstract: In this paper, a mathematical model related to a problem of phase-change process with periodic surface heat flux and space-dependent latent heat is considered. We have used the homotopy analysis approach to acquire the solution to the problem. To show the correctness of the calculated result, the comparisons have been discussed with the existing exact solution in a particular case. The effect of various parameters on the movement of the interface is also investigated.

Keywords: *homotopy analysis method; variable latent heat; periodic boundary condition; phase change problem.*

Mathematics Subject Classification (2010): 80A22, 35R37, 35R35, 80A20.

1 Introduction

In recent years, the phase change problem (the Stefan problem) involving diffusion process and variable latent heat is of great interest from mathematical and physical points of views. The research related to the diffusion process and its occurence can be found in many works [1–3]. Physically, a variable latent heat term arises in the Stefan problem governing the processes of movement of a shoreline in a sedimentary ocean basin due to changes in various parameters [4]. Some solutions of the Stefan problems including space-dependent latent heat have been reported in [5–7]. Zhou et al. [8] presented a phase change model (the Stefan problem) that contains a variable latent heat term and they discussed the similarity solution to the problem. After that Zhou and Xia [9] used the Kummer functions to present the similarity solution to a Stefan problem containing a more general variable latent heat term. Mathematically, the Stefan problem with periodic

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boundary is always interesting due to the difficulty associated with its solution. From the literature, it is found that the exact solution to the phase change problem with periodic heat-flux is not known even in its simplest form and a sophisticated scheme is required to solve these problems [10]. Therefore, various numerical [11–13] and approximate analytical techniques [7, 14] have been used by the researchers to solve the phase change problem containing the boundary conditions of periodic nature.

In this study, we consider a Stefan problem containing space-dependent latent heat and a periodic boundary condition. The solution of the problem is obtained by a well-known approximate technique, the homotopy analysis technique, introduced by Liao [12]. From the literature [16–22], it can be seen that this scheme is used by many researchers to solve various problems occurring in science and industries. In this paper, Wolfram Mathematica 8.0.1 has been used for all the computations with the aid of [23]. For the validity of proposed solution, the comparisons have been made with the analytical solution in a particular case. Dependence of movement of interface on some parameters is also analysed.

2 Mathematical Formulation

This section presents a phase change problem involving melting/freezing process in the half plane, i.e. $x > 0$. Motivated by the work of Zhou et al. [8] and Zhou and Xia [9], we have assumed that the latent heat is space-dependent. Moreover, a periodic surface heat flux is supposed in the problem. The mathematical model describing the process is given below:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t),\ t > 0, \tag{1}$$

$$T(s(t), t) = 0, \quad t > 0, \tag{2}$$

$$k \frac{\partial T(0,t)}{\partial x} = -q(1 + \epsilon \sin \omega t), \quad t > 0, \tag{3}$$

$$k \frac{\partial T(s(t),t)}{\partial x} = -\gamma s \frac{ds}{dt}, \quad t > 0, \tag{4}$$

$$s(0) = 0, \tag{5}$$

where $T(x,t)$ is the temperature profile, x represents the space variable, t is the time, α denotes the thermal diffusivity, $s(t)$ denotes the tracking of moving phase front, k is the thermal conductivity, ω is the oscillation frequency, ϵ is the amplitude, $q(1 + \epsilon \sin \omega t)$ is the periodic heat flux and γs is the latent heat term per unit volume which depends on space.

3 Solution of the Problem

According to the homotopy analysis method (HAM) [17, 18], we assume

$$N\left[\phi(x,t;p)\right] = \frac{\partial}{\partial t}\phi(x,t;p) - \alpha \frac{\partial^2}{\partial x^2}\phi(x,t;p), \tag{6}$$

and

$$L\left[\phi(x,t;p)\right] = \frac{\partial^2}{\partial x^2}\phi(x,t;p) \tag{7}$$

as the non-linear and linear operators, respectively. For equation (1), we first construct the following homotopy:

$$(1-p)L\left[\phi(x,t;p)-T_0(x,t)\right]=p\mu H(x,t)N\left[\phi(x,t;p)\right], \tag{8}$$

where $p\in[0,1]$ denotes the embedding parameter, $T_0(x,t)$ represents the initial guess, $\mu\neq 0$ is the auxiliary parameter, $H(x,t)\neq 0$ is the auxiliary function.

If we substitute $p=0$ and $p=1$ in equation (8), then we simply obtain $\phi(x,t;0)=T_0(x,t)$ and $\phi(x,t;1)=T(x,t)$, respectively. This indicates that when p tends to 1 from 0, the initial estimate $T_0(x,t)$ shifts towards $T(x,t)$ which satisfies the proposed problem.

For equation (1), we can get the $m-th$ order deformation equation [17, 18] as given below:

$$L\left[T_m(x,t)-\chi_m T_{m-1}(x,t)\right]=\mu H(x,t)R_m(\vec{T}_{m-1}), \tag{9}$$

where

$$R_m(\vec{T}_{m-1})=\frac{\partial T_{m-1}(x,t)}{\partial t}-\alpha\frac{\partial^2 T_{m-1}(x,t)}{\partial x^2}$$

and

$$\chi_m=\begin{cases}0, & m<2,\\ 1, & m\geq 2.\end{cases}$$

According to Rajeev et al. [3], we consider the following initial approximation of $T(x,t)$:

$$T_0(x,t)=\frac{q}{k}\left((1+\epsilon\sin\omega t)(s_0-x)\right), \tag{10}$$

where $s_0=\left(\frac{2q}{\gamma}\left(t-\frac{\epsilon}{\omega}\cos\omega t+\frac{\epsilon}{\omega}\right)\right)^{\frac{1}{2}}$.

Using equation (10) in equation (9), we obtain

$$\begin{aligned}T_1(x,t)=&\mu\left(\frac{q^2}{k\gamma}(1+\epsilon\sin\omega t)^2 s_0^{-1}\right)\frac{x^2}{2}+\mu\left(\frac{q}{k}\omega\epsilon\cos\omega t s_0\right)\frac{x^2}{2}\\&-\mu\left(\frac{q}{k}\omega\epsilon\cos\omega t\right)\frac{x^3}{6},\end{aligned} \tag{11}$$

$$\begin{aligned}T_2(x,t)=&T_1(x,t)-\frac{\alpha\mu^2q^2(1+\epsilon\sin\omega t)^2s_0^{-1}}{k\gamma}\frac{x^2}{2}-\frac{\alpha\mu^2q\omega\epsilon\cos\omega t s_0}{k}\frac{x^2}{2}\\&+\frac{\alpha\mu^2q\omega\epsilon\cos\omega t}{k}\frac{x^3}{6}+\frac{\mu^2q^2}{k\gamma}\left\{-\frac{q}{\gamma}(1+\epsilon\sin\omega t)^3s_0^{-3}\right.\\&\left.+2(1+\epsilon\sin\omega t)\omega\epsilon s_0^{-1}\cos\omega t\right\}\frac{x^4}{24}+\frac{\mu^2q}{k}\left\{\frac{\omega q}{\gamma}\epsilon\cos\omega t(1+\epsilon\sin\omega t)s_0^{-1}\right.\\&\left.-(\omega^2\epsilon\sin\omega t)s_0\right\}\frac{x^4}{24}+\frac{\mu^2q\omega^2\epsilon\sin\omega t}{k}\frac{x^5}{120}\end{aligned} \tag{12}$$

and similarly, other components of $T(x,t)$ can be calculated.

Now, the solution $T(x,t)$ to the problem can be given by

$$T(x,t)=T_0(x,t)+T_1(x,t)+T_2(x,t)+... \tag{13}$$

Now, by choosing the following linear and non-linear operators, we have

$$L\left[\psi(t;p)\right] = \frac{d\psi(t;p)}{dt}, \tag{14}$$

and

$$N\left[\psi(t;p)\right] = k\frac{\partial T(\psi(t;p),t)}{\partial x} + \gamma\psi(t;p)\frac{d\psi(t;p)}{dt}. \tag{15}$$

We construct the following homotopy for the equation (4):

$$(1-p)\left[\psi(t;p) - s_0(t)\right] = p\hbar N\left[\psi(t;p)\right]. \tag{16}$$

From equation (16), we can easily find

$$\psi(t;0) = s_0, \tag{17}$$

and

$$\psi(t;1) = s(t). \tag{18}$$

According to [17,18], the m-th order deformation equation in the context of equation (4) is

$$L\left[s_m(t) - \chi_m s_{m-1}(t)\right] = \hbar N\left[s_{m-1}(t)\right]. \tag{19}$$

By considering the expression of s_0 (the initial approximation for the moving interface) and equations (13), (19) and (17), the various components of $s(t)$, i.e. $s_1(t)$, $s_2(t)$,..., can be calculated. Hence, the approximate solution for $s(t)$ is given by

$$s(t) = s_0(t) + s_1(t) + ... \tag{20}$$

4 Comparisons and Discussions

To show the accuracy of the obtained solution, we discuss the comparisons of our results for the temperature profile $T(x,t)$ and the location of moving phase front $s(t)$ with the exact solution at $\epsilon = 0$ in Tables 1 and 2, respectively. In case of $\epsilon = 0$, the equations (1)-(5) become a shoreline problem with a fixed line flux and a constant ocean level [4]. In this paper, the comparisons of our calculated results have been made with the exact solution established by Voller et al. [4]. Table 1 represents relative errors of temperature distribution between the obtained results and the exact result (given in [4]) at $\alpha = 1$, $\epsilon = 0$, $k = 1$ and $t = 5.5$. The absolute errors and relative errors of moving phase front are depicted in Table 2 at $\alpha = 1$, $\epsilon = 0$ and $k = 1$. From both tables, it is clear that the obtained computational results agree well with the result of exact solution.

q	x	$T_N(x,t)$	$T_E(x,t)$	Absolute Error	Relative Error
0.5	0.1	0.212321	0.211090	1.20 e-03	5.80 e-03
	0.2	0.162679	0.160212	2.40 e-03	1.50 e-02
	0.3	0.113274	0.109579	3.60 e-03	3.30 e-02
	0.4	0.064106	0.059189	4.90 e-03	8.30 e-02
	0.5	0.015176	0.009037	6.10 e-03	6.70 e-02
1.0	0.1	0.641957	0.637125	4.80 e-03	7.50 e-03
	0.2	0.542968	0.533223	9.70 e-03	1.80 e-02
	0.3	0.444652	0.430042	1.40 e-02	3.30 e-02
	0.4	0.347007	0.327569	1.90 e-02	5.90 e-02
	0.5	0.250031	0.225792	2.40 e-02	1.00 e-01
1.5	0.1	1.213060	1.202430	1.00 e-02	8.80 e-03
	0.2	1.064920	1.043280	2.10 e-02	2.00 e-02
	0.3	0.918012	0.885505	3.20 e-02	3.60 e-02
	0.4	0.772339	0.729075	4.30 e-02	5.90 e-02
	0.5	0.627896	0.573966	5.30 e-02	9.30 e-02

Table 1: Comparison between the exact value $T_E(x,t)$ and the numerical value $T_N(x,t)$ of temperature distribution at $\gamma = 20$.

q	t	$s_N(t)$	$s_E(t)$	Absolute Error	Relative Error
0.5	1	0.199681	0.198055	1.60 e-03	8.20 e-03
	2	0.282205	0.280092	2.10 e-03	7.50 e-03
	3	0.345453	0.343041	2.40 e-03	7.00 e-03
	4	0.398724	0.396109	2.60 e-03	6.60 e-03
	5	0.445619	0.442864	2.70 e-03	6.20 e-03
1.0	1	0.281571	0.277484	4.00 e-03	1.40 e-02
	2	0.397457	0.392422	5.00 e-03	1.20 e-02
	3	0.486084	0.480616	5.40 e-03	1.10 e-02
	4	0.560600	0.554968	5.60 e-03	1.00 e-02
	5	0.626098	0.620473	5.60 e-03	0.90 e-02
2.0	1	0.394948	0.385578	9.30 e-03	2.40 e-02
	2	0.555582	0.545290	10.20 e-03	1.80 e-02
	3	0.677665	0.667841	9.80 e-03	1.40 e-02
	4	0.779793	0.771156	8.60 e-03	1.10 e-02
	5	0.869169	0.862179	6.90 e-03	0.80 e-02

Table 2: Comparison between the exact value $s_E(t)$ and the numerical value $s_N(t)$ of moving interface at $\gamma = 25$.

Figures 1 and 2 show the evolution of movement of phase front at the fixed value of thermal diffusivity ($\alpha = 1.0$), the oscillation amplitude ($\epsilon = 0.5$) and the oscillation frequency ($\omega = \dfrac{\pi}{2}$). In Figure 1 and Figure 2, the effect of periodic heat flux on the movement of phase front is depicted for different values of γ and q, respectively. From Figure 1, it can be seen that the phase front propagates periodically and the movement of phase front becomes slow when we enhance the parameter γ. However, Figure 2 depicts that the periodic propagation of moving boundary $s(t)$ becomes fast as the value of q rises. It is also observed that when we raise the value of q, it makes melting/freezing process fast.

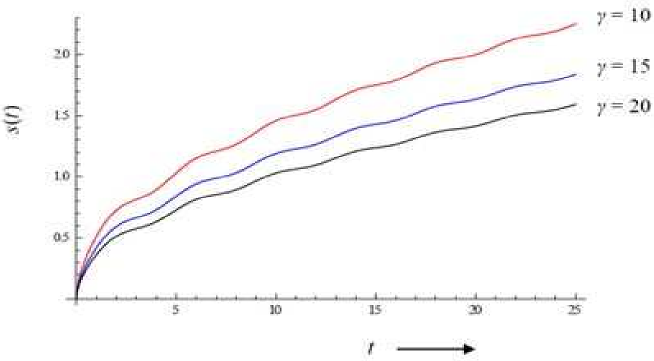

Figure 1: Plot of $s(t)$ vs. t at $\alpha = 1.0$, $q = 1.0$, $\epsilon = 0.5$, $\omega = \pi/2$.

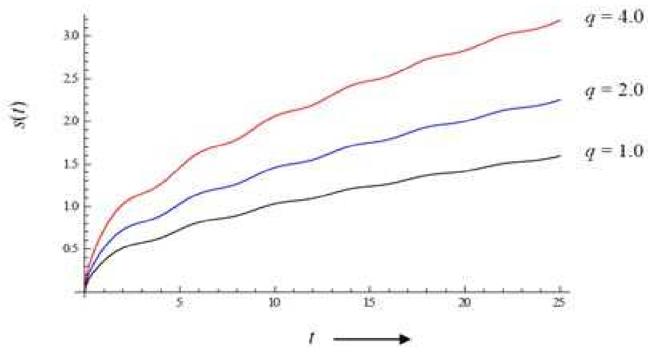

Figure 2: Plot of $s(t)$ vs. t at $\alpha = 1.0$, $\gamma = 20$, $\epsilon = 0.5$, $\omega = \pi/2$.

5 Conclusion

In this work, we study a complicated phase-change problem with a periodic heat flux and variable latent heat term. To the best of our knowledge, the exact solution to the proposed problem is not available in literature yet. Therefore, a homotopy analysis technique has been used to get an approximate analytical solution to the problem, and we have seen that our computed results are sufficiently close to the analytical solution when the surface heat flux is a constant, i.e. the oscillation amplitude is zero. In this paper, we have seen that the movement of interface/phase front is profoundly affected due to the change in various parameters like the oscillation amplitude, oscillation frequency, γ and q. It is also seen that the homotopy analysis technique is a straightforward method.

Moreover, this technique is sufficiently accurate and efficient to solve different types of phase-change problems arising in the various industries.

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Lie Group Classification of a Generalized Coupled Lane-Emden-Klein-Gordon-Fock System with Central Symmetry

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Abstract: In this paper, we perform a complete symmetry analysis of a generalized Lane-Emden-Klein-Fock system with central symmetry. Several cases for the non-equivalent forms of the arbitrary elements are obtained. Moreover, a symmetry reduction for some cases is performed.

Keywords: *Lie group classification; equivalent transformation; Lie point symmetries; similarity reduction.*

Mathematics Subject Classification (2010): 35J47, 35J61.

1 Introduction

In the recent paper [1], the author investigated both the Lie and Noether symmetries of a Lane-Emden-Klein-Fock system with central symmetry of the form

$$\begin{aligned} u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\gamma v^q}{r^n} &= 0, \\ v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\alpha u^p}{r^n} &= 0, \end{aligned} \tag{1}$$

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where p, n, γ, α, q are non-zero constants. If the constants $n = 2, \gamma = \alpha = 1$, system (1) reduces to

$$\begin{aligned} u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{v^q}{r^2} &= 0, \\ v_{tt} - v_{rr} - \frac{2}{r}v_r + \frac{u^p}{r^2} &= 0. \end{aligned} \tag{2}$$

Systems of this type occur in various physical phenomena, see, for example, [1–4] and references therein. Actually, system (1) can also be viewed as a natural extension of the well-known two-component generalization of the nonlinear wave equation, namely

$$u_{tt} - u_{rr} - \frac{m}{r}u_r - u^p = 0, \tag{3}$$

with the real-valued function $u = u(t, r)$, and p representing the interaction power while (t, r) denote time and radial coordinates, respectively, in $m \neq 0$ dimensions [4].

This system has been extensively studied in [2] for its Lie and Noether symmetries and the associated conservation laws for various values of the parameters p and q. More recently, hyperbolic versions of these types of system have also been investigated in [3]. Motivated by the recent results in [1–4], we study a generalized coupled Lane-Emden-Klein-Fock system with central symmetry of the form

$$\begin{aligned} u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} &= 0, \\ v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} &= 0, \end{aligned} \tag{4}$$

where $\Phi(v)$ and $\Psi(u)$ are arbitrary functions of v and u respectively.

The plan of this paper is as follows. In Section 2, we derive the equivalent generators of system (4). The Lie group classification of system (4) is performed in Section 3. In Section 4, we compute a symmetry reduction for some cases. Concluding remarks are given in Section 5.

2 Equivalence and Composition Transformations

In this section we employ the formulas derived in [5,6]. Applying the classical approach of group classification [7], we conclude that the generalized coupled Lane-Emden-Klein-Fock system (4) admits the following seven equivalence generators spanned by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}, \;\; X_2 = \frac{\partial}{\partial u}, \;\; X_3 = \frac{\partial}{\partial v}, \;\; X_4 = u\frac{\partial}{\partial u} + \Phi\frac{\partial}{\partial \Phi}, \;\; X_5 = v\frac{\partial}{\partial v} + \Psi\frac{\partial}{\partial \Psi}, \\ &X_6 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + (n-2)\Phi\frac{\partial}{\partial \Phi} + (n-2)\Psi\frac{\partial}{\partial \Psi}, \;\; X_7 = \frac{\partial}{\partial r} + \frac{n}{r}\Phi\frac{\partial}{\partial \Phi} + \frac{n}{r}\Psi\frac{\partial}{\partial \Psi} \end{aligned}$$

and the associated equivalence group is

$$\begin{aligned}
X_1 &: \bar{t}=a_1+t,\ \bar{r}=r, \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi,\\
X_2 &: \bar{t}=t,\ \bar{r}=r, \bar{u}=u+a_2,\ \bar{v}=v,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi,\\
X_3 &: \bar{t}=t,\ \bar{r}=r,\ \bar{u}=u,\ \bar{v}=v+a_3,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi,\\
X_4 &: \bar{t}=t,\ \bar{r}=r,\ \bar{u}=ue^{a_4},\ \bar{v}=v,\ \bar{\Phi}=\Phi e^{a_4}, \bar{\Psi}=\Psi,\\
X_5 &: \bar{t}=t,\ \bar{r}=r,\ \bar{u}=u,\ \ \bar{v}=ve^{a_5},\ \bar{\Phi}=\Phi, \bar{\Psi}=\Psi e^{a_5},\\
X_6 &: \bar{t}=te^{a_6},\ \bar{r}=re^{a_6},\ \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=\Phi e^{(n-2)a_6},\ \bar{\Psi}=\Psi e^{(n-2)a_6},\\
X_7 &: \bar{t}=t,\ \bar{r}=r+a_7,\ \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=(r+a_7)^n\frac{\Phi}{r^n},\ \bar{\Psi}=(r+a_7)^n\frac{\Psi}{r^n}.
\end{aligned}$$

Thus the corresponding composition of the above transformations is

$$\begin{aligned}
\bar{t} &= e^{a_6}(t+a_1),\\
\bar{r} &= e^{a_6}(r+a_7),\\
\bar{u} &= e^{a_4}(u+a_2),\\
\bar{v} &= e^{a_5}(v+a_3),\\
\bar{\Phi} &= e^{a_4+(n-2)a_6}\big[(r+a_7)^n r^{-n}\Phi\big],\\
\bar{\Psi} &= e^{a_5+(n-2)a_6}\big[(r+a_7)^n r^{-n}\Psi\big]. \qquad (5)
\end{aligned}$$

3 Group Classification of System (4)

A generalized coupled Lane-Emden-Klein-Fock system with central symmetry (4) is invariant under the group with the generator

$$X = \xi^1(t,r,u,v)\frac{\partial}{\partial t}+\xi^2(t,r,u,v)\frac{\partial}{\partial x}+\eta^1(t,r,u,v)\frac{\partial}{\partial u}+\eta^2(t,r,u,v)\frac{\partial}{\partial v} \qquad (6)$$

if and only if

$$X^{[2]}\left(u_{tt}-u_{rr}-\frac{n}{r}u_r+\frac{\Phi(v)}{r^n}=0\right)\Bigg|_{(4)}=0,\ X^{[2]}\left(v_{tt}-v_{rr}-\frac{n}{r}v_r+\frac{\Psi(u)}{r^n}=0\right)\Bigg|_{(4)}=0 \qquad (7)$$

with $X^{[2]}$ being the second extension of the generator (6) [4,6–9]. Expanding system (7) and solving the resulting determined system of partial differential equations for arbitrary $\Phi(v)$ and $\Psi(u)$ yield the one-dimensional principal Lie algebra spanned by

$$X_1=\frac{\partial}{\partial t} \qquad (8)$$

and the classifying relations are

$$\begin{cases} (\delta u+\theta)\Psi'(u)+\beta\Psi(u)+\alpha=0,\\ \\ (\lambda v+\gamma)\Phi'(v)+\psi\Phi(v)+\omega=0, \end{cases} \qquad (9)$$

where $\alpha,\beta,\gamma,\delta,\theta,\lambda$ and ω are constants. System (9) is invariant under the equivalence transformations (5) if

$$\begin{aligned}
&\bar{\delta}=\delta,\ \ \bar{\beta}=\beta,\ \ \bar{\lambda}=\lambda,\ \ \bar{\theta}=\delta a_2+\theta e^{-a_4},\ \ \bar{\psi}=\psi,\ \ \bar{\gamma}=\lambda a_3+\gamma e^{-a_5},\\
&\bar{\omega}=e^{(n-2)a_6-a_4}\left(\frac{r^n}{(r+a_7)^n}\right),\ \ \bar{\alpha}=e^{(n-2)a_6-a_5}\left(\frac{r^n}{(r+a_7)^n}\right).
\end{aligned}$$

A complete analysis of equation (9) yields the following cases for the non-equivalent forms of the arbitrary element $\Phi(v)$, $\Psi(u)$ and n:

Case 1: $\Phi(v)$ and $\Psi(u)$ are arbitrary, but not of the form as cases 2-8 given below. In this case, we obtain only the principal Lie algebra (8).

Case 1.1: $n = 2$.

The principal Lie algebra is extended by one symmetry, viz,

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$$

Case 2: $\Phi(v) = av^p$ and $\Psi(u) = bu^q$, where a, b, p and q are non-zero constants. This case reduces to the system studied in [1].

Case 3: $\Phi(v) = av^{-1}$ and $\Psi(u)$ is arbitrary, with a and n being non-zero constants. This case extends the principal Lie algebra by one symmetry, namely,

$$X_2 = v(n-2)\frac{\partial}{\partial v} - t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r}. \tag{10}$$

Case 4: $\Phi(v)$ is arbitrary and $\Psi(u) = bu^{-1}$, where b and n are non-zero constants. Again the algebra is two-dimensional and is spanned by (8) and

$$X_2 = u(n-2)\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r}.$$

Case 5: $\Phi(v) = av$ and $\Psi(u) = bu$, where a, b and n are constants.

Here the algebra extends by four, with the additional operators,

$$\begin{aligned}
&X_2 = \frac{\partial}{\partial u},\ X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_4 = av\frac{\partial}{\partial u} + bu\frac{\partial}{\partial v},\\
&X_5 = aH\frac{\partial}{\partial u} + \left[nr^{n-1}H_r + r^nH_{rr} - r^nH_{tt}\right]\frac{\partial}{\partial v},
\end{aligned}$$

where $H(t, r)$ is any solution of partial differential equation

$$\begin{aligned}
&br^3(c_1 + aH) + \left[4r^{2n}n^2 - 2r^{2n}n^3 - 2r^{2n}n\right]H_r + \left[3r^{2n+1}n - 5r^{2n+1}n^2\right]H_{rr}\\
&-4r^{2n+2}nH_{rrr} - r^{2n+3}H_{rrrr} + \left[2r^{2n+1}n^2 - r^{2n+1}n\right]H_{tt} + 4r^{2n+2}nH_{ttr}\\
&-r^{2n+3}H_{tttt} + 2r^{2n+3}H_{ttrr} = 0
\end{aligned}$$

and c_1 is an arbitrary constant.

Case 5.1: $n = 2$.

The Lie algebra extends by six additional generators,

$$\begin{aligned}
&X_2 = \frac{\partial}{\partial u},\ X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_4 = av\frac{\partial}{\partial u} + bu\frac{\partial}{\partial v},\\
&X_5 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad X_6 = 2tu\frac{\partial}{\partial u} + 2tv\frac{\partial}{\partial v} - (t^2 + r^2)\frac{\partial}{\partial t} - 2tr\frac{\partial}{\partial r},\\
&X_7 = aH\frac{\partial}{\partial u} + \left[2rH_r + r^2H_{rr} - r^2H_{tt}\right]\frac{\partial}{\partial v},
\end{aligned}$$

where $H(t, r)$ satisfies the partial differential equation

$$\begin{aligned}
&b(c_2 + aH) - 4rH_r - 14r^2H_{rr} - 8r^3H_{rrr} - r^4H_{rrrr} + 6r^2H_{tt} + 8r^3H_{ttr}\\
&-r^4H_{tttt} + 2r^4H_{ttrr} = 0
\end{aligned}$$

and c_2 is an arbitrary constant.

Case 6: $\Phi(v) = de^{-\lambda v}$ and $\Psi(u) = ke^{-au}$, where a, d, λ, k, n are constants.
Here the principle algebra enlarges by one operator,

$$X_2 = \lambda(n-2)\frac{\partial}{\partial u} + a(n-2)\frac{\partial}{\partial v} - \lambda at\frac{\partial}{\partial t} - \lambda ar\frac{\partial}{\partial r}. \tag{11}$$

Case 7: $\Phi(v) = mv^p$ and $\Psi(u) = ke^{-au}$, where p, a, m, k, n are arbitrary constants.
Again the Lie algebra extends by one generator,

$$X_2 = va(n-2)\frac{\partial}{\partial v} - (p+1)(n-2)\frac{\partial}{\partial u} + pat\frac{\partial}{\partial t} + par\frac{\partial}{\partial r}. \tag{12}$$

Case 8: $\Phi(v) = de^{-\lambda v}$ and $\Psi(u) = ku^q$, where λ, d, k, n are constants.
The principle algebra also enlarges by one generator,

$$X_2 = u\lambda(n-2)\frac{\partial}{\partial u} - (q+1)(n-2)\frac{\partial}{\partial v} + \lambda qt\frac{\partial}{\partial t} + \lambda qr\frac{\partial}{\partial r}.$$

4 Reduction of System (4)

This section aims to perform reduction of system (4) using some symmetries obtained in Section 3. To obtain the symmetry reduction of system (4), we begin with the principle Lie algebra (8) and take $\Phi(v)$ and $\Psi(u)$ arbitrary. Solving the invariant surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{du}{0} = \frac{dv}{0}$$

yields the following group invariant solution $u(t,r) = \phi(r)$, $v(t,r) = \psi(r)$ of system (4) where $\phi(r)$ and $\psi(r)$ satisfy

$$\begin{aligned} \phi'' + \frac{n}{r}\phi' - \frac{\Psi}{r^n} &= 0, \\ \psi'' + \frac{n}{r}\psi' - \frac{\Phi}{r^n} &= 0. \end{aligned} \tag{13}$$

We now choose case 3 with the generator (10). The integration of the invariant surface condition

$$\frac{dt}{-t} = \frac{dr}{-r} = \frac{du}{0} = \frac{dv}{v(n-2)}$$

gives the following invariant solution of system (4); $u(t,r) = \phi(z)$, $v(t,r) = r^{-(n-2)}\psi(z)$ with the similarity variable $z = \frac{t}{r}$. Substituting the values of u and v into system (4) we get

$$\begin{aligned} (z^2-1)\phi'' - (n-2)\phi' - \frac{a}{\psi} &= 0, \\ (z^2-1)\psi'' + (n-2)z\psi' - (n-2)\psi + \phi &= 0, \end{aligned} \tag{14}$$

where $\phi(z)$ and $\psi(z)$ are any solutions of the system of ordinary differential equations (14).

We now choose case 6 and the generator (11). After some straightforward but lengthy computations, we obtain the invariant $z = \frac{t}{r}$ and $u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a}$, $v(t,r) = \psi(z) + \frac{n\ln(r)}{\lambda} - \frac{2\ln(r)}{\lambda}$ as the group invariant solution of system (4), with $\phi(z)$ and $\psi(z)$ being any solutions of the system of ordinary differential equations

$$\begin{aligned}
&(z^2-1)\phi'' - (n-2)z\phi' - de^{-\lambda\phi} - \frac{(n-2)(n-1)}{a} = 0,\\
&(z^2-1)\psi'' - (n-2)z\psi' - ke^{-a\psi} - \frac{(n-2)(n-1)}{\lambda} = 0.
\end{aligned} \tag{15}$$

Another general group invariant solution of system (4) will be derived from case 7 with the operator (12). Considering the invariant surface condition

$$\frac{dt}{apt} = \frac{dr}{apr} = \frac{du}{(2-p)(p+1)} = \frac{dv}{av(n-2)}$$

we conclude that the group invariant solution of system (4) is $u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a} + \frac{n\ln(r)}{ap} - \frac{2\ln(r)}{ap}$, $v(t,r) = r^{\frac{-(n-2)}{p}}\psi(z)$ with the invariant $z = \frac{t}{r}$, where $\phi(z)$ and $\psi(z)$ satisfy the system of ordinary differential equations

$$\begin{aligned}
&(z^2-1)\phi'' - z(n-2)\phi' - m\psi^p - \frac{(p+1)(n-2)(n-1)}{ap} = 0,\\
&(z^2-1)\psi'' - \frac{(p+2)(n-2)}{p}z\psi' + \frac{(n-2)}{p^2}(p(n-1)+(n-2))\psi - ke^{-a\phi} = 0.
\end{aligned} \tag{16}$$

Following the aforementioned procedure, one can obtain more group invariant solutions for the generalized coupled Lane-Emden-Klein-Fock system with central symmetry system (4). It is worthy mentioning that all the cases that do not extend the principle Lie algebra have been excluded.

5 Conclusion

In this paper we performed a complete Lie symmetry classification of a generalized coupled Lane-Emden-Klein-Fock system with central symmetry (4). Several cases which resulted in Lie symmetries have been obtained. Moreover, some symmetry reductions for some cases were derived. In future, we would like to extend the results obtained in this manuscript by employing the techniques in [10–15].

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A Recursive Solution Approach to Linear Systems with Non-Zero Minors

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Abstract: In this paper, we introduce a recursive solution approach to linear systems of the form $Ax = b$, where A is non-singular and its corner minors are all non-zero. For the first time in the literature, we show how one can exploit (possible) useful information provided by corner sub-matrices of A towards an efficient solution approach to the linear system. This is going to initiate a thorough study of solution methods whose goals are to fully exploit available information within the given linear system.

Keywords: *linear system of equations; corner minors; matrix inversion; recursive methods.*

Mathematics Subject Classification (2010): 15A06, 15A09.

1 Introduction

The problem of solving a linear system $Ax = b$ is central to scientific computation [1], a subject which is used in most parts of modern mathematics. Computational solution methods of such system are often an important part of numerical linear algebra (see [2,3]), and play an important role in engineering, physics, chemistry, computer science, and economics [4]. Even more, systems of non-linear equations are often approximated by linear ones with the aim of linearization, a helpful technique while making a mathematical model or computer simulation of a relatively complex system. A reader interested in the applications of linear systems is referred to [4–7].

Iterative vs. direct solution methods for solving general linear systems have been gaining popularity in many areas of scientific computing [8, 9]. Until recently, direct

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solution methods were often preferred to iterative methods in real applications because of their robustness and predictable behavior [9]. However, to the best of our knowledge, none of the existing methods is capable of exploiting special information provided by the underlying linear system. This information could appear in an application setting within which a linear system with known solution is going to be expanded to a larger linear system. Other than that, simple matrix operations often reveal sub-matrices of A whose inverse are quickly computable. This paper initiates the study of linear systems when such information is available. We limit our attention to a special class of non-singular matrices and build necessary algebraic tools to study linear systems with such coefficient matrices.

The rest of the paper is organized as follows. In Section 2, we define and elaborate on the necessary notations and definitions needed in the paper. In Section 3, we build algebraic tools to derive matrix inverse while fully exploiting available information of inverse of a sub-matrix. We elaborate on the method by algorithmic restatement and also by giving an example. In Section 4, we explain how the result obtained in Section 3 can naturally result in a solution method to linear systems. Finally, in Section 5 we draw some conclusions and outline some possible avenues for further improvement.

2 Terminology

We consider a matrix $A = (a_{i,j})_{n\times m}$ of n rows and m columns. For any $1 \leq i \leq n$ and any $1 \leq j \leq m$, the i-th row and the j-th column of A are denoted by A^i and A_j, respectively. The index sequence of rows and columns of A are the sequence $\langle 1, 2, \cdots, n\rangle$ and $\langle 1, 2, \cdots, m\rangle$, respectively. Let us refer to A's index sequence of *rows* as A's *r-sequence* and A's index sequence of *columns* as A's *c-sequence*. Having a sub-sequence $\langle r_1, r_2, \cdots, r_p\rangle$ of the A's r-sequence and a sub-sequence $\langle c_1, c_2, \cdots, c_q\rangle$ of A's c-sequence, one can define a sub-matrix $S = (s_{i,j})_{p\times q}$ of A as $s_{i,j} = a_{r_i,c_j}$. Conversely, for any sub-matrix S of A, S's r-sequence and c-sequence are proper sub-sequences of A's r-sequence and A's c-sequence, respectively. In this setting, crossing off the i-th index in A's r-sequence defines a sub-matrix of A denoted by $\mathsf{del}^i(A)$. Similarly, crossing off the j-th index in A's c-sequence defines a sub-matrix denoted by $\mathsf{del}_j(A)$. If the deletion operations happen simultaneously, we get the sub-matrix $\mathsf{del}^i_j(A)$. We also need to define a matrix obtained by adding a new row and simultaneously a new column to A. Given indexes $1 \leq i \leq n+1$ and $1 \leq j \leq m+1$, and vectors $F_{1\times(n+1)}, G_{(m+1)\times 1}$ with $f_{1,j} = g_{i,1}$, the unique matrix B, defined by

$$B^i = F, \quad B_j = G, \quad \mathsf{del}^i_j(B) = A,$$

is denoted by $\mathsf{add}^i_j(A, F, G)$. The operators del and add will be extensively used in the following.

3 Computing A^{-1}

Given a non-singular $n \times n$ matrix A, suppose that there exists a square sub-matrix of A, say S, whose inverse is known (or quickly computable). The core question in this work asks: how can A^{-1} be computed using the available information on (the inverse of) the sub-matrix S? In this paper, we build our results on a special class of non-singular matrices for which every corner minor is non-zero.

Let us limit our attention and assume every corner minor of A is non-zero. Let $S = \mathsf{del}_n^n(A)$ and suppose that its inverse, S^{-1}, is known. Note that by the assumption on A, S^{-1} does exist. Define

$$B = \mathsf{add}_n^n(S, I^n, I_n) = \left(\begin{smallmatrix} S & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

whose inverse is simply

$$B^{-1} = \mathsf{add}_n^n(S^{-1}, I^n, I_n) = \left(\begin{smallmatrix} S^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

and consider the $n \times n$ square matrix C given by the equation $A = B.C$. Then C is simply given by

$$C = \left(\begin{smallmatrix} I_{(n-1)\times(n-1)} & V \\ a_{n,1}\cdots a_{n,n-1} & a_{n,n} \end{smallmatrix}\right) \text{ where } V = S^{-1}.(a_{1,n}, \cdots, a_{n-1,n})^T \tag{1}$$

and I is the identity matrix. Matrix C has the property that its inverse can be easily computed by means of the following lemma.

Lemma 3.1 Let $p = A^n.\left(\begin{smallmatrix} V \\ -1 \end{smallmatrix}\right)$, then p is non-zero and the $i-$th row of C^{-1} is given by

$$(C^{-1})^i = \begin{cases} \frac{1}{p}(A^n - (1 + a_{nn})I^n), & i = n \\ -v_i(C^{-1})^n + I^i, & i \neq n. \end{cases} \tag{2}$$

Proof. Knowing $C^{-1}C = I$, let us expand the equations obtained by $(C^{-1})^n C = I^n$:

$$\begin{aligned} & c^{-1}_{n,1} + c^{-1}_{nn} a_{n,1} = 0, \\ & c^{-1}_{n,2} + c^{-1}_{nn} a_{n,2} = 0, \\ & \vdots \\ & c^{-1}_{n,n-1} + c^{-1}_{n,n} a_{n,n-1} = 0, \\ & (c^{-1}_{n,1}, c^{-1}_{n,2}, \cdots, c^{-1}_{n,n-1}) \cdot V + c^{-1}_{n,n} a_{n,n} = 1. \end{aligned} \tag{3}$$

Now, the j-th equation gives $c^{-1}_{n,j} = -c^{-1}_{nn} a_{n,j}$ for each $j = 1, \cdots, n-1$. Then we write the last equation as

$$1 - c^{-1}_{n,n} a_{n,n} = -c^{-1}_{n,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,n-1}).V$$

and we get

$$\begin{aligned} & c^{-1}_{n,n}((a_{n,1} a_{n,2} \cdots a_{n,n-1}) \cdot V - a_{n,n}) = -1, \\ & c^{-1}_{n,n}\left((a_{n,1}, a_{n,2}, \cdots a_{n,n-1}, a_{n,n})\begin{pmatrix} V \\ -1 \end{pmatrix}\right) = -1, \\ & c^{-1}_{n,n}\left(A^n \begin{pmatrix} V \\ -1 \end{pmatrix}\right) = -1. \end{aligned} \tag{4}$$

This, in turn, implies that $c^{-1}_{n,n} = -\frac{1}{p}$, where $p = A^n\binom{V}{-1} \neq 0$. As a result

$$\begin{aligned}
(C^{-1})^n &= (c^{-1}_{n,1}, c^{-1}_{n,2}, \cdots, c^{-1}_{n,n-1}, c^{-1}_{n,n}) \\
&= (-c^{-1}_{n,n}a_{n,1}, -c^{-1}_{n,n}a_{n,2}, \cdots - c_{n,n-1}a_{n,n-1}, c^{-1}_{n,n}) \\
&= -c^{-1}_{n,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,n-1}, -1) \\
&= -c^{-1}_{n,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,n-1}, -1 - a_{nn} + a_{nn}) \\
&= \frac{1}{p}\left(A^n - (1 + a_{n,n}I^n)\right).
\end{aligned} \tag{5}$$

Now, in order to compute other rows of C^{-1}, let us expand the equations obtained by $(C^{-1})^iC = I^i, i \neq n$, as

$$\begin{aligned}
&c^{-1}_{i,1} + c^{-1}_{i,n}a_{n,1} = 0, \\
&c^{-1}_{i,2} + c^{-1}_{i,n}a_{n,2} = 0, \\
&\vdots \\
&c^{-1}_{i,i} + c^{-1}_{i,n}a_{n,i} = 1, \\
&\vdots \\
&c^{-1}_{i,n-1} + c^{-1}_{i,n}a_{n,n-1} = 0, \\
&(c^{-1}_{i,1}c^{-1}_{i,2} \cdots c^{-1}_{i,i} \cdots c^{-1}_{i,n-1}) \cdot V + c^{-1}_{i,n}a_{n,n} = 0.
\end{aligned} \tag{6}$$

One can write the first $n-1$ equations as

$$(c^{-1}_{i,1}, c^{-1}_{i,2}, \cdots, c^{-1}_{i,i}, \cdots, c^{-1}_{i,n-1}) = I^i - c^{-1}_{i,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,i}, \cdots, a_{n,n-1}).$$

Now, using the last equation in (6), we get

$$\begin{aligned}
&-c^{-1}_{i,n}a_{n,n} = I^i \cdot V - c^{-1}_{i,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,i}, \cdots, a_{n,n-1}) \cdot V, \\
&c^{-1}_{i,n}\left(A^n\begin{pmatrix} V \\ -1 \end{pmatrix}\right) = v_i.
\end{aligned}$$

This, in turn, implies that $c^{-1}_{i,n} = \frac{1}{p}.v_i$. As a result

$$\begin{aligned}
(C^{-1})^i &= (c^{-1}_{i,1}, c^{-1}_{i,2}, \cdots, c^{-1}_{i,i}, \cdots c^{-1}_{i,n-1}, c^{-1}_{i,n}) \\
&= -c^{-1}_{i,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,i}, \cdots, a_{n,n-1}, -1) + I^i \\
&= -c^{-1}_{i,n}(a_{n,1}, a_{n,2}, \cdots, a_{n,i}, \cdots, a_{n,n-1}, -1 + a_{n,n} - a_{n,n}) + I^i \\
&= -c^{-1}_{i,n}(A^n - (1 + a_{n,n})I^n) + I^i \\
&= -\frac{1}{p} \cdot v_i(A^n - (1 + a_{n,n})I^n) + I^i \\
&= -v_i.(C^{-1})^n + I^i.
\end{aligned}$$

This completes the proof. □

Having computed B^{-1} and C^{-1}, the inverse of A can be computed as $A^{-1} = C^{-1}B^{-1}$. Note how S^{-1} is used in computing A^{-1}. The equation also suggests a recursive procedure to obtain A^{-1} via its corner sub-matrices as described in Algorithm 3.1.

Algorithm 3.1 Computing A^{-1}

procedure $\textsc{Inverse}(A, n)$
  $S \leftarrow \mathsf{del}_n^n(A)$
  $S^{-1} \leftarrow \textsc{Inverse}(S, n-1)$
  $V \leftarrow S^{-1}.(a_{1,n}, \cdots, a_{n-1,n})^T$
  $B^{-1} \leftarrow \begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix}$
  $p = A^n \begin{pmatrix} V \\ -1 \end{pmatrix}$
  $(C^{-1})^i \leftarrow \begin{cases} \frac{1}{p}(A^n - (1 + a_{n,n}).I^n) & \text{if } i = n, \\ -v_i.(C^{-1})^n + I^i & \text{if } i \neq n. \end{cases}, \qquad \forall \quad i = 1, 2, \cdots, n$
  return $C^{-1}B^{-1}$

Example 3.1 Let

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 3 \end{array}\right)$$

and set $S = \mathsf{del}_4^4(A)$, which is simply $I_{3\times3}$. Then

$$\begin{aligned}
&V = I_{3\times3}.(2, -1, 1)^T = (2, -1, 1)^T, \\
&p = (1, 1, 0, 3).(2, -1, 1, -1)^T = -2, \\
&(C^{-1})^4 = -\frac{1}{2}\{(1, 1, 0, 3) - 4(0, 0, 0, 1)\} = (-0.5, 0.5, 0, 0.5), \\
&(C^{-1})^1 = -2.(\frac{-1}{2}, \frac{-1}{2}, 0, \frac{1}{2}) + (1, 0, 0, 0) = (2, 1, 0, -1), \\
&(C^{-1})^2 = +1.(\frac{-1}{2}, \frac{-1}{2}, 0, \frac{1}{2}) + (0, 1, 0, 0) = (\frac{-1}{2}, \frac{1}{2}, 0, \frac{1}{2}), \\
&(C^{-1})^3 = -1.(\frac{-1}{2}, \frac{-1}{2}, 0, \frac{1}{2}) + (0, 0, 1, 0) = (\frac{1}{2}, \frac{1}{2}, 1, \frac{-1}{2}).
\end{aligned}$$

Putting all together

$$C^{-1} = \begin{pmatrix} 2 & 1 & 0 & -1 \\ \frac{-1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

we have $A^{-1} = C^{-1}$ as computed above. □

4 Solving Linear System of Equations

Having a procedure to compute A^{-1}, as introduced above, automatically results in a solution procedure of the linear system $A.x = b$, where $x = (x_1, \cdots, x_n)^T$ and $b = (b_1, \cdots, b_n)^T$. Algorithm 3.1 will immediately translate to a recursive solution procedure of the linear system as follows.

Here again, we try to find a connection between the solution of the linear system and the solution of the subsystem $\mathsf{del}_n^n(A).y = \mathsf{del}^n(b)$. Recall that $S = \mathsf{del}_n^n(A)$.

Solution x of the linear system $Ax = b$ simply satisfies

$$S.\mathsf{del}^n(x) = \mathsf{del}^n(b) - x_n.\mathsf{del}^n(A_n), \tag{7}$$

$$\mathsf{del}_n(A^n).\mathsf{del}^n(x) = b_n - a_{n,n}.x_n. \tag{8}$$

Having S^{-1} available, one can rewrite (7) as

$$\mathsf{del}^n(x) = S^{-1}.\mathsf{del}^n(b) - x_n.S^{-1}.\mathsf{del}^n(A_n). \tag{9}$$

Note that the term $S^{-1}.\mathsf{del}^n(b)$ is the solution to the subsystem $S.y = \mathsf{del}^n(b)$. Then the solution to the system $Ax = b$ can be easily computed using equations (8) and (9). In this way, the solution process of the system $Ax = b$ can carefully make use of the information (possibly) available through the subsystem $S.y = \mathsf{del}^n(b)$.

Example 4.1 Let A be the matrix given in Example 1 and $b = (1, -2, 1, 4)^T$. In order to solve the system $Ax = b$, set $S = \mathsf{del}_4^4(A)$ which is simply $I_{3\times 3}$. Computing $\mathsf{del}^4(x)$ by equation (9) and putting it in equation (8) give

$$(1, 1, 0)\left(\left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right) - x_4\left(\begin{smallmatrix} 2 \\ -1 \\ 1 \end{smallmatrix}\right)\right) = 4 - 3x_4,$$

then $x_4 = 2.5$ and equation (9) computes

$$\mathsf{del}^4(x) = \left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right) - x_4\left(\begin{smallmatrix} 2 \\ -1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -4 \\ 0.5 \\ -1.5 \end{smallmatrix}\right).$$

So, we have $x = (\mathsf{del}^4(x), x_4)^T = (-4, 0.5, -1.5, 2.5)^T$. $\square$

Note that the way we solved the above linear system has an important capability with which different solution procedures of a linear system can be combined.

5 Conclusion

In this paper, we studied linear systems of the form $Ax = b$. When A admits non-zero corner minors, we showed a solution method could be devised capable of using available information provided by the corner submatrices of A. This, in turn, asks for a more detailed study of solution methods whose goals are to fully exploit available information within the given linear system having a general coefficient matrix.

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Numerical Solutions of Fractional Chemical Kinetics System

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Abstract: The aim of this paper was to investigate a fractional model of chemical kinetics system. The numerical solution of this fractional model is obtained by Bernstein polynomials. The basic idea is to apply operational matrices of fractional integration and multiplication of Bernstein polynomials. The important point to note here is the given problem turns into a set of algebraic equations by expanding the solution as Bernstein polynomials with unknown coefficients. Then, by solving algebraic equations, the numerical solutions are obtained. This result may be explained by the fact that the suggested technique is computationally efficient.

Keywords: *fractional model; chemical kinetics system; Caputo derivative; Bernstein polynomials.*

Mathematics Subject Classification (2010): 26A33, 34A08.

1 Introduction

One of the most significant current subjects in pure and applied mathematics is fractional calculus. Many applications have appeared in different areas of applied sciences such as physics and engineering [1–3]. A model is a simplified representation of a real world process. These models are an equation, a differential equation, an integral equation, a system of integral equations, etc. A chemical kinetics system is represented by a nonlinear system of ordinary differential equations.
Consider this model of a chemical process consisting of three species, which are denoted by A, B and C. The three reactions are:

$$A \longrightarrow B, \tag{1}$$

$$B + C \longrightarrow A + C, \tag{2}$$

$$B + B \longrightarrow C. \tag{3}$$

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We assume that the concentrations of A, B and C are indicated by ζ, η, and κ, respectively. We suppose that these concentrations are scaled so that the sum of three concentrations is one and that all of three constituent reactions are added with the concentration of some of the species accurately at the rate of the corresponding values of the reactants. We denote by θ_1 the the reaction rate of equation (1) . It indicates that the rate at which ζ decreases, and the rate at which η increases, because of this reaction, are equivalent to $\theta_1\zeta$. In the reaction showed by equation (2), C acts as a catalyst for the configuration of species A from B. The reaction rate is represented by using the symbol θ_2 which means the increase in the concentration ζ and the decrease in the concentration κ; this reaction has a rate and is equivalent to the product $\theta_2\eta\kappa$. Lastly, the formation of C from B has a constant rate equivalent to θ_3, which means the rate at which the mentioned reaction is occurring has to be equivalent to the product $\theta_3\eta^2$. We find the system of differential equations for the variation with time of the three concentrations to be:

$$
\begin{aligned}
\frac{d\zeta}{dt} &= -\theta_1\zeta(t) + \theta_2\eta(t)\kappa(t),\\
\frac{d\eta}{dt} &= \theta_1\zeta(t) - \theta_2\eta(t)\kappa(t) - \theta_3\eta^2(t),\\
\frac{d\kappa}{dt} &= \theta_3\eta^2(t).
\end{aligned} \tag{4}
$$

Since various materials and dynamical processes with memory and hereditary effects can be modeled by fractional order models better than integer-order models, we repleace the time-derivative in equation (4) by the Caputo fractional derivative:

$$
\begin{aligned}
{}_0D_t^\gamma\zeta(t) &= -\theta_1\zeta(t) + \theta_2\eta(t)\kappa(t),\\
{}_0D_t^\gamma\eta(t) &= \theta_1\zeta(t) - \theta_2\eta(t)\kappa(t) - \theta_3\eta^2(t),\\
{}_0D_t^\gamma\kappa(t) &= \theta_3\eta^2(t),
\end{aligned} \tag{5}
$$

with the initial conditions $\zeta(0) = 1$, $\eta(0) = 0$, $\kappa(0) = 0$.

In 2011, Aminikhah obtained the analytical approximation of chemical kinetics system using a homotopy perturbation method [4]. Two years later, Khader derived numerical solutions of this system using the Picard-Padé technique [5]. In 2017, Singh and coworkers considered the analysis of chemical kinetics system with a fractional derivative with the Mittag-Leffler type kernel [6] and numerous papers have been published on the analytical and numerical methods for solving nonlinear fractional differential equations such as [7–20]. In this paper, we apply Bernstein polynomials (Bps) for solving fractional chemical kinetics system. Here, we use operational matrices of fractional integration and multiplication of Bps. In equation (5), $D^\gamma\zeta(t)$ is indicated to be the Caputo fractional derivative of order γ which is defined as [1,3]:

$$
D_t^\gamma\zeta(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{\zeta(\tau)}{(t-\tau)^{1+\gamma-n}}d\tau, & n-1<\gamma<n, \quad n\in I\!N,\\[2ex] \dfrac{d^n\zeta(t)}{dt^n}, & \gamma = n. \end{cases} \tag{6}
$$

Note that

$$(i)\ {}_0D_t^\gamma\lambda = 0, \qquad (\lambda\ is\ a\ constant),$$

$$(ii) \qquad {}_0D_t^{\gamma} t^{\delta} = \begin{cases} 0, & \delta \in I\!N_0,\ \delta < \gamma, \\ \frac{\Gamma(\delta+1)}{\Gamma(1+\delta-\gamma)} t^{\delta-\gamma}, & Otherwise, \end{cases} \tag{7}$$

$$(iii) \qquad {}_0I_t^{\gamma}\, {}_0D_t^{\gamma} \zeta(t) = \zeta(t) - \sum_{l=0}^{n-1} \zeta^{(l)}(0^+) \frac{t^l}{l!}, \qquad n-1 < \gamma \le n. \tag{8}$$

In equation (8) the fractional Riemann-Liouville integral I_t^{γ} is described as [1,3]:

$${}_0I_t^{\gamma} \zeta(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\zeta(\tau)}{(t-\tau)^{1-\gamma}} d\tau, \qquad \gamma > 0. \tag{9}$$

The rest part of the present paper is organized as follows. The second section of this paper will impart Bernstein polynomials and approximation of function. Section 3 gives a brief overview of the operational matrix for fractional integration and multiplication of Bps. The suggested approach is used to approximate the fractional chemical kinetics system in the next Section 4. In Section 5, we assess the proposed technique with two examples. In the last section, conclusion is summarised.

2 Bernstein Polynomials and Approximation of Function

2.1 Definition of Bernstein polynomials

The Bernstein polynomials of the n-th degree on $[0,1]$ are presented as [21]:

$$\begin{aligned} B_{l,n}(t) &= \binom{n}{l} t^l (1-t)^{n-l} = \sum_{j=0}^{n-l} (-1)^j \binom{n}{l} \binom{n-l}{j} t^{l+j} \\ &= \sum_{j=l}^{n} (-1)^{j-l} \binom{n}{l} \binom{n-l}{j-l} t^j, \qquad l = 0, 1, \ldots, n. \end{aligned} \tag{10}$$

We can demonstrate $\phi(t) = \Lambda T_n(t)$, where $\phi(t) = [B_{0,n}(t), B_{1,n}(t), \cdots, B_{n,n}(t)]^T$, $T_n(t) = [1, t, \cdots, t^n]^T$ and $\Lambda = (\lambda_{l,j})_{l,j=1}^{n+1}$ is a matrix of order $(n+1)$ given in the form:

$$\lambda_{l+1,j+1} = \begin{cases} (-1)^{j-l} \binom{n}{l} \binom{n-l}{j-l}, & l \le j, \\ & l, j = 0, 1, \cdots, n, \\ 0, \quad l > j. & \end{cases} \tag{11}$$

2.2 Approximation of function

The set of Bernstein polynomials $\{B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)\}$ in Hilbert space $L^2[0,1]$ is a complete basis [22]. In consequence, we can indicate any function by BPs:

$$\zeta(t) = \sum_{l=0}^{n} z_l B_{l,n}(t) = Z^T \phi(t), \tag{12}$$

where $Z^T = [z_0, z_1, \ldots, z_n]$. Then, we can find Z^T as below:

$$Z^T = \left(\int_0^1 \zeta(t) \phi(t)^T dt \right) Q^{-1}. \tag{13}$$

In equation (13), Q is called the dual matrix of $\phi(t)$ and the Q is derived such that

$$Q = \int_0^1 \phi(t)\phi(t)^T dt. \tag{14}$$

3 Operational Matrix for Fractional Integration based on BPs

In this subsection, we want to investigate an operational matrix of fractional integration for Bps. Therefore, by fractional integration of the vector $\phi(t)$ as below, we get

$$ {}_0I_t^{\gamma}\phi(t) \simeq \mathbf{I}^{\gamma}\phi(t), \tag{15}$$

where $\mathbf{I}^{\gamma}$ is the $(n+1)\times(n+1)$ Riemann-Liouville fractional operational matrix of integration for BPs. Instead of using $\phi(t)$ we can substitute $\Lambda T_n(t)$, in consequence we get to:

$$\begin{aligned} {}_0I_t^{\gamma}\phi(t) &= {}_0I_t^{\gamma}\Lambda T_n(t) = \Lambda\ {}_0I_t^{\gamma}T_n(t) = \Lambda\left[{}_0I_t^{\gamma}1, {}_0I_t^{\gamma}t, \ldots, {}_0I_t^{\gamma}t^n\right]^T \\ &= \Lambda\left[\frac{0!}{\Gamma(\gamma+1)}t^{\gamma}, \frac{1!}{\Gamma(\gamma+2)}t^{\gamma+1}, \ldots, \frac{n!}{\Gamma(\gamma+n+1)}t^{\gamma+n}\right]^T = \Lambda\Theta\overline{T}_n(t), \end{aligned} \tag{16}$$

where Θ, being an $(n+1)\times(n+1)$ matrix, and $\overline{T}_n(t)$ are given by

$$\Theta_{i,j} = \begin{cases} \frac{i!}{\Gamma(\gamma+i+1)}, & i=j, \\ 0, & i\neq j. \end{cases} \qquad i,j = 0\cdots,n, \qquad \overline{T}_n = \left[t^{\gamma}, t^{\gamma+1}, \cdots, t^{\gamma+n}\right]^T. \tag{17}$$

In the same way as in Subsection 2.2, we approximate $t^{l+\alpha}$ as follows:

$$t^{\gamma+l} \simeq w_l^T\phi(t), \qquad l = 0,\cdots,n. \tag{18}$$

Therefore we have

$$\begin{aligned} w_l &= Q^{-1}\left(\int_0^1 t^{\gamma+l}\phi(t)dt\right) \\ &= Q^{-1}\left[\int_0^1 t^{\gamma+l}B_{0,n}(t)dt, \int_0^1 t^{\gamma+l}B_{1,n}(t)dt, \ldots, \int_0^1 t^{\gamma+l}B_{n,n}(t)dt\right]^T = Q^{-1}\overline{w}_l, \end{aligned} \tag{19}$$

where $\overline{w}_l = [\overline{w}_{l,0}, \overline{w}_{l,1}, \ldots, \overline{w}_{l,n}]^T$ and

$$\overline{w}_{l,k} = \int_0^1 t^{\gamma+l}B_{k,n}(t)dt = \frac{n!\Gamma(l+k+\gamma+1)}{k!\Gamma(l+n+\gamma+2)}, \qquad l,k = 0,1,\ldots,n, \tag{20}$$

where $w = [w_0, w_1, \cdots, w_n]^T$ is an $(n+1)\times(n+1)$ matrix that has vector $Q^{-1}\overline{w}_l$ for the i-th columns. Therefore, we can write

$$ {}_0I_t^{\gamma}\phi(t) \simeq \mathbf{I}^{\gamma}\phi(t) = \Lambda\Theta w^T\phi(t), \tag{21}$$

where $\mathbf{I}^{\gamma} = \Lambda\Theta w^T$ is called the fractional integration within the operational matrix.

4 Convergence Analysis

In the current section, we compute the error bounds of the operational matrices of fractional integrals for obtaining the convergence of the numerical approach introduced in the previous section.

Theorem 4.1 *Suppose that H is a Hilbert space and Y is a closed subspace of H such that $dimY < \infty$ and $\{y_1, y_2, \ldots, y_n\}$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then*

$$\|x - y_0\|^2 = \frac{G(x, y_1, y_2, \ldots, y_n)}{G(y_1, y_2, \ldots, y_n)}, \tag{22}$$

where

$$G(x, y_1, y_2, \ldots, y_n)= \begin{vmatrix} \langle x, x\rangle & \langle x, y_1\rangle & \ldots & \langle x, y_n\rangle \\ \langle y_1, x\rangle & \langle y_1, y_2\rangle & \ldots & \langle y_1, y_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, x\rangle & \langle y_n, y_1\rangle & \ldots & \langle y_n, y_n\rangle \end{vmatrix}.$$

Proof. See Kreyszig, 1978 [22].

Theorem 4.2 *Suppose that function $f \in L^2[0, 1]$ and $Y = Span\{B_{0,n}(t), B_{1,n}(t), \ldots, B_{n,n}(t)\}$, if $f(t)$ is approximated by*

$$f_n(t) = \sum_{l=0}^{n} c_l B_l(t) = C^T \phi(t), \tag{23}$$

where f_n is the best approximation of f out of Y.

Consider

$$L_n(f) = \int_0^1 [f(t) - f_n(t)]^2 dt,$$

then we have

$$\lim_{n \longrightarrow \infty} L_n(t) = 0.$$

Proof. For the proof see [19].

Now, by using these theorems, we compute the error upper bound of the operational matrix of the fractional integration $\mathbf{I}^\gamma$ based on Bernstein polynomials in the interval $[0, 1]$. Consider $E_{\mathbf{I}}^\gamma$ as the error vector of the operational matrix of fractional integration as

$$E_{\mathbf{I}}^\gamma = \mathbf{I}^\gamma \phi(t) - {}_0I_t^\gamma \phi(t), \tag{24}$$

where $E_{\mathbf{I}}^\gamma = \left[E_{\mathbf{I},0}^\gamma, E_{\mathbf{I},1}^\gamma, \cdots, E_{\mathbf{I},n}^\gamma\right]^T$.

The fractional integral of any Bernstein polynomial $B_{l,n}$ is given by

$$\begin{aligned}
{}_0I_t^{\gamma}B_{l,n} &= \sum_{j=l}^{n}(-1)^{j-l}\binom{n}{l}\binom{n-l}{j-l}{}_0I_t^{\gamma}t^j \\
&= \sum_{j=l}^{n}(-1)^{j-l}\binom{n}{l}\binom{n-l}{j-l}{}_0I_t^{\gamma}\frac{t^{j+\gamma}\Gamma(j+1)}{\Gamma(j+\gamma+1)} \\
&= \sum_{j=l}^{n}(-1)^{j-l}\frac{n!j!}{l!(j-l)!(n-2l-j)!\Gamma(j+\gamma+1)}t^{j+\gamma} = \sum_{j=l}^{n}b_{l,j}t^{j+\gamma}.
\end{aligned} \tag{25}$$

By virtue of (15), (24) and (25), we have

$$\begin{aligned}
\parallel E_{\mathbf{I},l}^{\gamma}\parallel_2 &= \parallel \mathbf{I}^{\gamma}B_{l,n}(t) - \sum_{k=0}^{n}(\sum_{j=l}^{n}b_{l,j}c_{j,k})B_{k,n}(t)\parallel \\
&\leq \sum_{j=l}^{n}(-1)^{j-l}\frac{n!j!}{l!(j-l)!(n-2l-j)!\Gamma(j+\gamma+1)}\parallel t^{j+\gamma} - \sum_{k=0}^{n}c_{j,k}B_{k,n}(t)\parallel \\
&\leq \sum_{j=l}^{n}b_{l,n}\left(\frac{G(t^{j+\gamma},B_{0,n}(t),B_{1,n}(t),\ldots,B_{n,n}(t))}{G(B_{0,n}(t),B_{1,n}(t),\ldots,B_{n,n}(t))}\right)^{\frac{1}{2}}.
\end{aligned} \tag{26}$$

We can conclude by Theorem 2 and equation (26) that by increasing the number of Bernstein bases, the error vector $E_{\mathbf{I},l}^{\gamma}$ tends to zero.

5 Numerical Results

In this section, we estimate the numerical results for the fractional chemical kinetics model for various values of γ by using the operational matrix of fractional integration and multiplication of Bps. For solving equation (5), we expand fractional derivatives by Bernstein polynomials as, say,

$$D_t^{\gamma}\zeta(t) = Z^T\phi(t), \qquad D_t^{\gamma}\eta(t) = N^T\phi(t), \qquad D_t^{\gamma}\kappa(t) = K^T\phi(t), \tag{27}$$

where

$$Z^T = [\zeta_0,\zeta_1,\cdots,\zeta_n]^T, \qquad N^T = [\eta_0,\eta_1,\cdots,\eta_n]^T, \qquad K^T = [\kappa_0,\kappa_1,\cdots,\kappa_n]^T.$$

Applying the fractional integral operator on the both sides of equation (27) and by replacing the initial condition in equation (28), then with the aid of equation (12) and equation (21) we can obtain the following result:

$$\begin{aligned}
&\zeta(t) = Z^T{}_0I_t^{\gamma}\phi(t) + \zeta(0) = Z^T\mathbf{I}^{\gamma}\phi(t) + d^T\phi(t) = G_1^T\phi(t), \\
&\eta(t) = N^T{}_0I_t^{\gamma}\phi(t) + \eta(0) = N^T\mathbf{I}^{\gamma}\phi(t) = G_2^T\phi(t), \\
&\kappa(t) = K^T{}_0I_t^{\gamma}\phi(t) + \kappa(0) = K^T\mathbf{I}^{\gamma}\phi(t) = G_3^T\phi(t).
\end{aligned} \tag{28}$$

Inserting equations (27) and (28) in equation (5), we have

$$\begin{aligned}
&Z^T\phi(t) = -\theta_1G_1^T\phi(t) + \theta_2G_3^T\hat{G_2}^T\phi(t), \\
&N^T\phi(t) = \theta_1G_1^T\phi(t) - \theta_2G_3^T\hat{G_2}^T\phi(t) - \theta_3G_2^T\hat{G_2}^T\phi(t), \\
&K^T\phi(t) = \theta_3G_2^T\hat{G_2}^T\phi(t),
\end{aligned} \tag{29}$$

where $\hat{G_2}$ is an operational matrix of product. For more information about an operational matrix of product, refer to [11]. Finally, we get the following set of algebraic equations as:

$$\begin{aligned} &Z^T + \theta_1 G_1^T - \theta_2 G_3^T \hat{G_2}^T = 0, \\ &N^T - \theta_1 G_1^T + \theta_2 G_3^T \hat{G_2}^T + \theta_3 G_2^T \hat{G_2}^T = 0, \\ &K^T - \theta_3 G_2^T \hat{G_2}^T = 0. \end{aligned} \tag{30}$$

By solving this system for the vectors ζ, η, κ, we can approximate $\zeta(t), \eta(t)$ and $\kappa(t)$ from (28). We have taken the values of parameters as $\theta_1 = 0.1$, $\theta_2 = 0.02$, and $\theta_3 = 0.009$. Comparisons between the exact solution and the numerical results obtained by this technique for $m = 6$ and different values of γ for $\zeta(t), \eta(t), \kappa(t)$ are shown in Fig. 1 respectively. Fig. 2 presents comparison between the exact and approximate solutions obtained by the help of BPs for $\zeta(t), \eta(t), \kappa(t)$ when $\gamma = 0.97$ and $m = 2, 3, 6$.

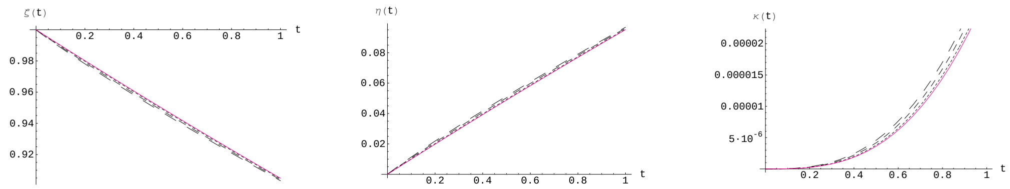

Figure 1: The exact solution: (red line) and approximation solutions $\zeta(t), \eta(t), \kappa(t)$ for $m = 6$ when $\gamma = 0.99$ (dotted), $\gamma = 0.97$ (dashed), $\gamma = 0.95$ (long-dashed).

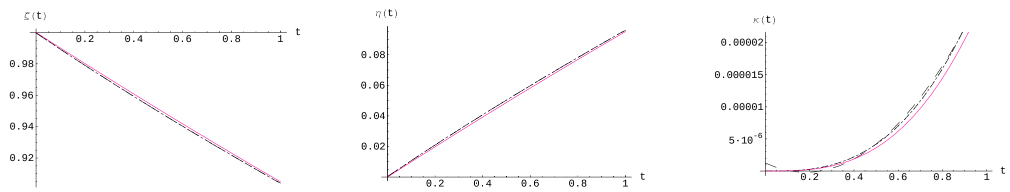

Figure 2: The exact solution: (red line) and approximation solutions $\zeta(t), \eta(t), \kappa(t)$ for $\gamma = 0.97$ when $m = 6$ (dotted), $m = 3$ (dashed), $m = 2$ (long-dashed).

6 Concluding Remarks and Discussion

In this work we have presented a numerical solution of the fractional chemical kinetics model using the operational matrices of fractional integration and multiplication based on BPs. The main advantage of this method is that the main problem reduces into a system of nonlinear algebraic equations. The obtained results demonstrate that only a small number of Bernstein polynomials bases is needed to obtain the accurate approximate solution via the present method. For the accuracy of the scheme we have given an example which shows that the results are much better.

The numerical simulations were carried out by Mathematica.

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Dual Phase Synchronization of Chaotic Systems Using Nonlinear Observer Based Technique

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Abstract: The present paper reports an investigation on dual phase synchronization results among chaotic systems with nonlinear observer controller. The dual phase synchronization is achieved using the nonlinear state observer technique and the stability theory. The Qi system and the Newton-Leipnik system are considered during the demonstration of dual phase synchronization. The nonlinear state observer technique is found to be very effective and convenient to achieve dual phase synchronization of various types of chaotic systems. Numerical simulation and graphical results demonstrate the effectiveness of the control technique during dual phase synchronization among chaotic systems.

Keywords: *dual synchronization, phase synchronization, chaotic systems, nonlinear state observer technique.*

Mathematics Subject Classification (2010): 34D06, 74H65, 34C28.

1 Introduction

Chaos theory is a developing field since 1970 and still the theory has not yet been understood very well. If a dynamical system is bounded and has infinite recurrences with dependency on initial conditions, then it is known as chaotic [1]. Several researchers have studied chaotic dynamical systems in various fields and effect of chaos in nonlinear dynamics is studied during the last few years. This effect is most common and has been detected in a number of dynamical systems of various types of physical nature. Chaos theory is also used to analyze the problems of dynamical and non-linear dynamical systems related with complex networks which are generally used in biological and social systems in ecology, medicine and in the field of business strategy. The most important achievement in the research of chaos is that chaotic systems can be made to synchronize with each other. The first idea of synchronization of two identical chaotic systems was

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analyzed by Pecora and Carrols [2]. In 2011, Runzi et al. [3] discussed combination synchronization using two master and one slave systems, before that synchronization was confined to one master and one slave systems. Yadav et al. [4] obtained dual function projective synchronization of fractional order complex chaotic systems.

In recent years, a lot of methods have been used to analyse synchronizations of the chaotic systems theoretically and experimentally, viz., the active control method, observer based method, backstepping method, nonlinear control method etc. Also, these methods are applied to study some new types of synchronizations, viz., combination synchronization, combination-combination synchronization, compound synchronization, multi-switching synchronization, compound-combination synchronization etc. ([5]- [9]). Juan and Xing-yuan [10] discussed nonlinear observer based phase synchronization of chaotic systems. Singh et. al. [11] explained dual combination synchronization of the fractional order complex chaotic systems.

The purpose of this paper is the investigation of dual phase synchronization of chaotic systems with nonlinear observer controllers. Dual synchronization is a special circumstance in synchronization in which two identical/non-identical pairs of chaotic systems are synchronized. The dual synchronization of systems plays an important role in many fields including chaotic secure communication. But it has received less attention of the researchers. There are only a few results available in the literature on dual synchronization between chaotic systems ([12]– [13]). In phase synchronization, the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. The phase synchronization is usually applied upon two waveforms of the same frequency with identical phase angles with each cycle. However it can be applied if there is an integer relationship of frequency such that the cyclic signals share a repeating sequence of phase angles over consecutive cycles. There are few results about the phase synchronizations for the chaotic systems ([14]– [17]). Observer design, having vital importance in the area of systems and control theory, arises whenever some components of the state are not directly measured. After the solution of multivariate problems in the linear time invariant case by Luenberger [18], many researchers were motivated to extend the basic ideas of his work to the nonlinear context. Though the applications of linear observer theory to nonlinear problems had been a success, still the researchers were reduced to construct nonlinear observers using tools from nonlinear systems theory. A brief introduction to some of these nonlinear approaches to the problem of observer design can be found in the paper of Primbs [19]. In 2012, Beikzadeh and Taghirad [20] presented a novel nonlinear continuous-time observer based on differential state-dependent Riccati equation filter with guaranteed exponential stability of the estimation error dynamics utilising Lyapunov stability analysis which is used to obtain the required conditions for exponential stability of the estimation error dynamics.

These results have motivated the authors to study the dual phase synchronization between two identical pairs of different chaotic systems with nonlinear state observer algorithm using stability theory.The numerical example is provided to illustrate the obtained results. Dual phase synchronization between the systems with time delays ([21]– [25]) using the similar method will be considered for future study.

2 Problem Formulation

Let us consider the following two chaotic systems:

$$\dot{x} = Ax + Bf(x), \tag{1}$$

$$\dot{y} = Cy + Dg(y), \tag{2}$$

where $x, y \in R^n$ are the state vectors of the systems (1) and (2). $A, B \in R^{n\times n}$,$C, D \in R^{n\times m}$ are the constant matrices and $f, g : R^n \to R^m$ are the nonlinear functions. Suppose the dynamical systems (1) and (2) with the output are represented as

$$s(x) = f(x) + K_j x, \tag{3}$$

$$S(y) = g(y) + K'_j y, \tag{4}$$

where $K_j, K'_j \in R^{m\times n}$ denote the feedback gain matrices. Let us define the observer as

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + B[s(x) - s(\hat{x})], \tag{5}$$

$$\dot{\hat{y}} = C\hat{y} + Dg(\hat{y}) + D[S(y) - S(\hat{y})]. \tag{6}$$

The synchronization errors among the systems (1), (2) and (5), (6) are defined as

$$e_{x\hat{x}} = x - \hat{x}, \tag{7}$$

$$e_{y\hat{y}} = y - \hat{y}. \tag{8}$$

Then the error systems can be obtained as

$$\dot{e}_{x\hat{x}} = \dot{x} - \dot{\hat{x}} = Ae_{x\hat{x}} + Bf(x) - Bf(\hat{x}) - B[s(x) - s(\hat{x})],$$

$$\dot{e}_{y\hat{y}} = \dot{y} - \dot{\hat{y}} = Ce_{y\hat{y}} + Dg(y) - Dg(\hat{y}) - D[S(y) - S(\hat{y})].$$

From equations (3) and (4), the error systems reduce in the following form

$$\dot{e}_{x\hat{x}} = [A - BK_j]e_{x\hat{x}}, \tag{9}$$

$$\dot{e}_{y\hat{y}} = [C - DK'_j]e_{y\hat{y}}. \tag{10}$$

In order to make systems (9) and (10) controllable with the controllable matrices $[B, AB, ...A^{n-1}B]$ and $[D, CD, ...C^{n-1}D]$ of full ranks, the choices of the feedback gain matrices,K_j, K'_j will be in such a way that the characteristic polynomials of the matrices $[A - BK_j]$ and $[C - DK'_j]$ must have all the eigenvalues with negative real parts. Then the error systems will be stabilized and the dual synchronization among the systems under consideration is achieved. If there is any eigenvalue of the error system equal to zero, then another type of synchronization phenomenon called the phase synchronization occurs, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant.

3 Systems' Descriptions

3.1 Qi chaotic system

Consider the following Qi system [26]:

$$\dot{x}_1 = a_1(x_2 - x_1) + x_2x_3; \ \ \dot{x}_2 = a_3x_1 - x_2 - x_1x_3; \ \ \dot{x}_3 = -a_2x_3 + x_1x_2, \tag{11}$$

where x_1, x_2, x_3 are the state variables. The phase portrait of the system (11) for the parameter values $a_1 = 35, a_2 = 8/3, a_3 = 80$ and the initial condition $(3, 2, 1)$ is depicted in Fig. 1(a).

3.2 Newton-Leipnik system

The Newton-Leipnik system [27] is defined as

$$\dot{y}_1 = -b_1 y_1 + y_2 + 10 y_2 y_3;\;\; \dot{y}_2 = -y_1 - 0.4 y_2 + 5 y_1 y_3;\;\; \dot{y}_3 = b_2 y_3 - 5 y_1 y_2. \tag{12}$$

The phase portrait of the Newton-Leipnik system (12) is depicted in Fig. 1(b) for the values of the parameters $b_1 = 0.4, b_2 = 0.175$ and the initial condition $(0.394, 0, -0.16)$.

4 Dual Phase Synchronization of Chaotic Systems

In this section we are taking two systems, viz., Qi and Newton-Leipnik, to perform dual phase synchronization. The systems (11) and (12) can be rewritten as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_2x_3\\ x_1x_3\\ x_1x_2\end{bmatrix} \tag{13}$$

and

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{bmatrix} = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix} + \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}\begin{bmatrix}y_2y_3\\ y_1y_3\\ y_1y_2\end{bmatrix}. \tag{14}$$

Comparing equations (13) and (14) with equations (1) and (2), we get

$$A = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}, B = \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}, C = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}, D = \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}.$$

The observers of the systems (11) and (12) are designed as

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\end{bmatrix} = \begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_2\hat{x}_3\\ \hat{x}_1\hat{x}_3\\ \hat{x}_1\hat{x}_2\end{bmatrix} + B[s(x) - s(\hat{x})], \tag{15}$$

$$\begin{bmatrix}\dot{\hat{y}}_1\\ \dot{\hat{y}}_2\\ \dot{\hat{y}}_3\end{bmatrix} = \begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix}\begin{bmatrix}\hat{y}_1\\ \hat{y}_2\\ \hat{y}_3\end{bmatrix} + \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix}\begin{bmatrix}\hat{y}_2\hat{y}_3\\ \hat{y}_1\hat{y}_3\\ \hat{y}_1\hat{y}_2\end{bmatrix} + D[S(y) - S(\hat{y})], \tag{16}$$

where $B[s(x)-s(\hat{x})], D[S(y)-S(\hat{y})]$ are outputs of the systems. Now by defining the error function towards dual synchronization as $e_{x_1\hat{x}_1} = x_1 - \hat{x}_1, e_{x_2\hat{x}_2} = x_2 - \hat{x}_2, e_{x_3\hat{x}_3} = x_3 - \hat{x}_3$, $e_{y_1\hat{y}_1} = y_1 - \hat{y}_1, e_{y_2\hat{y}_2} = y_2 - \hat{y}_2, e_{y_3\hat{y}_3} = y_3 - \hat{y}_3$, the error systems can be obtained as

$$\begin{bmatrix}\dot{e}_{x_1\hat{x}_1}\\ \dot{e}_{x_2\hat{x}_2}\\ \dot{e}_{x_3\hat{x}_3}\end{bmatrix} = \left\{\begin{bmatrix}-a_1 & a_1 & 0\\ a_3 & -1 & 0\\ 0 & 0 & -a_2\end{bmatrix} - \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix} K_1\right\}\begin{bmatrix}e_{x_1\hat{x}_1}\\ e_{x_2\hat{x}_2}\\ e_{x_3\hat{x}_3}\end{bmatrix}, \tag{17}$$

$$\begin{bmatrix}\dot{e}_{y_1\hat{y}_1}\\ \dot{e}_{y_2\hat{y}_2}\\ \dot{e}_{y_3\hat{y}_3}\end{bmatrix} = \left\{\begin{bmatrix}-b_1 & 1 & 0\\ -1 & -0.4 & 0\\ 0 & 0 & b_2\end{bmatrix} - \begin{bmatrix}10 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -5\end{bmatrix} K_1'\right\}\begin{bmatrix}e_{y_1\hat{y}_1}\\ e_{y_2\hat{y}_2}\\ e_{y_3\hat{y}_3}\end{bmatrix}. \tag{18}$$

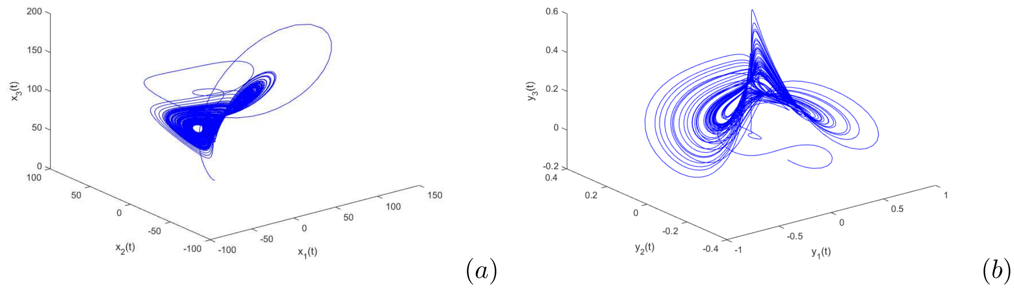


Figure 1: Phase portraits of chaotic systems: (a) the Qi system; (b) the Newton-Leipnik system.

The matrices $[B, AB, A^2B]$ and $[D, CD, C^2D]$ are in full ranks, so the systems (15) and (16) are the global observers of systems (13) and (14) through proper choices of the feedback gain matrices towards the synchronization

$$K_1 = \begin{bmatrix} -34 & 35 & 0 \\ -80 & 0 & 0 \\ 0 & 0 & -5/3 \end{bmatrix}, \quad K_1' = \begin{bmatrix} -3/50 & 1/10 & 0 \\ -1/5 & 3/25 & 0 \\ 0 & 0 & -0.235 \end{bmatrix}.$$

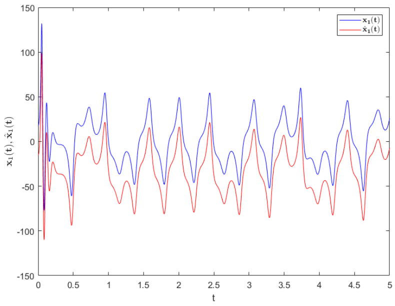
For phase synchronization of the above-mentioned systems, the feedback gain matrices are taken as

$$K_1 = \begin{bmatrix} -35 & 35 & 0 \\ -80 & 1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \quad K_1' = \begin{bmatrix} -2/50 & 1/10 & 0 \\ -1/5 & -2/25 & 0 \\ 0 & 0 & -0.035 \end{bmatrix}.$$

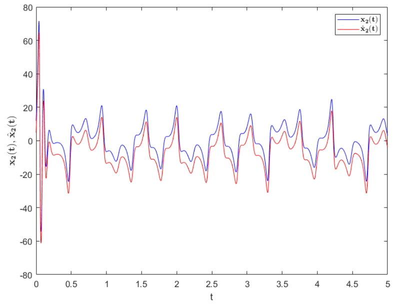
5 Numerical Simulation and Results

During numerical simulation the earlier considered parameters of the chaotic systems are taken. For the dual phase synchronization the initial conditions of the master systems I, II and slave systems I, II are taken as $(x_1(0), x_2(0), x_3(0)) = (18, 12, 10), (y_1(0), y_2(0), y_3(0)) = (0.349, 1.5, -0.16)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-15, 5, 1), (\hat{y}_1(0), \hat{y}_2(0), \hat{y}_3(0)) = (0.5, 2.5, 0.5)$, respectively. Hence the initial conditions of error system for dual phase synchronization will be $(33, 7, 9, -0.151, -1, -0.66)$. During dual synchronization of the systems, the time step size is taken as 0.005. Now, by choosing $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$, the phase synchronization between signals $x_1(t)$ and $\hat{x}_1(t)$ is achieved. It should be noted that, when $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$, the signals $x_2(t)$ and $\hat{x}_2(t)$ and $x_3(t)$ and $\hat{x}_3(t)$ and $y_1(t)$ and $\hat{y}_1(t)$ and $y_2(t)$ and $\hat{y}_2(t)$ and $y_3(t)$ and $\hat{y}_3(t)$ become synchronized. Similarly, if $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$; $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 0, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = -1$; $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = 0, \lambda_5 = -1, \lambda_6 = -1$; $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = 0, \lambda_6 = -1$ and $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -1, \lambda_6 = 0$ are taken, phase synchronizations between signals $x_2(t)$ and $\hat{x}_2(t)$ and $x_3(t)$ and $\hat{x}_3(t)$ and $y_1(t)$ and $\hat{y}_1(t)$ and

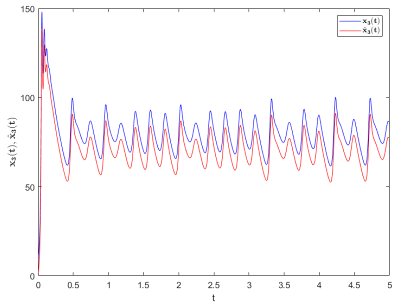
$y_2(t)$ and $\hat{y}_2(t)$ and $y_3(t)$ and $\hat{y}_3(t)$ are obtained, respectively. State trajectories of the dual phase synchronization of chaotic systems are depicted in Fig. 2(a)-(f). The plot of the error function for dual synchronization is depicted in Fig. 2(g), which shows that error states converge to zero when time becomes large. This implies that the dual phase synchronization between identical pairs of different chaotic systems consisting of the Qi and Newton-Leipnik systems occurs with the help of nonlinear observers.



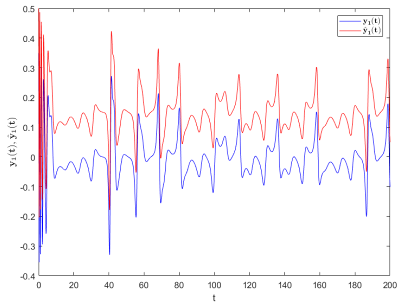
(a)



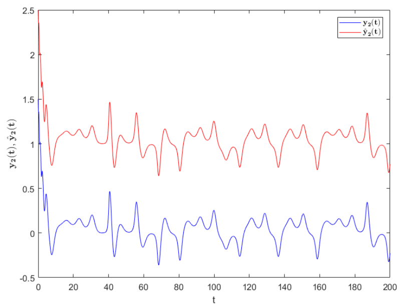
(b)



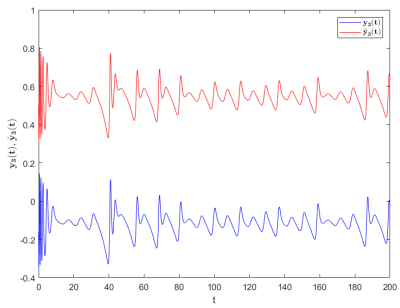
(c)



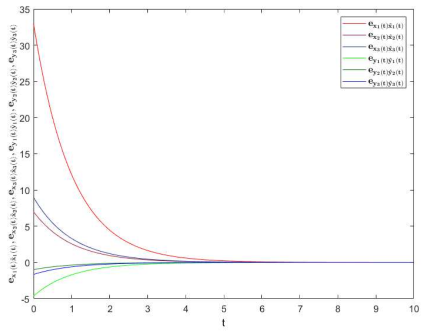
(d)



(e)



(f)



(g)

Figure 2: Phase synchronization for signals (a) between $x_1(t)$ and $\hat{x}_1(t)$, (b) between $x_2(t)$ and $\hat{x}_2(t)$, (c) between $x_3(t)$ and $\hat{x}_3(t)$, (d) between $y_1(t)$ and $\hat{y}_1(t)$, (e) between $y_2(t)$ and $\hat{y}_2(t)$, (f) between $y_3(t)$ and $\hat{y}_3(t)$, (g) The evolution of the error functions of chaotic systems during synchronization.

6 Conclusion

The present paper has successfully demonstrated the dual phase synchronization between the Qi and Newton-Leipnik systems using the nonlinear observer based technique. Based on the stability analysis, the dual phase synchronization of chaotic systems through nonlinear controller input parameters on the respective systems has been achieved and the components of the error system tend to zero as time becomes large, which helps to find the time required for dual phase synchronization between different chaotic systems. Numerical simulations are given to exhibit the reliability and effectiveness of the proposed dual combination synchronization scheme towards predicting the accuracy of the theory. The authors are optimistic that the outcome of this chapter will be utilized by the researchers involved in the field of chaotic systems.

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