Volume 18, Number 3, 2018 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 18 Number 3 2018

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 ISSN 1562-8353 Printed in Ukraine
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NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 18, No. 3, 2018

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An International Journal of Research and Surveys

InforMath Publishing Group
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NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 18 Number 3 2018

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Founded by A.A. Martynyuk in 2001.

Registered in Ukraine Number: KB 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2017: SNIP – 0.707, SJR – 0.316

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

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Nonlinear Dynamics and Systems Theory will have 4 issues in 2018, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (`mailto:journals@ebsco.com`), or website of the Journal: `http://e-ndst.kiev.ua`. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 213–224

Existence of Solution for Nonlinear Anisotropic Degenerated Elliptic Unilateral Problems

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Received: May 10, 2017; Revised: June 18, 2018

Abstract: In this paper, we prove the existence of entropy solutions of anisotropic elliptic equations $Au + \sum_{i=1}^{N} g_i(x, u, \nabla u) = f$, where the operator Au is a Leray-Lions anisotropic operator from $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ into its dual $W^{-1,\overrightarrow{p'}}(\Omega, \overrightarrow{\omega^*})$. The critical growth condition on g_i is with respect to ∇u and there is no the growth condition with respect to u and no the sign condition. The right-hand side f belongs to $L^1(\Omega)$.

Keywords: *nonlinear elliptic equations; quasilinear degenerated unilateral problems; non-variational inequalities.*

Mathematics Subject Classification (2010): 35J60, 35J70, 35J87.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary and let $Au = -\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u)$ be a degenerate anisotropic operator of Leray-Lions type defined in the weighted anisotropic Sobolev space $W^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, where $\overrightarrow{\omega} = (\omega_0, \omega_1, ..., \omega_N)$ is a vector of weight functions defined on Ω and $\overrightarrow{p} = (p_0, ..., p_N)$ is a vector of real number such that $p_i > 1$ for $i = 0, ..., N$.

We consider the following nonlinear elliptic anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u) + \sum_{i=1}^{N} g_i(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = 0, \quad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $g_i(x, s, \xi)$ is a Carathéodory function satisfying only the following growth condition $|g_i(x, s, \xi)| \leq \gamma(x) + \rho(s)|\xi_i|^{p_i}$ and where the right-hand side f belongs to $L^1(\Omega)$. In the

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particular case, where $\sum_{i=1}^{N} g_i(x,s,\xi) = -C_0|u|^{p-2}u$, the following degenerated equation $-\mathrm{div}(a(x,u,\nabla u)) - C_0|u|^{p-2}u = f(x,u,\nabla u)$ has been studied by Drabek-Nicolsi in [11] under more degeneracy and some additional assumptions on f and $a(x,u,\nabla u)$.

In the isotropic case, more precisely, when $p_0 = p_1 = ... = p_N = p$ and $\sum_{i=1}^{N} g_i(x,u,\nabla u) \equiv g(x,u,\nabla u)$, the existence result for the unilateral problem with $g(x,u,\nabla u)$ satisfying the following growth condition

$$|g(x,s,\xi)| \leq b(|s|)(C(x) + \sum_{i=1}^{N} \omega_i |\xi_i|^p) \tag{2}$$

and the sign condition

$$g(x,s,\xi)s \geq 0, \tag{3}$$

when f belongs to $W^{-1,p'}(\Omega,\omega^*)$, is studied by Akdim et al. in [7] under the following integrability condition

$$\sigma^{1-q'} \in L^1_{loc}(\Omega) \quad \text{with} \quad 1 < q < +\infty, \tag{4}$$

where σ is a weight function which is assumed satisfying the Hardy inequality

$$\int_\Omega |u|^q \sigma(x)dx \leq C\Big(\sum_{i=1}^{N} \int_\Omega |\partial_i u|^p \omega_i(x)dx\Big)^{\frac{1}{p}}. \tag{5}$$

Our aim in this paper is to prove the existence of entropy solution for the following weighted unilateral elliptic anisotropic problem

$$\begin{cases} u \geq \psi \ a.e. \text{ in } \Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}), \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\partial_i T_k(u-v) + \sum_{i=1}^{N} \int_\Omega g_i(x,u,\nabla u) T_k(u-v) \leq \int_\Omega f T_k(u-v), \\ \forall v \in K_\psi(\Omega,\overrightarrow{\omega}) \cap L^\infty(\Omega) \text{ and } \forall k > 0, \end{cases} \tag{6}$$

without the conditions (3) and (4).

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the Lipschitz continuous boundary and let $1 < p_0, p_1, ..., p_N < \infty$ be $N+1$ real numbers, $p^+ = \max\{p_1,...,p_N\}$, $p^- = \min\{p_1,...,p_N\}$. We denote $\partial_i = \dfrac{\partial}{\partial x_i}$, let ω_i be non negative functions on Ω such that $\omega_i > 0$ a.e. in Ω for all $i = 0,1,...,N$. We set $\overrightarrow{\omega} = (\omega_0,\omega_1,...,\omega_N)$ and $\overrightarrow{p} = (p_0,p_1,...,p_N)$. We suppose that for $i = 0,1,...,N$ and for $j = 0,1,...,N$

$$\omega_i \in L^1_{loc}(\Omega) \text{ and } \omega_i^{-\frac{1}{p_j-1}} \in L^1_{loc}(\Omega). \tag{7}$$

As the classical weighted Sobolev space in [10], we define the anisotropic weighted Sobolev space by

$$W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) = \Big\{u \in L^{p_0}(\Omega,\omega_0) : \partial_i u \in L^{p_i}(\Omega,\omega_i), i = 1,2,...,N\Big\}.$$

As in Theorem 1.11 in [13], by (7) the space $W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is a Banach space under the following norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})} = \|u\|_{L^{p_0}(\Omega,\omega_0)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega,\omega_i)}. \tag{8}$$

Since $\omega_i \in L^1_{loc}(\Omega)$, we have that $C_0^\infty(\Omega)$ is a subset of $W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ and we can introduce the space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ as the closure of $C_0^\infty(\Omega)$ with respect to norm (8). We recall that the dual space of weighted anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is equivalent to $W^{-1,\overrightarrow{p'}}(\Omega,\overrightarrow{\omega^*})$, where $\overrightarrow{\omega^*} = (\omega_1^*,...,\omega_N^*)$, $\omega_i^* = \omega_i^{1-p_i'}$, $\overrightarrow{p'} = (p_1',...,p_N')$ and $p_i' = \frac{p_i}{p_i-1}$, for all $i = 1,...,N$.

Now, we introduce the following assumptions:

Assumptions (H_1):

– The expression

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})} = \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega,\omega_i)} \tag{9}$$

is a norm defined on $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ and it is equivalent to the norm (8).

– There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality

$$\Big(\int_\Omega |u|^q \sigma dx\Big)^{\frac{1}{q}} \leq C \sum_{i=1}^{N} \Big(\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} w_i\Big)^{\frac{1}{p_i}} \tag{10}$$

holds for every $u \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$, where C is a positive constant independent of u.

– The embedding

$$W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) \hookrightarrow L^q(\Omega,\sigma) \tag{11}$$

expressed by (10) is compact.

Remark 2.1 Let us take $p_0 = p_1 = p_2 = ... = p_N = p$, $\omega_0(x) = \omega_1(x) = \omega_2(x) = ... = \omega_N(x) = [dist(x,\partial\Omega)]^\lambda$ and $\sigma(x) = [dist(x,\partial\Omega)]^\gamma$, $\lambda,\gamma \in \mathbb{R}$. In this case, the Hardy inequality reads

$$\Big(\int_\Omega |u|^q [dist(x,\partial\Omega)]^\gamma\Big)^{\frac{1}{q}} dx \leq \sum_{i=1}^{N} \Big(\int_\Omega |\partial_i u|^p [dist(x,\partial\Omega)]^\lambda dx\Big)^{\frac{1}{p}}.$$

The imbedding $W_0^{1,p}(\Omega, dist(x,\partial\Omega)) \hookrightarrow L^q(\Omega, dist(x,\partial\Omega))$ is compact (see Example 1.5 in [10]) if and only if either:

i) $1 < p \leq q < +\infty$, $\lambda < p-1$, $\frac{N}{q} - \frac{N}{p} + 1 \geq 0$, $\gamma \geq \lambda\frac{q}{p} - N + N\frac{q}{p} - q$ or

ii) $1 \leq q < p < +\infty$, $\lambda < p-1$, $\gamma \geq \lambda\frac{q}{p} - 1 + \frac{q}{p} - q$.

Similarly, in the isotropic case, see [1], we can construct an isometric from $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ in $\prod_{i=1}^N L^{p_i}(\Omega,\omega_i)$ which implies with (7) that the space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is a reflexive and separable Banach space. Moreover, we consider $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) = \{u \text{ measurable in } \Omega : T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}), \forall k > 0\}$.

3 Mains Results

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the Lipschitz continuous boundary $\partial\Omega$. The functions a: $\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ such that $a(x,s,\xi) = \Big(a_1(x,s,\xi),...,a_N(x,s,\xi)\Big)$ and g_i: $\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ with a_i and g_i are Carathéodory functions satisfying the following assumptions for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N, \xi^{'} \in \mathbb{R}^N$ and a. e. in Ω :

Assumptions H_2:

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \alpha \sum_{i=1}^{N} \omega_i |\xi_i|^{p_i}, \tag{12}$$

$$|a_i(x,s,\xi)| \leq \beta\omega_i^{\frac{1}{p_i}}[j_i(x) + \sigma^{\frac{1}{p_i'}}|s|^{\frac{q}{p_i'}} + \omega^{\frac{1}{p_i'}}|\xi_i|^{p_i-1}], \tag{13}$$

$$(a_i(x,s,\xi) - a_i(x,s,\xi^{'}))(\xi_i - \xi_i^{'}) > 0 \quad \text{for } \xi_i \neq \xi_i^{'}, \tag{14}$$

where α, β are some positive constants, j_i is a positive function in $L^{p_i^{'}}(\Omega)$.

Assumptions H_3:

$$|g_i(x,s,\xi)| \leq \gamma(x) + \rho(s)\omega_i|\xi_i|^{p_i} \ \forall i = 1,....,N, \tag{15}$$

where γ is a positive function in $L^1(\Omega)$ and $\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function in $L^1(\mathbb{R})$.

Moreover, we suppose that

$$f \in L^1(\Omega). \tag{16}$$

Let us define the convex set $K_\psi(\Omega,\overrightarrow{\omega}) = \{u \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}),\ u \geq \psi \ a.e. \text{ in } \Omega\}$, where ψ is a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) \cap L^\infty(\Omega). \tag{17}$$

3.1 Some technical lemmas

The following lemma generalizes to the anisotropic case the analogous Lemma 5 in [9]. We use the method of [7] and [9].

Lemma 3.1 *Assume that (12)-(14) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ and $\lim_{n\to+\infty}\int_\Omega \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)\nabla(u_n - u)$ $= 0$. Then $u_n \to u$ strongly in $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ for a subsequence.*

Definition 3.1 A function u is an entropy solution for problem (1) if it satisfies (6).

Theorem 3.1 *Assume that (12)-(17) hold. Then there exists at least one entropy solution in the sense of the definition (3.1) of problem (1).*

Proof of Theorem 3.1.
The proof of this theorem is done in four steps.
Step 1 : Approximate problems.
We consider the following approximate problems

$$\begin{cases} u_n \in K_\psi(\Omega, \overrightarrow{\omega}). \\ \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - v) + \sum_{i=1}^{N} \int_\Omega g_i^n(x, u_n, \nabla u_n)(u_n - v) \leq \int_\Omega f_n(u_n - v), \\ \forall v \in K_\psi(\Omega, \overrightarrow{\omega}), \end{cases} \tag{18}$$

where $g_i^n(x, s, \xi) = \frac{g_i(x,s,\xi)}{1+\frac{1}{n}|g_i(x,s,\xi)|} T_{\frac{1}{n}}(\sigma^{\frac{1}{q}}(x))$ and $f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|}$. We have $|g_i^n(x, s, \xi)| \leq |g_i(x, s, \xi)|$, $|g_i^n(x, s, \xi)| \leq n$, $|g_i^n(x, u, \nabla u)| \leq n^2\sigma^{\frac{1}{q}}(x)$, $|f_n(x)| \leq |f(x)|$ and $|f_n(x)| \leq n$.
For all u and v in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, we have

$$\begin{aligned} |\int_\Omega g_i^n(x, u, \nabla u)v dx| &\leq \Big(\int_\Omega |g_i^n(x, u, \nabla u)|^{q'} \sigma^{\frac{-q'}{q}} dx\Big)^{\frac{1}{q'}} \Big(\int_\Omega |v|^q \sigma dx\Big)^{\frac{1}{q}} \\ &\leq n^2 \Big(\int_\Omega \sigma^{\frac{q'}{q}} \sigma^{\frac{-q'}{q}} dx\Big)^{\frac{1}{q'}} \|v\|_{L^q(\Omega,\sigma)} \\ &\leq C_n \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})}. \end{aligned}$$

Proposition 3.1 *Under the conditions (12)-(17), there exists at least one solution of the problem (18).*

Proof of Proposition 3.1.
Thanks to the Leray-Lions theorem and Theorem 8.2 from Chapter 2 in [14], there exists at least one solution to problem (18).
Step 2 : A priori estimate.

Proposition 3.2 *Assume that (12)- (17) hold and if u_n is a solution of the approximate problem (18), then there exists a constant C such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} \omega_i \leq Ck \quad \forall k > 0.$$

Proof: Let $v = u_n - \eta \exp(G(u_n))T_k(u_n^+ - \psi^+)$, where $G(s) = \int_0^s \frac{\rho(t)}{\alpha} dt$ and $\eta \geq 0$. Since $v \in W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and for all η small enough, we have $v \in K_\psi(\Omega, \overrightarrow{\omega})$. We take v as a test function in problem (18), thanks to (12) and (15), we obtain

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) \exp(G(u_n))\partial_i T_k(u_n^+ - \psi^+) &\leq (\|f\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)}) \exp(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha})k \\ &\leq Ck. \end{aligned}$$

By (12) and Young's inequality, we have

$$\sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i dx \leq C'k \quad \forall k > 0. \tag{19}$$

Since $\{x \in \Omega,\ |u_n^+| \leq k\} \subset \{x \in \Omega,\ |u_n^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}$, we have

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n^+)|^{p_i} \omega_i dx = \sum_{i=1}^{N} \int_{\{|u_n^+| \leq k\}} |\partial_i u_n^+|^{p_i} \omega_i dx$$

$$\leq \sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}} |\partial_i u_n^+|^{p_i} \omega_i dx.$$

This implies, by (19), that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n^+)|^{p_i} \omega_i dx \leq C'k, \ \forall k > 0. \tag{20}$$

Similarly, taking $v = u_n + \exp(-G(u_n))T_k(u_n^-)$ as a test function in approximate problem (18), thanks to (12) and (15), we obtain

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} a_i(x, u_n, \nabla u_n) \exp(-G(u_n)) \partial_i T_k(u_n) \leq Ck.$$

By (12), we deduce that

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} |\partial_i T_k(u_n)|^{p_i} \omega_i \leq Ck. \tag{21}$$

Combining (20) and (21), we obtain $\displaystyle\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n)|^{p_i} \omega_i \leq Ck$. It yields

$$\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})} \leq Ck^{\frac{1}{p_-}}, \qquad \forall k > 1. \tag{22}$$

Step 3: Strong convergence of truncations.

Lemma 3.2 *There exist a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}).$$

Proof: By (22), the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to v_k a. e. in Ω, weakly in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and strongly in $L^q(\Omega, \sigma)$ as n tends to $+\infty$. Since $(u_n)_n$ is a Cauchy sequence in measure in Ω, there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a. e. in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \text{ and a. e. in } \Omega, \quad \forall k > 0. \tag{23}$$

Now, we prove that

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big) = 0. \tag{24}$$

Let us take $v = u_n + \exp(-G(u_n))T_1(u_n - T_m(u_n))^-$ in approximate problem (18), by (12) and (15), we have

$$\sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \exp(-G(u_n)) \partial_i u_n$$

$$\leq -\int_\Omega f_n \exp(-G(u_n))T_1(u_n - T_m(u_n))^- + \sum_{i=1}^N \int_\Omega \gamma(x) \exp(-G(u_n))T_1(u_n - T_m(u_n))^-. \tag{25}$$

By Lebesgue's theorem, we have the right-hand side in (25) tends to zero as n and m tend to ∞. Then, we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i=1}^N \int_{\{-(m+1)\leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n)\partial_i u_n = 0. \tag{26}$$

Similarly, taking $v = u_n - \eta \exp(G(u_n))T_1(u_n - T_m(u_n))^+$ as a test function in approximate problem (18), we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{i=1}^N \int_{\{m\leq u_n \leq m+1\}} a_i(x, u_n, \nabla u_n)\partial_i u_n = 0. \tag{27}$$

We consider the following function of one real variable:

$$h_m(s) = \begin{cases} 1, & \text{if } |s| \leq m, \\ 0, & \text{if } |s| \geq m+1, \\ m+1-|s|, & \text{if } m \leq |s| \leq m+1, \end{cases}$$

where $m > k$. Let $\varphi = u_n - \eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n)$ be a test function in approximate problem (18), using (12) and (15), we get

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n) \exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n) \\ &\quad + \sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n) \exp(G(u_n))(T_k(u_n) - T_k(u))^+ \partial_i u_n h_m^{'}(u_n) \\ &\qquad \leq \sum_{i=1}^N \int_\Omega \gamma(x) \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n) \\ &\qquad\quad + \int_\Omega f_n \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n). \end{aligned}$$

This implies that

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n) \exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n) \\ &\quad \leq \sum_{i=1}^N \int_{\{m\leq u_n\leq m+1\}} a_i(x, u_n, \nabla u_n) \exp(G(u_n))(T_k(u_n) - T_k(u))^+ \partial_i u_n \\ &\quad + \sum_{i=1}^N \int_\Omega \gamma(x) \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n) \\ &\quad + \int_\Omega f_n \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n). \end{aligned}$$

Thanks to Lebesgue's theorem and (27), we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n) \exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n) \leq 0,$$

which implies that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)$$

$$-\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u))^+h_m(u_n) \leq 0,$$

since $h_m(u_n)=0$ if $|u_n|>m+1$, we have

$$\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u))^+h_m(u_n)$$

$$=\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\exp(G(u_n))\partial_i(T_k(u))^+h_m(u_n).$$

By (13) and (22), we have $a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\rightharpoonup X_m^i$ in $L^{p_i'}(\Omega,\omega_i^*)$. It yields

$$\lim_{m,n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\exp(G(u_n))\partial_i(T_k(u))^+h_m(u_n)$$

$$=\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\{|u|>k\}} X_m^i\exp(G(u))\partial_i T_k(u)h_m(u)=0.$$

Using $a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ a. e. in Ω, we see that the sequence
$\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\Big)_n$ is equi-integrable in $L^{p_i'}(\Omega,\omega_i^*)$ and Vitali's theorem implies that

$$a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u) \textit{ in } L^{p_i'}(\Omega,\omega_i^*).$$

Since $\partial_i T_k(u_n)\rightharpoonup\partial_i T_k(u))$ weakly in $L^{p_i}(\Omega,\omega_i)$, we get

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,T_k(u_n),\nabla T_k(u))\exp(G(u_n))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0,$$

thus we conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\times$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{28}$$

Similarly, we take $\varphi=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(u))^-h_m(u_n)$ as a test function in approximating problem (18), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\times$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{29}$$

Combining (28) and (29), we deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{30}$$

Let $\varphi = u_n + \exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n))$ be a test function in approximate problem (18) and using (13) and (15), we get

$$\sum_{i=1}^{N}\int_{\{u_n\le 0\}} a_i(x,u_n,\nabla u_n)\exp(-G(u_n))\partial_i T_k(u_n)(1-h_m(u_n))$$
$$\le -\sum_{i=1}^{N}\int_{\{-(j+1)\le u_n\le -j\}} a_i(x,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)\partial_i u_n$$
$$+\sum_{i=1}^{N}\int_{\Omega}\gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n))$$
$$-\sum_{i=1}^{N}\int_{\Omega} f_n(x)\exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n)).$$

In view of (26) and Lebesgue's theorem, the integrals in the righthand side converge to zero as n and m tend to infinity. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{u_n\le 0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n)) = 0. \tag{31}$$

On the other hand, we take $\varphi = u_n - \eta\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n))$ as a test function in approximate problem (18) and using (13) and (15), we get

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))$$
$$\le \sum_{i=1}^{N}\int_{\{-(j+1)\le u_n\le -j\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))T_k((u_n)^+ - \psi^+)\partial_i u_n$$
$$+\sum_{i=1}^{N}\int_{\Omega}\gamma(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n))$$
$$+\sum_{i=1}^{N}\int_{\Omega} f_n(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n)).$$

By Lebesgue's theorem and (26), we deduce that

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i u_n^+(1-h_m(u_n))$$
$$\le \sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i\psi^+(1-h_m(u_n)) + \varepsilon_1(n,m). \tag{32}$$

Thanks to (13) and Young's inequality, we have

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i u_n^+(1-h_m(u_n)) \le \varepsilon_2(n,m),$$

where $\varepsilon_1(n,m)$ and $\varepsilon_2(n,m)$ converge to zero as n and m tend to infinity. Since $\rho\in L^1(\mathbb{R})$, we have $\exp(G(u_n))$ is bounded. It yields

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n)) \le \varepsilon_3(n,m).$$

Since $\{x\in\Omega,\ |u_n^+|\leq k\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\leq k+\|\psi^+\|_{L^\infty(\Omega)}\}$, hence $\displaystyle\sum_{i=1}^N\int_{\{|u_n^+|\leq k\}}a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n))\leq\varepsilon_3(n,m)$, which implies that, for all $k>0$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_{\{u_n\geq 0\}}a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))=0. \tag{33}$$

Combining (31) and (33), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))=0. \tag{34}$$

Moreover, we have

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n)-\partial_i T_k(u)\Big)=\\&\sum_{i=1}^N\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n)-\partial_i T_k(u)\Big)h_m(u_n)\\&+\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))\\&-\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u)(1-h_m(u_n))\\&-\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(\partial_i T_k(u_n)-\partial_i T_k(u))(1-h_m(u_n)).\end{aligned}$$

By (30) and (33), the first and the second integrals of the right-hand side converge to zero as $n,m\to+\infty$. Since $\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ is bounded in $L^{p_i'}(\Omega,\omega_i^*)$ and $\partial_i T_k(u)(1-h_m(u_n))$ converges to zero in $L^{p_i}(\Omega,\omega_i)$, the third integral converges to zero. So the fourth integral converges to zero while $\partial_i T_k(u_n)\rightharpoonup\partial_i T_k(u)$ weakly in $L^{p_i}(\Omega,\omega_i)$ and $a_i(x,T_k(u_n),\nabla T_k(u_n))(1-h_m(u_n))$ converges to $a_i(x,T_k(u),\nabla T_k(u))(1-h_m(u))$ strongly in $L^{p_i'}(\Omega,\omega_i^*)$. We conclude the proof of (24).
Using (23), (24) and Lemma 3.1, we deduce

$$T_k(u_n)\to T_k(u)\text{ strongly in }W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})\text{ and a. e. in }\Omega,\quad\forall k>0. \tag{35}$$

This implies that

$$\nabla u_n\to\nabla u\text{ a. e. in }\Omega, \tag{36}$$

which gives

$$a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u)\text{ in }L^{p_i'}(\Omega,\omega_i^*). \tag{37}$$

Step 4: Equi integrability of the non linearity sequence.
We shall prove that $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ in $L^1(\Omega)$.
We have $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ a. e. in Ω.
Let $v=u_n+\exp(-G(u_n))\displaystyle\int_{u_n}^0\rho(\nu)\chi_{\{\nu<-h\}}d\nu$. Since $v\in K_\psi(\Omega,\overrightarrow{\omega})$, we take v as a test

function in approximate problem (18). Then, by (12) and (15), we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\exp(-G(u_n))\partial_i u_n\rho(u_n)\chi_{\{u_n<-h\}}$$
$$\leq \sum_{i=1}^{N}\int_\Omega \gamma(x)\exp(-G(u_n))\int_{u_n}^{0}\rho(\nu)\chi_{\{\nu<-h\}}d\nu - \int_\Omega f_n\exp(-G(u_n))\int_{u_n}^{0}\rho(\nu)\chi_{\{\nu<-h\}}d\nu$$
$$\leq \exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big(\int_{-\infty}^{-h}\rho(s)ds\Big)\Big(N\|\gamma\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)}\Big).$$

Using again (12), we obtain $\sum_{i=1}^{N}\int_\Omega \alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)\chi_{\{u_n<-h\}} \leq c\int_{-\infty}^{-h}\rho(s)ds.$

Since $\rho\in L^1(\mathbb{R})$, we have

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{u_n<-h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{38}$$

Let h be such that $h\geq \exp(G(u_n))\int_0^{+\infty}\rho(\nu)d\nu+\|\psi^+\|_{L^\infty(\Omega)}$ and we take $v=u_n-\exp(G(u_n))\int_0^{u_n}\rho(\nu)\chi_{\{\nu>h\}}d\nu$ as a test function in approximate problem (18). Then, similarly as in (38), we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{u_n>h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{39}$$

Combining (38) and (39), we deduce

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{|u_n|>h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{40}$$

Using (35), (36), (40) and Vitali's theorem, we obtain

$$g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u) \text{ in } L^1(\Omega). \tag{41}$$

On the other hand, let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $v=u_n-T_k(u_n-\varphi)$ be a test function in approximate problem (18). We get

$$\begin{cases} u_n\in K_\psi. \\ \displaystyle\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n-\varphi)+\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n-\varphi) \\ \qquad\qquad \displaystyle\leq \int_\Omega f_nT_k(u_n-\varphi), \\ \forall\varphi\in K_\psi\cap L^\infty(\Omega) \text{ and } \forall k>0, \end{cases} \tag{42}$$

Using (35), (37) and (41), we can pass to the limit in (42).

4 Example

Let us consider the following case:

$$a_i(x,s,\xi)=\omega_i|\xi_i|^{p_i-1}sign(\xi_i) \text{ and } g_i(x,s,\xi)=\frac{1}{1+s^2}\omega_i|\xi_i|^{p_i}.$$

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Nonlinear Dynamics and Systems Theory, 18 (3) (2018) 225–232

Stability Analysis of Nonlinear Mechanical Systems with Delay in Positional Forces

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Received: March 12, 2018; Revised: June 15, 2018

Abstract: The paper is devoted to the problem of delay-independent stability for a class of nonlinear mechanical systems. Mechanical systems with linear velocity forces and essentially nonlinear positional ones are studied. It is assumed that there is a delay in the positional forces. With the aid of the decomposition method and original constructions of Lyapunov–Krasovskii functionals, conditions are found under which the trivial equilibrium positions of the considered systems are asymptotically stable for any constant nonnegative delay. An example is given to demonstrate the effectiveness of the obtained results.

Keywords: *mechanical system; delay; asymptotic stability, Lyapunov–Krasovskii functional, decomposition.*

Mathematics Subject Classification (2010): 34K20, 70K20, 93D30.

1 Introduction

An efficient approach to investigation of dynamical properties of complex systems is the decomposition method [15, 21]. The approach is successfully applied in various forms to the stability analysis of mechanical systems, see, for example, [15, 17, 20, 22, 24] and the bibliography therein.

An interesting and practically important result on the decomposition of mechanical system was obtained by V.I. Zubov [24]. He studied the stability of gyroscopic systems described by linear time-invariant second order systems and found conditions under which the stability problem for an original system can be reduced to that for two auxiliary independent first order subsystems. However, it should be noted that the Zubov approach

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is based on the Lyapunov first method, and it is inapplicable to nonstationary and nonlinear systems.

Another approach to derive the Zubov result has been proposed by A.A. Kosov [14]. He suggested to use a special transformation of variables and the Lyapunov direct method. This approach was further developed in [1, 4, 5], where it has been applied not only to linear time-invariant systems but also to switched systems and systems with nonlinear force fields. Furthermore, in [3, 7], with the aid of the Kosov approach and a special technique of using the Razumikhin theorem, new delay-independent stability conditions for some classes of mechanical systems were obtained.

In the present contribution, we consider mechanical systems with linear velocity forces and essentially nonlinear positional ones. It is assumed that there is a delay in the positional forces. We will look for conditions guaranteeing that the trivial equilibrium positions of the systems under consideration are asymptotically stable for any constant nonnegative delay.

Let us note that such conditions were derived in [7] with the aid of the decomposition method and Lyapunov–Razumikhin functions. In this paper, instead of Lyapunov–Razumikhin functions, we will use special constructions of Lyapunov–Krasovskii functionals. It will be shown that such an approach permits us to obtain less conservative delay-independent stability conditions than those in [7].

2 Notation

Throughout the paper the following notation is used:

- $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- $\|\cdot\|$ is the Euclidean norm of a vector.
- $P > 0$ ($P < 0$) means that the matrix P is symmetric and positive (negative) definite.
- A^T is the transpose of a matrix A.
- A matrix C is called Metzler [13] if all its off-diagonal entries are nonnegative.
- $\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is the diagonal matrix with the elements $\lambda_1, \ldots, \lambda_n$.
- A matrix C is called diagonally stable if there exists a diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} > 0$ such that $\Lambda C + C^T \Lambda < 0$.
- For a given positive number τ, let $C^1([-\tau, 0], \mathbb{R}^n)$ be the space of continuously differentiable functions $\varphi(\theta) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \max_{\theta \in [-\tau, 0]} \left(\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\| \right).$$

- Ω_Δ is the set of functions $\varphi(\theta) \in C^1([-\tau, 0], \mathbb{R}^n)$ satisfying the condition $\|\varphi\|_\tau < \Delta$, $0 < \Delta \leq +\infty$.

3 Problem Formulation

Consider the system

$$A\ddot{q}(t) + B\dot{q}(t) + Cf(q(t)) + Df(q(t - \tau)) = 0 \tag{1}$$

describing motions of a nonlinear mechanical system. Here $q(t), \dot{q}(t) \in \mathbb{R}^n$; A, B, C, D are constant matrices; vector function $f(q)$ is continuous for $\|q\| < \Delta$, $0 < \Delta \leq +\infty$; τ is a constant nonnegative delay.

Assume that $f(q)$ is a separable nonlinearity, i.e., $f(q) = (f_1(q_1), \ldots, f_n(q_n))^T$, and each scalar function $f_i(q_i)$ satisfies the sector-like condition $q_i f_i(q_i) > 0$ for $q_i \neq 0$, $i = 1, \ldots, n$. It is worth noting that such functions are widely used in models of automatic control systems and neural networks [2, 13, 16].

Hence, we consider a mechanical system with linear velocity forces and nonlinear positional ones. The term $-Df(q(t-\tau))$ can be treated as a control vector, and the presence of delay τ might be caused by a time lag between the moments of measuring of the state and the application of the corresponding control force, see [12, 19].

Let $q(t, t_0, \varphi, \dot{\varphi})$ stand for a solution of the system (1) with the initial conditions $t_0 \geq 0$, $\varphi(\theta) \in \Omega_\Delta$, and $q_t(t_0, \varphi, \dot{\varphi})$ denote the restriction of the solution to the segment $[t-\tau, t]$, i.e., $q_t(t_0, \varphi, \dot{\varphi}) : \theta \to q(t+\theta, t_0, \varphi, \dot{\varphi})$, $\theta \in [-\tau, 0]$.

In what follows, we will impose additional restrictions on the system (1).

Assumption 3.1 Let the matrices A and B be nonsingular.

Assumption 3.2 Let $f_i(q_i) = \alpha_i q_i^{\mu_i}$, where α_i are positive coefficients and $\mu_i > 1$ are rationals with odd numerators and denominators, $i = 1, \ldots, n$.

Remark 3.1 Without loss of generality, we will consider the case where $\alpha_i = 1$, $i = 1, \ldots, n$, and $\mu_1 \leq \ldots \leq \mu_n$.

Thus, the positional forces in (1) are essentially nonlinear ones. It should be noted that models with essentially nonlinear forces are widely used in contemporary mechanical and civil engineering, see, for instance, [8–10, 18].

The system (1) has the trivial equilibrium position

$$q = \dot{q} = 0. \tag{2}$$

We will look for conditions providing the asymptotic stability of the equilibrium position for an arbitrary constant nonnegative delay.

4 Main Results

According to the Zubov approach, consider two auxiliary isolated delay-free subsystems

$$\dot{y}(t) = Pf(y(t)), \tag{3}$$

$$\dot{z}(t) = -A^{-1}Bz(t). \tag{4}$$

Here $P = \{p_{ij}\}_{i,j=1}^n = -B^{-1}(C + D)$. It is worth mentioning that the subsystem (4) is linear, whereas the subsystem (3) belongs to the well-known class of Persidskii type systems [13].

Assumption 4.1 Let the subsystem (4) be asymptotically stable.

Define entries of the matrix $\bar{P} = \{\bar{p}_{ij}\}_{i,j=1}^n$ by the formulae $\bar{p}_{ii} = p_{ii}$, and $\bar{p}_{ij} = |p_{ij}|$ for $i \neq j$; $i, j = 1, \ldots, n$. The matrix $\bar{P}$ is Metzler, see [13].

In [7], with the aid of the decomposition method and Lyapunov–Razumikhin functions, it was proved that if Assumptions 3.1, 3.2, 4.1 are fulfilled, and the matrix $\bar{P}$ is Hurwitz, then the equilibrium position (2) of the system (1) is asymptotically stable for any $\tau \geq 0$.

To obtain less conservative stability conditions, we will use the original construction of Lyapunov–Krasovskii functionals for systems of the form (3) proposed in [6].

Theorem 4.1 *Let Assumptions 3.1, 3.2, 4.1 be fulfilled, and the matrix P be diagonally stable. Then the equilibrium position* (2) *of the system* (1) *is asymptotically stable for an arbitrary nonnegative delay.*

Proof. Introduce new variables by the formulae

$$z(t) = \dot{q}(t), \qquad y(t) = B^{-1}A\dot{q}(t) + q(t). \tag{5}$$

Then

$$\begin{aligned} B\dot{y}(t) &= -(C+D)f(y(t)) + C\left(f(y(t)) - f\left(y(t) - B^{-1}Az(t)\right)\right) \\ &+ D\left(f(y(t)) - f\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right), \\ A\dot{z}(t) &= -Bz(t) - Cf\left(y(t) - B^{-1}Az(t)\right) - Df\left(y(t-\tau) - B^{-1}Az(t-\tau)\right). \end{aligned} \tag{6}$$

Taking into account properties of the transformation (5), we obtain that the equilibrium position (2) of the system (1) is asymptotically stable if and only if the zero solution of (6) is asymptotically stable.

It is known, see [23], that under Assumption 4.1, for any number $\gamma > 1$, there exists a continuously differentiable for $z \in \mathbb{R}^n$ positive homogeneous of the order γ Lyapunov function $\widetilde{V}(z)$ such that the estimates

$$a_1\|z\|^\gamma \le \widetilde{V}(z) \le a_2\|z\|^\gamma, \qquad \left\|\frac{\partial\widetilde{V}(z)}{\partial z}\right\| \le a_3\|z\|^{\gamma-1}, \qquad \left(\frac{\partial\widetilde{V}(z)}{\partial z}\right)^T A^{-1}Bz \ge a_4\|z\|^\gamma$$

hold for $z \in \mathbb{R}^n$. Here $a_i > 0$, $i = 1, 2, 3, 4$.

The matrix P is diagonally stable. Therefore, one can choose a matrix $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\} > 0$ such that $\Lambda P + P^T\Lambda < 0$.

Using the approach proposed in [6], construct a Lyapunov–Krasovskii functional for the system (6) in the form

$$V(y_t, z_t) = \widetilde{V}(z) + \beta_1\int_{t-\tau}^{t}\|z(s)\|^\gamma ds + \sum_{i=1}^{n}\lambda_i\frac{y_i^{\mu_i+1}(t)}{\mu_i+1} + \beta_2\int_{t-\tau}^{t}\|f(y(s))\|^2 ds$$

$$+\beta_3\int_{t-\tau}^{t}(\tau+s-t)\|f(y(s))\|^2 ds - f^T(y(t))\Lambda B^{-1}D\int_{t-\tau}^{t}f(y(s))ds,$$

where $\beta_1, \beta_2, \beta_3$ are positive coefficients.

Differentiating functional $V(y_t, z_t)$ along the solutions of the system (6), we obtain

$$\dot{V} = -\left(\frac{\partial\widetilde{V}(z(t))}{\partial z}\right)^T A^{-1}Bz(t) + \beta_1\|z(t)\|^\gamma - \beta_1\|z(t-\tau)\|^\gamma$$

$$-\left(\frac{\partial\widetilde{V}(z(t))}{\partial z}\right)^T A^{-1}\left(Cf\left(y(t) - B^{-1}Az(t)\right) + Df\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right)$$

$$+f^T(y(t))\Lambda Pf(y(t)) + (\beta_2 + \tau\beta_3)\|f(y(t))\|^2 - \beta_2\|f(y(t-\tau))\|^2 - \beta_3\int_{t-\tau}^{t}\|f(y(s))\|^2 ds$$

$$+f^T(y(t))\Lambda B^{-1}\Big(C\left(f(y(t)) - f\left(y(t) - B^{-1}Az(t)\right)\right)$$

$$+D\left(f(y(t-\tau)) - f\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right)\Big)$$

$$+\int_{t-\tau}^{t} f^T(y(s))dsD^T\left(B^{-1}\right)^T\Lambda\frac{\partial f(y(t))}{\partial y}B^{-1}\Big(Cf\left(y(t) - B^{-1}Az(t)\right)$$

$$+Df\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\Big).$$

If $\|z(\xi)\| < 1$ for $\xi \in [t-\tau, t]$, then

$$\dot{V} \leq (-a_4 + \beta_1)\|z(t)\|^\gamma - \beta_1\|z(t-\tau)\|^\gamma + (\beta_2 + \tau\beta_3 - c_1)\|f(y(t))\|^2 - \beta_2\|f(y(t-\tau))\|^2$$

$$+c_2\left\|\frac{\partial f(y(t))}{\partial y}\right\|\int_{t-\tau}^{t}\|f(y(s))\|ds\left(\|f(y(t))\| + \|z(t)\|^{\mu_1} + \|f(y(t-\tau))\| + \|z(t-\tau)\|^{\mu_1}\right)$$

$$+c_3\|z(t)\|^{\gamma-1}\left(\|f(y(t))\| + \|z(t)\|^{\mu_1} + \|f(y(t-\tau))\| + \|z(t-\tau)\|^{\mu_1}\right)$$

$$-\beta_3\int_{t-\tau}^{t}\|f(y(s))\|^2ds + c_4\|f(y(t))\|\left\|f(y(t)) - f\left(y(t) - B^{-1}Az(t)\right)\right\|$$

$$+c_5\|f(y(t))\|\left\|f(y(t-\tau)) - f\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right\|,$$

where c_1, c_2, c_3, c_4, c_5 are positive constants.

Let $2 < \gamma < 2\mu_1$. Using homogeneous functions properties, see [23], it is easy to show that, for sufficiently small values of parameters $\beta_1, \beta_2, \beta_3$, there exist positive numbers $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \delta$ such that if $\|y(\xi)\| + \|z(\xi)\| < \delta$ for $\xi \in [t-\tau, t]$, then

$$\tilde{c}_1\left(\|z(t)\|^\gamma + \sum_{i=1}^{n} y_i^{\mu_i+1}(t)\right) \leq V(y_t, z_t)$$

$$\leq \tilde{c}_2\left(\|z(t)\|^\gamma + \int_{t-\tau}^{t}\|z(s)\|^\gamma ds + \sum_{i=1}^{n} y_i^{\mu_i+1}(t) + \int_{t-\tau}^{t}\|f(y(s))\|^2ds\right),$$

$$\dot{V} \leq -\tilde{c}_3\left(\|z(t)\|^\gamma + \|z(t-\tau)\|^\gamma + \|f(y(t))\|^2 + \|f(y(t-\tau))\|^2 + \int_{t-\tau}^{t}\|f(y(s))\|^2ds\right).$$

From the obtained estimates it follows [11] that the zero solution of the system (6) is asymptotically stable. This implies that the equilibrium position (2) of the original system (1) is asymptotically stable as well. □

Remark 4.1 On the one hand, it is well known, see [13], that if the matrix $\bar{P}$ is Hurwitz, then the matrix P is diagonally stable. On the other hand, the matrix

$$P = \begin{pmatrix} -1 & 10 \\ -10 & -1 \end{pmatrix}$$

is diagonally stable, but the corresponding matrix

$$\bar{P} = \begin{pmatrix} -1 & 10 \\ 10 & -1 \end{pmatrix}$$

is not Hurwitz. Hence, conditions of Theorem 4.1 are less conservative than those obtained in [7].

Next, together with (1), consider the perturbed system

$$A\ddot{q}(t) + B\dot{q}(t) + Cf(q(t)) + Df(q(t-\tau)) = G(t, q(t), q(t-\tau)). \tag{7}$$

Here vector function $G(t, q, u)$ is continuous for $t \geq 0$, $\|q\| < \Delta$, $\|u\| < \Delta$.

Assumption 4.2 The estimate $\|G(t, q, u)\| \leq \tilde{a}\,(\|f(q)\| + \|f(u)\|)^{\sigma}$ is valid for $t \geq 0$, $\|q\| < \Delta$, $\|u\| < \Delta$, where $\tilde{a}$ and σ are positive constants.

If Assumption 4.2 is fulfilled, then the system (7) admits the equilibrium position (2). We will look for conditions under which perturbations do not disturb the asymptotic stability of the equilibrium position.

Theorem 4.2 *Let Assumptions 3.1, 3.2, 4.1, 4.2 be fulfilled, and the matrix P be diagonally stable. If $\sigma > 1$, then the equilibrium position* (2) *of the system* (7) *is asymptotically stable for an arbitrary nonnegative delay.*

The proof of the theorem is similar to that of Theorem 4.1.

5 Example

Let system (1) be of the form

$$\begin{aligned} &\ddot{q}_1(t) + b\dot{q}_1(t) + g\dot{q}_2(t) - cq_1^3(t) = u_1, \\ &\ddot{q}_2(t) + b\dot{q}_2(t) - g\dot{q}_1(t) - cq_2^5(t) = u_2. \end{aligned} \tag{8}$$

Here $q_1(t), q_2(t) \in \mathbb{R}$, b, g, c are positive constants, u_1, u_2 are control variables.

If $u_1 = u_2 = 0$, then the equilibrium position

$$q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0 \tag{9}$$

of the system (8) is unstable, see [17]. We are going to design a feedback control providing the asymptotic stability of the equilibrium position.

Assume that the control law depends on q_1 and q_2, and is independent of $\dot{q}_1$ and $\dot{q}_2$. Moreover, we consider the case where there exists a delay τ in the control scheme.

It should be noted that for the linear control law

$$u_1 = a_{11}q_1(t-\tau) + a_{12}q_2(t-\tau), \qquad u_2 = a_{21}q_1(t-\tau) + a_{22}q_2(t-\tau),$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constants, the presence of delay might result in instability of the equilibrium position. Therefore, we choose a nonlinear control in the form

$$u_1 = -dq_2^5(t-\tau), \qquad u_2 = dq_1^3(t-\tau), \qquad d = \text{const} > 0.$$

Verifying the conditions of Theorem 4.1, it is easy to show that if $d > bc/g$, then the equilibrium position (9) of the corresponding closed-loop system is asymptotically stable for an arbitrary constant nonnegative delay.

6 Conclusion

In this paper, new delay-independent conditions of the asymptotic stability are found for a class of nonlinear mechanical systems. Compared with the results of [7], these conditions are less conservative. However, it is worth mentioning that in [7] it was assumed that the delay may be a continuous nonnegative and bounded function of time, whereas the results of the present paper are valid only for systems with constant delays.

It should be noted that the approach to construction of Lyapunov–Krasovskii functionals proposed in the paper not only permits us to prove the asymptotic stability but also can be used to derive estimates of the convergence rate of solutions to the equilibrium position.

Acknowledgment

This work was partially supported by the Russian Foundation for Basic Research, grant nos. 16-01-00587-a and 17-01-00672-a.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 233–240

Solitary Wave Solutions of the Phi-Four Equation and the Breaking Soliton System by Means of Jacobi Elliptic Sine-Cosine Expansion Method

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Received: February 12, 2018;   Revised: June 25, 2018

Abstract: The goal of this study is twofold. The Jacobi elliptic expansion method is used to extract new solutions for the phi-four equation and the breaking soliton system. Special values of the Jacobi elliptic module and other involved parameters are chosen to produce solutions of soliton type and singular periodic solutions. The obtained solutions are verified and presented graphically.

Keywords: *Jacobi elliptic sine-cosine expansion method; phi-four equation; breaking soliton system.*

Mathematics Subject Classification (2010): 74J35, 34G20, 93C10.

1 Introduction

Solitary waves occur due to nonlinear phenomena appearing in different fields of science and engineering. These nonlinear phenomena are interpreted as $(n + 1)$-dimensional nonlinear partial differential equations. Seeking the exact solutions to these equations provides essential information about the physical structure of such phenomena. Since there is no specific method that produces such solutions, researchers made all the efforts to construct and modify methods to retrieve different kind of solutions for the same nonlinear model. We may mention some of these well-known techniques such as: the simplified bilinear method [11, 18, 31], sine-cosine method [4, 5], rational trigonometric function method [6], tanh method [7], extended tanh method [12,27], Yan transformation

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method [33–35], sech-tanh method [8–10,32], exponential-function method [25], the first integral method [2,29], the (G'/G)-expansion method [3,23,24], etc.

In this work, we use the Jacobi elliptic expansion method to explore further new solutions for two physical models: the phi-four equation that reads [17]

$$u_{tt} - \alpha u_{xx} - \lambda u + \beta u^2 = 0, \tag{1}$$

and the breaking soliton system

$$\begin{aligned} u_t &= -\alpha u_{xxy} - 4\alpha(uv)_x, \\ u_y &= v_x. \end{aligned} \tag{2}$$

The phi-four equation is a mathematical model that is used in nuclear and particle physics. Many methods have been used to study the solutions of this model. In [13], the modified simple equation method is used and tanh-coth solutions are derived. The modified (G'/G)-expansion method is adopted in [26] and produced the same solutions as in [13]. In [28], $\tan^2$ and $\cot^2$ solutions are obtained by using the extended direct algebraic method. Finally, the exponential-function method is used and rational trigonometric solutions of the phi-four equation are obtained in [14].

Different versions of the breaking soliton model are also studied by many researchers. In [30], the mapping method is used to obtain propagating solutions. The tanh-coth method is implemented [15] to construct solitary and soliton solutions of the breaking soliton equations. Finally, the exponential-function method is used [16] to obtain multiple soliton solutions of $(2+1)$ and $(3+1)$-dimensional breaking soliton equations.

2 Jacobi Elliptic Sine-Cosine Expansion Method

Partial differential equations can be written as a polynomial of the unknown function and its partial derivatives, i.e.

$$f(u, u_t, u_x, u_{xt}, u_{xx}...) = 0, \qquad u = u(x,t). \tag{3}$$

By using the variable of the form $\xi = \mu(x - ct)$ and the chain rule, equation (3) is transformed into

$$g(u, -c\mu u', \mu u', -c\mu^2 u'', \mu^2 u'', ...) = 0, \qquad u = u(\xi). \tag{4}$$

For the Jacobi elliptic sine-cosine technique [1,19–22], we write the solution as a power series of order n in terms of either the Jacobi elliptic sine $sn(\xi, m)$ or cosine $cn(\xi, m)$. The index m is regarded as the Jacobi module and $0 \le m \le 1$, i.e.

$$u(\xi) = \sum_{i=0}^{n} a_i Y^i, \tag{5}$$

where

$$Y = sn(\xi, m), \tag{6}$$

or

$$Y = cn(\xi, m). \tag{7}$$

Then, we determine the value of n by matching the order of Y in the highest derivative term with its order in the other nonlinear terms of the equation. Once n is obtained, we substitute (5) in (4) and collect the coefficients of $Y^i: \ i = 0, 1, 2, ..., n, ...$ Setting these coefficients to zero and solving the resulting non algebraic system lead to identifying the required $a_0,\ a_1,\ ...,\ a_n,\ \mu$ and c.

3 The Phi-Four Equation

Consider the phi-four equation that reads

$$u_{tt} - \alpha u_{xx} - \lambda u + \beta u^2 = 0. \tag{8}$$

By the wave variable $\xi = k(x - ct)$, equation (8) is turned into the differential equation:

$$k^2(c^2 - \alpha)u'' - \lambda u + \beta u^2 = 0. \tag{9}$$

Balancing u'' with u^2 produces the algebraic equation $n+2 = 2n$ whose solution is $n = 2$. Thus, the solution of (8) in terms of the elliptic sine function will have the form

$$u(\xi) = a_0 + a_1\ sn(\xi, m) + a_2\ sn^2(\xi, m). \tag{10}$$

Substituting (10) into (9) and collecting the coefficients of the same power of sn lead to the nonlinear algebraic system

$$\begin{aligned}
0 &= 2a_2k^2(c^2 - \alpha) + a_0(a_0\beta - \lambda),\\
0 &= -a_1(c^2k^2(1 + m^2) - k^2(1 + m^2)\alpha - 2a_0\beta + \lambda),\\
0 &= a_1^2\beta - a_2(4c^2k^2(1 + m^2) - 4k^2(1 + m^2)\alpha - 2a_0\beta + \lambda),\\
0 &= 2a_1(c^2k^2m^2 - k^2m^2\alpha + a_2\beta),\\
0 &= a_2(6c^2k^2m^2 - 6k^2m^2\alpha + a_2\beta).
\end{aligned} \tag{11}$$

By solving the above system for the parameters a_0, a_1, a_2, c and k, we get

$$\begin{aligned}
a_0 &= \frac{\lambda}{2\beta}\left(1 - \frac{1 + m^2}{\sqrt{1 - m^2 + m^4}}\right),\\
a_1 &= 0,\\
a_2 &= \frac{3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}},\\
c &= \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}\ ,
\end{aligned} \tag{12}$$

where k is a free parameter. Thus, our first solution to the phi-four model is

$$\begin{aligned}
u(x,t) &= \frac{3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}}\ sn^2(k(x - \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}t), m)\\
&+ \frac{\lambda}{2\beta}\left(1 - \frac{1 + m^2}{\sqrt{1 - m^2 + m^4}}\right).
\end{aligned} \tag{13}$$

Substituting $m = 1$ in (13) produces the soliton solution

$$u(x,t) = -\frac{\lambda}{2\beta} + \frac{3\lambda}{2\beta}\ \tanh^2(k(x - \frac{1}{2}t\sqrt{4\alpha - \frac{\lambda}{k^2}})). \tag{14}$$

Now, replacing sn in (10) by cn will lead to a second solution, which is

$$\begin{aligned}
u(x,t) &= \frac{-3m^2\lambda}{2\beta\sqrt{1 - m^2 + m^4}}\ cn^2(k(x - \frac{1}{2}\sqrt{4\alpha - \frac{\lambda}{k^2\sqrt{1 - m^2 + m^4}}}t), m)\\
&+ \frac{\lambda}{2\beta}\left(1 + \frac{2m^2 - 1}{\sqrt{1 - m^2 + m^4}}\right).
\end{aligned} \tag{15}$$

Let $m = 1$ in (15), this produces the soliton solution

$$u(x,t) = \frac{\lambda}{\beta} - \frac{3\lambda}{2\beta}\ \text{sech}^{\,2}(k(x - \frac{1}{2}t\sqrt{4\alpha - \frac{\lambda}{k^2}})). \tag{16}$$

Remark 1 The obtained solution given in (16) can be obtained directly from (14) by using the identity $\text{sech}^{\,2}(x) = 1 - \tanh^2(x)$.

Remark 2 If we replace the free parameter k in (14) by $i\gamma$ with $i = \sqrt{-1}$, we obtain the singular periodic solution

$$u(x,t) = -\frac{\lambda}{2\beta} - \frac{3\lambda}{2\beta}\ \tan^2(\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}})). \tag{17}$$

Also, in (16), we obtain the singular periodic solution

$$u(x,t) = \frac{\lambda}{\beta} - \frac{3\lambda}{2\beta}\ \sec^2(\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}})), \tag{18}$$

where the singularities occur on the line characteristics $\gamma(x - \frac{1}{2}t\sqrt{4\alpha + \frac{\lambda}{\gamma^2}}) = \frac{\pi}{2} + n\pi$.

Proof: Use the fact that $\tanh(ix) = i\tan(x)$ and $\text{sech}\,(ix) = \sec(x)$.

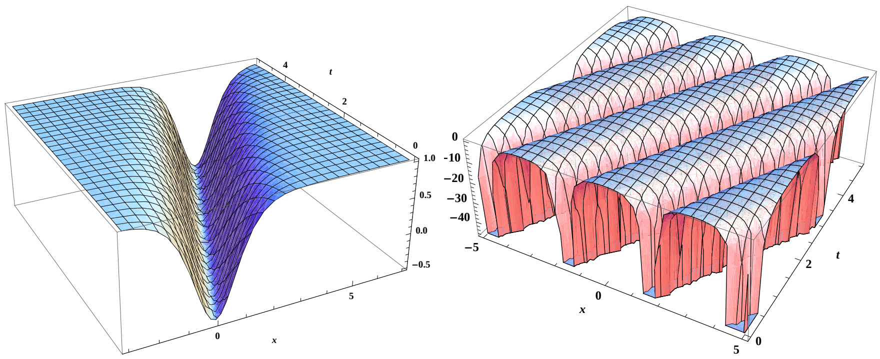

Figure 1: The obtained solutions given in (14) and (18) respectively, when $\lambda = \alpha = \beta = k = 1$.

4 $(2+1)$-Dimensional Breaking Soliton Equations

We recall the following (2+1)-dimensional breaking soliton equations

$$\begin{aligned} u_t &= -\alpha u_{xxy} - 4\alpha(uv)_x, \\ u_y &= v_x. \end{aligned} \tag{19}$$

Substituting $\xi = \mu x + \lambda y - ct$ into (19) yields

$$-cu + \alpha\mu^2\lambda u'' + 4\alpha\mu(uv) = 0 \tag{20}$$

and

$$\lambda u' = \mu v'. \tag{21}$$

From (21) we get

$$v = \frac{\lambda}{\mu} u. \tag{22}$$

Substituting (22) in (20) yields

$$-cu + 4\alpha\lambda u^2 + \alpha\lambda\mu^2 u'' = 0. \tag{23}$$

Balancing u'' with u^2 in (23) produces the algebraic equation $n+2 = 2n$ whose solution is $n = 2$. Thus, by the Jacobi elliptic sine expansion, the solution has the form

$$u(\xi) = a_0 + a_1\ sn(\xi, m) + a_2\ sn^2(\xi, m). \tag{24}$$

Substitute (24) into (23) to get the following algebraic system

$$\begin{aligned}
0 &= -a_0 c + 4a_0^2\alpha\lambda + 2a_2\alpha\lambda\mu^2,\\
0 &= -a_1(c + \alpha\lambda(-8a_0 + (1+m^2)\mu^2)),\\
0 &= 4a_1^2\alpha\lambda - a_2(c + 4\alpha\lambda(-2a_0 + (1+m^2)\mu^2)),\\
0 &= 2a_1\alpha\lambda(4a_2 + m^2\mu^2),\\
0 &= 2a_2\alpha\lambda(2a_2 + 3m^2\mu^2).
\end{aligned} \tag{25}$$

Solving the above system with respect to a_0, a_1, a_2, μ, λ and c, we get

$$\begin{aligned}
a_0 &= \frac{1}{2}\left(1 + m^2 \pm \sqrt{1 - m^2 + m^4}\right)\mu^2,\\
a_1 &= 0,\\
a_2 &= \frac{-3}{2} m^2\mu^2,\\
c &= \pm 4\alpha\lambda\mu^2\sqrt{1 - m^2 + m^4}.
\end{aligned} \tag{26}$$

Thus, the solution is

$$\begin{aligned}
u(x,y,t) &= \frac{1}{2}\mu^2\{1 + m^2 + A - 3m^2 sn^2(\mu x + \lambda y - 4A\alpha\lambda\mu^2 t, m)\},\\
v(x,y,t) &= \frac{1}{2}\lambda\mu\{1 + m^2 + A - 3m^2 sn^2(\mu x + \lambda y - 4A\alpha\lambda\mu^2 t, m)\},
\end{aligned} \tag{27}$$

where $A = \sqrt{1 - m^2 + m^4}$. When $m = 1$ in (27), we obtain

$$\begin{aligned}
u(x,y,t) &= \frac{3}{2}\mu^2\left(1 - \tanh^2(\mu x + \lambda y - 4\alpha\lambda\mu^2 t)\right),\\
v(x,y,t) &= \frac{3}{2}\lambda\mu\left(1 - \tanh^2(\mu x + \lambda y - 4\alpha\lambda\mu^2 t)\right).
\end{aligned} \tag{28}$$

Now, by the Jacobi elliptic cosine expansion, the solution has the form

$$u(\xi) = a_0 + a_1\ cn(\xi, m) + a_2\ cn^2(\xi, m) \tag{29}$$

Substituting (29) into (23) and solving the resulting algebraic system, we arrive at

$$\begin{aligned} u(x,y,t) &= \frac{1}{4}\mu^2\{1-2m^2-m^4+B+2(m^2+2m^4)\ cn^2(\mu x+\lambda y-2B\alpha\lambda\mu^2 t,m)\},\\ v(x,y,t) &= \frac{1}{4}\lambda\mu\{1-2m^2-m^4+B+2(m^2+2m^4)\ cn^2(\mu x+\lambda y-2B\alpha\lambda\mu^2 t,m)\}, \end{aligned} \tag{30}$$

where $B=\sqrt{1+6m^4-4m^6+m^8}$ and $\lambda,\ \mu$ are free variables. When $m=1$, the solution is

$$\begin{aligned} u(x,y,t) &= \frac{3}{2}\mu^2\ \mathrm{sech}^2(\mu x+\lambda y-4\alpha\lambda\mu^2 t),\\ v(x,y,t) &= \frac{3}{2}\lambda\mu\ \mathrm{sech}^2(\mu x+\lambda y-4\alpha\lambda\mu^2 t). \end{aligned} \tag{31}$$

Remark 3 If we replace λ by $\theta\lambda_1$ and μ by $\theta\mu_1$ and θ by $i\theta_1$ in both (28) and (31), where $i=\sqrt{-1}$, two singular periodic solutions are obtained

$$\begin{aligned} u(x,y,t) &= -\frac{3}{2}\theta_1^2\mu_1^2\left(1+\tan^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t))\right),\\ v(x,y,t) &= -\frac{3}{2}\theta_1^2\lambda_1\mu_1\left(1+\tan^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t))\right) \end{aligned} \tag{32}$$

and

$$\begin{aligned} u(x,y,t) &= -\frac{3}{2}\theta_1^2\mu_1^2\sec^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)),\\ v(x,y,t) &= -\frac{3}{2}\theta_1^2\lambda_1\mu_1\sec^2(\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)). \end{aligned} \tag{33}$$

The singularities of the last two solutions occur on the plane characteristics $\theta_1(\mu_1 x+\lambda_1 y+\theta_1^2 4\alpha\lambda_1\mu_1^2 t)=\frac{\pi}{2}+n\pi$.

5 Conclusion

In this paper, we used the Jacobi elliptic sine-cosine expansion method to study the solutions of two physical models, the phi-four equation and the $(2+1)$-dimensional breaking soliton system. Special values of the Jacobi elliptic module and the free parameters lead us to different types of solutions to these models such as soliton, singular-soliton and periodic solution. This work reveals that the proposed method is a reliable technique that provides different types of solutions and is relatively easy when applied to nonlinear equations.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 241–252

Average Edge Betweenness of a Graph

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Received: February 11, 2016;   Revised: June 19, 2018

Abstract: Vulnerability is an important concept in network analysis. When a failure occurs in some of the components of the network, vulnerability measures the ability of the network to disruption in order to avoid the external or internal effects. Graph theory is an important concept in network vulnerability analysis. If a network is modeled as an undirected and unweighted graph composed of processing vertices and communication links, there have been several proposals for measuring graph vulnerability under link or vertex failures. In this paper, we consider the concept of average edge betweenness of a graph in order to measure the network stability. The average edge betweenness is related to the edge betweenness of an edge. The edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. The average edge betweenness considers both the local and the global structure of the graph. In this paper, we obtain exact values for average edge betweenness and normalized average edge betweenness for some special graphs and E_p^t graph.

Keywords: *network vulnerability; network design and communication; stability; average edge betweenness.*

Mathematics Subject Classification (2010): 05C40, 68M10, 68R10.

1 Introduction

Many complex systems in the real world can be conceptually described as networks, where vertices represent the system constituents and edges depict the interaction between them, such as social networks (collaboration network), technological networks (communication networks, the Internet), information networks (the World Wide Web), biological networks (protein-protein interaction networks, neural networks) and etc. [10, 11] . A central issue in the analysis of complex networks is the assessment of their stability and vulnerability. Vulnerability is an important concept in network analysis related with the ability of

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the network to avoid intentional attacks or disruption when a failure is produced in some of its components. Often enough, the network is modeled as an undirected and unweighted graph. Different measures for graph vulnerability have been introduced so far to study different aspects of the graph behavior after removal of vertices or links such as connectivity, toughness, scattering number, integrity, residual closeness and exponential domination number [1, 4, 9, 12–15].

As an important parameter in the study of networks associated with complex systems in both modeling and measuring the reliability, the graph-theoretical concept of vertex betweenness was first proposed by Freeman [7] in 1977. Then, Girvan and Newman in [8] generalize this definition to edges and introduce the edge betweenness of an edge as the fraction of shortest paths between pairs of vertices that run along it. The edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. This measure considers both the local and the global structure of the graph. Since average edge betweenness gives information on which edge carries the most of the network vulnerability, it is important to determine the average edge betweenness of several graph classes.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. The *complement* $\overline{G}$ of a graph G is the graph with a vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. A *vertex dominating set* for a graph G is a set S of vertices such that every vertex G belongs to S or is adjacent to a vertex of S. The minimum cardinality of a vertex dominating set in a graph G is called the *vertex dominating number* of G and is denoted by $\gamma(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of the shortest path between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. In addition, the distance between the vertices u and v in G can be denoted by $d(u, v\,|G)$. The *diameter* of G, denoted by $diam(G)$, is the largest distance between two vertices in $V(G)$ [2].

The paper proceeds as follows. In Section 2, definitions and known results for average edge betweenness and normalized average edge betweenness are given. In Sections 3 and 4, average edge betweenness and normalized average edge betweenness of some special graphs are respectively determined and exact values are given. Conclusions are addressed in Section 5.

2 Average Edge Betweenness and Normalized Average Edge Betweenness

In this paper, we consider a simple finite undirected graph that has no self-loops and possesses no more than one edge between any two different vertices. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$.

Average edge betweenness of the graph G is defined as $b(G) = \frac{1}{|E|} \sum\limits_{e \in E} b_e$, where $|E|$ is the number of the edges, and b_e is the edge betweenness of the edge e, defined as $b_e = \sum\limits_{i \neq j} b_e(i, j)$, where $b_e(i, j) = n_{ij}(e)/n_{ij}$, $n_{ij}(e)$ is the number of geodesics (shortest paths) from vertex i to vertex j that contain the edge e, and n_{ij} is the total number of shortest paths [3, 5].

Let us compare two graphs G_1 and G_2. If $b(G_1) < b(G_2)$, then $G_1$is more stable than G_2. Since for a graph with a fixed number of vertices $b(G)$ decreases as the number of edges in the graph increases, it can be said that they represent how “well connected”the graph is. The higher the values of $b(G)$, the more vulnerable G is to the loss of edges. We consider the concept of average edge betweenness of a graph because when computing $b(G)$, we can gather information on which edge carries the most of the graph vulnerability.

A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. The complete graph with n vertices has $n(n-1)/2$ edges. For a complete graph, we have $b\left(G_{complete}\right) = 1$. A path graph is a particularly simple example of a tree, which is not branched at all, that is, it contains only vertices of degree two and one. In particular, two of its vertices have degree 1 and all others (if any) have degree 2. For a path graph with n vertices, $|E| = n-1$, and therefore: $b\left(G_{path}\right) = n\left(n+1\right)/6$.
It is easy to see that $b\left(G_{complete}\right) \leq b\left(G\right) \leq b\left(G_{path}\right)$. As a consequence, we can define the normalized average edge betweenness of a graph G

$$b_{nor}(G) = \frac{b(G) - b(G_{complete})}{b(G_{path}) - b(G_{complete})} = (b(G) - 1)/(n(n+1)/6 - 1).$$

Clearly $0 \leq b_{nor}\left(G\right) \leq 1$; if the normalized average edge betweenness is close to 0, it means that the network is more robust, when it is close to 1, then the graph is more vulnerable.

The following lemma provides some basic properties for the betweenness related parameters. Let us recall that for a graph G, b_e is the betweenness of edge e, $b(G)$ is the average edge betweenness of G.

Example 2.1 Let us find the edge betweenness value of each edge of the graph G with sixvertices and sevenedges given in Fig. 1. Let us find the average edge betweenness value of the graph G.

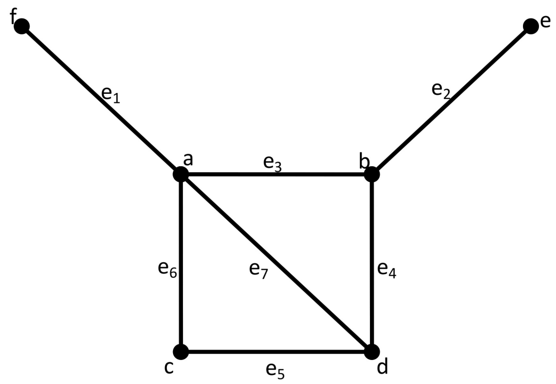

Figure 1: The graph G with sixvertices and sevenedges.

In Table 1, the shortest paths between all pairs of vertices of the graph G are found. According to these shortest paths, the edge betweenness values of each edge are calculated. Next, the normalization process is performed by finding the average edge betweenness value of the graph G.

As can be seen in line SUM of Table 1, the edge betweenness values of the edges $e_1, e_2, e_3, e_4, e_5, e_6$ *and* e_7 are found to be 5, 5, 5, 3, 2, 3 and 2, respectively. Here, the highest edge betweenness value is 5. This shows that the edges e_1, e_2 *and* e_3 are the most important positions in the graph. According to Table 1, the lowest value is 2. The edges

pairs of vertices	the shortest paths	e_1	e_2	e_3	e_4	e_5	e_6	e_7
a,b	e_3	0	0	1	0	0	0	0
a,c	e_6	0	0	0	0	0	1	0
a,d	e_7	0	0	0	0	0	0	1
a,e	e_3e_2	0	1	1	0	0	0	0
a,f	e_1	1	0	0	0	0	0	0
b,c	e_3e_6, e_4e_5	0	0	1/2	1/2	1/2	1/2	0
b,d	e_4	0	0	0	1	0	0	0
b,e	e_2	0	1	0	0	0	0	0
b,f	e_3e_1	1	0	1	0	0	0	0
c,d	e_5	0	0	0	0	1	0	0
c,e	$e_6e_3e_2$, $e_5e_4e_2$	0	2/2	1/2	1/2	1/2	1/2	0
c,f	e_6e_1	1	0	0	0	0	1	0
d,e	e_4e_2	0	1	0	1	0	0	0
d,f	e_7e_1	1	0	0	0	0	0	1
e,f	$e_2e_3e_1$	1	1	1	0	0	0	0
	SUM	5	5	5	3	2	3	2

Table 1: The edge betweenness values of the edges and the average edge betweenness value of the graph G.

e_5 and e_7 have this value. This fact shows that these edges play a more passive role than other edges of the graph. By using these values, the average edge betweenness value of G is obtained as

$$b(G) = \frac{1}{7}\sum_{i=1}^{7} b_{e_i} = \frac{25}{7} = 3,57.$$

For $n = 6$, the normalized average edge betweenness value of the G graph is as follows:

$$b_{nor}(G) = \frac{b(G) - 1}{\frac{n(n+1)}{6} - 1} = \frac{\frac{25}{7} - 1}{\frac{42}{6} - 1} = \frac{18}{42} = 0,4.$$

Lemma 2.1 *[5] Let G be a connected graph and let $e \in E$ be an edge with end vertices i, $j \in V$, then*

1. $b_e(i, j) = 1 = b_e(j, i)$.
2. $2 \le b_e \le n^2/2$ *if n is even and* $2 \le b_e \le (n-1)^2/2$ *if n is odd.*
3. $b_e = 2(n-1)$ *if one of the end vertices of e has degree* 1.

Lemma 2.2 *[5] Let G be a graph of order n, then*

1. *If e is an edge-bridge of the graph G connecting G_1 with $G \backslash G_1$, where $|V(G_1)| = n_1$, then $b_e = 2n_1(n - n_1)$.*
2. *If C is a cut-set of edges of the graph G connecting two sets of vertices X and $V(G) \backslash X$ and*

$$|X| = n_x \ , \quad \textit{then} \ \sum\nolimits_{e \in C} b_e = 2n_x(n - n_x) \ .$$

Theorem 2.1 *Let $\overline{G}$ be the complement graph of G. Then, if G has n vertices and m edges with domination number $\gamma(G) > 2$, then the average edge betweenness of $\overline{G}$ is*

$$b(\overline{G}) = (n(n-1) + 2m)/(n(n-1) - 2m).$$

Proof. Let i and j be the vertices of G and e be any edge of G. We have two cases according to $d(i,j)$:

Case 1. If $d(i,j) > 1$ is in G graph, then $d(i,j)=1$ is in $\overline{G}$. Therefore, there are $(n(n-1)/2 \; - m)$ paths with length 1 in $\overline{G}$. Thus, for all vertex pairs i and j, the summation of the values of edge betweenness of e is

$$\sum_{i \neq j} b_e(i,j) = (n(n-1)/2 \; - m)\,.$$

Case 2. If $d(i,j) = 1$ is in G graph, then $d(i,j) > 1$ is in $\overline{G}$. Let t be the number of vertices which are not adjacent to vertices i and j. Since $\gamma(G) > 2$, it is clear that $t \geq 1$. Thus, there are t paths with length 2 in $\overline{G}$. Hence, for all vertex pairs i and j , the summation of the values of edge betweenness of e is

$$\sum_{i \neq j} b_e(i,j) = t\,(1/t\,)\,2m = 2m.$$

By summing up *Cases 1* and *2*, we obtain

$$\sum_{e \in E} b_e = (n(n-1)/2 \; - m) + 2m = n(n-1)/2 \; + m.$$

As a consequence, the average edge betweenness of $\overline{G}$ is

$$b(\overline{G}) = 1/(n(n-1)/2 \; - m)\,(n(n-1)/2 \; + m) = (n(n-1) + 2m)/(n(n-1) - 2m).$$

The proof is completed. □

3 The Average Edge Betweenness of Some Special Graphs

In this section, we give some results on average edge betweennesses of some special graphs. These graphs are: C_n is a cycle graph, $S_{1,n}$ is a star graph, $W_{1,n}$ is a wheel graph, and $K_{m,n}$ is a complete bipartite graph. Finally we give average edge betweenness of E_p^t graph.

Lemma 3.1 *Label the vertices of C_n as $1, 2, 3, \ldots, n$ and the edges of C_n as e_1, e_2, e_3, $\ldots$, e_n, respectively. Let $d_{ij}(e_k)$ be the distance between i and j including the edge e_k. $n_{ij}(e_k)$ is the number of paths which include the edge e_k with length $d_{ij}(e_k)$ $(1 \leq i,j,k \leq n$ and $i \neq j)$. The relation between $d_{ij}(e_k)$ and $n_{ij}(e_k)$in graph C_n is the following*

$$If\, d_{ij}(e_k) = 1, \quad then \quad n_{ij}(e_k) = 1 \tag{1}$$
$$If\, d_{ij}(e_k) = 2, \quad then \quad n_{ij}(e_k) = 2 \tag{2}$$
$$If\, d_{ij}(e_k) = 3, \quad then \quad n_{ij}(e_k) = 3 \tag{3}$$
$$\vdots \tag{4}$$
$$If\, d_{ij}(e_k) = (n-1)/2, \quad then \quad n_{ij}(e_k) = (n-1)/2\,. \tag{5}$$
$$\tag{6}$$

Theorem 3.1 *If C_n is a cycle graph, then the average edge betweenness for the cycle graph C_n with n vertices is*

$$b(C_n) = \begin{cases} (n^2-1)/8\,, & n \text{ is odd} \\ n^2/8\,, & n \text{ is even.} \end{cases}$$

Proof. There exist two cases according to n:

Case 1. If n is odd, then $n_{ij} = 1$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \ne j)$, we get $b_{e_k} = \sum\limits_{i\ne j} \frac{n_{ij}(e_k)}{n_{ij}} = \sum\limits_{i\ne j} n_{ij}(e_k)$. From *Lemma* 3.1 and $d_{ij}(e_k) \le diam(C_n) = (n-1)/2$, we obtain $b_{e_k} = \sum\limits_{i\ne j} n_{ij}(e_k) = 1+2+3+\ldots+((n-1)/2\ ) = (n^2-1)/8$. By the definition of the average edge betweenness of a graph,

$$b(C_n) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i}\right) = (n^2-1)/8.$$

Case 2. If n is even, then we have two subcases for $d_{ij}(e_k)$.

Subcase 1. If $d_{ij}(e_k) < diam(C_n) = n/2$, then $n_{ij} = 1$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \ne j)$. In this case we proceed in a similar way as in Case 1 and

$$b_{e_k}(i,j) = 1+2+3+\ldots+[(n/2)-1] = \left(n^2-2n\right)/8$$

is obtained.

Subcase 2. If $d_{ij}(e_k) = diam(C_n) = n/2$, then $n_{ij}(e_k) = n/2$ and $n_{ij} = 2$ for $\forall i,j \quad (1 \le i,j,k \le n \text{ and } i \ne j)$, we get

$$b_{e_k}(i,j) = \sum_{i\ne j} n_{ij}(e)/n_{ij} = (n/2)(1/2) = n/4.$$

By Subcase 1 and Subcase 2, it is clear that

$$b_{e_k} = (n^2-2n)/8 + n/4 = n^2/8 \quad (\forall k = \overline{1,n}).$$

Consequently, we obtain the average edge betweenness of C_n

$$b(C_n) = \frac{1}{|E|}\sum_{i=1}^{n} b_{e_i} = n^2/8.$$

Thus, the proof is completed. □

Theorem 3.2 *If $S_{1,n}$ is a star graph, then the average edge betweenness for the star graph $S_{1,n}$ with $n+1$ vertices is $b(S_{1,n}) = n$.*

Proof. The vertices of $S_{1,n}$ are of two kinds: one vertex of degree n and n vertices of degree one. The vertices of degree one will be referred to as the minor vertices and the vertex of degree n as the center vertex. Label the minor vertices as $1,2,3,\ldots,n$, the center vertex as c, and the edges of $S_{1,n}$ as e_i $(i = \overline{1,n})$. We have two cases in order to find the shortest paths.

Case 1. The shortest path between central vertex c and any minor vertex i:
There is only one path e_i in this case. By the definition of the edge betweenness, we obtain the value of the edge e_i $(i = \overline{1, n})$

$$b_{e_i}(c, i) = 1.$$

Case 2. The shortest path between any two different minor vertices:
There is only one path $e_i e_j$ between the minor vertices i and j $(1 \leq i, j \leq n)$. By using *Lemma* 2.1, for $\forall i, j$, we get $n_{ij} = n_{ji} = 1$ and $n_{ij}(e_k) = 1$ $(k = i \vee j)$. Thus, we have $b_{e_k}(i, j) = 1/1 = 1$. There are $n - 1$ different pairs of vertices that include e_k. Hence, the value of the edge betweenness of e_k

$$b_{e_k}(i, j) = (n - 1)\ 1 = n - 1.$$

By summing up *Cases 1* and *2*, we clearly see that

$$b_{e_i} = 1 + n - 1 = n.$$

Consequently, the average edge betweenness of $S_{1,n}$ is

$$b(S_{1,n}) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i}\right) = n.$$

Thus, the proof is completed. □

Theorem 3.3 *If $W_{1,n}$ is a wheel graph, then the average edge betweenness for the wheel graph $W_{1,n}$ $(n \geq 5)$ with $n + 1$ vertices is $b(W_{1,n}) = (n - 1)/2$.*

Proof. The vertices of $W_{1,n}$ are of two kinds: n vertices which are of degree 3 will be referred to as the minor vertices and the vertex of degree n will be referred to as the central vertex. Label the minor vertices as $1, 2, 3, \ldots, n$, the central vertex as c, the edges between the central vertex and the minor vertices as $e_{c\,i}$ $(i = \overline{1, n})$ and the other remaining edges as e_i $(i = \overline{1, n})$. There exist two cases for the shortest paths between the pairs of vertices.

Case 1. If the pair of vertices includes the central vertex and the minor vertices:
There exists only one path e_{ci} between those vertices that has the length $d(c, i)$=1. It is clear that for the path e_{ci}, we have $n_{ci} = 1$ and $n_{ci}(e_{ci}) = 1$. Hence, the value of the edge betweenness of e_{ci}

$$b_{e_{ci}}(c, i) = 1.$$

Case 2. If the pair of vertices includes any two different minor vertices i and j:
We have three subcases for these minor vertices according to the length of the shortest path between the vertices:

Subcase 1. If $d(i, j) = 1$, then there is only one path e_k $(k = i \vee j)$ between i and j. It is clear that for this path e_k, we have $n_{ij} = 1$ and $n_{ij}(e_k) = 1$. Hence, the value of the edge betweenness of e_k

$$b_{e_k}(i, j) = 1.$$

Subcase 2. If $d(i,j) = 2$, then there are two paths: the paths $e_i e_j$ and $e_{ci} e_{cj}$ between the vertices i and j. The lengths of the paths between the vertices i and j including the edges e_k and e_{ck} $(k = i \vee j)$ are $d_{ij}(e_k) = 2$ and $d_{ij}(e_{ck}) = 2$, respectively. By Lemma 3.1, $n_{ij}(e_k) = 2$, $n_{ij}(e_{ck}) = 2$ and $n_{ij} = 2$. Thus we have

$$b_{e_k}(i,j) = 2/2 = 1, \quad b_{e_{ck}}(i,j) = 2/2 = 1.$$

Subcase 3. If $d(i,j) > 2$, then there is only one path between the vertices i and j with length 2, that is $e_{ci} e_{cj}$. It is clear that for this path $e_{ci} e_{cj}$, we have $n_{ij} = 1$ and $n_{ij}(e_{ck}) = 1$ $(k = i \vee j)$. Hence,

$$b_{e_{ck}}(i,j) = 1.$$

In this way, since there are $n - 5$ different pairs of vertices that include the edge e_{ck}, the value of the edge betweenness of e_{ck} is

$$b_{e_{ck}}(i,j) = 1(n-5) = n-5.$$

By summing up Subcases 1 and 2, we get the value of the edge betweenness of e_k as

$$b_{e_k} = 1 + 1 = 2.$$

By summing up Case 1 and Subcases 2 and 3, we get the value of the edge betweenness of e_{ck} as

$$b_{e_{ck}} = 1 + 1 + n - 5 = n - 3.$$

Consequently, the average edge betweenness of $W_{1,n}$ is

$$b(W_{1,n}) = \frac{1}{|E|}\left(\sum_{i=1}^{n} b_{e_i} + \sum_{i=1}^{n} b_{e_{ci}}\right) = (n-1)/2.$$

Thus, the proof is completed. □

Theorem 3.4 *If $K_{m,n}$ is a complete bipartite graph, then the average edge betweenness for the complete bipartite graph $K_{m,n}$ with $m+n$ vertices is $b(K_{m,n}) = (m^2 + n^2 - (m+n))/mn + 1$.*

Proof. Let $G = K_{m,n}$, where S_1 and S_2 are the partite sets of G with cardinality m and n respectively. The set of edges of $K_{m,n}$ is $E = \{e_{pk} \mid p \in S_1 \ \ and \ \ k \in S_2\}$ and $|E| = mn$. We have 3 cases in order to find the shortest paths according to the vertices being either in S_1 or in S_2 . Let i and j be the vertices of $K_{m,n}$.

Case 1. If $i \in S_1$ and $j \in S_2$, then there is only one path e_{ij} between the vertices i and j. Therefore, it is straightforward that $n_{ij} = 1$ and $n_{ij}(e_{ij}) = 1$. Thus

$$b_{e_{ij}}(i,j) = 1.$$

Case 2. If $i, j \in S_1$, then there are n paths $e_{ik} e_{jk}$ with length 2 between the vertices i and j $(k \in S_2)$. Clearly, $n_{ij} = n$, $n_{ij}(e_{pk}) = 1$ $(p = i \vee j)$. Hence,

$$b_{e_{pk}}(i,j) = 1/n.$$

There are $m - 1$ different pairs of vertices that include e_{pk}, the value of the edge betweenness of e_{pk} is

$$b_{e_{pk}}(i,j) = (m-1)/n.$$

Case 3. If $i, j \in S_2$, then there are m paths $e_{ik}e_{jk}$ with length 2 between the vertices i and j ($k \in S_1$). This case is similar to *Case 2*, and for $n-1$ different pairs of vertices that include e_{pk}, the value of the edge betweenness of e_{pk} is

$$b_{e_{pk}}(i,j) = (n-1)/m.$$

By summing up Cases 1, 2 and 3, we get the value of the edge betweenness of e_{pk} as

$$b_{e_{pk}} = (m^2 + n^2 - (m+n))/mn + 1.$$

Consequently, the average edge betweenness of $K_{m,n}$ is

$$b(K_{m,n}) = \frac{1}{|E|}\sum_{p=1}^{m}\sum_{k=1}^{n} b_{e_{pk}} = (m^2 + n^2 - (m+n))/mn + 1.$$

Hence the desired result holds. □

Definition 3.1 [6] The graph E_p^t has t legs and each leg has p vertices (Figure 2). Thus E_p^t has $pt+2$ vertices and $pt+1$ edges.

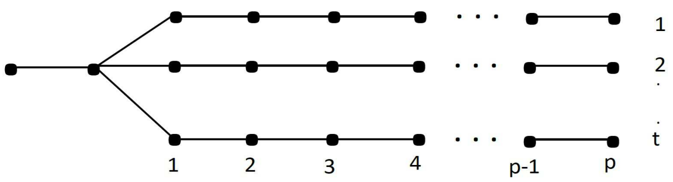

Figure 2: E_p^t graph with $pt+2$ vertices.

Theorem 3.5 *Let t and p be integers ($t \geq 2, p \geq 2$). The average edge betweenness of graph E_p^t is*

$$b(E_p^t) = [(pt\,(p+1))\,/6\,(pt+1)]\;[3pt - 2p + 5] + 1.$$

Proof. Label the vertex with degree $t+1$ as v, the neighbor of v with degree 1 as u, the vertices of jth leg as (i,j) ($i = \overline{1,p}$ *and* $j = \overline{1,t}$), the edge between the vertices u and v as e, the edge between the vertices v and (i,j) as bridge e_{ij}, where $i = 1$, and the edges of jth leg as e_{ij} respectively ($i = \overline{2,p}$ and $j = \overline{1,t}$).

This labeling is shown in Figure 3. Since E_p^t is a tree, there is only one path between any pairs of vertices. Clearly, $n_{ij} = 1$ and $b_e = \sum\limits_{i \neq j}(n_{ij}(e)/n_{ij}) = \sum\limits_{i \neq j} n_{ij}(e)$ ($i = \overline{1,p},\ j = \overline{1,t}$). We have four cases for the vertex pairs of E_p^t.

Case 1. Consider the shortest paths between the vertex u and the other vertices. There exist $(pt+1)$ paths. Each of these paths includes the edge e. The value of the edge betweenness of this edge e is

$$b_e = b_e(u,\,(i,j)) \,+\, b_e(u,v) = pt+1.$$

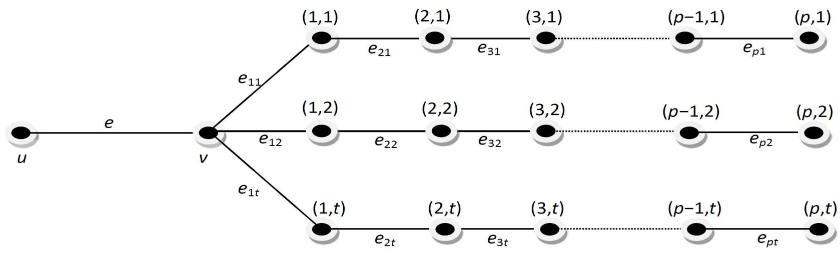

Figure 3: The labeling of vertices and edges of E_p^t graph.

Each of these paths also includes the edge e_{ij}. The edge e_{ij}, that is at distance i to the vertex v, is on $p+1-i$ different paths. The value of the edge betweenness of this edge e_{ij} that is between the vertices u and (k,m) ($k=\overline{1,p}$ and $m=\overline{1,t}$) is

$$b_{e_{ij}}\left(u,(k,m)\right)=p+1-i.$$

Case 2. Consider the shortest paths between the vertex v and the other vertices on the legs. Each of these paths includes only the edge e_{ij}. The value of the edge betweenness of this edge e_{ij} that is between the vertices v and (k,m) ($k=\overline{1,p}$ and $m=\overline{1,t}$) is

$$b_{e_{ij}}\left(v,(k,m)\right)=p+1-i.$$

Case 3. Consider the shortest paths between the vertices of any leg. The initial vertex is $(1,j)$ and the last vertex is (p,j) ($j=\overline{1,\,t}$) on a leg. Thus we have t paths with p vertices, that is P_p. Those paths include the edge e_{ij}. The value of the edge betweenness of this edge e_{ij} equals the number of the left-hand side vertices of e_{ij} multiplied by the number of the right-hand side vertices of e_{ij}. If the edge e_{ij} is between the vertices (k,m) and (k',m) ($k,k'=\overline{1,p}$ and $m=\overline{1,t}$), then we have

$$b_{e_{ij}}\left((k,m),(k',m)\right)=(i-1)(p+1-i).$$

Case 4. Consider the shortest paths between the vertices of any leg and the vertices of the other legs. This case is similar to Case 3. If the edge e_{ij} is between the vertices (k,m) and (i,j) ($k=\overline{1,p}$ and $m=\overline{1,t}$), then we get

$$b_{e_{ij}}\left((k,m),(i,j)\right)=\left[p(t-1)\right](p+1-i).$$

By summing up Cases 1, 2, 3, and 4, we obtain

$$b_{e_{ij}}=(p+1)(p(t-1)+1)+i(p(2-t))-i^2.$$

The summation for all the edges e_{ij} of the graph is

$$\sum_{i=1}^{p}\sum_{j=1}^{t}b_{e_{ij}}=(pt/6)\left[3p^2t+3pt+3p-2p^2+5\right].$$

Consequently, the average edge betweenness of E_p^t graph is

$$b(E_p^t) = [\,1/\,(1+pt)\,]\ \left[(pt+1) + (pt/6)\ \left(3p^2t + 3pt + 3p - 2p^2 + 5\right)\,\right]$$

$$b(E_p^t) = [(pt\,(p+1))\,/6\,(pt+1)]\ [3pt - 2p + 5] + 1.$$

Thus the proof is completed. □

4 The Normalized Average Edge Betweenness of Some Special Graphs

In this section, we give the normalized average edge betweennesses of some special graphs whose average edge betweennesses values are calculated in Section 3.

1. $b_{nor}(C_n) = \begin{cases} [3(n-3)]/[4(n-2)], & n \text{ is odd} \\ \left[3(n^2-8)\right]/\left[4(n^2+n-6)\right], & n \text{ is even.} \end{cases}$
2. $b_{nor}(W_{1,n}) = (3n-9)/(n^2+3n-4)$.
3. $b_{nor}(S_{1,n}) = 6/(n+4)$.
4. $K_{m,n}$ *and* $p = m+n$, $b_{nor}(K_{m,n}) = \left[6(m^2+n^2-p)\right]/\left[mn(p^2+p-6)\right]$.
5. $b_{nor}(E_p^t) = \left[3p^2t - 2p^2 + p + 6pt + 13\right]/[pt+5] - [2p-5]/[pt(pt+5)]$.
6. $\overline{G}$ is a complement graph of G with $\gamma(G) > 2$,

$$b_{nor}(\overline{G}) = 24m/\left[(n^2-n-2m)(n^2+n-6)\right].$$

5 Conclusion

In this paper, we evaluate the average edge betweenness and the normalized average edge betweenness of some special graphs and E_p^t graph. The average edge betweenness is a new characteristic for graph vulnerability introduced in [8]. Calculation of average edge betweenness for simple graph types is important because we can gather information on which edge is the most vulnerable. The average edge betweenness of a given edge is the fraction of shortest paths, counted over all pairs of vertices that pass through that edge. This measure considers both the local and the global structure of the graph.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 253–258

Coexistence of Different Types of Chaos Synchronization Between Non-Identical and Different Dimensional Dynamical Systems

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Received: April 30, 2018; Revised: July 12, 2018

Abstract: In this paper, based on Lyapunov stability theory, the coexistence of full state hybrid projective synchronization (FSHPS), $\Phi - \Theta$ synchronization, generalized synchronization (GS) and Q-S synchronization between different dimensional chaotic systems is studied. An application example and numerical simulations are presented to validate the main results of this paper.

Keywords: *chaos; full state hybrid projective synchronization; $\Phi - \Theta$ synchronization; generalized synchronization; Q-S synchronization.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C55, 93D05.

1 Introduction

Over the last few decades, a great deal of attention has been paid to the subject of chaotic dynamical systems and their synchronization control. Synchronization is an adaptive process that works to force the variables of a chaotic slave system to follow those of a corresponding master system [1]. This considerable interest has resulted in many synchronization types and schemes, see [2–5]. Among the most effective types of synchronization for chaotic and hyperchaotic systems are the full state hybrid projective synchronization (FSHPS) [6], Φ-Θ synchronization [7,8], generalized synchronization (GS) [9] and Q-S synchronization [10]. As a natural consequence of defining a variety

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of synchronization types, it became apparent that multiple types could coexist simultaneously, e.g. [11–13], a property that is of particular importance in the fields of secure communications and chaotic encryption schemes.

In this paper, we are concerned with the coexistence of the four types of synchronization mentioned above, i.e. FSHPS, Φ–Θ, GS and Q-S, in four dimensions between a three–dimensional chaotic master system and a four–dimensional hyperchaotic slave system. For this, we employ nonlinear control methods and make use of the well known direct Lyapunov method for establishing the global asymptotic convergence of synchronization errors towards zero. The resulting conditions are simple and their verification is trivial. Also, in order to put the reader's mind at ease and confirm the results of our study, we consider a numerical example, whereby the coexistence of FSHPS, Φ–Θ, GS and Q-S is illustrated for some typical chaotic and hyperchaotic systems. In Section 2 of this paper, the problem formulation and main result are given. Section 3 presents the numerical application of the proposed coexistence result with the aim of demonstrating the effectiveness of the approach developed herein. Section 4 summarizes the work reported in this paper.

2 Problem Formulation and Main Result

We consider the following master and slave systems

$$\dot{x}_i(t) = f_i(X(t)), \quad i = 1, 2, 3, \tag{1}$$

$$\dot{y}_i(t) = \textstyle\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{2}$$

where $X(t) = (x_i)_{1 \le i \le 3}$ and $Y(t) = (y_i)_{1 \le i \le 4}$ are the states of the master and the slave systems, respectively, $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, $(b_{ij})_{4\times 4} \in \mathbb{R}^{4\times 4}$, $g_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 1, 2, 3, 4$, are nonlinear functions, and $U = (u_1, u_2, u_3, u_4)^T$ is a vector–valued controller. The problem of coexistence of FSHPS, $\Theta - \Phi$ synchronization, GS and Q-S synchronization between master system (1) and slave system (2) is to find controllers u_i, $i = 1, 2, 3, 4$, such that the errors

$$\begin{aligned} e_1(t) &= y_1(t) - \Lambda \times X(t), \\ e_2(t) &= \Theta \times Y(t) - \Phi \times X(t), \\ e_3(t) &= y_3(t) - \phi\left(X(t)\right), \\ e_4(t) &= Q\left(Y(t)\right) - S\left(X(t)\right) \end{aligned} \tag{3}$$

satisfy

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3, 4,$$

where $\Lambda = (\Lambda_i)_{1 \le i \le 3}$, $\Theta = (\Theta_i)_{1 \le i \le 4}$, $\Phi = (\Phi_i)_{1 \le i \le 4}$ are constant matrices and $\phi : \mathbb{R}^3 \to \mathbb{R}$, $Q : \mathbb{R}^4 \to \mathbb{R}$, $S : \mathbb{R}^3 \to \mathbb{R}$ are differentiable functions. Here e_1 stands for the FSHPS error, e_2 stands for the $\Theta - \Phi$ synchronization error, e_3 denotes the GS error, and e_4 is the Q-S synchronization error.

Theorem 2.1 *FSHPS, $\Phi - \Theta$ synchronization, GS and Q-S synchronization coexist between master system (1) and slave system (2) under the following conditions:*

(i) $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ 0 & 0 & 1 & 0 \\ \frac{\partial Q}{\partial y_1} & \frac{\partial Q}{\partial y_2} & \frac{\partial Q}{\partial y_3} & \frac{\partial Q}{\partial y_4} \end{pmatrix}$ *is an invertible matrix and M^{-1} is its inverse.*

(ii) $U = M^{-1}\left((B-C)\,e\,(t) - R\right)$, *where* $C \in \mathbb{R}^{4\times 4}$ *is a control matrix and*

$$R = \begin{pmatrix} \sum_{j=1}^{4} b_{1j}y_j + g_1 - \sum_{j=1}^{3}\Lambda_j f_j \\ \sum_{i=1}^{4}\Theta_i\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) - \sum_{j=1}^{3}\Phi_j f_j \\ \sum_{j=1}^{4} b_{3j}y_j + g_3 - \sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j} f_j \\ \sum_{i=1}^{4}\frac{\partial Q}{\partial y_i}\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) - \sum_{j=1}^{3}\frac{\partial S}{\partial x_j} f_j \end{pmatrix}.$$

(iii) $(B-C) + (B-C)^T$ *is a negative definite matrix, where* $B = (b_{ij})_{4\times 4}$.

Proof. The error system (3) can be differentiated as follows:

$$\begin{aligned} \dot{e}_1(t) &= \sum_{j=1}^{4} b_{1j}y_j + g_1 + u_1 - \sum_{j=1}^{3}\Lambda_j f_j, \\ \dot{e}_2(t) &= \sum_{i=1}^{4}\Theta_i\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) + \sum_{j=1}^{4}\Theta_j u_j - \sum_{j=1}^{3}\Phi_j f_j, \\ \dot{e}_3(t) &= \sum_{j=1}^{4} b_{3j}y_j + g_3 + u_3 - \sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j} f_j, \\ \dot{e}_4(t) &= \sum_{i=1}^{4}\frac{\partial Q}{\partial y_i}\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) + \sum_{j=1}^{4}\frac{\partial Q}{\partial y_j}u_j - \sum_{j=1}^{3}\frac{\partial S}{\partial x_j} f_j. \end{aligned} \tag{4}$$

The error system (4) can be written in the following compact form

$$\dot{e}\,(t) = M \times U + R. \tag{5}$$

By substituting the control law (ii) into equation (5), the error system can be written as

$$\dot{e}(t) = (B-C)\,e(t). \tag{6}$$

Construct the candidate Lyapunov function in the form: $V\,(e(t)) = e^T(t)e(t)$, we obtain

$$\begin{aligned} \dot{V}\,(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= e^T(t)(B-C)^T e(t) + e^T(t)\,(B-C)\,e\,(t) \\ &= e^T(t)\left[(B-C)^T + (B-C)\right]e(t). \end{aligned}$$

From (iii), we get $\dot{V}\,(e(t)) < 0$. Thus, from the Lyapunov stability theory, the zero solution of the error system (6) is globally asymptotically stable and, therefore, systems (1) and (2) are globally synchronized.

3 Numerical Application

In this example, the master system is chosen as the following 3D system

$$\begin{aligned} \dot{x}_1 &= a_1\,(x_2 - x_1)\,, \\ \dot{x}_2 &= x_1 x_3, \\ \dot{x}_3 &= 50 - a_2 x_1^2 - a_3 x_3. \end{aligned} \tag{7}$$

When $a_1 = 2.9$, $a_2 = 0.7$ and $a_3 = 0.6$, system (7) exhibits chaotic attractors [14]. The salve system is described by

$$\begin{aligned} \dot{y}_1 &= \alpha\left(y_2 - y_1\right) + \gamma y_4 + u_1, \\ \dot{y}_2 &= -y_1 y_3 - y_2 + \gamma y_4 + u_2, \\ \dot{y}_3 &= y_1 y_2 - y_3 - \beta + u_3, \\ \dot{y}_4 &= -\delta\left(y_1 + y_2\right) + u_4. \end{aligned} \tag{8}$$

The uncontrolled system (8) (i.e. with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits strange hyperchaotic attractors for the parameter values $\alpha = 4$, $\beta = 20$, $\gamma = 0.2$ and $\delta = 0.5$ [15]. The linear part B and nonlinear part g of the slave system (8) can be formulated as

$$B = \begin{pmatrix} -4 & 4 & 0 & 0.2 \\ 0 & -1 & 0 & 0.2 \\ 0 & 0 & -1 & 0 \\ -0.5 & -0.5 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 - \beta \\ 0 \end{pmatrix}.$$

According to our approach, the error system between systems (7) and (8) is described by

$$\begin{aligned} e_1 &= y_1 - \Lambda \times \left(x_1, x_2, x_3\right)^T, \\ e_2 &= \Theta \times \left(y_1, y_2, y_3, y_4\right)^T - \Phi \times \left(x_1, x_2, x_3\right)^T, \\ e_3 &= y_4 - \phi\left(x_1, x_2, x_3\right), \\ e_4 &= Q\left(y_1, y_2, y_3, y_4\right) - S\left(x_1, x_2, x_3\right), \end{aligned} \tag{9}$$

where $\Lambda = (-1, 0, 2)$, $\Theta = (0, 2, 0, 0)$, $\Phi = (1, 2, 3)$, $\phi\left(x_1, x_2, x_3\right) = x_1 x_2 + x_3, Q\left(y_1, y_2, y_3, y_4\right) = 1 + 3y_4$ and $S\left(x_1, x_2, x_3\right) = x_1 x_2 x_3$. Based on the notations used in Section 2, the matrix M is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \tag{10}$$

and thus

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{11}$$

Then, the control matrix C can be selected as

$$C = \begin{pmatrix} 0 & 4 & 0 & 0.2 \\ 0 & 2 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{pmatrix}. \tag{12}$$

Using matrices (10), (11) and (12) we can easily construct the control law (ii) described in Theorem 1. We can see that $(B - C) + (B - C)^T$ is a negative–definite matrix and all conditions of Theorem 1 are satisfied. Therefore, systems (7) and (8) are globally synchronized in 4-D. The time evolution of errors between systems (7) and (8) is shown in Figure 1.

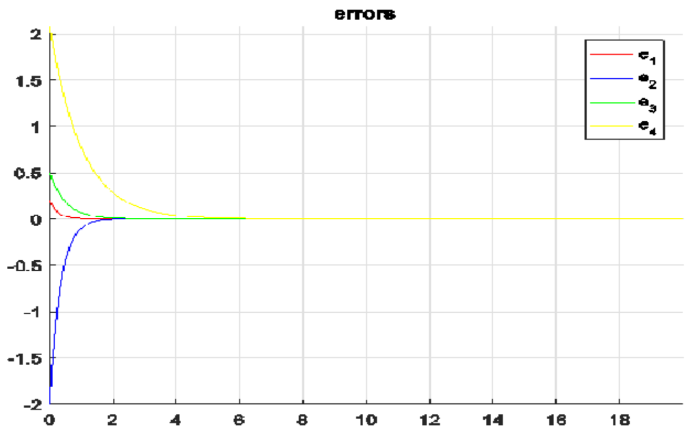

Figure 1: Time evolution of the synchronization errors e_1,e_2, e_3 and e_4 between the master system (7) and the slave system (8).

4 Conclusion

A new synchronization scheme has been used to achieve coexistence of several types of synchronization between an arbitrary 3-dimensional master and a 4-dimensional slave system. By using Lyapunov stability theory, the paper analysed the coexistence of full state hybrid projective synchronization (FSHPS), $\Phi - \Theta$ synchronization, generalized synchronization (GS) and Q-S synchronization based on the control of the linear part of the master system. The numerical example detailed in the previous section confirms the effectiveness of the theoretical analysis.

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Generalized Monotone Method for Riemann-Liouville Fractional Reaction Diffusion Equation with Applications

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Received: May 27, 2018; Revised: June 27, 2018

Abstract: Initially, we have obtained the integral representation for the solution of the linear Riemann-Liouville fractional reaction diffusion equation of order q, where $0 < q < 1$, in terms of Green's function. We have developed a generalized monotone method for the non-linear Riemann-Liouville reaction diffusion equation when the forcing term is the sum of an increasing and decreasing functions. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. Under uniqueness assumption, we prove the existence of a unique solution for the non-linear Riemann-Liouville fractional reaction diffusion equation.

Keywords: *Riemann-Liouville fractional derivative; representation form; eigenfunction expansion; Mittag-Leffler function; coupled upper and lower solutions; generalized monotone method.*

Mathematics Subject Classification (2010): 26A33, 26A48.

1 Introduction

Computation of explicit solutions of non-linear dynamic equation is rarely possible. It is more so with non-linear fractional dynamic equations with initial and boundary conditions. In general, the existence and uniqueness of solution of the fractional dynamic equation has been established mostly, using some kind of fixed point approach. See [1, 3, 7–9, 15–17, 28, 29, 31, 32] and the references therein for the existence, uniqueness and applications of fractional dynamic equations. The drawback of fixed point theorem results for the initial and/or boundary value problem is that they do not guarantee the

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interval of existence. The method of upper and lower solutions combined with the monotone iterative technique not only guarantees the interval of existence but also the method is both theoretical and computational. See [4,5,12–14,24–27] for monotone methods and generalized monotone methods for nonlinear dynamic equations. The monotone method is feasible only when the non-linear function is increasing or could be made increasing. In this case, we obtain a sequences of approximate solutions which are either monotonically increasing or monotonically decreasing if the approximation is the lower solution or the upper solution respectively. If the non-linear function is decreasing, the monotone method will yield alternating sequences. However, from practical application problems, the non-linear forcing term will be a sum of increasing and decreasing functions as in the population models and chemical combustion models, see [19]. In order to handle such problems, a generalized monotone method has been developed in [20,22,23,30].

In this work, we consider the non-linear Riemann Liouville fractional reaction diffusion equation where the forcing function is the sum of increasing and decreasing functions. We develop a generalized monotone method for the non-linear Riemann-Liouville fractional reaction diffusion equation using coupled lower and upper solutions. Initially, we have obtained a representation form for the solution of the linear Riemann-Liouville fractional reaction diffusion equation using the eigen function expansion method and Green's identity. We have also developed the maximum principle and comparision results relative to one dimentional time fractional parabolic equations. These results are useful in proving that the sequences developed in the generalized monotone method converge to the coupled minimal and maximal solutions of the non-linear fractional reaction diffusion equation. The convergence of the sequences is monotonic and uniform in the weighted norm. Finally, under the uniqueness assumption, we can prove that there exists a unique solution to the non-linear Riemann-Liouville fractional reaction diffusion equation.

2 Preliminary Results

In this section, we recall some known definitions and known results which are useful to develop our main results. Here and throughout, the notation $\Gamma(q)$ denotes the gamma function of order q.

Definition 2.1 The Riemann-Liouville fractional integral of $u(t)$ of order q is defined by

$$D_t^{-q}u = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}u(s)ds, \tag{1}$$

where $0 < q \leq 1$.

Definition 2.2 The Riemann-Liouville (left-sided) fractional derivative of $u(t)$ of order q, when $0 < q < 1$, is defined as:

$$D^q u(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{q-1}u(s)ds,\ t > 0. \tag{2}$$

Next we define the Mittag-Leffler function which is useful in computing the solution of linear fractional differential equation explicitly.

Definition 2.3 The two parameter Mittag-Leffler function is defined as

$$E_{q,r}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+r)}. \tag{3}$$

If $r = q$, (3) reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{4}$$

If $r = 1$, the Mittag-leffler function is defined as

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma q(k+1)}. \tag{5}$$

Further, if $q = r = 1$, $E_{1,1} = e^{\lambda t}$ is the exponential function.

For more details, see [6, 11, 18, 19, 21]. In our next definition we assume $p = 1 - q$, when $0 < q < 1$, $J = (0, T]$ and $J_0 = [0, T]$.

Definition 2.4 A function $\phi(t) \in C(J, R)$ is a C_p continuous function, if $t^{1-q}\phi(t) \in C(J_0, R)$. The set of C_p continuous functions is denoted by $C_p(J, R)$. Further, given a function $\phi(t) \in C_p(J, R)$, we call the function $t^{1-q}\phi(t)$ the continuous extension of $\phi(t)$.

Note that any continuous function in J_0 is also a C_p continuous function.

Consider the initial value problem for the linear Riemann-Liouville fractional reaction differential equation of order q as

$$D^q u = \lambda u + f(t),\ \Gamma(q)u(t)\ t^{1-q}|_{t=0} = u^0, \tag{6}$$

where λ is a real number and $f \in\ C[[0, T], \mathbb{R}]$. The integral representation of the solution of equation(6) is:

$$u(t) = u^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}[\lambda(t-s)^q] f(s) ds. \tag{7}$$

For details, see [10, 11, 21]. The next result is a basic comparison result involving the q^{th} order fractional Riemann-Liouville derivative with respect to time.

Lemma 2.1 *Let $m(t) \in C_p[[0, T], R]$ be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$, and $t^{1-q}m(t) \le 0$ on $[0, t_1]$, then $D^q m(t_1) \ge 0$. See more in [4, 5].*

Remark: In the above theorem, if m is a function of (x, t), then the conclusion is true with the partial fractional derivative of m with respect to t of order q. This is what we need in our work.

3 Auxiliary Results

In this section, we obtain a representation form for the solution of the linear Riemann-Liouville fractional reaction diffusion equation with the fractional time derivative. We achieve this by using the eigen function expansion method. Then we will develop comparison results for the non-linear Riemann-Liouville fractional reaction diffusion equation with initial and boundary conditions. The first comparison theorem is with respect to the natural lower and upper solutions when the non-linear term is of the form $F(x, t, u)$, where $F(x, t, u)$ satisfies the one sided Lipschitz condition. The second comparison theorem is relative to coupled lower and upper solutions. In this case, we assume the non-linear term as the sum of two functions $f(x,t,u)$ and $g(x,t,u)$, where $f(x,t,u)$ is a non-decreasing function in u and $g(x,t,u)$ is a non-increasing function in u for (x,t) in $[0,L]\times[0,T]$. In order to present our result, consider the linear Riemann-Liouville fractional reaction diffusion equation with initial and boundary conditions of the form

$$\begin{aligned}\partial_t^q u - k u_{xx} &= Q(x,t) \ on \ Q_T,\\ u(0,t)=A(t),\ u(L,t)&=B(t) \ in \ \Gamma_T,\\ \Gamma(q)t^{1-q}u(x,t)|_{t=0} &= f^0(x),\ x\in\ \overline{\Omega},\end{aligned} \tag{8}$$

where $\Omega=[0,L]$, $J=(0,T]$, $Q_T=J\times\Omega$, $k>0$ and $\Gamma_T=(0,T)\times\partial\Omega$. Here, ∂_t^q is the partial Riemann-Liuoville fractional derivative with respect to time 't' of order q, $0<q<1$.

In order for the initial boundary value problem to be compatible, we assume that $f^0(0)=A(0)=f^0(L)=B(0)=0$, $\Gamma(q)\ t^{1-q}u(x,t)|_{t=0}=f^0(x)$. Here and throughout this work, we assume the initial and boundary conditions satisfy the compatibility conditions. Using the method of eigenfunction expansion on equation (8), we have the solution of the form:

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\phi_n(x), \tag{9}$$

where the eigenfunctions of the related homogeneous problem are known to be $\phi_n(x)=sin\frac{n\pi x}{L}$ and its corresponding eigenvalues are $\lambda_n=(\frac{n\pi}{L})^2$. Using the same approach as in [22], we can compute $b_n(t)$, where $b_n(t)$ will be the solution of the ordinary linear Riemann-Liuoville differential equation.

Here, our aim is to find $b_n(t)$. Using the standard arguments, one can compute $b_n(t)$. The explicit form of $b_n(t)$ is

$$\begin{aligned} b_n(t) \quad &= \quad b_n^0 t^{q-1}E_{q,q}(-k\lambda_n t^q) \qquad (10)\\ &+\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)q_n(s)+k\frac{2n\pi}{L^2}(A(s)-(-1)^nB(s)ds,\end{aligned}$$

where

$$b_n^0=\frac{2}{L}\int_0^L f^0(y)\phi_n(y)dy \ and \tag{11}$$

$$q_n(t)=\frac{2}{L}\int_0^L Q(y,t)\phi_n(y)dy. \tag{12}$$

Therefore,

$$\begin{aligned} b_n(t) \quad &= \quad \frac{2}{L}\int_0^L f^0(y)\phi_n(y)dyt^{q-1}E_{q,q}(-k\lambda_n t^q) \qquad (13)\\ &+\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)\frac{2}{L}\int_0^L Q(y,s)\phi_n(y)dyds\\ &+k\frac{2n\pi}{L^2}\int_0^t (t-s)^{q-1}E_{q,q}(-k\lambda_n t^q)(A(s)-(-1)^nB(s))ds.\end{aligned}$$

So, using $b_n(t)$ in (9), we can get the solution $u(x,t)$ of the form

$$\begin{aligned} u(x,t) &= \int_0^L t^{q-1}[\sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n(t^q))\phi_n(x)\phi_n(y)]f^0(y)dy \qquad (14)\\ &+\int_0^t\int_0^L[\sum_{n=1}^{\infty}\frac{2}{L}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)]Q(y,s)dyds\\ &+k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]A(s)ds\\ &-k\int_0^t[\frac{2n\pi}{L^2}(t-s)^{q-1}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)]B(s)ds. \end{aligned}$$

Finally, we can write

$$\begin{aligned} u(x,t) = \int_0^L t^{q-1}G(x,y,t)f^0(y)dy + \int_0^t\int_0^L G(x,y,t-s)Q(y,s)dyds \qquad (15)\\ +k\int_0^t G_y(x,0,t-s)A(s)ds - k\int_0^t G_y(x,L,t-s)B(s)ds, \end{aligned}$$

where

$$G(x,y,t) = \sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n t^q)\phi_n(x)\phi_n(y).$$

This result will be useful in our main result when we are computing the linear approximations of the generalized monotone iterates.

Here, we can find the steady state condition with homogeneous boundary conditions in which the source term $Q(x,t) = Q(x)$ is independent of time:

$$ku_{xx} + Q(x) = 0.$$

Now the form $u_{xx} = g(x)$, in which $g(x) = -\frac{Q(x)}{k}$.
Therefore,

$$u(x,t) = \int_0^L f^0(y)t^{q-1}G(x,t;y,0)dy + \int_0^L -kg(y)\big[\int_0^t G(x,t;y,s)ds\big]dy, \qquad (16)$$

where
$t^{q-1}G(x,t;y,s) = t^{q-1}\sum_{n=1}^{\infty}\frac{2}{L}E_{q,q}(-k\lambda_n(t-s)^q)\phi_n(x)\phi_n(y)$.
As $t \to \infty$, $G(x,t;y,0) \to 0$ such that the effect of the initial condition $t^{1-q}u(x,t)|_{t=0} = f^0(x)$ vanishes as $t \to \infty$. But, as $t^{q-1}G(x,t;y,s) \to 0$ as $t \to \infty$, the steady source is still important as $t \to \infty$ since $\int_0^t E_{q,q}(-k\lambda_n(t-s)^q)ds = \frac{1-E_{q,q}(-k\lambda_n t^q)}{k(\frac{n\pi}{L})^2}$.
Thus, as $t \to \infty$,

$$u(x,t) \to u(x) = \int_0^L g(y)G(x,y)dy,$$

where

$$G(x,y) = -\sum_{n=1}^{\infty}\frac{2}{L}\phi_n(x)\phi_n(y).$$

Hence, we obtained the steady-state temperature distribution $u(x)$ by taking the limit as $t \to \infty$ of the time-dependent problem with a steady source $Q(x) = -kg(x)$.
We recall two known lemmas regarding the Mittag-Leffler functions series from [2].

Lemma 3.1 *Let $E_{q,1}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then, $\frac{E_{q,1}(-\lambda_1 t^q)}{E_{q,1}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.*

Lemma 3.2 *Let $E_{q,q}(-\lambda t^q)$ be the Mittag-Leffler function of order q, where $0 < q \leq 1$. Then $\frac{E_{q,q}(-\lambda_1 t^q)}{E_{q,q}(-\lambda_2 t^q)} < 1$, where $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 = \lambda_2 + k$ for $k > 0$.*

Now, we can show the convergence of the above solution using the two lemmas above, i.e Lemma 3.1 and Lemma 3.2. We can split the solution of (8) as $u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ respectively as follows:
(a) $u_1(x,t)$ is the solution of (8), when $Q(x,t) = 0, A(t) = 0 = B(t)$,
(b) $u_2(x,t)$ is the solution of (8), when $A(t) = 0 = B(t), f^0(x) = 0$,
(c) $u_3(x,t)$ is the solution of (8), when $Q(x,t) = 0, f^0(x) = 0$.

Theorem 3.1 *$u_1(x,t)$, $u_2(x,t)$ and $u_3(x,t)$ converge when $|f^0(x)| < N_1$, $N_1 > 0$, $|Q(x,t)| < N_2$, $N_2 > 0$, $|A(t)| < M_1$ and $|B(t)| < M_2$, $M_1, M_2 > 0$ respectively.*

Proof of the above theorem follows as an application of Lemma 3.1 and Lemma 3.2. The details of the proof can be found in [2]. Next we will consider the non-linear Riemann-Louiville fractional reaction diffusion equation of the type:

$$\begin{aligned} \partial_t^q u - k\frac{\partial^2 u}{\partial x^2} &= f(x,t,u) + g(x,t,u), \ (x,t) \in Q_T, \\ & \Gamma(q)(t)^{1-q} u(x,t)|_{t=0} = f^0(x), \ x \in \overline{\Omega}, \\ u(0,t) &= A(t), u(L,t) = B(t) \ on \ \Gamma_T, \\ & J = (0,T], \ Q_T = J \times \Omega, \ k > 0 \ and \ \Gamma_T = (0 \times T) \times \partial\Omega, \\ & f, g \in \ C^{2,q}[[0,L] \times J \times \mathbb{R}, \mathbb{R}]. \end{aligned} \tag{17}$$

In this work, we seek the classical solution such that $u(x,t) \in C_p^{2,q}$ *on* Q_T, *and* $u(x,t) \in C_p$ *on* $\overline{Q}_T$. In order to develop the generalized monotone method for (17), we need the following definitions.

Definition 3.1 $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$. Then
(a) $v(x,t)$ and $w(x,t)$ are called the natural lower and upper solutions of (17) if the following inequalities are satisfied:

$$\begin{aligned} \partial_t^q v(x,t) - k\frac{\partial^2 v(x,t)}{\partial x^2} \leq f(x,t,v(x,t)) + g(x,t,v(x,t)) \ on \ Q_T, \\ \Gamma(q)(t-t_0)^{1-q} v(x,t)|_{t=0} \leq f^0(x), \ x \in \ \overline{\Omega}, \\ v(x,0) \leq A(t), v(L,t) \leq B(t) \ in \ \Gamma_T, \end{aligned} \tag{18}$$

$$\begin{aligned} \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} \geq f(x,t,w(x,t)) + g(x,t,w(x,t)) \ on \ Q_T, \\ \Gamma(q)(t-t_0)^{1-q} w(x,t)|_{t=0} \geq f^0(x), \ x \in \ \overline{\Omega}, \\ w(x,0) \geq A(t), w(L,t) \geq B(t) \ in \ \Gamma_T. \end{aligned} \tag{19}$$

(b) $v(x,t)$ and $w(x,t)$ are called coupled lower and upper solutions of type I if the following inequalities are satisfied:

$$\begin{aligned} \partial_t^q v(x,t) - k\frac{\partial^2 v(x,t)}{\partial x^2} &\le f(x,t,v(x,t)) + g(x,t,w(x,t)) \ on\ Q_T, \\ \Gamma(q)(t-t_0)^{1-q}v(x,t)|_{t=0} &\le f^0(x),\ x \in \overline{\Omega}, \\ v(x,0) &\le A(t), v(L,t) \le B(t)\ in\ \Gamma_T, \end{aligned} \tag{20}$$

$$\begin{aligned} \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} &\ge f(x,t,w(x,t)) + g(x,t,v(x,t)) \ on\ Q_T, \\ \Gamma(q)(t-t_0)^{1-q}w(x,t)|_{t=0} &\ge f^0(x),\ x \in \overline{\Omega}, \\ w(x,0) &\ge A(t), w(L,t) \ge B(t)\ in\ \Gamma_T. \end{aligned} \tag{21}$$

The next result is a comparison result relative to lower and upper solutions of (17) of natural type. For that purpose, we write $F(x,t,u) = f(x,t,u) + g(x,t,u)$.

Theorem 3.2 *Assume that*
(i) $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T, \mathbb{R}]$ are natural lower and upper solutions of (17), respectively. Furthermore, $\Gamma(q)t^{1-q}v(x,t)|_{t=0} \le \Gamma(q)t^{1-q}w(x,t)|_{t=0}$, $v(0,t) \le w(0,t)$ and $v(L,t) \le w(L,t)$;
(ii) $F(x,t,u)$ satisfies the one sided Lipschitz condition of the form

$$F(x,t,u_1) - F(x,t,u_2) \le L(u_1 - u_2),$$

whenever $u_1 \ge u_2$ and $L > 0$. Then $v(x,t) \le w(x,t)$ on $J \times \Omega$.

Proof. Initially, we will prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m(x,t) = v(x,t) - w(x,t)$. We claim that $m(x,t) < 0$, $(x,t) \in [0,L] \times (0,T]$. Suppose that the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{1-q}m(x_1,t) < 0$ on $[0,t_1)$, $m(x_1,t_1) = 0$. It is easy to check $m_x(x_1,t_1) = 0$ and $m_{xx}(x_1,t_1) \le 0$.
Then, using Lemma 3.2, we get $\partial_t^q m(x_1,t_1) \ge 0$.
From the hypothesis, we also have

$$\begin{aligned} &\partial_t^q m(x_1,t_1) \\ &\quad = \partial_t^q v(x_1,t_1) - \partial_t^q w(x_1,t_1) \\ &\quad < k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + F(x_1,t_1,v(x_1,t_1)) - k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} - F(x_1,t_1,w(x_1,t_1)) \\ &\quad < F(x_1,t_1,v(x_1,t_1)) - F(x_1,t_1,w(x_1,t_1)) = 0, \end{aligned} \tag{22}$$

which is a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q}_T$.

In order to prove the theorem for the non strict inequalities, let

$$\overline{w}(x,t) = w(x,t) + \epsilon t^{q-1}E_{q,q}[2Lt^q],$$

$$\overline{v}(x,t) = v(x,t) - \epsilon t^{q-1}E_{q,q}[2Lt^q].$$

From this it follows

$$\overline{w}(0,t) > \overline{v}(0,t),$$

$$\overline{w}(L,t) > \overline{v}(L,t),$$

$$\Gamma(q)t^{1-q}\overline{w}(x,t)|_{t=0} > \Gamma(q)t^{1-q}w(x,t)|_{t=0} > \Gamma(q)t^{1-q}v(x,t)|_{t=0} > \Gamma(q)t^{1-q}\overline{v}(x,t)|_{t=0}.$$

Then,

$$\begin{aligned}
&\partial_t^q\overline{w}(x,t) - k\frac{\partial^2\overline{w}(x,t)}{\partial x^2} \\
&= \partial_t^q w(x,t) - k\frac{\partial^2 w(x,t)}{\partial x^2} + \partial_t^q \epsilon t^{q-1}E_{q,q}[2Lt^q] \\
&\geq F(x,t,w(x,t)) + \epsilon t^{q-1}E_{q,q}2LEq,q[2Lt^q] \\
&= F(x,t,w(x,t)) + 2\epsilon Lt^{q-1}E_{q,q}[2Lt^q] - F(x,t,\overline{w}(x,t)) + F(x,t,\overline{w}(x,t)) \\
&\geq -L(\overline{w}-w) + F(x,t,\overline{w}(x,t)) + \epsilon 2LE_{q,q}(2Lt^q) \\
&= -L\epsilon t^{q-1}E_{q,q}[2Lt^q] + F(x,t,\overline{w}(x,t)) + 2L\epsilon E_{q,q}(2Lt^q) \\
&= F(x,t,\overline{w}(x,t)) + \epsilon Lt^{q-1}E_{q,q}[2Lt^q] \\
&> F(x,t,\overline{w}(x,t)) \ on\ \overline{Q}_T.
\end{aligned} \tag{23}$$

Similarly,

$$\partial_t^q\overline{v}(x,t) - k\frac{\partial^2\overline{v}(x,t)}{\partial x^2} > F(x,t,\overline{v}(x,t)) \ on\ \overline{Q}_T. \tag{24}$$

By the strict inequality result, $\overline{v} < \overline{w}$ on $\overline{Q}_T$. Letting $\epsilon \to 0$, we have $v \leq w$ on $\overline{Q}_T$.

The next result is related to coupled lower and upper solutions of type I related to (17).

Theorem 3.3 *Assume that*
(i) $v(x,t)$, $w(x,t) \in C^{2,q}[Q_T,\mathbb{R}]$ are coupled lower and upper solutions of type I of (17), respectively.
(ii) Assume $F(x,t,u) = f(x,t,u) + g(x,t,u)$, where f is a nondecreasing function and g is a nonincreasing function respectively for $(x,t) \in \overline{Q}_T$ in u.
(iii) Let $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition of the form

$$f(x,t,u_1) - f(x,t,u_2) \leq L(u_1 - u_2),$$

$$g(x,t,u_1) - g(x,t,u_2) \geq -M(u_1 - u_2),$$

whenever $u_1 \geq u_2$ and $L, M > 0$. Then $v(x,t) \leq w(x,t)$ on $J \times \Omega$.

Proof. Initially, we will prove the theorem when one of the inequalities in (i) is strict. For that purpose, let $m(x,t) = v(x,t) - w(x,t)$. It is easy to see that $m(x,0) < 0$ on $[0,L]$. Also, $m(0,t) < 0$ and $m(L,t) < 0$, $t \in (0,T]$. Suppose the conclusion is not true, then there exists a $t_1 \in J$ and $x_1 \in \Omega$ such that $t^{1-q}m(x,t) < 0$ on $(0,t_1]$, $m(x_1,t_1) = 0$. This implies $v(x_1,t_1) = w(x_1,t_1)$ and $\frac{\partial^2 m(x_1,t_1)}{\partial x^2} \leq 0$, where $t_1 > 0$ and $x_1 \in (0,L)$. Using Lemma 3.2, $\partial_t^q m(x_1,t_1) \geq 0$.

From the hypothesis, we also have

$$\begin{aligned}\partial_t^q m(x_1,t_1)& \\ &= \partial_t^q v(x_1,t_1) - \partial_t^q w(x_1,t_1) \\ &< k\frac{\partial^2 v(x_1,t_1)}{\partial x^2} + f(x_1,t_1,v(x_1,t_1)) + g(x_1,t_1,w(x_1,t_1)) \\ &- k\frac{\partial^2 w(x_1,t_1)}{\partial x^2} - f(x_1,t_1,w(x_1,t_1)) - g(x_1,t_1,v(x_1,t_1)) \\ &\le 0,\end{aligned} \tag{25}$$

which leads to a contradiction. Therefore, $v(x,t) < w(x,t)$ on $\overline{Q}_T$.
In order to prove the theorem for the non strict inequalities, let

$$\overline{w}(x,t) = w(x,t) + \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q],$$

$$\overline{v}(x,t) = v(x,t) - \epsilon(t-t_0)^{q-1}E_{q,q}[2(L+M)(t-t_0)^q].$$

One can show $\overline{v}(x,t)$ *and* $\overline{w}(x,t)$ satisfy the hypothesis with strict inequalities. Using the strict inequality result, $\overline{v} < \overline{w}$ on $\overline{Q}_T$. Letting $\epsilon \to 0$, we have $v \le w$ on $\overline{Q}_T$.
The next result is the maximum principle for the Riemann-Liouville parabolic equation in one dimensional space which will be useful in proving the uniqueness of the solution.

Corollary 3.1 *Let*

$$\partial_t^q m(x,t) - k\frac{\partial^2 m(x,t)}{\partial x^2} \le 0 \text{ on } Q_T,$$

$$m(0,t) \le 0, m(L,t) \le 0 \text{ on } \Gamma_T,$$

$$\Gamma(q)t^{1-q}m(x,t)|_{t=0} \le 0 \text{ on } \overline{\Omega}.$$

Then $m(x,t) \le 0$ *on* Q_T.

Proof. Suppose $m(x,t)$ has a positive maximum at (x_1,t_1). Let $m(x_1,t_1) = K$. Let $\overline{m}(x,t) = m(x,t) - K$. Then, $t^{1-q}\overline{m}(x,t) \le 0$ *on* $(0,t_1]$ and $\overline{m}(x_1,t_1) = 0$. Using Lemma 2.1, we get $\partial_t^q\overline{m}(x_1,t_1) \ge 0$. Also, $\overline{m}_{xx}(x_1,t_1) \le 0$. Combining these two, we get $\partial_t^q\overline{m}(x_1,t_1) - K\overline{m}_{xx}(x_1,t_1) \ge 0$.
We can also observe

$$\partial_t^q\overline{m}(x,t) - K\overline{m}_{xx} = \partial_t^q m - Km_{xx} - K\frac{t^{q-1}}{\Gamma q} < \partial_t^q m - Km_{xx} < 0, \tag{26}$$

which gives a contradiction. Hence, $m(x,t) \le 0$.

We can also prove this corolary by other method. We can show it is true first for the strict inequality and then for the instrict inequality by using the strict inequality. The solution of the linear problem is unique which follows from this maximum principle. This maximum principle is used to show the uniqueness of iterates and the monotonicity of the iterates. In next section, we will develop a generalized monotone method for the nonlinear Riemann-Liuoville fractional reaction diffusion equation (17) using coupled lower and upper solutions of type I. The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (17). Further using the uniqueness condition, we prove the uniqueness of the solution of (17). The next result is a generalized monotone method for (17).

4 Main Results

Theorem 4.1 *(i) Let (v_0, w_0) be the coupled lower and upper solutions of (17) such that $t^{1-q}v_0 \le t^{1-q}w_0$ on $\overline{Q}_T$.*
(ii) Suppose $f(x,t,u)$ is nondecreasing and $g(x,t,u)$ is nonincreasing in u on $\overline{Q}_T$, respectively. Then there exist monotone sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ such that $t^{1-q}v_n(x,t) \to t^{1-q}\rho(x,t)$ and $t^{1-q}w_n(x,t) \to t^{1-q}r(x,t)$ uniformly and monotonically on $\overline{Q}_T$, where $\rho(x,t)$ and $r(x,t)$ are coupled minimal and maximal solutions of (17) respectively. That is, $\rho(x,t)$ and $r(x,t)$ satisfy

$$\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2} = f(x,t,\rho) + g(x,t,r) \text{ on } Q_T,$$
$$\rho(0,t) = A(t), \rho(L,t) = B(t) \text{ on } \Gamma_T,$$
$$\Gamma(q)t^{1-q}\rho(x,t)|_{t=0} = f^0(x) \text{ on } \Omega$$

and

$$\partial_t^q r(x,t) - k\frac{\partial^2 r(x,t)}{\partial x^2} = f(x,t,r) + g(x,t,\rho) \text{ on } Q_T,$$
$$r(0,t) = A(t), r(L,t) = B(t) \text{ on } \Gamma_T,$$
$$\Gamma(q)t^{1-q}r(x,t)|_{t=0} = f^0(x) \text{ on } \Omega$$

such that $t^{1-q}v_0(x,t) < t^{1-q}\rho(x,t) < t^{1-q}u(x,t) < t^{1-q}r(x,t) < t^{1-q}w_0(x,t)$.

Proof. We construct the sequences $\{v_n(x,t)\}$ and $\{w_n(x,t)\}$ as follows:

$$\begin{aligned} \partial_t^q v_n(x,t) - k\frac{\partial^2 v_n(x,t)}{\partial x^2} &= f(x,t,v_{n-1}) + g(x,t,w_{n-1}) \text{ on } Q_T, \\ v_n(0,t) &= A(t), v_n(L,t) = B(t), \\ \Gamma(q)t^{1-q}v_n(x,t)|_{t=0} &= f^0(x) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \partial_t^q w_n(x,t) - k\frac{\partial^2 w_n(x,t)}{\partial x^2} &= f(x,t,w_{n-1}) + g(x,t,v_{n-1}) \text{ on } Q_T, \\ w_n(0,t) &= A(t), w_n(L,t) = B(t), \\ \Gamma(q)t^{1-q}w_n(x,t)|_{t=0} &= f^0(x). \end{aligned} \tag{28}$$

It is easy to observe that $v_1(x,t)$ and $w_1(x,t)$ exist and are unique by the representation form of linear equation and Corollary 3.1. By induction and the assumptions on f and g, we can prove that the solutions $v_n(x,t)$ and $w_n(x,t)$ exist and are unique by Corollary 3.1, for any n.

Let us prove first that $v_0(x,t) \le v_1(x,t)$ and that $w_1(x,t) \le w_0(x,t)$ on $\overline{Q}_T$. Let $p(x,t) = v_0(x,t) - v_1(x,t)$. Then

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$

$$= \partial_t^q v_0(x,t) - k\frac{\partial^2 v_0(x,t)}{\partial x^2} - \left(\partial_t^q v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2}\right)$$

$$\leq f(x,t,v_0(x,t)) + g(x,t,w_0(x,t)) - (f(x,t,v_0(x,t)) + g(x,t,w_0(x,t))) = 0,$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$ and $t^{1-q}v_0(x,t) \leq t^{1-q}v_1(x,t)$ on $\overline{Q}_T$.
Similarly, we can show that $w_1(x,t) \leq w_0(x,t)$ on $\overline{Q}_T$.
Then, we prove that $v_1(x,t) \leq w_1(x,t)$. Let $p(x,t) = v_1(x,t) - w_1(x,t)$. Then from our hypothesis, we get

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$

$$= \partial_t^q v_1(x,t) - k\frac{\partial^2 v_1(x,t)}{\partial x^2} - (\partial_t^q w_1(x,t) - k\frac{\partial^2 w_1(x,t)}{\partial x^2})$$

$$\leq f(x,t,v_0(x,t)) + g(x,t,w_0(x,t)) - (f(x,t,v_0(x,t)) + g(x,t,w_0(x,t))) = 0,$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$ and $t^{1-q}v_1(x,t) \leq t^{1-q}w_1(x,t)$ *on* $\overline{Q}_T$. Hence,

$$t^{1-q}v_0(x,t) \leq t^{1-q}v_1(x,t) \leq t^{1-q}w_1(x,t) \leq t^{1-q}w_0(x,t) \ on\ \overline{Q}_T.$$

By mathematical induction, we have

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_n(x,t) \leq t^{1-q}w_n(x,t) \leq ... \leq t^{1-q}w_0(x,t) \ on\ \ \overline{Q}_T\ for\ all\ n.$$

Furthermore, if $t^{1-q}v_0(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w_0(x,t)$ *on* $\overline{Q}_T$, then for any $u(x,t)$ of (17), we establish the following inequality by the method of induction.

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_n(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w_n(x,t) \leq ... \leq t^{1-q}w_0(x,t) \quad (29)$$

$$on\ \overline{Q}_T\ for\ all\ n.$$

It is certainly true for $n = 0$, by hypothesis. Assume the inequality (29) to be true for $n = k$, that is

$$t^{1-q}v_0(x,t) \leq ... \leq t^{1-q}v_k(x,t) \leq t^{1-q}u(x,t) \leq t^{1-q}w - k(x,t) \leq ... \leq t^{1-q}w_0(x,t) \quad (30)$$

$$on\ \overline{Q}_T\ for\ all\ n.$$

Let $p(x,t) = v_{k+1}(x,t) - u(x,t)$. Then from our hypothesis, we get

$$\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2}$$

$$= \partial_t^q v_{k+1}(x,t) - k\frac{\partial^2 v_{k+1}(x,t)}{\partial x^2} - (\partial_t^q u(x,t) - k\frac{\partial^2 u(x,t)}{\partial x^2})$$

$$\leq f(x,t,v_k) + g(x,t,w_k) - (f(x,t,u) + g(x,t,u)) \leq 0,$$

$p(0,t) = 0, p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,0)|_{t=0} = 0$ on Γ_T. Therefore, by Corollary 3.1, it follows that $p(x,t) \leq 0$ on $\overline{Q}_T$. Therefore, $t^{1-q}v_{k+1}(x,t) \leq t^{1-q}u(x,t)$ on $\overline{Q}_T$. In a similar way, we can show that $t^{1-q}u(x,t) \leq t^{1-q}w_{k+1}(x,t)$ on $\overline{Q}_T$.

Hence we constructed the monotonic sequences. Using the integral representation of the linear problem and an appropriate computation process, we can show that the sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ are uniformly bounded and equicontinuous. Using the Ascoli-Arzela theorem, we obtain subsequences of $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ which converge uniformly and monotonically on $\overline{Q}_T$. Since the sequences

$\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ are monotone, the entire sequences $\{t^{1-q}v_n(x,t)\}$ and $\{t^{1-q}w_n(x,t)\}$ converge to $t^{1-q}\rho(x,t)$ and $t^{1-q}r(x,t)$, respectively. From this it follows that

$$t^{1-q}v_0(x,t) \le t^{1-q}v_1(x,t) \le ... \le t^{1-q}v_n(x,t) \le ... \le t^{1-q}\rho(x,t) \le t^{1-q}u(x,t) \le t^{1-q}r(x,t) \le ...t^{1-q}w_n(x,t) \le ... \le t^{1-q}w_0(x,t) \text{ on } \overline{Q}_T. \tag{31}$$

Consequently, $\rho(x,t)$ and $r(x,t)$ are coupled minimal and maximal solutions of (17) since

$$t^{1-q}v_0(x,t) \le t^{1-q}\rho(x,t) \le t^{1-q}u(x,t) \le t^{1-q}r(x,t) \le t^{1-q}w_0(x,t) \text{ on } \overline{Q}_T. \tag{32}$$

Since $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition, we prove the uniqueness of the solution of (17). The next result is precisely this.

Theorem 4.2 *Let all the assumptions of Theorem 4.1 hold. Further, let $f(x,t,u)$ and $g(x,t,u)$ satisfy the one sided Lipschitz condition of the form*

$$f(x,t,u_1) - f(x,t,u_2) \le L_1(u_1 - u_2),$$

$$g(x,t,u_1) - g(x,t,u_2) \ge -L_2(u_1 - u_2),$$

whenever $u_1 \ge u_2$ and $L_1, L_2 > 0$. Then the solution $u(x,t)$ of (17) exists and is unique.

Proof. We have already proved (ρ, r) are coupled minimal and maximal solutions of (17) on $\overline{Q}_T$. Hence it is enough to show that $r(x,t) \le \rho(x,t)$ on $\overline{Q}_T$.
It is known from Theorem 4.1 that $\rho(x,t) \le r(x,t)$ on $\overline{Q}_T$.
Let $p(x,t) = r(x,t) - \rho(x,t)$. By the hypothesis, we get

$$\begin{aligned} &\partial_t^q p(x,t) - k\frac{\partial^2 p(x,t)}{\partial x^2} \\ &= \partial_t^q r(x,t) - k\frac{\partial^2 r(x,t)}{\partial x^2} - \left(\partial_t^q \rho(x,t) - k\frac{\partial^2 \rho(x,t)}{\partial x^2}\right) \\ &\le f(x,t,r) + p(x,t,\rho) - (f(x,t,\rho) + g(x,t,r)) \\ &\le t^{1-q}L_1|r-\rho| + t^{1-q}L_2|r-\rho| \\ &\le (L_1 + L_2)|p|, \end{aligned}$$

$p(0,t) = 0,\ p(L,t) = 0$ on $\overline{\Omega}$ and $\Gamma(q)t^{1-q}p(x,t)|_{t=0} = 0$ on Γ_T. It follows from Corollary 3.1 that $p(x,t) \le 0$. This proves that $r(x,t) = \rho(x,t) = u(x,t)$ on $\overline{Q}_T$ and the proof is complete.

5 Conclusion

In this work, initially we have obtained an integral representation for the solution of the Riemann-Liouville reaction diffusion equation with $Q(x,t)$, $f^0(x)$, $A(t)$, $B(t)$ being the non-homogeneous term, the initial function and the boundary functions respectively. In addition, we assume that the boundary conditions and the initial function satisfy the compatibility condition. We also establish, when $Q(x,t)$, $f^0(x)$, $A(t)$ and $B(t)$ are bounded, the solution $u(x,t)$ converges, by using the convergence of the series involving the Mittag-Leffler function. In addition, when $Q(x,t) = Q(x)$ is independent of t and

$A(t) = B(t) = 0$, we have proved that the solution of the Riemann-Liouville fractional reaction diffusion equation converges to the steady state solution. We have proved the maximum principle and comparison theorem relative to the non-linear Riemann-Liouville fractional reaction diffusion equation of (17) on $\overline{Q}_T$. Using the comparison result as a tool, we have developed a generalized monotone method for the Riemann-Liouville fractional reaction diffusion equation of (17). The generalized monotone method yields monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (17). Under the uniqueness assumption, we have proved that the unique solution of $u(x,t)$ of (17) exists and is unique. In our future work, we plan to use our method relative to the physical application problem.

Acknowledgment

This research is partially supported by Louisiana Board of Regents Support Fund LEQSF(2016-17)-C-17.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 273–284

Generalized Synchronization Between Two Chaotic Fractional Non-Commensurate Order Systems with Different Dimensions

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Received: April 8, 2017; Revised: June 16, 2018

Abstract: This paper deals with the problem of generalized synchronization between two chaotic and hyperchaotic fractional non-commensurate order systems with different dimensions. By designing an active control technique, the sufficient conditions for achieving generalized synchronization are derived by using the Laplace transform technique and final value theorem. Numerical simulations are also given to illustrate and validate the generalized synchronization results derived in this paper.

Keywords: *chaos; generalized synchronization; fractional non-commensurate order; active control.*

Mathematics Subject Classification (2010): 34A34, 37B25, 35B35, 93C83, 37C25, 37N30.

1 Introduction

Chaos synchronization phenomena have received increasing attention in the study of dynamical systems, because they can be applied in vast areas of engineering and information science, in particular, in secure communication, control processing and cryptology [1–4]. Various methods in chaos synchronization have been proposed [5–7]. Most of the synchronization methods focus on integer order chaotic systems in both continuous and discrete time.

Recently, fractional calculus has attracted a lot of attention and has become an excellent instrument to describe the dynamics of complex systems. Based on the stability

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criterion of linear fractional systems, many fractional-order chaotic systems can be synchronized [8–13].

An interesting aspect is the generalized type of synchronization called $Q-S$ synchronization. It has been investigated for integer order chaotic dynamical systems as well [14–18]. However, to the best of our knowledge, there are few treatments in the literature of the general scheme for generalized $Q-S$ synchronization of fractional non commensurate order systems with different dimensions.

In view of this consideration, this paper investigates an active control technique [15] for generalized synchronization between two different dimensional chaotic fractional non-commensurate order systems, using two suitable real matrices. Based on the Laplace transform technique and final value theorem, the designed control makes the fractional non-commensurate-order chaotic system states asymptotically synchronized. Numerical examples are given to verify the capability of the method.

The rest of the paper is organized as follows. In the following section, we present some basic concepts of fractional calculus fundamentals. In Section 3, we motivate the problem and give the main results. In Section 4, two examples are used to verify the effectiveness of the proposed method. Finally, some concluding remarks are given in Section 5.

2 Fractional Calculus Fundamentals

The three definitions used for the general fractional derivative are the Grunwald–Letnikov (GL) definition, the Riemann– Liouville (RL) and the Caputo definition [19]. The Riemann–Liouville fractional integral of order $\alpha > 0$ is given by

$$J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \tag{1}$$

where Γ is the gamma function. The Riemann–Liouville fractional-order derivative ${}^{RL}_{a}d_t^{\alpha} f$ is defined by

$${}^{RL}_{a}d_t^{\alpha} f(t) = d^m J_a^{m-\alpha} f(t), \tag{2}$$

where $m = \lceil \alpha \rceil$ is the first integer greater then α.

The Caputo fractional-order derivative ${}_{a}d_t^{\alpha} f$ is defined by

$${}_{a}d_t^{\alpha} f(t) = J_a^{m-\alpha} d^m f(t), \ m = \lceil \alpha \rceil. \tag{3}$$

The Grűnwald–Letnikov fractional-order derivative ${}^{GL}_{a}d_t^{\alpha} f$ is given by

$${}^{GL}_{a}d_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{k=0}^{\frac{t-a}{h}} (-1)^k \left(\frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha-k+1))} \right) f(t-kh). \tag{4}$$

Recall that the Laplace transform of a function $f(t)$ is the function $F(s)$ defined as follows

$$F(s) = L\left\{f(t), s\right\} = \int_0^{+\infty} \exp(-st) f(t) dt, \tag{5}$$

$f(t)$ is called original which can be reconstituted from the inverse Laplace transform

$$f(t) = L^{-1}\{F(s), t\} = \int_{c-i\infty}^{c+i\infty} \exp(st)F(s)ds, \quad c = \Re(s) > 0. \tag{6}$$

Taking into account that the Laplace transform of the convolution is

$$L\{f(t) * g(t), s\} = F(s).G(s), \tag{7}$$

where $f(t)$ and $g(t)$ are two causal functions for $t < 0$, we see that $F(s)$ and $G(s)$ are their Laplace transforms.

Using the following property of the Laplace transform of conventional derivative

$$L\{f^m(t), s\} = s^m F(s) - \sum_{k=0}^{m-1} s^k f^{(m-k-1)}(0), \tag{8}$$

we obtain the Laplace transform of the Riemann-Liouville derivative

$$L\left\{{}_0^{RL}d_t^{\alpha} f(t), s\right\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^k \left[{}_0^{RL}d^{\alpha-k-1} f(t)\right]_{t=0} \tag{9}$$

with $m - 1 \leq \alpha < m$. This transform is well known. However its practical application is limited by the absence of the physical interpretation of the function at $t = 0$.

In view of the Laplace transform formula of the Riemann-Liouville integral, the Laplace transform of the Caputo fractional derivative is

$$L\{{}_0^c d^{\alpha} f(t), s\} = s^{\alpha}F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \tag{10}$$

with $m - 1 \leq \alpha < m$. Since the initial conditions for the fractional differential equations with the Caputo derivative are of the same form as for the integer-order derivatives, which have clear physical meaning, the Caputo derivative is used in this paper.

Theorem 2.1 *(Final value theorem) Let $F(s)$ be the Laplace transform of function $f(t)$. If the indicated limits exist, then*

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s). \tag{11}$$

Proof. See [20].

3 Problem of Synchronization and Analytical Results

Generally, we consider the following non-commensurate fractional order nonlinear system in the form

$$d_t^{\alpha} X = f(X). \tag{12}$$

We take (12) as the drive system. The controlled response system is given by

$$d_t^{\alpha} Y = g(Y) + U, \tag{13}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ is the vector of rational number between 0 and 1, d_t^α is the Caputo fractional derivative of order α, for $i = 1, 2, ..., m$, $X(t) \in \mathbb{R}^n$, $Y(t) \in \mathbb{R}^m$, $(m > n)$ are the state vectors of the drive system (12) and the response system (13), respectively, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^m$ are the non linear vector functions and $U \in \mathbb{R}^m$ is the control input vector.

Our goal is to design an appropriate active control U [15] such that the synchronization between the drive system (12) and the response system (13) is achieved for a given two suitable real matrices $Q = (q_{ij})$, $i = 1, 2, ..., d$, $j = 1, 2, ..., m$ and $S = (s_{kh})$, $k = 1, 2, ..., d$, $h = 1, 2, ..., n$. Particularly, Q and S are chosen such that $q_{ij} = s_{kh} = 0$, for all $i \neq j$ and $k \neq h$.

Hence, the error system is defined as

$$e(t) = QY(t) - SX(t), \tag{14}$$

which means that systems (12) and (13) are globally asymptotically synchronized, i.e.

$$\lim_{t \to +\infty} \|e(t)\| = \lim_{t \to +\infty} \|QY(t) - SX(t)\| = 0.$$

Most existing methods for synchronizing chaos with different dimensions are used only for reduced order or increased order. Motivated by the above idea, in this work, we discuss the two cases: $d = m$ and $d = n$.

3.1 Increased order

In this case assume that $d = m$. By submitting systems (12) and (13) into (14), the error system (14) can be expressed as

$$d_t^\alpha e(t) = Q d_t^\alpha Y(t) - S d_t^\alpha X(t), \tag{15}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$. Hence

$$\begin{aligned} d_t^\alpha e(t) &= Q\left[g(Y(t)) + U\right] - Sf(X(t)) \\ &= A_1 e(t) + K\left(Y(t), X(t)\right) + QU, \end{aligned} \tag{16}$$

where

$$K\left(Y(t), X(t)\right) = -A_1 e(t) + Qg(Y(t)) - Sf(X(t)), \tag{17}$$

and $A_1 \in \mathbb{R}^{m\times m}$ is the linear part of system (13).

We redefine the control function $U = (u_1, u_2, ...u_m)^T$ to eliminate all terms which cannot be shown in the form e such that

$$QU = -K\left(y(t), x(t)\right) + Be(t), \tag{18}$$

and $B \in \mathbb{R}^{m\times m}$ is a feedback gain matrix to be determined. We find the error system as

$$d_t^\alpha e(t) = (A_1 + B)e(t). \tag{19}$$

Applying the Laplace transform for the previous system, letting

$$F_i(s) = L(e_i(t)), \quad i = 1, 2, ..., m, \tag{20}$$

and using the formula

$$L\{d_t^{\alpha_i} e_i(t)\} = s^{\alpha_i} F_i(s) - s^{\alpha_i - 1} e_i(0), \quad i = 1, 2, ..., m, \tag{21}$$

we find a new system

$$s^{\alpha} F(s) = s^{\alpha-1} e(0) + (A_1 + B) F(s), \tag{22}$$

where

$$F = (F_1, F_2, ..., F_m)^T, \quad s^{\alpha} = (s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_m}).$$

Hence, we have the following result.

Theorem 3.1 *If the matrix B is chosen such that all poles of $sF_i(s)$ lie in the open left half plane, then the drive system (12) and response system (13) are globally generally synchronized.*

Proof. Suppose that the matrix B is chosen such that all poles of $sF_i(s)$ lie in the open left half plane. Using Theorem 2.1, we have

$$\lim_{t \to +\infty} e_i(t) = \lim_{s \to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, ..., m.$$

This means that the drive system (12)and the response system (13) achieve the synchronization. □

3.2 Reduced order

In this case assume that $d = n$. Using the notation (14), the error system can be derived as

$$d_t^{\alpha} e(t) = A_2 e(t) + H(Y(t), X(t)) + QU, \tag{23}$$

where

$$H(Y(t), X(t)) = -A_2 e(t) + Qg(Y(t)) - Sf(X(t)), \tag{24}$$

and $A_2 \in \mathbb{R}^{n \times n}$ is the linear part of system (12).

We redefine the control function $U = (u_1, u_2, ...u_n, 0, 0..., 0)^T$ to eliminate all terms which cannot be shown in the form $e = (e_1, e_2, ...e_n)^T$ such that

$$Q^0 U^0 = -H(Y(t), X(t)) + Ce(t), \tag{25}$$

where $U^0 = (u_1, u_2, ...u_n)^T$, $C \in \mathbb{R}^{n \times n}$ is a feedback gain matrix to be determined and $Q^0 = diag(Q_{11}, Q_{22}, ..., Q_{nn})$. Then the error system is changed to

$$d_t^{\alpha} e(t) = (A_2 + C) e(t). \tag{26}$$

Applying the Laplace transform for the previous system, letting

$$F_i(s) = L(e_i(t)), \quad i = 1, 2, ..., n, \tag{27}$$

and using the formula

$$L\{d_t^{\alpha_i} e_i(t)\} = s^{\alpha_i} F_i(s) - s^{\alpha_i - 1} e_i(0), \quad i = 1, 2, ..., n, \tag{28}$$

we find a new system

$$s^{\alpha} F(s) = s^{\alpha-1} e(0) + (A_1 + C) F(s), \tag{29}$$

where

$$F = (F_1, F_2, ..., F_n)^T, \quad s^{\alpha} = (s^{\alpha_1}, s^{\alpha_2}, ..., s^{\alpha_n}).$$

Hence, we have the following result.

Theorem 3.2 *If the matrix C is chosen such that all poles of $sF_i(s)$ lie in the open left half plane, then the drive system (12) and response system (13) are globally generally synchronized.*

Proof. The proof is similar to that of Theorem 3.1.

4 Numerical Examples

In this section, we present some simulation examples to illustrate our proposed general method.

4.1 Simulation results (Increased order)

In this case , we assume that the new memristor-based simplest chaotic circuit system of non-commensurate fractional order (MBSCCS) [21] is the drive system. The dynamic of the circuit is described by the mathematical model

$$\begin{cases} d_t^{\alpha_1} x_1(t) = a_1 x_2, \\ d_t^{\alpha_2} x_2(t) = -b_1(x_1 + M(x_3)x_2), \\ d_t^{\alpha_3} x_3(t) = -x_2 - c_1 x_3 + x_2^2 x_3. \end{cases} \tag{30}$$

In (30), x_1, x_2, x_3 are the states, $a_1, b_1, c_1, \beta, \gamma$ are the positive parameters, M is the memristor function defined by

$$M(x_3(t)) = \gamma x_3^2(t) - \beta, \tag{31}$$

and $\alpha_i, i = 1, 2, 3$ are rational numbers between 0 and 1.

For all numerical simulation, we take the initial states of system (30) as

$$x_1(0) = 0.1, x_2(0) = -0.5, x_3(0) = 1. \tag{32}$$

The parameters values are taken as

$$(a_1, b_1, c_1, \beta, \gamma) = (1, \frac{1}{3}, 0.9, 3, 0.4). \tag{33}$$

The proposed fractional orders are taken as

$$(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.98, 0.99). \tag{34}$$

The system (30) exhibits chaotic behaviour as shown in Figure 1.

The linear part A_2 of system (30) is given by

$$A_2 = \begin{pmatrix} 0 & a_1 & 0 \\ -b_1 & b_1\beta & 0 \\ 0 & -1 & -c_1 \end{pmatrix}.$$

Assume that the fractional-order hyperchaotic Lorenz system [22] is the response system. The controlled hyperchaotic Lorenz system is expressed by the mathematical model

$$\begin{cases} d_t^{\alpha_1} y_1(t) = a_2(y_2 - y_1) + y_4 + u_1, \\ d_t^{\alpha_2} y_2(t) = c_2 y_1 - y_2 - y_1 y_3 + u_2, \\ d_t^{\alpha_3} y_3(t) = y_1 y_2 - b_2 y_3 + u_3, \\ d_t^{\alpha_4} y_4(t) = -y_2 y_3 + r y_4 + u_4. \end{cases} \tag{35}$$

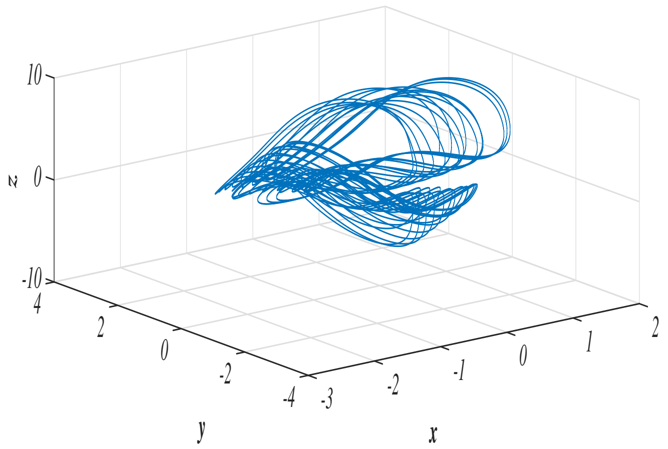

Figure 1: Chaotic Attractor of the Fractional Order MBSCCS System with $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 098, 0.99)$.

In (35), y_1, y_2, y_3, y_3 are the states, $a_2, c_2,\ b_2, r$ are the positive parameters and $\alpha_i, i = 1, 2, 3, 4$ are rational numbers between 0 and 1.

For all numerical simulation, we take the initial states of system (35) as

$$y_1(0) = 1, y_2(0) = 1, y_3(0) = 0, y_4(0) = -1. \tag{36}$$

The parameters values are taken as

$$(a_2, c_2,\ b_2, r\ ) = (10, 28, \frac{8}{3}, 1.3). \tag{37}$$

The proposed fractional orders are taken as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.97, 0.98, 0.99, 0.999). \tag{38}$$

The system (35) (with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits chaotic behaviour as shown in Figure 2.

The linear part A_1 of system (35) is given by

$$A_1 = \begin{pmatrix} -a_2 & a_2 & 0 & 1 \\ c_2 & -1 & 0 & 0 \\ 0 & 0 & -b_2 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}.$$

Here, we choose

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0. \end{pmatrix}.$$

Let us define the error variables between the slave system (35) to be controlled and the master system (30) as

$$e(t) = QY(t) - SX(t),$$

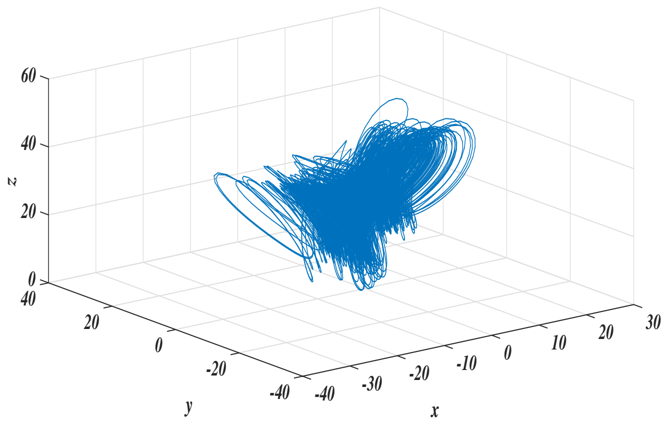

Figure 2: Chaotic Attractor of the Fractional Order Lorenz System with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.97, 0.98, 0.99, 0.999)$.

i.e.

$$\begin{cases} e_1 = -2x_1 + y_1, \\ e_2 = -2\left(x_2 - y_2\right) - x_3, \\ e_3 = -3x_3 + y_3, \\ e_4 = 2y_4. \end{cases} \tag{39}$$

For simplicity, choose the suitable feedback gain matrix B such that

$$A_1 + B = \begin{pmatrix} -a_2 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \\ 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & -r \end{pmatrix}. \tag{40}$$

Hence

$$\begin{cases} u_1 = -e_1 a_2 + 2a_1 x_2 - y_4 + a_2 y_1 - a_2 y_2, \\ u_2 = -\frac{1}{2} b_2 e_1 - M b_1 x_2 x_3 + \frac{1}{2} x_2^2 x_3 - \frac{1}{2} x_2 - \frac{1}{2} c_1 x_3 - b_1 x_1 + y_2 - c_2 y_1 + y_1 y_3, \\ u_3 = -c_2 e_1 + 3x_2^2 x_3 - 3x_2 - 3c_1 x_3 + b_2 y_3 - y_1 y_2, \\ u_4 = -\frac{1}{2} r e_1 - r y_4 + y_2 y_3. \end{cases} \tag{41}$$

The error system can be rewritten as

$$d_t^{\alpha_i} e_i(t) = (A_1 + B) e_i, \text{ for all } i = 1, 2, 3, 4. \tag{42}$$

To prove that the error system converges to 0, we apply the formulas (20) and (21), we obtain

$$\begin{cases} s^{\alpha_1} F_1(s) = s^{\alpha_1 - 1} e_1(0) - a_2 F_1(s), \\ s^{\alpha_2} F_2(s) = s^{\alpha_2 - 1} e_2(0) - b_2 F_2(s), \\ s^{\alpha_3} F_3(s) = s^{\alpha_3 - 1} e_3(0) - c_2 F_3(s), \\ s^{\alpha_4} F_4(s) = s^{\alpha_4 - 1} e_4(0) - r F_4(s). \end{cases} \tag{43}$$

It follows from the equations of the system(43) that

$$\begin{cases} F_1(s) = \dfrac{s^{\alpha_1 - 1} e_1(0)}{s^{\alpha_1} + a_2}, \\ F_2(s) = \dfrac{s^{\alpha_2 - 1} e_2(0)}{s^{\alpha_2} + b_2}, \\ F_3(s) = \dfrac{s^{\alpha_3 - 1} e_3(0)}{s^{\alpha_3} + c_2}, \\ F_4(s) = \dfrac{s^{\alpha_3 - 1} e_4(0)}{s^{\alpha_4} + r}. \end{cases} \tag{44}$$

Since a_2, b_2, c_2, r are positive parameters, we can conclude that all poles of $sF_i(s)$, $i = 1, 2, 3, 4$ lie in the open left half plane. Thus, by using Theorem 3.1, we get

$$\lim_{t \to +\infty} e_i(t) = \lim_{s \to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, 3, 4. \tag{45}$$

This means that the drive system (30) and the response system (35) achieve the synchronization. The error functions evolution, in this case, is shown in Figure 3.

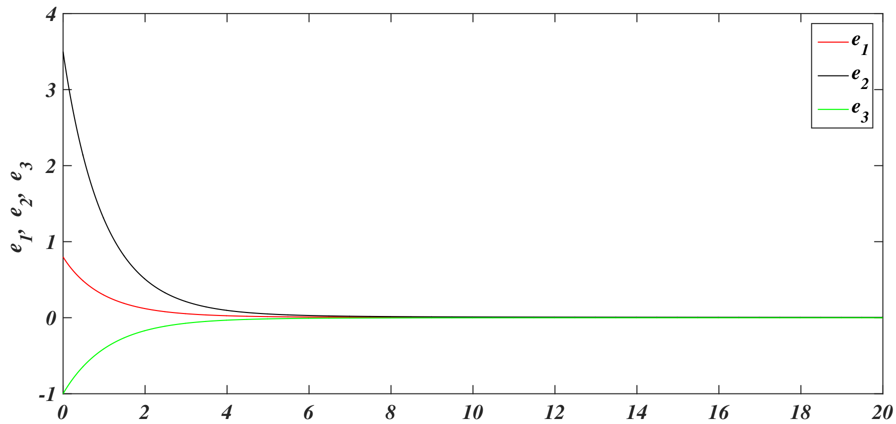

Figure 3: Error Functions Evolution of System (42).

4.2 Simulation results (Reduced order)

Let us take the same previous systems. Here, we choose

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{pmatrix}, \quad \left(Q^0\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1. \end{pmatrix}.$$

To investigate the generalized synchronization of the systems (30) and (35), we define the error states as

$$e(t) = QY(t) - SX(t), \tag{46}$$

i.e.

$$\left\{\begin{array}{l} e_1 = -2x_1 + y_1, \\ e_2 = -3x_2 + 2y_2, \\ e_3 = -x_3 + \frac{1}{2}y_3. \end{array}\right. \tag{47}$$

For simplicity, choose the suitable feedback gain matrix C such that

$$(A_2 + C) = \left(\begin{array}{ccc} -a_1 & 0 & 0 \\ 0 & -b_1\beta & 0 \\ 0 & 0 & -c_1. \end{array}\right). \tag{48}$$

Hence

$$\left\{\begin{array}{l} u_1 = -a_1e_1 + 2a_1x_2 - y_4 + a_2y_1 - a_2y_2, \\ u_2 = -\frac{1}{2}b_1\beta e_1 - \frac{3}{2}Mb_1x_2x_3 - \frac{3}{2}b_1x_1 + y_2 - c_2y_1 + y_1y_3, \\ u_3 = -2e_1 - 2x_2 - 2c_1x_3 + 2x_2^2x_3 + b_2y_3 - y_1y_2 - 2c_1, \\ u_4 = 0. \end{array}\right. \tag{49}$$

The error system can be rewritten as

$$d_t^{\alpha_i}e_i(t) = (A_2 + C)e_i, \text{ for all } i = 1, 2, 3. \tag{50}$$

To prove that the error system converges to 0, we apply the formulas (20) and (21), we

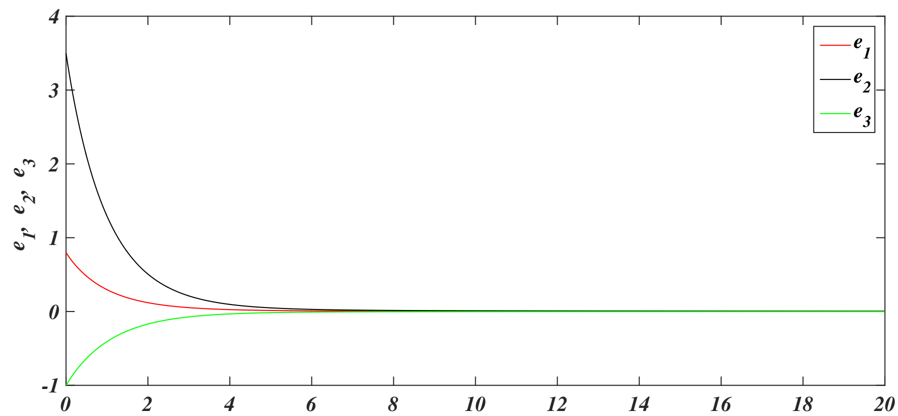

Figure 4: Error Functions Evolution of System (50).

obtain

$$\left\{\begin{array}{l} s^{\alpha_1}F_1(s) = s^{\alpha_1-1}e_1(0) - a_1F_1(s), \\ s^{\alpha_2}F_2(s) = s^{\alpha_2-1}e_2(0) - b_1\beta F_2(s), \\ s^{\alpha_3}F_3(s) = s^{\alpha_3-1}e_3(0) - c_1F_3(s)). \end{array}\right. \tag{51}$$

It follows from the equations of the system (51) that

$$\left\{\begin{array}{l} F_1(s) = \dfrac{s^{\alpha_1-1}e_1(0)}{s^{\alpha_1} + a_1}, \\ F_2(s) = \dfrac{s^{\alpha_2-1}e_2(0)}{s^{\alpha_2} + b_1\beta}, \\ F_3(s) = \dfrac{s^{\alpha_3-1}e_3(0)}{s^{\alpha_3} + c_1}. \end{array}\right. \tag{52}$$

Since a_1, b_1, c_1 are positive parameters, we can conclude that all poles of $sF_i(s),\ i = 1, 2, 3$ lie in the open left half plane. Thus, by using Theorem 3.2, we get

$$\lim_{t\to+\infty} e_i(t) = \lim_{s\to 0^+} sF_i(s) = 0, \quad \text{for all } i = 1, 2, 3, \tag{53}$$

which clearly demonstrates that the drive system (30) and the response system (35) achieve the generalized synchronization. The error functions evolution, in this case, is shown in Figure 4.

5 Conclusion

In this paper, we have investigated the generalized synchronization between two different dimensional chaotic fractional non-commensurate order systems. The analytical conditions for the synchronization between these chaotic systems are derived by using the Laplace transform technique and final value theorem. Numerical simulations of chaotic and hyperchaotic systems have been given to illustrate and validate the effectiveness of the proposed generalized synchronization.

Our future work is to develop some type of synchronization and we suggest some potential applications in secure communication.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 285–296

Feedback Control of Chaotic Systems by Using Jacobian Matrix Conditions

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Received: February 14, 2018; Revised: June 18, 2018

Abstract: In this work, we propose, for stabilizing chaotic systems at fixed points, new conditions based on the Jacobian matrix and its relation with the conditions of Routh-Hurwitz. We apply the results of feedback control method to the second type Rössler system, Liu system and Genesio system.

Keywords: *Routh-Hurwitz theorem; Jacobian matrix; feedback control; chaotic systems.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied over the past decades. After the pioneering work of Ott et al [1], and Pecora and Carroll [2], research efforts have been devoted to the chaos control problems in many physical systems [3–5]. The control problem attempts to stabilize a chaotic attractor to either a periodic orbit or an equilibrium point [20, 21]. Many potential applications have come true in securing communication, laser and biological systems, and other areas [6–9, 19]. Different control strategies for stabilizing chaos have been proposed, such as adaptive control, time delay control, and fuzzy control. Generally speaking, there are two main approaches for controlling chaos: feedback control and non-feedback control. The feedback control [10, 17, 18] approach offers many advantages such as robustness and computational complexity over the non-feedback control method. The aim of this paper is to apply the feedback control to chaotic systems, with new conditions for the stability at fixed points based on the Jacobian matrix. We present the numerical simulation studies for control of the Rössler, Liu and modified Genesio systems.

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2 Preliminaries

Suppose that A is an $n \times n$ matrix of real constants, its characteristic polynomial is

$$f(\lambda) = \lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + c\lambda^{n-3} + ..., n = 1, 2, 3, 4.$$

The Routh-Hurwitz theorem [10–13] is as follows.

Theorem 2.1 *All the roots of the caracteristic polynomial have negative real parts precisely when the given conditions are satisfied:*
$\lambda^2 + a\lambda + b : a > 0, b > 0.$
$\lambda^3 + a\lambda^2 + b\lambda + c : a > 0, c > 0, ab - c > 0.$
$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d : a > 0, ab - c > 0, (ab - c)c - a^2 d > 0, d > 0.$

3 Main Results

3.1 The case of third dimension

We assume A is the Jacobian matrix of the third dimension:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{1}$$

then the relation between the coefficients of characteristic polynomial and the Jacobian matrix is

$$\begin{cases} a = -trace(A), \\ b = A_{11} + A_{22} + A_{33}, \\ c = -\det(A), \end{cases} \tag{2}$$

where $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ and $A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{vmatrix}.$
Then $ab-c = -a_{11}(A_{22}+A_{33}) - a_{22}(A_{11}+A_{33}) - a_{33}(A_{22}+A_{11}) - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$.

Remark 3.1 We note that, if $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$ so that $t = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \geq 0$, then the coefficients of the characteristic polynomial are positive. On the other hand, we have $t = 0$ for the Rössler, Liu and other systems. So, we can ensure the stability of any chaotic systems with the following theorem.

We consider A is the Jacobian matrix at a fixed point, and $t = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$.

Theorem 3.1 *If $t \geq 0$, all the roots of the characteristic polynomial of A have negative real parts when the given conditions are satisfied:*
$\det(A) < 0$, $a_{ii} < 0$ *and* $A_{ii} > 0$ *for* $i = 1, 2, 3$.

Proof. We have

$$\begin{cases} a = -trace(A) > 0, \\ b = A_{11} + A_{22} + A_{33} > 0, \\ ab - c > 0, \end{cases}$$

then, by the Routh-Hurwitz theorem all the roots of the characteristic polynomial have negative real parts.

Remark 3.2 We can use the condition $t \geq 0$ as an additional condition with the condition of Routh-Hurwitz to get quickly the convergence to the fixed point.

4 Application to Chaotic Systems

4.1 The second type Rössler system

The Rössler system [14] is given by the following equations:

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + \alpha y, \\ \dot{z} = \beta x + xz - \gamma z, \end{cases}$$

where $\alpha = 0.38, \beta = 0.3, \gamma = 4.5$. The two equilibrium points of system are given by $E_1 = (0,0,0)$, $E_2 = (\gamma - \alpha\beta, \beta - \frac{\gamma}{\alpha}, \frac{\gamma}{\alpha} - \beta)$.

4.1.1 Control at the equilibrium point E_1

If the controlled Rössler system is given by the equations

$$\begin{cases} \dot{x} = -(y+z) - u_1, \\ \dot{y} = x + 0.38y - u_2, \\ \dot{z} = 0.3x + (x - 4.5)z - u_3, \end{cases} \tag{3}$$

where $u_1 = kx, u_2 = ky, u_3 = kz$, and k is the feedback coefficient; when $k > 0.38$, the system (3) will gradually converge to the equilibrium point $(0,0,0)$.

Proof. The Jacobian matrix of system (3) with regard to the equilibrium point $(0,0,0)$ is

$$A = \begin{pmatrix} -k & -1 & -1 \\ 1 & 0.38-k & 0 \\ 0.3 & 0 & -4.5-k \end{pmatrix},$$

where $a_{11} = -k, a_{22} = 0.38 - k, a_{33} = -4.5 - k$, $A_{11} = k^2 + 4.12k - 1.71$, $A_{22} = k^2 + 4.5k + 0.3$, $A_{33} = k^2 - 0.38k + 1$ and $\det(A) = -1k^3 - 4.12k^2 + 0.41k - 4.386$.We have $t = 0$, therefore

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0 \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > 0.38, \\ k > -4.5 \end{cases}$$

and

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -4.5[\cup]0.38, \infty[, \\ k \in]-6.7685 \times 10^{-2}, \infty[\cup]-\infty, -4.4323[, \\ k \in]-\infty, \infty[, \\ k \in]-4.4354, \infty[. \end{cases}$$

Obviously, if $k > 0.38$, then $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0$ and $A_{11} > 0, A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (3)will gradually converge to the unstable equilibrium point $(0,0,0)$, thus the proof is completed (see Figure 1).

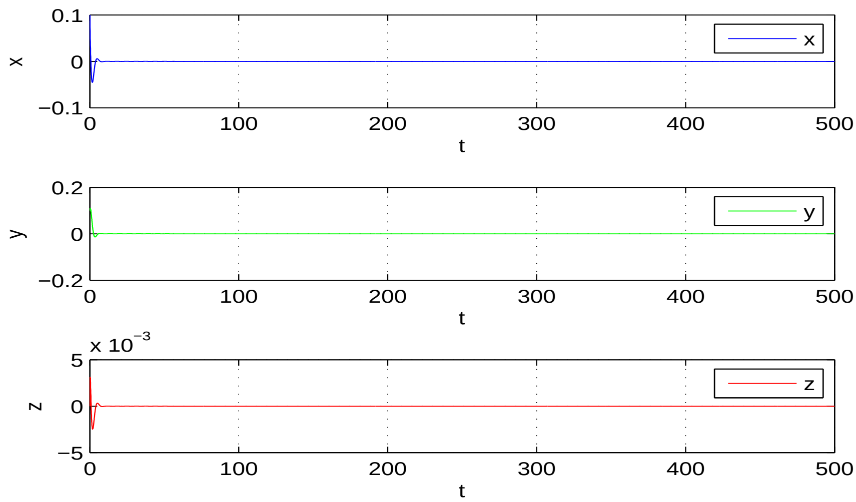

Figure 1: Control of the Rössler system at the equilibrium point E_1.

Remark 4.1 By using the Routh-Hurwitz theorem, we found $k > 0.776\,61$.

Similarly, the system can also be controlled at $E_2(4.386, -11.542, 11.542)$ by the similar control method. The controlled Rössler system is

$$\begin{cases} \dot{x} = -(y+z) - u_1, \\ \dot{y} = x + 0.38y - u_2, \\ \dot{z} = 0.3x + (x-4.5)z - u_3, \end{cases} \tag{4}$$

where $u_1 = k(x - 4.386),, u_2 = k(y + 11.542), u_3 = k(z - 11.542)$.

For demonstrating this conclusion, we do the following transformations: $x_1 = x - \beta$, $y_1 = y + \alpha$,$z_1 = z - \alpha$. When $\alpha = 11.542$, $\beta = 4.386$, then the system (4) has the following form:

$$\begin{cases} \dot{x}_1 = -(y_1+z_1) - kx_1, \\ \dot{y}_1 = x_1 + 0.38y_1 - ky_1, \\ \dot{z}_1 = 11.842x_1 + (x_1 - 0.114)z_1 - kz_1. \end{cases} \tag{5}$$

The Jacobian matrix of the system (5) is

$$A = \begin{pmatrix} -k & -1 & -1 \\ 1 & 0.38-k & 0 \\ 11.842 & 0 & -0.114-k \end{pmatrix},$$

where k is the feedback coefficient; when $k > 0.38$, we found that the system (5) will converge to the equilibrium point $E_2^{'}(0,0,0)$, that is system (4) will gradually converge to the equilibrium point $E_2(4.386, -11.542, 11.542)$.

Proof. We have $\det(A) = -1.0k^3 + 0.266\,k^2 - 12.799k + 4.386\,0$,

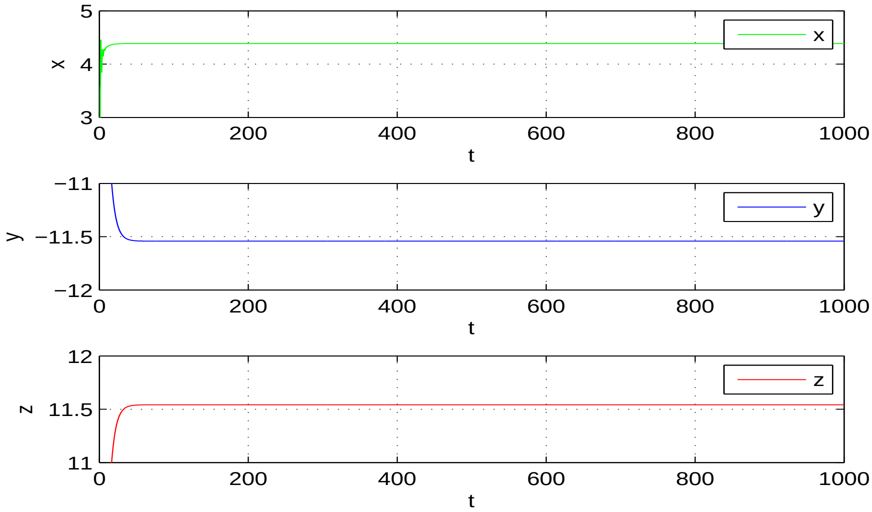

Figure 2: Control of the Rössler system at the equilibrium point E_2.

$a_{11} = -k, a_{22} = 0.38 - k, a_{33} = -0.114 - k,\ A_{11} = k^2 - 0.266\,k - 0.043\,32, A_{22} = k^2 + 0.114\,k + 11.842, A_{33} = 11.842k - 4.500\,0$, and $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > 0.38, \\ k > -0.114, \end{cases}$$

and

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in \left]-\infty, -0.114\right[\cup \left]0.38, \infty\right[, \\ k \in \mathbb{R}, \\ k \in \left]0.38, \infty\right[, \\ k \in \left]0.341\,99, \infty\right[. \end{cases}$$

.

When $k > 0.38$, we have $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0,\ A_{22} > 0$, and $A_{33} > 0$. According to Theorem 3.1, the system (5) will gradually converge to the unstable equilibrium point E_2. Hence the proof is completed (see Figure 2).

4.2 Control of the Liu system

The Liu system [15] is given by

$$\begin{cases} \dot{x} = \alpha(y - x), \\ \dot{y} = x(\lambda - \gamma z), \\ \dot{z} = \delta x^2 - \beta z, \end{cases}$$

where $\alpha = 10, \lambda = 40, \gamma = 1, \delta = 4, \beta = 2.5$. The fixed points are $E_1 : (0, 0, 0)$, $E_{2,3} : (\pm\sqrt{\frac{\beta\delta}{\gamma\lambda}}, \pm\sqrt{\frac{\beta\delta}{\gamma\lambda}}, \frac{\beta}{\gamma})$.

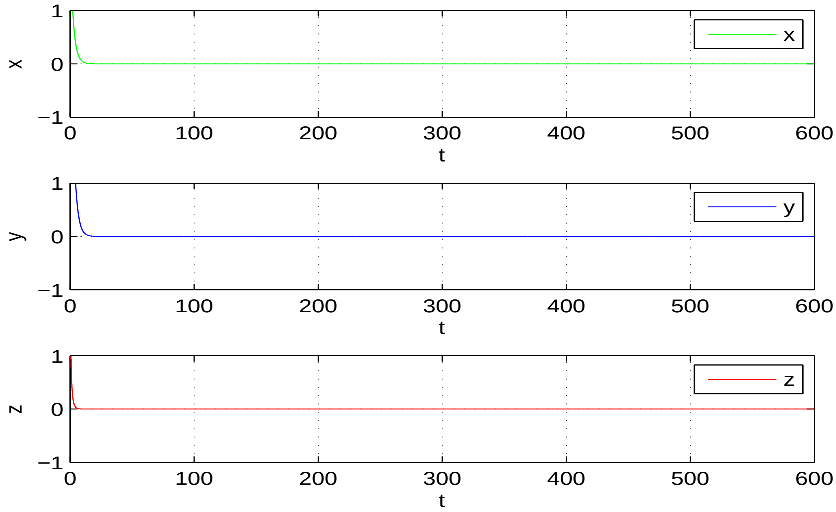

Figure 3: Control of the Liu system at the equilibrium point E_1.

4.2.1 Control at the equilibrium point E_1

The controlled Liu system is

$$\begin{cases} \dot{x} = \alpha(y - x) - u_1, \\ \dot{y} = x(\lambda - \gamma z) - u_2, \\ \dot{z} = \delta x^2 - \beta z - u_3, \end{cases} \tag{6}$$

where $u_1 = kx, u_2 = ky, u_3 = kz$
and k is the feedback coefficient; when we have $k > 15.616$, the system (6) will gradually converge to the equilibrium point $(0, 0, 0)$.

Proof. The Jacobian matrix of the system (6) with regard to the equilibrium point $(0, 0, 0)$ is

$$A = \begin{pmatrix} -10 - k & 10 & 0 \\ 40 & -k & 0 \\ 0 & 0 & -2.5 - k \end{pmatrix},$$

thus $\det(A) = -1k^3 - 12.5k^2 + 375k + 1000 < 0$,
$a_{11} = -10 - k, a_{22} = -k, a_{33} = -2.5 - k$,
$A_{11} = k^2 + 2.5k$,
$A_{22} = k^2 + 12.5k + 25$,
$A_{33} = k^2 + 10k - 400$
with $t = 0$. So,

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > -10, \\ k > 0, \\ k > -2.5. \end{cases} \quad \text{and}$$

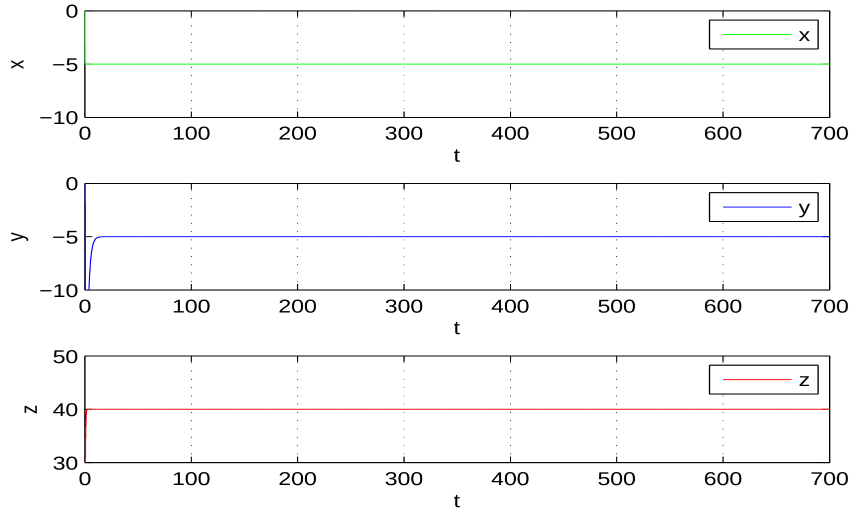

Figure 4: Control of the Liu system at the equilibrium point E_2.

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in \,]-\infty, -2.5[\cup]0, \infty[\,, \\ k \in \,]-\infty, -10.0[\cup]-2.5, \infty[\,, \\ k \in \,]-\infty, -25.616[\cup], 15.616, \infty[\,, \\ k \in \,]-10, -2.5[\cup]0, \infty[\,. \end{cases}$$

It can be easily seen when $k > 15.616$, so $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0$, and $A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (6) will gradually converge to the unstable equilibrium point $(0, 0, 0)$ (see Figure 3).

4.2.2 Control at the equilibrium point E_2

We consider the controlled Liu system given by

$$\begin{cases} \dot{x} = \alpha(y - x) - u_1, \\ \dot{y} = x(\lambda - \gamma z) - u_2, \\ \dot{z} = \delta x^2 - \beta z - u_3, \end{cases} \tag{7}$$

where $u_1 = k(x + 5) + 10(y + 5), u_2 = k(y + 5), u_3 = k(z - 40)$. Here k is the feedback coefficient; when $k > 0$, it can be demonstrated that system (7) will gradually converge to the equilibrium point $(-5, -5, 40)$.

Proof. The Jacobian matrix of the system(7) at $(-5, -5, 40)$ is

$$A = \begin{pmatrix} -10 - k & 0 & 0 \\ 0 & -k & 5 \\ -40 & 0 & -2.5 - k \end{pmatrix},$$

where $\det(A) = -1.0k^3 - 12.5k^2 - 25k$,

$a_{11} = -10 - k, a_{22} = -k, a_{33} = -2.5 - k,$
$A_{11} = k^2 + 2.5k,$
$A_{22} = k^2 + 12.5k + 25,$
$A_{33} = k^2 + 10k$
with $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > -10, \\ k > 0, \\ k > -2.5, \end{cases} \quad \text{and}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -2.5[\cup]0, \infty[, \\ k \in]-\infty, -10.0[\cup]-2.5, \infty[, \\ k \in]-\infty, -10[\cup]0, \infty[, \\ k \in]-10, -2.5[\cup]0, \infty[. \end{cases}$$

When $k > 0$, we have $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0$, and $A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (7) will gradually converge to the unstable equilibrium point $(-5, -5, 40)$ (see Figure 4).

Remark 4.2 Similarly, the system can also be controlled at $E_3(5, 5, 40)$ by the similar control method if $k > 0$.

4.3 The modified Genesio system

We have the modified Genesio system [16, 17] as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \alpha_1 y + z, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z, \end{cases}$$

where $\alpha_1 = -0.5, \alpha_2 = 3, \alpha_3 = -6, \alpha_4 = -2.85, \alpha_5 = -0.5$, and the fixed points are $E_1 = (0, 0, 0)$, $E_2 = (2, 0, 0)$.

4.3.1 Control at the equilibrium point E_1

The controlled modified Genesio system is given by

$$\begin{cases} \dot{x} = y - u_1, \\ \dot{y} = \alpha_1 y + z - u_2, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z - u_3, \end{cases} \tag{8}$$

where $u_1 = kx, u_2 = ky - z, u_3 = kz$. Here k is the feedback coefficient; when $k > 0$, we found that the system (8)will gradually converge to the equilibrium point $(0, 0, 0)$.

Proof. The Jacobian matrix of the system (8) with regard to the equilibrium point (0.0.0) is

$$A = \begin{pmatrix} -k & 1 & 0 \\ 0 & -0.5 - k & 0 \\ -6 & -2.85 & -0.5 - k \end{pmatrix},$$

where $\det(A) = -2k^3 - 1.5k^2 - 0.25k,$
$a_{11} = -k, a_{22} = -0.5 - k, a_{33} = -0.5 - k,$

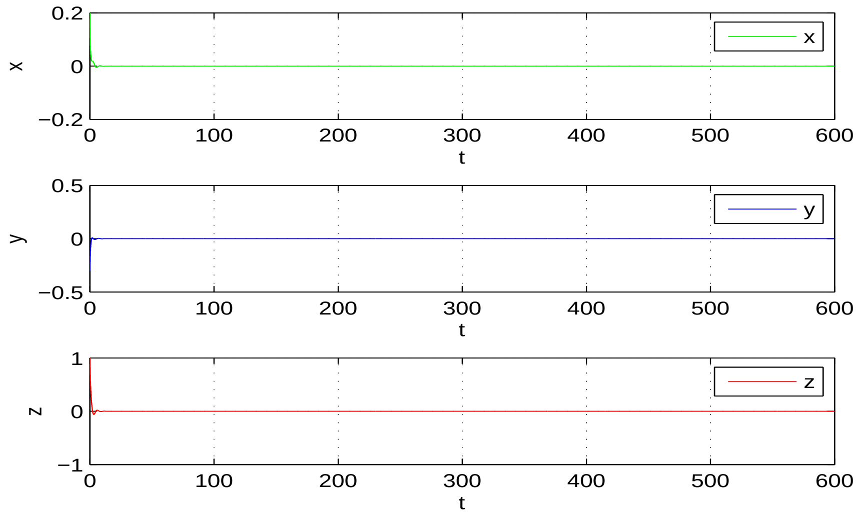

Figure 5: Control of the Modified Genesio System to the original equilibrium point.

$A_{11} = 2k^2 + 1.5k + 0.25$,
$A_{22} = k^2 + 0.5k$,
$A_{33} = k^2 + 0.5k$ and $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > -0.5, \quad \text{and} \\ k > -0.5, \end{cases}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in \,]-\infty, -0.5[\, \cup \,]-0.25, \infty[\,, \\ k \in \,]-\infty, -0.5[\, \cup \,]0, \infty[\,, \\ k \in \,]-\infty, -0.5[\, \cup \,]0, \infty[\,, \\ k \in \,]-0.5, -0.25[\, \cup \,]0, \infty[\,. \end{cases}$$

Obviously, when $k > 0$,then $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$. According to Theorem 3.1, the system (8) will gradually converge to the unstable equilibrium point $(0, 0, 0)$. Hence the proof is completed (see Figure 5).

4.3.2 Control at the equilibrium point $E_2 : (2, 0, 0)$

The controlled modified Genesio system is given by

$$\begin{cases} \dot{x} = y - u_1, \\ \dot{y} = \alpha_1 y + z - u_2, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z - u_3, \end{cases} \tag{9}$$

where $u_1 = k(x - 2), u_2 = ky - z, u_3 = kz$, and k is the feedback coefficient, if $k > 0$, the system (9) will gradually converge to the equilibrium point $(2, 0, 0)$.

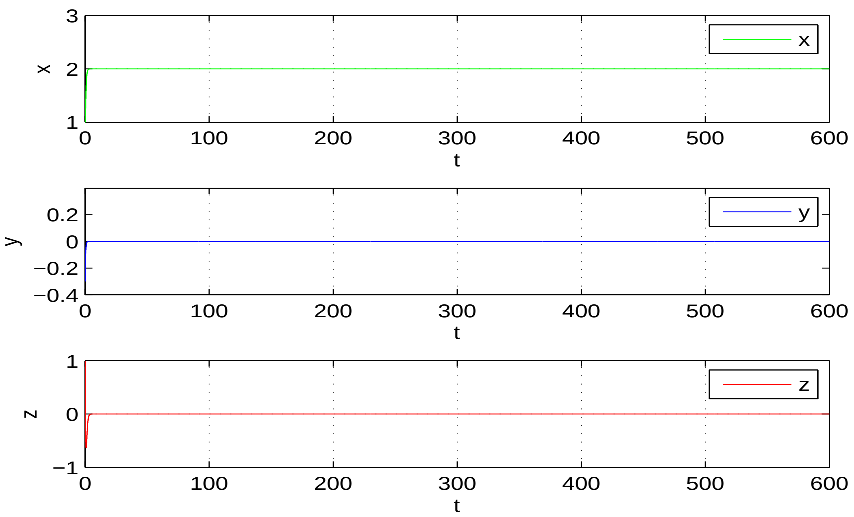

Figure 6: Control of the modified Genesio system at the equilibrium point E_2.

Proof. The Jacobian matrix of the system (9) at (2.0.0) is

$$A = \begin{pmatrix} -k & 1 & 0 \\ 0 & -0.5-k & 0 \\ 6 & -2.85 & -0.5-k \end{pmatrix},$$

where $\det(A) = -1k^3 - 1k^2 - 0.25k$,
$a_{11} = -k, a_{22} = -0.5 - k, a_{33} = -0.5 - k$,
$A_{11} = k^2 + k + 0.25$,
$A_{22} = k^2 + 0.5k$,
$A_{33} = k^2 + 0.5k$ with $t = 0$. So

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > -0.5, \\ k > -0.5, \end{cases} \text{and}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -0.5[\cup]-0.5, \infty[, \\ k \in]-\infty, -0.5[\cup]0, \infty[, \\ k \in]-\infty, -0.5[\cup]0, \infty[, \\ k \in]0, \infty[. \end{cases}$$

Obviously, when $k > 0$, we have, $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$. According to Theorem 3.1, the system (9) will gradually converge to the unstable equilibrium point $(2, 0, 0)$, thus the proof is completed (see Figure 6).

5 Conclusion

This work presents the feedback control at fixed points of the second type Rössler, Liu and modified Genesio chaotic systems. By using new conditions for the stability based on the Jacobian matrix, we simplified and modified the calculations for the Routh-Hurwitz coefficient.

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Solving Two-Dimensional Integral Equations of Fractional Order by Using Operational Matrix of Two-Dimensional Shifted Legendre Polynomials

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Received: November 4, 2017; Revised: June 16, 2018

Abstract: In this paper, we first present a new numerical method for solving two-dimensional integral equations of fractional order. The method is based upon two-dimensional shifted Legendre polynomials. Then we construct an operational matrix for two-dimensional fractional integral. Also, we give the error analysis. Finally, three examples are shown to confirm the theoretical results.

Keywords: *two-dimensional shifted Legendre polynomials; two-dimensional fractional integral equations; operational matrix.*

Mathematics Subject Classification (2010): 45AXX, 26A33.

1 Introduction

In this paper, we present a numerical method for the solution of two-dimensional Volterra integral equations of fractional order in the form

$$f(x,y) - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} k(x,y,s,t) f(s,t)\, dt ds = g(x,y),$$
$$(r_1,r_2) \in (0,\infty)\times(0,\infty), f \in L^1(\Omega), \Omega := [0,l_1]\times[0,l_2]. \tag{1}$$

In [1–3] the authors mentioned that equation (1) is a solution for a class of impulsive partial hyperbolic differential equations involving the Caputo fractional derivative. Therefore, researchers are interested in solving this kind of equations. In recent years, several numerical methods for solving two-dimensional integral equations of fractional

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order have been presented in the literature. Najafalizadeh and Ezzati in [4] used a two-dimensional block pulse operational matrix to solve two-dimensional nonlinear integral equations of fractional order. In [5], two-dimensional orthogonal triangular functions were used for solving two-dimensional integral nonlinear equations of fractional order. In [6], we see that the operational matrix of two-dimensional Bernstein polynomials is used for two-dimensional integral equations of fractional order. Here, we would like to use two-dimensional shifted Legendre polynomials for solving two-dimensional integral equations of fractional order. Firstly, we present some preliminaries in fractional calculus. In Section 3, we review some general concepts concerning one-dimensional and two-dimensional shifted Legendre polynomials, and derive an operational matrix of two-dimensional shifted Legendre polynomials for two-dimensional integration of fractional order. Section 4 is devoted to solving two-dimensional nonlinear fractional integral equations by applying the operational matrix of integration of fractional order. Section 5 represents an error estimation for the presented method. In Section 6, we show accuracy and efficiency of this method through several examples. Finally, a conclusion is given in Section 7.

2 Brief Review of Fractional Calculus

In this section, we present a short introduction of the fractional calculus which will be used in this paper.

Definition 2.1 [7]. The Riemann-Liouville fractional integral operator I^α of order $\alpha \geq 0$ is defined as

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)\, dt \ , \alpha > 0, \tag{2}$$

where $x > 0$ and $\Gamma(.)$ is the Euler gamma function.

The Riemann-Liouville integral satisfies the following properties:

- $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$,
- $I^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1+\beta)} x^{\alpha+\beta}$.

Definition 2.2 [8]. The left-sided mixed Riemann-Liouville integral of order r of f is defined as

$$I_\theta^r f(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s,t)\, dtds, \tag{3}$$

where $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$ and $\theta = (0,0)$.

Some properties of the left-sided mixed Riemann-Liouville integral are the following:

- $I_\theta^\theta f(x,y) = f(x,y)$,
- if $p, g \in (-1,\infty)$ then, $I_\theta^r x^p y^q = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+1+r_1)\Gamma(q+1+r_2)} x^{p+r_1} y^{q+r_2}$.

3 Shifted Legendre Polynomials

3.1 One-dimensional shifted Legendre polynomials

Let $L_i(x)$, $x \in [0.l]$, be the shifted Legendre polynomials. Then $L_i(x)$ can be obtained with the aid of the following recurrence formula [9]:

$$L_{i+1}(x) = \frac{2i+1}{i+1}(\frac{2x}{l} - 1)L_i(x) - \frac{i}{i+1}L_{i-1}(x), \quad i = 1, 2, 3, ...,$$

where $L_0(x) = 1, L_1(x) = \frac{2x}{l} - 1$.

The shifted Legendre polynomials on $[0, l]$ have the following properties:

- $L_i(x) = \sum_{k=0}^{i} \frac{(-1)^{(i+k)}(i+k)!}{(i-k)! l^k (k!)^2} x^k$,
- $\int_0^l L_i(x)L_j(x)\,dt = \begin{cases} \frac{l}{2i+1}, & for\ i = j, \\ 0, & for\ i \neq j, \end{cases}$
- A function $f(x) \in C[0, l]$ can be expanded by shifted Legendre polynomials in the following form:

$$f(x) \backsimeq \sum_{i=0}^{m} c_i L_i(x) = C^T \Phi(x),$$

where the coefficients c_i are given by

$$c_i = \frac{(2i+1)}{l} \int_0^l L_i(x)y(x)\,dt,$$

and the vectors C, $\Phi(x)$ are given by

$$C^T = [c_0, c_1, ..., c_m], \tag{4}$$
$$\Phi(x) = [L_0(x), L_1(x), ..., L_m(x)]^T. \tag{5}$$

Previously, in [10] the operational matrix of shifted Legendre polynomials for fractional integration in the interval $[0, 1]$ has been presented. Now we present the operational matrix of shifted Legendre polynomials for fractional integration in the interval $[0, l]$ as follows:

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} \Phi(x)\,dt = P^\alpha \Phi(x) \tag{6}$$

and

$$P^\alpha = \begin{pmatrix} \sum_{k=0}^{0} \theta_{0,0,k} & \sum_{k=0}^{0} \theta_{0,1,k} & . & . & . & \sum_{k=0}^{0} \theta_{0,m,k} \\ \sum_{k=0}^{1} \theta_{1,0,k} & \sum_{k=0}^{1} \theta_{1,1,k} & . & . & . & \sum_{k=0}^{1} \theta_{1,m,k} \\ \vdots & \vdots & . & . & . & \vdots \\ \sum_{k=0}^{i} \theta_{i,0,k} & \sum_{k=0}^{i} \theta_{i,1,k} & . & . & . & \sum_{k=0}^{i} \theta_{i,m,k} \\ \vdots & \vdots & . & . & . & \vdots \\ \sum_{k=0}^{m} \theta_{m,0,k} & \sum_{k=0}^{m} \theta_{m,1,k} & . & . & . & \sum_{k=0}^{m} \theta_{m,m,k} \end{pmatrix},$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = (2j+1)\sum_{s=0}^{j} \frac{(-1)^{(s+j+s+k)}(i+k)!(j+s)!l^{\alpha+s-k}}{(i-k)!k!\Gamma(k+\alpha+1)(j-s)!(s!)^2(k+s+\alpha+1)},$$

and P^{α} is called the shifted Legendre polynomials operational matrix for fractional integration.

The proof is similar to the proof of Theorem 3 in [10].

3.2 Two-dimensional shifted Legendre polynomials

The two-dimensional shifted Legendre polynomials are defined on $\Omega = [0, l_1] \times [0, l_2]$ as follows [11]:

$$\psi_{m,n}(x,y) = L_m(x)L_n(y), \quad m, n = 0, 1, 2, \cdots,$$

where $L_m(x)$ and $L_n(y)$ are shifted Legendre polynomials which are defined in the same way as on the intervals $[0, l_1]$ and $[0, l_2]$, respectively. In the following, we study the important properties of the two-dimensional shifted Legendre polynomials.

The two-dimensional shifted Legendre polynomials are orthogonal with each other

$$\int_0^{l_1}\int_0^{l_2} \psi_{m,n}(x,y)\psi_{i,j}(x,y)\,dydx = \begin{cases} (\frac{l_1 l_2}{(2m+1)(2n+1)}), & i=m, j=n, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $\Theta = L^2(\Omega)$, the inner product in this space is defined by

$\langle (f(x,y), g(x,y)\rangle = \int_0^{l_1}\int_0^{l_2} f(x,y)g(x,y)\,dydx,$

and the norm is as follows:

$\|f(x,y)\|_2 = \langle f(x,y), f(x,y)\rangle^{\frac{1}{2}} = (\int_0^{l_1}\int_0^{l_2} |f(x,y)|^2\,dydx)^{\frac{1}{2}}.$

For every $f(x,y) \in \Theta$, we have

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}\phi_{ij}(x,y). \tag{7}$$

If the infinite series in (7) is truncated, then we will have

$$f(x,y) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\phi_{ij}(x,y) = F^T\phi(x,y) = \phi^T(x,y)F, \tag{8}$$

where $\phi(x,y)$ and F are $(M+1)(N+1)\times 1$ vectors of the following form

$$F = [f_{00}, ..., f_{0N}, ..., f_{M0}, ..., f_{MN}]^T, \tag{9}$$

$$\phi(x,y) = [\phi_{00}(x,y), ..., \phi_{0N}(x,y), ..., \phi_{M0}(x,y), ..., \phi_{MN}(x,y)]^T \tag{10}$$

and $\phi_{i,j}(x,y) = \phi_i(x).\phi_j(y)$.

The two-dimensional shifted Legendre polynomials coefficients $f_{i,j}$ are obtained by

$$f_{i,j} = \frac{\langle f(x,y), \phi_{i,j}(x,y)\rangle}{\|\phi_{i,j}(x,y)\|_2^2}.$$

By using the Kronecker product of $\phi(x)$ and $\phi(y)$ we can show $\phi(x,y)$ as

$$\phi(x,y) = \phi(x) \otimes \phi(y), \tag{11}$$

where $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as

$$A \otimes B = (a_{i,j}B),$$

also it has the following two basic properties [12]:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \qquad (A+B) \otimes C = A \otimes C + B \otimes C. \tag{12}$$

Similarly, the function $k(x,y,s,t)$ in $L^2(\Omega \times \Omega)$ can be expanded in terms of two shifted Legendre polynomials as

$$k(x,y,s,t) \simeq \phi^T(x,y) K \phi(s,t), \tag{13}$$

where K is a block matrix of the form

$$K = [K^{(i,m)}]_{i,m=0}^{M}$$

in which

$$K^{(i,m)} = [k_{ijmn}]_{j,n=0}^{N}, \quad i,m = 0,1,...,M$$

and the two-shifted Legendre polynomials coefficient k_{ijmn} is given by

$$k_{ijmn} = \frac{\langle\langle k(x,y,s,t)\phi_{m,n}(s,t)\rangle, \phi_{i,j}(x,y)\rangle}{\|\phi_{i,j}(x,y)\|_2^2 \|\phi_{m,n}(s,t)\|_2^2}, \quad i,m = 0,1,...,M. \quad j,n = 0,1,...,N.$$

The product of two vectors $\phi(x,y)$ and $\phi^T(x,y)$ with the vector F is given by

$$\phi(x,y)\phi^T(x,y)F \simeq \tilde{F}\phi(x,y), \tag{14}$$

where F is defined by (9) and $\tilde{F}$ is an $(M+1)(N+1) \times (M+1)(N+1)$ matrix

$$\tilde{F} = [F^{(i,j)}]_{i,j=0,1,...M}, \tag{15}$$

where $F^{(i,j)}$, $i,j = 0,1,...,M$, are given by

$$F^{(i,j)} = \frac{2j+1}{l_2} = \sum_{m=0}^{M} W_{i,j,m}\Lambda_m,$$

in which $W_{i,j,m}$ is defined as

$$W_{i,j,m} = \int_0^{l_1} L_i(\frac{2}{l_1}x-1)L_j(\frac{2}{l_1}x-1)L_m(\frac{2}{l_1}x-1)\,dx.$$

and Λ_m, $m = 0,1,...,M$, are $(N+1) \times (N+1)$ matrices

$$[\Lambda_m]_{kh} = \frac{2h+1}{l_1} = \sum_{n=0}^{N} \acute{W}_{k,h,n} f_{mn}, \quad k,h = 0.,1,...,N,$$

where

$$\acute{W}_{k,h,n} = \int_0^{l_2} L_k(\frac{2}{l_2}y-1)L_h(\frac{2}{l_2}y-1)L_n(\frac{2}{l_2}y-1)\,dy.$$

3.3 Operational matrix of fractional order

Now, we construct an operational matrix of two-dimensional shifted Legendre polynomials for the fractional integration.

By using equations (10), (11) we have

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds =$$
$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s)\otimes\phi(t)\,dtds =$$
$$\frac{1}{\Gamma(r_1)}\int_0^x (x-s)^{r_1-1}\phi(s)\,ds\otimes\frac{1}{\Gamma(r_2)}\int_0^y (y-t)^{r_2-1}\phi(t)\,dt = *.$$

From equation (6) we get

$$\begin{aligned} * =& p^{r_1}\phi(x)\otimes p^{r_2}\phi(y)\\ =&(p^{r_1}\otimes p^{r_2})(\phi(x)\otimes\phi(y))\\ =&(p^{r_1}\otimes p^{r_2})\phi(x,y). \end{aligned}$$

Hence,

$$\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds = p^{r_1,r_2}\phi(x,y), \tag{16}$$

where

$$p^{r_1,r_2} = (p^{r_1}\otimes p^{r_2}).$$

4 Numerical Solution of Two-Dimensional Volterra Integral Equations of Fractional Order

In this section, we present an effective method to solve equation (1). For this purpose, by using the method mentioned in Section 3, the functions $f(x,y)$, $g(x,y)$ and $k(x,y,s,t)$ can be approximated by

$$\begin{aligned} f(x,y) &= \phi(x,y)^T F,\\ g(x,y) &= \phi(x,y)^T G,\\ k(x,y,s,t) &= \phi(x,y)^T K\phi(s,t), \end{aligned} \tag{17}$$

where $\phi(x,y)$ is defined in equation (10) and the vectors F,G and matrix K are two-dimensional shifted Legendre polynomials coefficients of $f(x,y), g(x,y)$ and $k(x,y,s,t)$, respectively. Now, substituting equation (17) in equation (1), we have

$$\phi^T(x,y)F - \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi^T(x,y)K\phi(s,t)\phi^T(s,t)F\,dtds$$
$$\simeq \phi^T(x,y)G. \tag{18}$$

By using equations (14) and (16) we conclude that

$$\phi^T(x,y)F - \frac{\phi^T(x,y)K\tilde{F}}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\phi(s,t)\,dtds \simeq \phi^T(x,y)G, \tag{19}$$

$$\phi^T(x,y)F - \phi^T(x,y)K\tilde{F}P^{r_1,r_2}\phi(x,y) \simeq \phi^T(x,y)G. \tag{20}$$

If in the above equation we substitute $\simeq$ with $=$, we get the following equation

$$F - K\tilde{F}P^{r_1,r_2}\phi(x,y) = G. \tag{21}$$

Now we collocate equation (21) in $(M+1)(N+1)$ Newton-Cotes nodes as

$$x_m = \frac{2m+1}{2(M+1)}, y_n = \frac{2n+1}{2(N+1)}, m = 0,1,...,M, n = 0,1,...,N.$$

We will have a linear system of algebraic equations

$$F - K\tilde{F}P^{r_1,r_2}\phi(x_m,y_n) = G, m = 0,1,...,M, n = 0,1,...,N. \tag{22}$$

It is clear that, by solving this system, we can obtain the approximate solution of equation (1) according to equation (8).

5 Error Analysis

Theorem 5.1 . *[11] Let $\tilde{f}(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} f_{ij}\phi_{ij}(x,y)$ be the two-dimensional shifted Legendre polynomials expansion of the real sufficiently smooth function $f(x,t)$ in Ω, then there exist real numbers C_1 , C_2 and C_3 such that*

$$\|f(x,y) - \tilde{f}(x,y)\|_2 \le C_1\frac{(\frac{l_1}{2})^{M+1}}{(M+1)!2^M} + C_2\frac{(\frac{l_2}{2})^{N+1}}{(N+1)!2^N} + C_3\frac{(\frac{l_1}{2})^{M+1}(\frac{l_2}{2})^{N+1}}{(M+1)!(N+1)!2^{M+N}}.$$

In the special case when $M = N$ and $l_1 = l_2 = 1$ we get

$$\|f(x,y) - \tilde{f}(x,y)\|_2 \le (C_1 + C_2 + C_3\frac{1}{(M+1)!2^{2M+1}})\frac{1}{(M+1)!2^{2M+1}},$$

hence

$$\|f(x,y) - \tilde{f}(x,y)\|_2 = O(\frac{1}{(M+1)!2^{2M+1}}).$$

Theorem 5.2 *Suppose $M = N$, $l_1 = l_2 = 1$ and $f(x,y)$ is an exact solution of the fractional integral equation (1) and $\tilde{f}(x,y)$ shows the approximate solution by the two-dimensional shifted Legendre polynomials. If $|(x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)| < C$, $f(x,y)$ and $k(x,y,s,t)$ are sufficiently smooth functions, then*

$$\|f(x,y) - \tilde{f}(x,y)\|_2^2 \le \frac{C^2}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1 + C_2 + C_3\frac{1}{(M+1)!2^{2M+1}})^2.$$

Proof.

$$\|f(x,y)-\tilde{f}(x,y)\|_2^2 =$$
$$\frac{1}{(\Gamma(r_1)\Gamma(r_2))^2}\|\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)(f(s,t)-\tilde{f}(s,t))\,dtds\|_2^2$$
$$\leq \frac{1}{(\Gamma(r_1)\Gamma(r_2))^2}\int_0^x\int_0^y \|(x-s)^{r_1-1}(y-t)^{r_2-1}k(x,y,s,t)(f(s,t)-\tilde{f}(s,t))\|_2^2\,dtds$$
$$\leq \frac{C^2}{(\Gamma(r_1)\Gamma(r_2))^2}\int_0^x\int_0^y \|(f(s,t)-\tilde{f}(s,t))\|_2^2\,dtds$$
$$\leq \frac{C^2xy}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1+C_2+C_3\frac{1}{(M+1)!2^{2M+1}})^2$$
$$\leq \frac{C^2}{(\Gamma(r_1)\Gamma(r_2)(M+1)!2^{2M+1})^2}(C_1+C_2+C_3\frac{1}{(M+1)!2^{2M+1}})^2. \square$$

6 Illustrative Examples

In this section we will implement our method by three examples. For justifying our method, we compare our computed results and those by other authors. Outcomes show the accuracy and the validity of the presented method. In these examples we let $l_1 = l_2 = 1$, $M = N$ and denote the following error function

$$e(x,y) = |f(x,y) - \tilde{f}_{M,N}(x,y)|,$$

where $f(x,y)$ and $\tilde{f}_{M,N}(x,y)$ are the exact and approximate solutions of the two-dimensional fractional integral equation, respectively.

Example 6.1 Consider the two-dimensional fractional integral equation given in [5]

$$f(x,y) - \frac{1}{\Gamma(3.5)\Gamma(3.5)}\int_0^x\int_0^y (x-s)^{2.5}(y-t)^{2.5}xyt^{\frac{1}{2}}f(s,t)\,dtds = \frac{1}{2}xy - \frac{x^{5.5}y^6}{9450}.$$

The exact solution of this equation is $f(x,y) = \frac{1}{2}xy$. Table 1 shows the absolute error obtained by using the present method and by using the 2D-Tf method [5].

Example 6.2 Consider the two-dimensional fractional integral equation given in [6]

$$f(x,y) - \frac{1}{\Gamma(3.5)\Gamma(2.5)}\int_0^x\int_0^y (x-s)^{2.5}(y-t)^{1.5}e^{-t}(y^2+s)f(s,t)\,dtds =$$
$$x^2e^y - \frac{1024x^{5.5}y^{2.5}(6x+13y^2)}{2027025\pi}$$

and the exact solution of the above equation is $f(x,y) = e^yx^2$. Table 2 shows the absolute error obtained by using the present method and by using the two-dimensional Bernstein polynomials method [6].

Example 6.3 As the last example, we have the two-dimensional fractional integral equation

$$f(x,y) - \frac{1}{\Gamma(3.5)\Gamma(3.5)}\int_0^x\int_0^y (x-s)^{3.5}(y-t)^{3.5}52\sqrt{tx}f(s,t)\,dtds = xy^2 - \frac{x^5y^5}{5670}$$

	Present method	Present method	Method [5]
x=y	$m=1$	$m=2$	m=8
0.1	2.2349×10^{-6}	$7.84825\ \times10^{-8}$	1.126×10^{-4}
0.2	$2.13487\ \times10^{-6}$	$3.48089\ \times10^{-8}$	$1.363\ \times10^{-4}$
0.3	2.03717×10^{-6}	1.47598×10^{-7}	6.22×10^{-5}
0.4	1.94179×10^{-6}	2.60141×10^{-7}	1.27×10^{-5}
0.5	1.84874×10^{-6}	3.7269×10^{-7}	1.983×10^{-4}
0.6	1.758×10^{-6}	4.8549×10^{-7}	4.6×10^{-5}
0.7	1.66959×10^{-6}	5.9879×10^{-7}	5.2×10^{-5}
0.8	1.58351×10^{-6}	7.1281×10^{-7}	6.8×10^{-4}
0.9	1.49975×10^{-6}	8.2781×10^{-7}	6.8×10^{-4}

Table 1: Absolute error for Example 1.

	Present method	Present method	Method [6]
x=y	$m=1$	$m=2$	m=4
0.0	1.1458×10^{-2}	2.4215×10^{-5}	4.086×10^{-4}
0.1	1.1130×10^{-2}	$2.1511\ \times10^{-5}$	4.181×10^{-4}
0.2	$1.0799\ \times10^{-2}$	$1.9207\ \times10^{-5}$	$4.471\ \times10^{-4}$
0.3	1.0466×10^{-2}	1.7355×10^{-5}	4.970×10^{-4}
0.4	1.0131×10^{-2}	1.6000×10^{-5}	5.656×10^{-4}
0.5	9.7937×10^{-3}	1.5188×10^{-5}	6.474×10^{-4}
0.6	9.4538×10^{-3}	1.4957×10^{-5}	7.316×10^{-4}
0.7	9.1117×10^{-3}	1.5342×10^{-5}	7.817×10^{-4}
0.8	8.7676×10^{-3}	1.6374×10^{-5}	6.788×10^{-4}
0.9	8.4215×10^{-3}	1.8082×10^{-5}	1.004×10^{-4}

Table 2: Absolute error for Example 2.

and the exact solution of the above equation is $f(x, y) = xy^2$. Table 3 illustrates the numerical results for this example.

7 Conclusion

In this paper a general formulation for the two-dimensional shifted Legendre polynomials operational matrix of two-dimensional fractional integral equations has been derived. This matrix is used to approximate numerical solution of the two-dimensional nonlinear fractional integral equations. The properties of two-dimensional shifted Legendre polynomials and the operational matrices are used to reduce the two-dimensional fractional integral equations to a system of algebraic equations that can be solved easily. Finally, illustrative examples are presented to show the validity and the accuracy of the proposed method.

x=y	$m = 1$	$m = 2$	m=3
0.0	5.9600×10^{-3}	1.3080×10^{-6}	2.7534×10^{-7}
0.1	6.3096×10^{-3}	$1.4595\ \times10^{-6}$	2.8692×10^{-7}
0.2	$6.6175\ \times10^{-3}$	$1.6017\ \times10^{-6}$	2.9819×10^{-7}
0.3	6.8844×10^{-3}	1.7349×10^{-6}	3.0917×10^{-7}
0.4	7.1110×10^{-3}	1.8596×10^{-6}	3.1989×10^{-7}
0.5	7.2982×10^{-3}	1.9763×10^{-6}	3.3038×10^{-7}
0.6	7.4466×10^{-3}	2.0852×10^{-6}	3.4065×10^{-7}
0.7	7.5570×10^{-3}	2.1868×10^{-6}	3.5073×10^{-7}
0.8	7.6301×10^{-3}	2.2815×10^{-6}	3.6063×10^{-7}
0.9	7.6668×10^{-3}	2.3696×10^{-6}	3.7036×10^{-7}

Table 3: Absolute error for Example 3.

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Nonlinear Dynamics and Systems Theory, **18** (3) (2018) 307–318

Mild Solutions for Multi-Term Time-Fractional Impulsive Differential Systems

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Received: June 20, 2017; Revised: June 28, 2018

Abstract: In this paper, we study the existence and uniqueness of mild solutions for multi-term time-fractional differential systems with non-instantaneous impulses and finite delay. We use the tools of the Banach fixed point theorem and condensing map along with generalization of the semigroup theory for linear operators and fractional calculus to come up with a new set of sufficient conditions for the existence and uniqueness of the mild solutions. An illustration is provided to demonstrate the established results.

Keywords: *fractional calculus, generalized semigroup theory, multi-term time-fractional differential system, (β, γ_j)-resolvent family, non-instantaneous impulses.*

Mathematics Subject Classification (2010): 34A08, 34G20, 35R12, 26A33, 34A12, 34A37.

1 Introduction

During the last few decades, the fractional differential equations (FDEs) including Riemann-Liouville and Caputo derivatives have attracted the interest of many researchers, motivated by demonstrated applications in widespread areas of science and engineering such as models of medicine (modeling of human tissue under mechanical loads), electrical engineering(transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) etc. In addition, due to the memory and hereditary properties of the materials and processes, in some areas of science such as identification systems, signal processing, robotics or control theory, the fractional differential operators

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seem more appropriate in modeling than the classical integer operators. For fundamental certainties regarding to fractional systems, one can make reference to the papers [6, 9, 14, 19–21, 25, 26], the monographs [10, 16, 24] and references therein. Moreover, fractional differential systems with delay are used frequently in many fields such as 3-D printing and oil drilling, modeling of equations, panorama of natural phenomena and porous media. For more details, see the cited papers [1, 3].

On the other hand, the theory of fractional impulsive differential equations (in short, FIDEs) also has generated a great interest among the researchers, because many real world processes and phenomena which are effected by abrupt changes in the state at certain moments are naturally described by FIDEs. These changes occur due to disturbances, changing operational conditions and component failures of the state. For example, mechanical and biological models subject to shocks. Generally, the abrupt changes in the state for instant period in evolution process are formulated by impulsive differential equations. However, it is not necessary that the dynamical systems with evolutionary processes always be characterized by instantaneous impulses. For example, pharmacotherapy [23], in which the hemodynamic equilibrium of a person is considered. The initiation of the drugs in the bloodstream and the resultant absorption for the body are gradual and continuous processes. Therefore, instantaneous impulses failed to describe such processes. To characterize these type of situations Hernández and O′Regan [8] introduce a new case of impulsive actions, which are triggered abruptly at an arbitrary instant and their action remains for a finite time interval. Meanwhile, Pierri et al. [22] extended the results of [8] with an α-normed Banach space. For the general theory of impulsive differential equations, we refer to the monographs [4, 12], research papers [5, 11, 13, 15, 17, 18, 28] and references therein.

Indeed, in [9, 14, 19, 27], the authors have obtained the existence and uniqueness results without impulsive conditions, and in [20], Pardo studied weighted pseudo almost automorphic mild solutions for two-term time-fractional order differential equations. In [21], Pardo and Lizama studied a nonlinear multi-term time-differential system of the form

$$^{c}D_t^{\gamma}y(t) + \sum_{j=1}^{d}\mu_j\,^{c}D_t^{\beta_j}y(t) = Ay(t) + f(t, y(t)), \quad \beta_j > 0,\, t \in [0,1], 0 < \gamma \leq 2, \tag{1}$$

$$y(0) = 0, \qquad y'(0) = g(y), \tag{2}$$

where $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator and f and g are suitable functions. In the foregoing cases, the initial value problems were considered, but the study of existence of mild solutions for the system modeled as $(1)-(2)$ involving non-instantaneous impulses and delay was left open. Anticipating a wide interest in the problems modeled as the system $(3) - (5)$, this paper contributes to fill this important gap.

This paper is organized as follows. Section 2 is devoted to recall basics of fractional calculus and mild solution which will be employed to attain our mains outcomes. In Section 3, the existence and uniqueness results for the system $(3) - (5)$ are analyzed under the Banach and condensing map fixed point theorems. In Section 4, as a final point, an example is provided to show the feasibility of the theory discussed in this paper.

2 Problem Formulation

Let $\mathbb{X}$ be a Banach space. Let $\mathcal{L}(\mathbb{X})$ be the space of all bounded and linear operators on $\mathbb{X}$ equipped with the norm $\|\cdot\|_{\mathcal{L}}$. Let $\mathbb{R}$ and $\mathbb{N}$ stand for real numbers and natural numbers, respectively. For a linear operator A on $\mathbb{X}$, $\mathcal{R}(A), \mathcal{D}(A)$ and $\varrho(A)$ represent the range, domain and resolvent of A respectively. To facilitate the discussion due to delay, we use the space $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], \mathbb{X})$ formed by the continuous functions from $[-\tau, 0]$ to $\mathbb{X}$ equipped with the norm $\|y\|_{\mathcal{PC}_0} = \sup_{t\in[-\tau,0]}\{\|y(t)\|_{\mathbb{X}} : y \in \mathcal{PC}_0\}$. To study the impulsive forces, we define a space $\mathcal{PC}_T := \mathcal{PC}([-\tau, T], \mathbb{X}), 0 \le t \le T$ of all functions $y : [-\tau, T] \to \mathbb{X}$, which are continuous everywhere except the points $t_k \in (0, T), k = 1, 2, ..., m$, at which $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$. Obviously, $\mathcal{PC}_T$ is a Banach space equipped with the norm $\|y\|_{\mathcal{PC}_T} = \sup_{t\in[-\tau,T]}\{\|y(t)\|_{\mathbb{X}} : y \in \mathcal{PC}_T\}$.

In this paper, we study the existence and uniqueness of mild solutions for the following class of multi-term time-fractional differential equations with non-instantaneous impulses

$$
\begin{aligned}
&{}^cD^{1+\beta}_{s_k} y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j}_{s_k} y(t) \\
&\qquad = Ay(t) + F\left(t, y_t, \int_0^t \mathfrak{K}(t,s)(y_s)ds\right), \quad t \in \cup_{k=0}^{m}(s_k, t_{k+1}], \quad (3)
\end{aligned}
$$

$$
y(t) = G_k(t,y_t), \quad y'(t) = H_k(t, y_t), \qquad t \in \cup_{k=1}^{m}(t_k, s_k], \quad (4)
$$

$$
y(t) + g_1(y) = \phi(t), \quad y'(t) + g_2(y) = \varphi(t), \quad t \in [-\tau, 0], \quad (5)
$$

where $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator. ${}^cD^{\eta}_{s_k}$ stands for the Caputo derivative of order $\eta > 0$ and $\mathcal{I} = [0, T] = \{0\} \cup_{k=0}^{m} (s_k, t_{k+1}] \cup_{k=1}^{m} (t_k, s_k], T < \infty$ such that $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$ are prefix numbers. All γ_j, $j = 1, 2, 3...n$, are positive real numbers such that $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$. G_k and H_k are continuous functions from $\cup_{k=1}^{m}(t_k, s_k] \times \mathcal{PC}_0$ into $\mathbb{X}$ for all $k = 1, 2, ..., m$. $F : \mathcal{I} \times \mathcal{PC}_0 \times \mathcal{PC}_0 \to \mathbb{X}$ is a suitable function. The history function $y_t : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $y_t(\theta) = y(t + \theta), \theta \in [-\tau, 0]$ and also $\phi, \varphi \in \mathcal{PC}_0$. y' denotes the usual derivative of y with respect to t. $\mathfrak{K}$ is a positive and continuous operator on $\Omega := \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $k^0 = \sup \int_0^t \mathfrak{K}(t, s)ds < \infty$. Here by non-instantaneous, we mean that the impulses start abruptly at t_k and their effect will continue on the interval $[t_k, s_k]$ for $k = 1, 2, 3, ..., m$.

Now, we recall some definitions and basic results on fractional calculus (for more details, see [24]). Define $g_\eta(t)$ for $\eta > 0$ by

$$
g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0; \\ 0, & t \le 0, \end{cases}
$$

where Γ denotes the gamma function. Let $(X * Y)(t)$ be the convolution of X and Y given by $(X * Y)(t) := \int_0^t X(t - s)Y(s)ds$.

Definition 2.1 The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}(\mathbb{R}^+, \mathbb{X})$ of order $\eta > 0$ with the lower limit $a \ge 0$ is defined as follows

$$
I^\eta_a f(t) = \int_a^t g_\eta(t - s)f(s)ds, \quad t > 0,
$$

and $I^0_a f(t) = f(t)$. This fractional integral satisfies the properties $I^\eta_0 \circ I^b_0 = I^{\eta+b}_0$ for $b > 0$ and $I^\eta_0 f(t) = (g_\eta * f)(t)$.

Definition 2.2 [21] Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m([0,\infty),\mathbb{R})$ with the lower limit $a \geq 0$ is given by

$$^cD_a^\eta f(t) = I_a^{m-\eta} D_a^m f(t) = \int_a^t g_{m-\eta}(t-s)\frac{d^m}{dt^m}f(s)ds,$$

and $^cD_a^0 f(t) = f(t)$. In addition, we have $^cD_0^\eta f(t) = (g_{m-\eta} * D^m f)(t)$.

To give an appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define the following family of operators.

Definition 2.3 [21] Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and $\beta > 0, \gamma_j, \alpha_j$ be the real positive numbers. Then A is called the generator of a $(\beta,\gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} : \operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ and

$$\lambda^\beta\left(\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{S}_{\beta,\gamma_j}(t)y dt, \quad \operatorname{Re}\lambda > \omega, y \in \mathbb{X}. \tag{6}$$

The following result provides the existence of $(\beta,\gamma_j)-$ resolvent family under some suitable conditions.

Theorem 2.1 *[21] Let $0 < \beta \leq \gamma_i \leq, \cdots, \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates a bounded $(\beta,\gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.*

Motivated by [21], we define a mild solution for the system $(3)-(5)$ as follows.

Definition 2.4 A function $y \in \mathcal{PC}_T$ is called a mild solution of the system $(3)-(5)$, if $y(t) = \phi(t) - g_1(y), y'(t) = \varphi(t) - g_2(y)$ for $[-\tau, 0]$ and $y(t) = G_k(t, y_t), y'(t) = H_k(t, y_t)$ for $t \in \cup_{k=1}^m (t_k, s_k]$ and satisfy the following integral equations

$$y(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ +\displaystyle\sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ + \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t - s_k)G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s - s_k)H_k(s_k, y_{s_k})ds \\ +\displaystyle\sum_{j=1}^n \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s - s_k)G_k(s_k, y_{s_k})ds \\ + \int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}], \end{cases} \tag{7}$$

where $K(y_s) = \int_0^s \mathfrak{K}(s,\xi)(y_\xi)d\xi$.

Theorem 2.2 *[7, Condensing theorem] Let $\mathcal{M}$ be a closed, bounded and convex subset of a Banach space $\mathbb{X}$ and assume that $Q : \mathcal{M} \to \mathcal{M}$ is a condensing map. Then Q admits a fixed point in $\mathcal{M}$.*

3 Main Results

In this section, we establish the existence and uniqueness of mild solution for the system (3) $-$ (5). We denote $S_0 = \sup_{t\in[0,T]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. In order to establish the existence and uniqueness result by the Banach fixed point theorem, we consider the following assumptions:

(A_1) There exist positive constants μ_F and μ_F^0 such that

$$\|F(t,\psi_1,\chi_1) - F(t,\psi_2,\chi_2)\|_{\mathbb{X}} \le \mu_F\|\psi_1-\psi_2\|_{\mathcal{PC}_0} + \mu_F^0\|\chi_1-\chi_2\|_{\mathcal{PC}_0},$$

where $\psi_i, \chi_i \in \mathcal{PC}_0$, $i = 1, 2$.

(A_2) There exist positive constants μ_G, μ_{g_i} and μ_H such that

$$\|G_k(t,\psi) - G_k(t,\chi)\|_{\mathbb{X}} \le \mu_G\|\psi-\chi\|_{\mathcal{PC}_0},\ \|H_k(t,\psi) - H_k(t,\chi)\|_{\mathbb{X}} \le \mu_H\|\psi-\chi\|_{\mathcal{PC}_0},$$
$$\|g_i(x) - g_i(y)\|_{\mathbb{X}} \le \mu_{g_i}\|x-y\|_{\mathbb{X}},$$

for all $\psi, \chi \in \mathcal{PC}_0$, $x, y \in \mathbb{X}$, $i = 1, 2$ and $k = 1, 2, 3, \dots, m$.

Theorem 3.1 *Assume that the assumptions $(A_1)-(A_2)$ are fulfilled, then the system (3) $-$ (5) has a unique mild solution in $\mathcal{I}$ if $\Theta < 1$, where*

$$\Theta = \max\left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n}\frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)}[\mu_F + \mu_F^0 k^0], \mu_G\right],$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$ and $T_0 = \max\limits_{0\le k\le m} |t_{k+1} - s_k|$.

Proof. To transform the problem into a fixed point problem, we define an operator $Q : \mathcal{PC}_T \to \mathcal{PC}_T$ by $Qy(t) = \phi(t)$ for $t \in [-\tau, 0]$ and $Qy(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^m (t_k, s_k]$, and

$$Qy(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ +\displaystyle\sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ +\int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t-s_k)G_k(s_k, y_{s_k}) \\ +\int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k)H_k(s_k, y_{s_k})ds \\ +\displaystyle\sum_{j=1}^{n}\alpha_j\int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s-s_k)G_k(s_k, y_{s_k})ds \\ +\int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{8}$$

Let $x, y \in \mathcal{PC}_T$. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
&\|Qx(t) - Qy(t)\|_{\mathbb{X}} \\
&\leq \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}\|g_1(x) - g_1(y)\|_{\mathbb{X}} + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}\|g_2(x) - g_2(y)\|_{\mathbb{X}} ds \\
&\quad + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}\|g_1(x) - g_1(y)\|_{\mathbb{X}} ds \\
&\quad + \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, x_s, K(x_s)(s)) - F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
&\leq \left[S_0\mu_{g_1} + T_0S_0\mu_{g_2} + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0T_0^{1+\beta}}{\Gamma(2+\beta)}[\mu_F + \mu_F^0 k^0]\right] \|x - y\|_{\mathcal{PC}_T}.
\end{aligned}
$$

For $t \in \cup_{k=1}^{m}(t_k, s_k]$, we get

$$\|Qx(t) - Qy(t)\|_{\mathbb{X}} \leq \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \leq \mu_G\|x - y\|_{\mathcal{PC}_T}, \quad k = 1, 2, 3, \ldots, m.$$

Similarly, for $t \in \cup_{k=1}^{m}(s_k, t_{k+1}]$ we get

$$
\begin{aligned}
&\|Qx(t) - Qy(t)\|_{\mathbb{X}} \\
&\leq \|\mathcal{S}_{\beta,\gamma_j}(t - s_k)\|_{\mathcal{L}}\|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \\
&\quad + \int_{s_k}^t \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}}\|H_k(s_k, x_{s_k}) - H_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
&\quad + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}}\|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
&\quad + \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, x_s, K(x_s)(s)) - F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
&\leq \left[S_0\mu_G + T_0S_0\mu_H + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0T_0^{1+\beta}}{\Gamma(2+\beta)}[\mu_F + \mu_F^0 k^0]\right] \|x - y\|_{\mathcal{PC}_T}.
\end{aligned}
$$

Gathering the above results, we have $\|Qx - Qy\|_{\mathcal{PC}_T} \leq \Theta\|x - y\|_{\mathcal{PC}_T}$. Now, by the Banach contraction theorem the system $(3) - (5)$ has a unique mild solution.

In order to establish the existence results by virtue of the condensing map, we consider the following assumptions:

(A_3) The functions G_k, H_k, g_1 and g_2 are continuous functions and F is compact and continuous, and there exist positive constants ν_F, ν_F^0, ν_G, ν_H, ν_{g_1}, ν_{g_2} such that

$$
\begin{aligned}
\|F(t, \psi, \chi)\|_{\mathbb{X}} \leq \nu_F\|\psi\|_{\mathcal{PC}_0} + \nu_F^0\|\chi\|_{\mathcal{PC}_0}, \quad \|g_i(x)\|_{\mathbb{X}} \leq \nu_{g_i}\|x\|_{\mathbb{X}}, \\
\|G_k(t, \psi)\|_{\mathbb{X}} \leq \nu_G\|\psi\|_{\mathcal{PC}_0}, \quad \|H_k(t, \psi)\|_{\mathbb{X}} \leq \nu_H\|\psi\|_{\mathcal{PC}_0}
\end{aligned}
$$

for all $x \in \mathbb{X}, \psi, \chi \in \mathcal{PC}_0$.

Theorem 3.2 *Assume that the assumptions $(A_2) - (A_3)$ are fulfilled, then the system $(3) - (5)$ has a mild solution in $\mathcal{I}$ if $\Delta < 1$, where*

$$\Delta = \max\left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}, \mu_G\right],$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$.

Proof. Consider the operator $Q : \mathcal{PC}_T \to \mathcal{PC}_T$ defined in Theorem 3.1. We show that Q has a fixed point. It is easy to see that $Qy(t) \in \mathcal{PC}_T$. Let $\mathcal{B}_{r_0} := \{y \in \mathcal{PC}_T : \|y\|_{\mathcal{PC}_T} \le r_0\}$, where

$$r_0 \ge \max\Bigg[S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n}\frac{\alpha_j S_0Y_1T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)},\ \nu_G r_0,\ S_0\nu_G r_0 + T_0S_0\nu_H r_0 \\ + \sum_{j=1}^{n}\frac{\alpha_j S_0\nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\Bigg] + \frac{S_0T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0]r_0, \tag{9}$$

where $Y_1 = \|\phi(0)\| + \nu_{g_1} r_0$, $Z_1 = \|\varphi(0)\| + \nu_{g_2} r_0$. It is clear that $\mathcal{B}_{r_0}$ is a closed, bounded and convex subset of $\mathcal{PC}_T$. Let $y \in \mathcal{B}_{r_0}$, then for $t \in [0, t_1]$, we have

$$\begin{aligned}\|Qy(t)\|_{\mathbb{X}} \le& \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}(\|\phi(0)\| + \|g_1(y)\|_{\mathbb{X}}) + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\varphi(0)\| + \|g_2(y)\|_{\mathbb{X}})ds \\ &+ \sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\phi(0)\| + \|g(y)\|_{\mathbb{X}})ds \\ &+ \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|_{\mathbb{X}}ds \\ \le& S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n}\frac{\alpha_j S_0Y_1T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0]r_0.\end{aligned}$$

For $t \in \cup_{k=1}^m (t_k, s_k]$, we get

$$\|Qy(t)\|_{\mathbb{X}} \le \|G_k(t, y_t)\|_{\mathbb{X}} \le \nu_G r_0, \quad k = 1, 2, 3, \dots, m.$$

Similarly, for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we get

$$\begin{aligned}\|Qy(t)\|_{\mathbb{X}} \le& \|\mathcal{S}_{\beta,\gamma_j}(t-s_k)\|_{\mathcal{L}}\|G_k(s_k, y_{s_k})\|_{\mathbb{X}} + \int_{s_k}^t \|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}}\|H_k(s_k, y_{s_k})\|_{\mathbb{X}}ds \\ &+ \sum_{j=1}^{n}\alpha_j\int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}}\|G_k(s_k, y_{s_k})\|_{\mathbb{X}}ds \\ &+ \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, y_s, K(y_s))\|_{\mathbb{X}}ds \\ \le& S_0\nu_G r_0 + T_0S_0\nu_H r_0 + \sum_{j=1}^{n}\frac{\alpha_j S_0\nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0]r_0.\end{aligned}$$

We conclude by (9) that $\|Qy\|_{\mathcal{PC}_T} \le r_0$. Thus we conclude that $Q(\mathcal{B}_{r_0}) \subseteq \mathcal{B}_{r_0}$. Next, we show that Q is a condensing operator. Let us decompose Q by $Q = Q_1 + Q_2$, where $Q_1y(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^m (t_k, s_k]$ and

$$Q_1y(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ +\sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds, \quad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t-s_k)G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k)H_k(s_k, y_{s_k})ds \\ +\sum_{j=1}^{n}\alpha_j\int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s-s_k)G_k(s_k, y_{s_k})ds,\ t \in \cup_{k=1}^m (s_k, t_{k+1}], \end{cases} \tag{10}$$

and

$$Q_2y(t) = \begin{cases} \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s,y_s,K(y_s))ds, & t \in [0,t_1]; \\ \int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s,y_s,K(y_s))ds, & t \in \cup_{k=1}^m (s_k,t_{k+1}]. \end{cases} \tag{11}$$

First, we show that Q_1 is continuous, so consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$, then for $t \in [0,t_1]$, we get

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\|_{\mathbb{X}} \leq & S_0\|g_1(y^n) - g_1(y)\|_{\mathbb{X}} + S_0T_0\|g_2(y^n) - g_2(y)\|_{\mathbb{X}} \\ & + \sum_{j=1}^n \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\|g_1(y^n) - g_1(y)\|_{\mathbb{X}}. \end{aligned}$$

For $t \in \cup_{k=1}^m (s_k,t_{k+1}]$, we obtain

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\|_{\mathbb{X}} \leq & S_0\|G_k(s_k,y^n_{s_k}) - G_k(s_k,y_{s_k})\|_{\mathbb{X}} \\ & + S_0T_0\|H_k(s_k,y^n_{s_k}) - H_k(s_k,y_{s_k})\|_{\mathbb{X}} \\ & + \sum_{j=1}^n \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\|G_k(s_k,y^n_{s_k}) - G_k(s_k,y_{s_k})\|_{\mathbb{X}}. \end{aligned}$$

By continuity of G_k, H_k, g_1 and g_2, we have $\|Q_1y^n - Q_1y\|_{\mathcal{PC}_T} \to 0$ as $n \to \infty$. Hence Q_1 is continuous. Let $x, y \in \mathcal{PC}_T$. As we have done in Theorem 3.1 for $t \in [0,t_1]$, we have

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \left[S_0\mu_{g_1} + T_0S_0\mu_{g_2} + \sum_{j=1}^n \frac{\alpha_j S_0 \mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right]\|x-y\|_{\mathcal{PC}_T}.$$

For $t \in \cup_{k=1}^m (t_k,s_k]$, we get

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \|G_k(s_k,x_{s_k}) - G_k(s_k,y_{s_k})\|_{\mathbb{X}} \leq \mu_G\|x-y\|_{\mathcal{PC}_T}, \quad k = 1,2,\ldots,m,$$

and for $t \in \cup_{k=1}^m (s_k,t_{k+1}]$, we obtain

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \left[S_0\mu_G + T_0S_0\mu_H + \sum_{j=1}^n \frac{\alpha_j S_0 \mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right]\|x-y\|_{\mathcal{PC}_T}.$$

Gathering the above results, we have $\|Q_1x - Q_1y\|_{\mathcal{PC}_T} \leq \Delta\|x-y\|_{\mathcal{PC}_T}$. Hence, Q_1 is a contraction mapping.

Next, we show that Q_2 is completely continuous. First, we verify that Q_2 is continuous, so we consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$ as $n \to \infty$, then for $t \in [0,t_1]$, we get

$$\begin{aligned} & \|Q_2y^n(t) - Q_2y(t)\|_{\mathbb{X}} \\ & \qquad \leq \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s,y^n_s,K(y^n_s)) - F(s,y_s,K(y_s))\|_{\mathbb{X}}ds, \end{aligned}$$

for $t \in \cup_{k=1}^m (s_k,t_{k+1}]$, we obtain

$$\begin{aligned} & \|Q_2y^n(t) - Q_2y(t)\|_{\mathbb{X}} \\ & \qquad \leq \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s,y^n_s,K(y^n_s)) - F(s,y_s,K(y_s))\|_{\mathbb{X}}ds. \end{aligned}$$

By continuity of F, we get $\|Q_2y^n - Q_2y\|_{\mathcal{PC}_T} \to 0$ as $n \to \infty$. Hence Q_2 is continuous. Further, we show that Q_2 is a family of equi-continuous functions. Let $l_2, l_1 \in [0, t_1]$ such that $0 \leq l_1 < l_2 \leq t_1$, we have

$$\begin{aligned}\|Q_2y(l_2)-&Q_2y(l_1)\|_{\mathbb{X}}\\ &\leq \int_0^{l_1} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s) - (g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_1-s)\|_{\mathcal{L}}\|F(s,y_s,K(y_s))\|_{\mathbb{X}}ds\\ &\quad + \int_{l_1}^{l_2} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s)\|_{\mathcal{L}}\|F(s,y_s,K(y_s))\|_{\mathbb{X}}ds\\ &\leq S_0\left[\int_0^{l_1}\left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)}\right)ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0\\ &\leq \frac{S_0}{\Gamma(2+\beta)}\left[\left|(l_2^{1+\beta} - l_1^{1+\beta}) - (l_2-l_1)^{1+\beta}\right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0.\end{aligned}$$

For $l_2, l_1 \in \cup_{k=1}^m (s_k, t_{k+1}]$ such that $s_k \leq l_1 < l_2 \leq t_{k+1}$, we have

$$\begin{aligned}&\|Q_2y(l_2) - Q_2y(l_1)\|_{\mathbb{X}}\\ &\leq \int_{s_k}^{l_1} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s) - (g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_1-s)\|_{\mathcal{L}}\|F(s,y_s,K(y_s))\|_{\mathbb{X}}ds\\ &\quad + \int_{l_1}^{l_2} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s)\|_{\mathcal{L}}\|F(s,y_s,K(y_s))\|_{\mathbb{X}}ds\\ &\leq S_0\left[\int_{s_k}^{l_1}\left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)}\right)ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0\\ &\leq \frac{S_0}{\Gamma(2+\beta)}\left[\left|((l_2-s_k)^{1+\beta} - (l_1-s_k)^{1+\beta}) - (l_2-l_1)^{1+\beta}\right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)}\right][\nu_F + \nu_F^0 k^0]r_0,\end{aligned}$$

from aforemention inequalities we conclude that $\|Q_2y(l_2) - Q_2y(l_1)\|_{\mathcal{PC}_T} \to 0$ as $l_2 \to l_1$ for $t \in [0, T]$. This shows that Q_2 is a family of equi-continuous functions.

Finally, we will show that $\mathbb{Y} = \{Q_2y(t) : y \in \mathbb{B}_{r_0}\}$ is precompact in $\mathbb{X}$. Let $t > 0$ be fixed and let $y^n \in \mathbb{B}_{r_0}$, $\{y^n\}$ be a bounded sequence in $\mathcal{PC}_T$. Let $\mathbb{Y} = \{Q_2y^n(t) : y^n \in \mathbb{B}_{r_0}\}$ be a bounded sequence in $\mathbb{B}_{r_0}$. Hence, for any $t^* \in \cup_{k=0}^m (s_k, t_{k+1}]$, the sequence $\{y^n(t^*)\}$ is bounded in $\mathbb{B}_{r_0}$. Since F is compact, it has a convergent subsequence such that

$$F(t^*, y^n_{t^*}, K(y^n_{t^*})) \to F(t^*, y_{t^*}, K(y_{t^*})),$$

or

$$\|F(t^*, y^n_{t^*}, K(y^n_{t^*})) - F(t^*, y_{t^*}, K(y_{t^*}))\|_{\mathbb{X}} \to 0 \text{ as } n \to \infty.$$

Using the bounded convergence theorem, we can conclude that

$$(Q_2y^n)(t) \to (Q_2y)(t), \text{ in } \mathbb{B}_{r_0}.$$

This proves that Q_2 is a compact operator. Therefore Q_1 is a continuous and contraction operator and Q_2 is a completely continuous operator, hence $Q = Q_1 + Q_2$ is a condensing map on $\mathcal{B}_{r_0}$. Finally, by Theorem 2.2, we infer that there exists a mild solution of the system $(3) - (5)$ in $\mathcal{B}_{r_0}$.

4 Example

In this section, we provide an example to illustrate the feasibility of the established results. Set $\mathbb{X} = L^2(\mathbb{R}^n)$, then $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], L^2(\mathbb{R}^n))$. Let $\beta, \gamma_J > 0$ for $j = 1, 2, 3, \dots, n$ be given, satisfying $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$ and $\tau \in \mathbb{R}$ such that $\tau > 0$. We consider the following system

$$\partial_t^{1+\beta} u(t,x) + \sum_{j=1}^{n} \alpha_j \partial_t^{\gamma_j} u(t,x) = \Delta u(t,x) + \frac{u_t(\theta,x)}{50} + \int_{-\tau}^{t} \cos(t-\xi)\frac{u_t(\theta,x)}{25} d\xi, \tag{12}$$

for all $(t,x) \in \cup_{k=0}^{m}(s_k, t_{k+1}] \times [0,1]$,

$$G_k(t, u_t(\theta,x)) = \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi,$$

$$H_k(t, u_t(\theta,x)) = \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi, \qquad t \in \cup_{k=1}^{m}(t_k, s_k], \tag{13}$$

$$u(\theta,x) + \sum_{r=1}^{q} a_r y(t_r) = \phi(\theta,x), \quad u'(\theta,x) + \sum_{r=1}^{q} b_r y(t_r) = \varphi(\theta,x), \tag{14}$$

where $a_r, b_r \in \mathbb{R}, \theta \in [-\tau, 0]$. The points $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = 1$ are prefix numbers, $\partial_t^{1+\beta}$ denotes the Caputo derivative of order $(1+\beta)$ and Δ is the Laplacian with a maximal domain $\{v \in \mathbb{X} : v \in H^2(\mathbb{R}^n)\}$. The history function $u_t(\theta,x) : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $u_t(\theta,x) = u(t+\theta,x), \theta \in [-\tau,0]$. Set $y(t)(x) = u(t,x)$, $g_1(x) = \sum_{r=1}^{p} a_r x(t_r)$, $g_2(x) = \sum_{r=1}^{p} b_r x(t_r)$ and $\phi(\theta)(x) = \phi(\theta,x)$, $(\theta,x) \in [-\tau,0] \times [0,1]$. Now, we have $F(t,\psi,K(\psi)) = \frac{\psi}{50} + \int_{-\tau}^{t} \cos(t-\xi)\frac{\psi}{5^2} d\xi$, $G_k(t,\psi) = \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$, $H_k(t,\psi) = \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$. Now, we observe that the system (12) – (14) has the abstract form of the system (3) – (5). Moreover, for $t \in [0,1], \psi_i, \chi_i \in \mathcal{PC}_0, i = 1,2$ and $x, y \in \mathbb{X}$, we have

$$\|F(t,\psi_1,K(\chi_1)) - F(t,\psi_2,K(\chi_2))\| \le \frac{1}{50}\|\psi_1 - \psi_2\| + \frac{1}{25}\|\chi_1 - \chi_2\|,$$

$$\|G_k(t,\chi_1) - G_k(t,\chi_2)\| \le \frac{2}{25}\|\chi_1 - \chi_2\|; \ \|H_k(t,\chi_1) - H_k(t,\chi_2)\| \le \frac{1}{25}\|\chi_1 - \chi_2\|,$$

$$\|g_1(x) - g_1(y)\|_{\mathbb{X}} \le qa\|x-y\|_{\mathbb{X}}; \ \|g_2(x) - g_2(y)\|_{\mathbb{X}} \le qb\|x-y\|_{\mathbb{X}},$$

where $a = \max_{1 \le r \le q} |a_r|$ and $b = \max_{1 \le r \le q} |b_r|$. Thus the assumptions (A_1) and (A_2) are satisfied. On the other hand, it follows from the theory of cosine families that Δ generates a bounded cosine function $\{C(t)\}_{t \ge 0}$ on $L^2(\mathbb{R}^n)$. Moreover, by Theorem 2.1 the operator Δ in equation (12) generates a bounded $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t \ge 0}$-resolvent family. Let $S_0 = \sup_{t\in[0,1]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. Now, by Theorem 3.1 if

$$\max\left[S_0 d + S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d}{\Gamma(2+\beta-\gamma_j)} + \frac{3S_0}{50\Gamma(2+\beta)}, \frac{1}{25}\right] < 1,$$

where $d = \max\{qa, \frac{2}{25}\}$, $e = \max\{qb, \frac{2}{25}\}$, then the system (12) – (14) admits a unique mild solution.

5 Conclusion

In this paper, an approach has been developed concerning the existence and uniqueness of mild solutions for the system $(3)-(5)$ using the Banach fixed point theorem and condensing map theorem. The system $(3)-(5)$ involves abrupt forces(impulsive effects), hence our results generalize the results of Pardo and Lizama studied in [21]. Thus, our results are more general and interesting.

Acknowledgment

The work of the first author is supported by the "Ministry of Human Resource and Development, India under grant number: MHR-02-23-200-44".

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