Volume 18, Number 1, 2018 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 18 Number 1 2018

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 ISSN 1562-8353 Printed in Ukraine
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NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 18, No. 1, 2018

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InforMath Publishing Group
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NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 18 Number 1 2018

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Founded by A.A. Martynyuk in 2001.
Registered in Ukraine Number: KB 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2017: SNIP – 0.707, SJR – 0.316

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

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PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 4 issues in 2018, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (mailto:journals@ebsco.com), or website of the Journal: http://e-ndst.kiev.ua. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 1–11

Weak Solutions to Implicit Differential Equations Involving the Hilfer Fractional Derivative

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Received: October 28, 2016; Revised: October 26, 2017

Abstract: In this paper, the authors present some existence results for weak solutions to some functional implicit fractional differential equations of Hilfer type, by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness.

Keywords: *functional differential equation; left-sided mixed Pettis Riemann-Liouville integral of fractional order; Hilfer fractional derivative; implicit; weak solution; fixed point.*

Mathematics Subject Classification (2010): 26A33, 36A08, 34A09.

1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, bio-engineering, and other applied sciences [15, 24]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas *et al.* [1–3], Samko *et al.* [23], Kilbas *et al.* [18], and Zhou [27].

The notion of a measure of weak noncompactness was introduced by De Blasi [13]. The strong measure of noncompactness was developed first by Banaś and Goebel [7] and subsequently developed and used in many papers; see, for example, Akhmerov *et al.* [5], Alvàrez [6], Benchohra *et al.* [11], Guo *et al.* [14], and the references therein. In [11, 21], the authors considered some existence results by applying the techniques of the measure

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of noncompactness. Recently, several researchers obtained other results by applying the technique of measure of weak noncompactness; see [3, 9, 10], and the references therein.

Implicit functional differential equations have been considered by many authors [4, 8, 26]. Our intention is to extend the results to implicit differential equations of fractional order. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with a Hilfer fractional derivative; see, for example, [15–17, 25]. In this paper, we discuss the existence of weak solutions to the problem of implicit Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = f(t, u(t), (D_0^{\alpha,\beta}u)(t)), \ t \in I := [0, T], \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta - \alpha\beta$, $T > 0$, $\phi \in E$, $f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual space E^*, such that E is the dual space of a weakly compactly generated Banach space X, $I_0^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the generalized Riemann-Liouville derivative operator of order α and type β introduced by Hilfer in [15]. Our goal in this work is to give some existence results for implicit Hilfer fractional differential equations in Banach spaces.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|_E.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. We denote by $AC^1(I)$, the space defined by

$$AC^1(I) := \{w : I \to E : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we mean the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0, T] \to E : t^{1-\gamma}w(t) \in C\}$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t \in I} \|t^{1-\gamma}w(t)\|_E,$$

and

$$C_\gamma^1(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\}$$

with the norm

$$\|w\|_{C_\gamma^1} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

In what follows, we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$. Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology.

Definition 2.1 A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $\{u_n\}$ in E with $u_n \to u$ in (E, w), we have $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3 ([22]) The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue (by definition, $u_J = \int_J u(s)ds$).

Let $P(I, E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I, E)$ be the Banach space of Lebesgue integrable functions $u : I \to E$. Define the class $P_1(I, E)$ by

$$P_1(I, E) = \{u \in P(I, E) : \varphi(u) \in L^1(I, E) \ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \leq 1} \int_0^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [22, Theorem 3.4 and Corollary 3.41]).

Proposition 2.1 ([22]) *If $u \in P_1(I, E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in P_1(I, E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral. Now, we give some results and properties of fractional calculus.

Definition 2.4 ([2,18,23]) The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_\theta^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} w(s) ds \ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt,\ \xi > 0.$$

Notice that for all r, r_1, $r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t)\ for\ a.e.\ t \in I.$$

Definition 2.5 ([2,18,23]) The Riemann-Liouville fractional derivative of order $r \in (0, 1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t - s)^{-r} w(s) ds\ for\ a.e.\ t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$, and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator given by

$$(D_0^r I_0^r w)(t) = w(t) \; for \; all \; t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the composition

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r}w)(0^+)}{\Gamma(r)} t^{r-1} \text{ for all } t \in (0,T]$$

is proved in [23].

Definition 2.6 ([2,18,23]) The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} ({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w \right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s) ds \; for \; a.e. \; t \in I. \end{aligned}$$

In [15], Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as special cases (see also [16,17,25]).

Definition 2.7 (Hilfer derivative) Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)} \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w \right)(t) \quad for \; a.e. \; t \in I. \tag{2}$$

Properties of the Hilfer derivative. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w \right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w \right)(t) \quad for \; a.e. \; t \in I.$$

   Moreover, the parameter γ satisfies

$$\gamma \in (0,1], \; \gamma \geq \alpha, \; \gamma > \beta, \quad \text{and} \quad 1-\gamma < 1 - \beta(1-\alpha).$$

2. The generalization (2) for $\beta = 0$ coincides with the Riemann-Liouville derivative, and for $\beta = 1$, with the Caputo derivative:

$$D_0^{\alpha,0} = D_0^{\alpha} \quad \text{and} \quad D_0^{\alpha,1} = {}^c D_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t) \; for \; a.e. \; t \in I.$$

   Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t) \; for \; a.e. \; t \in I.$$

4. If $D_0^{\gamma} w$ exists and is in $L^1(I)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1} \; for\ a.e.\ t \in I.$$

Corollary 2.1 *Let $h \in C_{\gamma}(I)$. A function $u \in L^1(I, E)$ is a solution of the problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t), & t \in I := [0, T], \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

if and only if u satisfies the Volterra integral equation

$$w(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

From the above corollary and Lemma 5.1 in [4], we have the following lemma.

Lemma 2.1 *Let $f : I \times E \times E \to E$ be such that $f(\cdot, u(\cdot), v(\cdot)) \in C_{\gamma}(I)$ for any u, $v \in C_{\gamma}(I)$. Then problem (1) is equivalent to the problem of obtaining the solution of the equation*

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} g)(t), g(t)\right);$$

moreover, if $g(\cdot) \in C_{\gamma}$ is the solution of this equation, then

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} g)(t).$$

Remark 2.1 Let $h \in P_1([I, E)$. For every $\varphi \in E^*$, we have

$$\varphi(I_0^{\alpha} h)(t) = (I_0^{\alpha} \varphi h)(t) \; for\ a.e.\ t \in I.$$

Definition 2.8 ([13]) Let E be a Banach space, Ω_E be the class of all bounded subsets of E, and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact } \Omega \subset E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A \subset B$ implies $\beta(A) \le \beta(B)$;

(b) $\beta(A) = 0$ if and only if A is weakly relatively compact;

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(d) $\beta(\overline{A}^{\omega}) = \beta(A)$ where $\overline{A}^{\omega}$ denotes the weak closure of A;

(e) $\beta(A + B) \le \beta(A) + \beta(B)$;

(f) $\beta(\lambda A) = |\lambda| \beta(A)$;

(g) $\beta(conv(A)) = \beta(A)$;

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.2 *Let E be a normed space and let $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$, let

$$V(t) = \{v(t) : v \in V\},\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 2.2 ([14]) *Let H be a bounded and equicontinuous subset of C. Then the function $t \to \beta(H(t))$ is continuous on I,*

$$\beta_C(H) = \max_{t\in I} \beta(H(t))$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purposes, we will need the following fixed point theorem.

Theorem 2.1 ([20]) *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(I, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \text{ implies } V \text{ is relatively weakly compact}, \tag{3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of Weak Solutions

Let us start by defining what we mean by a weak solution of the problem (1).

Definition 3.1 By a weak solution of the problem (1) we mean a measurable function $u \in C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+) = \phi$ and the equation $(D_0^{\alpha,\beta}u)(t) = f(t, u(t), (D_0^{\alpha,\beta}u)(t))$ on I.

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I$, the functions $v \to f(t, v, w)$ and $w \to f(t, v, w)$ are weakly sequentially continuous.

(H_2) For each v, $w \in E$, the function $t \to f(t, v, w)$ is Pettis integrable a.e. on I.

(H_3) There exists $p \in C(I, [0, \infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t, u, v))| \leq \frac{p(t)\|\varphi\|}{1 + \|\varphi\| + \|u\|_E + \|v\|_E} \text{ for a.e. } t \in I \text{ and each } u,\ v \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f(t, B, D_0^{\alpha,\beta} B) \leq t^{1-r} p(t)\beta(B),$$

where $D_0^{\alpha,\beta} B = \{D_0^{\alpha,\beta} w : w \in B\}$.

Set

$$p^* = \sup_{t \in I} p(t).$$

Theorem 3.1 *Assume that conditions (H_1)–(H_4) hold. If*

$$L := \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{4}$$

then the problem (1) has at least one weak solution defined on I.

Proof. Consider the operator $N : C_\gamma \to C_\gamma$ defined by

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), \tag{5}$$

where $g \in C_\gamma$ is given by

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

First notice that by hypothesis, for each $g \in C_\gamma$, the function

$$t \mapsto (t-s)^{\alpha-1} g(s)$$

is Pettis integrable over I, and the function

$$t \mapsto f\left(t, \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right) \ for\ a.e.\ t \in I,$$

is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that

$$R > \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)},$$

and consider the set

$$Q = \Big\{ u \in C_\gamma : \|u\|_C \leq R \text{ and } \|t_2^{1-\gamma} u(t_2) - t_1^{1-\gamma} u(t_1)\|_E \\ \leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} (t_2 - t_1)^\alpha + \frac{p^*}{\Gamma(\alpha)} \int_0^{t_1} |t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}| ds \Big\}.$$

Clearly, the subset Q is closed, convex, and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.1. The proof will be given in several steps.

Step 1. *N maps Q into itself.*

Let $u \in Q$ and $t \in I$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|t^{1-\gamma}(Nu)(t)\|_E = |\varphi(t^{1-\gamma}(Nu)(t))|$. Thus,

$$\|t^{1-\gamma}(Nu)(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds\right),$$

where $g \in C_\gamma$ is given by

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

Then,

$$\begin{aligned}
\|t^{1-\gamma}(Nu)(t)\|_E &\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\varphi(g(s))|ds \\
&\leq \frac{p^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\
&< R.
\end{aligned}$$

Next, take t_1, $t_2 \in I$ with $t_1 < t_2$, and let $u \in Q$ with

$$t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1) \neq 0.$$

Then, there exists $\varphi \in E^*$ such that

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))|$$

and $\|\varphi\| = 1$. Hence,

$$\begin{aligned}
&\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))| \\
&\leq \varphi\left(t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds\right),
\end{aligned}$$

where $g \in C_\gamma$ satisfies

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

Therefore,

$$\begin{aligned}
&\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|\varphi(g(s))|}{\Gamma(\alpha)}ds \\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|\varphi(g(s))|}{\Gamma(\alpha)}ds \\
&\leq t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{p(s)}{\Gamma(\alpha)}ds \\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p(s)}{\Gamma(\alpha)}ds \\
&\leq \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha + \frac{p^*}{\Gamma(\alpha)}\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|ds.
\end{aligned}$$

Hence, $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*

Let $\{u_n\}$ be a sequence in Q such that $\{u_n(t)\} \to u(t)$ in (E, ω) for each $t \in I$. Fix $t \in I$; since f satisfies (H_1), we have $f(t, u_n(t), (D_0^{\alpha,\beta} u_n)(t))$ converges weakly uniformly to $f(t, u(t), (D_0^{\alpha,\beta} u)(t))$. Hence, by the Lebesgue dominated convergence theorem for Pettis integrals, $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E, ω) for each $t \in I$. Thus, $N(u_n) \to N(u)$, and so $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (3) holds.*

Let V be a subset of Q such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Clearly,

$$V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}) \text{ for each } \ t \in I.$$

Furthermore, since V is bounded and equicontinuous, by Lemma 3 in [12] the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2.2, and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned} t^{1-\gamma} v(t) &\leq \beta(t^{1-\gamma}(NV)(t) \cup \{0\}) \\ &\leq \beta(t^{1-\gamma}(NV)(t)) \\ &\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} p(s) \beta(V(s)) ds \\ &\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} s^{1-\gamma} p(s) v(s) ds \\ &\leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \|v\|_C. \end{aligned}$$

Thus,

$$\|v\|_C \leq L \|v\|_C.$$

From (4), we see that $\|v\|_C = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$. By [19, Theorem 2], V is weakly relatively compact in C. Applying Theorem 2.1, we conclude that N has a fixed point that is a weak solution of the problem (1).

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \dots, u_n, \dots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be our Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

As an application of our results, we consider the Hilfer fractional differential equation

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t, u(t), (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t)), & t \in [0,1], \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = (0, 0, \dots, 0, \dots), \end{cases} \tag{6}$$

where

$$f_n(t, u(t), v(t)) = \frac{ct^2}{1 + \|u(t)\|_E + \|v(t)\|_E} \frac{u_n(t)}{e^{t+4}}, \quad t \in [0, 1],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots), \text{ and } c := \frac{e^4}{8}\Gamma\left(\frac{1}{2}\right).$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots).$$

Clearly, the function f is continuous.

For each u, $v \in E$ and $t \in [0, 1]$, we have

$$\|f(t, u, v)\|_E \leq ct^2 \frac{1}{e^{t+4}}.$$

Hence, condition (H_3) is satisfied with $p^* = ce^{-4}$. We shall show that condition (4) holds with $T = 1$. In fact,

$$\frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{2ce^{-4}}{\Gamma(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all the conditions of Theorem 3.1 are satisfied, and so problem (6) has at least one weak solution defined on $[0, 1]$.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 12–21

Monoaxial Attitude Stabilization of a Rigid Body under Vanishing Restoring Torque

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Received: June 19, 2017;   Revised: December 14, 2017

Abstract: The paper deals with the problem of monoaxial attitude stabilization of a rigid body. The possibility of implementing such a control system in which the restoring torque tends to zero as time increases is studied. With the aid of the Lyapunov direct method and the differential inequalities theory, conditions under which an equilibrium position of the body is stable with respect to all variables as well as with respect to a part of variables are derived. The results of a numerical modeling are presented to demonstrate the effectiveness of the proposed approaches.

Keywords: *rigid body; monoaxial attitude stabilization; dissipation; asymptotic stability; Lyapunov function; differential inequality.*

Mathematics Subject Classification (2010): 34H15, 70Q05, 93C10.

1 Introduction

In problems of a rigid body attitude control, restoring torques are usually the basis of control system functioning. However, attitude stabilization of a body is impossible without damping torques ensuring suppression of a body oscillations in a neighborhood of a stable equilibrium position. Therefore, the question how to create a damping torque and to design a specific damping mechanism is one of the main problems that should be solved for practical realization of attitude control systems [6, 7, 9, 14, 20, 24]. At the same time, due to limited resources of control systems based on jet propulsion, there arises a natural question on the possibility of implementing such a control system in which the restoring torque tends to zero as time increases.

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A more general formulation of the problem suggests that a mechanical system with dissipative and potential forces is given. Let the system admit an asymptotically stable equilibrium position. Consider the case of an evolution of the potential forces. We assume that the evolution consists of the appearance of a scalar positive time-varying multiplier at the vector of these forces. The issue of preservation of stability of the equilibrium position despite the evolution of potential forces is stated.

The stability problem in mechanical systems with a nonstationary parameter at potential forces was considered in many works, see, for example, [1, 3, 10, 13, 15, 22, 23, 26, 28] and the references cited therein. However, it should be noted that a few results were obtained for the case of vanishing potential forces.

In this contribution, the issue of monoaxial attitude stabilization of a rigid body is studied. It is assumed that the body is under the action of a time-invariant essentially nonlinear dissipative torque and a time-varying restoring torque that vanishes as time increases. Using the differential inequalities theory [11, 16–18] and approaches proposed in [1, 3, 23], conditions providing stability with respect to all variables as well as with respect to a part of variables of an equilibrium position of the body are derived.

2 Statement of the Problem

Consider a rigid body rotating about its mass center O with angular velocity $\boldsymbol{\omega}$. Assume that the axes $Oxyz$ are principal central axes of inertia of the body. Differential equations governing the attitude motion of the body under control torque $\mathbf{M}$ have the following form

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{M}, \tag{1}$$

where $\mathbf{J} = \mathrm{diag}\{A, B, C\}$ is a body inertia tensor in the axes $Oxyz$.

Let unit vectors $\mathbf{s}$ and $\mathbf{r}$ be given, the vector $\mathbf{s}$ be constant in the inertial space and the vector $\mathbf{r}$ be constant in the body-fixed frame. Then the vector $\mathbf{s}$ rotates with respect to the coordinate system $Oxyz$ with angular velocity $-\boldsymbol{\omega}$. Hence,

$$\dot{\mathbf{s}} = -\boldsymbol{\omega} \times \mathbf{s}. \tag{2}$$

Thus, we will consider the differential system consisting of the Euler dynamic equations (1) and the Poisson kinematic equations (2).

Let the torque $\mathbf{M}$ be a sum of the dissipative component $\mathbf{M}_d$ and the restoring one $\mathbf{M}_r$: $\mathbf{M} = \mathbf{M}_d + \mathbf{M}_r$. We will assume that the dissipative torque is defined by the formula $\mathbf{M}_d = -\partial W(\boldsymbol{\omega})/\partial \boldsymbol{\omega}$, where $W(\boldsymbol{\omega})$ is a continuously differentiable for $\boldsymbol{\omega} \in \mathbb{R}^3$ positive definite homogeneous function of the order $\nu + 1$, $\nu > 1$. It should be noted that mechanical systems with essentially nonlinear dissipative forces were considered, for instance, in [19, 21]. In particular, such type forces arise when a body rotates in a viscous medium [19]. Moreover, it is worth mentioning that essentially nonlinear control laws are more robust with respect to the impact of delay and nonstationary perturbations than linear ones, see [2, 5].

The restoring torque $\mathbf{M}_r$ should be chosen such that the torque $\mathbf{M}$ ensures monoaxial stabilization of a rigid body [29]: the system of equations (1), (2) should admit the asymptotically stable equilibrium position

$$\boldsymbol{\omega} = \mathbf{0}, \qquad \mathbf{s} = \mathbf{r}. \tag{3}$$

From the results of [25, 29] it follows that the torque $\mathbf{M}_r$ can be determined by the formula

$$\mathbf{M}_r = -a\|\mathbf{s} - \mathbf{r}\|^{\mu-1}\mathbf{s} \times \mathbf{r},$$

where $\mu \geq 1$, $a > 0$, and $\|\cdot\|$ denotes the Euclidean norm of a vector.

Next, consider the case where the restoring torque evolves with time, and the evolution is expressed in the appearance of a scalar multiplier $h(t)$ at the vector of the torque. Thus, system (1) can be rewritten as follows

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = -\frac{\partial W(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} - h(t)a\|\mathbf{s} - \mathbf{r}\|^{\mu-1}\mathbf{s} \times \mathbf{r}. \tag{4}$$

Assume that $h(t)$ is a positive and continuously differentiable for $t \geq 0$ function, and $h(t) \to 0$ as $t \to +\infty$. Hence, the restoring torque vanishes as time increases. We will look for conditions under which the equilibrium position (3) of system (2), (4) is stable with respect to all or a part of variables.

3 Main Results

First, according to the approach proposed in [23], construct a Lyapunov function in the form

$$V_1 = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega} + \frac{ah(t)}{\mu+1}\|\mathbf{s} - \mathbf{r}\|^{\mu+1}.$$

Differentiating the function with respect to system (2), (4), we obtain

$$\dot{V}_1 = -(\nu+1)W(\boldsymbol{\omega}) + \frac{a\dot{h}(t)}{\mu+1}\|\mathbf{s} - \mathbf{r}\|^{\mu+1} \leq \varphi(t)V_1,$$

where $\varphi(t) = \max\left\{0;\, \dot{h}(t)/h(t)\right\}$.

Thus, on the basis of the theory of differential inequalities, see [11, 16], we arrive at the following theorem.

Theorem 3.1 *If there exists a constant $L > 0$ such that $\int_0^t \varphi(\tau)d\tau \leq L$ for $t \geq 0$, then the equilibrium position* (3) *of system* (2), (4) *is stable with respect to $\boldsymbol{\omega}$.*

Corollary 3.1 *If $\dot{h}(t) \leq 0$ for $t \geq 0$, then the equilibrium position* (3) *of system* (2), (4) *is stable with respect to $\boldsymbol{\omega}$.*

Next, we will show that with the aid of more precise estimates of the derivative of V_1, conditions of the asymptotic stability with respect to $\boldsymbol{\omega}$ of the equilibrium position (3) can be derived.

Let $\dot{h}(t) \leq 0$ for $t \geq 0$. Denote

$$z_1 = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega}, \quad z_2 = \frac{ah(t)}{\mu+1}\|\mathbf{s} - \mathbf{r}\|^{\mu+1}.$$

Then $V_1 = z_1 + z_2$.

Choose a positive number Δ. We obtain

$$\dot{V}_1 \leq -cz_1^{\nu+1} - \psi(t)z_2^{\nu+1}$$

for $t \geq 0$, $z_1 \geq 0$, $0 \leq z_2 \leq \Delta$, where

$$c = (\nu+1) \min_{\|\boldsymbol{\omega}\|=1} \frac{W(\boldsymbol{\omega})}{(\boldsymbol{\omega}^\top \mathbf{J} \boldsymbol{\omega}/2)^{\frac{\nu+1}{2}}} > 0, \qquad \psi(t) = -\frac{\dot{h}(t)}{h(t)} \Delta^{\frac{1-\nu}{2}}.$$

Hence, the differential inequality

$$\dot{V}_1 \leq -\tilde{\varphi}(t) V_1^{\frac{\nu+1}{2}} \tag{5}$$

holds in a neighborhood of the equilibrium position (3) and for all $t \geq 0$. Here

$$\tilde{\varphi}(t) = \min_{u_1 \geq 0,\ u_2 \geq 0,\ u_1+u_2=1} \left(c\, u_1^{\nu+1} + \psi(t)\, u_2^{\nu+1} \right).$$

It can be shown that

$$\tilde{\varphi}(t) = \frac{c\psi(t)}{\left(c^{\frac{2}{\nu-1}} + \psi^{\frac{2}{\nu-1}}(t) \right)^{\frac{\nu-1}{2}}}.$$

Assume that for a solution $(\boldsymbol{\omega}^\top(t), \mathbf{s}^\top(t))^\top$ of (2), (4) the condition

$$\frac{ah(t)}{\mu+1} \|\mathbf{s}(t) - \mathbf{r}\|^{\mu+1} \leq \Delta$$

is fulfilled on an interval $[t_0, t_1]$, where $0 \leq t_0 < t_1$. Then, integrating differential inequality (5), we obtain

$$\begin{gathered} \frac{1}{2} \boldsymbol{\omega}^\top(t) \mathbf{J} \boldsymbol{\omega}(t) + \frac{ah(t)}{\mu+1} \|\mathbf{s}(t) - \mathbf{r}\|^{\mu+1} = \hat{V}_1(t) \\ \leq \hat{V}_1(t_0) \left(1 + \frac{\nu-1}{2} \hat{V}_1^{\frac{\nu-1}{2}}(t_0) \int_{t_0}^{t} \tilde{\varphi}(\tau) d\tau \right)^{-\frac{2}{\nu-1}} \end{gathered} \tag{6}$$

for $t \in [t_0, t_1]$. Here $\hat{V}_1(t) = V_1(t, \boldsymbol{\omega}(t), \mathbf{s}(t))$.

Thus, we arrive at the following theorem.

Theorem 3.2 *If $\dot{h}(t) \leq 0$ for $t \geq 0$ and*

$$\int_0^t \tilde{\varphi}(\tau) d\tau \to +\infty \quad \text{as} \quad t \to +\infty, \tag{7}$$

then the equilibrium position (3) *of system* (2), (4) *is asymptotically stable with respect to $\boldsymbol{\omega}$.*

Example 3.1 Let the nonstationary multiplier $h(t)$ in system (4) be defined by the formula $h(t) = e^{-\beta t}$, where $\beta = \text{const} > 0$. Then, for any $\beta > 0$ and any $\Delta > 0$, we obtain $\tilde{\varphi}(t) \equiv \text{const} > 0$. Hence, the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to $\boldsymbol{\omega}$.

Remark 3.1 Function $\tilde{\varphi}(t)$ depends on the chosen number Δ. To guarantee that the equilibrium position is asymptotically stable with respect to $\boldsymbol{\omega}$, it is sufficient to find at least one value of Δ for which condition (7) is fulfilled.

Remark 3.2 It is easy to verify that, the smaller the value of Δ, the more precise estimate (6). However, decreasing the value of Δ, we narrow the domain of initial conditions of solutions of system (2), (4) for which the estimate can be applied.

Remark 3.3 The use of estimate (6) does not permit us to obtain conditions of stability with respect to $\mathbf{s}$.

Really, for any $\Delta > 0$, the inequality $\tilde{\varphi}(t) \leq \psi(t)$ holds for $t \geq 0$. Hence,

$$\frac{1}{h(t)}\left(1+\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\int_{t_0}^{t}\tilde{\varphi}(\tau)d\tau\right)^{-\frac{2}{\nu-1}}$$

$$\geq \frac{1}{h(t)}\left(1-\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\Delta^{\frac{1-\nu}{2}}\int_{t_0}^{t}\frac{\dot{h}(\tau)}{h(\tau)}d\tau\right)^{-\frac{2}{\nu-1}}$$

$$=\frac{1}{h(t)}\left(1-\frac{\nu-1}{2}\hat{V}_1^{\frac{\nu-1}{2}}(t_0)\Delta^{\frac{1-\nu}{2}}\log\frac{h(t)}{h(t_0)}\right)^{-\frac{2}{\nu-1}} \to +\infty \quad \text{as} \quad t\to+\infty.$$

Finally in this section, we consider one more approach to a Lyapunov function construction for system (2), (4) which permits us to find stability conditions not only with respect to $\boldsymbol{\omega}$, but also with respect to all variables.

Let

$$V_2 = \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega} + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \gamma h^\sigma(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top\mathbf{J}(\mathbf{s}\times\mathbf{r}),$$

where $\gamma > 0$, $\beta \geq 1$, $\sigma > 0$. Then there exist positive numbers α_1, α_2, α_3 such that

$$\alpha_1\|\boldsymbol{\omega}\|^2 + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} - \alpha_3\gamma h^\sigma(t)\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^\beta \leq V_2$$

$$\leq \alpha_2\|\boldsymbol{\omega}\|^2 + \frac{ah(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \alpha_3\gamma h^\sigma(t)\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^\beta.$$

Differentiating function V_2 with respect to system (2), (4), we obtain

$$\dot{V}_2 = -(\nu+1)W(\boldsymbol{\omega}) + \frac{a\dot{h}(t)}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1} + \sigma\gamma h^{\sigma-1}(t)\dot{h}(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top\mathbf{J}(\mathbf{s}\times\mathbf{r})$$

$$+\gamma h^\sigma(t)\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}(\mathbf{s}\times\mathbf{r})^\top\left(-\boldsymbol{\omega}\times\mathbf{J}\boldsymbol{\omega} - \frac{\partial W(\boldsymbol{\omega})}{\partial\boldsymbol{\omega}} - h(t)a\|\mathbf{s}-\mathbf{r}\|^{\mu-1}(\mathbf{s}\times\mathbf{r})\right)$$

$$+\gamma h^\sigma(t)\boldsymbol{\omega}^\top\mathbf{J}\,\frac{\partial\left(\|\mathbf{s}\times\mathbf{r}\|^{\beta-1}(\mathbf{s}\times\mathbf{r})\right)}{\partial\mathbf{s}}\left(-\boldsymbol{\omega}\times\mathbf{s}\right).$$

Assume that $\dot{h}(t) \leq 0$ for $t \geq 0$. It is easy to verify that one can choose positive constants α_4, α_5, α_6 and δ such that the inequality

$$\dot{V}_2 \leq -\alpha_4\left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma h^{\sigma+1}(t)\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}\right) + \alpha_5\gamma h^{\sigma-1}(t)\dot{h}(t)\|\mathbf{s}-\mathbf{r}\|^\beta\|\boldsymbol{\omega}\|$$

$$+\alpha_6\gamma h^\sigma(t)\left(\|\mathbf{s}-\mathbf{r}\|^\beta\|\boldsymbol{\omega}\|^2 + \|\mathbf{s}-\mathbf{r}\|^\beta\|\boldsymbol{\omega}\|^\nu + \|\mathbf{s}-\mathbf{r}\|^{\beta-1}\|\boldsymbol{\omega}\|^2\right)$$

holds for $t \geq 0$, $\boldsymbol{\omega} \in \mathbb{R}^3$, $\|\mathbf{s}-\mathbf{r}\| < \delta$.

With the aid of the substitution $\xi = h^{\frac{1}{\mu+1}}\|\mathbf{s}-\mathbf{r}\|$, we arrive at the estimates

$$\alpha_1\|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1}\xi^{\mu+1} - \alpha_3\gamma h^{\sigma-\frac{\beta}{\mu+1}}(t)\|\boldsymbol{\omega}\|\xi^{\beta} \le V_2$$

$$\le \alpha_2\|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1}\xi^{\mu+1} + \alpha_3\gamma h^{\sigma-\frac{\beta}{\mu+1}}(t)\|\boldsymbol{\omega}\|\xi^{\beta},$$

$$\dot V_2 \le -\alpha_4\left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma h^{\sigma-\frac{\beta-1}{\mu+1}}(t)\xi^{\beta+\mu}\right) + \alpha_5\gamma h^{\sigma-1-\frac{\beta}{\mu+1}}(t)\dot h(t)\|\boldsymbol{\omega}\|\xi^{\beta}$$

$$+\alpha_6\gamma h^{\sigma-\frac{\beta}{\mu+1}}(t)\left(\|\boldsymbol{\omega}\|^2 + \|\boldsymbol{\omega}\|^{\nu}\right)\xi^{\beta} + \alpha_6\gamma h^{\sigma-\frac{\beta-1}{\mu+1}}(t)\|\boldsymbol{\omega}\|^2\xi^{\beta-1}.$$

Hence, if $\beta \ge \mu\nu$, $\sigma \ge \beta/\mu$, γ is sufficiently small, $\|\mathbf{s}-\mathbf{r}\| < \delta$, and

$$|\dot h(t)| \le L h^{1+\frac{\beta-\sigma\mu}{\beta+\mu}}(t) \quad \text{for} \quad t \ge 0, \tag{8}$$

where $L = \text{const} > 0$, then

$$\frac{1}{2}\left(\alpha_1\|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1}\xi^{\mu+1}\right) \le V_2 \le 2\left(\alpha_2\|\boldsymbol{\omega}\|^2 + \frac{a}{\mu+1}\xi^{\mu+1}\right), \tag{9}$$

$$\dot V_2 \le -\frac{1}{2}\alpha_4 h^{\sigma-\frac{\beta-1}{\mu+1}}(t)\left(\|\boldsymbol{\omega}\|^{\nu+1} + \gamma\xi^{\beta+\mu}\right) \le -\alpha_7 h^{\sigma-\frac{\beta-1}{\mu+1}}(t)V_2^{\frac{\beta+\mu}{\mu+1}}. \tag{10}$$

Here α_7 is a positive constant.

Using estimates (9) and (10), we obtain that if there exist numbers β and σ such that $\beta \ge \mu\nu$, $\sigma \ge \beta/\mu$, inequality (8) is valid, and

$$h^{\frac{\beta-1}{\mu+1}}(t)\int_0^t h^{\sigma-\frac{\beta-1}{\mu+1}}(\tau)\,d\tau \to +\infty \quad \text{as} \quad t \to +\infty, \tag{11}$$

then the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables.

Denote $\theta = \sigma - \beta/\mu$. Then conditions (8) and (11) can be rewritten as follows

$$|\dot h(t)| \le L h^{1-\frac{\theta\mu}{\beta+\mu}}(t) \quad \text{for} \quad t \ge 0,$$

$$h^{\frac{\beta-1}{\mu+1}}(t)\int_0^t h^{\theta+\frac{\beta+\mu}{\mu(\mu+1)}}(\tau)\,d\tau \to +\infty \quad \text{as} \quad t \to +\infty.$$

It is easy to see that, to derive less conservative stability conditions, we should take $\beta = \mu\nu$. As a result, we obtain the following theorem.

Theorem 3.3 *If $\dot h(t) \le 0$ for $t \ge 0$, and there exist positive numbers θ and L such that*

$$|\dot h(t)| \le L h^{1-\frac{\theta}{\nu+1}}(t) \quad \textit{for} \quad t \ge 0, \tag{12}$$

$$h^{\frac{\mu\nu-1}{\mu+1}}(t)\int_0^t h^{\theta+\frac{\nu+1}{\mu+1}}(\tau)\,d\tau \to +\infty \quad \textit{as} \quad t \to +\infty,$$

then the equilibrium position (3) *of system* (2), (4) *is asymptotically stable with respect to all variables.*

Corollary 3.2 *If $\dot{h}(t) \leq 0$ for $t \geq 0$, there exist positive numbers θ and L such that condition* (12) *is valid, and*

$$h^{\frac{\mu\nu-1}{\mu+1}}(t)\left(1+\int_0^t h^{\theta+\frac{\nu+1}{\mu+1}}(\tau)\,d\tau\right) \geq \rho \quad for \;\; t \geq 0,$$

where $\rho = \mathrm{const} > 0$, then the equilibrium position (3) *of system* (2), (4) *is stable with respect to all variables and asymptotically stable with respect to $\boldsymbol{\omega}$.*

Example 3.2 Let the nonstationary multiplier $h(t)$ in system (4) be defined by the formula $h(t) = (t+1)^\alpha$, where $\alpha < 0$. In this case Theorem 3.3 and Corollary 3.2 provide less conservative stability conditions for $\theta = 0$.

We obtain that if $\alpha > -1/\nu$, then the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables, whereas if $\alpha = -1/\nu$, then the equilibrium position is stable with respect to all variables and asymptotically stable with respect to $\boldsymbol{\omega}$.

Remark 3.4 Recently, attention was paid to the problems of synchronization in various nonlinear systems such as dumbbell satellites [8], coupled systems [27], dissimilar and uncoupled rotating systems [12]. As the stability properties are important in studying oscillations in such systems, it seems that the results obtained in this paper may be extended to the mentioned classes of nonlinear systems.

4 Results of a Numerical Simulation

In this section, we demonstrate the previous theoretical results by means of a numerical simulation. Consider the monoaxial attitude stabilization of a rigid body with the inertia tensor $\mathbf{J} = \mathrm{diag}\{1.0,\, 1.2,\, 0.8\}$ in the equilibrium position (3). Denote the unit vectors of the body-fixed frame $Oxyz$ by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ and the direction cosines of the unit vector $\mathbf{r}$ in the body-fixed frame $Oxyz$ by $\gamma_1, \gamma_2, \gamma_3$. Let $\mathbf{r}$ be chosen as $\mathbf{r} = \frac{1}{\sqrt{3}}\mathbf{r}_1 + \frac{1}{\sqrt{3}}\mathbf{r}_2 + \frac{1}{\sqrt{3}}\mathbf{r}_3$. So, in the equilibrium position (3) the direction cosines $\gamma_1, \gamma_2, \gamma_3$ are equal to $1/\sqrt{3}$.

Assume that a positive definite homogeneous dissipative function W is defined by the formula

$$W = \frac{3}{8}\left(\omega_x^{8/3} + \omega_y^{8/3} + \omega_z^{8/3}\right).$$

Here ω_x, ω_y, ω_z are components of the vector $\boldsymbol{\omega}$. In this case $\nu = 5/3$, and the dissipative torque is $\mathbf{M}_d = -\left(\omega_x^{5/3},\, \omega_y^{5/3},\, \omega_z^{5/3}\right)^\top$.

Choose the restoring torque as a linear function of $\mathbf{s}$ ($\mu = 1$). Such approach is commonly used for satellite attitude stabilization, see [25, 29]. In particular, in [4], it was applied to the problem of monoaxial satellite stabilization in the orbital frame. Let

$$\mathbf{M}_r = -\frac{h(t)}{5\sqrt{3}}\,\mathbf{s} \times \mathbf{r},$$

where $h(t) = (t+0.1)^\alpha$, $\alpha = \mathrm{const} < 0$. We will consider two values of the parameter α: 1) $\alpha = -1/5$ and 2) $\alpha = -12/5$.

In the first case, in accordance with Theorem 3.3, the equilibrium position (3) of system (2), (4) is asymptotically stable with respect to all variables $\gamma_1, \gamma_2, \gamma_3, \omega_x, \omega_y, \omega_z$ (see Figs. 1 and 2).

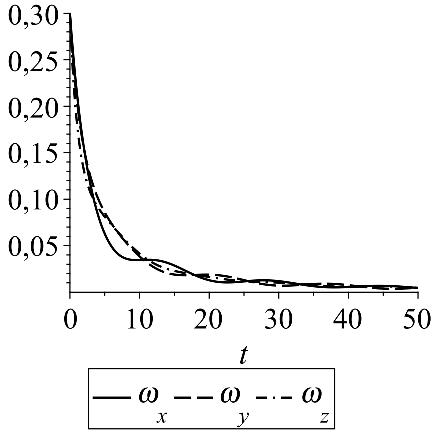

Figure 1: Angular velocity for $\alpha = -1/5$.

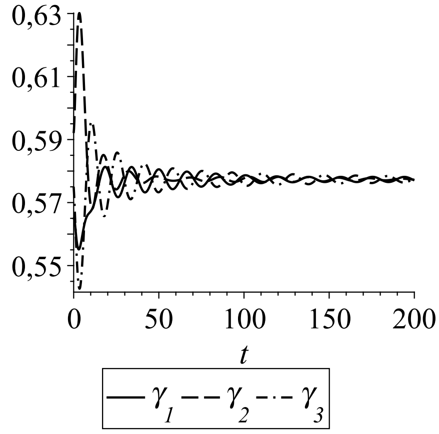

Figure 2: Direction cosines for $\alpha = -1/5$.

In the second case, in accordance with Theorem 3.2, the equilibrium position is asymptotically stable with respect to $\omega_x, \omega_y, \omega_z$ (see Fig. 3). At the same time, Fig. 4 demonstrates that there is no asymptotic stability with respect to $\gamma_1, \gamma_2, \gamma_3$.

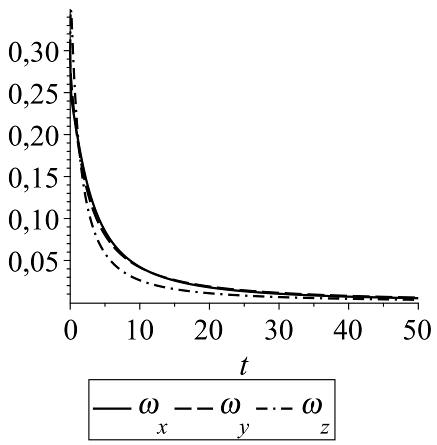

Figure 3: Angular velocity for $\alpha = -12/5$.

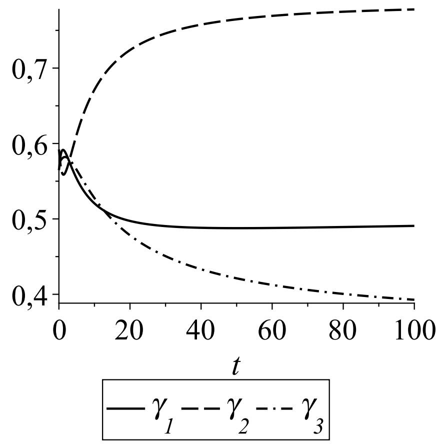

Figure 4: Direction cosines for $\alpha = -12/5$.

In both cases one and the same set of initial conditions was taken. The initial values of "aircraft" angles $\varphi(0) = 0.8$, $\psi(0) = 1.0$, $\theta(0) = -0.6$ result in the following initial values of direction cosines: $\gamma_1(0) = 0.5646424737$, $\gamma_2(0) = 0.5920595303$, $\gamma_3(0) = 0.5750168603$. The initial values of angular velocity projections are $\omega_x(0) = \omega_y(0) = \omega_z(0) = 0.3$.

5 Conclusion

The method of differential inequalities is a powerful tool for the stability analysis of nonlinear systems. In the present paper, the method is used for the investigation of the problem of monoaxial attitude stabilization of a rigid body. The possibility of implementing such a control system in which the restoring torque tends to zero as time increases is studied. The practical use of the investigation is connected with the challenge of propellant economy in control systems. With the aid of the Lyapunov direct method and the differential inequalities theory, stability conditions of an equilibrium position of the body are derived. It should be noted that Theorem 3.1 provides conditions of stability with respect to the angular velocity, in Theorem 3.2 conditions of the asymptotic stability with respect to the angular velocity are given, whereas, under the conditions of Theorem 3.3, we can guarantee the asymptotic stability with respect to all variables.

An interesting direction for further research is the application of the proposed approaches to the problem of three-axial stabilization of a rigid body.

Acknowledgment

This work was supported by the Russian Foundation for Basic Research, grant nos. 16-01-00587-a, 16-08-00997-a and 17-01-00672-a.

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Lie Symmetry Reductions of a Coupled Kdv System of Fractional Order

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Received: February 2, 2017; Revised: December 15, 2017

Abstract: In this paper, we investigate the coupled KdV system of fractional order, which describes a resonant interaction of two wave modes in shallow stratified liquid. The Lie group analysis method is applied for this coupled system. Then the corresponding invariant solutions are obtained using infinitesimal generators. Finally, we determined the reduced fractional ODE system corresponding to the fractional PDE system.

Keywords: *coupled KdV system; Lie symmetry method; Riemann-Liouville derivative; group- invariant solutions; reduced fractional system.*

Mathematics Subject Classification (2010): 76M60, 34A08, 35R11.

1 Introduction

Fractional partial differential equations (FPDEs) are becoming increasingly popular due to their practical applications in various fields of science and engineering, such as polymer physics, viscoelasticity materials, control theory, signal processing, systems identification and electrochemistry [1–5].

So it is necessary to obtain exact solutions or numerical solutions for FPDEs. During last few decades several analytical numerical and semi-analytical methods have been used for solving FPDEs [6, 7, 9, 10, 20].

Lie group analysis originally advocated by Sophus Lie has proven to be an efficient approach for PDEs [8], with the increasing applications of FPDEs, principle procedure of Lie group analysis was extended to FPDEs for finding the exact solution of the equation [11–13].

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Jafari et al. [14, 15] applied Lie group method to solve the time-fractional Kaup-Kupershmidt equation and time-fractional Boussinesq equation. In [18], Adem and Khalique have applied Lie symmetry analysis for Korteweg-de Vries(KdV) system given by

$$\begin{cases} u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \\ v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0. \end{cases} \tag{1}$$

The result for time fractional KdV-type equation has been obtained by Hu et al. [16]. Chen and Jiang [17] have applied the methods to simplify successfully two classes of FPDEs.

In this paper, we study Lie group method for solving the KdV system of fractional order

$$\begin{cases} D_t^{\alpha}u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \\ D_t^{\alpha}v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0, \end{cases} \tag{2}$$

where α $(0 < \alpha \leqslant 1)$ is a parameter describing the order of the fractional derivative, when $\alpha = 1$, the KdV system (2) becomes the KdV system (1).

The paper is organized as follows. In Section 2, we present the analysis of the Lie symmetry group of FPDEs system. We obtain the Lie point symmetries of fractional KdV system in Section 3. Then, in Section 4, we obtain invariant solutions and reduced equations of this system. Finally, conclusions are given in Section 5.

2 Preliminaries

We give some basic definitions and properties of the fractional Lie group method for finding infinitesimal function of the PDE system of fractional order.

Definition 2.1 The Riemann-Liouville fractional derivative of order α [2, 19], is defined by

$$D_t^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{\partial^n u(x,t)}{\partial t^n}; & n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau; & n-1 < \alpha < n. \end{cases}$$

For fractional PDE system with two independent variables we have

$$\begin{aligned} \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} &= F(x,t,u,v,u_{(1)},v_{(1)},\cdots), \qquad 0 < \alpha < 1, \\ \frac{\partial^{\alpha}v(x,t)}{\partial t^{\alpha}} &= G(x,t,u,v,u_{(1)},v_{(1)},\cdots). \end{aligned}$$

According to Lie's algorithm, the infinitesimal generator of the symmetry group admitted by (2) is given by

$$X = \xi^x(x,t,u,v)\frac{\partial}{\partial x} + \xi^t(x,t,u,v)\frac{\partial}{\partial t} + \eta^u(x,t,u,v)\frac{\partial}{\partial u} + \eta^v(x,t,u,v)\frac{\partial}{\partial v}, \tag{3}$$

in which ξ^x,ξ^t,η^u,η^v are infinitesimal functions of the group variables.

Since the KdV system of fractional order has at most α-order derivatives, the α-prolongation of the generator should be considered in the form

$$\begin{aligned}
X^{(\alpha)} &= \xi^x(x,t,u,v)\frac{\partial}{\partial x} + \xi^t(x,t,u,v)\frac{\partial}{\partial t} + \eta^u(x,t,u,v)\frac{\partial}{\partial u} + \eta^v(x,t,u,v)\frac{\partial}{\partial v} \\
&+ \eta_i^{(1)u}(x,t,u,v,u_{(i)},v_{(i)})\frac{\partial}{\partial u_i} + \eta_i^{(1)v}(x,t,u,v,u_{(i)},v_{(i)})\frac{\partial}{\partial v_i} + \cdots \\
&+ \eta_{i_1\cdots i_k}^{(k)u}(x,t,u,v,u_{(1)},v_{(1)},\cdots,u_{(k)},v_{(k)})\frac{\partial}{\partial u_{i_1,\cdots,i_k}} \\
&+ \eta_{i_1\cdots i_k}^{(k)v}(x,t,u,v,u_{(1)},v_{(1)},\cdots,u_{(k)},v_{(k)})\frac{\partial}{\partial v_{i_1,\cdots,i_k}} \\
&+ \eta_t^{(\alpha)u}(x,t,u,v,\cdots,u_{(\alpha)},\cdots)\frac{\partial}{\partial u_t^\alpha} + \eta_t^{(\alpha)v}(x,t,u,v,\cdots,v_{(\alpha)},\cdots)\frac{\partial}{\partial v_t^\alpha},
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\eta_t^{(\alpha)u} &= D_{1t}^\alpha(\eta^u) + \xi^x D_{1t}^\alpha(u_x) - D_{1t}^\alpha(\xi^x u_x) + D_{1t}^\alpha(D_{1t}(\xi^t)u) - D_{1t}^{\alpha+1}(\xi^t u) + \xi^t D_{1t}^{\alpha+1}u, \\
\eta_t^{(\alpha)v} &= D_{2t}^\alpha(\eta^v) + \xi^x D_{2t}^\alpha(v_x) - D_{2t}^\alpha(\xi^x v_x) + D_{2t}^\alpha(D_{2t}(\xi^t)v) - D_{2t}^{\alpha+1}(\xi^t v) + \xi^t D_{2t}^{\alpha+1}v.
\end{aligned}$$

D_{1t} and D_{2t} are the total derivative operators defined as

$$\begin{aligned}
D_{1t} &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{xxt}\frac{\partial}{\partial u_{xx}} + \cdots, \\
D_{2t} &= \frac{\partial}{\partial t} + v_t\frac{\partial}{\partial v} + v_{xt}\frac{\partial}{\partial v_x} + v_{tt}\frac{\partial}{\partial v_t} + v_{xxt}\frac{\partial}{\partial v_{xx}} + \cdots.
\end{aligned}$$

Definition 2.2 A vector X given by (3) is said to be Lie point symmetry vector field for system (2), if

$$\begin{aligned}
X^{(\alpha)}\Big[D_t^\alpha u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\Big] &= 0, \\
X^{(\alpha)}\Big[D_t^\alpha v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\Big] &= 0.
\end{aligned}$$

3 Lie Symmetry for Coupled KdV System of Fractional Order

In this section, we investigate the infinitesimal generator of the KdV system of fractional order (2).

Theorem 3.1 *Lie symmetries of (2) are*

1. *If $\alpha \neq \frac{1}{2}, \frac{1}{3}$, then we have:*

$$\begin{aligned}
\xi^x(x,t,u,v) &= c_1 + c_2\alpha x, & \xi^t(x,t,u,v) &= 3c_2 t, \\
\eta^u(x,t,u,v) &= -2c_2\alpha u, & \eta^v(x,t,u,v) &= -2c_2\alpha v,
\end{aligned}$$

where c_1 and c_2 are two arbitrary constants. Hence, the infinitesimal generators are given by

$$X_{a1} = \frac{\partial}{\partial x}, \qquad X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

2. *If $\alpha = \frac{1}{2}$, then we have:*

$$\xi^x(x,t,u,v) = c_1 - c_2 x, \qquad \xi^t(x,t,u,v) = -6c_2 t,$$
$$\eta^u(x,t,u,v) = 2c_2\alpha u, \qquad \eta^v(x,t,u,v) = 2c_2\alpha v,$$

where c_1 and c_2 are two arbitrary constants. Hence

$$X_{b1} = \frac{\partial}{\partial x}, \qquad X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2\alpha u\frac{\partial}{\partial u} + 2\alpha v\frac{\partial}{\partial v}.$$

3. *If $\alpha = \frac{1}{3}$, then we have:*

$$\xi^x(x,t,u,v) = c_1 + c_2 x, \qquad \xi^t(x,t,u,v) = 9c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v,$$

where c_1 and c_2 are two arbitrary constants. Hence

$$X_{c1} = \frac{\partial}{\partial x}, \qquad X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

Proof. Let us consider a one parameter Lie group of infinitesimal transformation in x, t, u, v given by

$$x \longrightarrow x + \epsilon\xi^x(x,t,u,v), \quad t \longrightarrow t + \epsilon\xi^t(x,t,u,v),$$
$$u \longrightarrow u + \epsilon\eta^u(x,t,u,v), \quad v \longrightarrow v + \epsilon\eta^v(x,t,u,v), \tag{5}$$

with a small parameter $\epsilon \ll 1$, and the symmetry group of KdV system will be generated by the vector field (3), now we find the coefficient functions $\xi^x, \xi^t, \eta^u, \eta^v$ in (5).

By applying the $X^{(\alpha)}$ to both sides of (2), we have

$$X^{(\alpha)}\Big[D_t^\alpha u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\Big] = 0,$$
$$X^{(\alpha)}\Big[D_t^\alpha v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\Big] = 0. \tag{6}$$

Expanding (6), and solving the obtained system using a mathematical software, we obtain the Lie point symmetries.

1. If $\alpha \neq \frac{1}{2}, \frac{1}{3}$, then we have:

$$\xi^x(x,t,u,v) = c_1 + c_2\alpha x, \qquad \xi^t(x,t,u,v) = 3c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v.$$

Hence, the infinitesimal generators are given by

$$X_{a1} = \frac{\partial}{\partial x}, \qquad X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

2. If $\alpha = \frac{1}{2}$, then

$$\xi^x(x,t,u,v) = c_1 - c_2 x, \qquad \xi^t(x,t,u,v) = -6c_2 t,$$
$$\eta^u(x,t,u,v) = 2c_2\alpha u, \qquad \eta^v(x,t,u,v) = 2c_2\alpha v.$$

Therefore

$$X_{b1} = \frac{\partial}{\partial x}, \qquad X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2\alpha u\frac{\partial}{\partial u} + 2\alpha v\frac{\partial}{\partial v}.$$

3. If $\alpha = \frac{1}{3}$, then

$$\xi^x(x,t,u,v) = c_1 + c_2 x, \qquad \xi^t(x,t,u,v) = 9c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v.$$

Therefore

$$X_{c1} = \frac{\partial}{\partial x}, \qquad X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

4 Symmetry Reduction

In the previous section, we obtained the infinitesimal generators X_{ij} $(i = a,b,c, j = 1,2)$. Here we want to obtain similarity variables and their reduction equations. Then by using these variables the system (2) transforms into a system of fractional ODE.

One has to solve the associated Lagrange equations

$$\frac{dx}{\xi^x(x,t,u,v)} = \frac{dt}{\xi^t(x,t,u,v)} = \frac{du}{\eta^u(x,t,u,v)} = \frac{dv}{\eta^v(x,t,u,v)}.$$

We consider the following cases.

- Case 1: $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $X_{a1} = \frac{\partial}{\partial x}$.
  In this case the symmetry X_{a1} gives rise to the group-invariant solution:
  $$r = t, \qquad u = F(r), \qquad v = G(r), \tag{7}$$
  substituting (7) into (2) results in the fact that $F(r)$ and $G(r)$ satisfy the following differential equations:
  $$\frac{d^\alpha F(t)}{dt^\alpha} = 0, \qquad \frac{d^\alpha G(t)}{dt^\alpha} = 0,$$
  by using Laplace transformation we get
  $$F(t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \qquad G(t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1},$$
  where k is a constant, therefore
  $$u(x,t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \qquad v(x,t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}.$$

- Case 2: $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}$.
  In this case, the group-invariant solution is:
  $$r = tx^{\frac{-3}{\alpha}}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}, \tag{8}$$
  substituting (8) into (2) leads to the following fractional ODE system:
  $$\begin{cases} D_r^\alpha F + k_1F(r) + k_2 rF'(r) + k_3r^2F''(r) + k_4r^3F^{(3)}(r) + k_5F^2(r) \\ +k_6rF(r)F'(r) + k_7G^2(r) + k_8rG(r)G'(r) + k_9F(r)G(r) \\ +k_{10}rF'(r)G(r) + k_{11}rF(r)G'(r) = 0, \\ D_r^\alpha G + k_1'G(r) + k_2'rG'(r) + k_3'r^2G''(r) + k_4'r^3G^{(3)}(r) + k_5'F^2(r) \\ +k_6'rF(r)F'(r) + k_7'G^2(r) + k_8'rG(r)G'(r) + k_9'F(r)G(r) \\ +k_{10}'rF'(r)G(r) + k_{11}'rF(r)G'(r) = 0, \end{cases}$$
  where $k_i = h_i(\alpha)$ and $k_i' = g_i(\alpha)$, $(i = 1,2,\cdots,11)$ are constants.

- Case 3:$\alpha = \frac{1}{2}$, $X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$.
  For this case, the group-invariant solution is:

$$r = tx^{-6}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}. \tag{9}$$

  Again by substituting (9) into (2), we have:

$$\begin{cases} D_r^{\frac{1}{2}}F - 24F(r) - 696rF'(r) - 405r^2F''(r) - 216r^3F^3(r) + \frac{7}{2}F^2(r) \\ +\frac{21}{2}rF(r)F'(r) + 2G^2(r) + 6rG(r)G'(r) - 5F(r)G(r) \\ -\frac{15}{2}rF'(r)G(r) - \frac{15}{2}rF(r)G'(r) = 0, \\ D_r^{\frac{1}{2}}G - 24G(r) - 696rG'(r) - 405r^2G''(r) - 216r^3G^3(r) + \frac{5}{2}F^2(r) \\ +\frac{15}{2}rF(r)F'(r) + \frac{7}{2}G^2(r) + \frac{21}{2}rG(r)G'(r) - 8F(r)G(r) \\ -12rF'(r)G(r) - 12rF(r)G'(r) = 0. \end{cases}$$

- Case 4:$\alpha = \frac{1}{3}$, $X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}$.
  In this case, the group-invariant solution is:

$$r = tx^{-9}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}, \tag{10}$$

  substituting (10) into (2) results in the fact that $F(r)$ and $G(r)$ satisfy the following fractional ODE system

$$\begin{cases} D_r^{\frac{1}{3}}F - 24F(r) - 1692.09rF'(r) - 2430r^2F''(r) - 729r^3F^3(r) + \frac{7}{2}F^2(r) \\ +\frac{63}{4}rF(r)F'(r) + 2G^2(r) + 9rG(r)G'(r) - 5F(r)G(r) - \frac{45}{4}rF'(r)G(r) \\ -\frac{45}{4}rF(r)G'(r) = 0, \\ D_r^{\frac{1}{3}}G - 24G(r) - 1692.09rG'(r) - 2430r^2G''(r) - 729r^3G^3(r) + \frac{5}{2}F^2(r) \\ +\frac{45}{4}rF(r)F'(r) + \frac{7}{2}G^2(r) + \frac{63}{4}rG(r)G'(r) - 8F(r)G(r) \\ -18rF'(r)G(r) - 18rF(r)G'(r) = 0. \end{cases}$$

Note. For $\alpha = 1$, the Lie point symmetries provide is similar results to those obtained by Adem and Khalique in [18].

5 Conclusion

In this paper, we carry out the Lie symmetry group analysis for a fractional PDE system. First, we apply Lie symmetries method for the KdV system of fractional order (2), and get its infinitesimal generators. Then, we use similarity variables to obtain reduction equations. Finally, we have shown that the KdV system of fractional order can be transformed into a fractional ODE system.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 29–41

Stability, Boundedness and Square Integrability of Solutions to Certain Third-Order Vector Differential Equations

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Received: November 10, 2016; Revised: December 12, 2017

Abstract: In this paper, we establish some new sufficient conditions which guarantee the stability and the boundedness of solutions of certain third order vector differential equations. Sufficient conditions are also established for square integrability of solutions and their derivatives. By this work, we extend and improve some stability and boundedness results in the literature.

Keywords: *Lyapunov functional; third-order vector differential equation; boundedness; stability; square integrability.*

Mathematics Subject Classification (2010): 34C10, 34C11, 34D05, 34D20, 34D40.

1 Introduction

In recent years much attention has been drawn to the stability and boundedness of solutions of ordinary scalar and vector nonlinear differential equations of third order. See Afuwape [1, 2],Omeike [9, 10] Ezeilo [4, 5], Remili [11–14] and the references cited therein for a comprehensive treatment of the subject. Lyapunov's second (direct) method has been used as a basic tool to verify the results established in these works.

In 2009, Tunç [17] proved two results, for the cases P = 0 and P $\neq$ 0, respectively, on the stability and boundedness of solutions to the vector differential equations of third order

$$X'''(t) + \Psi(X'(t))X''(t) + BX'(t) + cX(t) = P(t). \tag{1}$$

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Recently, in 2014, for the same cases, Omeike [9] discussed the global asymptotic stability and boundedness of solutions to nonlinear vector differential equations of third order

$$X'''(t) + \Psi(X'(t))X''(t) + \Phi(X(t))X'(t) + cX(t) = P(t). \tag{2}$$

The purpose of this paper is to study the uniform asymptotic stability, boundedness and square integrability of solutions of the third order nonlinear vector differential equations of the form

$$(\Omega(X(t)))X'(t))'' + \Psi(X'(t))X''(t) + G(X(t))X'(t) + cX(t) = P(t), \tag{3}$$

where $X \in \mathbb{R}^n$, $t \in \mathbb{R}$ and c is a positive constant, Ψ and G are $n \times n-$symetric and differentiable matrix functions; Ω is an $n \times n-$symetric differentiable and inversible matrix function. $P : \mathbb{R} \to \mathbb{R}^n$ is a continuous function with respect to t. Let

$$\Omega' = \Omega'(X(t)) = \frac{d}{dt}(\mu_{i,j}(X(t)), \text{and } G' = G'(X(t)) = \frac{d}{dt}(g_{i,j}(X(t)) \quad (i, j = 1, 2, ..., n),$$

where $\mu_{i,j}(X(t))$ and $g_{i,j}(X(t))$ are the components of $\Omega(X)$ and $G(X)$ respectively. On the other hand $X(t)$, $Y(t)$, $Z(t)$, $\Omega(X(t)))$, $G(X(t)))$ and $\Psi(X'(t)))$ are, respectively, abbreviated as X, Y, Z, Ω, G and Ψ throughout the paper. Additionally, the symbol $\langle X, Y\rangle$ corresponding to any pair X and Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$, Thus $\langle X, X\rangle = \|X\|^2$.

Let us, for convenience, replace (3) by the equivalent differential system

$$\begin{cases} X' = \Omega^{-1}(X)Y, \\ Y' = Z, \\ Z' = -\Psi\Omega^{-1}(X)Z - \Psi\theta Y - G\Omega^{-1}(X)Y - cX + P(t), \end{cases} \tag{4}$$

which was obtained by setting

$$\begin{aligned} X' &= \Omega^{-1}(X)Y, \\ X'' &= \theta(t)Y + \Omega^{-1}(X)Z, \end{aligned}$$

where

$$\theta(t) = \left(\Omega^{-1}(X)\right)' = -\Omega^{-1}(X)\Omega'(X)\Omega^{-1}(X). \tag{5}$$

This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3 we give stability results. In Section 4 boundedness of solutions is discussed. Finally, in Section 5 sufficient conditions for the square integrability of solutions are given.

2 Preliminaries

In order to reach our main results, we dispose some well-known algebraic results which will be required in the proofs.

Lemma 2.1 *[4] Let D be a real symmetric positive definite $n \times n$ matrix. Then for any X in $\mathbb{R}^n$, we have*

$$\delta_d \parallel X \parallel^2 \leq \langle DX, X\rangle \leq \Delta_d \parallel X \parallel^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 *[4] Let Q, D be any two real $n \times n$ commuting matrices. Then*

(i) *The eigenvalues $\lambda_i (QD) \, (i = 1, 2..., n)$ of the product matrix QD are all real and satisfy*

$$\min_{1 \leq j,k \leq n} \lambda_j (Q) \lambda_k (D) \leq \lambda_i (QD) \leq \max_{1 \leq j,k \leq n} \lambda_j (Q) \lambda_k (D) .$$

(ii) *The eigenvalues $\lambda_i (Q + D) \, (i = 1, 2..., n)$ of the sum of matrix Q and D are all real and satisfy*

$$\min_{1 \leq j \leq n} \lambda_j (Q) + \min_{1 \leq k \leq n} \lambda_k (D) \leq \lambda_i (Q + D) \leq \max_{1 \leq j \leq n} \lambda_j (Q) + \max_{1 \leq k \leq n} \lambda_k (D) .$$

Lemma 2.3 *[4] Let H be a continuous matrix function with $H(0) = 0$. Then*

$$\frac{d}{dt} \int_0^1 \sigma \langle H(\sigma X)X, X \rangle d\sigma = \langle H(X), \frac{dX}{dt} \rangle.$$

Lemma 2.4 *Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then*

$$\delta_h \parallel X \parallel^2 \leq \int_0^1 \langle H (\sigma X) , X \rangle \, d\sigma \leq \Delta_h \parallel X \parallel^2,$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

Definition 2.1 We define the spectral radius $\rho (A)$ of a matrix A by

$$\rho (A) = \max \{ |\lambda| : \quad \lambda \text{ is the eigenvalue of } A \} .$$

Lemma 2.5 *For any $A \in \mathbb{R}^{n \times n}$, we have the norm $\|A\| = \sqrt{\rho (A^T A)}$. If A is symmetric, then $\|A\| = \rho (A)$.*

We shall note all the equivalent norms by the same notation $\|X\|$ for $X \in \mathbb{R}^n$ and $\|A\|$ for a matrix $A \in \mathbb{R}^{n \times n}$.

In the sequel we will assume :

H_1) There are positive constants $\omega_0, \ \omega_1, \ a_0, \ a_1, \ b_0, \ b_1$ such that the following conditions are satisfied

$$b_0 \leq \lambda_i(G) \leq b_1, \quad a_0 \leq \lambda_i(\Psi) \leq a_1, \quad \omega_0 \leq \lambda_i(\Omega) \leq \omega_1.$$

H_2) The $n \times n$ differentiable matrices $\Omega, \ \Omega^{-1}, \ \Psi$ and G are symmetric, associative and commute pairwise.

3 Stability

Our study of (3) here is concerned primarily with the problems of the stability for the case $P(t) = 0$. For the ease of exposition throughout this paper we will adopt the following notation :

$$\delta(t) = \parallel \Omega'(X(t)) + G'(X(t)) \parallel . \tag{6}$$

Theorem 3.1 *In addition to the fundamental assumptions imposed on* Ω, Ψ *and* G, *we suppose there exist positive constants* β *and* δ_0 *such that*
i) $\dfrac{c}{a_0 b_0} < \beta < \dfrac{1}{\omega_1}$,
ii) $\int_0^{+\infty} \delta(s)ds \leq \delta_0 < \infty$.

Then every solution of (4) satisfies

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

Proof. To prove this theorem, we define a Lyapunov functional $W = W(t, X, Y, Z)$ as

$$W = V \exp(-\mu(t)), \tag{7}$$

where

$$\mu(t) = \frac{1}{d}\int_0^t \delta(s)ds,$$

$$\begin{aligned} V \;=\; & \frac{1}{2}\langle cX, cX\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\Omega^{-1}Y\right\rangle + \beta\frac{b_0}{2}\langle\, Z, Z\rangle + \langle c\Omega^{-1}Y, Z\rangle \\ & +\beta\langle cX, b_0\; Y\rangle + \int_0^1 \sigma\langle c\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma, \end{aligned} \tag{8}$$

d is some positive constant which will be specified later. It is clear by (8) that $W(t, 0, 0, 0) = 0$. Note that $\omega_0 \leq \lambda_i(\Omega) \leq \omega_1$ implies that $\frac{1}{\omega_1} \leq \lambda_i(\Omega^{-1}) \leq \frac{1}{\omega_0}$. Hence by (H_1), Lemma 2.1 and Lemma 2.2, we have

$$c\int_0^1 \sigma\langle\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma \;\geq\; \frac{ca_0}{2\omega_1^2} \parallel Y \parallel^2$$

and

$$\frac{1}{2}\beta b_0 \left\langle Y, G\Omega^{-1}Y\right\rangle \geq \frac{\beta b_0^2}{2\omega_1} \parallel Y \parallel^2 .$$

Hence

$$V \geq \frac{c^2}{2} \parallel X \parallel^2 + \beta\langle cX, b_0 Y\rangle + \beta\frac{b_0}{2} \parallel Z \parallel^2 + \langle c\Omega^{-1}Y, Z\rangle + \Big(\frac{\beta b_0^2}{2\omega_1} + \frac{ca_0}{2\omega_1^2}\Big) \parallel Y \parallel^2 .$$

Thus, we clearly have

$$\frac{c^2}{2} \parallel X \parallel^2 + \beta\langle cX, b_0 Y\rangle = \frac{1}{2}\|cX + \beta b_0 Y\|^2 - \frac{\beta^2 b_0^2}{2} \parallel Y \parallel^2$$

and

$$\begin{aligned} \frac{\beta b_0}{2} \parallel Z \parallel^2 + \langle c\Omega^{-1}Y, Z\rangle \;&=\; \frac{\beta b_0}{2}\|Z + \frac{c}{\beta b_0}\Omega^{-1}Y\|^2 - \frac{c^2}{2\beta b_0}\langle\Omega^{-1}Y, \Omega^{-1}Y\rangle \\ &\geq\; \frac{\beta b_0}{2}\|Z + \frac{c}{\beta b_0}\Omega^{-1}Y\|^2 - \frac{c^2}{2\beta\omega_1^2 b_0} \parallel Y \parallel^2 . \end{aligned}$$

Combining the preceding estimates, we find

$$V \geq \frac{1}{2} \parallel cX + \beta b_0 Y \parallel^2 + \frac{\beta b_0}{2} \parallel Z + \frac{c}{\beta b_0} \Omega^{-1} Y \parallel^2 + \Delta \parallel Y \parallel^2,$$

where

$$\Delta = \frac{\beta b_0^2}{2\omega_1} + \frac{c a_0}{2\omega_1^2} - \frac{\beta^2 b_0^2}{2} - \frac{c^2}{2\beta\omega_1^2 b_0}.$$

Condition (i) implies

$$\Delta = c\frac{\beta\ a_0 b_0 - c}{2\beta b_0 \omega_1^2} + \beta b_0^2 (\frac{1}{2\omega_1} - \frac{\beta}{2}) \geq \frac{c}{2\beta b_0 \omega_1^2} (\beta\ a_0 b_0 - c) > 0.$$

It is evident, from the terms included in the last inequality, that there exists a sufficiently small positive constant k_0 such that

$$V \geq k_0 \big(\parallel X \parallel^2 + \parallel Y \parallel^2 + \parallel Z \parallel^2 \big). \tag{9}$$

Finally, by condition (ii) and (7) we get

$$W \geq K_0 \big(\parallel X \parallel^2 + \parallel Y \parallel^2 + \parallel Z \parallel^2, \tag{10}$$

where $K_0 = k_0 \exp(-\frac{\delta_0}{d})$.

Now, we show that $W'_{(4)}$ is negative definite function.
First, by Lemma 2.3, from the integral term in (8) we have the following derivative

$$\frac{d}{dt}\int_0^1 \sigma \langle c\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma = c\langle \Psi\Omega^{-1}Y, \theta Y + \Omega^{-1}Z\rangle.$$

Hence, the time derivative of functional V along the system (4) leads to

$$V'_{(4)} = V_1 + V_2 + V_3,$$

where

$$\begin{aligned} V_1 &= \beta c b_0 \langle \Omega^{-1}Y, Y\rangle - c\left\langle Y, G\Omega^{-2}Y\right\rangle, \\ V_2 &= c\left\langle \Omega^{-1}Z, Z\right\rangle - \beta b_0 \langle Z, \Psi\Omega^{-1}Z\rangle, \\ V_3 &= c\left\langle \theta Y, Z\right\rangle - \beta b_0 \langle Z, \Psi\theta Y\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\theta Y\right\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G'\Omega^{-1}Y\right\rangle. \end{aligned}$$

By virtue of (H_1), Lemma 2.1 and Lemma 2.2 it follows

$$\begin{aligned} V_1 &= \left\langle Y, (\beta c b_0 I - cG\Omega^{-1})\Omega^{-1}Y\right\rangle \leq -\frac{cb_0}{\omega_0}(\frac{1}{\omega_1} - \beta) \parallel Y \parallel^2, \\ V_2 &= \left\langle Z, (cI - \beta b_0 \Psi)\Omega^{-1}Z\right\rangle \leq -\frac{1}{\omega_0}(\beta a_0 b_0 - c) \parallel Z \parallel^2. \end{aligned}$$

Finally, by (5), Lemma 2.5 and the inequality $2 \parallel UV \parallel \leq \parallel U \parallel^2 + \parallel V \parallel^2$ we get

$$\begin{aligned} \parallel \theta(t) \parallel &= \parallel \Omega^{-1}(X)\Omega'(X)\Omega^{-1}(X) \parallel \leq \frac{1}{\omega_0^2} \parallel \Omega'(X) \parallel, \qquad (11) \\ V_3 &= c\left\langle \theta Y, Z\right\rangle - \beta b_0 \langle Z, \Psi\theta Y\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\theta Y\right\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G'\Omega^{-1}Y\right\rangle \\ &\leq \Big[\frac{1}{\omega_0^2}\left(\frac{c}{2k_0} + \frac{\beta b_0 a_1}{2k_0} + \frac{1}{2k_0}\beta b_0 b_1\right) \parallel \Omega' \parallel + \frac{1}{2k_0}\beta b_0 \parallel G' \parallel \Big] V \\ &\leq K_1 \delta(t) V, \end{aligned}$$

where $K_1 = \max\left\{\frac{1}{2k_0\omega_0^2}\big(c+\beta b_0 a_1+\beta b_0 b_1\big); \frac{\beta b_0}{2k_0}\right\}$. Hence, we conclude that

$$V'_{(4)} \leq -M \parallel Z \parallel^2 - N \parallel Y \parallel^2 + K_1\delta(t)V. \tag{12}$$

Clearly, from condition (i) of Theorem 3.1 we have

$$N = \frac{cb_0}{\omega_0}\Big(\frac{1}{\omega_1}-\beta\Big) > 0 \quad \text{and} \quad M = \frac{1}{\omega_0}(\beta a_0 b_0 - c) > 0.$$

Now, from (7) and (12) we obtain

$$\begin{aligned} W'_{(4)} &= \left[V' - \frac{1}{d}\delta(t)V\right]\exp(-\mu(t)) \\ &\leq \left[-M \parallel Z \parallel^2 - N \parallel Y \parallel^2 + (K_1 - \frac{1}{d})\delta(t)V\right]\exp(-\mu(t)). \end{aligned}$$

Choosing $K_1 - \dfrac{1}{d} = 0$, the last inequality becomes

$$W'_{(4)} \leq -C(\parallel Z \parallel^2 + \parallel Y \parallel^2), \tag{13}$$

where $C = \exp(-\frac{\delta_0}{d})\min\{M, \ N\}$. In view of (10) and (13), it follows that the solution $\big(X(t), Y(t), Z(t)\big)$ of (4) is uniformly stable.

Now $E = \{(X,Y,Z) : \ W'_{(4)}(X,Y,Z) = 0\} = \{(X,0,0) : X \in \mathbb{R}^n\}$ and the largest invariant set contained in E is $F = \{(0,0,0)\}$. By LaSalle's invariance principe

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

This fact completes the proof of Theorem 3.1.

4 Boundedness

Our main theorem in this section is stated with respect to $P(t) \neq 0$ as follows :

Theorem 4.1 *Assume that all the conditions of Theorem 3.1 are satisfied and there exist positive constants d_1 and D_1 such that :*

$I_1)$ $\parallel P(t) \parallel \leq \lambda(t) < d_1$,

$I_2)$ $\int_0^t \lambda(s)ds < D_1$,

$I_3)$ $\lim\limits_{t\to\infty} \parallel \Omega'(X(t)) \parallel$ *exists.*

Then there exists a positive constant D_5 such that any solution $X(t)$ of (3) and their derivatives $X'(t)$, and $X''(t)$ satisfy

$$\parallel X(t) \parallel \leq D_5, \ \parallel X'(t) \parallel \leq D_5, \ \parallel X''(t) \parallel \leq D_5. \tag{14}$$

Proof. For the case $P(t) \neq 0$, on differentiating (8) along the system (4) we obtain

$$\begin{aligned} V'_{(4)} &\leq -J + K_1\delta(t)V + c\langle \Omega^{-1}Y, P(t)\rangle + \langle \beta b_0 Z, P(t)\rangle \\ &\leq -J + K_1\delta(t)V + \lambda(t)\Big(c \parallel \Omega^{-1} \parallel \parallel Y \parallel + \beta b_0 \parallel Z \parallel\Big). \end{aligned}$$

Using Lemma 2.5 we get

$$V'_{(4)} \leq -J + K_1\delta(t)V + K_2\lambda(t)(\| Y \| + \| Z \|),$$

where $K_2 = \max\left\{\dfrac{c}{\omega_0}, \beta b_0\right\}$ and $J = M \| Z \|^2 + N \| Y \|^2$.

Now, the inequalities $\| Y \| \leq \| Y \|^2 + 1$ and $\| Z \| \leq \| Z \|^2 + 1$ lead to

$$V'_{(4)} \leq -J + K_1\delta(t)V + K_2\lambda(t)(\| Y \|^2 + \| Z \|^2 + 2). \tag{15}$$

From (7) we have

$$W'_{(4)} = \left[V' - \frac{1}{d}\delta(t)V\right]\exp(-\mu(t)). \tag{16}$$

Since $K_1 - \dfrac{1}{d} = 0$, it follows that

$$W'_{(4)} \leq \left[-J + K_2\lambda(t)(\| Y \|^2 + \| Z \|^2 + 2)\right]\exp(-\mu(t)).$$

In view of (13) and (10), the above estimates imply that

$$W'_{(4)} \leq -C(\| Y \|^2 + \| Z \|^2) + \frac{K_2}{K_0}\lambda(t)\ W + K_3\lambda(t), \tag{17}$$

with $K_3 = 2K_2$. Integrating both sides (17) from 0 to t, one can easily obtain

$$W(t) - W(0) \leq K_3\int_0^t \lambda(s)ds + \frac{K_2}{K_0}\int_0^t W(s)\lambda(s)ds.$$

Let

$$D_2 = W(0) + K_3 D_1. \tag{18}$$

Thus

$$W(t) \leq D_2 + \frac{K_2}{K_0}\int_0^t W(s)\lambda(s)ds.$$

By the Gronwall inequality it follows

$$W(t) \leq D_2 \exp\left(\frac{K_2}{K_0}\int_0^t \lambda(s)ds\right) \leq D_3, \tag{19}$$

where $D_3 = D_2\exp\left(\frac{K_2}{K_0}D_1\right)$. This result implies that there exists a constant D_4 such that

$$\| X(t) \| \leq D_4,\ \| Y(t) \| \leq D_4, \| Z(t) \| \leq D_4.$$

From (4) we have

$$\begin{aligned} \| X'(t) \| &= \| \Omega^{-1}Y(t) \| \\ &\leq \left\|\Omega^{-1}\right\| \| Y(t) \| \\ &\leq \frac{D_4}{\omega_0}. \end{aligned}$$

Since $\lim_{t\to\infty} \| \Omega'(X(t)) \|$ exists, we have

$$\| \Omega'\big(X(t)\big) \| < q_1, \tag{20}$$

for some positive constant q_1. So, from (11) we get

$$\| \theta(t) \| \leq \frac{q_1}{\omega_0^2}. \tag{21}$$

Hence

$$\begin{aligned} \| X^{''}(t) \| &= \| \theta(t)Y(t) + \Omega^{-1}Z(t) \| \\ &\leq \| \theta(t)Y(t) \| + \| \Omega^{-1}Z(t) \| \\ &\leq (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0})D_4. \end{aligned}$$

Therefore, there exists a positive constant D_5 such that

$$\| X(t) \| \leq D_5, \ \| X'(t) \| \leq D_5, \ \| X''(t) \| \leq D_5, \tag{22}$$

for all $t \geq 0$, where $D_5 = \max\big\{(\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0})D_4, \ D_4\big\}$. This completes the proof of Theorem 4.1.

5 Square Integrability

Our next result concerns the square integrability of solutions of equation (3).

Theorem 5.1 *In addition to the assumptions of Theorem 4.1, we assume that*

$I_4)$ $c - (\frac{a_1 + b_1}{2}) > 0.$

Then all the solutions of (3) and their derivatives are elements of $L^2[0, +\infty)$.

Proof. Define H(t) as

$$H(t) = W(t) + \varepsilon \int_0^t (\| Z(s) \|^2 + \| Y(s) \|^2) ds, \tag{23}$$

where $\varepsilon > 0$ is a constant to be specified later. By differentiating H(t) and using (17) we obtain

$$H'(t) \leq (\varepsilon - C)(\| Z(t) \|^2 + \| Y(t) \|^2) + \Big(K_2 W + K_3\Big)\lambda(t).$$

If we choose $\varepsilon - C < 0$, then from (19) we get

$$H'(t) \leq K_4 \lambda(t), \tag{24}$$

where $K_4 = K_2 D_3 + K_3$. Integrating (24) from 0 to t, $t \geq 0$, and using condition (I_2) of Theorem 4.1 we obtain

$$H(t) - H(0) = \int_0^t H'(s) ds \leq K_4 D_1.$$

Using (18) and equality $H(0) = W(0)$ we get

$$H(t) \leq K_4 D_1 + D_2 - K_3 D_1.$$

We can conclude by (23) that

$$\int_0^t (\| Z(s) \|^2 + \| Y(s) \|^2) ds < \frac{K_4 D_1 + D_2 - K_3 D_1}{\varepsilon},$$

which implies the existence of positive constants σ_1 and σ_2 such that

$$\int_0^t \| Z(s) \|^2 \, ds \leq \sigma_2 \text{ and } \int_0^t \| Y(s) \|^2 \, ds \leq \sigma_1.$$

From (4) we have

$$\begin{aligned} \int_0^t \| X'(s) \|^2 \, ds &= \int \| \Omega^{-1} Y(s) \|^2 \, ds \\ &\leq \int \left\| \Omega^{-1} \right\|^2 \| Y(s) \|^2 \, ds \\ &\leq \frac{\sigma_1}{\omega_0^2} = \beta_1. \end{aligned} \tag{25}$$

Also

$$\begin{aligned} \int_0^t \| X^{''}(s) \|^2 \, ds &= \int_0^t \Big(\| \theta(s) Y(s) + \Omega^{-1} Z(s) \|^2 \Big) ds \\ &\leq \int_0^t \big(\| \theta(s) \|^2 + \| \theta(s) \| \, \| \Omega^{-1} \| \big) \| Y(s) \|^2 \, ds \\ &\quad + \int_0^t \big(\| \Omega^{-1} \|^2 + \| \theta(s) \| \, \| \Omega^{-1} \| \big) \| Z(s) \|^2 \, ds. \end{aligned}$$

From (21) and (20) we have

$$\begin{aligned} \int_0^t \big(\| \theta(s) \|^2 + \| \theta(s) \| \, \| \Omega^{-1} \| \big) \| Y(s) \|^2 \, ds &\leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \int_0^t \| Y(s) \|^2 \, ds \\ &\leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \sigma_1, \end{aligned}$$

and

$$\begin{aligned} \int_0^t \big(\| \Omega^{-1} \|^2 + \| \theta(s) \| \, \| \Omega^{-1} \| \big) \| Z(s) \|^2 \, ds &\leq \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \int_0^t \| Y(s) \|^2 \, ds \\ &\leq \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \sigma_2. \end{aligned}$$

It follows

$$\int_0^t \| X^{''}(s) \|^2 \, ds \leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \sigma_1 + \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \sigma_2 \; = \beta_2. \tag{26}$$

Next, multiplying (3) by $X(t)$, we obtain

$$\Big\langle \big(\Omega(X)X'\big)'', X\Big\rangle + \langle \Psi(X')X'', X\rangle + \langle G(X)X', X\rangle + c \parallel X \parallel^2 = \langle X, P(t)\rangle. \tag{27}$$

Integrating (27) from 0 to t we have

$$c\int_0^t \parallel X(s) \parallel^2 ds = L_1(t) + L_2(t) + L_3(t), \tag{28}$$

where

$$\begin{aligned} L_1(t) &= -\int_0^t \Big\langle \big(\Omega(X(s))X'(s)\big)'', X(s)\Big\rangle ds, \\ L_2(t) &= -\int_0^t \Big\langle \Big(\Psi(X'(s))X''(s) + G(X(s))X'(s)\Big), X(s)\Big\rangle ds, \\ L_3(t) &= \int_0^t \langle X(s), P(s)\rangle ds. \end{aligned}$$

Integrating by parts and using (25) and (26), we obtain

$$\begin{aligned} L_1(t) &= -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle \\ &\quad + \langle \Omega X'(t), X'(t)\rangle - \int_0^t \langle \Omega X'(s), X''(s)\rangle\, ds \\ &\le \mid -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle \mid \\ &\quad + \int_0^t \frac{\omega_1}{2}\big(\parallel X'(s) \parallel^2 + \parallel X''(s) \parallel^2\big)\, ds \\ &\le \mid -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle \mid + \frac{\omega_1}{2}(\beta_1 + \beta_2). \end{aligned}$$

In view of (20) and (22) we get

$$|-\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle| \le D_5^2\big(q_1 + 2\omega_1\big),$$

for all $t \ge 0$. Consequently, there exists a constant l_1 such that $L_1(t) < l_1$, with $l_1 = D_5^2\big(q_1 + 2\omega_1\big) + \frac{\omega_1}{2}(\beta_1 + \beta_2)$. Similarly we have

$$\begin{aligned} L_2(t) &= -\int_0^t \big\langle \big(\Psi X''(s) - G X'(s)\big), X(s)\big\rangle\, ds \\ &\le \int_0^t \Big(\parallel \Psi \parallel \parallel X''(s) \parallel + \parallel G \parallel \parallel X'(s) \parallel \Big) \parallel X(s) \parallel ds \\ &\le \int_0^t \parallel \Psi \parallel \parallel X''(s) \parallel \parallel X(s) \parallel ds + \int_0^t \parallel G \parallel \parallel X'(s) \parallel \parallel X(s) \parallel ds \\ &\le \frac{a_1}{2}\int_0^t \parallel X''(s) \parallel^2 ds + (\frac{a_1 + b_1}{2})\int_0^t \parallel X(s) \parallel^2 ds + \frac{b_1}{2}\int_0^t \parallel X'(s) \parallel^2 ds \\ &\le \frac{a_1}{2}\beta_2 + \frac{b_1}{2}\beta_1 + (\frac{a_1 + b_1}{2})\int_0^t \parallel X(s) \parallel^2 ds. \end{aligned}$$

Next

$$\begin{aligned} L_3(t) &\leq \int_0^t \parallel X(s) \parallel \parallel P(s) \parallel ds \\ &\leq D_5 \int_0^t \lambda(s) ds \\ &\leq D_1 D_5. \end{aligned}$$

By (28) and condition (I_4) of the Theorem 5.1 we obtain

$$\left(c - (\frac{a_1 + b_1}{2})\right) \int_0^t \parallel X(s) \parallel^2 ds \leq K,$$

where $K = l_1 + \frac{a_1}{2}\beta_2 + \frac{b_1}{2}\beta_1 + D_1 D_5$. This fact completes the proof of theorem.

Example 5.1 As a special case consider the following equation

$$(\Omega(X(t))X'(t))'' + \Psi(X')X''(t) + G(X)X'(t) + cX(t) = P(t), \tag{29}$$

where

$$\begin{aligned} \Omega(X) &= \begin{pmatrix} \frac{\sin x}{1+x^2} + 2 & 0 \\ 0 & \frac{2}{10}\frac{\cos y}{1+y^2} + 2 \end{pmatrix}, \quad \Psi(Y) = \begin{pmatrix} 9 + \frac{1}{1+y^2} & 1 \\ 1 & 9 + \frac{1}{1+y^2} \end{pmatrix}, \\ G(X) &= \begin{pmatrix} \frac{1}{3+x^2} + 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad P(t) = \begin{pmatrix} \frac{\sin t}{1+t^2} \\ \frac{\cos t}{1+t^2} \end{pmatrix}, c = 7 \quad . \end{aligned}$$

Clearly, $\Psi(Y), G(X)$ and $\Omega(X)$ are symmetric matrices and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices $\Psi(Y), G(X)$ and $\Omega(X)$ as follows:

$$\begin{aligned} \omega_0 &= 1 \leq \lambda_i(\Omega(X)) \leq 2.2 = \omega_1, \\ a_0 &= 8 \leq \lambda_i(\Psi(Y)) \leq 11 = a_1, \\ b_0 &= 2 \leq \lambda_i(G(X)) \leq \frac{7}{3} = b_1, \end{aligned}$$

for $i \in \{1, 2\}$. For $t \in [0, +\infty)$ a straightforward calculation gives

$$\begin{aligned} \int_0^t \parallel \Omega'(X(s)) \parallel du &= \int_0^t \left| (\frac{\cos x}{1+x^2} - \frac{2x \sin x}{(1+x^2)^2}) x'(s) \right| ds \\ &\quad + \int_0^t \left| (\frac{-\sin y}{1+y^2} - \frac{2y \cos y}{(1+y^2)^2}) y'(s) \right| ds \\ &\leq \int_{\theta_1(t)}^{\theta_2(t)} \left| (\frac{\cos u}{1+u^2} - \frac{2u \sin u}{(1+u^2)^2}) \right| du \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left| (\frac{-\sin v}{1+v^2} - \frac{2v \cos v}{(1+v^2)^2}) \right| dv \\ &< \left(\int_{-\infty}^{+\infty} \left| \frac{1+u^2+2u}{(1+u^2)^2} \right| du + \int_{-\infty}^{+\infty} \left| \frac{1+u^2+2u}{(1+u^2)^2} \right| du \right) \\ &= (\pi + 2), \end{aligned}$$

where

$$\theta_1(t) = \min\{x(0), x(t)\}, \qquad \theta_2(t) = \max\{x(0), x(t)\},$$
$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

Similarly

$$\int_{-\infty}^{+\infty} \|G'(X(s))\| ds = \int_{-\infty}^{+\infty} \left| \frac{-2u}{(3+u^2)^2} \right| du = \frac{2}{3}.$$

Now, we have

$$\| P(t) \| = \sqrt{\frac{\sin^2 t}{1+t^2} + \frac{\cos^2 t}{1+t^2}} = \frac{1}{1+t^2} < \frac{2}{1+t^2} = \lambda(t) < 2 = d_1.$$

So,

$$\int_0^t \| \lambda(s) \| \, ds = \int_0^t \frac{2}{1+s^2} ds < \int_0^{+\infty} \frac{2}{1+s^2} ds = \pi = D_1.$$

By taking $\beta = 0.44$, it follows easily that

$$0.4375 = \frac{7}{16} = \frac{c}{a_0 b_0} < \beta < \frac{1}{\omega_1} = 0.45455.$$

We have also

$$c - \frac{a_1 + b_1}{2} = \frac{1}{3} > 0.$$

Thus, all the conditions of Theorem 5.1 are satisfied.

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On Efficient Chaotic Optimization Algorithm Based on Partition of Data Set in Global Research Step

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Received: April 18, 2017; Revised: December 18, 2017

Abstract: The great difficulty facing the optimization algorithms is the easiness of trapping into local optima. Many researchers have benefited from the good characteristics of chaotic mappings to overcome this difficulty, but for some complex functions the problem persists. In this paper, we attempt to avoid this problem by proposing a new chaos optimization technique based on partition of data set in global research step. The numerical results show that the proposed algorithm provides the best results as compared to other ones.

Keywords: *chaos optimization; test functions; probability density function; Lozi map.*

Mathematics Subject Classification (2010): 65P20, 65Q10, 37N40, 65K10, 80M50, 37E05.

1 Introduction

Chaos theory has been successfully developed since its early years through wide applications in other sciences such as physics, mechanics, electronics, biology, economy, astronomy, meteorology, optimization, secure communication, ... etc [1–7]. As far as optimization problems of some usual functions that are continuously differentiable are concerned, some traditional optimization algorithms such as the Newton method, the gradient method and the Hessians method [8, 9] can get their global optimal points with the advantage of speed convergence and high precision. However, these traditional optimization algorithms will easily trap into local optimum when solving optimization problems of some multi-modal functions.

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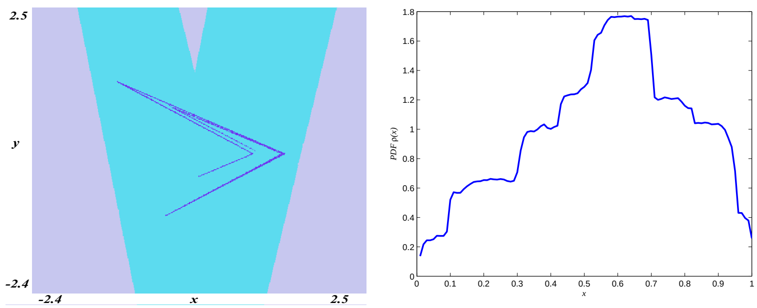
This is due to the several important dynamical characteristics of chaos, namely: the sensitive dependence on initial conditions, ergodicity, pseudo-randomness, and strange attractor with self-similar fractal pattern. Many researchers use the chaotic mappings in the optimization algorithm in order to avoid falling into local optimum [10, 11].

Recently, researchers have focused on developing the hybrid algorithms by combining heuristic algorithms with chaos searching technique to solve non linear system of equations and optimization problems such as chaotic Monte Carlo optimization, chaotic BFGS, chaotic particle swarm optimization, chaotic genetic algorithms, chaotic harmony search algorithm, chaotic simulated annealing, gradient based methods and so on [12–14].

Among those who tried to find a solution to the problem of trapping in local minima are L.S. Coelho in [15] and T. Hamaizia et al in [16]. They have resolved this problem for a large range of objective functions but for some complex functions the problem persists as we will explain later. In this paper, we recall the algorithm proposed by T. Hamaizia et al in [16] and we propose some modifications in order to improve it. The chaotic variables are generated by using the Lozi map [17] defined by the function L as follows:

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1\begin{pmatrix} x \\ y \end{pmatrix} \\ L_2\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 - a \mid x \mid + by \\ x \end{pmatrix}. \tag{1}$$

It is a $2-d$ invertible iterated map that gives a chaotic attractor called the Lozi attractor which is obtained for $a = 1.4$ and $b = 0.3$ as shown in Figure 1 (a). Numerical computation of the density $\rho(s)$ of iterated values $x(k)$ is displayed in Figure 1 (b). In this figure, the iterated values $x(k)$ are normalized in the range $[0, 1]$ i.e. $\int_0^1 \rho(x)\,\mathrm{d}x = 1$ and we notice that the highest value of $\rho(x)$ is approximately 1.8 when x is in the neighbourhood of 0.6.



(a) (b)

Figure 1: (a) Chaotic attractor of Lozi map (1) and its attractive basin obtained for $a = 1.7$ and $b = 0.5$. (b) Density of $x(k)$ in (1) over the interval $[0, 1]$ splitted into 100 boxes for 10, 000, 000 iterated values.

2 The ICOLM Algorithm

In [16] T. Hamaizia and R. Lozi have used a sampling mechanism to coordinate the research methods based on chaos theory, and they refined the final solution using a second method of local search. The obtained results show that the ICOLM algorithm is fast and converges to a good optimum compared with the COLM algorithm. But for some complex functions the problem persists. In order to avoid this problem, we will give some modifications of this method so that to improve it. We can describe this algorithm as follows:

Firstly, we choose a map and adopt it to have a chaotic behavior in order to use it to generate several sequences of points by using different initial conditions.

Secondly, every sequence $\{y(i), i = 1, 2, ..., n\}$ is normalized in the range $[0, 1]$ as follows:

$$z(i) = \frac{y(i) - \alpha}{\beta - \alpha}$$

for all $i = 1, 2, ...n$, where $\alpha = \min\{(y(i), i \geq 1\}$, $\beta = \max\{(y(i), i \geq 1\}$. The rest are:

Algorithm 2.1 Inputs:
M_g**:** max number of iterations of chaotic global search.
Mgl_1**:** max number of iterations of first chaotic local search in global search.
Mgl_2**:** max number of iterations of second chaotic local search in global search.
M_l**:** max number of iterations of chaotic local search.
$Mt = M_g(Mgl_1 + Mgl_2) + M_l$**:** stopping criterion of chaotic optimization method in iterations.
λ_{gl1}**:** step size in first global-local search.
λ_{gl2}**:** step size in second global-local search.
λ**:** step size in chaotic local search.

Outputs:
$\bar{x}$**:** best solution from current run of chaotic search.
$\bar{f}$**:** best objective function (minimization problem).

Step 1: Initialization of the numbers M_g, Mgl_1, Mgl_2, M_l of steps of chaotic search and initialization of parameters λ_{gl1}, λ_{gl1}, λ and initial conditions. Set $k = 1$, $y_1(1)$, $y_2(1)$, $a = 1.7$ and $b = 0.3$. Set the initial best objective function $\bar{f} = +\infty$.

-Step 2: Algorithm of chaotic global search:
while $k \leq M_g$ **do**
$x_i(k) = L_i + z_i(k)(U_i - L_i)$, $i = 1, 2, ..., n$
if $f(x(k)) < \bar{f}$, **then**
$\bar{x} = x(k)$, $\bar{f} = f(x(k))$
end if

-Step 2-1: Sub algorithm of first chaotic global-local search:
while $j \leq M_{gl1}$ **do**
for $i = 1$ to n **do**
if $r \leq 0.5$ **then (where** r **is a uniformly distributed random variable with range** $[0, 1]$**)**
$x_i(j) = \bar{x_i} + \lambda_{gl1} z_i(j)(U_i - \bar{x_i})$
else
$x_i(j) = \bar{x_i} - \lambda_{gl1} z_i(j)(\bar{x_i} - L_i)$
end if
end for
if $f(x(j)) < \bar{f}$, **then**
$\bar{x} = x(j)$, $\bar{f} = f(x(j))$
end if

$j = j + 1$
end while
- Step 2-2: Sub algorithm of second chaotic global-local search:
while $s \leq M_{gl2}$ **do**
for $i = 1$ to n **do**
if $r \leq 0.5$ **then**
$x_i(s) = \bar{x_i} + \lambda_{gl2} z_i(s)(U_i - \bar{x_i})$
else
$x_i(s) = \bar{x_i} - \lambda_{gl2} z_i(s)(\bar{x_i} - L_i)$
end if
end for
if $f(x(s)) < \bar{f}$, **then**
$\bar{x} = x(s)$, $\bar{f} = f(x(s))$
end if
$s = s + 1$
end while
$k = k + 1$
end while
- Step 3: Algorithm of chaotic local search:
while $k \leq M_l$ **do**
for $i = 1$ to n **do**
if $r \leq 0.5$ **then**
$x_i(k) = \bar{x_i} + \lambda z_i(k)(U_i - \bar{x_i})$
else
$x_i(k) = \bar{x_i} - \lambda z_i(k)(\bar{x_i} - L_i)$
end if
end for
if $f(x(k)) < \bar{f}$, **then**
$\bar{x} = x(k)$, $\bar{f} = f(x(k))$
end if
$k = k + 1$
end while.

Although this method was developed to find a solution to trapping into local optimization when solving optimization problems of some multi-modal functions, the success was partial because if the objective function is not smooth, this method will easily trap into local minima as we are going to clarify. If in step k in global search the optimal solution of our problem is $f(x^*)$, then all the points $x(s), s > k$ in the red part of Figure 2 will be ignored during the search; but it is possible that the global minima will be in the neighbourhood of one point of the red part. To solve this problem we suggest to divide the number of iterations in global search into packs and at the beginning of each pack we set the best objective function $\bar{f} = +\infty$.

On the other hand, due to the non-repetition of chaos, the chaotic research can carry out overall searches at higher speed than stochastic ergodic searches that depend on probabilities. Motivated by this idea, we will replace the step of local search (random step) by a chaotic local search as we will explain later. This is why we will call this new method Pure Chaotic Optimization Algorithm (PCOA).

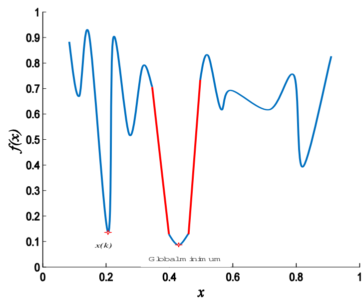


Figure 2: Example of the trapping into local minima.

3 Pure Chaotic Optimization Algorithm

As mentioned in the previous section, the fundamental changes that will be undertaken on the ICOLM are:

At first, we divide the data set that will be used in global search into packs and at the beginning of each pack we set the best objective function $\bar{f} = +\infty$ in order to go out of the local minima.

The second change is in the global local search and local search where we use chaotic search instead of random search. To apply the global local search, we use a linear transformation to project the points of chaotic sequences in the neighbourhood of the point of global search and the same idea will be used in the local search. In the following we give an example to illustrate this idea.

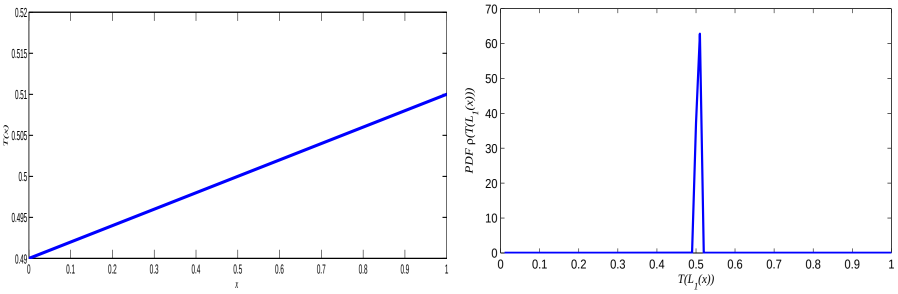
Example 3.1 In order to facilitate the process suppose that the search domain is $[l, u] = [0, 1]$ and we need to do a local search in the neighbourhood of the point $x^* = 0.5$ (i.e. the interval of local search is $[x^* - \lambda, x^* + \lambda]$, but if $x^* - \lambda < l$ (resp $x^* + \lambda > u$), the interval of local search is $[l, x^* + \lambda]$ (resp $[x^* - \lambda, u]$)). To project all the points in the neighbourhood of the point $x^* = 0.5$ we use the following linear transformation:

$$T(x) = \frac{2\lambda}{u - l} x + (x^* - \lambda).$$

Figure 3 (a) shows the plot of transformation T where we see that all the points of the interval $[l, u]$ are transformed into the interval $[x^* - \lambda, x^* + \lambda]$ ($\lambda = 0.01$) and Figure 3 (b) shows the probability density function of $T(L_1)$.

In the following we are going to describe the pure chaotic optimization algorithm.

Algorithm 3.1 Inputs:
N**:** max number of iterations of chaotic global search.
N_p**:** max number of packets of global search.
M_g**:** max number of iterations of chaotic global search for any packets.
M_{gl}**:** max number of iterations of chaotic local search in global search.
M_l**:** max number of iterations of chaotic local search.
$Mt = N_p(M_g M_{gl} + M_l)$**:** stopping criterion of chaotic optimization method in iterations.



(a) (b)

Figure 3: (a) Transformation T. (b) Probability density function of $T(L_1)$.

λ_{gl}**:** the width of the interval in chaotic local search in global search.
λ**:** the width of the interval in chaotic local search.

Outputs:

$\bar{x}$**:** best solution from current run of chaotic search.
$\bar{f}$**:** best objective function (minimization problem).

Step 1: Initialization of the numbers M_g, M_{gl}, M_l of steps of chaotic search and initialization of parameters λ_{gl}, λ and initial conditions. The Lozi map (1) is adopted to have a chaotic behavior in order to use it for generating several sequences of points by using different initial conditions (the number of sequences is equal to dimension of the objective function) after every sequence $\{y(i), i = 1, 2, ...n\}$ is normalized in the range $[0, 1]$ as follows:

$$z(i) = \frac{y(i) - \alpha}{\beta - \alpha}$$

for all $i = 1, 2, ...n$, where $\alpha = min\{(y(i), i \geq 1\}$, $\beta = max\{(y(i), i \geq 1\}$.

-Step 2-1: Algorithm of chaotic global search:

for $t = 1 : N_p$
Set the initial best objective function $\bar{f(t)} = +\infty$.
while $k \leq M_g$ **do**
$x_i(k) = L_i + z_i(k)(U_i - L_i)$, $i = 1, 2, ..., n$
if $f(x(k)) < \bar{f}$, **then**
$\bar{x} = x(k)$, $\bar{f} = f(x(k))$

- Step 2-2: Sub algorithm of chaotic global-local search:

Transform the points generated by Lozi map in the neighbourhood of the point $\bar{x}$ and we begin the search
while $j \leq M_{gl}$ **do**
if $f(x(j)) < \bar{f}$, **then**
$\bar{x} = x(j)$, $\bar{f} = f(x(j))$
end if
$j = j + 1$
end while
end if
$k = k + 1$

end while
end for

- Step 3: Algorithm of chaotic local search:

Transform the points generated by logistic map in the neighbourhood of the point $\bar{x}$ and we begin the search
while $k \leq M_l$ **do**
if $f(x(k)) < \bar{f}$, **then**
$\bar{x} = x(k)$, $\bar{f} = f(x(k))$
end if
$k = k + 1$
end while.

During the chaotic local search, the step size λ (resp λ_{gl}) is an important parameter in convergence behavior of optimization method which adjusts small ergodic ranges around X*. The step sizes λ and λ_{gl} are employed to control the impact of the current best solution on generating a new trial solution. The small λ and λ_{gl} tend to perform exploitation to refine results by local search, while the large ones tend to facilitate a global exploration of search space.

4 Numerical Examples and Discussion

In order to test this new method vs the previous one in very tough conditions, the simulation results are obtained with the following four objective functions.

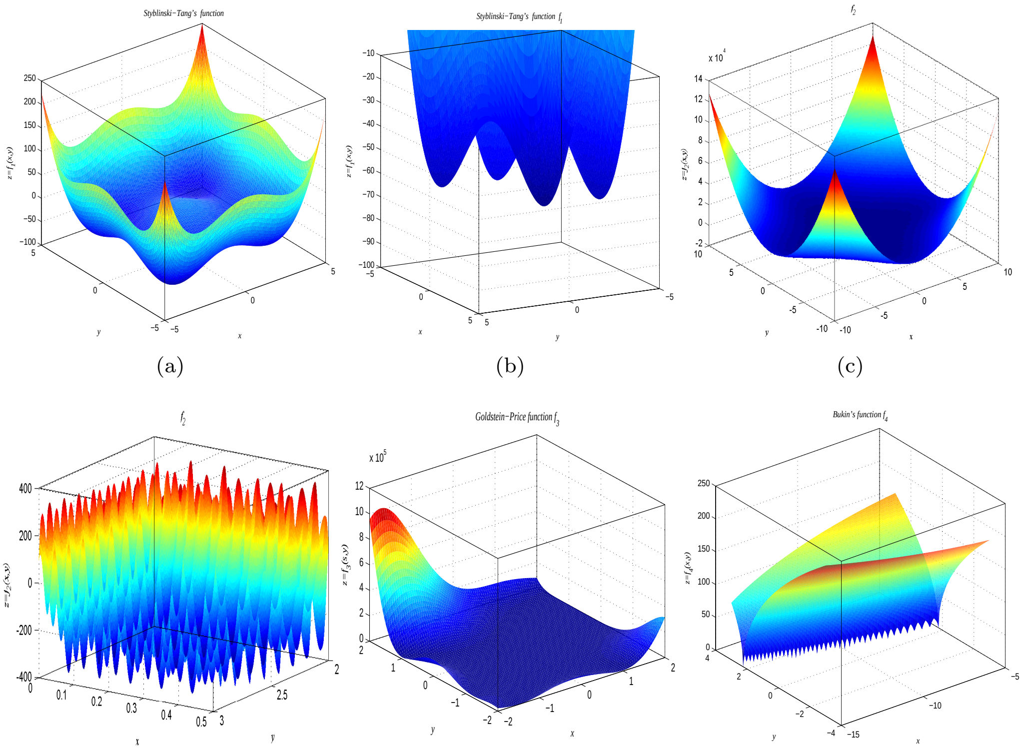
4.1 Some test functions

1. $$f_1(x_1, x_2, ..., x_n) = \frac{\sum_{i=1}^{n}(x_i^4 - 16x_i^2 + 5x_i)}{2},$$
   where $-5 \leq x_i \leq 5$ for $1 \leq i \leq n$.

2. $$f_2(x_1, x_2) = x_1^4 - 7x_1^2 + x_2^4 - 9x_2^2 - 5x_2 + 11x_1^2x_2^2 + 99\sin(71x_1) + 137\sin(97x_1x_2) + 131\sin(51x_2),$$
   where $-10 \leq x_1 \leq 10$ and $-10 \leq x_2 \leq 10$.

3. $$f_3(x_1, x_2) = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)]\times [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)],$$
   where $-2 \leq x_1 \leq 2$ and $-2 \leq x_2 \leq 2$.

4. $$f_4(x_1, x_2) = 100\sqrt{|x_2 - 0.01x_1^2|} + 0.01\,|x_1 + 10|\,,$$
   where $-15 \leq x_1 \leq -5$ and $-3 \leq x_2 \leq 3$.



(a) (b) (c)

(d) (e) (f)

Figure 4: (a) Styblinski-Tang's function f_1. (b) Magnification of Styblinski-Tang's function f_1. (c) Function f_2. (d) Magnification of function f_2. (e) Goldstein-Price function f_3. (f) Bukin function f_4.

Figures 4 (a) and (b) show the 3D plots of the Styblinski-Tang function f_1 which is a d-dimensional function, usually evaluated on the hypercube $x_i \in [-5, 5]$, for all $i = 1, ..., d$. It has a global minimum

$$-39.16617 \times d \leq f_4(-2.903534, ..., -2.903534) \leq -39.16616 \times d.$$

Concerning f_2 shown in Figures 4 (c) and (d), it possesses hundreds of local minima [16], but its global minimum is not yet theoretically known.

f_3 is the Goldstein-Price function usually evaluated on the rectangle

$$(x_1, x_2) \in [-2, 2] \times [-2, 2],$$

it has a lot of local minima and one global minimum $f_3(0, -1) = 3$ and the 3D plot of this function is in Figure 4 (e).

f_4 is the Bukin function which is usually evaluated on the rectangle

$$(x_1, x_2) \in [-15, -5] \times [-3, 3],$$

it has a lot of local minima and one global minimum $f_4(-10, 1) = 0$, see Figure 4 (f).

4.2 Numerical experiments

In order to enrich our study, we are going to use different values of step sizes and different values of the number of iterations for both methods that are presented in Tables 1 and 2. Each optimization method was implemented in Matlab (MathWorks). All the programs were run on a 2.53 GHz, $i3$ processor with 4 GB of random access memory. Since the ICOLM algorithm gives random results, in each case study 50 independent runs are made involving 50 different initial trial conditions and all the results are summarised in Table 3; however the pure chaotic optimization algorithm is a deterministic method, therefore one run is made involving 50 different initial trial conditions and all the results are summarised in Table 4.

We generally believe that the use of large number of steps will lead us closer to the global minimum for all test functions. But this is not true as shown in Table 3 because of the trap of local minima mentioned in Section 2.

Concerning the optimization results by using the PCOA we have:

- For the function f_3 the global minimum is easily reached in few steps and little time compared with the ICOLM algorithm as explained in Tables 3 and 4.
- Concerning f_1, the global minimum is obtained by using configurations C3.
- For f_2 which possesses hundreds of local minima, the best result is obtained using configurations C3 and the global minimum is not yet theoretically known.
- Finally, the best result for f_4 is obtained using configurations C3.
  We note that the PCOA converges faster than the ICOLM as shown in Tables 3 and 4.

	λ	λ_{gl1}	λ_{gl2}	M_g	M_{gl1}	M_{gl2}	M_l
C1	0.01	0.04	0.01	30	5	5	20
C2	0.01	0.04	0.01	100	5	5	50
C3	0.001	0.04	0.01	500	10	10	100

Table 1: The set of parameter values for every run of the ICOLM algorithm.

	λ	λ_{gl}	N_p	M_g	M_{gl}	M_l
C1	0.001	0.01	100	10	100	100
C2	0.002	0.05	100	100	200	200
C3	0.005	0.08	1000	100	200	200

Table 2: The set of parameter values for every run of the PCOA algorithm.

T Fun	Case	Op So	Op Pts	Mean val	Std.Dev	T/s
f_1	C1	-103.3610	(2.7455, -2.8977,-2.9069)	-103.3383	0.0136	6.461655
	C2	-117.4956	(-2.8970,2.9005,-2.8926)	-117.4806	0.0114	20.458741
	C3	-117.4983	(-2.9046, -2.9000,-2.9038)	-117.4867	0.0082	180.311540
f_2	C1	-392.9923	(0.2443,2.0614)	-383.8462	7.6147	7.619080
	C2	-395.8094	(0.2434,2.0632)	-389.7800	5.6617	27.695888
	C3	-395.7769	(0.2434,2.0640)	-387.4540	6.1347	253.251734
f_3	C1	3.0669	(0.0108,-1.0068)	3.7525	0.2849	3.561953
	C2	3.0004	(-0.0007, -1.0010)	3.0064	0.0052	11.280039
	C3	3.0001	(0.0006, -1.0001)	3.0039	0.0026	105.905089
f_4	C1	0.1027	(-9.4415,0.8914)	0.7547	0.4245	3.562340
	C2	0.02794	(-9.4132,0.8861)	0.4295	0.1159	12.257843
	C3	0.0487	(-9.5870,0.9191)	0.3587	0.2091	109.698371

Table 3: Optimization results over 50 runs for 3 parameter configurations using ICOLM algorithm.

Test Function	Cases	Optimal solution	Optimal point	T/s
f_1	C1	-117.4772	(-2.8830, -2.8759, -2.9111)	2.648436
	C2	-117.4924	(-2.8869,-2.8949,-2.9014)	8.289850
	C3	-117.4985	(-2.9034, -2.9026,-2.8952)	47.761714
f_2	C1	-390.2672	(0.0622,1.8189)	2.237673
	C2	-395.8622	(0.2433,2.0638)	5.072239
	C3	-395.8742	(0.2432,2.0636)	49.7400
f_3	C1	3.0000	(-0.0001,-0.9999)	1.202800
	C2	3.0000	(-0.0000, -1.0000)	3.343877
	C3	3.0000	(-0.0000, -1.0000)	18.763473
f_4	C1	0.0322	(-10.7807,1.1622)	1.122277
	C2	0.0108	(-9.6809, 0.9372)	2.754711
	C3	0.0086	(-10.2723,1.0552)	25.725318

Table 4: Optimization results over one run for 3 parameter configurations using PCOA algorithm.

5 Conclusion

In this paper, we have presented a new technique of chaotic optimization algorithm inspired by ICOLM methods [16]. In order to test the numerical performance of this new technique, the four non linear multi modal benchmark functions are employed. More detailed analysis on this new technique by using other maps and testing them on a large number of test functions in higher dimension will be provided in near future.

Acknowledgment

The authors wish to thank the editor and anonymous reviewers for their valuable suggestions and comments.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 53–66

Some Gronwall Lemmas Using Picard Operator Theory: Application to Dynamical Systems

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Received: January 12, 2017; Revised: November 21, 2017

Abstract: In this paper we derive optimal explicit bounds for the solutions to integral inequalities. We rewrite the inequalities in terms of integral operators and we get the bound as a fixed point of the corresponding operator. As application, we study the stability of ceratin dynamical systems.

Keywords: *Gronwall lemma; dynamical system; fixed point; Picard operators.*

Mathematics Subject Classification (2010): 26D10, 26D15, 26D20, 34A40.

1 Introduction

Integral inequalities are a necessary tool in the study of various classes of equations. In 1919, Gronwall [10] introduced the famous Gronwall inequality in the study of the solutions of differential equations. Since then, many contributions have been made (see [1]-[3]). The applications of integral inequalities were developed in a remarkable way in the study of the existence, the uniqueness, the comparison, the stability and continuous dependence of the solution in respect to data. In the last few years, a series of generalizations of these inequalities appeared. The problem of stability can be solved by Lyapunov techniques for differential equations (see [12]- [14]), or in terms of nonlinear integral inequalities. These inequalities can be used in the analysis of various problems in the theory of nonlinear differential equations and control systems (see [3] and references therein). There is an extensive literature on the inequalities, for example, the Barbalats lemma is an integral inequality used in applied nonlinear control. The second Lyapunov method has long played an important role in the history of stability theory, and it will with no doubt continue to serve as an indispensable tool in future research

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papers (see [14]). V. I. Zubov studied the boundaries of the asymptotic stability domain in which he proved the theorem of the asymptotic stability domain. This result is now known as Zubov's theorem (see [17], [6]). The concepts of stability and boundedness of solutions have been studied extensively by Taro Yoshizawa (see [18], [19], [7]).

In a recent paper [11], I.A. Rus has formulated ten problems of interest in the theory of Gronwall lemmas. One of them concerns finding examples of Gronwall-type lemmas in which the upper bounds are fixed points of the corresponding operator A (**Problem 5**). The new inequalities, derived in this paper, are useful in many applications, in particular to the stability of dynamical systems. We propose new sufficient conditions to ensure the global uniform asymptotic stability of time-varying systems described by the following equation:

$$\dot{x} = f(t,x) + g(t,x), \tag{1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $\mathbb{R}_+ \times \mathbb{R}^n$, and the associated nominal system is given by:

$$\dot{x} = f(t,x). \tag{2}$$

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_+$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution at time t_0 starting from the point x_0.

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [4]: $\mathcal{K}$ is the class of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. A positive definite function $\mathbb{R}_+ \to \mathbb{R}_+$ is the one that is zero at the origin and positive otherwise. We define the closed ball $B_r := \Big\{x \in \mathbb{R}^n : \ \|x\| \leq r\Big\}$.

2 Abstract Gronwall Lemma

To present our problem we need some standard notations of Nonlinear Analysis. Let X be a nonempty set and $A : X \to X$ be an operator. We denote by $F_A = \{x \in X / Ax = x\}$ the fixed point set of the operator A. The symbol, $F_A = \{x_A^*\}$, has the following meaning: the operator A has a unique fixed point and we denote this unique fixed point by x_A^*. In general, throughout this paper we follow the notation and terminology from I.A. Rus [15] and [16].

Definition 2.1 (I.A. Rus [15]). Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator if:
i) $F_f = \{x^*\}$.
ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

In terms of the Picard operators, a classical result in metric fixed point theory has the following form ([13], [9]).

Proposition 2.1 *(Contraction principle). Let (X, d) be a complete metric space and let $f : X \to X$ be an a-contraction, i.e., $a \in]0, 1[$ and $d(f(x), f(y)) \leq a.d(x, y)$, for each $x, y \in X$. Then f is a Picard operator.*

Proposition 2.2 *(Abstract Gronwall lemma). Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:*
i) A is a Picard operator ($F_A = \{x_A^*\}$).
ii) A is an increasing operator.
Then we have:
a) $x \in X$, $x \leq A(x) \Rightarrow x \leq x_A^*$.
b) $x \in X$, $x \geq A(x) \Rightarrow x \geq x_A^*$.

3 Main Results

In this section we point out some Gronwall-type inequalities using some results concerning Picard operator theory.

The following result is well known from the book of A.N. Filatov (see [8]), here we will give a new proof of it using the theory of operators.

Theorem 3.1 *Let $x \in \mathcal{C}([a, b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \delta_2(t-a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad \forall t \in [a, b], \tag{3}$$

where $\delta_1 > 0$, δ_2 and δ_3 are real numbers, then

$$x(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1(t-a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a, b]. \tag{4}$$

Proof. Let $(X, \to, \leq) = (\mathcal{C}[a, b], \xrightarrow{\|.\|_\tau}, \leq)$, where $\|.\|_\tau$ is the Bielecki norm on $\mathcal{C}[a, b]$, i.e., τ is a positive real number and

$$\|x\|_\tau = \max_{a \leq t \leq b} \left(|x(t)| \exp(-\tau(t-a))\right).$$

We consider on $X = \mathcal{C}[a, b]$ the operator $A : X \to X$ defined by

$$A(x)(t) = \delta_2(t-a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad t \in [a, b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \delta_2(t-a) + \delta_1 \int_a^t x(s)ds + \delta_3, \quad t \in [a, b].$$

By differentiation, we get

$$x'(t) = \delta_1 x(t) + \delta_2,$$

which is an ordinary differential equation (added to an initial condition, this ODE admits a unique solution according to the Cauchy-Lipschitz theorem). Since $x(a) = \delta_3$, it comes out that

$$x(t) = \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1(t-a) - \frac{\delta_2}{\delta_1}.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp \delta_1 (t-a) - \frac{\delta_2}{\delta_1}, \quad t \in [a, b].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a, b]$, if $x \leq y$, then $A(x) \leq A(y)$. The last point is to show that A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned}
|A(x)(t) - A(y)(t)| e^{-\tau(t-a)} &\leq \delta_1 e^{-\tau(t-a)} \int_a^t |x(s) - y(s)| ds \\
&\leq \|x - y\|_\tau \delta_1 e^{-\tau(t-a)} \int_a^t e^{\tau(s-a)} ds \\
&\leq \|x - y\|_\tau \frac{\delta_1}{\tau} \left[1 - e^{-\tau(t-a)}\right] \\
&\leq \|x - y\|_\tau \frac{\delta_1}{\tau} \left[1 - e^{-\tau(b-a)}\right].
\end{aligned}$$

Then $\|A(x) - A(y)\|_\tau \leq \|x - y\|_\tau \frac{\delta_1}{\tau} \left[1 - e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. $\square$

Remark 3.1 If $\delta_2 \geq 0$, then there is a direct proof for this well known Gronwall-type lemma.

Theorem 3.2 *Let $x \in \mathcal{C}([a, b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \delta_2 (t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad \forall t \in [a, b], \tag{5}$$

where $\delta_1 > 0$, δ_2, δ_3 are real numbers and φ is a continuous function on $[a, b]$, then

$$x(t) \leq \frac{\delta_2}{\delta_1} \exp \delta_1 (t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1 (t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a, b]. \tag{6}$$

Proof. We use the same notations as in the last proof. Let the operator A be defined by

$$A(x)(t) = \delta_2 (t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad t \in [a, b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \delta_2 (t-a) + \delta_1 \int_a^t x(s) ds + \varphi(t), \quad t \in [a, b].$$

By differentiation, we get

$$x'(t) = \delta_1 x(t) + \delta_2 + \varphi'(t),$$

which is an ordinary differential equation. Since $x(a) = \varphi(a)$, it comes out that

$$x(t) = \frac{\delta_2}{\delta_1} \exp \delta_1 (t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1 (t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \frac{\delta_2}{\delta_1} \exp \delta_1 (t-a) + \delta_1 \int_a^t \varphi(s) \exp \delta_1 (t-s) ds + \varphi(a) - \frac{\delta_2}{\delta_1}, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. On the other hand, by the same calculation as in the previous theorem, one can easily check that A is a contraction with respect to $\|.\|_\tau$, with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Remark 3.2 If the function φ is a constant, then we get the particular case of Theorem 3.1.

Theorem 3.3 *Let $x \in \mathcal{C}([a,b], \mathbb{R}_+)$ be such that*

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad \forall t \in [a,b], \tag{7}$$

where α is continuous and β is a continuous function on $[a,b]$, then

$$x(t) \leq \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp \left(\int_s^t \beta(u) du \right) ds, \quad \forall t \in [a,b]. \tag{8}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t) = \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad t \in [a,b].$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$x(t) = \alpha(t) + \beta(t) \int_a^t x(s) ds, \quad t \in [a,b].$$

By differentiation, we get

$$\beta(t) x'(t) = \left[\beta'(t) + \beta^2(t) \right] x(t) + \alpha'(t)\beta(t) - \beta'(t)\alpha(t),$$

which is an ordinary differential equation. Since $x(a) = \alpha(a)$, it comes out that

$$x(t) = \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp \left(\int_s^t \beta(u) du \right) ds.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \alpha(t) + \beta(t) \int_a^t \alpha(s) \exp \left(\int_s^t \beta(u) du \right) ds, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. The last point is to show that A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned}|A(x)(t)-A(y)(t)|e^{-\tau(t-a)} &\leq \beta(t)e^{-\tau(t-a)}\int_a^t |x(s)-y(s)|ds\\ &\leq \|x-y\|_\tau \beta(t)e^{-\tau(t-a)}\int_a^t e^{\tau(s-a)}ds\\ &\leq \|x-y\|_\tau \frac{\beta(t)}{\tau}\left[1-e^{-\tau(t-a)}\right]\\ &\leq \|x-y\|_\tau \frac{\|\beta\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right].\end{aligned}$$

Then $\|A(x)-A(y)\|_\tau \leq \|x-y\|_\tau \frac{\|\beta\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. $\square$

Remark 3.3 If $\alpha(t)=\delta_3$ and $\beta(t)=\delta_1$, then we get the particular case of Theorem 3.1.

The following result is well known from the book of Filatov and Scharova (1976), here we will give a new proof of it using the theory of operators.

Theorem 3.4 *Let $x \in \mathcal{C}([a,b],\mathbb{R}_+)$ be such that*

$$x(t) \leq \alpha(t)+\beta(t)\int_a^t k(s)x(s)ds, \quad \forall t \in [a,b], \tag{9}$$

where α is continuous, β and k are continuous functions on $[a,b]$, then

$$x(t) \leq \alpha(t)+\beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds, \quad \forall t \in [a,b]. \tag{10}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t)=\alpha(t)+\beta(t)\int_a^t k(s)x(s)ds, \quad t \in [a,b].$$

Suppose that x is a fixed point of A, then $A(x)=x$ or, equivalently,

$$x(t)=\alpha(t)+\beta(t)\int_a^t k(s)x(s)ds, \quad t \in [a,b].$$

By differentiation, we get

$$x'(t)=\left[\beta(t)k(t)+\frac{\beta'(t)}{\beta(t)}\right]x(t)+\alpha'(t)-\alpha(t)\frac{\beta'(t)}{\beta(t)},$$

which is an ordinary differential equation. The solutions of the homogenous equation are

$$x(t)=\lambda\beta(t)\exp\left(\int_a^t \beta(s)k(s)ds\right).$$

A particular solution can be obtained using the method of variation of the constant, then the solutions of our ODE are

$$x(t) = \lambda\beta(t)\exp\left(\int_a^t \beta(s)k(s)ds\right) + \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds.$$

Since $x(a) = \alpha(a)$, it comes out that

$$x(t) = \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we get

$$x_A^*(t) = \alpha(t) + \beta(t)\int_a^t \alpha(s)k(s)\exp\left(\int_s^t \beta(u)k(u)du\right)ds, \quad \forall t \in [a,b].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. On the other hand, A is a contraction with respect to $\|.\|_\tau$. We have

$$\begin{aligned}
|A(x)(t) - A(y)(t)|e^{-\tau(t-a)} &\leq \beta(t)e^{-\tau(t-a)}\int_a^t k(s)|x(s)-y(s)|ds \\
&\leq \|x-y\|_\tau \beta(t)e^{-\tau(t-a)}\int_a^t k(s)e^{\tau(s-a)}ds \\
&\leq \|x-y\|_\tau \frac{\beta(t)}{\tau}\|k\|_\infty\left[1-e^{-\tau(t-a)}\right] \\
&\leq \|x-y\|_\tau \frac{\|\beta\|_\infty\|k\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right].
\end{aligned}$$

Then $\|A(x) - A(y)\|_\tau \leq \|x-y\|_\tau \dfrac{\|\beta\|_\infty\|k\|_\infty}{\tau}\left[1-e^{-\tau(b-a)}\right]$ and A is a contraction with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

Theorem 3.5 *Let $x(t)$ be continuous and nonnegative on $[0,h]$ and satisfy*

$$x(t) \leq a(t) + \int_0^t (a_1(s)x(s) + b(s))\,ds, \tag{11}$$

where $a_1(t)$ and $b(t)$ are nonnegative integrable functions. Then, on $[0,h]$

$$x(t) \leq a(t) + \int_0^t (a_1(s)a(s) + b(s))\exp\left(\int_s^t a_1(\xi)d\xi\right)ds. \tag{12}$$

Proof. Using the same notations, let the operator A be defined by

$$A(x)(t) = a(t) + \int_0^t (a_1(s)x(s) + b(s))\,ds, \quad t \in [0,h].$$

We note that $F_A = \{x_A^*\}$, where

$$x_A^*(t) = a(t) + \int_0^t (a_1(s)a(s) + b(s))\exp\left(\int_s^t a_1(\xi)d\xi\right)ds, \quad \forall t \in [0,h].$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a,b]$, if $x \leq y$, then $A(x) \leq A(y)$. On the other hand, A is a contraction with respect to $\|.\|_\tau$, with τ suitably chosen. Finally, the proof follows from Proposition 2.2. □

For the following results we will use another norm, the rest of data are the same.

Theorem 3.6 *Let $x(t)$ be bounded continuous in $J = [\alpha, \infty)$, and suppose*

$$x(t) \leq ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds, \quad t \in J, \tag{13}$$

where $a \geq 0$, $b \geq 0$, and $\gamma > 0$ are constants and $b < \frac{\gamma}{2}$. Then

$$x(t) \leq \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}, \quad t \in J, \tag{14}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

Proof. Let $(X, \rightarrow, \leq) = (\mathcal{C}(J), \xrightarrow{\|.\|}, \leq)$, where $\mathcal{C}(J)$ is the Banach space of functions x which are bounded and continuous in $J = [\alpha, \infty)$ with norm $\|x\| = \sup\limits_{t\in J}|x(t)|$. Using the same notations, let the operator A be defined by

$$A(x)(t) = ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds, \quad t \in J,$$

Suppose that x is a fixed point of A, then $A(x) = x$ or, equivalently,

$$\begin{aligned} x(t) &= ae^{-\gamma(t-\alpha)} + \int_\alpha^\infty be^{-\gamma|t-s|}x(s)ds \\ &= ae^{-\gamma(t-\alpha)} + be^{-\gamma t}\int_\alpha^t e^{\gamma s}x(s)ds + be^{\gamma t}\int_t^\infty e^{-\gamma s}x(s)ds. \end{aligned}$$

By differentiation, we get

$$x'(t) = -2a\gamma e^{-\gamma(t-\alpha)} + \gamma x(t) - 2b\gamma e^{-\gamma t}\int_\alpha^t e^{\gamma s}x(s)ds,$$

we derive once again, it comes out that

$$x''(t) = (\gamma^2 - 2b\gamma)x(t),$$

which is an ordinary differential equation. Using $x(\alpha)$ and $x'(\alpha)$, we get

$$x(t) = \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}.$$

Conversly, we can easily verify that $A(x) = x$ and using the fact that A admits a unique fixed point, we arrive at

$$x_A^*(t) = \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}, \quad t \in J.$$

One can easily check that A is an increasing operator: let x, $y \in \mathcal{C}[a, b]$, if $x \leq y$, then $A(x) \leq A(y)$. If $x, y \in \mathcal{C}(J)$ and $\|x - y\| = L$, it is easy to see that

$$|A(x)(t) - A(y)(t)| \leq \int_{\alpha}^{t} bLe^{-\gamma(t-s)}ds + \int_{t}^{\infty} bLe^{\gamma(t-s)}ds \leq \frac{2b}{\gamma}L = \frac{2b}{\gamma}\|x - y\|,$$

whence we conclude that $A(x) \in \mathcal{C}(J)$ and A is a contraction. Finally, the proof follows from Proposition 2.2. $\square$

Theorem 3.7 *Let $x(t)$ be a continuous function for $\alpha \leq t \leq \beta$, and suppose*

$$x(t) \leq ae^{-\gamma(\beta-t)} + \int_{\alpha}^{\beta} be^{-\gamma|t-s|}x(s)ds, \quad \alpha \leq t \leq \beta, \tag{15}$$

where $a \geq 0$, $b \geq 0$, and $\gamma > 0$ are constants and $b < \frac{\gamma}{2}$. Then

$$x(t) \leq \frac{a}{b}(\gamma - \delta)e^{-\delta(\beta-t)}, \quad \alpha \leq t \leq \beta, \tag{16}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

Proof. Since the proof of this result follows by the similar arguments as in the last theorem, we omit the details. $\square$

Remark 3.4 We use the condition $b < \frac{\gamma}{2}$ to prove that the operator A is a contraction but we can omit this condition and use the Gronwall lemma to prove the last proposition.

4 Application to Stability of Dynamical Systems

We consider the following system:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \tag{17}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t and locally Lipschitz in x. We begin by giving the definition of uniform boundedness and uniform stability (see [14], [18], [19], [7]).

Definition 4.1 (uniform boundedness) A solution of (17) is said to be globally uniformly bounded if for every $\alpha > 0$ there exists $c = c(\alpha)$ such that, for all $t_0 \geq 0$,

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| \leq c(\alpha), \quad \forall\ t \geq t_0.$$

Definition 4.2 (uniform stability)
(i) The origin $x = 0$ is uniformly stable if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0.$$

(ii) The origin $x = 0$ is globally uniformly stable if it is uniformly stable and the solutions of system (17) are globally uniformly bounded.

We recall in the following definition the notion of practical stability (see [5]).

Definition 4.3 (practical stability) The system (17) is said to be $(PS1)$ uniformly practically stable if, given (λ, A) with $0 < \lambda < A$, we have

$$\|x_0\| < \lambda \Rightarrow \|x(t)\| < A, \quad t \geq t_0, \quad \forall t_0 \in \mathbb{R}_+.$$

$(PS2)$ quasi-uniformly asymptotically stable (in the large) if $\forall \varepsilon > 0$, $\alpha > 0$, $t_0 \in \mathbb{R}_+$, there exists a positive number $T = T(\varepsilon, \alpha)$ such that

$$\|x_0\| \leq \alpha \Rightarrow \|x(t)\| < \varepsilon, \quad t \geq t_0 + T.$$

$(PS3)$ uniformly practically asymptotically stable if $(PS1)$ and $(PS2)$ hold at the same time.

As application to stability, let us consider the nonlinear dynamical system:

$$\dot{x} = A(t)x + g(t, x), \tag{18}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, the matrix $A(.)$ is continuous and bounded, $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in (t, x), locally Lipschitz in x such that $g(t, 0) = 0$. We suppose that $x = 0$ is globally uniformly asymptotically stable equilibrium point for the nominal system $\dot{x} = A(t)x$, this is equivalent to saying that

$$\|\Phi(t, t_0)\| \leq k \exp -\gamma(t - t_0), \ \ \forall t \geq t_0, k > 0, \gamma > 0, \tag{19}$$

where $\Phi(t, t_0)$ is the state transition matrix associated to $A(t)$. The solution of this system with the initial condition (t_0, x_0) is given by:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)g(s, x(s))ds. \tag{20}$$

We have

$$\|x(t)\| \leq k \exp -\gamma(t - t_0)\|x(t_0)\| + \int_{t_0}^{t} ke^{-\gamma(t-s)}\|g(s, x(s))\|ds. \tag{21}$$

It follows that

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\|g(s, x(s))\|ds. \tag{22}$$

We will impose a restriction on g to study the practical stability.

If we suppose that for all (t, x),

$$\|g(t, x)\| \leq \rho(t),$$

with ρ being a nonnegative continuous function which tends to zero as $t \to \infty$, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\rho(s)ds.$$

The assumption on ρ means that: $\forall \varepsilon > 0, \quad \exists T > 0/ \quad \forall t \geq t_0 + T, \quad \rho(t) < \varepsilon$. We have also $\exists \beta/ \quad \forall t \in [t_0, t_0 + T], \quad \rho(t) \leq \beta$.

Then, $\forall t \geq t_0 + T$,

$$\begin{aligned} e^{\gamma t}\|x(t)\| &\leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t_0+T} ke^{\gamma s}\rho(s)ds + \int_{t_0+T}^{t} ke^{\gamma s}\rho(s)ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + k\beta\int_{t_0}^{t_0+T} e^{\gamma s}ds + k\varepsilon\int_{t_0+T}^{t} e^{\gamma s}ds \\ &\leq ke^{\gamma t_0}\|x(t_0)\| + \frac{k\beta}{\gamma}\left[e^{\gamma(t_0+T)} - e^{\gamma t_0}\right] + \frac{k\varepsilon}{\gamma}\left[e^{\gamma t} - e^{\gamma(t_0+T)}\right], \end{aligned}$$

or equivalently, $\forall t \geq t_0 + T$

$$\|x(t)\| \leq ke^{-\gamma(t-t_0)}\|x(t_0)\| + \frac{k\beta}{\gamma}e^{-\gamma(t-t_0)}\left[e^{\gamma T} - 1\right] + \frac{k\varepsilon}{\gamma}.$$

We see that the function : $t \mapsto ke^{-\gamma(t-t_0)}\|x(t_0)\| + \frac{k\beta}{\gamma}e^{-\gamma(t-t_0)}\left[e^{\gamma T} - 1\right]$ vanishes, then

$$\|x(t)\| \leq M\varepsilon, \quad \forall t \geq t_0 + T',$$

for a certain $T' > T > 0$, this shows the practical stability of the system.

Another approach is to study the asymptotic behavior of the system in a small neighborhood of the origin. For the rest of our presentation, we need the following definitions which are related to stability.

Definition 4.4 (uniform stability of B_r)
(i) B_r is uniformly stable if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $t_0 \geq 0$,

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (4.1) are globally uniformly bounded.

Definition 4.5 (uniform attractivity) The origin $x = 0$ is globally uniformly attractive if for all $\epsilon > 0$ and $c > 0$, there exists $T(\epsilon, c) > 0$, such that for all $t_0 \geq 0$,

$$\|x(t)\| < \epsilon, \quad \forall\ t \geq t_0 + T(\epsilon, c), \quad \|x_0\| < c.$$

Definition 4.6 (Class $\mathcal{K}$ function) A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 4.7 (Class $\mathcal{KL}$ function) A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$, if for each fixed point s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.

The following proposition provides a characterization of global uniform attractivity and global uniform stability.

Proposition 4.1 *If there exists a class $\mathcal{KL}$ function β, a class $\mathcal{K}_\infty$ α, a constant $r > 0$ such that, given any initial state x_0, the solution satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + r, \quad \forall t \geq 0,$$

then B_r is globally uniformly attractive and globally uniformly stable.

Note that, if the class $\mathcal{KL}$-function β in the above relation is of the form $\beta(r,s) = kre^{-\lambda t}$, with $\lambda, k > 0$ we say that the ball B_r is globally uniformly exponentially stable. It is also worth noting that if, in the above definitions, we take $r = 0$, then one deals with the standard concept of GUAS and GUES of the origin (see [4] for more details). Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| x(t) \| - r$, so that if $r = 0$, we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point.

Other applications to stability will be done in the following example by considering the system (18), we keep the same assumptions.

Example 4.1 1) Suppose that condition (22) holds and for all (t,x),

$$\|g(t,x)\| \leq \eta(t)\|x\|,$$

with η being an integrable function, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} k\eta(s)e^{\gamma s}\|x(s)\|ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} k\eta(s)u(s)ds,$$

using Theorem 3.4 we get

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} k^2 u(t_0)\eta(s)\left(\exp\int_{s}^{t} k\eta(u)du\right)ds,$$

then

$$u(t) \leq kMu(t_0), \quad \text{where} \quad M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}.$$

One can obtain an estimation on the trajectories as follows, for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)}.$$

Then the origin is a globally uniformly exponentially stable equilibrium point for the system.

2) If we suppose that for all (t,x),

$$\|g(t,x)\| \leq \eta(t)\|x\| + \eta',$$

with η being an integrable function and $\eta' > 0$, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(s)\|x(s)\| + \eta'\}ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} \{k\eta(s)u(s) + k\eta' e^{\gamma s}\}ds,$$

using Theorem 3.5 we get

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} \{k^2 u(t_0)\eta(s) + k\eta' e^{\gamma s}\} \left(\exp \int_{s}^{t} k\eta(u)du \right) ds,$$

then $u(t) \leq kMu(t_0) + re^{\gamma t}$, where $M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}$ and $r = \frac{k\eta'}{\gamma} e^{k\|\eta\|_1}$.

One can obtain an estimation on the trajectories as follows, for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)} + r.$$

Then B_r is globally uniformly exponentially stable.

In the following example $g(t, 0)$ is not necessarily zero, in such a situation $x = 0$ is no longer an equilibrium point.

3) We suppose that for all (t, x),

$$\|g(t, x)\| \leq \eta(t)\|x\| + \eta'(t),$$

with η being integrable and η' being a piecewise continuous function, then (22) becomes

$$e^{\gamma t}\|x(t)\| \leq ke^{\gamma t_0}\|x(t_0)\| + \int_{t_0}^{t} ke^{\gamma s}\{\eta(s)\|x(s)\| + \eta'(s)\}ds.$$

Let $u(t) = e^{\gamma t}\|x(t)\|$, then the last inequality becomes

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} \{k\eta(s)u(s) + k\eta'(s)e^{\gamma s}\}ds,$$

using Theorem 3.5 we get

$$u(t) \leq ku(t_0) + \int_{t_0}^{t} \{k^2 u(t_0)\eta(s) + k\eta'(s)e^{\gamma s}\} \left(\exp \int_{s}^{t} k\eta(u)du \right) ds,$$

then $u(t) \leq kMu(t_0) + \varepsilon(t)$, where $M = 1 + k\|\eta\|_1 e^{k\|\eta\|_1}$ and $\varepsilon(t) = ke^{k\|\eta\|_1} \int_{t_0}^{t} \eta'(s)e^{\gamma s}ds$.

Finally, we get for all $t \geq t_0$,

$$\|x(t)\| \leq kM\|x(t_0)\|e^{-\gamma(t-t_0)} + \varepsilon(t)e^{-\gamma t}.$$

If we suppose that the function : $t \mapsto \varepsilon(t)e^{-\gamma t}$ vanishes, we obtain that the system (18) is uniformly practically asymptotically stable.

5 Conclusion

In this paper we have reduced the study of various integral inequalities to fixed point problems. We have also derived some general Gronwall-type results and have given examples of such results in the particular case of the Banach space $\mathcal{C}(J)$ using two different norms.

Acknowledgment

The authors wish to thank the reviewers for their valuable and careful comments.

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Weak Heteroclinic Solutions of Discrete Nonlinear Problems of Kirchhoff Type with Variable Exponents

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Received: November 21, 2016; Revised: December 17, 2017

Abstract: We prove the existence of weak heteroclinic solutions for discrete nonlinear problems of Kirchhoff type. The proof of the main result is based on a minimization method.

Keywords: *nonlinear difference equation, heteroclinic solution, anisotropic problems, Kirchhoff, critical points.*

Mathematics Subject Classification (2010): 47A75; 35B38; 35P30; 34L05; 34L30.

1 Introduction

In this paper, we study the following nonlinear discrete anisotropic problem

$$\begin{cases} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) \\ +\alpha(k)|u(k)|^{p(k)-2}u(k) = \delta(k)f(k,u(k)),\ k\in\mathbb{Z}^*, \\ u(0)=0, \quad \lim\limits_{k\to-\infty}u(k)=-1, \quad \lim\limits_{k\to+\infty}u(k)=1, \end{cases} \tag{1}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\mathbb{Z}^* = \{k \in \mathbb{Z} : k \neq 0\}$ and $M, a, \alpha, \delta, f, p$ are functions to be defined later.

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Note that difference equations can be seen as a discrete counterpart of partial differential equations and are usually studied in connection with numerical analysis. In this way, the main operator in problem (1)

$$\Delta(a(k-1,\Delta u(k-1)))$$

can be seen as a discrete counterpart of the anisotropic operator

$$\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a\Big(x,\frac{\partial}{\partial x_i}u\Big).$$

The first study in this direction for constant exponents was done by Cabada et *al.* [2] and for variable exponent by Mihailescu et *al.* [8] (see also [6]). In [6], the authors studied the following problem

$$\begin{cases}-\Delta\big(a\big(k-1,\Delta u(k-1)\big)\big)+\alpha(k)g\big(k,u(k)\big)=\delta(k)f\big(k,u(k)\big), \quad k\in\mathbb{Z}^*,\\ u(0)=0,\ \lim\limits_{k\to-\infty}u(k)=-1,\ \lim\limits_{k\to+\infty}u(k)=1,\end{cases} \tag{2}$$

where

$$g(k,\xi)=|\xi-1|^{p(k)-2}(\xi-1)\chi_{\mathbb{Z}^+}(k)+|\xi+1|^{p(k)-2}(\xi+1)\chi_{\mathbb{Z}^-}(k).$$

The authors in [6] proved an existence result of weak heteroclinic solutions of problem (2).

In this paper, we consider the same boundary conditions as in [6], but the function

$$M(A(k-1,\Delta u(k-1)))$$

which appears in the left-hand side of problem (1) is more general than the one which appears in [6]. Indeed, if we take $M(t)=1$ in the problem (1), we obtain the probem studied by Guiro et *al* in [6].

To prove an existence result of problem (1), we define other new spaces and new associated norms and we adapt the classical minimization methods used for the study of anisotropic PDEs. The idea is to transfer the problem of the existence of solutions for (1) into the problem of the existence of a minimizer for some associated energy functional.

The study of heteroclinic connections for boundary value problems got a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such as phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations. Indeed, heteroclinic solutions are often called transitional solutions (see [3,7]). Problem (1) involves variable exponents due to their use in image restoration (see [4]), in electrorheological and thermorheological fluids dynamic (see [5,9,10]).

The remaining part of this paper is organized as follows: Section 2 is devoted to mathematical preliminaries. The main existence result is stated and proved in Section 3.

2 Preliminaries and Assumptions

We use the notations

$$p^+=\sup_{k\in\mathbb{Z}}p(k), \qquad p^-=\inf_{k\in\mathbb{Z}}p(k)$$

and we set

$$\mathbb{Z}^+ := \{k \in \mathbb{Z} : k \geq 0\}; \qquad \mathbb{Z}^- := \{k \in \mathbb{Z} : k \leq 0\};$$
$$\mathbb{Z}^+_* := \{k \in \mathbb{Z} : k > 0\}; \qquad \mathbb{Z}^-_* := \{k \in \mathbb{Z} : k < 0\}.$$

In order to present the main result, for each $p(.) : \mathbb{Z} \to (0, +\infty)$ and $\beta \geq 1$, we introduce the following spaces:

$$\mathcal{L}^1 := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \sum_{k\in\mathbb{Z}} |u(k)| < +\infty \Big\},$$

$$\mathcal{L}^\infty := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \sup_{k\in\mathbb{Z}} |u(k)| < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_0 := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p(.)}(u) := \sum_{k\in\mathbb{Z}} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_{0,+} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_{0,-} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} |u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_{0,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p(.)}(u) := \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_{0,+,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p_+(.)}(u) := \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\mathcal{L}^{p(.)}_{0,-,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ \ u(0) = 0 \text{ and } \ \rho_{\alpha(.),p_-(.)}(u) := \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} < +\infty \Big\},$$

$$\begin{aligned} \mathcal{W}^{1,p(.)}_{0,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}} |\Delta u(k)|^{p(k)} \Big)^\beta < +\infty \Big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p_+(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}^+} |\Delta u(k)|^{p(k)} \Big)^\beta < +\infty \Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)} := \Big\{ u : \mathbb{Z} \to \mathbb{R}; \ u(0) = 0, \ \rho_{1,\alpha(.),p_-(.)}(u) \ &:= \ \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} \\ &+ \ \Big(\sum_{k\in\mathbb{Z}^-} |\Delta u(k)|^{p(k)} \Big)^\beta < +\infty \Big\}. \end{aligned}$$

For the data $a,\ f,\ \alpha$ and δ, we assume the following.

$$(H_1):\begin{cases} a(k,.):\mathbb{R}\to\mathbb{R},\quad k\in\mathbb{Z} \text{ and there exists a mapping } \ A(.,.):\mathbb{Z}\times\mathbb{R}\to\mathbb{R} \text{ which}\\ \text{satisfies } \ a(k,\xi)=\frac{\partial}{\partial\xi}A(k,\xi) \ \text{ and } \ A(k,0)=0, \quad \text{for all } k\in\mathbb{Z}. \end{cases}$$

$(H_2): |\xi|^{p(k)} \leq a(k,\xi)\xi \leq p(k)A(k,\xi)$, for all $k\in\mathbb{Z}$ and $\xi\in\mathbb{R}$.

(H_3): There exists a positive constant C_1 such that $|a(k,\xi)| \leq C_1(j(k)+|\xi|^{p(k)-1})$, for all
$k\in\mathbb{Z}$ and $\xi\in\mathbb{R}$, with $j\in\mathcal{L}^{q(.)}_{0,\alpha(.)}$, where $\frac{1}{p(k)}+\frac{1}{q(k)}=1$.

(H_4) : $\big(a_i(x,\xi)-a_i(x,\eta)\big)(\xi-\eta)>0$ for all $k\in\mathbb{Z}$ and $\xi,\eta\in\mathbb{R}$ such that $\xi\neq\eta$.

$(H_5): f:\mathbb{Z}\times\mathbb{R}\longrightarrow\mathbb{R}$ and there exists a constant $C_2>0$ such that

$$|f(k,t)| \leq C_2\big(1+|t-1|^{p(k)-1}\big)\chi_{\mathbb{Z}^+}(k)+C_2\big(1+|t+1|^{p(k)-1}\big)\chi_{\mathbb{Z}^-_*}(k),$$

for all $k\in\mathbb{Z},\ t\in\mathbb{R}$, where $\chi_A(k)=1$ if $k\in A$ and $\chi_A(k)=0$ if $k\notin A$.

Assumption (H_5) implies that

$$\begin{cases} |f(k,t+1)| \leq C_2(1+|t|^{p(k)-1}) & \text{if } \ k\geq 0,\\ |f(k,t-1)| \leq C_2(1+|t|^{p(k)-1}) & \text{if } \ k<0. \end{cases} \tag{3}$$

So by denoting

$$F(k,t)=\int_0^t f(k,s)ds \quad \text{for } \ k\in\mathbb{Z},\ t\in\mathbb{R},$$

we deduce that there exists a positive constant $C_3>1$ such that

$$\begin{cases} |F(k,t+1)| \leq C_3(1+|t|^{p(k)}) & \text{if } \ k\geq 0,\\ |F(k,t-1)| \leq C_3(1+|t|^{p(k)}) & \text{if } \ k<0. \end{cases} \tag{4}$$

$$(H_6):\begin{cases} \alpha:\mathbb{Z}\to\mathbb{R} \text{ and } \delta:\mathbb{Z}\to\mathbb{R} \text{ are such that } \alpha(k)\geq\alpha_0>0 \ \text{ for all } k\in\mathbb{Z},\\ 0<\delta(k)\leq\bar{\delta}=\sup\limits_{k\in\mathbb{Z}}|\delta(k)|<+\infty \text{ and } \delta\in\mathcal{L}^1. \end{cases}$$

$(H_7): \alpha_0>\bar{\delta}p^+C_3$.

This condition means that the parameter $\alpha(.)$ should be bigger than the parameter $\bar{\delta}$ and is called the competition phenomenon between $\alpha(.)$ and $\delta(.)$.

We also assume that
$(H_8): p:\mathbb{Z}\longrightarrow(1,+\infty)$ with $1<p^-\leq p^+<+\infty$.

$(H_9): M:(0,+\infty)\longrightarrow(0,+\infty)$ is continuous, nondecreasing and there exist three positive real numbers $B_1,\ B_2,\ \beta$ with $B_1\leq B_2$, and $\beta\geq 1$ such that

$$B_1t^{\beta-1}\leq M(t)\leq B_2t^{\beta-1}, \quad \text{for all} \quad t>0.$$

Example 2.1 We can give the following functions which satisfy assumptions (H_1) – (H_4):

- $A(k,\xi) = \dfrac{1}{p(k)}|\xi|^{p(k)}$, where $a(k,\xi) = |\xi|^{p(k)-2}\xi$, $\forall k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.
- $A(k,\xi) = \dfrac{1}{p(k)}\Big(\big(1+|\xi|^2\big)^{p(k)/2}-1\Big)$, where $a(k,\xi) = \big(1+|\xi|^2\big)^{(p(k)-2)/2}\xi$, $\forall\, k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

We introduce on $\mathcal{L}^{p(.)}_{0,+}$ and $\mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ the Luxemburg norms

$$||u||_{p+(.)} := \inf\Big\{\lambda > 0; \quad \sum_{k\in\mathbb{Z}^+}\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} \le 1\Big\},$$

$$||u||_{\alpha(.),p+(.)} := \inf\Big\{\lambda > 0; \quad \sum_{k\in\mathbb{Z}^+}\alpha(k)\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} \le 1\Big\}$$

and we define, on the space $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, the norm

$$||u||_{1,\alpha(.),p+(.)} := \inf\Big\{\lambda > 0; \ \sum_{k\in\mathbb{Z}^+}\alpha(k)\Big|\frac{u(k)}{\lambda}\Big|^{p(k)} + \Big(\sum_{k\in\mathbb{Z}^+}\Big|\frac{\Delta u(k)}{\lambda}\Big|^{p(k)}\Big)^{\beta} \le 1\Big\}.$$

We replace $\mathbb{Z}^+$ by $\mathbb{Z}^-$ to get the norms on $\mathcal{L}^{p(.)}_{0,-}$, $\mathcal{L}^{p(.)}_{0,-,\alpha(.)}$ and $\mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$.

Remark 2.1 We have the following:

$$\mathcal{L}^{p(.)}_{0,+,\alpha(.)} \supset \mathcal{L}^{p(.)}_{0,\alpha(.)},\quad \mathcal{L}^{p(.)}_{0,-,\alpha(.)} \supset \mathcal{L}^{p(.)}_{0,\alpha(.)},\quad \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)} \supset \mathcal{W}^{1,p(.)}_{0,\alpha(.)} \text{ and } \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)} \supset \mathcal{W}^{1,p(.)}_{0,\alpha(.)}.$$

Indeed, $\alpha(k)|u(k)|^{p(k)}$ *is nonnegative for all* $k \in \mathbb{Z}$. *Therefore, if* $\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)} < +\infty$, *then* $\sum_{k\in\mathbb{Z}^+}\alpha(k)|u(k)|^{p(k)} < +\infty$.

In the sequel, we will use the following result.

Proposition 2.1 *([6], Proposition 2.5). If* $u \in \mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ *and* $p^+ < \infty$, *then the following properties hold:*

1. $||u||_{\alpha(.),p+(.)} < 1 \implies ||u||^{p+}_{\alpha(.),p+(.)} \le \rho_{\alpha(.),p+(.)}(u) \le ||u||^{p^-}_{\alpha(.),p+(.)}$;
2. $||u||_{\alpha(.),p+(.)} > 1 \implies ||u||^{p-}_{\alpha(.),p+(.)} \le \rho_{\alpha(.),p+(.)}(u) \le ||u||^{p^+}_{\alpha(.),p+(.)}$;
3. $||u||_{\alpha(.),p+(.)} < 1\ (= 1; > 1) \iff \rho_{\alpha(.),p+(.)}(u) < 1\ (= 1; > 1)$;
4. $||u||_{\alpha(.),p+(.)} \longrightarrow 0 \iff \rho_{\alpha(.),p+(.)}(u) \longrightarrow 0$.

Lemma 2.1 *([6], Lemma 2.8)(discrete Hölder type inequality). Let* $u \in \mathcal{L}^{p(.)}_{0,+,\alpha(.)}$ *and* $v \in \mathcal{L}^{q(.)}_{0,+,\alpha(.)}$ *with* $\frac{1}{p(k)} + \frac{1}{q(k)}$ *for any* k *in* $\mathbb{Z}$. *Then*

$$\sum_{k\in\mathbb{Z}^+}|uv| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)||u||_{\alpha(.),p+(.)}||v||_{\alpha(.),q+(.)}. \tag{5}$$

As in [6], we have the following results.

Proposition 2.2

1. $\rho_{1,\alpha(.),p_+(.)}(u+v) \leq 2^{\beta p^+-1}\big(\rho_{1,\alpha(.),p_+(.)}(u) + \rho_{1,\alpha(.),p_+(.)}(v)\big), \quad \forall\ u,\ v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.
2. *Let* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$. *Then:*

   i) if $\lambda > 1$, *we have*

$$\lambda^{p^-}\rho_{1,\alpha(.),p_+(.)}(u) \leq \rho_{1,\alpha(.),p_+(.)}(\lambda u) \leq \lambda^{\beta p^+}\rho_{1,\alpha(.),p_+(.)}(u); \tag{6}$$

   ii) if $0 < \lambda < 1$, *we have*

$$\lambda^{\beta p^+}\rho_{1,\alpha(.),p_+(.)}(u) \leq \rho_{1,\alpha(.),p_+(.)}(\lambda u) \leq \lambda^{p^-}\rho_{1,\alpha(.),p_+(.)}(u). \tag{7}$$

Theorem 2.1 *Let* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)} \setminus \{0\}$. *Then*

$$||u||_{1,\alpha(.),p_+(.)} = a \quad \textit{if and only if} \quad \rho_{1,\alpha(.),p_+(.)}(u/a) = 1.$$

Proposition 2.3 *If* $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ *and* $p^+ < \infty$, *then the following properties hold:*

1. $||u||_{1,\alpha(.),p_+(.)} < 1 \implies ||u||^{\beta p^+}_{1,\alpha(.),p_+(.)} \leq \rho_{1,\alpha(.),p_+(.)}(u) \leq ||u||^{p^-}_{1,\alpha(.),p_+(.)}$;
2. $||u||_{1,\alpha(.),p_+(.)} > 1 \implies ||u||^{p-}_{1,\alpha(.),p_+(.)} \leq \rho_{1,\alpha(.),p_+(.)}(u) \leq ||u||^{\beta p^+}_{1,\alpha(.),p_+(.)}$;
3. $||u||_{1,\alpha(.),p_+(.)} < 1\ (= 1; > 1) \iff \rho_{1,\alpha(.),p_+(.)}(u) < 1\ (= 1; > 1)$;
4. $||u||_{1,\alpha(.),p_+(.)} \longrightarrow 0 \iff \rho_{1,\alpha(.),p_+(.)}(u) \longrightarrow 0$.

3 Existence of Weak Heteroclinic Solutions

In this section we investigate the existence of weak heteroclinic solutions of problem (1) in the following sense.

Definition 3.1 A weak heteroclinic solution of problem (1) is a function $u \in \mathcal{W}^{1,p(.)}_{0,\alpha(.)}$ such that

$$\begin{cases} M\Big(\displaystyle\sum_{k\in\mathbb{Z}} A\big(k-1,\Delta u(k-1)\big)\Big)\displaystyle\sum_{k\in\mathbb{Z}} a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \\ +\displaystyle\sum_{k\in\mathbb{Z}}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k) = \displaystyle\sum_{k\in\mathbb{Z}}\delta(k)f\big(k,u(k)\big)v(k), \end{cases} \tag{8}$$

for any $v \in \mathcal{W}^{1,p(.)}_{0,\alpha(.)}$, with $u(0) = 0$, $\lim\limits_{k\to-\infty} u(k) = -1$ and $\lim\limits_{k\to+\infty} u(k) = 1$.

The main result is the following.

Theorem 3.1 *Assume that assumptions* (H_1)-(H_9) *hold true. Then, there exists at least one weak heteroclinic solution of problem (1).*

Proof. We first consider the following problem

$$\begin{cases} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) \\ \\ +\alpha(k)|u(k)|^{p(k)-2}u(k)=\delta(k)f\big(k,u(k)+1\big),\ k\in\mathbb{Z}_*^+, \\ \\ u(0)=0, \quad \lim\limits_{k\to+\infty}u(k)=0. \end{cases} \tag{9}$$

Definition 3.2 A weak solution of problem (9) is a function $u\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ such that

$$\begin{cases} M\Big(\sum\limits_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)\sum\limits_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \\ +\sum\limits_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)=\sum\limits_{k=1}^{+\infty}\delta(k)f\big(k,u(k)+1\big)v(k), \end{cases} \tag{10}$$

for any $v\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.

Theorem 3.2 *Assume that hypotheses (H_1)-(H_9) hold. Then, there exists at least one weak solution of problem (9).*

To prove Theorem 3.2, we consider the energy functional corresponding to problem (9) defined by $J:\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}\longrightarrow\mathbb{R}$ such that

$$J(u)=\widehat{M}\Big(\sum_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)+\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}-\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big), \tag{11}$$

where $\widehat{M}(t)=\displaystyle\int_0^t M(s)\,ds$ and we present some basic properties of the functional J.

Proposition 3.1 *The functional J is well defined on $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ and is of class $C^1\big(\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)},\mathbb{R}\big)$ with the derivative given by*

$$\begin{cases} \langle J'(u),v\rangle=M\Big(\sum\limits_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)\sum\limits_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \\ \qquad +\sum\limits_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)-\sum\limits_{k=1}^{+\infty}\delta(k)f\big(k,u(k)+1\big)v(k), \end{cases} \tag{12}$$

for all $u,v\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$.

Indeed, we denote

$$I(u)=\widehat{M}\Big(\sum_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big),\qquad L(u)=\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}$$

and

$$\Lambda(u)=\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big).$$

We have, by using (H_9), that

$$\begin{aligned}
|I(u)| &= \Big|\int_0^{\sum_{k=1}^{+\infty}A(k-1,\Delta u(k-1))} M(t)dt\Big|\\
&\leq B_2\Big|\int_0^{\sum_{k=1}^{+\infty}A(k-1,\Delta u(k-1))} t^{\beta-1}dt\Big|\\
&\leq \frac{B_2}{\beta}\Big(\sum_{k=1}^{+\infty}\big|A(k-1,\Delta u(k-1))\big|\Big)^{\beta}.
\end{aligned}$$

According to $(H_1),(H_3)$ and the *discrete Hölder type inequality*, we write

$$\begin{aligned}
\sum_{k=1}^{+\infty}\big|A(k-1,\Delta u(k-1))\big| &\leq \sum_{k=1}^{+\infty}\int_0^{\Delta u(k-1)}|a(k-1,t)|dt\\
&\leq C_1\sum_{k=1}^{+\infty}\Big(j(k-1)+\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)-1}\Big)\Delta u(k-1)\\
&\leq C_1\sum_{k=1}^{+\infty}j(k-1)|\Delta u(k-1)|+\frac{C_1}{p^-}\sum_{k=1}^{+\infty}|\Delta u(k-1)|^{p(k-1)}\\
&\leq C_1\Big(\frac{1}{q^-}+\frac{1}{p^-}\Big)||j||_{q_+(.)}||\Delta u||_{p_+(.)}+\frac{C_1}{p^-}||\Delta u||_{p_+(.)}\\
&< +\infty
\end{aligned}$$

and we deduce that $|I(u)|<+\infty$. We have

$$|L(u)|=\Big|\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}\Big|\leq\frac{1}{p^-}\Big|\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)}\Big|<+\infty$$

and

$$\begin{aligned}
|\Lambda(u)| &= \Big|\sum_{k=1}^{+\infty}\delta(k)F\big(k,u(k)+1\big)\Big|\\
&\leq\sum_{k=1}^{+\infty}|\delta(k)||F\big(k,u(k)+1\big)|\\
&\leq\sum_{k=1}^{+\infty}C_3|\delta(k)|\big(1+|u(k)|^{p(k)}\big)\\
&\leq C_3\sum_{k=1}^{+\infty}|\delta(k)|+C_3\bar{\delta}\sum_{k=1}^{+\infty}|u(k)|^{p(k)}\\
&<+\infty.
\end{aligned}$$

Hence, J is well-defined. Clearly, the functionals I, L and Λ are in $C^1\big(\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}, \mathbb{R}\big)$.

In what follows we prove (12). Let $u, v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$. Since

$$\begin{cases} \lim\limits_{\lambda\to 0^+} \dfrac{I(u+\lambda v)-I(u)}{\lambda} \\ = \lim\limits_{\lambda\to 0^+} \dfrac{\widehat{M}\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)+\lambda\Delta v(k-1)\big)\Big) - \widehat{M}\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)\big)\Big)}{\lambda} \\ = M\Big(\sum\limits_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)\big)\Big) \sum\limits_{k=1}^{+\infty} a\big(k-1, \Delta u(k-1)\big)\Delta v(k-1), \end{cases}$$

$$\begin{aligned} \lim_{\lambda\to 0^+} \frac{L(u+\lambda v)-L(u)}{\lambda} &= \lim_{\lambda\to 0^+} \sum_{k=1}^{+\infty} \frac{|u(k)+\lambda v(k)|^{p(k)} - |u(k)|^{p(k)}}{p(k)\lambda} \\ &= \sum_{k=1}^{+\infty} \lim_{\lambda\to 0^+} \frac{|u(k)+\lambda v(k)|^{p(k)} - |u(k)|^{p(k)}}{p(k)\lambda} \\ &= \sum_{k=1}^{+\infty} |u(k)|^{p(k)-2} u(k) v(k) \end{aligned}$$

and

$$\begin{aligned} \lim_{\lambda\to 0^+} \frac{\Lambda(u+\lambda v)-\Lambda(u)}{\lambda} &= \lim_{\lambda\to 0^+} \sum_{k=1}^{+\infty} \delta(k) \frac{F\big(k, u(k)+\lambda v(k)+1\big) - F\big(k, u(k)+1\big)}{\lambda} \\ &= \sum_{k=1}^{+\infty} \delta(k) \lim_{\lambda\to 0^+} \frac{F\big(k, u(k)+\lambda v(k)+1\big) - F\big(k, u(k)+1\big)}{\lambda} \\ &= \sum_{k=1}^{+\infty} \delta(k) f\big(k, u(k)+1\big) v(k), \end{aligned}$$

we obtain the relation (12). □

Proposition 3.2 *The functional J is weakly lower semi-continuous.*

Indeed, by (H_1), (H_4) and (H_9) we have that J is convex. Thus, it is enough to show that J is lower semi-continuous. For this, we fix $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ and $\epsilon > 0$. Since J is convex, we deduce that for any $v \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$,

$$\begin{aligned} J(v) &\geq J(u) + \langle J'(u), v-u\rangle \\ &\geq J(u) + R(u,v) + S(u,v) + T(u,v), \end{aligned}$$

with

$$R(u,v) = M\Big(\sum_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)\big)\Big) \sum_{k=1}^{+\infty} a(k-1, \Delta u(k-1))\big(\Delta v(k-1) - \Delta u(k-1)\big),$$

$$S(u,v)=\sum_{k=1}^{+\infty}|u(k)|^{p(k)-2}u(k)\big(v(k)-u(k)\big)$$

and

$$T(u,v)=\sum_{k=1}^{+\infty}\delta(k)f\big(k,u(k)+1\big)\big(u(k)-v(k)\big).$$

Using the *discrete Hölder type inequality*, there exists three nonnegative constants C_4, C_5 and C_6 such that

$$\begin{aligned}R(u,v)&\geq -M\Big(\sum_{k=1}^{+\infty}A\big(k-1,\Delta u(k-1)\big)\Big)\sum_{k=1}^{+\infty}|a(k-1,\Delta u(k-1))||\Delta v(k-1)-\Delta u(k-1)|\\&\geq -C_4'||\Delta u-\Delta v||_{\alpha(.),p_+(.)}\\&\geq -C_4||u-v||_{1,\alpha(.),p_+(.)},\end{aligned}\tag{13}$$

$$T(u,v)\geq -C_5||u-v||_{1,\alpha(.),p_+(.)}\tag{14}$$

and

$$\begin{aligned}S(u,v)\quad&\geq -\sum_{k=1}^{+\infty}|u(k)|^{p(k)-1}|v(k)-u(k)|\\&\geq -\Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)\big|\big||u|^{p(.)-1}\big|\big|_{\alpha(.),q_+(.)}||u-v||_{\alpha(.),p_+(.)}\\&\geq -C_6||u-v||_{1,\alpha(.),p_+(.)}.\end{aligned}\tag{15}$$

Then, combining (13), (14) and (15), we get

$$J(v)\geq J(u)-K||u-v||_{1,\alpha(.),p_+(.)},\tag{16}$$

with $K=C_4+C_5+C_6$. Finally, for all $v\in\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$ with $||v-u||_{1,\alpha(.),p_+(.)}<\tau=\dfrac{\epsilon}{K}$, we get

$$J(v)\geq J(u)-\epsilon.$$

Then J is lower semi-continuous and by [1], Corollary III.8, J is weakly lower semi-continuous. $\square$

Proposition 3.3 *The functional J is coercive and bounded from below.*

Indeed, according to $(H_2), (H_5) - (H_9)$, we have

$$
\begin{aligned}
J(u) &= \widehat{M}\Big(\sum_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)\big)\Big) + \sum_{k=1}^{+\infty} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \sum_{k=1}^{+\infty} \delta(k) F\big(k, u(k)+1\big) \\
&\geq \frac{B_1}{\beta}\Big(\sum_{k=1}^{+\infty} A\big(k-1, \Delta u(k-1)\big)\Big)^{\beta} + \frac{1}{p^+}\sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)} \\
&-C_3 \sum_{k=1}^{+\infty} \delta(k)|u(k)|^{p(k)} - C_7 \\
&\geq \frac{B_1}{\beta}\Big(\sum_{k=1}^{+\infty} \frac{1}{p(k-1)}\big|\Delta u(k-1)\big|^{p(k-1)}\Big)^{\beta} + \frac{1}{p^+}\sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)} \\
&-\frac{C_3\bar{\delta}}{\alpha_0}\sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)} \\
&\geq \frac{B_1}{\beta(p^+)^{\beta}}\Big(\sum_{k=1}^{+\infty} \big|\Delta u(k-1)\big|^{p(k-1)}\Big)^{\beta} + \Big(\frac{1}{p^+} - \frac{C_3\bar{\delta}}{\alpha_0}\Big)\sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)} - C_7 \\
&\geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}}; \frac{1}{p^+} - \frac{C_3\bar{\delta}}{\alpha_0}\Big\}\rho_{1,\alpha(.),p_+(.)}(u) - C_7.
\end{aligned}
$$

To prove the coerciveness of the functional J, we may assume that $||u||_{1,\alpha(.),p_+(.)} > 1$ and, using Proposition 2.3, we deduce from the above inequality that

$$
J(u) \geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}}; \frac{1}{p^+} - \frac{C_3\bar{\delta}}{\alpha_0}\Big\}||u||^{p^-}_{1,\alpha(.),p_+(.)} - C_7.
$$

Thus, by assumption (H_7),

$$
J(u) \longrightarrow +\infty \ \text{ as } \ ||u||_{1,\alpha(.),p_+(.)} \longrightarrow +\infty,
$$

namely J is coercive. Besides, for $||u||_{1,\alpha(.),p_+(.)} \leq 1$, we have

$$
\begin{aligned}
J(u) \qquad &\geq \min\Big\{\frac{B_1}{\beta(p^+)^{\beta}}; \frac{1}{p^+} - \frac{C_3\bar{\delta}}{\alpha_0}\Big\}\rho_{1,\alpha(.),p_+(.)}(u) - C_7. \\
&\geq -C_7 > -\infty.
\end{aligned}
$$

Thus J is bounded from below. □

Since J is weakly lower semi-continuous, bounded from below and coercive on $\mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, using the relation between critical points of J and problem (9), we deduce that J has a minimizer which is a weak solution of (9).

We will show that every weak solution u of (9) is such that $u(k) \to 0$ as $k \to +\infty$. Let u be a weak solution of problem (9). Since $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, we have $\sum_{k=1}^{+\infty} |u(k)|^{p(k)} < +\infty$. Denote

$$
S_1 = \{k \in \mathbb{Z}^+_*; |u(k)| < 1\} \text{ and } \ S_2 = \{k \in \mathbb{Z}^+_*; |u(k)| \geq 1\}.
$$

Since $u \in \mathcal{W}^{1,p(.)}_{0,+,\alpha(.)}$, S_2 is necessary a finite set and $|u(k)| < +\infty$ for any $k \in S_2$. As S_2 is a finite set, then $\sum_{k\in S_2} |u(k)|^{p^+} < +\infty$.

On the other hand, we have $\sum_{k\in S_1} |u(k)|^{p^+} \leq \sum_{k\in S_1} |u(k)|^{p(k)} \leq \sum_{k=1}^{+\infty} |u(k)|^{p(k)} < +\infty$.

Therefore,

$$\sum_{k=1}^{+\infty} |u(k)|^{p^+} = \sum_{k\in S_1} |u(k)|^{p^+} + \sum_{k\in S_2} |u(k)|^{p^+} < +\infty.$$

Thus, $\lim_{k\to+\infty} |u(k)| = 0$, which completes the proof of Theorem 3.2. $\square$

To end the proof of Theorem 3.1, let us consider the following problem

$$\begin{cases} -M(A(k-1,\Delta u(k-1)))\Delta(a(k-1,\Delta u(k-1))) \\ \\ +\alpha(k)|u(k)|^{p(k)-2}u(k) = \delta(k)f\big(k, u(k)-1\big),\ k \in \mathbb{Z}^-_*, \\ \\ u(0) = 0, \quad \lim_{k\to-\infty} u(k) = 0. \end{cases} \tag{17}$$

Definition 3.3 A weak solution of problem (17) is a function $u \in \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$ such that

$$\begin{cases} M\Big(\sum_{k=-\infty}^{0} A\big(k-1,\Delta u(k-1)\big)\Big) \sum_{k=-\infty}^{0} a(k-1,\Delta u(k-1))\Delta v(k-1) \\ \\ + \sum_{k=-\infty}^{0} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) = \sum_{k=-\infty}^{0} \delta(k)f\big(k,u(k)-1\big)v(k), \end{cases} \tag{18}$$

for any $v \in \mathcal{W}^{1,p(.)}_{0,-,\alpha(.)}$.

By mimicking the proof of Theorem 3.2, we prove the following result.

Theorem 3.3 *Assume that assumptions* (H_1)-(H_9) *hold true. Then, there exists at least one weak solution of (17).*

Now, we end the proof of Theorem 3.1. For this, we define $v_1 = u_1 + 1$, where u_1 is a weak solution of problem (9) and $v_2 = u_2 - 1$, where u_2 is a weak solution of problem (17). Therefore, according to (H_5), we deduce that

$$u = v_1\chi_{\mathbb{Z}^+} + v_2\chi_{\mathbb{Z}^-} \tag{19}$$

is a weak heteroclinic solution of problem (1). $\square$

Acknowledgment

The authors express their deepest thanks to the editor and anonymous referee for their comments and suggestions on the paper.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 80–91

Homoclinic Orbits for Damped Vibration Systems with Small Forcing Terms

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Received: May 10, 2016; Revised: December 23, 2017

Abstract: We study the existence of homoclinic orbits for second order non-autonomous damped vibration system

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f(t),$$

where B is a skew-symmetric constant matrix, $t \in \mathbb{R}$, $q \in \mathbb{R}^N$ and $\mathrm{V} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $V(t, q) = -K(t, q) + W(t, q)$ is T-periodic with respect to t, $T > 0$. We assume that $W(t, q)$ satisfies an assumption weaker than the global Ambrosetti-Rabinowitz condition and that the norm of B is sufficiently small. This homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a certain sequence of second order differential equations. This result generalizes and improves some existing findings in the known literature.

Keywords: *vector field; homoclinic orbits; damped vibration systems; mountain pass theorem; critical points; minimax methods.*

Mathematics Subject Classification (2010): 34C37.

1 Introduction and Main Results

We consider the following system

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f(t), \qquad (DS)$$

where B is a skew-symmetric constant matrix, $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(t, x) \to V(t, x)$ is a continuous function, T-periodic in the first variable with $T > 0$ and differentiable with respect to the second variable such that $V'(t, x) = \frac{\partial V}{\partial x}(t, x)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$

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and f: $\mathbb{R} \to \mathbb{R}^N$ is a continuous and bounded function. We say that a solution $x(t)$ of (DS) is a nontrivial homoclinic(to 0) if $x \not\equiv 0$ and $x(t) \to 0$ as $t \to \pm\infty$. The importance of the study of the existence of homoclinic orbits for damped vibration systems has been recognized by Poincaré at the end of the 19th century. Therefore, the existence of homoclinic orbits has become one of the most important problems in the research of damped vibration systems. Firstly, when $B \equiv 0$ and $f \equiv 0$ the system (DS) is just the following second order non-autonomous Hamiltonian system:

$$\ddot{q}(t) + V'(t, q(t)) = 0. \tag{1}$$

In 1990, Rabinowitz [14] showed the existence of homoclinic orbits for system (1) by taking the limit of $2kT$-periodic solutions of approximating problems under the well known Ambrosetti-Rabinowitz condition: there exists a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^N \backslash \{0\}$

$$0 < \mu V(t, q) \leq V'(t, q).q.$$

By using the same approach, the existence of homoclinic orbits for the system (1) has been intensively studied by many mathematicians via variational methods in critical point theory, see([4], [5], [6], [8], [9], [13], [14], [16]) and the references therein. Particularly, in [10], Izydorek and Janczewska considered a more general Hamiltonian system

$$\ddot{q}(t) + V'(t, q(t)) \quad = \quad f(t), \tag{2}$$

where $V(t, q) = -K(t, q) + W(t, q)$. If V is neither autonomous nor periodic in t, the problem of the existence of homoclinic orbits of (1) is more complicated because the compactness arguments derived from Sobolev imbedding theorem are not available for the study of (1), see, for example, ([1], [4], [5], [6], [8], [10], [11], [15]). Secondly, if $B \not\equiv 0$, $f \not\equiv 0$ and $V = -K + W$ the existence of homoclinic orbits for system (DS) has not been previously studied. Our aim in this paper is to study the existence of homoclinic orbits for the system (DS), where K is a quadratic growth function and W satisfies an assumption weaker than the global Ambrosetti-Rabinowitz condition. Here and subsequently, $(.,.)$: $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denotes the standard inner product and $|.|$ is the induced norm in $\mathbb{R}^N$.

Definition 1.1 A vector field v defined on $\mathbb{R}^N$ is called positive if $v(x).x > 0$ for all $x \in \mathbb{R}^N \backslash \{0\}$. We call v a normalized positive vector field if v is positive, linear and satisfies the following condition:

$$v(x).x = x.x, \ \ \forall x \in \mathbb{R}^N. \qquad (v_1)$$

Our basic hypotheses on V and f are the following:
(V_1) There exist normalized positive vector field v and constant $b_1, b_2 > 0$ such that

$$b_1|x|^2 \leq K(t, x) \leq b_2|x|^2, \quad K(t, x) \leq K'(t, x).v(x) \leq 2K(t, x)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,
(V_2) $W'(t, x) = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$,
(V_3) There exists a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \backslash \{0\}$

$$0 < \mu W(t, x) \leq W'(t, x).v(x),$$

(V_4) $W(t,x) \le M|x|^\mu$, for all $t \in \mathbb{R}$ and $|x| \le 1$, where $M = \sup\limits_{t\in\mathbb{R},|x|=1} W(t,x)$.

(V_5) $\bar{b}_1 = \min\{1, 2b_1\} > 2M$ and $\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} \le \frac{\beta}{2C}$, where $0 < \beta < \bar{b}_1 - 2M$, and C is a positive constant defined in [10].

Remark 1.1 We see that if $v(x) = x$, then (V_1) becomes (H_3) and (V_3) becomes (H_5) in [10]. From (V_2)-(V_4) we see that $W(t,x) = o(|x|^2)$ as $|x| \to 0$ uniformly in t and $W(t,0) = 0, W'(t,0) = 0$. Moreover, from (V_1) we conclude that $K(t,0) = 0$, $K'(t,0) = 0$. Example 1.1 below shows that (V_3) is weaker than the global Ambrosetti-Rabinowitz condition.

In addition, we need the following hypothesis on B.

(V_6) $\|B\| < \min\left\{\bar{b}_1 - \beta - 2M, \frac{\mu-2}{\mu+2b}\bar{b}_1, \frac{1}{b}, \frac{b_1}{b}\right\}$, where $b = \|v\|$ is the norm of the operator v.

Now, we state our existence result of homoclinic orbits for problem (DS).

Theorem 1.1 *Suppose that K and W are T-periodic with respect to t, $T > 0$ satisfying $(V_1) - (V_6)$, then the system (DS) possesses a nontrivial homoclinic solution $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^N)$ such that $\dot{q}(t) \to 0$, as $t \to \pm\infty$.*

Example 1.1 Let $\theta(x)$ be the argument of $x = (\xi_1, \xi_2) \in \mathbb{R}^2 \backslash \{0\}$ defined by

$$\theta(x) = \begin{cases} \arctan(\xi_2/\xi_1), if \ \ \xi_1 > 0, \xi_2 \ge 0, \\ \frac{\pi}{2}, \ \ if \ \xi_1 = 0, \xi_2 > 0, \\ \arctan(\xi_2/\xi_1) + \pi, \ \ if \ \ \xi_1 < 0, \\ \frac{3\pi}{2}, \ \ if \ \xi_1 = 0, \xi_2 < 0, \\ \arctan(\xi_2/\xi_1) + 2\pi, if \ \ \xi_1 > 0, \xi_2 < 0. \end{cases}$$

Define a function $K \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows:

$$K(t,x) = \begin{cases} \frac{|x|^2}{\exp(2\sin 4(\ln|x| + \theta(x)))}, \ \ if \ x \ne 0, \\ 0, \ \ if \ x = 0. \end{cases}$$

Define a normalized positive vector field v by

$$v(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x.$$

An easy computation shows that K satisfies (V_1).

For any $\mu > 2$, define a function $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ as follows:

$$W(t,x) = \begin{cases} \frac{|x|^\mu}{\exp(\mu(2\sin 4(\ln|x| + \theta(x))))}, \ \ if \ \ x \ne 0, \\ 0, \ \ if \ \ x = 0. \end{cases}$$

A direct computation (see [3]) shows that W satisfies (V_2), (V_3) and (V_4). Moreover, W does not satisfy the global Ambrosetti-Rabinowitz condition.

2 Variational Setting and Preliminaries

Similarly to [10] and [14], we will prove the existence of homoclinic orbits for (DS) as the limit of $2kT$-periodic solutions of the following systems of differential equations:

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) = f_k(t), \qquad (DS_k)$$

where $f_k : \mathbb{R} \to \mathbb{R}^N$ is a bounded continuous function, $2kT$-periodic extension of f to the interval $[-kT, kT]$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let L^2_{2kT} be the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ equipped with the norm

$$\|q\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |q(t)|^2 dt \right)^{\frac{1}{2}},$$

and L^∞_{2kT} be the space of $2kT-$periodic essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^N$ equipped with the norm

$$\|q\|_{L^\infty_{2kT}} = esssup \left\{ |q(t)| : t \in [-kT, kT] \right\}.$$

Denote by $E_k := W^{1,2}_{2kT}$ the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|q\|_{E_k} = \left[\int_{-kT}^{kT} |q(t)|^2 dt + \int_{-kT}^{kT} |\dot{q}(t)|^2 dt \right]^{1/2}.$$

Next, we need the following lemma.

Lemma 2.1 *([10]). There is a positive constant C such that for each $k > 0$ and $q \in E_k$ the following inequality holds:*

$$\|q\|_{L^\infty_{2kT}} \leq C \|q\|_{E_k}. \tag{3}$$

Let $\eta_k : E_k \to [0, +\infty[$ be given by

$$\eta_k(q) = \left(\int_{-kT}^{kT} [|\dot{q}(t)|^2 + 2K(t, q(t))] dt \right)^{1/2}. \tag{4}$$

By using (V_1), we have

$$\bar{b}_1 \|q\|^2_{E_k} \leq \eta^2_k(q) \leq \bar{b}_2 \|q\|^2_{E_k}, \tag{5}$$

where $\bar{b}_2 = \max\{1, 2b_2\}$. Let $I_k : E_k \to \mathbb{R}$ be the functional defined by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \left[\frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2} Bq(t).\dot{q}(t) + K(t, q(t)) - W(t, q(t)) + f_k(t).q(t) \right] dt \\ &= \frac{1}{2}\eta^2_k(q) + \int_{-kT}^{kT} \left[\frac{1}{2} Bq(t).\dot{q}(t) - W(t, q(t)) + f_k(t).q(t) \right] dt. \end{aligned} \tag{6}$$

It is easy to check that $I_k \in C^1(E_k, \mathbb{R})$ and by using the skew-symmetry of B, we have for every $q, v \in E_k$

$$I'_k(q)v = \int_{-kT}^{kT} [\dot{q}(t).\dot{v}(t) - B\dot{q}(t).v(t) - V'(t, q(t)).v(t) + f_k(t).v(t)] dt. \tag{7}$$

It is known that the critical points of I_k in E_k are the classical $2kT$-periodic solution of (DS_k). We will obtain a critical point of I_k by using a standard version of the mountain pass theorem:

Lemma 2.2 *([13]). Let H be a real Banach space and $I \in C^1(H, \mathbb{R})$ satisfying the Palais-Smale condition. If I satisfies the following conditions:*
(i) $I(0) = 0$,
(ii) there exist constants ρ, $\alpha > 0$ such that $I_{|\partial B_\rho(0)} \geq \alpha$,
(iii) there exists $e \in H \backslash \overline{B}_\rho(0)$ such that $I(e) \leq 0$.
Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], H) : g(0) = 0, g(1) = e\}.$$

Lemma 2.3 *([4]). There exist $a_1, a_2 > 0$ such that*

$$W(t,x) \geq a_1 |x|^\mu - a_2, \ \forall \, t \in \mathbb{R}, \ \forall x \in \mathbb{R}^N. \tag{8}$$

Let v be the normalized positive vector field in (V_1) and (V_3) of Theorem 1.1. Then v is an invertible linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $a = \frac{1}{\|v^{-1}\|}$, $b = \|v\|$, where $\|v\|$ and $\|v^{-1}\|$ are operator norms. For any $x \in \mathbb{R}^N$, one has

$$a|x| \leq |v(x)| \leq b|x|. \tag{9}$$

Define a vector field $\tilde{v}$ on E_k by

$$(\tilde{v}(x))(t) = v(x(t)). \tag{10}$$

Using condition (v_1) and Fourier series, we perform direct computation to show the following lemma.

Lemma 2.4 *([4]). For any $x \in E_k$,*

$$\int_{-kT}^{kT} |\dot{x}(t)|^2 dt = \int_{-kT}^{kT} \dot{x}(t).\dot{\overbrace{v(x(t))}} dt. \tag{11}$$

$$a\|x\|_{E_k} \leq \|\tilde{v}(x)\|_{E_k} \leq b\|x\|_{E_k}. \tag{12}$$

From (V_1), (7), (10) and (11) we have

$$\begin{aligned} I_k'(q).\tilde{v}(q) &\leq \eta_k^2(q) - \int_{-kT}^{kT} \left[B\dot{q}(t).v(q(t)) - W'(t,q(t)).v(q(t))\right] dt \\ &+ \int_{-kT}^{kT} f_k(t).v(q(t)) dt. \end{aligned} \tag{13}$$

Lemma 2.5 *Under the assumptions (V_1)–(V_6), for every $k \in \mathbb{N}$ the system (DS_k) possesses a $2kT$-periodic solution $q_k \in E_k$.*

Proof. **Step 1.** We will show that I_k satisfies the Palais-Smale condition. Assume that $\{q_j\}_{j\in\mathbb{N}} \subset E_k$, $\{q_j\}_{j\in\mathbb{N}}$ has a convergent subsequence if $\{I_k(q_j)\}_{j\in\mathbb{N}}$ is bounded and $I_k'(q_j) \to 0$ as $j \to +\infty$. Then there exists a constant $M_k > 0$ such that

$$|I_k(q_j)| \le M_k, \quad \|I_k'(q_j)\|_{E_k^*} \le M_k \tag{14}$$

for every $j \in \mathbb{N}$. We firstly prove that $\{q_j\}_{j\in\mathbb{N}}$ is bounded in E_k. Without loss of generality, we may assume that $\|q_j\|_{E_k} \neq 0$. Then by (V_3) and (6), it follows that

$$\begin{aligned} \eta_k^2(q_j) &\le 2I_k(q_j) + \int_{-kT}^{kT} B\dot{q}_j(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} W'(t,q_j(t)).v(q_j(t))dt \\ &- 2\int_{-kT}^{kT} f_k(t).q_j(t)dt. \end{aligned} \tag{15}$$

From (13) and (15) we obtain

$$\begin{aligned} (1-\frac{2}{\mu})\eta_k^2(q_j) \le 2I_k(q_j) - \frac{2}{\mu}I_k'(q_j).\tilde{v}(q_j(t)) + \int_{-kT}^{kT} B\dot{q}_j(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} Bq_j(t).\dot{\overbrace{v(q_j(t))}}dt \\ -2\int_{-kT}^{kT} f_k(t).q_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} f_k(t).v(q_j(t))dt. \end{aligned} \tag{16}$$

Moreover, by (5), (9) and (16) it follows that

$$\begin{aligned} \left[\left(1-\frac{2}{\mu}\right)\bar{b}_1 - \left(1+\frac{2b}{\mu}\right)\|B\|\right]\|q_j\|_{E_k}^2 \le 2I_k(q_j) + \frac{2b}{\mu}\|I_k'(q_j)\|_{E_k^*}\|q_j\|_{E_k} \\ +2\left(\int_{-kT}^{kT}|f_k(t)|^2dt\right)^{\frac{1}{2}}\|q_j\|_{E_k} + \frac{2b}{\mu}\left(\int_{-kT}^{kT}|f_k(t)|^2dt\right)^{\frac{1}{2}}\|q_j\|_{E_k}. \end{aligned} \tag{17}$$

By (14), (17) and (V_5) we get

$$\left[\left(1-\frac{2}{\mu}\right)\bar{b}_1 - \left(1+\frac{2b}{\mu}\right)\|B\|\right]\|q_j\|_{E_k}^2 \le 2M_k + \left(\frac{2bM_k}{\mu} + \frac{\beta}{C}(1+\frac{b}{\mu})\right)\|q_j\|_{E_k}. \tag{18}$$

Since $\mu > 2$ and (V_6) imply that $\left[(1-\frac{2}{\mu})\bar{b}_1 - (1+\frac{2b}{\mu})\|B\|\right] > 0$, inequality (18) shows that $\{q_j\}_{j\in\mathbb{N}}$ is bounded in E_k. Going if necessary to a subsequence, we can assume that there exists $q \in E_k$ such that $q_j \rightharpoonup q$, as $j \to +\infty$ in E_k, which implies that $q_j \to q$ as $j \to +\infty$ uniformly on $[-kT, kT]$. By Proposition 4.3 in [17], we can prove that $\{q_j\}_{j\in\mathbb{N}}$ has a convergent subsequence in E_k. Hence, I_k satisfies the Palais-Smale condition.

Step 2. We prove that there exist constants $\rho, \alpha > 0$ independent of k such that I_k satisfies the assumption (ii) of Lemma 2.2. Letting $\rho = \frac{1}{C}$ and $\|q\|_{E_k} = \rho$, we have $\|q\|_{L^\infty_{2kT}} \le 1$, where $C > 0$ is defined in (3). It follows from (V_4) that

$$\int_{-kT}^{kT} W(t,q)dt \le M\int_{-kT}^{kT}|q(t)|^2dt \le M\|q\|_{E_k}^2. \tag{19}$$

In consequence, combining this with (5), (V_5) and Hölder's inequality, we obtain

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT}\left[\frac{1}{2}|\dot{q}(t)|^2+\frac{1}{2}Bq(t).\dot{q}(t)+K(t,q(t))-W(t,q(t))+f_k(t).q(t)\right]dt \\ &\geq \frac{1}{2}\bar{b}_1\|q\|_{E_k}^2-\frac{1}{2}\|B\|\|q\|_{E_k}^2-M\|q\|_{E_k}^2-\|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}} \\ &\geq \left(\frac{1}{2}\bar{b}_1-\frac{1}{2}\|B\|-M\right)\|q\|_{E_k}^2-\frac{\beta}{2C}\|q\|_{E_k} \\ &\geq \frac{1}{2}\left(\bar{b}_1-\beta-2M-\|B\|\right)\|q\|_{E_k}^2+\frac{\beta}{2}\|q\|_{E_k}^2-\frac{\beta}{2C}\|q\|_{E_k}. \end{aligned} \tag{20}$$

Note that (V_6) implies $\left(\bar{b}_1-\beta-2M-\|B\|\right)>0$.
We set $\alpha=\frac{\bar{b}_1-\beta-2M-\|B\|}{2C^2}$, than the inequality (20) implies that

$$I_{k_{|\partial B_\rho}}\geq\alpha>0 \ \ for \ \ k\in\mathbb{N}.$$

Step 3. It remains to show that I_k satisfies assumption (iii) of Lemma 2.2. By (5), (6) and (8), for every $s\in\mathbb{R}\backslash\{0\}$ and $q\in\mathbb{R}^N\backslash\{0\}$, we have

$$\begin{aligned} I_k(sq) &\leq \frac{\bar{b}_2 s^2}{2}\|q\|_{E_k}^2+s^2\|B\|\|q\|_{E_k}^2-a_1|s|^\mu\int_{-kT}^{kT}|q(t)|^\mu dt \\ &+ |s|\|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}}+2kTa_2. \end{aligned} \tag{21}$$

Take $Q\in E_1$ such that $Q(T)=Q(-T)=0$. Since $\mu>2$ and $a_1>0$, (21) implies that there exists $s_0\in\mathbb{R}\backslash\{0\}$ such that $\|s_0Q\|_{E_1}>\rho$ and $I_1(s_0Q)<0$. Set $e_1(t)=s_0Q(t)$ and

$$e_k(t)=\begin{cases} e_1(t) & for \ \ |t|\leq T, \\ 0 & for \ \ T<|t|\leq kT, \end{cases} \tag{22}$$

for $k>0$. Then $e_k\in E_k$, $\|e_k\|_{E_k}=\|e_1\|_{E_1}>\rho$ and $I_k(e_k)=I_1(e_1)<0$ for every $k\in\mathbb{N}$.
By Lemma 2.2, I_k possesses a critical value $c_k\geq\alpha$ given by

$$c_k=\inf_{g\in\Gamma_k}\max_{s\in[0,1]}I_k(g(s)), \tag{23}$$

where

$$\Gamma_k=\left\{g\in C([0,1],E_k):g(0)=0,g(1)=e_k\right\}.$$

Hence for every $k\in\mathbb{N}$, there exists $q_k\in E_k$ such that

$$I_k(q_k)=c_k, \ \ I_k'(q_k)=0. \tag{24}$$

The function q_k is a desired classical $2kT-$periodic solution of (DS_k) for $k\in\mathbb{N}$. Since $c_k>0$, q_k is a nontrivial solution even if $f\equiv 0$. The proof of Lemma 2.5 is complete.

Lemma 2.6 *Let $(q_k)_{k\in\mathbb{N}}$ be the solution of system (DS_k) which satisfies (24) for $k\in\mathbb{N}$. Then there exists a positive constant M_1 independent of k such that*

$$\|q_k\|_{E_k}\leq M_1, \ \ \forall k\in\mathbb{N}. \tag{25}$$

Proof. For $k \in \mathbb{N}$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (22). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (23)

$$c_k = \inf_{g\in\Gamma_k} \max_{s\in[0,1]} I_k(g(s)) \leq \max_{s\in[0,1]} I_k(g_1(s)) \equiv M_0, \ \ \forall k \in \mathbb{N}, \tag{26}$$

where M_0 is independent of k. Since $I'_k(q_k) = 0$, we get from (V_1), (V_3), (6) and (11)

$$c_k = I_k(q_k) - \frac{1}{2}I'_k(q_k).\tilde{v}(q_k) \geq (\frac{\mu}{2} - 1)\int_{-kT}^{kT} W(t,q_k(t))dt + \frac{1}{2}\int_{-kT}^{kT} Bq_k(t).\dot{q}_k(t)dt$$
$$+\frac{1}{2}\int_{-kT}^{kT} B\dot{q}_k(t).v(q_k(t))dt + \int_{-kT}^{kT} f_k(t).q_k(t))dt - \frac{1}{2}\int_{-kT}^{kT} f_k(t).v(q_k(t))dt.$$

Then we have

$$\begin{aligned}\int_{-kT}^{kT} W(t,q_k(t))dt \ &\leq \ \frac{1}{\mu-2}\int_{-kT}^{kT} B\dot{q}_k(t).q_k(t)dt + \frac{1}{\mu-2}\int_{-kT}^{kT} Bq_k.\dot{\overbrace{v(q_k(t))}}\\ &- \ \frac{2}{\mu-2}\int_{-kT}^{kT} f_k(t).q_k(t)dt + \frac{2c_k}{\mu-2}\\ &+ \ \frac{1}{\mu-2}\int_{-kT}^{kT} f_k(t).v(q_k(t)dt.\end{aligned} \tag{27}$$

Combining (27) with (5), (6), (12), (26), (V_5) and (V_6) we obtain

$$\left(\frac{\bar{b}_1}{2} - \frac{1+b}{\mu-2}\|B\|\right)\|q_k\|^2_{E_k} \leq \frac{\mu M_0}{\mu-2} + \frac{\beta(\mu+b)}{2C(\mu-2)}\|q_k\|_{E_k}. \tag{28}$$

Since (V_6) implies that $\frac{\bar{b}_1}{2} - \frac{1+b}{\mu-2}\|B\| > 0$ and all coefficients of (28) are independent of k, there exists a constant $M_1 > 0$ independent of k such that

$$\|q_k\|_{E_k} \leq M_1, \ \ \forall k \in \mathbb{N}. \tag{29}$$

The proof of Lemma 2.6 is complete. □

Let $C^p_{loc}(\mathbb{R},\mathbb{R}^N)$ $(p \in \mathbb{N})$ denote the space of C^p functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the topology of almost uniformly convergence on compact subintervals of $\mathbb{R}$ and all derivatives up to order p. Using the Arzelà-Ascoli theorem, we can prove the following lemma.

Lemma 2.7 *Let $\{q_k\}_{k\in\mathbb{N}}$ be the $2kT-$periodic solution of problem* (1) *which satisfies* (29) *for $k \in \mathbb{N}$. Then there exists a subsequence $\{q_{k_j}\}$ convergent to q in $C^1_{loc}(\mathbb{R},\mathbb{R}^N)$.*

Proof. Arguing as in Theorem 2.1 in [11], we conclude from the fact

$$|q_k(t_2) - q_k(t_1)| \leq \int_{t_1}^{t_2} |\dot{q}_k(t)|dt \leq (t_2 - t_1)^{1/2}\left(\int_{t_1}^{t_2} |\dot{q}_k(t)|^2 dt\right)^{1/2}$$

that the sequence (q_k) is equicontinuous on every interval $[-lT,lT] \subset [-kT,kT]$. By (29) and Arzelà-Ascoli theorem, the sequence (q_k) has a uniformly convergent subsequence on each $[-lT,lT]$.

Let $(q^1_{k_m})$ be a subsequence of (q_k) that converges on $[-T,T]$. Then $(q^1_{k_m})$ is equicontinuous and uniformly bounded on $[-2T,2T]$. So we can choose a subsequence $(q^2_{k_m})$ of $(q^1_{k_m})$ that converges uniformly on $[-2T,2T]$. Repeat this procedure for all k and take the diagonal sequence $(q^m_{k_m})$. It is obvious that $(q^m_{k_m})_m$ is a subsequence of $(q^i_{k_m})$ for any $1 \le i \le m$. Hence, it converges uniformly to a function $q(t)$ on any bounded interval. In the following, for simplicity, we also denote the subsequence $(q^m_{k_m})$ by (q_k). The proof of Lemma 2.7 is complete. □

Lemma 2.8 *Let $q : \mathbb{R} \to \mathbb{R}^N$ be the function given in Lemma 2.7. Then q is the desired nontrivial homoclinic solution of (DS) such that $\dot{u}(t) \to 0$, as $t \to \pm\infty$.*

Proof. Firstly, we will show that q is a solution of (1). Let $\{q_{k_j}\}_{k\in\mathbb{N}}$ be defined in Lemma 2.7, then we have

$$\ddot{q}_{k_j}(t) + B\dot{q}_{k_j}(t) + V'(t, q_{k_j}(t)) \quad = \quad f_{k_j}(t) \tag{30}$$

for every $j \in \mathbb{N}$ and $t \in [-k_jT, k_jT]$. Take $a, b \in \mathbb{R}$ such that $a < b$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and $t \in [a,b] \subset [-k_jT, k_jT]$, we have

$$\ddot{q}_{k_j}(t) \quad = \quad -B\dot{q}_{k_j}(t) - V'(t, q_{k_j}(t)) + f_{k_j}(t). \tag{31}$$

Hence, $\ddot{q}_{k_j}(t)$ is continuous in $[a,b]$ and $\ddot{q}_{k_j}(t)$ is a classical derivative of $\dot{q}_{k_j}(t)$ in $[a,b]$ for every $j > j_0$. Moreover, since $\dot{q}_{k_j} \to \dot{q}$ uniformly on $[a,b]$ and

$$\ddot{q}_{k_j}(t) \quad = \quad -B\dot{q}_{k_j}(t) - V'(t, q_{k_j}(t)) + f_{k_j}(t) \tag{32}$$

we obtain

$$\ddot{q}(t) + B\dot{q}(t) + V'(t, q(t)) \quad = \quad f(t), \tag{33}$$

for every $t \in [a,b]$. Since a and b are arbitrary, we conclude that q satisfies (DS). □

3 Proof of Theorem 1.1.

We have shown that q satisfies (1). It remains to prove that q is nontrivial and homoclinic to 0. First, we show that q is nontrivial. Obviously, this will be the case if $f \not\equiv 0$. Consider the function $\varphi : [0,+\infty[\to [0,+\infty[$ defined by

$$\varphi(s) = \begin{cases} \max\limits_{t\in\mathbb{R}, 0<|x|\le s} \dfrac{W'(t,x).v(x)}{|x|^2}, & s > 0, \\ 0, & s = 0. \end{cases}$$

Then by (V_2), (V_3), (8) and (9) φ is a continuous, nondecreasing function and $\varphi(s) \ge 0$ for $s \ge 0$. The definition of φ implies that

$$\int_{-kT}^{kT} W'(t, q_k(t)).v(q_k(t))dt \le \varphi(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k} \tag{34}$$

for every $n \in \mathbb{N}$. Since $I'_k(q_k).v(q_k) = 0$, we have

$$\int_{-kT}^{kT} W'(t, q_k(t)).v(q_k(t))dt =$$

$$\int_{-kT}^{kT} |\dot{q}_k(t)|^2 dt - \int_{-kT}^{kT} B\dot{q}_k(t).v(q_k(t))dt + \int_{-kT}^{kT} K'(t, q_k(t)).v(q_k(t))dt. \tag{35}$$

From (34), (35), (V_1) and (V_6), we obtain

$$\begin{aligned}\varphi(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k} &\geq \int_{-kT}^{kT}|\dot{q}_k(t)|^2 dt - \int_{-kT}^{kT}B\dot{q}_k(t).v(q_k(t)))dt + \int_{-kT}^{kT}K'(t,q_k(t))v(q_k(t))dt \\ &\geq (\min\{1,b_1\} - b\|B\|)\,\|q_k\|^2_{E_k}.\end{aligned}$$

Since $\|q_k\|_{E_k} > 0$, it follows that

$$\varphi(\|q_k\|_{L^\infty_{2kT}}) \geq (\min\{1,b_1\} - b\|B\|) > 0.$$

If $\|q_k\|_{L^\infty_{2kT}} \to 0$ as $k \to \infty$, we have $\varphi(0) \geq (\min\{1,b_1\} - b\|B\|) > 0$, which is a contradiction. Passing to a subsequence of (q_k) if necessary, we see that there is a constant $C_1 > 0$ such that

$$\|q_k\|_{L^\infty_{2kT}} \geq C_1 \tag{36}$$

for every $k \in \mathbb{N}$. Moreover, for all $j \in \mathbb{N}$, $t \mapsto q_k^j(t) = q_k(t + jT)$ is also a $2kT$-periodic solution of system (3). Hence, if the maximum of $|q_k|$ occurs in $\theta_k \in [-kT, kT]$ then the maximum of $|q_k^j|$ occurs in $\tau_k^j = \theta_k - jT$. Then there exists a $j_k \in \mathbb{Z}$ such that $\tau_k^{j_k} \in [-T, T]$. Consequently,

$$\|q_k^{j_k}\|_{L^\infty([-kT,kT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |q_k^{j_k}(t)|.$$

Suppose the contrary to our claim, that $q \equiv 0$. Then

$$\|q_k^{j_k}\|_{L^\infty([-kT,kT],\mathbb{R}^N)} = \max_{t\in[-T,T]} |q_k^{j_k}(t)| \to 0,$$

which contradicts (36).

Second, we now prove that $q(t) \to 0$ as $t \to \pm\infty$. We have, from (29)

$$\int_{-kT}^{kT} (|q_k(t)|^2 + |\dot{q}_k(t)|^2)dt \leq \|q_k\|^2_{E_k} \leq M_1^2.$$

Obviously, for each $i \in \mathbb{N}$ there is $k_i \in \mathbb{N}$ such that for all $k \geq k_i$

$$\int_{-iT}^{iT} (|q_k(t)|^2 + |\dot{q}_k(t)|^2)dt \leq \|q_k\|^2_{E_k} \leq M_1^2.$$

Letting $k \to +\infty$, we obtain

$$\int_{-iT}^{iT} (|q(t)|^2 + |\dot{q}(t)|^2)dt \leq M_1^2.$$

As $i \to +\infty$, we have

$$\int_{-\infty}^{+\infty} (|q(t)|^2 + |\dot{q}(t)|^2)dt \leq M_1^2.$$

Hence we get

$$\int_{|t|\geq r}(|q(t)|^2+|\dot{q}(t)|^2)dt \to 0 \text{ as } r\to+\infty. \tag{37}$$

By Corollary 2.2 in [16], we have

$$|q(t)|^2 \leq \int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds \tag{38}$$

for every $t\in\mathbb{R}$. Then, by (37) and (38) we conclude that

$$q(t)\to 0\, as\ |t|\to\infty.$$

Finally, we have to show that $\dot{q}(t)\to 0$ as $t\to\pm\infty$. From Corollary 2.2 in [16] we have

$$|\dot{q}(t)|^2 \leq \int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds + \int_{t-1}^{t+1}|\ddot{q}(s)|^2 ds,$$

for every $t\in\mathbb{R}$. Since $q\in W^{1,2}(\mathbb{R},\mathbb{R}^N)$, we get

$$\int_{t-1}^{t+1}(|q(s)|^2+|\dot{q}(s)|^2)ds \to 0 \text{ as } |t|\to\infty.$$

Hence, it suffices to prove that

$$\int_{t-1}^{t+1}|\ddot{q}(s)|^2 ds \to 0 \text{ as } |t|\to\infty. \tag{39}$$

Since q is a solution of (DS), we obtain

$$\begin{aligned}
\int_{t-1}^{t+1}|\ddot{q}(s)|^2 ds &\leq \|B\|^2\int_{t-1}^{t+1}|\dot{q}(s)|^2 ds + \int_{t-1}^{t+1}|V'(t,q(s))|^2 ds + \int_{t-1}^{t+1}|f(s)|^2 ds\\
&+ 2\|B\|\left(\int_{t-1}^{t+1}|\dot{q}(s)|^2 ds\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|V'(s,q(s))|^2 ds\right)^{\frac{1}{2}}\\
&+ 2\|B\|\left(\int_{t-1}^{t+1}|\dot{q}(s)|^2 ds\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|f(s)|^2 ds\right)^{\frac{1}{2}}\\
&+ 2\left(\int_{t-1}^{t+1}|f(s)|^2\right)^{\frac{1}{2}}\left(\int_{t-1}^{t+1}|V'(s,q(s))|^2 ds\right)^{\frac{1}{2}} ds.
\end{aligned}$$

By (V_5), we get

$$\int_{t-1}^{t+1}|f(s)|^2 ds \to 0,\ \ as\ |t|\to\infty. \tag{40}$$

Since $\displaystyle\int_{t-1}^{t+1}|\dot{q}(s)|^2 ds \to 0$ as $|t|\to\infty$, $q(t)\to 0$ as $|t|\to\infty$ and $V'(t,q)\to 0$ as $|q|\to 0$ uniformly in $t\in\mathbb{R}$, then (39) follows. The proof of Theorem 1.1 is complete. □

Acknowledgment

I wish to thank the anonymous referee for his/her suggestions and interesting remarks.

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Nonlinear Dynamics and Systems Theory, **18** (1) (2018) 92–106

Robust Output Feedback Stabilization and Optimization of Discrete-Time Control Systems

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Received: March 20, 2017; Revised: December 28, 2017

Abstract: The paper is devoted to the problems of output feedback stabilization, robust stabilization, quadratic optimization and generalized H_∞-control for a class of affine discrete-time systems. The solution of robust stabilization problem and evaluation of the quadratic performance criterion for a family of nonlinear nonautonomous control systems are proposed. Methods for construction of control laws providing a robust stability and specified evaluation of the weighted damping level of input signals and initial perturbations are proposed for linear systems with controllable and observable outputs. The application of the main results reduces to solving the systems of linear matrix inequalities.

Keywords: *discrete-time system, output feedback; robust stability; linear matrix inequality; quadratic Lyapunov function, H_∞-control.*

Mathematics Subject Classification (2010): Primary: 93C10, 93C55, 93D09, 93D15, 93D21; Secondary: 34D20, 37N35.

1 Introduction

State and output feedback controllers design for dynamic systems with the prescribed and desired properties is a key problem of control theory. At the same time, the properties of control systems such as asymptotic stability, robustness and optimality of the performance indexes are in the foreground. The main problem in H_∞-control theory for continuous systems is connected with suppression of external and initial perturbations (see, e.g., [1–6] as well as review papers [7–9]). Practical applications of many modern methods for control systems design reduce to solving the linear matrix inequalities (LMI) [10, 11].

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In this paper, we consider classes of linear and affine discrete-time control systems for which closed loop systems can be represented in the pseudolinear form

$$x_{t+1} = M(x_t, t)\, x_t, \quad x_t \in \mathbb{R}^n, \quad t \in \mathcal{T} = \{0, 1, 2, \dots\},$$

besides, a matrix function $M(x,t)$ can contain uncertain quantities belonging to certain sets. Intervals, polytopes, affine families of matrices and other objects may serve as the uncertainty sets. To define uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [6, 12, 13]. The applied control laws are of the form of static or dynamic output feedback. It should be noted that at the solution of many control problems the dynamic controllers have great potential as compared with the static controllers.

Our consideration includes the following types of problems:

• output feedback stabilization of discrete-time control systems (Section 2);

• robust stabilization and optimization of discrete-time control systems with polyhedral uncertainties (Section 3);

• robust stabilization and weighted perturbation suppression in discrete-time control systems (Section 4).

Throughout the paper, the following notations are used: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^T > 0$ (≥ 0) is the symmetric positive definite (semidefinite) matrix X; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the inertia of Hermitian matrix $X = X^*$ consisting of the numbers of positive $(\mathrm{i}_+(X))$, negative $(\mathrm{i}_-(X))$ and zero $(\mathrm{i}_0(X))$ eigenvalues (taking into account the multiplicities); $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of A, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix X, respectively; A^+ is the pseudoinverse matrix; W_A is a matrix whose columns make up the bases of the kernel $\mathrm{Ker}\, A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\|w\|_P$ denotes the weighted l_2-norm of a vector sequence w_t, $t \in \mathcal{T}$; $\mathrm{Co}\{A_1, \dots, A_\nu\}$ stands for a polytope in a matrix space described as a convex hull of the set $\{A_1, \dots, A_\nu\}$, i.e.

$$\mathrm{Co}\{A_1, \dots, A_\nu\} = \{\alpha_1 A_1 + \dots + \alpha_\nu A_\nu : \ \alpha_1 + \dots + \alpha_\nu = 1, \ \ \alpha_i \geq 0, \ i = \overline{1, \nu}\}.$$

Note that matrix intervals and affine sets are described in terms of polytopes.

2 Output Feedback Stabilization of Nonlinear Systems

Consider the affine discrete-time control system

$$x_{t+1} = A(x_t)x_t + B(x_t)u_t, \quad y_t = C(x_t)x_t + D(x_t)u_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is a state vector, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x_t = 0$, $t \in \mathcal{T}$. Assume that $\mathrm{rank}\, B(x) \equiv m$ and $\mathrm{rank}\, C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t, \tag{2}$$

with $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^\perp$ and $C^\perp$ be the orthogonal complements of B and C, respectively, i.e. $B^T B^\perp = 0$, $\det\left[B, B^\perp\right] \neq 0$, $C^\perp C^T = 0$, $\det\left[C^T, C^{\perp T}\right] \neq 0$.

2.1 Static controllers

Formulate stabilizability conditions of the zero state $x_t = 0$ for systems (1) and (2) through the static output-feedback controller

$$u_t = Ky_t, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \{K \in \mathbb{R}^{m\times l} : \det(I_m - KD) \neq 0\}$. Closed loop system (2), (3) has the form

$$x_{t+1} = Mx_t, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K$ is a nonlinear operator with the following properties:

- if $K \in \mathcal{K}_D$, then $\mathbf{D}(K) \equiv K(I_l - DK)^{-1}$ and $I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}$;
- if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{5}$$

where $\mathbf{D}_1(K_2) = (I_m - K_2D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;

- if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and $\mathbf{D}(K) = K_0$.

Definition 2.1 System (4) is ρ-stable if the spectrum $\sigma(M)$ lies inside the circle $\{\lambda : |\lambda| < \rho\}$, where $0 < \rho \leq 1$.

Theorem 2.1 *Let* $\operatorname{rank} B = m < n$ *and* $\operatorname{rank} C = l < n$. *Then the following statements are equivalent:*

1) *There exists a static controller* (3) *ensuring ρ-stability of system* (4).

2) *There exists a matrix $X = X^T > 0$ satisfying the relations*

$$B^{\perp T}(AXA^T - \rho^2 X)B^{\perp} < 0, \tag{6}$$

$$\mathrm{i}(H) = \{l, m, 0\}, \quad H = \begin{bmatrix} H_0 & H_1^T \\ H_1 & H_2 \end{bmatrix}, \tag{7}$$

where $H_0 = B^+(L - LRL)B^{+T}$, $H_1 = CXA^T(I_n - RL)B^{+T}$, $H_2 = C(X - XA^TRAX)C^T$, $L = AXA^T - \rho^2 X$, $R = B^{\perp}S^{-1}B^{\perp T}$, $S = B^{\perp T}LB^{\perp}$;

3) *There exists a matrix $X = X^T > 0$ satisfying the matrix inequalities* (6) *and*

$$AXA^T - \rho^2 X < AXC^T(CXC^T)^{-1}CXA^T. \tag{8}$$

4) *There exist mutually inverse matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying the relations* (6) *and*

$$C^{\perp}(A^TYA - \rho^2 Y)C^{\perp T} < 0. \tag{9}$$

5) *There exists a matrix $Y = Y^T > 0$ satisfying the matrix inequalities* (9) *and*

$$A^TYA - \rho^2 Y < A^TYB(B^TYB)^{-1}B^TYA. \tag{10}$$

When one of the statements 2) – 4) *is true, then the controller*

$$u_t = Ky_t, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{11}$$

where K_0 is a solution of one of the equivalent LMI

$$P_1^TK_0Q_1 + Q_1^TK_0^TP_1 < \begin{bmatrix} \rho^2 X & AX \\ XA^T & X \end{bmatrix}, \quad P_2^TK_0Q_2 + Q_2^TK_0^TP_2 < \begin{bmatrix} -H_0 & 0 \\ 0 & H_2^{-1} \end{bmatrix}, \tag{12}$$

with $P_1 = \big[-B^T, 0\big]$, $Q_1 = \big[0, CX\big]$, $P_2 = \big[I_m, 0\big]$ *and* $Q_2 = \big[H_1, I_l\big]$ *ensures ρ-stability of closed loop system* (4).

For the equivalence of the statements 1) and 2) in Theorem 2.1, see [6]. Equivalence of the statements 2) and 3) follows from the correlations (see [12, p. 147])

$$H = \widehat{H}_0 - \widehat{H}_1^T \widehat{H}_2^{-1} \widehat{H}_1, \ \ \mathrm{i}_+(\widehat{H}) = \mathrm{i}_+(H) = \mathrm{i}_+(\Delta), \ \ \mathrm{i}_-(\widehat{H}) = \mathrm{i}_-(H) + n - m = \mathrm{i}_-(\Delta),$$

where

$$\widehat{H} = \begin{bmatrix} \widehat{H}_0 & \widehat{H}_1^T \\ \widehat{H}_1 & \widehat{H}_2 \end{bmatrix} = \left[\begin{array}{cc|c} B^+LB^{+T} & B^+AXC^T & B^+LB^\perp \\ CXA^TB^{+T} & CXC^T & CXA^TB^\perp \\ \hline B^{\perp T}LB^{+T} & B^{\perp T}AXC^T & S \end{array}\right] = W\Delta W^T,$$

$$\Delta = \begin{bmatrix} AXA^T - \rho^2 X & AXC^T \\ CXA^T & CXC^T \end{bmatrix}, \quad W^T = \begin{bmatrix} B^{+T} & 0 & B^\perp \\ 0 & I_l & 0 \end{bmatrix}, \quad \det W \neq 0.$$

For the equivalence of the statements 1) and 4), see also [5] and [6, Therem 6.1.2].

Theorem 2.2 *Let one of the statements* 2) – 4) *of Theorem* 2.1 *hold for linear system* (2). *Then relations* (11) *and* (12) *determine static controller ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(x) = x^T X^{-1} x$ *of nonlinear closed loop system* (1), (11).

2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) is to find, if possible, a dynamic control law described by

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t, \quad t \in \mathcal{T}, \tag{13}$$

where $\xi_t \in \mathbb{R}^r$ and $r \le n$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (13) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller

$$\widehat{x}_{t+1} = \widehat{A}(\widehat{x}_t)\widehat{x}_t + \widehat{B}(\widehat{x}_t)\widehat{u}_t, \quad \widehat{y}_t = \widehat{C}(\widehat{x}_t)\widehat{x}_t + \widehat{D}(\widehat{x}_t)\widehat{u}_t, \quad \widehat{u}_t = \widehat{K}\widehat{y}_t, \tag{14}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{y}_t = \begin{bmatrix} y_t \\ \xi_t \end{bmatrix}, \quad \widehat{u}_t = \begin{bmatrix} u_t \\ \xi_{t+1} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0 \\ 0 & I_r \end{bmatrix}, \widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0 \\ 0 & I_r \end{bmatrix}, \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0 \\ 0 & 0 \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (13) has the form

$$\widehat{x}_{t+1} = \widehat{M}\,\widehat{x}_t, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{15}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$, $\widehat{\mathbf{D}}(\widehat{K}) = (I_{m+r} - \widehat{K}\widehat{D})^{-1}\widehat{K}$, and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

Theorem 2.3 *The following statements are equivalent:*

1) *There exists a dynamic controller* (13) *of order* $r \le n$ *ensuring* ρ*-stability of closed loop system* (15).

2) *There exist matrices* X *and* X_0 *satisfying the relations* (6) *and*

$$X \ge X_0 > 0, \ \ \mathrm{rank}\,(X - X_0) \le r, \ \ AX_0A^T - \rho^2 X_0 < AX_0C^T(CX_0C^T)^{-1}CX_0A^T. \quad (16)$$

3) *There exist matrices* X *and* Y *satisfying the relations* (6), (9) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0, \quad \mathrm{rank}\, W \le n + r. \quad (17)$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (15) (see [6]).

Remark 2.1 The coefficient matrices of stabilizing controller (13) in Theorem 2.3 may be defined in the form

$$\begin{aligned} K = (I_m + K_0D)^{-1}K_0, \quad U = (I_m + K_0D)^{-1}U_0, \\ V = V_0(I_l + DK_0)^{-1}, \quad Z = Z_0 - V_0(I_l + DK_0)^{-1}DU_0, \end{aligned} \quad (18)$$

using the solution $\widehat{K}_0$ of the LMI

$$\widehat{P}^T\widehat{K}_0\widehat{Q} + \widehat{Q}^T\widehat{K}_0^T\widehat{P} < \widehat{F}, \quad (19)$$

where $\widehat{P} = \big[-\widehat{B}^T, 0\big]$, $\widehat{Q} = \big[0, \widehat{C}\widehat{X}\big]$, $X - X_0 = X_1^T X_2^{-1} X_1 \ge 0$, $K_0 \in \mathcal{K}_D$, $0 < \rho \le 1$,

$$\widehat{F} = \begin{bmatrix} \rho^2\widehat{X} & \widehat{A}\widehat{X} \\ \widehat{X}\widehat{A}^T & \widehat{X} \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0.$$

For example, one can use the Cholesky factorization $X - X_0 = X_1^T X_1 \ge 0$ with $X_2 = I_r$.

Remark 2.2 Note, that matrices X and X_0 satisfy statement 2) iff matrices X and $Y = X_0^{-1}$ satisfy statement 3). From (17) it follows that matrices X and Y are positive definite. The rank restriction in (17) always holds in case of full order $r = n$ dynamic regulator.

Theorem 2.4 *Let one of the statements* 2) *or* 3) *of Theorem* 2.3 *hold for linear system* (2). *Then relations* (18) *and* (19) *determine dynamic controller* (13) *ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(\widehat{x}) = \widehat{x}^T\widehat{X}^{-1}\widehat{x}$ *of nonlinear closed loop system* (1), (13).

3 Robust Stabilization and Optimization of Nonlinear Systems

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider a nonlinear operator

$$\mathbf{F}(K) = W + U^T\mathbf{D}(K)V + V^T\mathbf{D}^T(K)U + V^T\mathbf{D}^T(K)\, R\, \mathbf{D}(K)V \quad (20)$$

with $\mathbf{D}(K) = (I_m - KD)^{-1}K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \big\{K \in \mathbb{R}^{m\times l} :\, K^TPK \le Q\big\}, \quad (21)$$

where $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T \ge 0$, $W = W^T \le 0$, U, V and D are matrices of suitable sizes.

Lemma 3.1 [14] *Suppose that the following matrix inequalities hold:*

$$D^TQD + R < P, \quad \Omega = \begin{bmatrix} W & U^T & V^T \\ U & R-P & D^T \\ V & D & -Q^{-1} \end{bmatrix} \leq 0 \ (< 0). \tag{22}$$

Then $\mathbf{F}(K) \leq 0$ (< 0) *for every matrix* $K \in \mathcal{K}$.

Note that Lemma 3.1 is a generalization of the sufficiency statement for a criterion known as Petersen's lemma on matrix uncertainty [15] (see also [16]).

Consider a nonlinear control system in the vector-matrix form

$$x_{t+1} = A(x_t,t)x_t + B(x_t,t)u_t, \ \ y_t = C(x_t,t)x_t + D(x_t,t)u_t, \tag{23}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$. We construct a set of the static controllers

$$u_t = K(x_t,t)\, y_t, \quad K(x_t,t) = K_*(x_t,t) + \widetilde{K}(x_t,t), \quad \widetilde{K}(x_t,t) \in \mathcal{K}, \tag{24}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (21). We assume that the matrices A, B, C, D, K and K_* depend on x_t and t continuously and the equilibrium state $x_t \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leq h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (23), (24) can be represented as

$$x_{t+1} = M(x_t,t)x_t, \quad M(x_t,t) = A + B\mathbf{D}(K)C. \tag{25}$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix K_* in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (25) is asymptotically Lyapunov stable for every matrix $\widetilde{K}(x_t,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [6,14]).

Theorem 3.1 *Let for some matrix functions* $X_t = X_t^T$ *and* $K_*(x,t)$ *the correlations*

$$\varepsilon_1 I_n \leq X_t \leq \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leq \varepsilon_2, \tag{26}$$

$$\begin{bmatrix} M_*^T X_{t+1} M_* - X_t + \varepsilon_0 I_n & M_*^T X_{t+1} B_* & C_*^T \\ B_*^T X_{t+1} M_* & B_*^T X_{t+1} B_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{27}$$

hold with $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$ *and* $D_* = D(I_m - K_*D)^{-1}$, $x_t = 0$ *and* $t \in \mathcal{T}$. *Then any control* (24) *ensures asymptotic stability of the zero state* $x_t \equiv 0$ *for system* (25) *and a common Lyapunov function* $v(x,t) = x^T X_t x$.

Consider control system (23) with quadratic quality functional

$$J_u(x_0) = \sum_0^\infty \varphi_t, \quad \varphi_t = \left[x_t^T \ u_t^T\right] \Phi_t \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \tag{28}$$

where

$$\Phi_t = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix}, \quad S \geq NR^{-1}N^T + \eta\, I_n, \quad R > 0, \quad \eta > 0, \quad t \in \mathcal{T}.$$

Theorem 3.2 *Let for some matrix functions $X_t = X_t^T$ and $K_*(x_t, t)$ the correlations* (26) *and*

$$\begin{bmatrix} M_*^T X_{t+1} M_* - X_t + \Phi_* + \varepsilon_0 I_n & M_*^T X_{t+1} B_* + N_* + C^T K_*^T R_* & C_*^T \\ B_*^T X_{t+1} M_* + N_*^T + R_* K_* C & B_*^T X_{t+1} B_* + R_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0 \quad (29)$$

hold with $\Phi_* = L_*^T \Phi L_*$, $L_*^T = \left[I_n, C^T \mathbf{D}^T(K_*)\right]$, $R_* = (I_m - K_* D)^{-1T} R (I_m - K_* D)^{-1}$, $N_* = N(I_m - K_* D)^{-1}$, $\varepsilon_0 > 0$, $x_t = 0$ and $t \in \mathcal{T}$. Then any control (24) *ensures asymptotic stability of the zero state $x_t \equiv 0$ for system* (25)*, a common Lyapunov function $v(x,t) = x^T X_t x$ and a bound on the functional $J_u(x_0) \leq v(x_0, 0) = x_0^T X_0 x_0$.*

Corollary 3.1 *Let for some matrix $X_t = X_t^T > 0$ and K_* the system of LMI*

$$\begin{bmatrix} M_{ijk}^T X_{t+1} M_{ijk} - X_t + \Phi_k + \varepsilon_0 I_n & M_{ijk}^T X_{t+1} B_{*j} + N_* + C_k^T K_*^T R_* & C_{*k}^T \\ B_{*j}^T X_{t+1} M_{ijk} + N_*^T + R_* K_* C_k & B_{*j}^T X_{t+1} B_{*j} + R_* - P & D_*^T \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0,$$

hold with $M_{ijk} = A_i + B_j \mathbf{D}(K_*) C_k$, $B_{*j} = B_j (I_m - K_* D)^{-1}$, $\Phi_k = L_k^T \Phi L_k$, $L_k^T = \left[I_n, C_k^T \mathbf{D}^T(K_*)\right]$, $C_{*k} = (I_l - D K_*)^{-1} C_k$, $i = \overline{1,\alpha}$, $j = \overline{1,\beta}$, $k = \overline{1,\gamma}$, $\varepsilon_0 > 0$, $x_t = 0$, $t \in \mathcal{T}$. Then any control (24) *ensures asymptotic stability of the zero state $x_t \equiv 0$ for system* (25) *with uncertainties $A(0,t) \in \mathrm{Co}\{A_1, \ldots, A_\alpha\}$, $B(0,t) \in \mathrm{Co}\{B_1, \ldots, B_\beta\}$ and $C(0,t) \in \mathrm{Co}\{C_1, \ldots, C_\gamma\}$, a common Lyapunov function $v(x,t) = x^T X_t x$ and a bound on the functional $J_u(x_0) \leq v(x_0, 0) = x_0^T X_0 x_0$.*

Note that the proofs of Theorems 3.1 and 3.2 follow directly from Lemma 3.1 and the Lyapunov theorem on asymptotic stability taking into account representation of the first difference of Lyapunov function $v(x,t)$ with respect to system (25) in the form of a quadratic function with matrix of the form (20) and application of formula (5) (see [6,14]).

4 Generalized H_∞-control

4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$x_{t+1} = f(x_t, w_t, t), \quad y_t = g(x_t, w_t, t), \quad t \in \mathcal{T}, \tag{30}$$

where $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are the state, the l_2-norm-limited external perturbations and the output vector, respectively.

Definition 4.1 The dynamical system (30) is called *nonexpansive*, if for all square-integrable functions w_t and $\tau > 0$

$$\sum_{t=0}^{\tau} y_t^T Q y_t \leq \sum_{t=0}^{\tau} w_t^T P w_t + x_0^T X_0 x_0,$$

where Q, P and X_0 are weight symmetric positive definite matrices.

We introduce the performance criterion of system (30) with respect to output y:

$$J = \sup_{0 < \|w\|_P^2 + x_0^T X_0 x_0 < \infty} \varphi(w, x_0), \tag{31}$$

where

$$\varphi(w, x_0) = \frac{\|y\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|y\|_Q^2 = \sum_{t=0}^{\infty} y_t^T Q y_t, \quad \|w\|_P^2 = \sum_{t=0}^{\infty} w_t^T P w_t.$$

In case of $x_0 = 0$, we denote J by J_0. It is obvious that $J_0 \leq J$ and $J \leq 1$ for a nonexpansive system. The value J describes the weighted level of external and initial perturbation suppression in system (30). If $P = I_s$, $Q = I_l$ and $X_0 = \rho I_n$, then J and J_0 coincide with known performance criteria of dynamical systems [17].

Consider the class of linear systems

$$x_{t+1} = Ax_t + Bw_t, \quad y_t = Cx_t + Dw_t, \quad t \in \mathcal{T}. \tag{32}$$

Lemma 4.1 *Let $\rho(A) < 1$. Then an evaluation $J_0 < \gamma$ for system* (32) *holds iff the LMI*

$$\Psi_\gamma = \begin{bmatrix} A^T XA - X + C^T QC & A^T XB + C^T QD \\ B^T XA + D^T QC & B^T XB + D^T QD - \gamma^2 P \end{bmatrix} < 0 \tag{33}$$

has a solution $X = X^T > 0$. To perform the evaluation $J < \gamma$ it is necessary and sufficient that LMI (33) *has a solution X such that*

$$0 < X < \gamma^2 X_0. \tag{34}$$

The sufficiency assertion of Lemma 4.1 follows from the relation

$$\Delta v(x_t) + y_t^T Q y_t - \gamma^2 w_t^T P w_t = \left[x_t^T, w_t^T\right] \Psi_\gamma \begin{bmatrix} x_t \\ w_t \end{bmatrix} < 0,$$

where $\Delta v(x_t) = v(x_{t+1}) - v(x_t)$ is the first difference of Lyapunov function $v(x) = x^T X x$ with respect to system (32). The necessity assertion of Lemma 4.1 may be established via representation of functional $\varphi(w, x_0)$ by similar expression with the identity weight matrices (see the proof of Lemma 5.1.1 in [6] and [17]).

Remark 4.1 If $J_0 < \gamma$, then system (32) with a structured uncertainty

$$w_t = \frac{1}{\gamma}\Theta y_t, \quad \Theta^T P \Theta \leq Q, \quad t \in \mathcal{T}, \tag{35}$$

is robust stable and has a common Lyapunov function $v(x) = x^T X x$. This fact follows from Lemma 4.1 and Theorem 3.1. The functional $\varphi(w, x_0)$ on the set of functions (35) takes the minimum value, if $\Theta^T P \Theta = Q$.

It follows from Lemma 4.1 that the performance criteria J and J_0 of system (32) may be computed as the solutions of the corresponding optimization problems:

$$J_0 = \inf\left\{\gamma : \Psi_\gamma < 0,\ X > 0\right\}, \quad J = \inf\left\{\gamma : \Psi_\gamma < 0,\ 0 < X < \gamma^2 X_0\right\}. \tag{36}$$

Consider the affine system with norm-limited external perturbations

$$x_{t+1} = A(x_t)x_t + B(x_t)w_t, \quad y_t = C(x_t)x_t + D(x_t)w_t, \quad t \in \mathcal{T}, \tag{37}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in $\mathcal{S}_0$. We can formulate the following statement.

Lemma 4.2 *Suppose that there exists a matrix $X = X^T > 0$ satisfying the matrix inequality*

$$\begin{bmatrix} A^T(x)XA(x) - X + C^T(x)QC(x) & A^T(x)XB(x) + C^T(x)QD(x) \\ B^T(x)XA(x) + D^T(x)QC(x) & B^T(x)XB(x) + D^T(x)QD(x) - \gamma^2 P \end{bmatrix} < 0 \tag{38}$$

for all $x \in \mathcal{S}_0$. Then $J_0 \leq \gamma$ and the zero state $x_t \equiv 0$ of system (37) *with a structured uncertainty* (35) *is robust stable with a common Lyapunov function $v(x) = x^T X x$. In addition, if the restriction $0 < X \leq \gamma^2 X_0$ holds, then $J \leq \gamma$.*

4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria J and J_0 of the form (31). We are interested in control laws that ensure nonexpansivity property of close loop system and minimize J and J_0. A control law is said to be J-*optimal*, if the corresponding close loop system has minimum performance criteria J.

Primarily, we consider the static output-feedback controller

$$u_t = K_* y_t + w_t, \quad t \in \mathcal{T}, \tag{39}$$

where $w_t \in \mathbb{R}^m$ is a vector of l_2-bounded perturbations and $K_* \in \mathcal{K}_D$ is an unknown matrix. Assuming that $\det\big[I_m - K_* D(x)\big] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding close loop systems in the form

$$x_{t+1} = A_*(x_t)x_t + B_*(x_t)w_t, \quad y_t = C_*(x_t)x_t + D_*(x_t)w_t, \tag{40}$$

$$x_{t+1} = A_* x_t + B_* w_t, \quad y_t = C_* x_t + D_* w_t, \tag{41}$$

where $A_*(x) = A(x) + B(x)\big[I_m - K_* D(x)\big]^{-1} K_* C(x)$, $B_*(x) = B(x)\big[I_m - K_* D(x)\big]^{-1}$, $C_*(x) = \big[I_l - D(x)K_*\big]^{-1} C(x)$, $D_*(x) = \big[I_l - D(x)K_*\big]^{-1} D(x)$, $A_* = A_*(0)$, $B_* = B_*(0)$, $C_* = C_*(0)$, $D_* = D_*(0)$.

Theorem 4.1 *For linear system* (2), *there exists an output-feedback controller* (39) *such that $J < \gamma$ iff the following correlations are feasible:*

$$W_R^T \begin{bmatrix} A^T XA - X + C^T QC & A^T XB + C^T QD \\ B^T XA + D^T QC & B^T XB + D^T QD - \gamma^2 P \end{bmatrix} W_R < 0, \tag{42}$$

$$W_L^T \begin{bmatrix} AYA^T - Y + BP^{-1}B^T & AYC^T + BP^{-1}D^T \\ CYA^T + DP^{-1}B^T & CYC^T + DP^{-1}D^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{43}$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \tag{44}$$

where $R = [C, D]$, $L = [B^T, D^T]$. Herewith, the zero states $x \equiv 0$ of systems (40) *and* (41) *with uncertainty* (35) *are robust stable with common Lyapunov function $v(x) = x^T X x$.*

Remark 4.2 The gain matrix K_* in Theorem 4.1 may be constructed in the form

$$K_* = K_0(I_l + DK_0)^{-1}, \quad -K_0 \in \mathcal{K}_D, \tag{45}$$

Here K_0 is an arbitrary solution of the LMI

$$L_0^T K_0 R_0 + R_0^T K_0^T L_0 + \Omega < 0, \tag{46}$$

where

$$\Omega = \begin{bmatrix} -X & 0 & A^T & C^T \\ 0 & -\gamma^2 P & B^T & D^T \\ A & B & -X^{-1} & 0 \\ C & D & 0 & -Q^{-1} \end{bmatrix}, \quad R_0^T = \begin{bmatrix} C^T \\ D^T \\ 0 \\ 0 \end{bmatrix}, \quad L_0^T = \begin{bmatrix} 0 \\ 0 \\ B \\ D \end{bmatrix}.$$

Lemma 4.3 [3] *LMI* (46) *has a solution* K_0 *if and only if*

$$W_{L_0}^T \Omega W_{L_0} < 0, \quad W_{R_0}^T \Omega W_{R_0} < 0, \tag{47}$$

where W_{L_0} (W_{R_0}) *is a matrix whose columns make up the bases of the kernel* $\operatorname{Ker} L_0$ ($\operatorname{Ker} R_0$).

4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\xi_{t+1} = Z\xi_t + V y_t, \quad u_t = U\xi_t + K y_t + w_t, \quad t \in \mathcal{T}, \tag{48}$$

where $\xi_0 = 0$, $w_t \in \mathbb{R}^m$ is a vector of bounded perturbations, Z, V, U and K are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear close loop system (2), (48) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_* \widehat{x}_t + \widehat{B}_* w_t, \quad y_t = \widehat{C}_* \widehat{x}_t + \widehat{D}_* w_t, \tag{49}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{A}_* = \widehat{A} + \widehat{B}\widehat{K}_0\widehat{C}, \quad \widehat{B}_* = \widehat{B}_1 + \widehat{B}\widehat{K}_0\widehat{D}_1, \quad \widehat{C}_* = \widehat{C}_1 + \widehat{D}_2\widehat{K}_0\widehat{C}, \quad \widehat{D}_* = D + \widehat{D}_2\widehat{K}_0\widehat{D}_1,$$

$$\widehat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \; \widehat{C}_1 = \begin{bmatrix} C & 0 \end{bmatrix}, \; \widehat{D}_1 = \begin{bmatrix} D \\ 0 \end{bmatrix}, \; \widehat{D}_2 = \begin{bmatrix} D & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},$$

$K_0 = \mathbf{D}(K)$, $U_0 = (I_m - KD)^{-1}U$, $V_0 = V(I_l - DK)^{-1}$, $Z_0 = Z + VD(I_m - KD)^{-1}U$.

We give the following auxiliary statement (see also [18] in the case of $\gamma = 1$).

Lemma 4.4 *Given the matrices* $X > 0$, $Y > 0$ *and the number* $\gamma > 0$, *there are matrices* $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ *and* $Y_2 \in \mathbb{R}^{r\times r}$ *such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{50}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{51}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (49), we get the following result.

Theorem 4.2 *There exists a dynamic controller* (48) *such that the evaluation* $J < \gamma$ *holds for linear system* (49), *iff the LMI system* (34), (42), (43) *and* (51) *is solvable with respect to* $X = X^T > 0$ *and* $Y = Y^T > 0$. *In addition, a close loop system* (49) *with a structured uncertainty* (35) *is robust stable.*

Remark 4.3 The coefficient matrices of dynamic controller (48) in Theorem 4.2 may be constructed in the form (18) by solving LMI with respect to $\widehat{K}_0$:

$$\widehat{L}^T \widehat{K}_0 \widehat{R} + \widehat{R}^T \widehat{K}_0^T \widehat{L} + \widehat{\Omega} < 0, \tag{52}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^T & \widehat{C}_1^T \\ 0 & -\gamma^2 P & \widehat{B}_1^T & D^T \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D & 0 & -Q^{-1} \end{bmatrix}, \quad \widehat{R}^T = \begin{bmatrix} \widehat{C}^T \\ \widehat{D}_1^T \\ 0 \\ 0 \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} 0 \\ 0 \\ \widehat{B} \\ \widehat{D}_2 \end{bmatrix}.$$

Here $\widehat{X}$ is a block matrix determined in Lemma 4.4 for X and Y satisfying Theorem 4.2.

If $K \in \mathcal{K}_D$, then $\det\left[I_m - KD(x)\right] \neq 0$ for all $x \in \mathcal{S}_0$, where $\mathcal{S}_0$ is some neighbourhood of the point $x = 0$, and nonlinear close loop system (1), (48) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t)w_t, \quad y_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t)w_t, \tag{53}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (48), (18) ensures robust stability of the zero state $\widehat{x}_t \equiv 0$ of system (53) with uncertainty (35) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics J_0 and J of system (53), we can apply Lemma 4.2.

4.4 Control systems with controlled and observed outputs

Consider the linear control system

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_t, \end{aligned} \tag{54}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathbb{R}^s$, $z_t \in \mathbb{R}^k$ and $y_t \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively, and $t \in \mathcal{T}$. We are interested in static and dynamic control laws that ensure nonexpansivity property of close loop system and minimize the performance criteria J and J_0 with respect to controlled output z of the form (31), where

$$\varphi(w, x_0) = \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|z\|_Q^2 = \sum_{t=0}^{\infty} z_t^T Q z_t, \quad \|w\|_P^2 = \sum_{t=0}^{\infty} w_t^T P w_t.$$

4.4.1 Static controllers

If we use the static output feedback controller

$$u_t = K y_t, \quad K \in \mathcal{K}_{D_{22}}, \quad t \in \mathcal{T}, \tag{55}$$

then closed loop system (54), (55) has the form

$$x_{t+1} = A_* x_t + B_* w_t, \quad z_t = C_* x_t + D_* w_t, \tag{56}$$

where $A_* = A + B_2K_0C_2$, $B_* = B_1 + B_2K_0D_{21}$, $C_* = C_1 + D_{12}K_0C_2$, $D_* = D_{11} + D_{12}K_0D_{21}$ and $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^T \begin{bmatrix} A^TXA - X + C_1^TQC_1 & A^TXB_1 + C_1^TQD_{11} \\ B_1^TXA + D_{11}^TQC_1 & B_1^TXB_1 + D_{11}^TQD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{57}$$

$$W_L^T \begin{bmatrix} AYA^T - Y + B_1P^{-1}B_1^T & AYC_1^T + B_1P^{-1}D_{11}^T \\ C_1YA^T + D_{11}P^{-1}B_1^T & C_1YC_1^T + D_{11}P^{-1}D_{11}^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{58}$$

where $R = \begin{bmatrix} C_2, D_{21} \end{bmatrix}$, $L = \begin{bmatrix} B_2^T, D_{12}^T \end{bmatrix}$.

Theorem 4.3 *For system* (54)*, there exists an output feedback controller* (55) *such that $J < \gamma$ iff the system of correlations* (44)*,* (57) *and* (58) *is feasible. Herewith, closed loop system* (56) *with a structured uncertainty*

$$w_t = \frac{1}{\gamma}\Theta z_t, \quad \Theta^T P \Theta \leq Q, \quad t \in \mathcal{T}, \tag{59}$$

is robust stable with common Lyapunov function $v(x) = x^TXx$.

If we use a static state feedback $u_t = Kx_t$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case the correlations (44) and (57) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad \begin{bmatrix} P - \gamma^{-2}D_{11}^TQD_{11} & B_1^T \\ B_1 & Y \end{bmatrix} > 0. \tag{60}$$

Corollary 4.1 *For system* (54)*, there exists a state feedback controller $u_t = Kx_t$ such that $J < \gamma$ iff the LMI system* (58) *and* (60) *is solvable for some matrix $Y = Y^T > 0$. Herewith, closed loop system* (56) *with uncertainty* (59) *is robust stable with common Lyapunov function $v(x) = \gamma^2 x^TY^{-1}x$.*

Remark 4.4 The gain matrix K in Theorem 4.3 and Corollary 4.1 may be constructed as

$$K = K_0(I_l + D_{22}K_0)^{-1}, \quad -K_0 \in \mathcal{K}_{D_{22}}, \tag{61}$$

where K_0 is an arbitrary solution of LMI:

$$L_0^TK_0R_0 + R_0^TK_0^TL_0 + \Omega < 0,$$

where

$$\Omega = \begin{bmatrix} -X & 0 & A^T & C_1^T \\ 0 & -\gamma^2P & B_1^T & D_{11}^T \\ A & B_1 & -X^{-1} & 0 \\ C_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \quad R_0^T = \begin{bmatrix} C_2^T \\ D_{21}^T \\ 0 \\ 0 \end{bmatrix}, \quad L_0^T = \begin{bmatrix} 0 \\ 0 \\ B_2 \\ D_{12} \end{bmatrix}.$$

4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t, \quad t \in \mathcal{T}, \tag{62}$$

with $\xi_0 = 0$ and $K \in \mathcal{K}_{D_{22}}$, then closed loop system (54), (62) has the form

$$\widehat{x}_{t+1} = \widehat{A}_*\widehat{x}_t + \widehat{B}_* w_t, \quad z_t = \widehat{C}_*\widehat{x}_t + \widehat{D}_* w_t, \tag{63}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I_r \end{bmatrix}, \quad \widehat{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2, \ \widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21}, \ \widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2, \ \widehat{D}_* = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \ \widehat{C}_1 = \big[C_1, 0\big], \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix}, \ \widehat{D}_{12} = \big[D_{12}, 0\big], \ \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

Here the blocks of matrix $\widehat{K}_0$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$
$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U,$$

are unknown, and

$$\begin{aligned} K &= (I_m + K_0D_{22})^{-1}K_0, \quad U = (I_m + K_0D_{22})^{-1}U_0, \\ V &= V_0(I_l + D_{22}K_0)^{-1}, \quad Z = Z_0 - V_0D_{22}(I_m + K_0D_{22})^{-1}U_0. \end{aligned} \tag{64}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (63), we get the following result.

Theorem 4.4 *For linear system* (54)*, there exists a dynamic controller* (62) *such that* $J < \gamma$ *iff the system of correlations* (34), (51), (57) *and* (58) *is feasible. Herewith, closed loop system* (63) *with uncertainty* (59) *is robust stable.*

Remark 4.5 The coefficient matrices of dynamic controller (62) in Theorem 4.4 may be constructed in the form (64) by solving the LMI

$$\widehat{L}^T\widehat{K}_0\widehat{R} + \widehat{R}^T\widehat{K}_0^T\widehat{L} + \widehat{\Omega} < 0, \tag{65}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^T & \widehat{C}_1^T \\ 0 & -\gamma^2 P & \widehat{B}_1^T & D_{11}^T \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \quad \widehat{R}^T = \begin{bmatrix} \widehat{C}_2^T \\ \widehat{D}_{21}^T \\ 0 \\ 0 \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} 0 \\ 0 \\ \widehat{B}_2 \\ \widehat{D}_{12} \end{bmatrix}.$$

Herewith, system (63) with uncertainty (59) has common Lyapunov function $v(\widehat{x}) = \widehat{x}^T\widehat{X}\widehat{x}$. Here $\widehat{X}$ is a block matrix determined in Lemma 4.4 for X and Y satisfying Theorem 4.4.

We give the following algorithm for constructing stabilizing dynamic controller (62) satisfying Theorem 4.4.

Algorithm 4.1 1) calculate the matrices W_R and W_L, where $R = \left[\, C_2, D_{21} \,\right]$ and $L = \left[\, B_2^T, D_{12}^T \,\right]$;
2) find the matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying (34), (51), (57) and (58);
3) construct the expansion $Z = Y - \gamma^2 X^{-1} = S^T S$, $S \in \mathbb{R}^{r \times n}$, $\ker S = \ker Z$ and form the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad X_1 = \frac{1}{\gamma} S X, \quad X_2 = \frac{1}{\gamma^2} S X S^T + I_r;$$

4) solve the LMI (65) under restriction $\det(I_m + K_0 D_{22}) \neq 0$;
5) calculate the coefficient matrices of dynamic controller (62) by formula (64).

Static and dynamic output-feedback controllers (55) and (62) with $K \in \mathcal{K}_{D_{22}}$ may be applied to a class of affine systems

$$\begin{aligned} x_{t+1} &= A(x_t)x_t + B_1(x_t)w_t + B_2(x_t)u_t, \\ z_t &= C_1(x_t)x_t + D_{11}(x_t)w_t + D_{12}(x_t)u_t, \\ y_t &= C_2(x_t)x_t + D_{21}(x_t)w_t + D_{22}(x_t)u_t. \end{aligned} \tag{66}$$

So, close loop system (62), (66) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t)w_t, \quad z_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t)w_t. \tag{67}$$

As a result, the dynamic controller (62), (64) ensures robust stability of the zero state $\widehat{x}_t \equiv 0$ of system (67) with uncertainty (59) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics J_0 and J of system (67), we can apply Lemma 4.2.

Remark 4.6 Note that we have necessary and sufficient conditions for an evaluation $J_0 < \gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without using additional restriction $X < \gamma^2 X_0$. With the use of static state feedback or full order dynamic controllers the problems under consideration are reduced to the solution of LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with uncertain coefficient matrices

$$A \in \mathrm{Co}\big\{A^1, \ldots, A^{\nu_1}\big\}, \ B_1 \in \mathrm{Co}\big\{B_1^1, \ldots, B_1^{\nu_2}\big\},$$

$$C_1 \in \mathrm{Co}\big\{C_1^1, \ldots, C_1^{\nu_3}\big\}, \ D_{11} \in \mathrm{Co}\big\{D_{11}^1, \ldots, D_{11}^{\nu_4}\big\}.$$

In addition, sufficient statements of these theorems may be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

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