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NONLINEAR DYNAMICS & SYSTEMS THEORY

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# Nonlinear Dynamics and Systems Theory

## An International Journal of Research and Surveys



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# Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys






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# NONLINEAR DYNAMICS AND SYSTEMS THEORY

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# On the Global Asymptotic Stability of a Class of Nonlinear Switched Systems

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**Abstract:** In this paper, a class of nonlinear switched systems with separable nonlinearities is studied. With the aid of multiple Lyapunov functions method, conditions on switching law are derived under which the zero solutions of the considered systems are globally asymptotically stable. Some examples are presented to illustrate the obtained results.

**Keywords:** *hybrid systems; switching law; global asymptotic stability; multiple Lyapunov functions.*

**Mathematics Subject Classification (2010):** 34A38, 34D23.

## 1 Introduction

Switched systems represent a subclass of hybrid systems and have strong engineering background in various applications. A significant number of real systems can be modeled as switched systems such as mechanical systems, chemical processes, vehicle control, traffic control, automotive industry, etc. [3, 11, 18, 23, 24].

A switched system has two components: a family of subsystems and a switching signal. Subsystems in the family are described by a set of indexed equations. The switching signal selects an active subsystem at every instant of time, i.e., the subsystem from the family that is currently being followed [18]. Switching signals are usually classified as time-dependent or state-dependent. Note that qualitative behaviour of a switched system depends not only on the behaviour of individual subsystems in the family, but also on the switching signal [24].

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In the past decades, different methods of analysis for switched systems were developed, and many significant results were obtained (see, for instance, [2–7, 9–11, 18, 24]). In particular, with the aid of the Lyapunov function approach, various conditions of asymptotic stability were derived. Stability is one of the fundamental concepts, and it plays the most important role in control systems design.

For the stability problem, the first question is whether a switched system is stable when there are no restrictions on switching signal (stability analysis under arbitrary switching). On the other hand, many switched systems may fail to preserve stability under arbitrary switching, but may be stable under restricted switching signals. In the second case, it is required to find corresponding restrictions.

Many constructive approaches were developed for the stability analysis of switched systems, for example, the method of differential inequalities (scalar, vector or matrix) [4, 12, 20], the dwell time approach [6, 10, 11], the method of common or multiple Lyapunov functions [6, 7, 9–11, 14, 18, 24], etc. These methods are powerful and effective tool for the finding switching signals providing the required properties.

Stability analysis is complicated if the considered system is essentially nonlinear or/and contains some uncertainties [1, 2, 7, 8]. Along with the asymptotic stability, the problems of ultimate boundedness and finite-time stability are considered in many papers [3–5, 20].

In addition to the solving the problem of stability, it is important to estimate the attraction domain of a given equilibrium position [16]. It should be noted that the size of the region of attraction depends, generally, on switchig law [4]. Of a particular interest is the situation where the equilibrium position is globally asymptotically stable.

In this paper, the problem of global asymptotic stability for a class of nonlinear switched systems with separable nonlinearities is studied. It is assumed that every subsystem from the considered family admits globally asymptotically stable zero solution. We will look for conditions on switching law which guarantee the preservation of global asymptotic stability for the corresponding switched system. We will employ multiple Lyapunov functions in our analysis. As an additional result, estimates of the convergence rate of solutions to the origin will be obtained.

## 2 Statement of the Problem

Consider the system with separable nonlinearities

$$\dot{\mathbf{x}} = \mathbf{P}_\sigma \mathbf{f}(\mathbf{x}). \tag{1}$$

Here $\mathbf{x} = (x_1, \dots, x_n)^T$; $\mathbf{f}(\mathbf{x}) = (f_1(x_1), \dots, f_n(x_n))^T$, scalar functions $f_i(x_i)$ are defined and continuous for $x_i \in (-\infty, +\infty)$ and satisfy the conditions $x_i f_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \dots, n$; $\sigma = \sigma(t)$ is a piecewise constant function defining the switching law, $\sigma(t)$ : $[0, +\infty) \to Q = \{1, \dots, N\}$; $\mathbf{P}_s = \{p_{ij}^{(s)}\}_{i,j=1}^n$ are constant matrices, $s = 1, \dots, N$.

Thus, at each time instant one of the subsystems

$$\dot{\mathbf{x}} = \mathbf{P}_s \mathbf{f}(\mathbf{x}), \quad s = 1, \dots, N, \tag{2}$$

is active. Subsystems of the form (2) belong to well-known class of the Persidskii type systems [21]. They are widely used for modeling of various practical systems and processes, see [2, 3, 13, 15, 17].

Let $\theta_i$, $i = 1, 2, \dots$, be the switching times, $0 < \theta_1 < \theta_2 < \dots$, and $\theta_0 = 0$. Assume that the function $\sigma(t)$ is right-continuous. Without loss of generality, we suppose that the interval $(0, +\infty)$ contains the infinite number of switching instants. Hereinafter, we consider non Zeno sequences [18], i.e., sequences that switch at most a finite number of times in any finite time interval.

From the properties of functions $f_1(x_1), \dots, f_n(x_n)$ it follows that system (1) has the zero solution. We will look for conditions providing global asymptotic stability of the solution.

In [7], the problem of the existence of a common Lyapunov function for family (2) was studied. Several approaches to the construction of such function were proposed. It is known [18, 24] that the existence of a common Lyapunov function guarantees the asymptotic stability of the zero solution of (1) for any switching law.

In the present contribution, we will assume that we failed to construct a common Lyapunov function for subsystems (2). In this case, to prove stability of a switched system, one should restrict the class of admissible switching signals [10, 11, 18, 24]. The general approach for finding such restrictions is based on the use of multiple Lyapunov functions [10, 11].

In what follows we will impose some additional conditions on the right-hand sides of subsystems (2).

**Assumption 2.1** For every $s \in \{1, \dots, N\}$, there exist positive constants $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ such that the matrix $\mathbf{P}_s^T \Lambda_s + \Lambda_s \mathbf{P}_s$ is negative definite. Here $\Lambda_s = \mathrm{diag}\{\lambda_1^{(s)}, \dots, \lambda_n^{(s)}\}$.

**Remark 2.1** Conditions of the existence of required values of $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ were investigated in [5, 7, 9, 14].

**Remark 2.2** If Assumption 2.1 is fulfilled, then for every $s \in \{1, \dots, N\}$ the zero solution of the $s$-th subsystem from (2) is asymptotically stable for any admissible functions $f_1(x_1), \dots, f_n(x_n)$, and for this subsystem the function

$$V_s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i^{(s)} \int_0^{x_i} f_i(\tau)\, d\tau \tag{3}$$

satisfies the requirements of the Lyapunov asymptotic stability theorem. If it is possible to choose values of $\lambda_1^{(s)}, \dots, \lambda_n^{(s)}$ the same for all $s = 1, \dots, N$, then a common Lyapunov function can be constructed for subsystems (2). However, conditions of the existence of such common Lyapunov function are more conservative than those of the existence of a partial Lyapunov function of the form (3) for every subsystem.

**Assumption 2.2** Let functions $f_j(x_j)$ be of the form $f_j(x_j) = \beta_j x_j^{\mu_j}$, $j = 1, \dots, n$, where $\beta_j$ be positive constants, and $\mu_j$ be positive rationals with odd numerators and denominators.

**Remark 2.3** Without loss of generality, we will assume that $\beta_j = 1$, $j = 1, \dots, n$, and $\mu_1 \le \dots \le \mu_n$.

Thus, under Assumption 2.2, we consider the family of subsystems

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}^{(s)} x_j^{\mu_j}, \quad i = 1, \dots, n, \quad s = 1, \dots, N, \tag{4}$$

and the corresponding switched system

$$\dot{x}_i = \sum_{j=1}^{n} p_{ij}^{(\sigma)} x_j^{\mu_j}, \quad i = 1, \ldots, n. \tag{5}$$

**Remark 2.4** System (5) can be treated as a system of the first, in the broad sense [25], approximation for a nonlinear hybrid system.

If Assumption 2.1 is fulfilled, then for subsystems from family (4) there exist Lyapunov functions of the form

$$V_s(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i^{(s)} \frac{x_i^{\mu_i+1}}{\mu_i + 1}, \qquad s = 1, \ldots, N, \tag{6}$$

and the zero solutions of these subsystems are globally asymptotically stable.

Our goal is to find classes of switching signals for which we can guarantee the global asymptotic stability of the zero solution of system (5).

**Remark 2.5** The case where $\mu_1 = \ldots = \mu_n$ was investigated in [4, 6, 11, 18]. Therefore, in the present paper we will assume that $\mu_1 < \mu_n$.

## 3 Preliminary Results

Let

$$c = \max_{s,j=1,\ldots,N} \max_{i=1,\ldots,n} (\lambda_i^{(s)} / \lambda_i^{(j)}).$$

Then $c \geq 1$, and

$$V_s(\mathbf{x}) \leq c V_j(\mathbf{x}), \quad s, j = 1, \ldots, N, \tag{7}$$

for $\mathbf{x} \in \mathbb{R}^n$.

**Remark 3.1** If $c = 1$, then $V_1(\mathbf{x}) \equiv \ldots \equiv V_N(\mathbf{x})$, i.e., for subsystems (4) a common Lyapunov function is constructed. In this case the zero solution of (5) is globally asymptotically stable for any admissible switching law. Therefore, in what follows we assume that $c > 1$.

Denote $T_i = \theta_i - \theta_{i-1}$, $i = 1, 2, \ldots$. Construct auxiliary sequences. Let $\psi_1(b, m) = \chi_1(m) = \varphi_1(b, m) = 0$,

$$\psi_k(b, m) = \sum_{i=1}^{k-1} T_{m+i} b^{k-i}, \qquad \chi_k(m) = \frac{1}{k} \sum_{i=1}^{k-1} T_{m+i}, \qquad \varphi_k(b, m) = \sum_{i=1}^{k-1} T_{m+i} b^{-i}$$

for $k = 2, 3, \ldots$. Here $b = \text{const} > 0$; $m = 1, 2, \ldots$.

Consider the conditions

$$\psi_k(b, m) \to +\infty \quad \text{as} \quad k \to \infty, \tag{8}$$

$$\chi_k(m) \to +\infty \quad \text{as} \quad k \to \infty, \tag{9}$$

$$\varphi_k(b, m) \to +\infty \quad \text{as} \quad k \to \infty. \tag{10}$$

It is worth mentioning that condition (8) needs to be checked only for $0 < b < 1$, and condition (10) only for $b > 1$.

**Lemma 3.1** *If any of conditions* (8)–(10) *is fulfilled for* $m = 1$, *then it is fulfilled for all* $m = 1, 2, \ldots$.

To prove the lemma, it is sufficient to note that the equalities

$$\psi_{m+k-1}(b,1) = \psi_k(b,m) + b^k \sum_{j=2}^{m} T_j\, b^{m-j},$$

$$\chi_{m+k-1}(1) = \frac{k}{m+k-1}\chi_k(m) + \frac{1}{m+k-1}\sum_{j=2}^{m} T_j,$$

$$\varphi_{m+k-1}(b,1) = b^{1-m}\varphi_k(b,m) + \sum_{j=2}^{m} T_j\, b^{1-j}$$

hold for $m = 1, 2, \ldots$ and $k = 2, 3, \ldots$ .

**Lemma 3.2** *Let* $0 < b < 1$. *If condition* (8) *is fulfilled, then condition* (9) *is also fulfilled. In addition, if condition* (8) *is fulfilled uniformly with respect to* $m = 1, 2, \ldots$, *then condition* (9) *is also fulfilled uniformly with respect to* $m = 1, 2, \ldots$.

***Proof.*** The equality $\psi_{k+1}(b,m) = b(\psi_k(b,m) + T_{m+k})$ holds for $k, m = 1, 2, \ldots$. Hence,

$$T_{m+k} = b^{-1}\psi_{k+1}(b,m) - \psi_k(b,m) = b^{-1}\left(\psi_{k+1}(b,m) - \psi_k(b,m)\right) + (b^{-1} - 1)\psi_k(b,m).$$

We obtain

$$\chi_k(m) = \frac{1}{bk}\sum_{i=1}^{k-1}\left(\psi_{i+1}(b,m) - \psi_i(b,m)\right) + \frac{1-b}{bk}\sum_{i=1}^{k-1}\psi_i(b,m)$$

$$= \frac{\psi_k(b,m)}{bk} + \frac{1-b}{bk}\sum_{i=1}^{k-1}\psi_i(b,m) \geq \frac{1-b}{b}\left(\frac{1}{k}\sum_{i=1}^{k}\psi_i(b,m)\right).$$

Let condition (8) be fulfilled. Then, for any $M > 0$, one can choose $N > 0$ such that $\psi_k(b,m) > M$ for $k \geq N$. Hence, $\chi_k(m) \geq (1-b)M/(2b)$ for $k \geq 2N$, and condition (9) is fulfilled.

If condition (8) is fulfilled uniformly with respect to $m = 1, 2, \ldots$, then the value of $N$ can be chosen independent of $m$. Therefore, condition (9) is also fulfilled uniformly with respect to $m = 1, 2, \ldots$. This completes the proof.

Assume that the inequalities

$$\dot{V}_s \leq -\beta V_s^{1+\rho}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{11}$$

hold in a domain $G \subset \mathbb{R}^n$. Here $\beta > 0$, $\rho > -1$, and $\dot{V}_s$ is the derivative of the function $V_s(\mathbf{x})$ with respect to the $s$-th subsystem from (4), $s = 1, \ldots, N$. Denote $b = c^{-\rho}$.

Let a switching law $\sigma(t)$ be given. Construct the multiple Lyapunov function $V_{\sigma(t)}(\mathbf{x})$ corresponding to the switching law. Choose $t_0 \geq 0$ and $\mathbf{x}_0 \in G$, and consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0$ from the point $\mathbf{x}_0$. Find a positive integer $m$ such that $t_0 \in [\theta_{m-1}, \theta_m)$.

Assume that a number $\tilde{t}$ satisfies the conditions $\tilde{t} > t_0$ and $\mathbf{x}(t) \in G$ for $t \in [t_0, \tilde{t}]$. Integrating differential inequalities (11) and taking into account formulae (7), we arrive at the following estimates:

(i) if $\rho > 0$, then

$$\begin{gathered} V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t})) \geq V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}_0) + \beta\rho\left(\tilde{t} - t_0\right) \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ V^{-\rho}_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) \geq b^k V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}_0) + \beta\rho\Big(\left(\tilde{t} - \theta_{m+k-1}\right) \\ + \psi_k(b, m) + b^k\left(\theta_m - t_0\right)\Big) \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1; \end{gathered} \tag{12}$$

(ii) if $\rho = 0$, then

$$\begin{gathered} V_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t})) \leq V_{\sigma(\theta_{m-1})}(\mathbf{x}_0)\mathrm{e}^{-\beta\left(\tilde{t} - t_0\right)} \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ V_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) \leq V_{\sigma(\theta_{m-1})}(\mathbf{x}_0)\mathrm{e}^{k\ln c - \beta\left(\tilde{t} - t_0\right)} \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1; \end{gathered} \tag{13}$$

(iii) if $-1 < \rho < 0$ and $\mathbf{0} \notin G$, then

$$\begin{gathered} V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}(\tilde{t})) \leq V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}_0) + \beta\rho\left(\tilde{t} - t_0\right) \quad \text{for } \tilde{t} \in [t_0, \theta_m), \\ V^{-\rho}_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) \leq b^k\Big(V^{-\rho}_{\sigma(\theta_{m-1})}(\mathbf{x}_0) + \beta\rho\big(b^{-k}\left(\tilde{t} - \theta_{m+k-1}\right) \\ + \varphi_k(b, m) + (\theta_m - t_0)\big)\Big) \quad \text{for } \tilde{t} \in [\theta_{m+k-1}, \theta_{m+k}),\ k \geq 1. \end{gathered} \tag{14}$$

## 4 Conditions of the Global Asymptotic Stability

Let Assumption 2.1 be fulfilled. Consider the partial Lyapunov functions (6) constructed for subsystems (4). It is easy to show that, for any positive numbers $\bar{H}$ and $\hat{H}$, one can find constants $\bar{\beta} > 0$ and $\hat{\beta} > 0$ such that

$$\dot{V}_s \leq -\bar{\beta} V_s^{1+\rho_n}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{15}$$

for $\|\mathbf{x}\| < \bar{H}$, and

$$\dot{V}_s \leq -\hat{\beta} V_s^{1+\rho_1}(\mathbf{x}), \quad s = 1, \ldots, N, \tag{16}$$

for $\|\mathbf{x}\| > \hat{H}$. Here $\rho_n = (\mu_n - 1)/(\mu_n + 1)$, $\rho_1 = (\mu_1 - 1)/(\mu_1 + 1)$, and $\|\cdot\|$ is the Euclidean norm of a vector.

Denote $\bar{b} = c^{-\rho_n}$, $\hat{b} = c^{-\rho_1}$.

**Theorem 4.1** *Let* $1 \leq \mu_1 < \mu_n$. *If*

$$\psi_k(\bar{b}, m) \to +\infty \quad as \quad k \to \infty \tag{17}$$

*uniformly with respect to* $m = 1, 2, \ldots$, *then the zero solution of system* (5) *is globally asymptotically stable.*

***Proof.*** Choose a positive number $\varepsilon$, and find $\bar{\beta} > 0$ such that estimates (15) hold in the domain $G_1 = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| < \varepsilon\}$.

The inequalities

$$\bar{a}_1\|\mathbf{x}\|^{\mu_n+1} \le V_s(\mathbf{x}) \le \bar{a}_2\|\mathbf{x}\|^{\mu_1+1}, \quad s = 1,\dots,N, \tag{18}$$

are valid for $\mathbf{x} \in G_1$. Here $\bar{a}_1$ and $\bar{a}_2$ are positive constants.

Using estimates (12) with $G = G_1$, $\beta = \bar{\beta}$, $\rho = \rho_n$, $b = \bar{b}$ and taking into account inequalities (18), it is easy to prove (see [6]) that under the assumptions of Theorem 4.1 one can choose $\delta > 0$ such that if $t_0 \ge 0$ and $\|\mathbf{x}_0\| < \delta$, then for a solution $\mathbf{x}(t)$ of (5) starting at $t = t_0$ from the point $\mathbf{x}_0$ the condition $\|\mathbf{x}(t)\| < \varepsilon$ hold for $t \ge t_0$, and $\|\mathbf{x}(t)\| \to 0$ as $t - t_0 \to +\infty$ uniformly with respect to $t_0 \ge 0$ and $\|\mathbf{x}_0\| < \delta$. Hence, the zero solution of system (5) is uniformly asymptotically stable.

Let us show that the attraction domain of the zero solution coincides with the space $\mathbb{R}^n$.

Choose an arbitrary number $\varepsilon > 0$, and find the corresponding value of $\delta > 0$ according to the definition of uniform asymptotic stability. Let $\hat{H} \in (0, \delta)$. Then there exists $\hat{\beta} > 0$ such that estimates (16) are fulfilled in the domain $G_2 = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| > \hat{H}\}$.

The inequalities

$$\hat{a}_1\|\mathbf{x}\|^{\mu_1+1} \le V_s(\mathbf{x}) \le \hat{a}_2\|\mathbf{x}\|^{\mu_n+1}, \quad s = 1,\dots,N, \tag{19}$$

hold for $\mathbf{x} \in G_2$, where $\hat{a}_1$ and $\hat{a}_2$ are positive constants.

Consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0 \ge 0$ from a point $\mathbf{x}_0 \in G_2$. There exists a positive integer $m$ such that $t_0 \in [\theta_{m-1}, \theta_m)$.

First, assume that $\mu_1 > 1$. Then $\hat{b} > \bar{b}$, and $\psi_k(\hat{b}, m) > \psi_k(\bar{b}, m)$ for all $k, m = 1, 2, \dots$. Therefore, $\psi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \dots$.

Using estimates (12) with $G = G_2$, $\beta = \hat{\beta}$, $\rho = \rho_1$, $b = \hat{b}$ and taking into account inequalities (19), one can find $\hat{T} \ge 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$. Hence, $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$.

Next, consider the case where $\mu_1 = 1$. Applying Lemma 3.2, we obtain that $\chi_k(m) \to +\infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \dots$. Note that $t - t_0 = (t - \theta_{m+k-1}) + k\chi_k(m) + (\theta_m - t_0))$ for $t \in [\theta_{m+k-1}, \theta_{m+k})$, $k \ge 1$.

Using estimates (13) with $G = G_2$, $\beta = \hat{\beta}$ and taking into account inequalities (19), it is easy to show the existence of a number $\hat{T} \ge 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$. Hence, $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This completes the proof.

**Remark 4.1** If $1 < \mu_1 < \mu_n$, then the value of $\hat{T}$ in the proof of Theorem 4.1 is independent of $t_0$ and $\mathbf{x}_0$. Therefore, under the assumptions of Theorem 4.1, for any given neighborhood of the origin, one can find an estimate of the transient time of all solutions into the neighborhood, and this estimate will be independent of initial conditions of solutions. In the case where $1 = \mu_1 < \mu_n$, the value of $\hat{T}$ is independent of $t_0$, but it depends on $\mathbf{x}_0$.

**Corollary 4.1** *Let $1 \le \mu_1 < \mu_n$. If $T_i \to +\infty$ as $i \to \infty$, then the zero solution of system* (5) *is globally asymptotically stable.*

**Remark 4.2** In the case where $1 \le \mu_1 < \mu_n$ and condition (17) is fulfilled nonuniformly with respect to $m = 1, 2, \dots$, we can guarantee only local and nonuniform asymptotic stability of the zero solution of system (5).

**Theorem 4.2** *Let $0 < \mu_1 < 1 < \mu_n$. If condition* (17) *is fulfilled uniformly with respect to $m = 1, 2, \ldots$, and*

$$\varphi_k(\hat{b}, 1) \to +\infty \quad as \quad k \to \infty, \tag{20}$$

*then the zero solution of system* (5) *is globally asymptotically stable.*

***Proof.*** In a similar way as in the proof of Theorem 4.1, we obtain that the zero solution of system (5) is uniformly asymptotically stable.

For an arbitrary chosen $\varepsilon > 0$, find constant $\delta > 0$ according to the definition of uniform asymptotic stability. Let $\hat{H} \in (0, \delta)$. Then there exists a constant $\hat{\beta} > 0$ such that estimates (16) hold in the domain $G = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| > \hat{H}\}$.

Consider a solution $\mathbf{x}(t)$ of system (5) starting at $t = t_0 \geq 0$ from a point $\mathbf{x}_0 \in G$. Find positive integer $m$ such that $t_0 \in [\theta_{m-1}, \theta_m)$.

Assume that $\mathbf{x}(t) \in G$ for all $t \geq t_0$. Then, for any $\tilde{t} > t_0$, estimates (14) are valid with the following specialization of parameters: $\beta = \hat{\beta}$, $\rho = \rho_1$, $b = \hat{b}$.

According to Lemma 3.1, condition (20) implies that $\varphi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ for any $m = 1, 2, \ldots$. Hence, from (14) it follows that if $\tilde{t}$ is sufficiently large, then $V^{-\rho_1}_{\sigma(\theta_{m+k-1})}(\mathbf{x}(\tilde{t})) < 0$. Thus, we arrive at the contradiction.

Therefore, there exists $\hat{T} \geq 0$ such that $\|\mathbf{x}(t_0 + \hat{T})\| < \delta$, and $\|\mathbf{x}(t)\| \to 0$ as $t \to +\infty$. This completes the proof.

**Remark 4.3** The value of $\hat{T}$ in the proof of Theorem 4.2 depends on $\mathbf{x}_0$, and if $\varphi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ nonuniformly with respect to $m = 1, 2, \ldots$, then it depends on $t_0$ as well. Thus, the proof of Theorem 4.2 permits us to obtain an estimate of transient time of all solutions into a given neighborhood of the origin. However, this estimate depends on initial conditions of solutions.

**Remark 4.4** If $0 < \mu_1 < 1 < \mu_n$, then $0 < \bar{b} < 1$ and $\hat{b} > 1$. In this case the fulfillment of condition (17), generally, does not guarantee the fulfillment of condition (20). Really, let $T_j = \hat{b}^{j/2}$, $j = 1, 2, \ldots$. Then, for any $0 < \bar{b} < 1$, condition (17) is fulfilled uniformly with respect to $m = 1, 2, \ldots$, whereas condition (20) is not fulfilled. Thus, condition (20) of Theorem 4.2 is not excessive one, and it can not be dropped.

**Remark 4.5** In the case where $\mu_1 = \ldots = \mu_n = 1$, one can find a constant $L > 0$ such that if $T_i \geq L$, $i = 1, 2, \ldots$, then the zero solution of the corresponding switched system is globally asymptotically stable [11, 18]. Theorems 4.1 and 4.2 do not permit us to obtain a similar result for $\mu_n > 1$. For instance, if $T_i = L = \text{const} > 0$, $i = 1, 2, \ldots$, then the conditions of Theorems 4.1 and 4.2 are not fulfilled for any value of $L$.

**Theorem 4.3** *Let $0 < \mu_1 < \mu_n = 1$. If condition* (20) *is fulfilled, and condition* (9) *is fulfilled uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system* (5) *is globally asymptotically stable.*

The proof of Theorem 4.3 is similar to those of Theorems 4.1 and 4.2.

**Theorem 4.4** *Let $0 < \mu_1 < \mu_n < 1$. Then the zero solution of system* (5) *is asymptotically stable for any admissible switching law. Furthermore, if condition* (20) *is fulfilled, and there exist a constant $\varphi^* > 0$ and a positive integer $\bar{k} > 0$ such that $\varphi_k(\bar{b}, m) \geq \varphi^*$ for $k \geq \bar{k}$, $m = 1, 2, \ldots$, then the zero solution of system* (5) *is globally asymptotically stable.*

***Proof.*** Let an admissible switching law and a positive number $\varepsilon$ be given. Find $\bar{\beta} > 0$ such that inequalities (15) hold in the domain $G = \{\mathbf{x} \in \mathbb{R}^n : \ 0 < \|\mathbf{x}\| < \varepsilon\}$.

Using estimates (14) with the following specialization of parameters: $\beta = \bar{\beta}$, $\rho = \rho_n$, $b = \bar{b}$, it is easy to prove that, for any $t_0 \geq 0$, one can choose numbers $\delta > 0$ and $\tilde{T} > 0$ such that if $0 < \|\mathbf{x}_0\| < \delta$, then $\|\mathbf{x}(t)\| = 0$ for $t \geq t_0 + \tilde{T}$. Here $\mathbf{x}(t)$ is a solution of (5) starting at $t = t_0$ from the point $\mathbf{x}_0$. Hence, the zero solution of system (5) is asymptotically stable.

Next, assume that condition (20) is fulfilled, and there exist a constant $\varphi^* > 0$ and a positive integer $\bar{k} > 0$ such that $\varphi_k(\bar{b}, m) \geq \varphi^*$ for $k \geq \bar{k}$, $m = 1, 2, \ldots$. In this case, $\delta$ and $\tilde{T}$ can be chosen independent of $t_0$. Thus, the zero solution of (5) is uniformly asymptotically stable. The subsequent proof is similar to those of Theorems 4.1–4.3.

**Corollary 4.2** *Let $0 < \mu_1 < \mu_n < 1$. If $\varphi_k(\hat{b}, m) \to +\infty$ as $k \to \infty$ uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system* (5) *is globally asymptotically stable.*

To prove the corollary, it is sufficient to note that if $0 < \mu_1 < \mu_n < 1$, then $\varphi_k(\bar{b}, m) \geq \varphi_k(\hat{b}, m)$ for $k, m = 1, 2, \ldots$.

**Remark 4.6** Theorem 4.4 does not guarantee the existence of a constant $L > 0$ such that if $T_i \geq L$, $i = 1, 2, \ldots$, then the zero solution of system (5) is globally asymptotically stable. However, for an arbitrary given bounded subset of $\mathbb{R}^n$, an appropriate choice of $L$ permits us to guarantee that the subset is contained in the attraction domain of the zero solution.

**Corollary 4.3** *Let $0 < \mu_1 < \mu_n < 1$. For any $\Delta > 0$, one can find a constant $L > 0$ such that if $T_i \geq L$, $i = 1, 2, \ldots$, then the set $\{\mathbf{x}_0 \in \mathbb{R}^n : \ \|\mathbf{x}_0\| < \Delta\}$ is contained in the attraction domain of the zero solution of system* (5) *for all $t_0 \geq 0$.*

**Example 4.1** Consider the switched indirect control system

$$\begin{aligned} \dot{y}_1 &= a_1^{(\sigma)} y_1 + b_1^{(\sigma)} \eta^3, \\ \dot{y}_2 &= a_2^{(\sigma)} y_2 + b_2^{(\sigma)} \eta^3, \\ \dot{\eta} &= d_1^{(\sigma)} y_1 + d_2^{(\sigma)} y_2 + b_3^{(\sigma)} \eta^3 \end{aligned} \tag{21}$$

and the corresponding family of subsystems

$$\begin{aligned} \dot{y}_1 &= a_1^{(s)} y_1 + b_1^{(s)} \eta^3, \\ \dot{y}_2 &= a_2^{(\sigma)} y_2 + b_2^{(s)} \eta^3, \qquad\qquad s = 1, 2. \\ \dot{\eta} &= d_1^{(s)} y_1 + d_2^{(s)} y_2 + b_3^{(s)} \eta^3, \end{aligned} \tag{22}$$

Thus, $\sigma(t) : [0, +\infty) \to Q = \{1, 2\}$. Let $a_1^{(1)} = -7$, $a_2^{(1)} = -3$, $b_1^{(1)} = 1$, $b_2^{(1)} = 2$, $b_3^{(1)} = -4$, $d_1^{(1)} = 4$, $d_2^{(1)} = 5$, $a_2^{(2)} = -6$, $a_2^{(2)} = -3$, $b_1^{(2)} = 6$, $b_2^{(2)} = 1$, $b_3^{(2)} = -5$, $d_1^{(2)} = 2$, $d_2^{(2)} = 7$.

System (21) is a special case of system (1). Here $n = 3$, $N = 2$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = \eta$, $f_1(x_1) = x_1$, $f_2(x_2) = x_2$, $f_3(x_3) = x_3^3$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 3$,

$$\mathbf{P}_1 = \begin{pmatrix} -7 & 0 & 1 \\ 0 & -3 & 2 \\ 4 & 5 & -4 \end{pmatrix}, \qquad \mathbf{P}_2 = \begin{pmatrix} -6 & 0 & 6 \\ 0 & -3 & 1 \\ 2 & 7 & -5 \end{pmatrix}.$$

Let

$$\Lambda_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then the matrices $\mathbf{P}_s^T \Lambda_s + \Lambda_s \mathbf{P}_s$, $s = 1, 2$, are negative definite. Hence, partial Lyapunov functions for subsystems (22) can be chosen in the form

$$V_1 = \frac{3y_1^2}{2} + y_2^2 + \frac{\eta^4}{4}, \quad V_2 = \frac{y_1^2}{2} + 3y_2^2 + \frac{\eta^4}{2}. \tag{23}$$

At the same time, there is no a positive definite diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ for which matrices

$$\mathbf{P}_s^T \Lambda + \Lambda \mathbf{P}_s, \qquad s = 1, 2, \tag{24}$$

are negative definite.

Really, without loss of generality, we may assume that $\lambda_3 = 1$. Then for the negative definiteness of matrices (24), it is necessary and sufficient the fulfilment of the conditions

$$\frac{48}{\lambda_1} + 3\lambda_1 + \frac{175}{\lambda_2} + 28\lambda_2 < 172, \quad \frac{2}{\lambda_1} + 18\lambda_1 + \frac{49}{\lambda_2} + \lambda_2 < 34.$$

Adding corresponding sides of these inequalities, we arrive at

$$\frac{50}{\lambda_1} + 21\lambda_1 + \frac{224}{\lambda_2} + 29\lambda_2 < 206.$$

However,

$$\frac{50}{\lambda_1} + 21\lambda_1 \geq 10\sqrt{42}, \quad \frac{224}{\lambda_2} + 29\lambda_2 \geq 8\sqrt{406}$$

for all $\lambda_1 > 0$, $\lambda_2 > 0$.

Thus, we can not construct a common Lyapunov function for family (22) in the form

$$V = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 \frac{\eta^4}{2}.$$

For Lyapunov functions (23), the estimates $V_i \leq 3V_j$, $i, j = 1, 2$, holds for $y_1, y_2, \eta \in (-\infty, +\infty)$. Hence, in this case, $c = 3$, $\bar{b} = 1/\sqrt{3}$. Applying Theorem 4.1, we obtain that if

$$\sum_{i=1}^{k-1} 3^{(i-k)/2} T_{m+i} \to +\infty \quad \text{as} \quad k \to \infty$$

uniformly with respect to $m = 1, 2, \ldots$, then the zero solution of system (21) is globally asymptotically stable.

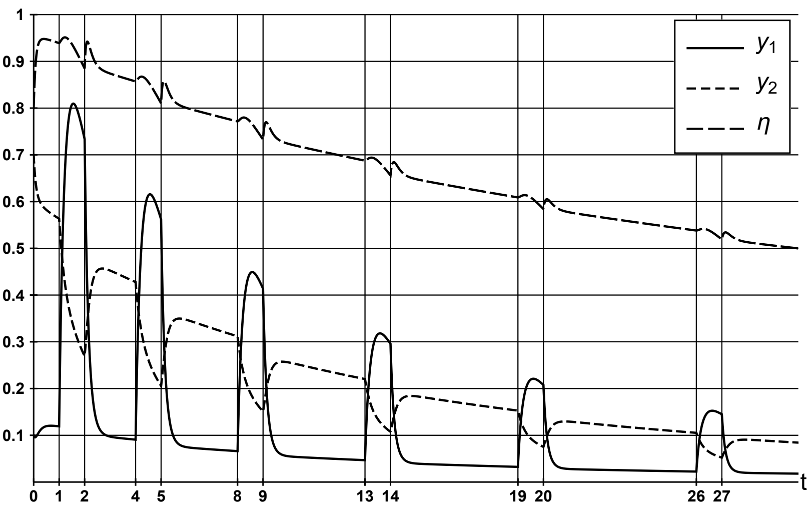


**Figure 1**: The state response of system (21).

The results of a computer simulation are presented in Figure 1. It is assumed that $T_i = i$ for $i = 1, 3, 5, \ldots$, and $T_i = 1$ for $i = 2, 4, 6, \ldots$. In this case,

$$\sum_{i=1}^{k-1} 3^{(i-k)/2} T_{m+i} > (T_{m+k-2} + T_{m+k-1})/3 \geq (k-1)/3 \to +\infty \quad \text{as} \quad k \to \infty$$

uniformly with respect to $m = 1, 2, \ldots$.

We consider the solution of (21) starting at $t = 0$ from the point $(y_1, y_2, \eta)^T = (0.1, 0.7, 0.8)^T$. In Fig. 1, the dependence of components of the solution on time is presented.

Finally in this section, consider the case where Assumption 2.2 is replaced by the following one.

**Assumption 4.1** Functions $f_j(x_j)$ in system (1) can be represented in the form $f_j(x_j) = \beta_j x_j^{\mu_j} + h_j(x_j)$, where $\beta_j$ are positive constants, $\mu_j$ are positive rationals with odd numerators and denominators, functions $h_j(x_j)$ are continuous for $x_j \in (-\infty, +\infty)$ and satisfy the condition $h_j(x_j)/x_j^{\mu_j} \to 0$ as $x_j \to 0$, $j = 1, \ldots, n$.

**Remark 4.7** As well as for Assumption 2.2, we will suppose that $\beta_j = 1$, $j = 1, \ldots, n$, and $\mu_1 \leq \ldots \leq \mu_n$.

**Theorem 4.5** *Let Assumptions 1.1 and 4.1 be fulfilled. Then under the conditions of any of Theorems 4.1–4.4 the zero solution of system* (1) *is asymptotically stable.*

**Remark 4.8** Theorem 4.5 guarantees only local asymptotic stability. However, if the estimates $|h_j(x_j)| \leq \eta_j |x_j|^{\mu_j}$ hold for $x_j \in (-\infty, +\infty)$, where $\eta_j$ are positive constants, $j = 1, \ldots, n$, then, for sufficiently small values of $\eta_j$, the fulfilment of conditions of any of Theorems 4.1–4.4 provides global asymptotic stability of the zero solution of system (1).

## 5 An Optimization of the Choice of Lyapunov Functions

Conditions of the global asymptotic stability obtained in the previous section depend on the value of constant $c$ in inequalities (7). The smaller the value of $c$, the less conservative are restrictions on switching law determined in Theorems 4.1–4.4. Therefore, the problem of finding Lyapunov functions for which value of $c$ is smallest is actual.

Let Lyapunov functions $V_1(\mathbf{x})$, …, $V_N(\mathbf{x})$ of the form (6) be constructed for subsystems (4). Then the estimates

$$V_s(\mathbf{x}) \leq c_{sj} V_j(\mathbf{x}), \quad s, j = 1, \dots, N,$$

hold for $\mathbf{x} \in \mathbb{R}^n$, where $c_{sj} = \max\limits_{i=1,\dots,n} (\lambda_i^{(s)}/\lambda_i^{(j)})$. Hence, the value of constant $c$ in inequalities (7) is defined by the formula $c = \max\limits_{s,j=1,\dots,N} c_{sj}$.

It should be noted that, for arbitrary positive constants $b_1, \dots, b_N$, functions $\widetilde{V}_s(\mathbf{x}) = b_s V_s(\mathbf{x})$, $s = 1, \dots, N$, are also Lyapunov functions for the considered subsystems. For these functions estimates (7) take the form

$$\widetilde{V}_s(\mathbf{x}) \leq \tilde{c} \widetilde{V}_j(\mathbf{x}), \quad s, j = 1, \dots, N,$$

where $\tilde{c} = \max\limits_{s,j=1,\dots,N} (c_{sj} b_s)/b_j$. As a result, we arrive at the optimization problem: it is required to choose positive constants $b_1, \dots, b_N$ for which value of $\tilde{c}$ is minimal. This problem can be reduced to the following nonlinear programming problem [19]:

$$\text{Minimize}: \ \tilde{c},$$

$$\text{subject to}: \ \frac{c_{sj} b_s}{b_j} \leq \tilde{c}, \quad s, j = 1, \dots, N. \tag{25}$$

Conditions of the existence of positive constants $b_1, \dots, b_N$ satisfying inequalities of the form (25) were investigated in [22]. According to the results of this paper, system (25) admits a positive solution if and only if, for any set of indices $i_1, \dots, i_k$ ($i_m \in \{1, \dots, N\}$, $i_m \neq i_l$ for $m \neq l$; $m, l = 1, \dots, k$, $1 \leq k \leq N$), the condition $c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_k i_1} \leq \tilde{c}^k$ is fulfilled. Hence, $\min \tilde{c} = \max(c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_k i_1})^{1/k}$, where the maximum is calculated on all pointed out sets of indices $i_1, \dots, i_k$.

It is worth mentioning that in [22] a constructive procedure for finding required constants $b_1, \dots, b_N$ was proposed.

**Example 5.1** Let family (4) consist of three subsystems of the second order. Hence, $N = 3$ and $n = 2$. Assume that the following Lyapunov functions

$$V_1(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + \frac{x_2^{\mu_2+1}}{\mu_2+1}, \ \ V_2(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + 2\frac{x_2^{\mu_2+1}}{\mu_2+1}, \ \ V_3(\mathbf{x}) = \frac{x_1^{\mu_1+1}}{\mu_1+1} + 3\frac{x_2^{\mu_2+1}}{\mu_2+1} \tag{26}$$

are constructed for these subsystems.

The estimates

$$V_1(\mathbf{x}) \leq V_2(\mathbf{x}), \quad V_1(\mathbf{x}) \leq V_3(\mathbf{x}),$$

$$V_2(\mathbf{x}) \leq 2V_1(\mathbf{x}), \quad V_2(\mathbf{x}) \leq V_3(\mathbf{x}),$$

$$V_3(\mathbf{x}) \leq 3V_1(\mathbf{x}), \quad V_3(\mathbf{x}) \leq \frac{3}{2} V_2(\mathbf{x})$$

are valid for $\mathbf{x} \in \mathbb{R}^2$. Therefore, $c = 3$.

Applying the proposed approach, we obtain

$$\min \tilde{c} = \max \left\{ \sqrt{2};\ \sqrt{3};\ \sqrt{\frac{3}{2}};\ \sqrt[3]{3} \right\} = \sqrt{3}.$$

In this case inequalities (25) take the form

$$\frac{b_1}{b_2} \le \sqrt{3}, \quad \frac{b_1}{b_3} \le \sqrt{3}, \quad \frac{2b_2}{b_1} \le \sqrt{3}, \quad \frac{b_2}{b_3} \le \sqrt{3}, \quad \frac{3b_3}{b_1} \le \sqrt{3}, \quad \frac{(3/2)b_3}{b_2} \le \sqrt{3}.$$

Choose, for instance, $b_1 = \sqrt{3}$, $b_2 = b_3 = 1$. As a result, we find the Lyapunov functions

$$\widetilde{V}_1(\mathbf{x}) = \sqrt{3}V_1(\mathbf{x}), \quad \widetilde{V}_2(\mathbf{x}) = V_2(\mathbf{x}), \quad \widetilde{V}_3(\mathbf{x}) = V_3(\mathbf{x}).$$

With the aid of these functions, one can derive less conservative stability conditions than those which can be obtained with the use of functions (26).

## 6 Conclusion

In this paper, the problem of global asymptotic stability for a class of nonlinear switched systems with separable nonlinearities was investigated. Sufficient conditions on the switching law which garantee the required property for the given equilibrium position are obtained.

It is worth mentioning that the approaches proposed in the paper can be used as well for the analysis of hybrid models of population dynamics and neutral networks. This will be our future work.

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# Global Existence of Weak Solutions to a Fractional Landau-Lifshitz-Gilbert Equation

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**Abstract:** We discuss global existence of weak solutions to a one dimensional periodical fractional Landau-Lifshitz-Gilbert equation. A Faedo-Galerkin/penalization method is employed to get approximate solutions and a fractional calculus inequality is used to deal with the convergence of nonlinear terms. We also study the asymptotic behavior of the obtained solutions when the vertical spin stiffness parameter tends to zero.

**Keywords:** *fractional Landau-Lifshitz-Gilbert equation; Zygmund operator; fractional calculus; global existence; weak solutions.*

**Mathematics Subject Classification (2010):** 35D30, 78A25, 35B40, 82D40.

## 1 Introduction

In the last decades the study of magnetization processes in magnetic materials has been the focus of considerable research for its application to magnetic recording technology. In fact, the design of currently widespread magnetic storage devices, such as the hard-disks, requires the knowledge of the microscopic phenomena occurring within magnetic media. In this respect, it is known that ferromagnetic materials present spontaneous magnetization which is the result of spontaneous alignment of the elementary magnetic moments that constitute the medium. The magnetic recording technology exploits the magnetization of ferromagnetic media to store information. The first example of magnetic storage device was the magnetic core memory prototype, realized by IBM in 1952. After magnetic core memories, magnetic tapes have been used, but the most widespread

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magnetic storage device is certainly the hard-disk. The progress made by research activity performed worldwide in this subject has led to exponential decay of magnetic device dimensions. For more details, we refer for example to [10, 13].

The Landau-Lifshitz (LL) equation [14] and its modification, the Landau-Lifshitz-Gilbert (LLG) equation [8], are the basic equations for studying the magnetization dynamics in ferromagnetic materials. Though these equations are equivalent from the mathematical point of view [7] (specifically, the LL equation reduces to the LLG one by a simple rescaling of the gyromagnetic ratio and damping parameter), the latter is more preferable from the physical point of view and widely used for studying the nonlinear effects in the magnetization dynamics, regimes of forced precession, magnetization switching, etc.

In this paper, we study the following one-dimensional fractional Landau-Lifshitz-Gilbert equation

$$\partial_t \boldsymbol{m} = \gamma \boldsymbol{m} \times \partial_t \boldsymbol{m} + (1+\gamma^2)\boldsymbol{m} \times \mathcal{H}_{\text{eff}}(\boldsymbol{m}). \tag{1}$$

The unknown $\boldsymbol{m}$, the magnetization vector, is an application of $Q = (0,T) \times \Omega$ ($T > 0$ and $\Omega$ is a bounded set of $\mathbb{R}$) into $S^2$ (the unit sphere of $\mathbb{R}^3$), $\partial_t \boldsymbol{m}$ denotes its derivative with respect to time, $\mathcal{H}_{\text{eff}}(\boldsymbol{m})$ is the effective field,"$\times$" is the three dimensional cross product and the magnitude of magnetization (which is constant in space and time) has been scaled to one

$$|\boldsymbol{m}(t,x)| = 1. \tag{2}$$

In (1), the positive constant $\gamma$ is the damping coefficient, and

$$\mathcal{H}_{\text{eff}}(\boldsymbol{m}) = -\frac{\partial \mathcal{E}}{\partial \boldsymbol{m}} \tag{3}$$

is the opposite of the functional derivative of the free energy $\mathcal{E}$. Typical expressions for $\mathcal{E}$ that are usually used in practice take into account several different physical phenomena, and can be found in [10] for instance. In this work, we will focus on the case where $\mathcal{H}_{\text{eff}}(\boldsymbol{m})$ is given by

$$\mathcal{H}_{\text{eff}}(\boldsymbol{m}) = a\Lambda^{2\alpha}\boldsymbol{m} + b\,\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m}, \tag{4}$$

when $\alpha \in (\frac{1}{2}, 1)$ and $a, b > 0$. The operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the square root of the Laplacian and called also Zygmund operator which can be defined for example via Fourier transformation [21].

Equation (1) has broad connections with other well-known equations appearing in mathematics and physics. When $\alpha = 1$ and $b = 0$, equation (1) becomes a standard LLG equation and global existence of weak solutions and nonuniqueness is proved in [1]. When $\alpha \in (\frac{1}{2}, 1)$ and $b = 0$, the existence of weak solutions for (1) is obtained using Faedo-Galerkin/penalization (FGP) method and fractional calculus for the convergence of nonlinear terms, see [18]. When $\alpha = 1$ and $b > 0$, Eq. (1) becomes a standard LLG equation with vertical spin stiffness and global existence of weak solutions is proved in [3].

The equation (1) is subject to the periodic boundary and initial conditions

$$\boldsymbol{m}(0,.) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in} \quad \Omega. \tag{5}$$

A simplified model can be obtained by assuming that $\Omega$ is a subset of $\mathbb{R}$. Specifically, we consider one dimensional domain $\Omega = (-\pi, \pi)$ and assume periodic boundary conditions.

Throughout this paper, for $k \in \mathbb{N}^*$, $\mathbb{L}^k(\Omega) = (L^k(\Omega))^3$ and $\mathbb{H}^k(\Omega) = (H^k(\Omega))^3$ are the usual Hilbert-type Lebesgue and Sobolev spaces, respectively. $\dot{\mathbb{H}}^\alpha(\Omega)$ denotes the homogenous Sobolev-Slobodetskii space and $\mathbb{H}^\alpha(\Omega)$ denotes the inhomogeneous one.

**Lemma 1.1** If $\boldsymbol{m}$ is a regular solution of the problem (1)-(5) then we have for all $t \in (0, T)$ the following energy estimate

$$\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}(t)|^2 \, \mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \, \mathrm{d}x,$$

where $\beta = a(1+\gamma^2)$ at $\lambda = b(1+\gamma^2)$.

***Proof.*** Using the saturation constraint $|\boldsymbol{m}| = 1$, the LLG equation (1) can be written in the following form

$$\gamma \partial_t \boldsymbol{m} + \boldsymbol{m} \times \partial_t \boldsymbol{m} + \beta \Lambda^{2\alpha} \boldsymbol{m} + \lambda \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} - \beta(\boldsymbol{m} \cdot \Lambda^{2\alpha} \boldsymbol{m})\boldsymbol{m} = 0. \tag{6}$$

Taking the inner product of (6) by $\partial_t \boldsymbol{m}$ and $\Lambda^{2\alpha}\boldsymbol{m}$ respectively, we get

$$\gamma \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x + \frac{\beta}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \, \mathrm{d}x + \lambda \int_\Omega \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} \cdot \partial_t \boldsymbol{m} \, \mathrm{d}x = 0 \tag{7}$$

and

$$\begin{aligned} \frac{\gamma}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \, \mathrm{d}x + \int_\Omega \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \Lambda^{2\alpha} \boldsymbol{m} \, \mathrm{d}x + \beta \int_\Omega |\Lambda^{2\alpha} \boldsymbol{m}|^2 \, \mathrm{d}x \\ -\beta \int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha} \boldsymbol{m})^2 \, \mathrm{d}x = 0. \end{aligned} \tag{8}$$

Adding (7) and (8) multiplied by $\lambda$, we obtain

$$\begin{aligned} \gamma \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x + \frac{\beta + \gamma\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}|^2 \, \mathrm{d}x + \lambda\beta \int_\Omega |\Lambda^{2\alpha} \boldsymbol{m}|^2 \, \mathrm{d}x \\ = \lambda\beta \int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha} \boldsymbol{m})^2 \, \mathrm{d}x. \end{aligned}$$

Since

$$\int_\Omega (\boldsymbol{m} \cdot \Lambda^{2\alpha} \boldsymbol{m})^2 \, \mathrm{d}x \leq \int_\Omega |\Lambda^{2\alpha} \boldsymbol{m}|^2 \, \mathrm{d}x,$$

and integrating from 0 to $t$, we obtain

$$\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}(t)|^2 \, \mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \, \mathrm{d}x$$

for all $t \in (0, T)$. □

In this work, we are mainly interested in studying the global existence of weak solutions for (1)-(5). To this end, we first give the definition of weak solutions.

**Definition 1.1** Let $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$ with $|\boldsymbol{m}_0| = 1$ a.e., we say that a three dimensional vector $\boldsymbol{m}$ is a weak solution of the problem (1)-(5) if

- for all $T > 0$, $\boldsymbol{m} \in L^\infty(0, T, \mathbb{H}^\alpha(\Omega))$ and $\partial_t \boldsymbol{m} \in \mathbb{L}^2(Q)$ with $|\boldsymbol{m}| = 1$ a.e.;

- For all $\boldsymbol{\phi} \in \mathcal{C}^{\infty}(\overline{Q})$, such that $\boldsymbol{\phi}(0,.) = \boldsymbol{\phi}(T,.)$

$$\begin{aligned} &\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \\ &= -\beta \int_Q \Lambda^{\alpha} \boldsymbol{m} \cdot \Lambda^{\alpha}(\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t - \lambda \int_Q (\boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m}) \cdot (\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{9}$$

- $\boldsymbol{m}(0,x) = \boldsymbol{m}_0(x)$ in the trace sense.
- For all $t \in (0,T)$

$$\gamma \int_0^t \int_{\Omega} |\partial_t \boldsymbol{m}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_{\Omega} |\Lambda^{\alpha} \boldsymbol{m}(t)|^2 \, \mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_{\Omega} |\Lambda^{\alpha} \boldsymbol{m}_0|^2 \, \mathrm{d}x. \tag{10}$$

**Remark 1.1** We will show in subsect.2.2 that $\boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m}$ makes sense in $\mathbb{L}^2(Q)$, and for this reason, it will be clear that (9) makes sense.

The rest of the paper is organized as follows. In the next section, we prove a global existence of weak solutions result by using Faedo-Galerkin/penalization method. Section 3 is devoted to revealing the relationships between the fractional LLG equation we have studied in this paper, and the classical fractional LLG equation (i.e., in the case $b = 0$). The last section concludes the paper and provides future directions for this work.

## 2 Global Existence of Weak Solutions

The purpose of the present section is to prove the following result

**Theorem 2.1** Let $\boldsymbol{m}_0 \in \mathbb{H}^{\alpha}(\Omega)$ with $|\boldsymbol{m}_0| = 1$ a.e., then there exists a weak solution of the problem (1)-(5) in the sense of Definition 1.1.

To prove Theorem 2.1, we proceed as in [1, 5, 18, 23].

### 2.1 The penalty problem

Let $\varepsilon > 0$. We introduce the following penalty problem. For an initial datum $\boldsymbol{m}_0 \in \mathbb{H}^{\alpha}(\Omega)$, and for each positive number $T$, find a vector field $\boldsymbol{m}_{\varepsilon}$ such as to satisfy the equation

$$\gamma \partial_t \boldsymbol{m}^{\varepsilon} + \boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon} + \beta \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon} + \lambda \boldsymbol{m}^{\varepsilon} \times \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon} + \frac{1}{\varepsilon}(|\boldsymbol{m}^{\varepsilon}|^2 - 1)\boldsymbol{m}^{\varepsilon} = 0. \tag{11}$$

subject to the periodic boundary and initial conditions

$$\boldsymbol{m}^{\varepsilon}(0,.) = \boldsymbol{m}_0, \quad |\boldsymbol{m}_0| = 1 \quad \text{in} \quad \Omega. \tag{12}$$

The last term of equation (11) was introduced at the end to represent the constraint $|\boldsymbol{m}| = 1$.

We have the following result.

**Proposition 2.1** *For each fixed positive $\varepsilon$, there is a weak solution $\boldsymbol{m}^\varepsilon$ of problem (11)-(12) such that*

$$\gamma\int_Q \partial_t \boldsymbol{m}^\varepsilon\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t+\int_Q(\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon)\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t+\beta\int_Q\Lambda^\alpha\boldsymbol{m}^\varepsilon\cdot\Lambda^\alpha\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t$$
$$-\lambda\int_Q\Lambda^\alpha\boldsymbol{m}^\varepsilon\cdot\Lambda^\alpha(\boldsymbol{m}^\varepsilon\times\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\int_Q\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\boldsymbol{m}^\varepsilon\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t=0$$

*for any $\boldsymbol{\varphi}$ in $\mathbb{L}^2(0,T,\mathbb{H}^\alpha(\Omega))$. Moreover, the following energy estimate holds*

$$\gamma\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t+\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^\varepsilon(t)|^2\ \mathrm{d}x\mathrm{d}t$$
$$+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^\varepsilon|^2-1)^2(t)\ \mathrm{d}x\le\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\ \mathrm{d}x$$

*for all $t\in(0,T)$.*

***Proof.*** We show the existence of solutions for the penalty problem by using Faedo-Galerkin method. Let $\{\chi_i\}_{i\in\mathbb{N}}$ be a complete orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of $\Lambda^{2\alpha}$

$$\Lambda^{2\alpha}\chi_i=\lambda_i\chi_i,\quad i=1,2,\dots \tag{13}$$

under periodic boundary conditions. The existence of such a basis can be proved as in Temam [22]. For fixed $\varepsilon>0$, we seek approximate solutions $\boldsymbol{m}^{\varepsilon,N}$ for equation (11) of the form

$$\boldsymbol{m}^{\varepsilon,N}(t,x)=\sum_{i=1}^N\boldsymbol{a}_i(t)\chi_i(x),$$

where $\boldsymbol{a}_i(t)$ are $\mathbb{R}^3$-valued vectors. We obtain the following approached problem

$$\begin{aligned}\gamma\partial_t\boldsymbol{m}^{\varepsilon,N}+\boldsymbol{m}^{\varepsilon,N}\times\partial_t\boldsymbol{m}^{\varepsilon,N}+\beta\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}+\lambda\boldsymbol{m}^{\varepsilon,N}\times\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}\\+\frac{1}{\varepsilon}\big(|\boldsymbol{m}^{\varepsilon,N}|^2-1\big)\boldsymbol{m}^{\varepsilon,N}=0\end{aligned} \tag{14}$$

with the following initial conditions

$$\boldsymbol{m}^{\varepsilon,N}(0,.)=\boldsymbol{m}^N(0,.)\text{ in }\Omega$$

and

$$\int_\Omega\boldsymbol{m}^N(0,.)\chi_i\ \mathrm{d}x=\int_\Omega\boldsymbol{m}_0(0,.)\chi_i\ \mathrm{d}x.$$

Multiplying the equation (14) by $\chi_i$ and integrating over $\Omega$, we get an ordinary differential system.

Note that

$$\gamma\partial_t\boldsymbol{m}^{\varepsilon,N}+\boldsymbol{m}^{\varepsilon,N}\times\partial_t\boldsymbol{m}^{\varepsilon,N}=\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})\partial_t\boldsymbol{m}^{\varepsilon,N},$$

where

$$\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})=\begin{pmatrix}\gamma & -m_3^{\varepsilon,N} & m_2^{\varepsilon,N}\\ m_3^{\varepsilon,N} & \gamma & -m_1^{\varepsilon,N}\\ -m_2^{\varepsilon,N} & m_1^{\varepsilon,N} & \gamma\end{pmatrix}.$$

We can write equation (14) in the form

$$\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})\partial_t \boldsymbol{m}^{\varepsilon,N} = -\beta\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} - \lambda \boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} - \frac{1}{\varepsilon}\big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big)\boldsymbol{m}^{\varepsilon,N}.$$

Since $\mathbb{A}(\boldsymbol{m}^{\varepsilon,N})$ is invertible, then the resulting system is locally Lipschitz. There exists a unique local solution for the approximate problem that can extend on $[0,T]$ using a priori estimate. To get bounds on the solutions, we multiply equation (14) by $\partial_t \boldsymbol{m}^{\varepsilon,N}$ and $\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}$ respectively and integrate over $\Omega$. We obtain

$$\begin{aligned} &\gamma \int_\Omega |\partial_t \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &+\lambda \int_\Omega \boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \cdot \partial_t \boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x = 0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\int_\Omega \boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x = 0. \end{aligned} \tag{16}$$

Multiplying (16) by $\lambda$ and make the sum with (15), we obtain

$$\begin{aligned} &\gamma \int_\Omega |\partial_t \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x \\ &+\lambda\beta \int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\lambda\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \\ &= -\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x. \end{aligned} \tag{17}$$

On the other hand, Young's inequality gives

$$\begin{aligned} &-\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x \\ &\le \frac{\lambda}{2d\varepsilon^2}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2|\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x + \frac{\lambda d}{2}\int_\Omega |\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x \end{aligned} \tag{18}$$

for any constant $d > 0$.

We multiply equation (14) by $(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N}$ and integrate over $\Omega$, we obtain

$$\begin{aligned} &\beta \int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x + \frac{\gamma}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x \\ &+\frac{1}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2|\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x = 0. \end{aligned}$$

Hence

$$\begin{aligned} &-\frac{\lambda}{\varepsilon}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)\boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N} \,\mathrm{d}x \\ &= \frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2 \,\mathrm{d}x + \frac{\lambda}{\beta\varepsilon^2}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2 - 1)^2|\boldsymbol{m}^{\varepsilon,N}|^2 \,\mathrm{d}x. \end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+\frac{\lambda}{\beta\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\\&\le\frac{\lambda}{2d\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\lambda d}{2}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x.\end{aligned}$$

That is

$$\begin{aligned}\frac{\gamma\lambda}{4\beta\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x+\frac{\lambda}{\varepsilon^2}(\frac{1}{\beta}-\frac{1}{2d})\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\\\le\frac{\lambda d}{2}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x.\end{aligned}$$

So for $d>\frac{\beta}{2}$

$$\begin{aligned}&\frac{\lambda}{2d\beta\varepsilon^2}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2|\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\\&\le\frac{\lambda d}{2(2d-\beta)}\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x-\frac{\gamma\lambda}{4\beta\varepsilon(2d-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x.\end{aligned}$$

Therefore from (18)

$$\begin{aligned}&-\frac{\lambda}{\varepsilon}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)\boldsymbol{m}^{\varepsilon,N}\cdot\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}\ \mathrm{d}x\\&\le\frac{\lambda d}{2}(1+\frac{\beta}{2d-\beta})\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x-\frac{\gamma\lambda}{4\varepsilon(2d-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x.\end{aligned}$$

Then from (17)

$$\begin{aligned}&\gamma\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\beta+\gamma\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\\&+\lambda(\beta-\frac{d^2}{2d-\beta})\int_\Omega|\Lambda^{2\alpha}\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{2d-\beta})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x\le0.\end{aligned}$$

Choose $d=\beta$, we get $\beta-\frac{d^2}{2d-\beta}=0$ and therefore

$$\begin{aligned}\gamma\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x+\frac{\beta+\gamma\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\\+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x\le0.\end{aligned}$$

We integrate from 0 to $t$ and we get

$$\begin{aligned}&\gamma\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x\mathrm{d}t+\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^{\varepsilon,N}(t)|^2\ \mathrm{d}x\\&+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2(t)\ \mathrm{d}x\le\frac{\beta+\gamma\lambda}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^N|^2(0)\ \mathrm{d}x\\&+\frac{1}{4\varepsilon}(1+\frac{\gamma\lambda}{\beta})\int_\Omega(|\boldsymbol{m}^N|^2-1)^2(0)\ \mathrm{d}x.\end{aligned}\tag{19}$$

The right-hand side is uniformly bounded. Indeed $\mathbb{H}^\alpha(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$ with continuous embedding, therefore

$$\begin{aligned}\int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2(0)\ \mathrm{d}x &= \int_\Omega |\boldsymbol{m}^N(0)|^4\ \mathrm{d}x - 2\int_\Omega |\boldsymbol{m}^N(0)|^2\ \mathrm{d}x + \mathrm{meas}(\Omega)\\ &\le \|\boldsymbol{m}^N(0)\|^4_{\mathbb{L}^4(\Omega)} + \mathrm{meas}(\Omega)\\ &\le C_1\|\boldsymbol{m}^N(0)\|^4_{\mathbb{H}^\alpha(\Omega)} + C_2,\end{aligned}$$

where $C_1$ and $C_2$ are two constants independent of $\varepsilon$ and $N$. Furthermore, note that $\boldsymbol{m}^{\varepsilon,N}(0) = \boldsymbol{m}^N(0)$, and since $\boldsymbol{m}^N(0)$ has the same components as $\boldsymbol{m}_0$ in the basis $\{\chi_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{m}_0 \in \mathbb{H}^\alpha(\Omega)$, we have $\|\boldsymbol{m}_0\|_{\mathbb{H}^\alpha(\Omega)} \le C_3$ with $C_3$ being a constant independent of $\varepsilon$ and $N$. Hence

$$\|\boldsymbol{m}^N(0)\|_{\mathbb{H}^\alpha(\Omega)} \le C_3.$$

Therefore,

$$\|\Lambda^\alpha \boldsymbol{m}^N(0)\|_{\mathbb{L}^2(\Omega)} \le C_3.$$

Thus for $\varepsilon$ fixed, we have

$$(|\boldsymbol{m}^{\varepsilon,N}|^2-1)_N \text{ is bounded in } L^\infty(0,T,\mathbb{L}^2(\Omega)),$$

$$(\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N})_N \text{ is bounded in } L^\infty(0,T,\mathbb{L}^2(\Omega)).$$

By Young's inequality

$$\int_\Omega |\boldsymbol{m}^{\varepsilon,N}|^2\ \mathrm{d}x \le C + \int_\Omega (|\boldsymbol{m}^{\varepsilon,N}|^2-1)^2\ \mathrm{d}x,$$

with $C$ being a constant which does not depend on $N$. Therefore,

$$(\boldsymbol{m}^{\varepsilon,N})_N \ \text{ is bounded in }\ L^\infty(0,T,\mathbb{H}^\alpha(\Omega)),$$

$$(\partial_t \boldsymbol{m}^{\varepsilon,N})_N \ \text{ is bounded in }\ L^2(0,T,\mathbb{L}^2(\Omega)) := \mathbb{L}^2(Q),$$

and we will need a compactness lemma due to Simon [20].

**Lemma 2.1** *Assume $B_0, B, B_1$ are three Banach spaces and satisfy $B_0 \subset B \subset B_1$ with compact embedding $B_0 \hookrightarrow B$. Let $W$ be bounded in $L^\infty(0,T;B_0)$ and $W_t := \{w_t; w \in W\}$ be bounded in $L^q(0,T;B_1)$ where $q > 1$. Then $W$ is relatively compact in $C([0,T];B)$.*

The proof can be found in Simon [20]. Then we have the following convergences to a subsequence further notes that $\boldsymbol{m}^{\varepsilon,N}$ for any $(1 < p < \infty)$

$$\boldsymbol{m}^{\varepsilon,N} \rightharpoonup \boldsymbol{m}^\varepsilon \ \text{ weakly in }\ L^p(0,T,\mathbb{H}^\alpha(\Omega)), \tag{20}$$

$$\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^\varepsilon \ \text{ strongly in }\ C([0,T],\mathbb{H}^\delta(\Omega)) \text{ and a.e for } 0 \le \delta < \alpha, \tag{21}$$

$$\partial_t \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \partial_t \boldsymbol{m}^\varepsilon \ \text{ weakly in }\ \mathbb{L}^2(Q), \tag{22}$$

$$|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \rightharpoonup \zeta \ \text{ weakly in }\ L^p(0,T,\mathbb{L}^2(\Omega)). \tag{23}$$

The convergence (21) is a consequence of (20) and by compactness embedding of $L^2(0,T,\mathbb{H}^\alpha(\Omega))$ in $L^2(0,T,\mathbb{L}^2(\Omega))$. On the other hand $\zeta = |\boldsymbol{m}^\varepsilon|^2 - 1$. This is provided by the following lemma.

**Lemma 2.2** *Let $\Theta$ be a bounded open subset of $\mathbb{R}^d_x \times \mathbb{R}_t$, $h_n$ and $h$ are functions of $L^q(\Theta)$ with $1 < q < \infty$ such as $\|h_n\|_{L^q(\Theta)} \leq C$, $h_n \to h$ a.e in $\Theta$ then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

The proof of Lemma 2.2 can be found in [15]. In our case $\Theta = Q$, $h_N = |\boldsymbol{m}^{\varepsilon,N}|^2 - 1$, $h = |\boldsymbol{m}^{\varepsilon}|^2 - 1$ and $q = 2$ and from (21) $|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \longrightarrow |\boldsymbol{m}^{\varepsilon}|^2 - 1$ a.e, and we have in particular $|\boldsymbol{m}^{\varepsilon,N}|^2 - 1 \in L^2(\Theta)$, $|\boldsymbol{m}^{\varepsilon}|^2 - 1 \in L^2(\Theta)$ and $\big\||\boldsymbol{m}^{\varepsilon}|^2 - 1\big\|_{L^2(\Theta)} \leq C$.

Now, we pass to the limit as $N \to \infty$. Multiplying the equation (14) by $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$ and integrating on $Q$ yield

$$\begin{aligned}\gamma \int_Q \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \mathrm{d}x\mathrm{d}t + \beta \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\ -\lambda \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon} \int_Q \big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big) \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{24}$$

We have

$$\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^{\varepsilon} \text{ strongly in } \mathbb{L}^2(Q).$$

Furthermore

$$\partial_t \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \partial_t \boldsymbol{m}^{\varepsilon} \text{ weakly in } \mathbb{L}^2(Q).$$

Thus

$$\int_Q (\boldsymbol{m}^{\varepsilon,N} \times \partial_t \boldsymbol{m}^{\varepsilon,N}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to \int_Q (\boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t.$$

On the other hand

$$\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \rightharpoonup \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \text{ weakly in } \mathbb{L}^2(Q).$$

Therefore

$$\int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \cdot \Lambda^\alpha \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t,$$

and

$$\int_Q \partial_t \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to \int_Q \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t.$$

Taking into account (23), we obtain

$$\int_Q \big(|\boldsymbol{m}^{\varepsilon,N}|^2 - 1\big) \boldsymbol{m}^{\varepsilon,N} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \to \int_Q \big(|\boldsymbol{m}^{\varepsilon}|^2 - 1\big) \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t.$$

For the third term of (24) we set

$$D_N = \int_Q (\boldsymbol{m}^{\varepsilon,N} \times \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon,N}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \quad \text{and} \quad D = \int_Q (\boldsymbol{m}^{\varepsilon} \times \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t.$$

We have

$$D_N = -\int_Q \Lambda^{2\alpha} \boldsymbol{m}^{\varepsilon,N} \cdot (\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t = -\int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t.$$

Then we define the commutator

$$[\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m} := \Lambda^\alpha (\boldsymbol{\varphi} \times \boldsymbol{m}) - \boldsymbol{\varphi} \times \Lambda^\alpha \boldsymbol{m}.$$

Since $\Lambda^\alpha$ is a nonlocal operator, the following fractional calculus inequality will play a critical role in the convergence of approximate solutions, see [6] for the proof.

**Lemma 2.3** *Suppose that $s > 0$ and $p \in (1, +\infty)$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|g\|_{\dot{H}^{s-1,p_2}} + \|f\|_{\dot{H}^{s,p_3}}\|g\|_{L^{p_4}})$$

*and*

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|g\|_{\dot{H}^{s,p_2}} + \|f\|_{\dot{H}^{s,p_3}}\|g\|_{L^{p_4}})$$

*with $p_2, p_3 \in (1, +\infty)$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

*and $f, g$ are such that the right-hand side terms make sense.*

We have

$$\begin{aligned}&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(\Omega)}\\ &\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\dot{W}^{\alpha-1,p_2}(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{L}^{p_4}(\Omega)}\bigg).\end{aligned}$$

We choose $p_1 = \frac{1}{1-\alpha}$, $p_2 = \frac{2}{2\alpha-1}$ and $p_3, p_4 \in (2, +\infty)$. This is justified by the fact that $\dot{W}^{k,p} \hookrightarrow L^q$ for $0 \leq k < \frac{n}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, in our case $n = 1$ and $k = 1 - \alpha$ and we want $\dot{W}^{k,p} \hookrightarrow L^2$. Therefore it is sufficient that $\frac{1}{2} = \frac{1}{p} - (1-\alpha)$ that is $\frac{1}{p} = \frac{3}{2} - \alpha = \frac{1}{p_2^*}$ where $\frac{1}{p_2} + \frac{1}{p_2^*} = 1$ and therefore $\dot{W}_0^{s,p_2^*} \hookrightarrow L^2 = (L^2)' \hookrightarrow (\dot{W}_0^{k,p})' \hookrightarrow \dot{W}^{-k,p_2}$. Thus for $\delta = \frac{1}{2} - \frac{1}{p_4} < \frac{1}{2} < \alpha$

$$\begin{aligned}&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(\Omega)}\\ &\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|_{\mathbb{H}^\delta(\Omega)}\bigg)\\ &\leq C\bigg(\|\nabla\boldsymbol{\varphi}\|^2_{\mathbb{L}^{p_1}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{L}^2(\Omega)} + \|\boldsymbol{\varphi}\|^2_{\dot{W}^{\alpha,p_3}(\Omega)}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{H}^\delta(\Omega)}\bigg).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left\|[\Lambda^\alpha, \boldsymbol{\varphi}](\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon)\right\|_{\mathbb{L}^2(Q)} \leq C\bigg(\|\nabla\boldsymbol{\varphi}\|^2_{L^\infty(0,T,\mathbb{L}^{p_1}(\Omega))}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{\mathbb{L}^2(Q)}\\ &+\|\boldsymbol{\varphi}\|^2_{L^\infty(0,T,\dot{W}^{\alpha,p_3}(\Omega))}\|\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^\varepsilon\|^2_{L^2(0,T,\mathbb{H}^\delta(\Omega))}\bigg).\end{aligned}$$

The right-hand side of the last inequality tends to 0 due to strong convergence of $\boldsymbol{m}^{\varepsilon,N} \to \boldsymbol{m}^\varepsilon$ in $\mathbb{L}^2(Q)$ and in $L^2(0, T, \mathbb{H}^\delta(\Omega))$. Moreover by the preceding lemma $[\Lambda^\alpha, \boldsymbol{\varphi}]\boldsymbol{m}^\varepsilon \in$

$\mathbb{L}^2(Q)$. Thus

$$
\begin{aligned}
&|D_N - D| = \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon,N} \, \mathrm{d}x\mathrm{d}t - \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right| \\
&= \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \, \mathrm{d}x\mathrm{d}t + \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \left| \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon,N} \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \, \mathrm{d}x\mathrm{d}t \right| + \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \|\Lambda^\alpha \boldsymbol{m}^{\varepsilon,N}\|_{\mathbb{L}^2(Q)} \left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \right\|_{\mathbb{L}^2(Q)} \\
&+ \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right| \\
&\leq \|\boldsymbol{m}^{\varepsilon,N}\|_{L^2(0,T,\mathbb{H}^\alpha(\Omega))} \left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \right\|_{\mathbb{L}^2(Q)} \\
&+ \left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right|.
\end{aligned}
$$

Since $\|\boldsymbol{m}^{\varepsilon,N}\|_{L^2(0,T,\mathbb{H}^\alpha(\Omega))} \leq C$ and

$$
\begin{aligned}
\left\| [\Lambda^\alpha, \boldsymbol{\varphi}] (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \right\|_{\mathbb{L}^2(Q)} &\to 0, \\
\left| \int_Q \Lambda^\alpha (\boldsymbol{m}^{\varepsilon,N} - \boldsymbol{m}^{\varepsilon}) \cdot [\Lambda^\alpha, \boldsymbol{\varphi}] \boldsymbol{m}^{\varepsilon} \, \mathrm{d}x\mathrm{d}t \right| &\to 0,
\end{aligned}
$$

this implies that

$$D_N \to D. \tag{25}$$

Using the previous convergences and passing to the limit ($N \to \infty$) in (24), we get

$$
\begin{aligned}
&\gamma \int_Q \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^{\varepsilon} \times \partial_t \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t + \beta \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \cdot \Lambda^\alpha \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\
&- \lambda \int_Q \Lambda^\alpha \boldsymbol{m}^{\varepsilon} \cdot \Lambda^\alpha (\boldsymbol{m}^{\varepsilon} \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon} \int_Q \big(|\boldsymbol{m}^{\varepsilon}|^2 - 1\big) \boldsymbol{m}^{\varepsilon} \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = 0
\end{aligned}
\tag{26}
$$

for all $\boldsymbol{\varphi}$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$ by density of $\mathbf{C}^\infty(\overline{Q})$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$.

Now back to (19) and taking into account the previous convergences in $N$ and using Fatou lemma, we get

$$
\begin{aligned}
&\gamma \int_0^t \int_\Omega |\partial_t \boldsymbol{m}^{\varepsilon}|^2 \, \mathrm{d}x\mathrm{d}t + \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}^{\varepsilon}(t)|^2 \, \mathrm{d}x\mathrm{d}t \\
&+ \frac{1}{4\varepsilon}(1 + \frac{\gamma\lambda}{\beta}) \int_\Omega (|\boldsymbol{m}^{\varepsilon}|^2 - 1)^2(t) \, \mathrm{d}x \leq \frac{\beta + \gamma\lambda}{2} \int_\Omega |\Lambda^\alpha \boldsymbol{m}_0|^2 \, \mathrm{d}x
\end{aligned}
\tag{27}
$$

for all $t \in (0,T)$. □

We are now in a position to prove Theorem 2.1.

## 2.2 Convergence of the approximate solutions

To pass to the limit in $\varepsilon$ ($\varepsilon \to 0$), we need estimate (19) and the following result

**Lemma 2.4** *If $\boldsymbol{m}^\varepsilon$ satisfies* (26) *then* $|\boldsymbol{m}^\varepsilon| \leq 1$ *a.e. on* $Q$.

***Proof.*** We choose $\boldsymbol{\varphi} = \psi_{\mathcal{B}}\boldsymbol{m}^\varepsilon$ with $\mathcal{B} = \{|\boldsymbol{m}^\varepsilon| > 1\}$ and $\psi_{\mathcal{B}}$ is the indicator function of the set $\mathcal{B}$. We have $\boldsymbol{\varphi}$ in $L^2(0,T,\mathbb{H}^\alpha(\Omega))$, and replacing $\boldsymbol{\varphi}$ by $\psi_{\mathcal{B}}\boldsymbol{m}^\varepsilon$ in (26), we obtain

$$\gamma\int_0^t\int_{\mathcal{B}}\partial_t\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t+\beta\int_0^t\int_{\mathcal{B}}|\Lambda^\alpha\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t+\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)|\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t=0.$$

Then

$$\begin{gathered}\frac{\gamma}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\ \mathrm{d}x\mathrm{d}t+\beta\int_0^t\int_{\mathcal{B}}|\Lambda^\alpha\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t\\+\frac{1}{\varepsilon}\int_0^t\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)|\boldsymbol{m}^\varepsilon|^2\ \mathrm{d}x\mathrm{d}t=0.\end{gathered}$$

Hence

$$\frac{\gamma}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon|^2-1\big)\ \mathrm{d}x\mathrm{d}t\leq 0.$$

We integrate from 0 to $t$, we get

$$\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon(t)|^2-1\big)\ \mathrm{d}x\leq\int_{\mathcal{B}}\big(|\boldsymbol{m}^\varepsilon(0)|^2-1\big)\ \mathrm{d}x=0.$$

Hence $|\boldsymbol{m}^\varepsilon| \leq 1$ a.e. on $Q$. □

Now we will look for an estimate of the term $\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon$. Multiplying equation (11) by $\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon$ and integrating over $\Omega$ we obtain

$$\begin{gathered}\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x+\beta\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x\\+\lambda\int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x=0.\end{gathered}\tag{28}$$

Multiply this time equation (11) by $\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon$ and integrating over $\Omega$, we get

$$\begin{gathered}\gamma\int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x+\int_\Omega\boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x\\+\lambda\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x=0.\end{gathered}\tag{29}$$

Multiplying equation (29) by $\lambda$ and making the sum with (28), we get

$$\begin{gathered}\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x+(\beta+\gamma\lambda)\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x\\-\lambda^2\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x=0.\end{gathered}$$

Then

$$\begin{gathered}\lambda^2\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x=\int_\Omega|\ \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ |^2\ \ \mathrm{d}x\\+(\beta+\gamma\lambda)\int_\Omega\Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\cdot\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\ \mathrm{d}x.\end{gathered}\tag{30}$$

Multiplying (11) by $\partial_t \boldsymbol{m}^\varepsilon$, integrating over $\Omega$, replacing $\int_\Omega \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \, \mathrm{d}x$ by its value in (30) and using Lemma 2.4, we obtain

$$\begin{aligned}
&\lambda^2 \int_\Omega |\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x = \int_\Omega |\boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x + \frac{\gamma(\beta+\gamma\lambda)}{\lambda} \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x \\
&+ \frac{\beta(\beta+\gamma\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x + \frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (|\boldsymbol{m}^\varepsilon|^2 - 1)^2 \, \mathrm{d}x \\
&\leq \int_\Omega |\boldsymbol{m}^\varepsilon|^2 |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x + \frac{\gamma(\beta+\gamma\lambda)}{\lambda} \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x + \frac{\beta(\beta+\gamma\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x \\
&+ \frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (|\boldsymbol{m}^\varepsilon|^2 - 1)^2 \, \mathrm{d}x \\
&\leq (1 + \frac{\gamma(\beta+\gamma\lambda)}{\lambda}) \int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x + \frac{\beta(\beta+\gamma\lambda)}{2\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\Lambda^\alpha \boldsymbol{m}^\varepsilon|^2 \, \mathrm{d}x \\
&+ \frac{(\beta+\gamma\lambda)}{4\varepsilon\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (|\boldsymbol{m}^\varepsilon|^2 - 1)^2 \, \mathrm{d}x.
\end{aligned}$$

We integrate from 0 to $t$, and using the previous lemma, we get

$$\lambda^2 \int_0^t \int_\Omega |\ \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon\ |^2 \ \mathrm{d}x\mathrm{d}t \leq C, \tag{31}$$

where $C$ is a constant independent of $\varepsilon$. Hence

$$(\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ \mathbb{L}^2(Q). \tag{32}$$

Consequently,

$$\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \rightharpoonup \Phi \ \text{ weakly in } \ \mathbb{L}^2(Q). \tag{33}$$

By (27), we have

$$\begin{gathered}
(\partial_t \boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ \mathbb{L}^2(Q), \\
(|\ \boldsymbol{m}^\varepsilon\ |^2 - 1)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;\mathbb{L}^2(\Omega)), \\
(\boldsymbol{m}^\varepsilon)_\varepsilon \ \text{ is bounded in } \ L^\infty(0,T;\mathbb{H}^\alpha(\Omega)).
\end{gathered}$$

Then we have the following convergences to a subsequence further notes that $(\boldsymbol{m}^\varepsilon)_\varepsilon$ for $(1 < p < \infty)$:

$$\begin{gathered}
\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \ \text{ weakly in } \ L^p(0,T;\mathbb{H}^\alpha(\Omega)), \\
\partial_t \boldsymbol{m}^\varepsilon \rightharpoonup \partial_t \boldsymbol{m} \ \text{ weakly in } \ \mathbb{L}^2(Q), \\
|\ \boldsymbol{m}^\varepsilon\ |^2 - 1 \to 0 \ \text{ strongly in } \ L^2(0,T;\mathbb{L}^2(\Omega)) \ \textit{ and } \ \ |\ \boldsymbol{m}\ | = 1 \text{ a.e.}
\end{gathered}$$

By compactness embedding of $\mathbb{H}^\alpha(Q)$ into $\mathbb{L}^4(Q)$, we have

$$\boldsymbol{m}^\varepsilon \to \boldsymbol{m} \ \text{ strongly in } \ \mathbb{L}^4(Q). \tag{34}$$

In the following, we show that

$$\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} = \Phi \in \mathbb{L}^2(Q). \tag{35}$$

Let $\boldsymbol{\varphi} \in \mathbb{H}^\alpha(\Omega)$. We have

$$\int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = -\int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\varphi}) \, \mathrm{d}x\mathrm{d}t.$$

On the other hand, using commutator estimate together with the same reasonings that lead to (25), we have

$$\begin{aligned}\int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t &\to \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t \\ &= -\int_Q (\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m}) \cdot \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t,\end{aligned}$$

and therefore (35) is proved. In particular, we have

$$\boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \rightharpoonup \boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m} \text{ weakly in } \mathbb{L}^2(Q).$$

Now back to (26) and taking $\boldsymbol{\varphi} = \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}$ with $\boldsymbol{\phi} \in \mathbf{C}^\infty(\overline{Q})$, we have

$$\begin{aligned}\gamma \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ +\beta \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha(\boldsymbol{m}^\varepsilon \times \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t + \lambda \int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha}\boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = 0.\end{aligned} \tag{36}$$

For the first term of (36), we set $\Theta_\varepsilon = \displaystyle\int_Q \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t$.

We have

$$\Theta_\varepsilon = \int_Q \mid \boldsymbol{m}^\varepsilon \mid^2 \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t - \int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t.$$

On the one hand

$$\begin{aligned}\int_Q |\boldsymbol{m}^\varepsilon|^2 \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t = \int_Q (|\boldsymbol{m}^\varepsilon|^2 - 1)\partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t + \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ \to \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t &= \frac{1}{2}\int_Q \partial_t(\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\left[\int_\Omega (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\right]_0^T \\ &\quad -\frac{1}{2}\int_Q (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\partial_t(\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now choose $\boldsymbol{\phi}$ so that $\boldsymbol{\phi} = 0$ in $t = 0$ and $t = T$. Then

$$\left[\int_\Omega (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\right]_0^T = 0.$$

Therefore,

$$\begin{aligned}\int_Q (\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{m}^\varepsilon\ \mathrm{d}x\mathrm{d}t &= -\frac{1}{2}\int_Q (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\partial_t(\boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi})\ \mathrm{d}x\mathrm{d}t \\ &= -\frac{1}{2}\int_Q (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \\ &\quad -\frac{1}{2}\int_Q (\mid \boldsymbol{m}^\varepsilon \mid^2 -1)\boldsymbol{m}^\varepsilon \cdot \partial_t \boldsymbol{\phi}\ \mathrm{d}x\mathrm{d}t \to 0.\end{aligned}$$

Hence

$$\Theta_\varepsilon \to \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

For the second term of (36)

$$\beta \int_Q \Lambda^\alpha \boldsymbol{m}^\varepsilon \cdot \Lambda^\alpha (\boldsymbol{m}^\varepsilon \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t \to \beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha (\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t.$$

For the third term of (36)

$$\lambda \int_Q \boldsymbol{m}^\varepsilon \times \Lambda^{2\alpha} \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \phi \, \mathrm{d}x\mathrm{d}t \to \lambda \int_Q \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

For the last term of (36)

$$\gamma \int_Q \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \to \gamma \int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t.$$

Let $\varepsilon$ tends to 0 in (36), we obtain

$$\begin{aligned} &\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t \\ +\beta &\int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha (\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t + \lambda \int_Q \boldsymbol{m} \times \Lambda^{2\alpha} \boldsymbol{m} \cdot \boldsymbol{m} \times \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t = 0 \end{aligned}$$

for all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$. Furthermore, the inequality (10) follows from (27) and we finish the proof of Theorem 2.1.

## 3 The Limit as $b \to 0$

The main purpose of this section is to reveal to relationships between the fractional LLG equation we have studied in this paper, and the classical fractional LLG equation (i.e., in the case $b = 0$). We will prove the following result.

**Proposition 3.1** *Let $b \to 0$. The weak solution $\boldsymbol{m}^b$ obtained in section 2 weakly converges, up to a subsequence, to a solution of the classical fractional LLG equation in the following sense.*
*For all $\boldsymbol{\phi} \in \mathcal{C}^\infty(\overline{Q})$ with $\boldsymbol{\phi}(0,.) = \boldsymbol{\phi}(T,.) = 0$,*

$$\int_Q \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t - \gamma \int_Q \boldsymbol{m} \times \partial_t \boldsymbol{m} \cdot \boldsymbol{\phi} \, \mathrm{d}x\mathrm{d}t = -\beta \int_Q \Lambda^\alpha \boldsymbol{m} \cdot \Lambda^\alpha (\boldsymbol{m} \times \boldsymbol{\phi}) \, \mathrm{d}x\mathrm{d}t.$$

***Proof.*** Using the fact that $|\boldsymbol{m}^b| = 1$ a.e in $Q$ and estimate (10), we deduce that

$$(\boldsymbol{m}^b)_b \text{ is bounded in } L^\infty(0, T, \mathbb{H}^\alpha(\Omega)),$$

and

$$(\partial_t \boldsymbol{m}^b)_b \text{ is bounded in } \mathbb{L}^2(Q).$$

Hence, up to a subsequence, we have

$$\boldsymbol{m}^b \rightharpoonup \boldsymbol{m} \text{ weakly in } L^p(0,T,\mathbb{H}^\alpha(\Omega)) \text{ for } 1<p<\infty,$$

$$\boldsymbol{m}^b \to \boldsymbol{m} \text{ strongly in } C([0,T],\mathbb{H}^\delta(\Omega)) \text{ and a.e for } 0\le\delta<\alpha,$$

$$\partial_t\boldsymbol{m}^b \rightharpoonup \partial_t\boldsymbol{m} \text{ weakly in } \mathbb{L}^2(Q).$$

Then $|\boldsymbol{m}|=1$ a.e in $Q$. On the other hand, we have

$$\gamma\partial_t\boldsymbol{m}^b+\boldsymbol{m}^b\times\partial_t\boldsymbol{m}^b+\beta\Lambda^{2\alpha}\boldsymbol{m}^b+\lambda\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b-\beta(\Lambda^{2\alpha}\boldsymbol{m}^b\cdot\boldsymbol{m}^b)\boldsymbol{m}^b=0 \text{ a.e. in } Q.$$

Multiplying this equation by $\partial_t\boldsymbol{m}^b$ and $\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b$ respectively and integrating over $\Omega$, we get

$$\gamma\int_\Omega |\partial_t\boldsymbol{m}^b|^2 \,\mathrm{d}x+\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^b|^2 \,\mathrm{d}x+\lambda\int_\Omega \boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b\cdot\partial_t\boldsymbol{m}^b\,\mathrm{d}x=0 \tag{37}$$

and

$$\lambda\int_\Omega|\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b|^2\,\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^b|^2\,\mathrm{d}x=-\gamma\int_\Omega\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b\cdot\partial_t\boldsymbol{m}^b\,\mathrm{d}x. \tag{38}$$

The equalities (37), (38) allow to get

$$\lambda^2\int_\Omega|\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b|^2\,\mathrm{d}x=\gamma^2\int_\Omega|\partial_t\boldsymbol{m}^b|^2\,\mathrm{d}x+(\frac{\gamma\beta-\lambda}{2})\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\Lambda^\alpha\boldsymbol{m}^b|^2\,\mathrm{d}x.$$

We integrate from 0 to $t$ to get

$$\begin{aligned}\lambda^2\int_0^t\int_\Omega|\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b|^2\,\mathrm{d}x\mathrm{d}t+(\frac{\gamma\beta-\lambda}{2})\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\,\mathrm{d}x\\ =\gamma^2\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^b|^2\,\mathrm{d}x\mathrm{d}t+(\frac{\gamma\beta-\lambda}{2})\int_\Omega|\Lambda^\alpha\boldsymbol{m}^b|^2\,\mathrm{d}x\end{aligned} \tag{39}$$

for all $t\in(0,T)$.

Recall that

$$\beta=a(1+\gamma^2) \text{ and } \lambda=b(1+\gamma^2).$$

Since $b$ is small enough, we assume that $b<a\gamma$ *i.e.,* $\lambda<\gamma\beta$. Using estimate (10), we have

$$\int_\Omega|\Lambda^\alpha\boldsymbol{m}^b|^2\,\mathrm{d}x\le\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\,\mathrm{d}x$$

and

$$\gamma^2\int_0^t\int_\Omega|\partial_t\boldsymbol{m}^b|^2\,\mathrm{d}x\mathrm{d}t\le\frac{\gamma\beta(1+\gamma^2)}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\,\mathrm{d}x.$$

Then, (39) implies that

$$b^2\int_0^t\int_\Omega|\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b|^2\,\mathrm{d}x\mathrm{d}t\le\frac{\gamma a}{2}\int_\Omega|\Lambda^\alpha\boldsymbol{m}_0|^2\,\mathrm{d}x.$$

Hence

$$(b\,\boldsymbol{m}^b\times\Lambda^{2\alpha}\boldsymbol{m}^b)_b \text{ is bounded in } \mathbb{L}^2(Q).$$

Therefore,

$$b\,\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b \rightharpoonup \xi \text{ weakly in } \mathbb{L}^2(Q).$$

Let $\boldsymbol{\psi} \in \mathbb{H}^\alpha(Q)$. We have

$$\int_Q b\,\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b \cdot \boldsymbol{\psi}\,\mathrm{d}x\mathrm{d}t = -b\int_Q \Lambda^\alpha \boldsymbol{m}^b \cdot \Lambda^\alpha(\boldsymbol{m}^b \times \boldsymbol{\phi})\,\mathrm{d}x\mathrm{d}t,$$

which tends to zero as $b$ goes to zero. We conclude that $\xi = 0$.

Now, we can pass to the limit as $b \to 0$ in the weak formulation

$$\int_Q \partial_t\boldsymbol{m}^b \cdot \boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t - \gamma\int_Q \boldsymbol{m}^b \times \partial_t\boldsymbol{m}^b \cdot \boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t$$
$$= -\beta\int_Q \Lambda^\alpha\boldsymbol{m}^b \cdot \Lambda^\alpha(\boldsymbol{m}^b \times \boldsymbol{\phi})\,\mathrm{d}x\mathrm{d}t - (1+\alpha^2)\int_Q b\,\boldsymbol{m}^b \times \Lambda^{2\alpha}\boldsymbol{m}^b \cdot \boldsymbol{m}^b \times \boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t.$$

We obtain

$$\int_Q \partial_t\boldsymbol{m} \cdot \boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t - \alpha\int_Q \boldsymbol{m} \times \partial_t\boldsymbol{m} \cdot \boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}t = -\beta\int_Q \Lambda^\alpha\boldsymbol{m} \cdot \Lambda^\alpha(\boldsymbol{m} \times \boldsymbol{\phi})\,\mathrm{d}x\mathrm{d}t.$$

Then Proposition 3.1 is proved. □

## 4 Concluding Remarks

In this paper, global existence of weak solutions to a modified fractional LLG equation is proved. The modification lies in the presence in the effective field of the term $b\,\boldsymbol{m} \times \Lambda^{2\alpha}\boldsymbol{m}$ describing fractional vertical spin stiffness. Due to nonlocal nonlinearities in the model, special structures of the equation, the commutator estimate and some calculus inequalities of fractional order are exploited to get the convergence of the approximating solutions. The relationship between the model and the classical fractional LLG equation is also revealed by discussing the limit of the obtained solutions when the vertical spin stiffness parameter $b$ tends to zero.

Let us mention that important progress has been made in the design of schemes constructing weak solutions to classical LLG equation. Several schemes were proposed, and their convergence to weak solutions was proved (see for examples [2, 4]). An interesting direction of future research is to propose numerical scheme for the fractional LLG equation. This will be helpful to give a strategy for efficient computer implementation which may reflect the true nature of the augmentation of the LLG model considered in this paper.

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# Exponential Domination and Bondage Numbers in Some Graceful Cyclic Structure

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**Abstract:** The domination number is an important vulnerability parameter that it has become one of the most widely studied topics in graph theory, and also the bondage number which is related by domination number the most often studied property of vulnerability of communication networks. Recently, Dankelmann et al. defined the exponential domination number denoted by $\gamma_e(G)$ in [17]. In 2016, the exponential bondage number, denoted by $b_{exp}(G)$, is defined by $b_{exp}(G) = min\{|B_e| : B_e \subseteq E(G), \gamma_e(G - B_e) > \gamma_e(G)\}$, where $\gamma_e(G)$ is the exponential domination number of $G$ [24]. In this paper, the above mentioned parameters is has been examined. Then exact formulas are obtained for the families of cyclic structures tend to have graceful subfamilies such as helm graph, windmill graph, circular necklace and friendship graph.

**Keywords:** *graph vulnerability; connectivity; domination number; bondage number; exponential domination number; exponential bondage number.*

**Mathematics Subject Classification (2010):** 05C40, 05C69, 68M10, 68R10.

## 1 Introduction

Graph theory plays vital role in various fields. One of the important areas in graph theory is graph labeling. Interest in graph labeling began in mid-1960s with a conjecture by Kotzig-Ringel and a paper by Rosa [5]. In 1967, Rosa published a pioneering paper on graph labeling problems. Graph labeling is powerful tool that makes things ease in various fields of networking. Graph labeling is very important major areas of computer science like data mining image processing, cryptography, software testing, information

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security, communication network etc. Also, there are many applications of graph labelling in the literature such as coding theory, radar, astronomy, circuit design, missile guidance, communication network addressing, xray crystallography, data base management [5,13].

We begin by recalling some standard definitions that we need throughout this paper. Let $G = (V, E)$ be a simple undirected graph of order $n$. For any vertex $v \in V$, the *open neighborhood* of $v$ is $N_G(v) = \{u \in V | uv \in E\}$ and *closed neighborhood* of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. The degree of $v$ in $G$ denoted by $deg(v)$, is the size of its open neighborhood. A vertex $v$ is said to be pendant vertex if $deg(v) = 1$ [7,18]. A vertex $u$ is called support vertex if $u$ is adjacent to a pendant vertex. The graph $G$ is called $r$-regular graph if $deg(v) = r$ for every vertex $v \in V$. The *distance* $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the length of a shortest path between them [7,18].

Given a graph $G = (V, E)$, the set $N$ of non-negative integers and a commutative binary operation $* : N \times N \to N$, every vertex $f : V \to N$ induces an edge function $f* : E \to N$such that $f * (uv) = |f(u) - f(v)|$, for all $uv \in E$. A function $f$ is called graceful labeling of a graph $G$ if $f : V \to 0, 1, 2, ..., q$ is injective and the induced function $f* : E \to 1, 2, ..., q$ is bijective. A graph which admits graceful labeling is called graceful graph.

A set $S \subseteq V$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in $S$. The minimum cardinality taken over all dominating sets of $G$ is called the *domination number* of $G$ is denoted by $\gamma(G)$ [7,18]. There are different application of domination problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research [18] or domination number is the one of the most important vulnerability parameter for networks [18,23]. When investigating the domination number of a given graph $G$, one may want to learn the answer of the following question: How does the domination number increases in a graph $G$? or How many edges need to be added to decrease the domination number of the original graph? One of the vulnerability parameters known as *bondage number* in a graph $G$ answers the former question. The bondage number $b(G)$ was introduced by Fink et al. [12] and is defined as follows:

$$b(G) = \min\{|B| : B \subseteq E, \gamma(G - B) > \gamma(G)\}.$$

We call such an edge set $B$ that $\gamma(G - B) > \gamma(G)$ the *bondage set* and the minimum one the *minimum bondage set.* If $b(G)$ does not exist, for example empty graphs, then $b(G) = \infty$ is defined.

In 2009, Dankelmann introduced the concept of *exponential domination* [17]. This new parameter is closely in relation with distance of each pair of vertices. The exponential domination number is the theoretical vulnerability parameters for a network that is represented by a graph [1, 17]. An exponential dominating set of graph $G$ is a kind of distance domination subset $S \subseteq V(G)$ such that $\sum_{v \in S} (1/2)^{\overline{d}(u,v)-1} \geq 1, \forall v \in V$, where $\overline{d}(u, v)$ is the length of a shortest path in $\langle V - (S - \{u\}) \rangle$ if such a path exist, and $\infty$ otherwise. The minimum exponential domination number, $\gamma_e(G)$ is the smallest cardinality of an exponential dominating set. We call such an edge set is a minimum exponential set which is denoted by $\gamma_e$-set.

Aytac et al. has defined exponential bondage number [24]. It is defined as follows:

$$b_{exp}(G) = \min\{|B_e| : B_e \subseteq E, \gamma_e(G - B_e) > \gamma_e(G)\},$$

where $\gamma_e(G)$ is the exponential domination number of the graph $G$. We call such an edge set $B_e$ that $\gamma_e(G - B_e) > \gamma_e(G)$ the *exponential bondage set* and the minimum one the *minimum exponential bondage set*.

There are many advantages of creating a communications network that is analogous a graceful graph. One advantage is that if a link goes out, a simple algorithm could detect which two centers are no longer linked, since each connection is labeled with the difference between the two communication centers. Another advantage is that this network also would have all the same properties as a graceful graph; such as having cyclic decompositions [5,13]. Many structures that have been studied in recent years are structures that involve cycles. One reason for this is that Rosa proved that all cycles that are of lengths $n \equiv 0, 3(mod4)$ are graceful. Hence, many families of cyclic structures tend to have graceful subfamilies. We will now investigate some of these structures such as: helm graph, windmill graph, circular necklace and friendship graph.

Calculation of exponential domination and bondage numbers for simple cyclic graph types is important because if one can break a more complex network into smaller networks, then under some conditions the solutions for the optimization problem on the smaller networks can be combined to a solution for the optimization problem on the larger network.

In Section 2, some well-known basic results are given for exponential domination and bondage numbers. In Section 3, examples of the exponential dominating and the exponential bondage sets of a graph are are given. In Section 4, the exponential domination numbers have been computed for helm graph, windmill graph, circular necklace and friendship graph. In Section 5, the exponential bondage numbers have been calculated for same structures.

## 2 Basic Results

In this section some well-known basic results are given with regard to exponential domination number and bondage number.

**Theorem 2.1** *[17] The exponential domination number of*

*a) the path graph $P_n$ of order $n \geq 2$ is $\gamma_e(P_n) = \lceil \frac{n+1}{4} \rceil$.*

*b) the cycle graph $C_n$ of order $n \geq 4$ is* $\gamma_e(C_n) = \left\{ \begin{array}{ll} 2 & , \text{if } n = 4; \\ \lceil \frac{n}{4} \rceil & , \text{if } n \neq 4. \end{array} \right\}$

**Theorem 2.2** *[17] For every graph $G$, $\gamma_e(G) \leq \gamma(G)$, and also $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$.*

**Theorem 2.3** *Let $G$ be any connected graph with $n$ vertices and $\exists v \in V(G)$ such that $deg(v) = n - 1$. Then $\gamma_e(G) = 1$.*

**Theorem 2.4** *[12] If $G$ is a connected graph of order $n \geq 2$, then $b(G) \leq n - \gamma(G) + 1$.*

**Theorem 2.5** *[12] The bondage number of*

*a) the path graph $P_n$ of order $n \geq 2$ is* $b(P_n) = \left\{ \begin{array}{ll} 2, & \text{if } n \equiv 1(mod\ 3); \\ 1, & \text{otherwise.} \end{array} \right.$

*b) the cycle graph $C_n$ of order $n \geq 3$ is* $b(C_n) = \left\{ \begin{array}{ll} 3, & \text{if } n \equiv 1(mod\ 3); \\ 2, & \text{otherwise.} \end{array} \right.$

*c) the complete graph $K_n$ of order $n \geq 2$ is $b(K_n) = \lceil \frac{n}{2} \rceil$.*

*d) the star graph $S_n$ of order $n \geq 3$ is $b(S_n) = 1$.*

**Theorem 2.6** *[22] If $G$ is a nonempty graph with a unique minimum dominating set, then $b(G) = 1$.*

**Theorem 2.7** *[24] Let $G$ be a connected graph of order $n$. If $G$ includes only one pendant vertex, then $b_{exp}(G) = 1$.*

## 3 Example

a) Let's find the exponential dominating sets of the given graph in Figure 1.

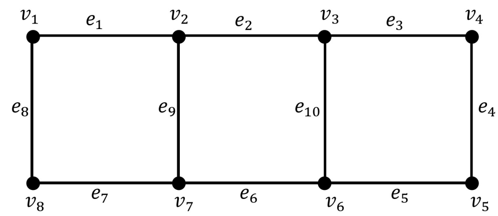


**Figure 1**: Graph $G$.

- For the set $S_1 = \{v_1,\ v_3,\ v_7,\ v_5\} \subseteq V(G)$, Table 1 is obtained.

**Table 1**: The weight values of $S_1$ at $v$.

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $w_{S_1}(v)$ | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 |

  From Table 1, it is easy to see that $w_{S_1}(v) \geq 1$. Hence, the set $S_1 \subseteq V(G)$ is an exponential dominating set of the graph G.

- For the set $S_2 = \{v_2, v_6, v_8\} \subseteq V(G)$, Table 2 is obtained.

**Table 2**: The weight values of $S_2$ at $v$.

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $w_{S_2}(v)$ | 2 | 2 | 2 | 1 | 5/4 | 2 | 3 | 2 |

  From Table 2, it is easy to see that $w_{S_2}(v) \geq 1$. Hence, the set $S_2 \subseteq V(G)$ is an exponential dominating set of the graph G.

- For the set $S_3 = \{v_1, v_5\} \subseteq V(G)$, Table 3 is obtained.

  From Table 3, it is easy to see that $w_{S_3}(v) \geq 1$. Hence, the set $S_3 \subseteq V(G)$ is an exponential dominating set of the graph G.

**Table 3**: The weight values of $S_3$ at $v$.

| $v$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $w_{S_3}(v)$ | 2 | 5/4 | 1 | 5/4 | 2 | 5/4 | 1 | 5/4 |

Among some of the exponential dominating sets discussed above, the set having minimum element is the set $S_3$. There is not a set that is exponential dominating and $|S| < |S_3|$ of the graph G. Namely $\exists S \subseteq V(G)$ can not be found. In this case, exponential domination number of the graph G is $\gamma_e(G) = |S_3| = 2$.

b) Let's find the exponential bondage sets of the given graph in Figure 1.

- Let's consider the set $B_e^1 = \{e_1\} \subseteq E(G)$. In this case, we examine exponential domination number of the $E(G) - B_e^1$ graph. Here, it is easy to see that $S = \{v_1, v_8\} \subseteq E(G) - B_e^1$ is a member of any minimum exponential dominating set. $B_e^1$ is not an exponential bondage set because $\gamma_e\left(E(G) - B_e^1\right) = \gamma_e(G) = 2$.
- Let's consider the set $B_e^2 = \{e_3, e_6\} \subseteq E(G)$. In this way, we examine exponential domination number of the $E(G) - B_e^2$ graph. Here, it can be easily seen that the set $S = \{v_1, v_3, v_5\} \subseteq E(G) - B_e^2$ is a minimum exponential dominating set. $B_e^2$ is an exponential bondage set because $\gamma_e\left(E(G) - B_e^2\right) = 3 > \gamma_e(G) = 2$.
- Let's consider the set $B_e^3 = \{e_2, e_6\} \subseteq E(G)$. The $E(G) - B_e^3$ graph consists of two components. In this case, we examine exponential domination number of the $E(G) - B_e^3$ graph. Here, it can be easily seen that the set $S = \{v_1, v_3, v_5, v_7\} \subseteq E(G) - B_e^3$ is a member of any minimum exponential dominating set. $B_e^3$ is an exponential bondage set because $\gamma_e\left(E(G) - B_e^3\right) = 4 > \gamma_e(G) = 2$.
- Let's consider the set $B_e^4 = \{e_3, e_5, e_{10}\} \subseteq E(G)$. The $E(G) - B_e^4$ graph consists of two components. In this case, we examine exponential domination number of the $E(G) - B_e^4$ graph. Here, it can be easily seen that the set $S = \{v_1, v_7, v_4\} \subseteq E(G) - B_e^4$ is a member of any minimum exponential dominating set. $B_e^4$ is an exponential bondage set because $\gamma_e\left(E(G) - B_e^4\right) = 3 > \gamma_e(G) = 2$.

Among some of the exponential bondage sets discussed above, the set having minimum element is the set $B_e^2$. There is not a set that is exponential bondage and $|B_e| < \left|B_e^2\right|$ of the graph G. Namely $\exists B_e \subseteq E(G)$ can not be found. In this case, exponential bondage number of the graph G is $b_{exp}(G) = \left|B_e^2\right| = 2$.

## 4 The Exponential Domination Number of Some Graceful Cyclic Structure

In this section, we give definition of well-known graceful cyclic structure. Then we calculate the exponential domination number of them.

**Definition 4.1** [15] A helm graph is denoted by $H_n$ is a graph obtained by attaching a single edge and vertex of the outer circuit of a wheel graph $W_n$. The number of vertices of $H_n$ is $2n+1$ and the number of edges is $3n$. We display the graph $H_4$ in Figure 2.

**Theorem 4.1** *If $H_n$ is a helm graph, then $\gamma_e(H_n) = 4$.*

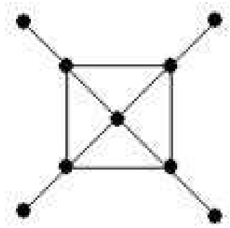

**Figure 2**: The Helm Graph $H_4$.

***Proof.*** The Helm $H_n$ consist of the vertex set $V(H_n) = \{v_i | 0 \leq i \leq n-1\} \cup \{a_i | 0 \leq i \leq n-1\} \cup \{c\}$. Let $c$ be the central vertex of $H_n$. The degree of central vertex is $n$. The vertices of $H_n \setminus \{c\}$ are two kinds: vertices of degree four and one, respectively. Clearly, $deg(v_i) = 4$ and $deg(a_i) = 1$.

Let $S$ be $\gamma_e$-set of $H_n$. If $S$ consists of only one central vertex c, then this vertex is exponentially dominated all vertices except that the pendant vertices $a_i$. Therefore, the vertices $v_i$ must be added to $S$.

If $c \in S$ and $v_i$ is not adjacent $a_i$, then $\overline{d}(v_i, a_i) \geq 2$. If $c \notin S$ and $v_i$ is not adjacent $a_i$, then $\overline{d}(v_i, a_i) = 2$ or $\overline{d}(v_i, a_i) = 3$.

Due to distance between $a_i$ and $v_i$ and because $S$ is $\gamma_e$-set, $S$ must not contain the central vertex $c$. In this case, the set $S$ must consist only of the vertices $v_i$. The geodesic(shortest) distances from the vertices $v_i$ to the other vertices of $H_n$ are as follows: $\overline{d}(v_i, a_i) \leq 3$, $\overline{d}(v_i, v_i) \leq 3$ and $\overline{d}(v_i, c) = 1$.

Accordingly, any vertex $x \in V(H_n)$ is at most 3 distance away from the vertex $v_i \in S$.

Initially, let's assume that $S$ is only one vertex $v_i$. Let $x$ be the vertex in $V(H_n) \setminus S$ such that $\overline{d}(v_i, x) = 3$. To dominate the exponentially the vertex $x$ by set $S$, the number of vertices that must be in $S$ is

$$w_s(x) = \sum_{v_i \in S} \frac{1}{2^{\overline{d}(v_i, x)}} \geq 1,$$

$$\frac{m}{2^2} \geq 1 \Rightarrow m \geq 4,$$

where $m = |S|$.

Thus, there must be at least 4 for vertices $v_i$ in the set $S$. Consequently, the exponential domination of $H_n$ is $\gamma_e(H_n) = 4$. The proof is completed. □

**Definition 4.2** [11] The windmill graph $Wd(k, n)$ can be constructed by joining $n$ copies of the complete graph $K_k$ with a common vertex. It has $(k-1)n + 1$ vertices and $nk(k-1)/2$ edges. We display the graph $Wd(5, 4)$ in Figure 3.

**Theorem 4.2** *If $Wd(k, n)$ is a windmill graph, then $\gamma_e(Wd(k, n)) = 1$.*

***Proof.*** By the Theorem 2.3, the proof is clear. □

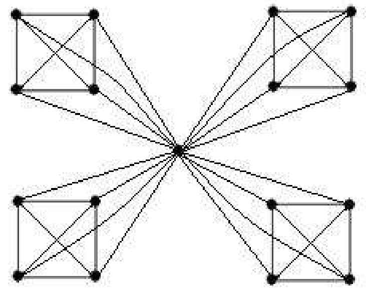

**Figure 3**: The Windmill graph $Wd(5,4)$.

**Definition 4.3** [11] Let $K_m$ and $K_{t_i}$ be complete graphs on $m$(say $v_1, v_2, ..., v_m$) and $t_i$ vertices, respectively. Let $t_i = 2^{r_i}$, $1 \leq i \leq m$, and $r_1 = r_2$, $r_{i+1} = r_i + 1$ for all $2 \leq i \leq m-1$ such that $K_m \uplus K_{t_i}$ has just $v_i$ as a cut vertex, where $r_i$ is an integer and $1 \leq i \leq m$. The resultant graph $K_m \uplus (\cup_{i=1}^{m} K_{t_i})$ is a circular necklace denoted by $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$. We display the graph $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$ in Figure 4.

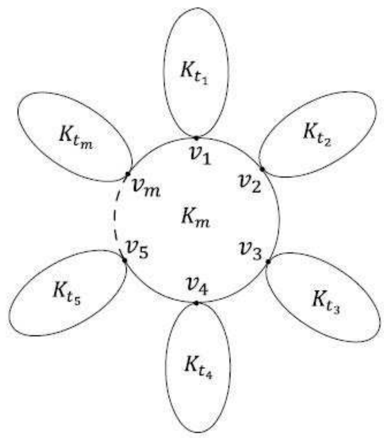


**Figure 4**: The Circular Necklace $CN(K_m; K_{t_1}, K_{t_2}, ..., K_{t_m})$.

**Theorem 4.3** *If $G$ is a circular necklace graph, then $\gamma_e(G) = 2$.*

***Proof.*** By the definition of circular necklace graph, both $K_m$ and $K_{t_i}$ are complete graphs. Any vertex exponentially dominates all the remaining vertices in complete graph. Let $v_1$, $v_2$,..., $v_m$ be vertices of $K_m$. Let $S$ be $\gamma_e$- set of the graph $G$. If $S$ consists of exactly one vertex $v_x$ of $K_m$, where $1 \leq x \leq m$. Then all vertices of $K_m$ and $K_{t_x}$ in $G$ are exponentially dominated. For the all remaining vertices $u \in V(G - V(K_m) - V(K_{t_x}))$, we get $\overline{d}(v_x, u) = 2$. Thus, the vertex $v_x$ contributes 1/2 to $w_s(u)$. To exponentially dominate all the remaining vertices $u$, only one vertex $v_i$ of $K_m$, also must be added to $S$. Hence, we get $\gamma_e(G) = 2$. The proof is completed. □

**Definition 4.4** [15] The friendship graph $F_n$ can be constructed by joining $n$ copies of the cycle graph $C_3$ with a common vertex. We display the graph $F_4$ in Figure 5.

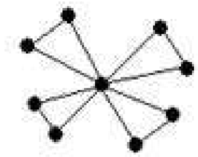

**Figure 5**: The Friendship graph $F_4$.

**Theorem 4.4** *If $F_n$ is a friendship graph, then $\gamma_e(F_n) = 1$.*

***Proof.*** By the Theorem 2.3, the proof is clear. □

## 5 The Exponential Bondage Number of Some Graceful Cyclic Structure

In this section, we calculate the exponential bondage number of well-known graceful cyclic structure.

**Theorem 5.1** *If $H_n$ is a helm graph, then $b_{exp}(H_n) = 1$.*

***Proof.*** The proof is easy to see by the Theorem 2.7. □

**Theorem 5.2** *If $Wd(k, n)$ is a windmill graph, then $b_{exp}(Wd(k, n)) = 1$.*

***Proof.*** Let $c$ be the central vertex of $Wd(k, n)$. Clearly, $deg(c) = n(k-1)$. The removal of an edge $e$ which is incident to $c$ leaves a graph $H$. The graph $H$ is connected graph with $(k-1)n+1$- vertices. It is easy to see that $|V(Wd(k, n))| = |V(H)|$ and $deg(c) = n(k-1) - 1$ in the graph $H$. Now, we determine the exponential domination number of $H$. Let $D$ be a $\gamma_e$- set of the graph $H$. If $D = \{c\}$, then $D$ exponentially dominates $(k-1)n$ vertices. Thus, there remains only one vertex $v$ exponentially dominated by $D$. The vertex $v$ is the end vertex of removed edge. The vertex $c$ contributes 1/2 to $w_D(v)$. Therefore, the vertex $v$ or any vertex at 1/2 distance to the vertex $v$ must be in $D$. Then we get $\gamma_e(H) = 2$.

Since $\gamma_e(H) > \gamma_e(Wd(k, n))$, the exponential bondage number of the windmill graph is $b_{exp}(Wd(k, n)) = 1$. The proof is completed. □

**Theorem 5.3** *If $G$ is a circular necklace graph, then*

$$b_{exp}(G) = \begin{cases} 2^{r_1} - 1, & \text{if } m > 2^{r_1}; \\ m - 1, & \text{otherwise.} \end{cases}$$

***Proof.*** By the definition of a circular necklace graph, $K_m$ and $K_{t_i}$ are complete graphs and $r_1 = r_2$, where $1 \leq i \leq m$. It is the graph $K_{t_1}$ or $K_{t_2}$ which has the least vertices on the graph $G$. Let $r_1 = r_2$ be an integer value of $r$. Thus, $|V(K_{t_1})| = |V(K_{t_2})| = 2^r$ and $|V(K_m)| = m$. Let $v_1$, $v_2$,..., $v_m$ and $v_i = u_{i1}, u_{i2}, ..., u_{i2^r}$ be vertices of graphs $K_m$ and $K_{t_i}$, where $1 \leq i \leq m$, respectively. For every $v \in V(K_m)$, we have $deg(v) = m - 1$ in the graph $K_m$. Similarly, for every $u_{1j} \in V(K_{t_1})$, we have $deg(u_{1j}) = 2^r - 1$ in the graph $K_{t_1}$, where $1 \leq j \leq 2^r$. There are two cases depending on the degrees of the vertices of $v$ and $u_{1j}$.

Case 1. $deg_{K_m}(v) > deg_{K_{t_i}}(u_{1j}) \Rightarrow m > 2^r$.
The removal of all edge incident to the vertex $u_{1j}$ in $G$ leaves a graph $H$ consisting of two components. One of these is an isolated vertex and the other is connected graph $CN(K_m; K_{t_1-1}, K_{t_2}, ..., K_m)$. Thus by the Theorem 4.3 we get

$$\gamma_e(H) = \gamma_e(CN(K_m; K_{t_1-1}, K_{t_2}, ..., K_m) + 1 = 2 + 1 > \gamma_e(G).$$

Since $\gamma_e(H) > \gamma_e(G)$ is obtained, we have $b_{exp}(G) = 2^r - 1$.

Case 2. $deg_{K_m}(v) < deg_{K_{t_i}}(u_{1j}) \Rightarrow m < 2^r$.
The removal of all edge incident to vertex $v$ in $G$ leaves a graph $H$ consisting of $K_{t_1}$ and $CN(K_{m-1}; K_{t_1}, K_{t_2}, ..., K_{t_m})$. Thus by the Theorem 4.3 and 2.3 we get

$$\gamma_e(H) = \gamma_e(CN(K_{m-1}; K_{t_1}, K_{t_2}, ..., K_{t_m}) + \gamma_e(K_{t_1}) = 2 + 1 > \gamma_e(G).$$

Since $\gamma_e(H) > \gamma_e(G)$ is obtained, we have $b_{exp}(G) = m - 1$.

By combining these two cases, the exponential domination number of the circular necklace graph is

$$b_{exp}(G) = \begin{cases} 2^{r_1} - 1, & \text{if } m > 2^{r_1}; \\ m - 1, & \text{otherwise.} \end{cases}$$

The proof is completed. □

**Theorem 5.4** *If $F_n$ is a friendship graph, then $b_{exp}(F_n) = 1$.*

***Proof.*** The vertices of $F_n$ are two kinds. Let $u$ and $v_i$ be vertices of $F_n$, where $i \in \{1, ..., 2n\}$. Since $deg(u) = 2n$ in $F_n$, the vertex $u$ is the central vertex of $F_n$. Furthermore, $deg(v_i) = 2$ for every $v_i \in V(F_n)$. If we remove the only one edge $e_{uv_i}$ incident with the vertex $u$, then remaining graph is $H$.

Now we determine the exponential domination number of $H$. In the graph $H$, $deg_H(u) = 2n - 1$. Let $D$ be a $\gamma_e$- set of the graph $H$. If $D = \{u\}$, then the set $D$ exponentially dominates $(2n - 1)$- vertices. Thus, the remains only one vertex exponentially dominated by $D$. The vertex $v_i$ is the end vertex of removed edge $e_{uv_i}$. The vertex $u$ contributes $1/2$ to $w_D(v_i)$. Therefore, the vertex $v_i$ or the vertex in $N(v_i) - \{u\}$ must be in $D$. Then we get $\gamma_e(H) = 2$.

Since $\gamma_e(H) > \gamma_e(F_n)$ is obtained, the exponential bondage number of the friendship graph is $b_{exp}(F_n) = 1$. The proof is completed. □

## 6 Conclusion

In this paper we determine the exact values of exponential domination and bondage numbers of a wheel helm graph, windmill graph, circular necklace and friendship graph. The problem of finding the exponential domination and bondage numbers of architecture such as Pyramid networks, Circulant networks are under investigation.

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# On the Hyers-Ulam Stability of Certain Partial Differential Equations of Second Order

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**Abstract:** In this paper, we obtain two new results on the Hyers-Ulam stability of the linear partial differential equation of second order with constant coefficients

$$Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y = 0$$

and the partial Euler differential equation of the form

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz = 0.$$

Our findings make a contribution to the topic and complete those in the relevant literature.

**Keywords:** *partial differential equation; Hyers-Ulam stability; second order.*

**Mathematics Subject Classification (2010):** 39B82, 35B35, 35F05.

## 1 Introduction

The stability theory is an important research area in the qualitative analysis of differential equations and partial differential equations. It follows from the relevant literature that the investigation of the Hyers-Ulam and Hyers-Ulam-Rassias stability of equations with partial derivatives started recently. We should mention the earliest results on the topic or some results obtained for the linear partial differential equations of first or second order by Alsina and Ger [1], Cimpean and Popa [2], Gordji et al. [3], Hyers [4], Jung ([5], [6], [7], [8]), Li and Huang [9], Liu and Zhao [10], Lungu and Popa ([11], [12]), Rassias

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[13], Tunç and Biçer [14], Ulam [15] and the references therein. We shall now give the details of some works done on the topic. In 2009, Jung [8] investigated the Hyers-Ulam stability of linear partial differential equations of first order

$$au_x(x,y) + bu_y(x,y) + g(y)u(x,y) + h(y) = 0$$

and

$$au_x(x,y) + bu_y(x,y) + g(x)u(x,y) + h(x) = 0,$$

in the cases of $a \leq 0, b > 0$ and $a > 0$ ,$b \leq 0$, $(a, b \in \Re)$, respectively.

Later, in 2011, Gordji et al. [3] proved the Hyers-Ulam-Rassias stability of the following nonlinear partial differential equations

$$\begin{aligned}
&\gamma_x(x,t) = f(x,t,\gamma(x,t)),\\
&a\gamma_x(x,t) + b\gamma_t(x,t) = f(x,t,\gamma(x,t)),\\
&p(x,t)\gamma_{xx}(x,t) + q(x,t)\gamma_x(x,t) = f(x,t,\gamma(x,t))
\end{aligned}$$

and

$$p(x,t)\gamma_{xt}(x,t) + q(x,t)\gamma_t(x,t) + p_t(x,t)\gamma_x(x,t) - p_x(x,t)\gamma_t(x,t) = f(x,t,\gamma(x,t)),$$

respectively, by using Banach's contraction mapping principle.

After that, in 2012, Lungu and Popa [11] discussed the Hyers-Ulam stability of first order partial differential equation of the form

$$p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} = p(x,y)r(x)u + f(x,y).$$

Finally, in 2014, Li and Huang [9] proved the Hyers-Ulam stability of the first order linear partial differential equations in n-dimensional space of the form

$$\sum_{i=1}^{n} a_i x_{x_i}(x_1, x_2, ..., x_n) + g(x_j)u(x_1, x_2, ..., x_n) + h(x_j) = 0,$$

where $a_i \in \Re$ are arbitrarily given constants.

In this paper, we investigate the Hyers-Ulam stability of the partial differential equation of second order with constant coefficients

$$Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y = 0 \tag{1}$$

and the partial Euler differential equation

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz = 0, \tag{2}$$

where $z = z(x,y)$ $(x,y) \in D$, $D = [a,b) \times \Re$, $D$ is a subset of $\Re^2$ and $A, B, m$ are real constants with $m > 0$ and $A > 0$. Let $\varepsilon > 0$ be a given number. Equation (1) is said to be stable in Hyers-Ulam sense if there exists $K > 0$ such that for every function $z : [a,b] \times \Re \to C$ satisfying

$$|Az_{xx} + (A+B)z_{xy} + Bz_{yy} + Az_x + Bz_y| < \varepsilon$$

for all $(x, y) \in D$ there exists a solution $z_0 : [a, b] \times \Re \to C$ of Eq. (1) with the property

$$|z(x, y) - z_0(x, y)| \leq K\varepsilon.$$

This work has been inspired basically by the papers of Gordji et al. [3], Jung [8], Li and Huang [9], Lungu and Popa [11], Vlasov [16], Vasundhara Devi [1] and those listed above. The results obtained here are different from those in the literature, new and original, and they have simple forms. They can be easily checked and applicable, and complete the previous ones in the literature. Hence the novelty and originality of the present paper.

## 2 Hyers-Ulam Stability

In this section, we give two theorems and two examples to show the Hyers-Ulam stability of equation (1) and equation (2).Our first Hyers-Ulam stability result is the following theorem.

**Theorem 1.** *Let $\varepsilon$ be a positive constant. If the function $z$ satisfies the differential inequality*

$$|Az_{xx} + (A + B)z_{xy} + Bz_{yy} + Az_x + Bz_y| < \varepsilon \tag{3}$$

*for all $(x, y) \in D$, then there exists a solution $z_0 : D \to \Re$ of equation (1) such that*

$$|z(x, y) - z_0(x, y)| \leq K\varepsilon, K > 0, K \in \Re.$$

**Proof.** Let $u(x, y) = Az_x + Bz_y$ for any $(x, y) \in D$. Then, it follows that

$$|u_x + u_y + u| = |Az_{xx} + (A + B)z_{xy} + Bz_{yy} + Az_x + Bz_y|$$

so that

$$|u_x + u_y + u| \leq \varepsilon.$$

Consider the change of coordinates

$$\zeta = x,$$
$$\eta = y - x.$$

Then, we have

$$|u_x + u_y + u| = |u_\zeta + u| < \varepsilon. \tag{4}$$

It is clear from (4) that

$$-\varepsilon \leq u_\zeta + u \leq \varepsilon.$$

Multiplying the above estimate by the function $exp(\zeta - a)$, we have

$$-\varepsilon e^{\zeta - a} \leq u_\zeta e^{\zeta - a} + ue^{\zeta - a} \leq \varepsilon e^{\zeta - a}.$$

Let $c \in [a, b]$. For any $\zeta \in [a, b]$ integrating the above inequality from $c$ to $\zeta$, we obtain

$$\int_c^\zeta -\varepsilon e^{s-a} ds \leq \int_c^\zeta \frac{\partial}{\partial s}[u(s, \eta)e^{s-a}]ds \leq \int_c^\zeta \varepsilon e^{s-a} ds.$$

Then

$$-\varepsilon e^{\zeta - a} \leq u(\zeta, \eta)e^{\zeta - a} - (u(c, \eta) + \varepsilon)e^{c-a} + f(\eta) \leq \varepsilon e^{\zeta - a}.$$

Hence, it is clear that

$$-\varepsilon \leq u(\zeta,\eta) - (u(c,\eta)+\varepsilon)e^{c-\zeta} + f(\eta)e^{-(\zeta-a)} \leq \varepsilon.$$

Let

$$v(\zeta,\eta) = (u(c,\eta)+\varepsilon)e^{c-\zeta} - f(\eta)e^{-(\zeta-a)}.$$

Then $v(\zeta,\eta)$ satisfies $v_\zeta + v = 0$ and $|u(\zeta,\eta) - v(\zeta,\eta)| \leq \varepsilon$, respectively.

Taking into account the change of coordinates, we can write

$$|u(x,y) - v(x,y)| \leq \varepsilon.$$

Since $u(x,y) = Az_x + Bz_y$, we have

$$-\varepsilon \leq Az_x + Bz_y - v(x,y) \leq \varepsilon.$$

Consider the change of coordinates

$$r = x,$$
$$s = Ay - Bx.$$

Hence

$$Az_x + Bz_y - v(x,y) = Az_r - v(r,s).$$

From this, it follows that

$$-\varepsilon \leq Az_r - v(r,s) \leq \varepsilon.$$

Multiplying the above estimate by $\frac{1}{A}$, $(A \neq 0)$, we obtain

$$-\frac{\varepsilon}{A} \leq z_r - \frac{v(r,s)}{A} \leq \frac{\varepsilon}{A}.$$

Select $k \in [a,b]$. For any $r \in [k,b]$ with $r > 2k$, integrating the above inequality from $k$ to $r$, we have

$$-\frac{\varepsilon}{A}(r-k) \leq z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du \leq \frac{\varepsilon}{A}(r-k).$$

Then, it follows that

$$-\frac{\varepsilon}{A}r \leq z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du - \frac{\varepsilon k}{A} \leq \frac{\varepsilon}{A}(r-2k)$$

so that

$$-\frac{\varepsilon}{A}r \leq z(r,s) - z(k,s) - \int_k^r \frac{v(u,s)}{A}du - \frac{\varepsilon k}{A} \leq \frac{\varepsilon}{A}r.$$

Let

$$z_0(r,s) = z(k,s) + \int_k^r \frac{v(u,s)}{A}du + \frac{\varepsilon k}{A}.$$

Then $v(\zeta,\eta)$ satisfies

$$A(z_0)_r - v(r,s) = 0.$$

Hence, we can conclude that

$$|z(r,s) - z_0(r,s)| \leq \frac{\varepsilon r}{A}, K = \frac{r}{A}, A \neq 0.$$

This result completes the proof of Theorem 1.

Our second and last Hyers-Ulam stability result is the following theorem.

**Theorem 2.** *Let $\varepsilon$ be a positive constant. If the function $z$ satisfies the differential inequality*

$$|x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz(x,y)| \leq \varepsilon \tag{5}$$

*for all $(x,y) \in D$, then there exists a solution $z_0 : D \to \Re$ of equation (2) such that*

$$|z(x,y) - z_0(x,y)| \leq \frac{\varepsilon}{m} M, (m > 0, M > 0).$$

**Proof.** For any $(x,y) \in D$ let

$$g(x,y) = xz_x + yz_y + mz.$$

Then

$$xg_x(x,y) + yg_y(x,y) - g(x,y) = x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mxz_x + myz_y - mz.$$

Therefore, inequality (5) implies

$$|xg_x(x,y) + yg_y(x,y) - g(x,y)| \leq \varepsilon.$$

Consider the change of coordinates

$$\zeta = x,$$
$$\eta = \frac{y}{x},\ x \neq 0.$$

Then we have

$$|\zeta g_\zeta - g| \leq \varepsilon.$$

Assume that $\zeta > 0$. Making use of the former inequality, we arrive at

$$-\varepsilon \leq \zeta g_\zeta - g \leq \varepsilon.$$

Multiplying the above estimate by $\frac{a}{\zeta^2}$, we have

$$-\frac{\varepsilon a}{\zeta^2} \leq \frac{a}{\zeta} g_\zeta - \frac{a}{\zeta^2} g \leq \frac{\varepsilon a}{\zeta^2}.$$

Select $c_1 \in [a,b]$. For any $\zeta \in [c_1, b]$, $c_1 > 0$, integrating the above inequality from $c_1$ to $\zeta$, we can write

$$\int_{c_1}^{\zeta} -\frac{\varepsilon a}{s^2} ds \leq \int_{c_1}^{\zeta} \frac{\partial}{\partial s}[\frac{a}{s} g(s,\eta)] ds \leq \int_{c_1}^{\zeta} \frac{\varepsilon a}{s^2} ds.$$

Hence

$$\frac{\varepsilon}{\zeta} - \frac{\varepsilon}{c_1} \leq \frac{1}{\zeta} g(\zeta,\eta) - \frac{1}{c_1} g(c_1,\eta) + f(\eta) \leq -\frac{\varepsilon}{\zeta} + \frac{\varepsilon}{c_1}.$$

From this, it is clear that

$$\frac{-\varepsilon}{c_1} \leq \frac{1}{\zeta} g(\zeta,\eta) - \frac{1}{c_1} g(c_1,\eta) + f(\eta) - \frac{\varepsilon}{\zeta} \leq \frac{\varepsilon}{c_1}.$$

Since $\zeta > 0$, if we multiply the above inequality by $\zeta$, we get

$$-\frac{\varepsilon}{c_1}\zeta \leq g(\zeta,\eta) - \frac{\zeta}{c_1} g(c_1,\eta) + \zeta f(\eta) - \varepsilon \leq \frac{\varepsilon}{c_1}\zeta.$$

Let

$$v(\zeta,\eta) = \frac{\zeta}{c_1} g(c_1,\eta) - \zeta f(n) + \varepsilon.$$

Thus $v(\zeta,\eta)$ satisfies the following equation

$$\zeta v_\zeta - v = 0$$

and the inequality

$$|g(\zeta,\eta) - v(\zeta,\eta)| \leq M\varepsilon,$$

where $M = \frac{\zeta}{c_1}$. In view of the fact that

$$g(x,y) = xz_x + yz_y + mz_y,$$

it is clear that

$$-\varepsilon M \leq xz_x + yz_y + mz(x,y) - v(x,y) \leq M\varepsilon.$$

Consider the change of coordinates

$$\begin{aligned} r &= x, \\ n &= \frac{y}{x}, x \neq 0. \end{aligned}$$

Then, from the previous inequality, we have

$$-\varepsilon M \leq rz_r + mz - v \leq M\varepsilon.$$

Multiplying the above estimate by the function $\frac{r^{m-1}}{a^m}$, $(r > 0, (\frac{r}{a})^m > 0)$, we get

$$-\varepsilon M \frac{r^{m-1}}{a^m} \leq \frac{r^m}{a^m} z_r + m\frac{r^{m-1}}{a^m} z - \frac{r^{m-1}}{a^m} v \leq \varepsilon M \frac{r^{m-1}}{a^m}.$$

Select $k \in [a,b]$. For any $r \in [k,b]$ with $\frac{k^m}{ma^m} > 0$, integrating above inequality from $k$ to $r$, we obtain

$$-\varepsilon(\frac{r^m}{ma^m} - \frac{k^m}{ma^m})M \leq \frac{r^m}{a^m} z(r,n) - \frac{k^m}{a^m} z(k,n) - \int_k^r \frac{s^{m-1}}{a^m} v(s,n)ds \leq \varepsilon(\frac{r^m}{ma^m} - \frac{k^m}{ma^m})M.$$

From the last inequality, it may be seen that

$$-\varepsilon M \frac{r^m}{ma^m} \leq \frac{r^m}{a^m} z(r,n) - \frac{k^m}{a^m} z(k,n) - \int_k^r \frac{s^{m-1}}{a^m} v(s,n)ds - \varepsilon\frac{k^m}{ma^m} \leq \varepsilon M \frac{r^m}{ma^m}$$

so that

$$-\frac{\varepsilon}{m}M \leq z(r,n) - \frac{k^m}{r^m}z(k,n) - r^{-m}\int_k^r s^{m-1}v(s,n)ds - \varepsilon\frac{k^m}{mr^m} \leq \frac{\varepsilon}{m}M.$$

Let

$$z_0(r,\eta) = \frac{k^m}{r^m}z(k,n) + r^{-m}\int_k^r s^{m-1}v(s,n)ds + \varepsilon\frac{k^m}{mr^m}.$$

Then

$$|z(r,s) - z_0(r,s)| \leq \frac{\varepsilon}{m}M.$$

This completes the proof of Theorem 2.

**Example 1.** We consider the following linear partial differential equation of second order with constant coefficients

$$z_{xx} + 2z_{xy} + z_{yy} + z_x + z_y = 0.$$

Let $s = y - x$ and $f(s) > 0$. It can be seen that $z(x,y) = (e^{-x} - 1)f(y-x)$ is a solution of this equation and

$$|z_{xx} + 2z_{xy} + z_{yy} + z_x + z_y| \leq \varepsilon.$$

Let $[a,b] = [0,\infty)$ and $k = 0$, $c = 2$, $r = \frac{5}{2}$. Then, from Theorem 1, we have

$$|z - z_0| \leq \frac{5}{2}\varepsilon$$

and

$$z_0(r,s) = z(k,s) + \int_k^r \frac{v(u,s)}{A}du + \frac{\varepsilon k}{A}.$$

Thus, we can write

$$\begin{aligned} z_0(r,s) = \int_0^r v(u,s)du &= \int_0^r [(u(c,s)+\varepsilon)e^{c-m} - f(s)e^{-(m-a)}]dm \\ &= -(u(c,s)+\varepsilon)e^{c-r} + (u(c,s)+\varepsilon)e^{c} + f(s)(e^{-r} - 1). \end{aligned}$$

At the end, we can conclude that $|z - z_0| \leq \varepsilon r$. This inequality shows that the result of Theorem 1 is true.

**Example 2.** Consider the partial Euler differential equation of the form

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y - z = 0.$$

Then, it may be followed that

$$z(x,y) = xf\left(\frac{y}{2x}\right) + \frac{x}{2}g\left(\frac{y}{x}\right)$$

is a solution of the former equation, and we can find

$$|x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + xz_x + yz_y - z| \leq \varepsilon.$$

For $k = 0$, from Theorem 2, we have

$$z_0 = \frac{1}{r}\int_0^r (\frac{s}{c_1}g(c_1,n) - sf(n) + \varepsilon)ds = \frac{r}{2c_1}g(c_1,n) - \frac{r}{2}f(n) + \frac{\varepsilon}{m}$$

and

$$|z - z_0| \leq \frac{\varepsilon}{m}.$$

Hence, we can conclude that the result of Theorem 2 is correct.

## 3 Conclusion

We consider a linear partial differential equation of second order with constant coefficients and a partial Euler differential equation of second order. We study the Hyers-Ulam stability of these equations. We give two examples to verify the obtained results and for illustrations. Our results are contributions to the topic and the related literature.

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# Multiplicity of Periodic Solutions for a Class of Second Order Hamiltonian Systems

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**Abstract:** In this paper, we study the multiplicity of periodic solutions for two classes of sublinear nonlinearity second order Hamiltonian systems by the use of minimax methods, in critical point theory. Our results improve and generalize those in some known literatures.

**Keywords:** *Hamiltonian system; periodic solutions; sublinear nonlinearity; saddle point theorem.*

**Mathematics Subject Classification (2010):** 34C37.

## 1 Introduction

Consider the following Hamiltonian system with unbounded nonlinearities

$$\begin{cases} \ddot{u}(t) + Au(t) - \nabla F(t, u(t)) = e(t), & a.e.\ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{$HS$}$$

where $A$ is a $(N \times N)$-symmetric matrix, $e \in L^1(0, T; \mathbb{R}^N)$, $T > 0$, and $F : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function, $T$-periodic in the first variable and differentiable with respect to the second variable with continuous derivative $\nabla F(t, x) = \frac{\partial F}{\partial x}(t, x)$.

The study of the existence and multiplicity of periodic solutions of Hamiltonian systems plays a very important role to solve many problems of natural sciences such as chemistry, biology and physics. For physics problem, we can cite planetry systems and fluid dynamic problem.

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When $A = 0$ and $e(t) = 0$ for all $t \in \mathbb{R}$, problem (HS) is just the following second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \ \ a.e. \ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1}$$

During the last decades, many authors studied the existence and multiplicity of periodic solutions for system (1) via critical point theory and variational methods, we refer the readers to [1]- [21] and references therein. Many solvability conditions are given such as the coercive condition (see [2]), the periodicity condition (see [18]), the convexity condition (see [4]) and the subadditive condition (see [13]).

For the case $A \neq 0$ and $e \neq 0$, Mawhin and Willem [5] proved that problem (HS) has at least one solution by using the saddle point theorem under the following bounded conditions: There exists $g \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, u)| \leq g(t), \ |\nabla F(t, u)| \leq g(t), \ \forall u \in \mathbb{R}^N, \ a.e. \ t \in [0, T]. \tag{2}$$

Precisely they obtained the following result.

**Theorem 1.1 ( [5], Theorem 4.9)** *Suppose $F$ satisfies (2) and the following assumptions:*
$(C_1)$ $\dim N(A) = m \geq 1$ *and $A$ has no eigenvalue of the form $k^2 w^2$ $(k \in \mathbb{N}^*)$, where* $w = \frac{2\pi}{T}$,
$(C_2)$ $\displaystyle\int_0^T (e(t), \alpha_j) dt = 0$ $(1 \leq j \leq m)$ *where $(\alpha_1, \alpha_2, ......, \alpha_m)$ is a basis of $N(A)$.*
$(\tilde{F}_0)$ *There exists $T_j > 0$ such that $F(t, u + T_j \alpha_j) = F(t, u)$ $(1 \leq j \leq m)$, $\forall u \in \mathbb{R}^N$, a.e. $t \in [0, T]$.*
*Then problem (HS) has at least one solution.*

In 2006, Feng and Han [6] generalized Mawhin and Willem's result as follows:

**Theorem 1.2 ( [6], Theorem 2.1)** *Suppose $F$ satisfies $(C_1)$, $(C_2)$, $(\tilde{F}_0)$ and the following conditions: There exist $a, b \in L^1(0, T; \mathbb{R}^+), 0 \leq \alpha < 1$ such that*

$$|\nabla F(t, x)| \leq a(t)|x|^\alpha + b(t), \ \ \forall x \in \mathbb{R}^N, \ a.e. \ \ t \in [0, T]. \tag{3}$$

*Then problem (HS) has at least one solution.*

**Theorem 1.3 ( [6], Theorem 2.2)** *Suppose $F$ satisfies $(C_1)$, $(C_2)$, (3) and*

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to +\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A), \tag{4}$$

*or*

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to -\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A). \tag{5}$$

*Then problem (HS) has at least one solution.*

**Theorem 1.4 ( [6], Theorem 2.3)** *Suppose $F$ satisfies $(C_1)$, $(C_2)$, (3)*
*$(F_0)$ and*

$$|u|^{-2\alpha} \int_0^T F(t, u) dt \to +\infty \ \ as \ \ |u| \to \infty, \ \ u \in N(A) \ominus span(\alpha_1, ...., \alpha_r), \tag{6}$$

*or*

$$|u|^{-2\alpha}\int_0^T F(t,u)dt \to -\infty \;\; as \;\; |u| \to \infty, \;\; u \in N(A) \ominus span(\alpha_1, ...., \alpha_r). \tag{7}$$

*Then problem (HS) has at least* $r+1$ *solutions in* $H_T^1$.

In 2012, Li Xiao [8] generalized Theorem 1.3. Precisely he proved that problem (HS) possesses at least one solution when the nonlinearity $\nabla F(t,u)$ may grow slightly slower than a control function $h(|u|)$ instead of $|u|^\alpha$.

A natural question is whether there exists a result which contains the corresponding results in [5], [6], [8] as a special case.

Motivated by [6] and [8], we give this question a positive answer by the minimax methods in critical point theory and we obtain some results ( Theorems 1.5 and 1.6), unify and generalize Theorems 1.2, 1.3 and 1.4 in [6], and Theorems 1.4 and 1.5 in [8].

Our basic hypotheses on $A$ and $F$ are the following:

$(C_1)$ $\dim N(A) = m \geq 1$ and $A$ has no eigenvalue of the form $k^2w^2$ $(k \in \mathbb{N}^*)$, where $w = \frac{2\pi}{T}$,

$(C_2)$ $\displaystyle\int_0^T (e(t), \alpha_j)dt = 0 \quad (1 \leq j \leq m)$ where $(\alpha_1, \alpha_2, ..., \alpha_m)$ is a basis of $N(A)$.

$(F_0)$ There exists $0 \leq r \leq m$, $T_j > 0$ such that $F(t, u + T_j\alpha_j) = F(t,u)$ $(1 \leq j \leq r)$ $\forall u \in \mathbb{R}^N$, *a.e.* $t \in [0,T]$.

$(F_1)$ There exist constants $C_0 \geq 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0,1[$, $a \in L^1(0,T;\mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ and a function $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties:

(i) $h(s) \leq h(t)$ $\quad \forall s \leq t, s,t \in \mathbb{R}^+$,
(ii) $h(s+t) \leq C_0(h(t) + h(s))$ $\quad \forall s,t \in \mathbb{R}^+$,
(iii) $0 \leq h(t) \leq K_1 t^\alpha + k_2$ $\quad \forall t \in \mathbb{R}^+$,
(iv) $h(t) \to +\infty$ $\quad$ as $t \to +\infty$,
such that

$$|\nabla F(t,x)| \leq a(t)h(|x|) + b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$,

$(F_1')$ There exist constants $C_0^* \geq 0$, $C^* > 0$ and a function $h^* \in C(\mathbb{R}^+, \mathbb{R}^+)$ with the properties:

(i) $h^*(s) \leq h^*(t) + C_0^*$ $\quad \forall s \leq t, s,t \in \mathbb{R}^+$,
(ii) $h^*(s+t) \leq C^*(h^*(t) + h^*(s))$ $\quad \forall s,t \in \mathbb{R}^+$,
(iii) $th^*(t) - 2H^*(t) \to -\infty$ $\quad$ as $t \to +\infty$,
(iv) $\frac{H^*(t)}{t^2} \to 0$ $\quad$ as $t \to +\infty$,
where $H^*(t) = \displaystyle\int_0^t h^*(s)ds$. Moreover, there exist $f \in L^1(0,T;\mathbb{R}^+)$ and $g \in L^1(0,T;\mathbb{R}^+)$ such that

$$|\nabla F(t,x)| \leq f(t)h^*(|x|) + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Now we state our main results.

**Theorem 1.5** *Suppose that conditions* $(C_1)$, $(C_2)$, $(F_0)$, $(F_1)$ *and the following assumption hold*
$(F_2)$

$$(i)\ \lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=-\infty, \quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

*or*

$$(ii)\ \lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=+\infty, \quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

*then problem (HS) has at least* $r+1$ *T-periodic solutions in* $H_T^1$.

**Theorem 1.6** *Suppose that conditions* $(C_1)$, $(C_2)$, $(F_0)$, $(F_1')$ *and the following assumption hold*
$(F_2')$

$$(i)\ \lim_{|x|\to+\infty}\frac{1}{H^*(|x|)}\int_0^T F(t,x)dt=-\infty, \quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

*or*

$$(ii)\ \lim_{|x|\to+\infty}\frac{1}{H^*(|x|)}\int_0^T F(t,x)dt=+\infty, \quad x\in N(A)\ominus span(\alpha_1,....,\alpha_r),$$

*then problem (HS) has at least* $r+1$ *T-periodic solutions in* $H_T^1$.

**Example 1.1** Let

$$A=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Then $\dim N(A)=2$ and N(A)=span$\{\alpha_1,\alpha_2\}$, where $\alpha_1=(0,1,0),\alpha_2=(0,0,1)$. So $(C_1)$ holds.

Let

$$\begin{aligned}F(t,x)&=(0.4T-t)\ln^{\frac{3}{2}}[98+x_1^2+\sin^2(x_2)+\cos^2(x_3)]\\&+\ d(t)\ln[100+x_1^2+\sin^2(x_2)+\cos^2(x_3)]\end{aligned}\tag{8}$$

for all $x=(x_1,x_2,x_3)\in\mathbb{R}^3,\ t\in[0,T]$, where $d\in C([0,T];\mathbb{R}^+)$. We have

$$F(t,x+\pi\alpha_j)=F(t,x),\quad j=1,2.$$

Let $e$ satisfy $\int_0^T e(t)dt=0$, then $\int_0^T(e(t),\alpha_j)dt=0,\ j=1,2$ and

$$|\nabla F(t,x)|\le 3|0.4T-t|\ln^{\frac{1}{2}}(100+|x|^2)+d(t).$$

Let $h(t)=\ln^{\frac{1}{2}}(100+|t|^2)$. Similar to the argument in [17], we know that $(F_1)$ holds. Moreover,

$$\lim_{|x|\to+\infty}\frac{1}{h^2(|x|)}\int_0^T F(t,x)dt=-\infty.$$

Hence, $(F_2)i)$ holds and then by Theorem 1.5 , problem (HS) has at least three solutions. On the other hand, for any $\alpha \in (0,1)$,

$$\lim_{|x|\to+\infty} \frac{1}{|x|^{2\alpha}} \int_0^T F(t,x)dt = 0,$$

so (8) does not satisfy Theorem 1.3 in [6].

**Example 1.2** Consider the function
$F(t,x) = (\frac{2}{3}T - t)\ln(100 + |x|^2) + l(t)\sqrt{100 + |x|^2}, \quad where\ l \in C([0,T], \mathbb{R}^+)$.
It is easy to see that $|\nabla F(t,x)| \leq 2\left|\frac{2}{3}T - t\right| \frac{|x|}{100+|x|^2} + l(t)$ for all $x \in \mathbb{R}^3$ and $t \in [0,T]$. Let $h^*(t) = \frac{t}{100+t^2}, H^*(t) = \int_0^t \frac{s}{100+s^2}ds, C_0^* = 2, C^* = 1, f(t) = 2\left|\frac{2}{3}T - t\right|\ and\ g(t) = l(t)$, we infer

(i) $h^*(s) \leq h^*(t) + 2 \qquad \forall s \leq t, s, t \in \mathbb{R}^+,$
(ii) $h^*(s+t) = \frac{s+t}{100+(s+t)^2} \leq (h^*(t) + h^*(s)) \qquad \forall s, t \in \mathbb{R}^+,$
(iii) $th^*(t) - 2H^*(t) = \frac{t^2}{100+t^2} - 2\left[\frac{1}{2}\ln(100+t^2) - \frac{1}{2}\ln(100)\right] \to -\infty$ as $t \to +\infty,$
(iv) $\frac{H^*(t)}{t^2} = \frac{\int_0^t \frac{s}{100+s^2}ds}{t^2} \to 0 \qquad$ as $t \to +\infty.$

Let $e$ satisfy $\int_0^T e(t)dt = 0$, then $\int_0^T (e(t), \alpha_j)dt = 0,\ j = 1,2$, we have $\lim_{|x|\to+\infty} \frac{1}{H^*(|x|)} \int_0^T F(t,x)dt \to +\infty$. So, by Theorem 1.6, problem (HS) has at least one solution in $H_T^1$.

**Remark 1.1** Unlike the control functions in $(F_1)$, where $h(t)$ is nondecreasing, here control function $h^*(t) = \frac{t}{100+t^2}$ is bounded but not increasing.

**Remark 1.2** (i) Theorem 1.5. is a generalization of the main results in [ [15], Theorems 2 and 3] and in [ [6], Theorems 2.1, 2.2, 2.3]. Obviously , our theorems, as $r = m$, contain Theorems 1.4 and 1.5 in [8].
(ii) If we let $h(t) = t^\alpha$, it is easy to see that $(F_1)$ generalizes (3).

## 2 Preliminaries.

Let
$H_T^1 = \left\{u : \mathbb{R} \to \mathbb{R}^N /\ u\ is\ absolutely\ continuous, u(t) = u(t+T), \dot{u} \in L^2(0,T;\mathbb{R}^N)\right\}.$
Then $H_T^1$ is a Hilbert space with the inner product

$$< u, v >= \int_0^T \left[(u(t), v(t)) + (\dot{u}(t), \dot{v}(t))\right] dt$$

and the associated norm

$$\|u\| = \left(\int_0^T \left[|u(t)|^2 + |\dot{u}(t)|^2\right] dt\right)^{\frac{1}{2}}$$

for each $u, v \in H_T^1$. Let

$$\bar{u} = \frac{1}{T}\int_0^T u(t)dt, \quad \tilde{u}(t) = u(t) - \bar{u}.$$

Then one has

$$\int_0^T |\tilde{u}(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt\ ,\ (Wirtinger's\ inequality)$$

and

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt\ ,\ (Sobolev's\ inequality).$$

(see Proposition 1.3 in [5]) which implies that

$$\|u\|_\infty \leq C \|u\| \tag{9}$$

for some $C > 0$ and all $u \in H_T^1$, where $\|u\|_\infty = \max\limits_{t\in[0,T]} |u(t)|$. It is well known that the functional $\varphi$ defined on $H_T^1$ by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 dt - \frac{1}{2}\int_0^T (A(t)u(t), u(t))dt + \int_0^T F(t, u(t))dt + \int_0^T (e(t), u(t))dt$$

is continuously differentiable and its critical points are the solutions of problem (HS). Moreover, one has

$$< \varphi'(u), v >= \int_0^T [(\dot{u}(t), \dot{v}(t)) - (A(t)u(t), v(t)) + (\nabla F(t, u(t)), v(t)) + (e(t), v(t)]dt$$

for $u, v \in H_T^1$. Let

$$q(u) = \frac{1}{2}\int_0^T \Big(|\dot{u}|^2 - (A(t)u(t), u(t))\Big)\, dt.$$

It is easy to see that

$$q(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_0^T ((A(t) + I)u(t), u(t))dt = \frac{1}{2} < (I - K)u, u >,$$

where $K : H_T^1 \to H_T^1$ is the self-adjoint operator defined, using Riesz representation theorem, by

$$\int_0^T ((A(t) + I)u(t), v(t))dt =< (Ku, v) >, \forall u, v \in H_T^1.$$

The compact embedding of $H_T^1$ into $C(0, T; \mathbb{R}^N)$ implies that $K$ is compact. By classical spectral theory, we can decompose $H_T^1$ into the orthogonal sum of invariant subspaces for $I - K$

$$H_T^1 = H^- \oplus H^0 \oplus H^+,$$

where $H^0 = Ker(I - K)$ and $H^-$, $H^+$ are such that, for some $\delta > 0$,

$$q(u) \leq -\frac{\delta}{2}\|u\|^2 \quad if\ u \in H^-, \tag{10}$$

$$q(u) \geq \frac{\delta}{2}\|u\|^2 \quad if\ u \in H^+. \tag{11}$$

Moreover, by $(C_1)$, it is well known that $H^0 = Ker(I - K) = N(A)$ (see [5]).

In the proofs, we mainly use the following generalized saddle point theorem from [9].

**Theorem 2.1** *Let $X$ be a Banach space and have a decomposition: $X = W + Z$ where $W$ and $Z$ are two subspaces of $X$ with $\dim Z < +\infty$. Let $V$ be a finite-dimensional, compact $C^2$-manifold without boundary. Let $f : X \times V \to \mathbb{R}$ be a $C^1$-function and satisfy the (PS) condition. Suppose that $f$ satisfies* $\inf_{u \in W \times X} f(u) \geq \alpha$, $\sup_{u \in S \times X} f(u) \leq \beta < \alpha$, *where $S = \partial D, D = \{u \in Z \ / \ \|u\| \leq R\}$ and $R$, $\alpha$, $\beta$ are constants. Then the function $f$ has at least cuplength$(V)$+1 critical points.*

Let $PH^0 = span(\alpha_1, ...., \alpha_r), QH^0 = N(A) \ominus PH^0 = span(\alpha_{r+1}, ...., \alpha_m)$. Then $u = u^- + u^+ + Pu^0 + Qu^0$, where $Pu^0 = \sum_{j=1}^{r} c_j \alpha_j$. Let $G = \{\sum_{j=1}^{r} k_j T_j \alpha_j / \ k_j \in \mathbb{N}\}$. Use the canonical mapping $\pi : H_T^1 \to H_T^1/G$. Let $H_T^1/G = X \times V = (W \oplus Z) \times V$, $W = H^+$, $Z = H^- \oplus QH^0$, $V = PH^0/G$. It is easy to see that $\dim Z < +\infty, \dim V < +\infty$, and $V$ is a compact $C^2$-manifold without boundary as it is diffeomorphic to the $r$-torus $T^r$. Element in $V$ can be represented as $P\hat{u}^0 = \sum_{j=1}^{r} \hat{c}_j \alpha_j$, where $\hat{c}_j = c_j - k_j T_j$ $(0 \leq \hat{c}_j < T_j)$.

Let $u = u^- + u^+ + P\hat{u}^0 + Qu^0$. Define the functional $\psi$ on $H_T^1/G$ by $\psi(\pi(u)) = \varphi(u)$. As $F(t, u + T_j \alpha_j) = F(t, u)$ $(1 \leq j \leq r)$, we can see that $\psi$ is well-defined, and $\psi$ is continuously differentiable on $H_T^1/G$.

## 3 Proof of the Main Results.

**Proof of Theorem 1.5.**

For the sake of convenience, we will denote various positive constants as $C_i$, $i = 1, 2, ...$ We only prove the case where $(F_2)(i)$ holds. The other case can be similarly given.

**Lemma 3.1** *[Lemma 3.1, [8]] Assume that $(F_1)$ holds. Then for any (PS) sequence $(u_n) \subset H_T^1$ of the functional $\varphi$, we have*

$$\|\tilde{u}_n\|^2 \leq C_1 h^2(|u_n^0|) + C_1, \tag{12}$$

*where $u_n = u_n^+ + u_n^- + u_n^0$ and $\tilde{u}_n = u_n^+ + u_n^-$.*

**Lemma 3.2** *Suppose that $(F_1)$ and $(F_2)(i)$ hold, Then every (PS) sequence $(u_n) \subset H_T^1$ such that $(Pu_n^0)$ is bounded contains a convergent subsequence.*

***Proof.*** By (12), we have

$$\|\tilde{u}_n\|^2 \quad \leq C_1 h^2(|u_n^0|) + C_1.$$

As $(Pu_n^0)$ is bounded, we have the inequality

$$\|\tilde{u}_n\|^2 \leq C_2 h^2(|Qu_n^0|) + C_2. \tag{13}$$

It follows from (9), $(F_1)$, (13), the mean value theorem and Young's inequality that

$$
\begin{aligned}
&\left|\int_0^T \big(F(t,u_n(t)) - F(t,Qu_n^0)\big)\,dt\right| \\
= \;& \left|\int_0^T\int_0^1 \big(\nabla F(t,Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0)), \tilde{u}_n(t)+Pu_n^0\big)\,dsdt\right| \\
\leq \;& \int_0^T\int_0^1 \left|\nabla F(t,Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0)\right| \left|\tilde{u}_n(t)+Pu_n^0\right| dsdt \\
\leq \;& \int_0^T\int_0^1 \big(a(t)h(\left|Qu_n^0 + s(\tilde{u}_n(t)+Pu_n^0)\right|) + b(t)\big)\left|\tilde{u}_n(t)+Pu_n^0\right| dsdt \\
\leq \;& \int_0^T \left[C_0(C_0+1)a(t)\left(h(|Qu_n^0|) + h(\|\tilde{u}_n\|_\infty) + h(|Pu_n^0|)\right)\right]\left(\|\tilde{u}_n\|_\infty + |Pu_n^0|\right) dt \\
+ \;& \int_0^T b(t)\left(\|\tilde{u}_n\|_\infty + |Pu_n^0|\right) dt \\
\leq \;& C_3\|\tilde{u}_n\|_\infty h(\|\tilde{u}_n\|_\infty) + C_3\|\tilde{u}_n\|_\infty h(|Qu_n^0|) + C_4\|\tilde{u}_n\|_\infty + C_5 h(|Qu_n^0|) \\
+ \;& C_5 h(\|\tilde{u}_n\|_\infty) + C_6 \\
\leq \;& C_3\|\tilde{u}_n\|_\infty (K_1\|\tilde{u}_n\|_\infty^\alpha + K_2) + C_3\|\tilde{u}_n\|_\infty h(\left|Qu_n^0\right|) + C_4\|\tilde{u}_n\|_\infty \\
+ \;& C_5 h(|Qu_n^0|) + C_5(K_1\|\tilde{u}_n\|_\infty^\alpha + K_2) + C_6 \\
\leq \;& C_7\|\tilde{u}_n\|^{\alpha+1} + C_8\|\tilde{u}_n\|^\alpha + C_9\|\tilde{u_n}\| \\
+ \;& C_{10}\|\tilde{u}_n\| h(|Qu_n^0|) + C_5 h(|Qu_n^0|) + C_{11} \\
\leq \;& C_{12}\|\tilde{u}_n\|^2 + C_{13}h^2(\left|Qu_n^0\right|) + C_{14} \\
\leq \;& C_{15}h^2(\left|Qu_n^0\right|) + C_{16}.
\end{aligned}
\tag{14}
$$

Hence, by (14) and the boundedness of $\varphi(u_n)$ we obtain

$$
\begin{aligned}
-C_{17} \;\leq\;& \varphi(u_n) = \frac{1}{2}\left((I-K)u_n, u_n\right) + \int_0^T \big(F(t,u_n(t)) - F(t,Qu_n^0)\big)\,dt \\
+\;& \int_0^T F(t,Qu_n^0)dt + \int_0^T (e(t),u_n(t))dt \\
\leq\;& C_{18}\|\tilde{u}_n\|^2 + C_{15}h^2(\left|Qu_n^0\right|) + C_{16} + \int_0^T F(t,Qu_n^0)dt + C_{19}\|\tilde{u}_n\| \\
\leq\;& C_{20}h^2(\left|Qu_n^0\right|) + \int_0^T F(t,Qu_n^0)dt + C_{21} \\
=\;& h^2(|Qu_n^0|)\left(C_{20} + \frac{1}{h^2(|Qu_n^0|)}\int_0^T F(t,Qu_n^0)dt\right) + C_{21}.
\end{aligned}
\tag{15}
$$

It follows from $(F_2)(i)$ and (15) that $(Qu_n^0)$ is bounded. Combining (13) and the boundedness of $(Pu_n^0)$, we obtain that $(u_n)$ is bounded. Arguing as in [Proposition 4.1, [5]] we conclude that $(u_n)$ contains a convergent subsequence. Thus we complete the proof.

Now we are ready to prove Theorem 1.5. First, we prove that $\psi$ satisfies the (PS) condition. Let $(u_n) \subset H_T^1$ be a (PS) sequence of $\psi$, that is $(\psi(\pi(u_n)))$ is bounded and $\psi'(\pi(u_n)) \to 0$.

We have $q(u) = \frac{1}{2}\left((I-K)u, u\right)$ so $q'(u) = (I-K)u$ and since $u_k - \hat{u}_k = \sum_{j=1}^{r} k_j T_j \alpha_j \in$

$N(I-K)$, we obtain that $q(u_n) = q(\hat{u}_n)$ and $q'(u_n) = q'(\hat{u}_n)$. Moreover, by conditions $(F_0)$ and $(C_2)$, we have $F(t, u_n(t)) = F(t, \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_j \alpha_j) = F(t, \hat{u}_n(t)$ and

$$\int_0^T (e(t), u_n(t))dt = \int_0^T (e(t), \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_i \alpha_j)dt = \int_0^T (e(t), \hat{u}_n(t))dt.$$

Hence, we obtain that $\varphi(u_n) = \varphi(\hat{u}_n)$. Consequently $\psi(\pi(u_n)) = \psi(\pi(\hat{u}_n))$. It follows from $(F_0)$ that $\nabla F(t, u + T_j\alpha_j) = \nabla F(t, u) \quad (1 \leq j \leq r)$. Hence $\varphi'(u_n) = \varphi'(\hat{u}_n)$, namely, $\psi'(\pi(u_n)) = \psi'(\pi(\hat{u}_n))$. As $(P\hat{u}_n)$ is bounded, we obtain by Lemma 3.2 that $(\hat{u}_n)$ contains a convergent subsequence: $\hat{u}_{n_k} \to \hat{u}$. Then

$$\lim_{k\to+\infty} \psi(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi(\pi(\hat{u}_{n_k})) = \psi(\pi(\hat{u})),$$
$$\lim_{k\to+\infty} \psi'(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi'(\pi(\hat{u}_{n_k})) = \psi'(\pi(\hat{u})).$$

Hence $\psi$ satisfies the (PS) condition.

In order to use the generalized saddle point theorem we only need to verify the following conditions:

$$(\psi_1) \quad \psi(\pi(u)) \to +\infty, \quad as \ \|u\| \to +\infty \quad in \quad W \times V,$$

$$(\psi_2) \quad \psi(\pi(u)) \to -\infty, \quad as \ \|u\| \to +\infty \quad in \quad Z \times V.$$

By (9), $(F_1)$, the mean value theorem and the boundedness of $(P\hat{u}_n)$, we have $\forall \pi(u) \in W \times V, \ \ u = u^+ + Pu^0$,

$$\begin{aligned}
&\int_0^T (F(t, \hat{u}(t)) - F(t, 0))\, dt \\
&= \int_0^T \int_0^1 \left(\nabla F(t, s(u^+(t) + P\hat{u}^0), u^+(t) + P\hat{u}^0\right) dsdt \\
&\leq \int_0^T \int_0^1 \left|\nabla F(t, s(u^+(t) + P\hat{u}^0)\right| \left|u^+(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T \int_0^1 \left(a(t)h(\left|u^+(t) + P\hat{u}^0\right|) + b(t)\right) \left|u^+(t) + P\hat{u}^0\right| dsdt \\
&\leq \int_0^T \left[C_0 a(t) \left(h(\|u^+\|_\infty) + h(|P\hat{u}^0|)\right) + b(t)\right] \left(\|u^+\|_\infty + |P\hat{u}^0|\right) dt \\
&\leq \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \left(C_0 K_1 \|u^+\|_\infty^\alpha \int_0^T a(t)dt + C_0 K_2 \int_0^T a(t)dt\right) \\
&+ \left(\|u^+\|_\infty + |P\hat{u}^0|\right) h(|P\hat{u}^0|) C_0 \int_0^T a(t)dt + \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \int_0^T b(t)dt \\
&\leq C_{22} \left\|u^+\right\|_\infty^{\alpha+1} + C_{23} \left\|u^+\right\|_\infty^\alpha + C_{24} \left\|u^+\right\|_\infty + C_{25} \\
&\leq C_{26} \left\|u^+\right\|^{\alpha+1} + C_{27} \left\|u^+\right\|^\alpha + C_{28} \left\|u^+\right\| + C_{25}.
\end{aligned} \tag{16}$$

It follows from (11) and (16) that

$$\begin{aligned}
\psi(\pi(u)) &= \varphi(u)=\varphi(\hat{u}) \\
&= \frac{1}{2}\big((I-K)u^+,u^+\big)+\int_0^T (F(t,\hat{u}(t))-F(t,0))\,dt \\
&+ \int_0^T F(t,0)dt+\int_0^T (e(t),\hat{u}(t))dt \\
&\geq \frac{\delta}{2}\|u^+\|^2-C_{26}\left\|u^+\right\|^{\alpha+1}-C_{27}\left\|u^+\right\|^{\alpha}-C_{29}\left\|u^+\right\|-C_{30}. \qquad (17)
\end{aligned}$$

Since $\alpha+1<2$, then by (17), $(\psi_1)$ is verified.
On the other hand, by (9), $(F_1)$, the mean value theorem, the boundedness of $(P\hat{u}_n)$ and Young's inequality we obtain for $\pi(u)\in Z\times V$, $\; u=u^-+Qu^0+Pu^0$,

$$\int_0^T \big(F(t,\hat{u}(t))-F(t,Qu^0)\big)\,dt$$

$$\begin{aligned}
&= \int_0^T\int_0^1 \big(\nabla F(t,Qu^0+s(u^-(t)+P\hat{u}^0)),u^-(t)+P\hat{u}^0\big)\,dsdt \\
&\leq \int_0^T\int_0^1 \left|\nabla F(t,Qu^0+s(u^-(t)+P\hat{u}^0)\right|\left|u^-(t)+P\hat{u}^0\right|dsdt \\
&\leq \int_0^T\int_0^1 \big(a(t)h(\left|Qu^0+s(u^-(t)+P\hat{u}^0)\right|)+b(t)\big)\left|u^-(t)+P\hat{u}^0\right|dsdt \\
&\leq \int_0^T C_0(C_0+1)a(t)\big(h(|Qu^0|)+h(\|u^-\|_\infty)+h(|P\hat{u}^0|)\big)\big(\|u^-\|_\infty+|P\hat{u}^0|\big)\,dt \\
&+ \int_0^T b(t)\big(\|u^-\|_\infty+|P\hat{u}^0|\big)\,dt \\
&\leq C_{31}\|u^-\|_\infty h(\|u^-\|_\infty)+C_{31}\|u^-\|_\infty h(|Qu^0|) \\
&+ C_{32}h(\|u^-\|_\infty)+C_{32}h(|Qu^0|)+C_{33}\|u^-\|_\infty+C_{34} \\
&\leq C_{31}\left\|u^-\right\|_\infty\big(K_1\left\|u^-\right\|_\infty^\alpha+K_2\big)+C_{31}\left\|u^-\right\|_\infty h(\left|Qu^0\right|)+C_{33}\|u^-\|_\infty \\
&+ C_{32}h(|Qu^0|)+C_{32}\big(K_1\left\|u^-\right\|_\infty^\alpha+K_2\big)+C_{34} \\
&\leq C_{35}\left\|u^-\right\|_\infty^{\alpha+1}+C_{36}\left\|u^-\right\|_\infty^\alpha+C_{37}\left\|u^-\right\|_\infty \\
&+ C_{31}\left\|u^-\right\|_\infty h(|Qu^0|)+C_{32}h(|Qu^0|)+C_{38} \\
&\leq C_{39}\left\|u^-\right\|^{\alpha+1}+C_{40}\left\|u^-\right\|^\alpha+C_{41}\left\|u^-\right\| \\
&+ C_{42}\left\|u^-\right\|h(|Qu^0|)+C_{32}h(|Qu^0|)+C_{38} \\
&\leq \varepsilon\left\|u^-\right\|^2+C_{39}\left\|u^-\right\|^{\alpha+1}+C_{40}\left\|u^-\right\|^\alpha \\
&+ C_{41}\left\|u^-\right\|+C_{43}h^2(\left|Qu^0\right|)+C_{44} \qquad (18)
\end{aligned}$$

for any $\varepsilon > 0$. Hence, by (10) and (18) we obtain

$$\begin{aligned}
\psi(\pi(u)) &= \varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^-, u^-\right) + \int_0^T \left(F(t,u(t)) - F(t,Qu^0)\right) dt \\
&+ \int_0^T F(t,Qu^0)dt + \int_0^T (e(t), u^-(t))dt \\
&\leq \frac{-\delta}{2}\|u^-\|^2 + \varepsilon \left\|u^-\right\|^2 + C_{39}\left\|u^-\right\|^{\alpha+1} \\
&+ C_{40}\left\|u^-\right\|^{\alpha} + C_{45}\left\|u^-\right\| + C_{43}h^2(|Qu^0|) + \int_0^T F(t,Qu^0)dt + C_{44} \\
&= (\frac{-\delta}{2} + \varepsilon)\left\|u^-\right\|^2 + C_{39}\left\|u^-\right\|^{\alpha+1} + C_{40}\left\|u^-\right\|^{\alpha} + C_{45}\left\|u^-\right\| \\
&+ h^2(|Qu^0|)\left(C_{43} + \frac{1}{h^2(|Qu^0|)}\int_0^T F(t,Qu^0)dt\right) + C_{44}. \qquad (19)
\end{aligned}$$

Fixing $\varepsilon < \frac{\delta}{2}$, by (19), $(F_2)(i)$ and since $\alpha + 1 < 2$, we obtain $\varphi(u) \to -\infty$ *as* $\|u\| \to +\infty$ *in* $Z \times V$. Thus $(\psi_2)$ is verified. The proof is completed. □

**Proof of Theorem 1.6.**
We only prove the case where $(F_2')(i)$ holds. The other case can be similarly given.

**Lemma 3.3 (Lemma 2.1, [19])** *Suppose that there exists a positive function $h^*$ satisfying the conditions (i), (ii), (iv) of $(F_1')$, then we have the following estimates:*
*(1)* $0 < h^*(t) < \varepsilon t + C_0$ *for any* $\varepsilon > 0, C_0 > 0, t \in \mathbb{R}^+$,
*(2)* $\frac{h^{*2}(t)}{H^*(t)} \longrightarrow 0$ *as* $t \to +\infty$,
*(3)* $H^*(t) \longrightarrow +\infty$ *as* $t \to +\infty$.

**Lemma 3.4** *Assume that $(F_1')$ holds. Then for any (PS) sequence $(u_n) \subset H_T^1$ of the functional $\varphi$, we have*

$$\|\tilde{u}_n\|^2 \leq C_{45}h^{*2}(\left|u_n^0\right|) + C_{45}, \qquad (20)$$

*where $u_n = u_n^+ + u_n^- + u_n^0$ and $\tilde{u}_n = u_n^+ + u_n^-$.*

***Proof.*** Assume that $(u_n) \subset H_T^1$ is a (PS) sequence for $\varphi$. Then

$$|\varphi(u_n)| \leq C_{46}, \quad |\varphi'(u_n)| \leq C_{46}, \forall n \in \mathbb{N}.$$

It follows from $(F_1')$, (9), Lemma 3.3 and Young's inequality that

$$
\begin{aligned}
&\left|\int_0^T \nabla F(t,u_n(t)), u_n^+(t)-u_n^-(t))dt\right| \\
\leq\ & \int_0^T |\nabla F(t,u_n(t))| \left|u_n^+(t)-u_n^-(t)\right| dt \\
\leq\ & \int_0^T f(t)h^*(\left|u_n^0+\tilde{u}_n(t)\right|) \left|u_n^+(t)-u_n^-(t)\right| dt + \int_0^T g(t)\left|u_n^+(t)-u_n^-(t)\right| dt \\
\leq\ & \left\|u_n^+-u_n^-\right\|_\infty \int_0^T f(t)[C_0^*+h^*(\left|u_n^0\right|+\|\tilde{u}_n\|_\infty)]dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T g(t)dt \\
=\ & \left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|+\|\tilde{u}_n\|_\infty)\int_0^T f(t)dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\|\tilde{u}_n\|_\infty)\int_0^T f(t)dt + C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|)\int_0^T f(t)dt \\
+\ & \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & \varepsilon C^*\left\|u_n^+-u_n^-\right\|_\infty \|\tilde{u}_n\|_\infty \int_0^T f(t)dt + C_0C^*\left\|u_n^+-u_n^-\right\|_\infty \int_0^T f(t)dt \\
+\ & C^*\left\|u_n^+-u_n^-\right\|_\infty h^*(\left|u_n^0\right|)\int_0^T f(t)dt + \left\|u_n^+-u_n^-\right\|_\infty \int_0^T (C_0^*f(t)+g(t))dt \\
\leq\ & \varepsilon C_{47}\|\tilde{u}_n\|^2 + C_{48}h^*(\left|u_n^0\right|)\|\tilde{u}_n\| + C_{49}\|\tilde{u}_n\| \\
\leq\ & 3\varepsilon C_{47}\|\tilde{u}_n\|^2 + C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) + C_{51}(\varepsilon)
\end{aligned} \tag{21}
$$

for any $\varepsilon > 0$.

Thus, we have

$$
\begin{aligned}
C_{46}\left\|u_n^+-u_n^-\right\| &= C_{46}\|\tilde{u}_n\| \\
&\geq \left(\varphi'(u_n), u_n^+-u_n^-\right) \\
&= \left((I-K)u_n, u_n^+-u_n^-\right) + \int_0^T \left(\nabla F(t,u_n(t))+e(t), u_n^+(t)-u_n^-(t)\right) dt \\
&\geq \delta\|\tilde{u}_n\|^2 - 3\varepsilon C_{47}\|\tilde{u}_n\|^2 - C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) - C_{51}(\varepsilon) \\
&- \left\|u_n^+-u_n^-\right\|_\infty \int_0^T |e(t)|\,dt \\
&\geq (\delta-3\varepsilon C_{47})\|\tilde{u}_n\|^2 - C_{50}(\varepsilon)h^{*2}(\left|u_n^0\right|) - C_{51}(\varepsilon) - C_{52}\|\tilde{u}_n\|.
\end{aligned}
$$

Hence, we obtain

$$
(\delta-5\varepsilon C_{47})\|\tilde{u}_n\|^2 \leq C_{50}h^{*2}(\left|u_n^0\right|) + C_{53}, \tag{22}
$$

if we fix $\varepsilon < \frac{\delta}{5C_{47}}$, then by (22) we have

$$
\|\tilde{u}_n\|^2 \leq C_{54}h^{*2}(\left|u_n^0\right|) + C_{55}.
$$

Take $C_{45} = \max\{C_{54}, C_{55}\}$, the proof is complete.

**Lemma 3.5** *Suppose that $(F_1')$ and $(F_2')(i)$ hold, Then every (PS) sequence $(u_n) \subset H_T^1$ such that $(Pu_n^0)$ is bounded contains a convergent subsequence.*

***Proof.*** By (20), we have

$$\|\tilde{u}_n\|^2 \leq C_{45} h^{*2}(|u_n^0|) + C_{45}.$$

As $(Pu_n^0)$ is bounded, we have the inequality

$$\|\tilde{u}_n\|^2 \leq C_{56} h^{*2}(|Qu_n^0|) + C_{56}. \tag{23}$$

It follows from (9), $(F_1')$, (23), the mean value theorem and Young's inequality that
$\left|\int_0^T \left(F(t, u_n(t)) - F(t, Qu_n^0)\right) dt\right|$

$$\begin{aligned}
&= \left|\int_0^T \int_0^1 \left(\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0), \tilde{u}_n(t) + P\hat{u}_n^0\right) dsdt\right| \\
&\leq \int_0^T \int_0^1 \left|\nabla F(t, Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0)\right| \left|\tilde{u}_n(t) + P\hat{u}_n^0\right| dsdt \\
&\leq \int_0^T \int_0^1 \left(f(t) h^*(\left|Qu_n^0 + s(\tilde{u}_n(t) + P\hat{u}_n^0)\right|) + g(t)\right) \left|\tilde{u}_n(t) + P\hat{u}_n^0\right| dsdt \\
&\leq \int_0^T \left[f(t)\left(h^*(|Qu_n^0| + \|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|) + C_0^*\right) + g(t)\right] \left(|\tilde{u}_n(t)| + |P\hat{u}_n^0|\right) dt \\
&\leq C^*(C^*+1)\left(h^*(|Qu_n^0|) + h^*(\|\tilde{u}_n\|_\infty) + h^*(|P\hat{u}_n^0|)\right)\left(\|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|\right) \int_0^T f(t) dt \\
&+ \left(\|\tilde{u}_n\|_\infty + |P\hat{u}_n^0|\right) \int_0^T (g(t) + C_0^* f(t)) dt \\
&\leq C_{57} \|\tilde{u}_n\|_\infty h^*(\|\tilde{u}_n\|_\infty) + C_{58} \|\tilde{u}_n\|_\infty h^*(|Qu_n^0|) + C_{59} \|\tilde{u}_n\|_\infty + C_{60} h^*(|Qu_n^0|) \\
&+ C_{61} h(\|\tilde{u}_n\|_\infty) + C_{62} \\
&\leq C_{63} h^{*2}(|Qu_n^0|) + C_{64}.
\end{aligned} \tag{24}$$

It follows from the boundedness of $\varphi(u_n)$ and (24) that

$$\begin{aligned}
-C_{65} &\leq \varphi(u_n) \\
&= \frac{1}{2}((I-K)u_n, u_n) + \int_0^T \left(F(t, u_n(t)) - F(t, Qu_n^0)\right) dt \\
&+ \int_0^T F(t, Qu_n^0) dt + \int_0^T (e(t), u_n(t)) dt \\
&\leq C_{66} \|\tilde{u}_n\|^2 + C_{63} h^{*2}(|Qu_n^0|) + C_{64} + \int_0^T F(t, Qu_n^0) dt + C_{67} \|\tilde{u}_n\| \\
&\leq C_{68} h^{*2}(|Qu_n^0|) + \int_0^T F(t, Qu_n^0) dt + C_{69} \\
&= H^*(|Qu_n^0|) \left(\frac{C_{68} h^{*2}(|Qu_n^0|)}{H^*(|Qu_n^0|)} + \frac{1}{H^*(|Qu_n^0|)} \int_0^T F(t, Qu_n^0) dt\right) \\
&+ C_{69}.
\end{aligned} \tag{25}$$

Hence, by $(F_2')(i)$, (25) and Lemma 3.3 we deduce that $(Qu_n^0)$ is bounded. Combining (20) and the boundedness of $(Pu_n^0)$, we obtain that $(u_n)$ is bounded. Arguing as in [Proposition 4.1, [5]] we conclude that $(u_n)$ contains a convergent subsequence. We complete the proof.

Now we are ready to prove Theorem 1.6. First, we prove that $\psi$ satisfies the (PS) condition. Let $(u_n) \subset H_T^1$ be a (PS) sequence of $\psi$, that is $(\psi(\pi(u_n)))$ is bounded and $\psi'(\pi(u_n)) \to 0$. We have $q(u) = \frac{1}{2}((I-K)u, u)$ so $q'(u) = (I-K)u$ and since $u_k - \hat{u}_k = \sum_{j=1}^{r} k_j T_j \alpha_j \in N(I-K)$, we obtain that $q(u_n) = q(\hat{u}_n)$ and $q'(u_n) = q'(\hat{u}_n)$. Therefore, by conditions $(F_0)$ and $(C_2)$, we have

$$F(t, u_n(t)) = F(t, \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_j \alpha_j) = F(t, \hat{u}_n(t),$$

$$\int_0^T (e(t), u_n(t))dt = \int_0^T (e(t), \hat{u}_n(t) + \sum_{j=1}^{r} k_j T_i \alpha_j)dt = \int_0^T (e(t), \hat{u}_n(t))dt.$$

Hence, we obtain that $\varphi(u_n) = \varphi(\hat{u}_n)$. Consequently $\psi(\pi(u_n)) = \psi(\pi(\hat{u}_n))$. It follows from $(F_1)$ that $\nabla F(t, u + T_j\alpha_j) = \nabla F(t, u) \quad (1 \le j \le r)$. Hence $\varphi'(u_n) = \varphi'(\hat{u}_n)$, namely, $\psi'(\pi(u_n)) = \psi'(\pi(\hat{u}_n))$. As $(P\hat{u}_n)$ is bounded, we obtain by Lemma 3.5 that $(\hat{u}_n)$ contains a convergent subsequence. Let $\hat{u}_{n_k} \to \hat{u}$.
Then

$$\lim_{k\to+\infty} \psi(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi(\pi(\hat{u}_{n_k})) = \psi(\pi(\hat{u})),$$
$$\lim_{k\to+\infty} \psi'(\pi(u_{n_k})) = \lim_{k\to+\infty} \psi'(\pi(\hat{u}_{n_k})) = \psi'(\pi(\hat{u})).$$

It implies that $\psi$ satisfies the (PS) condition.

In order to use the generalized saddle point theorem we only need to verify the following conditions:

$(\psi_1)$ There exists $\alpha \in \mathbb{R}$ such that $\psi(\pi(u)) \ge \alpha, \quad$ on $W \times V$,

$(\psi_2)$ There exists $\beta < \alpha$ such that $\psi(\pi(u)) \le \beta, \quad$ on $Z \times V$.

It follows from (9), $(F_1')$, the mean value theorem and the boundedness of $(P\hat{u}_n)$, that $\forall \pi(u) \in W \times V, u = u^+ + Pu^0$,

$$\begin{aligned}
&\int_0^T (F(t,\hat{u}(t)) - F(t,0))\,dt \\
&= \int_0^T \int_0^1 \left(\nabla F(t, s(u^+(t) + P\hat{u}^0), u^+(t) + P\hat{u}^0\right) ds dt \\
&\le \int_0^T \int_0^1 \left|\nabla F(t, s(u^+(t) + P\hat{u}^0)\right| \left|u^+(t) + P\hat{u}^0\right| ds dt \\
&\le \int_0^T \int_0^1 \left(f(t)h^*(\left|s(u^+(t) + P\hat{u}^0)\right|) + g(t)\right) \left|u^+(t) + P\hat{u}^0\right| ds dt \\
&\le \int_0^T \left[f(t)\left(h^*(\|u^+\|_\infty + |P\hat{u}^0|) + C_0^*\right) + g(t)\right] \left(\|u^+\|_\infty + |P\hat{u}^0|\right) dt \\
&\le C^* \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \left(h^*(\|u^+\|_\infty) + h^*(|P\hat{u}^0|)\right) \int_0^T f(t)dt
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left(\|u^+\|_\infty + |P\hat{u}^0|\right) \int_0^T (g(t) + C_0^* f(t))dt \\
&\leq \quad \varepsilon C_{70} \left\|u^+\right\|_\infty h^*(\left\|u^+\right\|_\infty) + C_{71} h^*(\left\|u^+\right\|_\infty) + C_{72} \left\|u^+\right\|_\infty + C_{73} \\
&\leq \quad \varepsilon C_{70} \left\|u^+\right\|_\infty^2 + C_{74} \left\|u^+\right\|_\infty + C_{75} \\
&\leq \quad \varepsilon C_{76} \left\|u^+\right\|^2 + C_{77} \left\|u^+\right\| + C_{75} \qquad (26)
\end{aligned}
$$

for any $\varepsilon > 0$.
Hence, we deduce from (11) and (26) that

$$
\begin{aligned}
\psi(\pi(u)) &= \varphi(u) = \varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^+, u^+\right) + \int_0^T \left(F(t, \hat{u}(t)) - F(t,0)\right) dt \\
&+ \int_0^T F(t,0)dt + \int_0^T (e(t), \hat{u}(t))dt \\
&\geq (\frac{\delta}{2} - \varepsilon C_{76}) \left\|u^+\right\|^2 - C_{80} \left\|u^+\right\| - C_{81}. \qquad (27)
\end{aligned}
$$

Choosing $\varepsilon < \frac{\delta}{2C_{76}}$, by (27) $\psi$ is bounded below on $W \times V$, and $(\psi_1)$ is verified.

On the other hand, by (9), $(F_1')$, the mean value theorem, the boundedness of $(P\hat{u}_n)$ and Young's inequality we have
$\forall \pi(u) \in Z \times V, \quad u = u^- + Qu^0 + Pu^0,$

$$
\begin{aligned}
&\int_0^T \left(F(t, \hat{u}(t)) - F(t, Qu^0)\right) dt \\
&\quad = \int_0^T \int_0^1 \left(\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0), u^-(t) + P\hat{u}^0\right) dsdt \\
&\quad \leq \int_0^T \int_0^1 \left|\nabla F(t, Qu^0 + s(u^-(t) + P\hat{u}^0)\right| \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\quad \leq \int_0^T \int_0^1 \left[f(t) h^*(\left|Qu^0 + s(u^-(t) + P\hat{u}^0)\right|) + g(t)\right] \left|u^-(t) + P\hat{u}^0\right| dsdt \\
&\quad \leq \int_0^T f(t) \left(h^*(|Qu^0| + \|u^-\|_\infty + |P\hat{u}^0|) + C_0^*\right) \left(\|u^-\|_\infty + |P\hat{u}^0|\right) dt \\
&\quad + \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T g(t)dt \\
&\leq C^*(C^*+1) \left(h^*(|Qu^0|) + h^*(\|u^-\|_\infty) + h^*(|P\hat{u}^0|)\right) \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T f(t)dt \\
&+ \left(\|u^-\|_\infty + |P\hat{u}^0|\right) \int_0^T (g(t) + C_0^* f(t))dt \\
&\leq C_{82} \|u^-\|_\infty h^*(\|u^-\|_\infty) + C_{82} \|u^-\|_\infty h^*(|Qu^0|) \\
&+ C_{83} h^*(\|u^-\|_\infty) + C_{83} h^*(|Qu^0|) + C_{84} \|u^-|_\infty + C_{85} \\
&\leq C_{86} \|u^-\|_\infty^2 + C_{86} h^{*2}(|Qu^0|) + C_{87} \|u^-\|_\infty + C_{88} \\
&\leq C_{89} \|u^-\|_\infty^2 + C_{90} h^*(|Qu^0|) + C_{91} \|u^-\|_\infty + C_{88} \\
&\leq C_{91} h^{*2}(\left|Qu^0\right|) + C_{92}. \qquad (28)
\end{aligned}
$$

Hence, by (10) and (28) we obtain

$$\begin{aligned}
\psi(\pi(u)) &= \varphi(u)=\varphi(\hat{u}) \\
&= \frac{1}{2}\left((I-K)u^-,u^-\right)+\int_0^T\left(F(t,u(t))-F(t,Qu^0)\right)dt \\
&+ \int_0^T F(t,Qu^0)dt+\int_0^T(e(t),u^-(t))dt \\
&\leq \frac{-\delta}{2}\|u^-\|^2+C_{91}h^{*2}(\left|Qu^0\right|)+C_{92}+C_{93}\|u^-\|+\int_0^T F(t,Qu^0)dt \\
&= H^*(|Qu^0|)\left(\frac{C_{91}h^{*2}(\left|Qu^0\right|)}{H^*(|Qu^0|)}+\frac{1}{H^*(|Qu^0|)}\int_0^T F(t,Qu^0)dt\right) \\
&+ \frac{-\delta}{2}\left\|u^-\right\|^2+C_{93}\left\|u^-\right\|+C_{92}. \qquad (29)
\end{aligned}$$

Hence, by (29), $(F_2')(i)$ we obtain that $\varphi(u)\to-\infty \quad as \quad \|u\|\to+\infty \quad in \quad Z\times V$.

Thus, $(\psi_2)$ is verified and we complete the proof.

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# Capacity, Theorem of H. Brezis and F.E. Browder Type in Musielak–Orlicz–Sobolev Spaces and Application

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**Abstract:** The second section of this paper is devoted to the study of the capacity theory in Musielak–Orlicz–Sobolev space, we study basic's properties, including monotonicity, countable subadditivity and several convergence results, we prove that each Musielak-Orlicz-Sobolev function has a quasi-continuous representative. In the third section, we generalize the Theorem of H. Brezis and F.E. Browder in the setting of Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$, which extends the previous result of H. Brezis and F.E. Browder [10]. In the fourth section, we make an application to an unilateral problem.

**Keywords:** *Musielak-Orlicz-Sobolev spaces; capacity; theorem of H. Brezis and F.E. Browder; unilateral problem.*

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## 1 Introduction

The theory of capacity and non-linear potential in the classical Lebesgue space $L^p(\Omega)$, was mainly studied by Maz'ya and Khavin in [17] and Meyers in [21]. These authors in their previous works have introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis and in the theory of partial differential equations.

When we replace the spaces $L^p(\Omega)$ by the general one $L_A(\Omega)$ generated by an $N$-function, some fundamental properties are not satisfied, in particular, the reflexivity of

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spaces (obviously for an $N$-function which doses not satisfying the $\triangle_2$ condition). In this case, we found some works, in particular In [3] and [4].

When we replace $A(t)$ by some Musielak–Orlicz function $\varphi(x,t)$, the situation belong more difficult and the Musielak–Orlicz spaces obtained is $L_\varphi(\Omega)$ which has lost many interest functional properties. In this case, we refer the reader [13] and [18].

Thus, the first goal of this paper is to extend the theory of capacity in the setting of Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$. Moreover, we generalize the Theorem 1 of [1], in the setting of Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$, this generalisation is an extension of the corresponding result of H.Brezis and F.E.Browder(see [10] and [15]).

Now, let give and comment the following theorem:

**Theorem 1.1** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, $m \in \mathbb{N}$ and $1 < p, p' < +\infty$, such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Consider $u$ in $W_0^{m,p}(\Omega)$, $u \geqslant 0$ a.e in $\Omega$ and $T$ in $W_0^{-m,p'}(\Omega)$, such that $T = \mu + h$, where $\mu$ is a positive Radon measure and $h$ an $L^1_{loc}(\Omega)$ function; Assume moreover that*

$$h(x)u(x) \geqslant -|\Phi(x)| \quad a.e \quad x \in \Omega, \ for \ some \ \Phi \ in \ L^1(\Omega).$$

*Then:*

$$hu \in L^1(\Omega), \ u \in L^1(\Omega, d\mu) \ and \ < T, u >= \int_\Omega u d\mu + \int_\Omega h u dx. \tag{1}$$

This result is proved by L. Boccardo, D. Giachetti and F. Murat in [15], and extends previous Theorem of H. Brezis and F. Browder in [10], who considered the cases where either $\mu \equiv 0$ or $h \equiv 0$. the main tool in order to prove these results is the Hedberg's approximation (in $W_0^{m,p}(\Omega)$ norm) of function $u \in W_0^{m,p}(\Omega)$ by a sequence of functions $(u_n)_n$ which belong to $L^\infty(\Omega) \cap W_0^{m,p}(\Omega)$, have compact support in $\Omega$ and satisfy $u_n u \geqslant 0$, $|u_n| \leqslant u \quad a.e.$ in $\Omega$.

Note that an application of the previous theorem to study the following nonlinear variational inequality:

$$u \in K_\Phi, \ g(. \ , \ u) \in L^1(\Omega), \ ug(. \ , \ u) \in L^1(\Omega),$$

$$< Au, v - u > + \int_\Omega g(., u)(v - u) dx \geqslant < f, v - u >, \ \forall v \in K_\Phi \cap L^\infty(\Omega), \tag{2}$$

where $A$ is a pseudo-monotone operator acting on $W_0^{m,p}(\Omega)$, $f \in W^{-m,p'}(\Omega)$, $K_\Phi = \{v : v \in W_0^{m,p}(\Omega), v \geqslant \Psi \ a.e \ in \ \Omega\}$, $\Psi \in W_0^{m,p}(\Omega) \cap L^\infty(\Omega)$ and $g$ satisfies the sign condition $sg(x,s) \geqslant 0$ but no growth restriction with respect to $s$.

Let us mention that a generalization of the Theorem1.1 and the problem ( 2 ) in the setting of Orlicz-Sobolev space $W^mL_A(\Omega)$ is studied by A.Benkirane in [1].

Hence, our second purpose is to extend the above Theorem1.1 in the general setting of Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$ and also, we give an application of this generalized result in order to study the previous unilateral problem (2) in the Musielak–Orlicz–Sobolev space $W^mL_\varphi(\Omega)$.

## 2 Preliminary

### 2.1 Musielak–Orlicz function

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$, and let $\varphi$ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:

a) $\varphi(x,.)$ is an N-function [convex; increasing; continuous; $\varphi(x,0) = 0$; $(\forall t > 0)$ $\varphi(x,t) > 0$ ; $\dfrac{\varphi(x,t)}{t} \to 0$ as $t \to 0$; $\dfrac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].

b) $\varphi(.,t)$ is a measurable function.

A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. Equivalently, $\varphi$ admits the representation: $\varphi(y,t) = \int_0^t a(y,\tau)d\tau$, for all $y \in \Omega$ and $t \geqslant 0$, where $a(y,.) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with for all $y \in \Omega$: $a(y,0) = 0, a(y,t) > 0$ for $t > 0$ and $\lim_{t\to+\infty} a(y,t) = +\infty$. The function a(y , .) is called the derivative of $\varphi(y,.)$. The Musielak–Orlicz function $\varphi$ is said to satisfy the $\Delta_2$-condition if there exists $K \geqslant 2$ such that

$$\varphi(y,2t) \leqslant K\varphi(y,t), \quad for\ all\ y \in \Omega \quad and \quad t \geqslant 0.$$

The smallest K is called the $\Delta_2$-constant of $\varphi$. When the last inequality holds only for $t \geqslant$ *some* $t_0 > 0$ then $\varphi$ is said to satisfy the $\Delta_2$-condition near infinity.

### 2.2 Musielak–Orlicz spaces

Let $\varphi$ be a Musielak–Orlicz function, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ a Lebesgue measurable function. In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability.

The set

$$K_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(u) < \infty\}$$

is called the Musielak–Orlicz class. The Musielak–Orlicz spaces $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < +\infty,\ \text{for some } \lambda > 0\}.$$

$K\varphi(\Omega)$ is a convex subset of $L\varphi(\Omega)$. If $\Omega = \mathbb{R}^N$ then $L_\varphi(\mathbb{R}^N)$ is denoted by $L_\varphi$.

Let

$$\varphi^*(x,s) = sup\{st - \varphi(x,t) \ \ /t \geqslant 0\}.$$

That is, $\varphi^*$ is the Musielak–Orlicz function complementary to $\varphi$ in the sense of Young with respect to the variable $s$. For two complementary Musielak–Orlicz functions $\varphi$ and $\varphi^*$ the following inequality is called the Young inequality [20]

$$t.s \leqslant \varphi(x,t) + \varphi^*(x,s) \text{ for all s, t} \geqslant 0\ , x \in \Omega. \tag{3}$$

If $s = a(x,t)$, then

$$t.a(x,t) = \varphi(x,t) + \varphi^*(x,a(x,t)) \text{ for all t} \geqslant 0\ , x \in \Omega. \tag{4}$$

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_{\varphi^*,\Omega} \leqslant 1} \int_\Omega |u(x)v(x)|dx,$$

where $\varphi^*$ is the Musielak–Orlicz function complementary to $\varphi$. These two norms are equivalent [20].

For two complementary Musielak–Orlicz functions $\varphi$ and $\varphi^*$ let $u \in L_\varphi(\Omega)$ and $v \in L_{\varphi^*}(\Omega)$, we have the *Hölder* inequality [20]

$$|\int_\Omega u(x)v(x)dx| \leqslant ||u||_{\varphi,\Omega}\, |||v|||_{\varphi^*,\Omega}. \tag{5}$$

In $L_\varphi(\Omega)$ we have the relation with the norm and the modular:

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) + 1, \tag{6}$$

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}(u) \ \ , \text{if}\, ||u||_{\varphi,\Omega} > 1, \tag{7}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}(u) \ \ , \text{if}\, ||u||_{\varphi,\Omega} \leqslant 1. \tag{8}$$

If $\Omega = \mathbb{R}^N$ then $||u||_{\varphi,\mathbb{R}^N}$ , $|||u|||_{\varphi,\mathbb{R}^N}$ and $\varrho_{\varphi,\mathbb{R}^N}(u)$ are denoted respectively by $||u||_\varphi$, $|||u|||_\varphi$ and $\varrho_\varphi(u)$ $(\forall u \in L_\varphi)$.

We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to+\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

If $\varphi$ satisfies the $\triangle_2$ condition, then modular convergence coincides with norm convergence. The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$ and it is a separable space. The equality $K_\varphi(\Omega) = E_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if $\varphi$ satisfies the $\triangle_2$ condition, for all $t$ or for large $t$ according to whether $\Omega$ has infinite measure or not. The dual of $E_\varphi(\Omega)$ can be identified with $L_{\varphi^*}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ and the dual norm on $L_{\varphi^*}(\Omega)$ is equivalent to $||.||_{\varphi^*,\Omega}$. The space $L_\varphi(\Omega)$ is reflexive if and only if $\varphi$ and $\varphi^*$ satisfies the $\triangle_2$ condition, for all $t$ or for large $t$ according to whether $\Omega$ has infinite measure or not.

**Lemma 2.1** *[12] Let $\varphi$ be a Musielak-Orlicz function and $f_n, f, g$ are measurable functions.*

*(a) If $f_n \longrightarrow f$ almost everywhere, then* $\varrho_{\varphi,\Omega}(f) \leqslant \liminf_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.

*(b) If $|f_n| \nearrow |f|$ almost everywhere, then* $\varrho_{\varphi,\Omega}(f) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.

*(c) If $f_n \longrightarrow f$ almost everywhere, $|f_n| \leqslant |g|$ almost everywhere, and $\varrho_{\varphi,\Omega}(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ strongly in $L_\varphi(\Omega)$.*

**Theorem 2.1** *[12] Let $\varphi$ be a Musielak-Orlicz function.*

*(a) $||f||_{\varphi,\Omega} = ||\ |f|\ ||_{\varphi,\Omega}$ for all $f \in L_\varphi(\Omega)$.*

*(b) If $f \in L_\varphi(\Omega)$, $g$ a measurable function, and $0 \leqslant |g| \leqslant |f|$ almost everywhere, then:*

$$g \in L_\varphi(\Omega) \ \ \text{and} \ \ ||g||_{\varphi,\Omega} \leqslant ||f||_{\varphi,\Omega}.$$

*(c) If* $f_n \to f$ *almost everywhere, then:* $||f||_{\varphi,\Omega} \leqslant \liminf_{n\to+\infty} ||f_n||_{\varphi,\Omega}$ .

*(d) If* $|f_n| \nearrow |f|$ *almost everywhere, with* $f_n \in L_\varphi(\Omega)$ *and* $\sup_n ||f_n||_{\varphi,\Omega} < \infty$ *then:*

$$f \in L_\varphi(\Omega) \ \ \textit{and} \ \ ||f_n||_{\varphi,\Omega} \nearrow ||f||_{\varphi,\Omega} .$$

### 2.3 Musielak–Orlicz–Sobolev spaces

For any fixed non-negative integer $m$ we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall \ |\alpha| \leqslant m, \ D^\alpha u \in L_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \alpha_2, ....., \alpha_n)$ with non-negative integer $\alpha_i$, $|\alpha| = |\alpha_1| + |\alpha_2| + ....|\alpha_n|$ and $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} .....\partial x_n^{\alpha_n}}$ denote the distributional derivatives of u. The $W^m L_\varphi(\Omega)$ is called the Musielak–Orlicz–Sobolev space.

For $u \in W^m L_\varphi(\Omega)$ let:

$$\bar{\varrho}^m_{\varphi,\Omega}(u) = \sum_{|\alpha|\leqslant m} \varrho_{\varphi,\Omega}(D^\alpha u)$$

and

$$||u||^m_{\varphi,\Omega} = inf\{\lambda > 0 : \bar{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}.$$

These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $(W^m L_\varphi(\Omega),||u||^m_{\varphi,\Omega})$ is a Banach space if $\varphi$ satisfies the following condition [20]:

$$(\exists c > 0) : \inf_{x\in\Omega} \varphi(x, 1) \geqslant c. \tag{9}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that:

$$\lim_{n\to+\infty} \bar{\varrho}^m_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

If $\Omega = \mathbb{R}^N$ then $W^m L_\varphi(\Omega)$, $\bar{\varrho}^m_{\varphi,\Omega}(u)$ and $||u||^m_{\varphi,\Omega}$ are denoted respectively by $W^m L_\varphi$ , $\bar{\varrho}^m_\varphi(u)$ and $||u||^m_\varphi$, $\forall u \in W^m L_\varphi$.

**Theorem 2.2** *[7] Let $\varphi$ and $\varphi^*$ be two complementary Musielak–Orlicz functions such that $\varphi$ satisfies the conditions (9) and there exists a constant $A > 0$ such that for all $x, y \in \Omega :$ $|x - y| \leqslant \dfrac{1}{2}$ we have:*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{\dfrac{A}{log(\dfrac{1}{|x-y|})}} \tag{10}$$

*for all $t \geq 1$. If $D \subset \Omega$ is a bounded measurable set, then $\displaystyle\int_D \varphi(x, 1)dx < \infty$. $\varphi^*$ satisfies the following condition :*

$$\exists C > 0 \ : \ \ \varphi^*(x, 1) \leqslant C \ almost \ everywhere \ in \ \ \Omega. \tag{11}$$

*Under the previous conditions, $D(\overline{\Omega})$ is dense in $W^m L_\varphi(\Omega)$ with respect to the modular topology.*

**Theorem 2.3** *[7] Let $\varphi$ be a Musielak–Orlicz functions which satisfies the assumptions of theorem2.2, with $\Omega = \mathbb{R}^N$. Then $D(\mathbb{R}^N)$ is dense in $W^m L_\varphi(\mathbb{R}^N)$ with respect to the modular topology.*

### 2.4 Capacity

**Definition 2.1** Let $T$ the classe of Borel sets in $\mathbb{R}^N$, and a function $C : T \to [0, +\infty]$.

1) C is called capacity if the following axioms are satisfied:

i) $C(\emptyset) = 0$.

ii) $X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all X and Y in T.

iii) For all sequences $(X_n) \subset T$:

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

2) C is called outer capacity if for all $X \in T$ :

$$C(X) = \inf\{C(O) : O \supset X, \quad O \ is \ \ open\}.$$

3) C is called an interior capacity if for all $X \subset T$ :

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ \ is \ compact\}.$$

4) A property, that holds true except perhaps on a set of capacity zero, is said to be true C-quasi-everywhere, ( abbreviated C-q.e).

5) $f$ *and* $(f_n)$ are real-valued finite functions C-q.e. We say that $(f_n)$ converges to $f$ in C-capacity if:

$$\forall \varepsilon > 0, \quad \lim_{n\to+\infty} C(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) $f$ *and* $(f_n)$ are real-valued function finite C-q.e. We say that $(f_n)$ converges to $f$ C-quasi- uniformly, (abbreviated C-q.u) if

$$(\forall \varepsilon > 0), (\exists \ X \in T) \ : \ C(X) < \varepsilon \ \ and \ (f_n) \ \ converges \ to \ f \ uniformly \ on \ X^c.$$

## 3 The Main Results

### 3.1 Preliminary lemma

**Lemma 3.1** *Let $\varphi$ be a Musielak-Orlicz function which satisfies the condition (9). If $u, \ v \in W^m L_\varphi(\Omega)$, then $\max\{u, v\}$ and min$\{u, v\}$ are in $W^m L_\varphi(\Omega)$ with $\forall \ |\alpha| \leqslant m$:*

$$D^\alpha \max\{u, v\}(x) = \begin{cases} D^\alpha u(x), & for \ almost \ \ every \ \ x \in \{u \geqslant v\}; \\ D^\alpha v(x), & for \ almost \ \ every \ x \in \{v \geqslant u\}; \end{cases}$$

*and*

$$D^\alpha \min\{u, v\}(x) = \begin{cases} D^\alpha u(x), & for \ almost \ \ every \ \ x \in \{u \leqslant v\}; \\ D^\alpha v(x), & for \ almost \ \ every \ x \in \{v \leqslant u\}. \end{cases}$$

*In particular, $|u|$ belongs to $W^m L_\varphi(\Omega)$.*

***Proof.*** It suffices to prove the assertions for $\max\{u,v\}$ since $\min\{u,v\} = -\max\{-u,-v\}$ . We have $\max\{u,v\} \leqslant |u|+|v|$ almost everywhere in $\Omega$, and $(|u|+|v|) \in L_\varphi(\Omega)$, then by Theorem 2.1 we obtain $\max\{u,v\} \in L_\varphi(\Omega)$.

On the other hand we have $|D^\alpha \max(u,v)| \leqslant |D^\alpha u| + |D^\alpha v|$ almost everywhere in $\Omega$, and $(|D^\alpha u| + |D^\alpha v|) \in L_\varphi(\Omega)$, then by Theorem 2.1 we obtain $D^\alpha \max\{u,v\} \in L_\varphi(\Omega)$.

Thus

$$\max\{u,v\} \in W^m L_\varphi(\Omega).$$

For $|u| \in W^m L_\varphi(\Omega)$ it suffices to note that $|u| = \max\{u,0\} - \min\{u,0\}$.

### 3.2 Capacity in Musielak–Orlicz–Sobolev space

In this section, $\Omega = \mathbb{R}^N$ and $\varphi$ is a Musielak-Orlicz function which satisfies the condition (9).

**Definition 3.1** The Sobolev $\varphi$-capacity of the set, $E \subset \mathbb{R}^N$ is defined by :

$$C_\varphi(E) = \inf_{u \in A_\varphi(E)} \bar{\rho}_{m,\varphi}(u),$$

where

$$A_\varphi(E) = \{u \in W^m L_\varphi \ : \ u \geqslant 1 \textit{ on an open set containing } E \textit{ and } u \geqslant 0\}.$$

If $A_\varphi(E) = \emptyset$ we set $C_\varphi(E) = \infty$. Functions belonging to $A_\varphi(E)$ are called admissible functions for $E$.

**Remark 3.1** In the definition of the capacity $C_\varphi$, we can restrict ourselves to those admissible functions u for which, $0 \leqslant u \leqslant 1$. Indeed, if $A'_\varphi(E) = \{u \in A_\varphi(E) \ : \ 0 \leqslant u \leqslant 1\}$, then $A'_\varphi(E) \subset A_\varphi(E)$ implies

$$C_\varphi(E) \leqslant \inf_{u \in A'_\varphi(E)} \bar{\rho}_{m,\varphi}(u).$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Then by Lemma 3.1 , we have $v = max(0, min(u,1))$ belongs to $A'_\varphi(E)$.

Therefore,

$$\inf_{\omega \in A'_\varphi(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant \bar{\rho}_{m,\varphi}(v) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, we obtain

$$\inf_{\omega \in A'_\varphi(E)} \bar{\rho}_{m,\varphi}(\omega) \leqslant C_\varphi(E).$$

This completes the proof.

**Theorem 3.1** *Let $E \subset \mathbb{R}^N$. If there exists $f \in W^m L_\varphi$ such that $f = +\infty$ on $E$, then $C_\varphi(E) = 0$.*

***Proof.*** If there exists $f \in W^m L_\varphi$ such that $f = +\infty$ on $E$, then $f \geqslant \alpha$ $on$E for all $\alpha > 0$. Therefore, $\forall\ \alpha > 0 \quad : C_\varphi(E) \leqslant \bar{\rho}_{m,\varphi}(\dfrac{f}{\alpha})$.

Let $\alpha > 1$, we have $\quad \bar{\rho}_{m,\varphi}(\dfrac{f}{\alpha}) \leqslant \dfrac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$, then $0 \leqslant C_\varphi(E) \leqslant \dfrac{1}{\alpha}\bar{\rho}_{m,\varphi}(f)$.

Letting $\alpha \longrightarrow +\infty$, we obtain $C_\varphi(E) = 0$.

**Theorem 3.2** *Let us consider the following propositions:*

*i)* $f_n \longrightarrow f$ *in* $W^m L_\varphi$.

*ii)* $f_n \longrightarrow f$ *in* $C_\varphi -$ *capacity.*

*iii) there is a subsequence* $(f_{n_j})$ *such that :* $f_{n_j} \longrightarrow f, \quad C_\varphi - q.u.$

*iv)* $f_{n_j} \longrightarrow f, \quad C_\varphi - q.e.$

*We have* $i) \Rightarrow\ ii) \Rightarrow\ iii) \Rightarrow iv)$.

***Proof.*** Let show that $i) \Rightarrow ii)$. By Theorem 3.1 we have $f$ and $f_n$ are finite for every $n$; $C_\varphi - q.e.$

Let $\varepsilon > 0$, we have

$$C_\varphi(\{x \quad : \quad |f_n - f|(x) > \varepsilon\}) \leqslant \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}).$$

Since $\quad f_n \longrightarrow f \ \ in \ \ W^m L_\varphi(\Omega)$,

$$(\forall \varepsilon > 0) : \quad \bar{\rho}_{m,\varphi}(\frac{f_n - f}{\varepsilon}) \longrightarrow 0.$$

Therefore,

$$\lim_{n \to +\infty} C_\varphi(\{x \quad : \quad |f_n - f|(x) > \varepsilon\}) = 0.$$

Let show that $ii) \Rightarrow\ iii)$. Let $\varepsilon > 0\ \ \exists\ f_{n_j}$ such that $C_\varphi(\{x\ :\ |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}$.

We put

$$E_j = \{x\ :\ |f_{n_j} - f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j,$$

we have $C_\varphi(G_m) \leqslant \displaystyle\sum_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.

On the other hand,

$$(\forall x \in (G_m)^c)\ :\ |f_{n_j} - f|(x) \leqslant 2^{-j}, (\forall j \geqslant m).$$

Thus

$$f_{n_j} \longrightarrow f \ \ C_\varphi - q.u.$$

Let show that $iii) \Rightarrow\ iv)$. We have $\forall j \in \mathbb{N}, \exists X_j : C_\varphi(X_j) \leqslant \dfrac{1}{j}$ and $f_{n_j} \longrightarrow f \ \ on \ \ (X_j)^c$.

We put $X = \displaystyle\bigcap_j X_j$, then $C_\varphi(X) = 0 \ \ and \ \ f_{n_j} \longrightarrow f \ \ on\ X^c$.

**Theorem 3.3** *Let* $\varphi$ *be a Musielak-Orlicz function, uniformly convex that satisfies the* $\triangle_2$ *condition. If* $f_n,\ f \in W^m L_\varphi$ *such that* $f_n \rightharpoonup f$ *weakly in* $W^m L_\varphi$, *then*

$$\liminf (f_n)(x) \leqslant f(x) \leqslant \limsup f_n(x) \quad C_\varphi - q.e.$$

***Proof.*** $(W^mL_\varphi\ ,\ ||.||)$ is uniformly convex, therefore reflexive. By the Banach–Saks theorem, there is a subsequence denoted again by $(f_n)$ such that the sequence $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converges to f strongly in $W^mL_\varphi$. By Theorem 3.2, there is a subsequence of $(g_n)$ denoted again $(g_n)$ such that

$$\lim_{n\to+\infty} g_n(x) = f(x) \quad C_\varphi - q.e.$$

On the other hand,

$$\liminf f_n(x) \leqslant \lim_{n\to+\infty} g_n(x).$$

Therefore,

$$\liminf_{n\to+\infty} f_n(x) \leqslant f(x) \quad C_\varphi - q.e.$$

For the second inequality, it suffices to replace $f_n$ by $(-f_n)$ in the first inequality.

**Theorem 3.4** *Let $\varphi$ be a Musielak–Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. Let $(X_n)$ be an increasing sequence of sets and $X = \bigcup_n X_n$. Then*

$$\lim_{n\to+\infty} C_\varphi(X_n) = C_\varphi(X).$$

***Proof.*** We have $\lim_{n\to+\infty} C_\varphi(X_n) \leqslant C_\varphi(X)$. For the reverse inequality, if $\lim_{n\to+\infty} C_\varphi(X_n) = +\infty$, there is nothing to show.

Assuming that $\lim_{n\to+\infty} C_\varphi(X_n) < +\infty$, we have

$$\forall n \in \mathbb{N},\ \ \exists f_n \in W^mL_\varphi\ :\ \ f_n \geqslant 1 \ on\ \ X_n\ \ and\ \ \bar{\varrho}_{m,\varphi}(f_n) \leqslant C_\varphi(X_n) + \frac{1}{n}.$$

Now $(f_n)$ is a bounded sequence in $W^mL_\varphi$, hence there exists a subsequence, which we denote again by $(f_n)$, which converges weakly to a function $f \in W^mL_\varphi$. Thus

$$\bar{\rho}_{m,\varphi}(f) \leqslant \liminf_n \bar{\varrho}_{m,\varphi}(f_n).$$

On the other hand by Theorem 3.3, we have

$$\forall n \in \mathbb{N}\ : f \geqslant 1 \ \ on\ \ X_n\ \ , C_\varphi - q.e.$$

Therefore, $f \geqslant 1\ \ on\ \ X\ \ C_\varphi - q.e.$

Let $Y$ be a subset of $X$ where $f \geqslant 1$, then $C_\varphi(X) = C_\varphi(Y)$. Thus,

$$\bar{\rho}_{m,\varphi}(f) \leqslant \lim_n (C_\varphi(X_n) + \frac{1}{n}).$$

Hence

$$C_\varphi(X) \leqslant \lim_n (C_\varphi(X_n).$$

**Theorem 3.5** *Let $\varphi$ be a Musielak–Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. $C_\varphi$ is an outer capacity.*

***Proof.*** It is obvious that $C_\varphi(\emptyset) = 0$ and $C_\varphi(X) \leqslant C_\varphi(Y)$ if $X \subset Y$.

To prove the countable sub-additivity, suppose that $E_i$ , $i = 1, 2, \ldots,$ subsets of $\mathbb{R}^N$, let $\varepsilon > 0$. We may assume that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then

$$C_{k,\varphi}(X_i) < +\infty; \quad \forall i \in \mathbb{N}.$$

Next we choose $u_i \in A_\varphi(E_i)$ so that

$$\bar{\rho}_{m,\varphi}(u_i) \leqslant C_\varphi(E_i) + \varepsilon.2^{-i}; \quad \forall i \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and $v_k = \max_{1 \leqslant i \leqslant k} u_i$. By Lemma 3.1 we have $v_k \in A_\varphi(\bigcup_{i=1}^{k} E_i)$.

Thus,

$$\bar{\rho}_{m,\varphi}(v_k) \leqslant \sum_{i=1}^{k} \bar{\rho}_{m,\varphi}(u_i) \leqslant \sum_{i=1}^{k} (C_\varphi(E_i) + \varepsilon.2^{-i}) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + (\varepsilon(1 - (\frac{1}{2})^k).$$

Then,

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$C_\varphi(\bigcup_{i=1}^{k} E_i) \leqslant \sum_{i=1}^{k} C_\varphi(E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

Since $(\bigcup_{i=1}^{k} E_i)$ increase to $(\bigcup_{i=1}^{\infty} E_i)$, by Theorem 3.4 we obtain:

$$C_\varphi(\bigcup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} C_\varphi(E_i).$$

It remains to prove that $C_\varphi$ is outer. Indeed, by monotonicity we have:

$$(\forall\ E \subset \mathbb{R}^N\ ) \ : C_\varphi(E) \leqslant \inf\{C_\varphi(O) : O \supset E, \ \ O \ is\ open\}.$$

For the reverse inequality, if $C_\varphi(E) = +\infty$, there is nothing to show.

Assume that $C_\varphi(E) < +\infty$, let $\varepsilon > 0$ and take $u \in A_\varphi(E)$ such that

$$\bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

Since $u \in A_\varphi(E)$, there is an open set O containing E such that $u \geqslant 1$ *on* $O$, which implies that

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u) \leqslant C_\varphi(E) + \varepsilon.$$

The inequality follows by letting $\varepsilon \to 0$.

**Theorem 3.6** *Let $(K_n)$ be a decreasing sequence of compacts and $K = \bigcap_n K_n$. Then,*

$$\lim_{n\to+\infty} C_\varphi(K_n) = C_\varphi(K).$$

***Proof.*** First, we observe that $C_\varphi(K) \leqslant \lim_{n\to+\infty} C_\varphi(K_n)$. On the other hand, let $O$ be an open set containing $K$. By the compactness of $K$, $K_i \subset O$ for all sufficiently large $i$. Therefore $\lim_{n\to+\infty} C_\varphi(K_n) \leqslant C_\varphi(O)$, and since $C_\varphi$ is an outer capacity, we obtain the claim by taking infimum over all open set $O$ containing $K$.

**Theorem 3.7** *Let $\varphi$ be a Musielak–Orlicz function.*

$$(\exists c > 0)(\forall X \subset \mathbb{R}^N): \quad |X| \leqslant c.C_\varphi(X),$$

*where $|X|$ is the Lebesgue's measure of $X$.*

***Proof.*** Let $u \in A_\varphi(X)$, we have $u \geqslant 1$ on $X$ and $\varrho_\varphi(u) \leqslant \bar{\rho}_{m,\varphi}(u)$. But $\varrho_\varphi(u) = \int_{\mathbb{R}^N} \varrho_\varphi(y, |u(y)|dy$, then

$$\varrho_\varphi(u) \geqslant \int_X \varrho_\varphi(y, |u(y)|dy \geqslant \int_X \varrho_\varphi(y, 1)dy.$$

By the inequality (9) there exists a constant $c > 0$ such that $\inf_{y\in\mathbb{R}^N} \varphi(y, 1) \geqslant c$. Therefore, $\varrho_\varphi(u) \geqslant c.|X|$. Thus,

$$c.|X| \leqslant \bar{\rho}_{m,\varphi}(u).$$

The claim follows by passing to inf on $u \in A_\varphi(X)$.

**Corollary 3.1** *Let $\varphi$ be a Musielak–Orlicz function. If $(f_n)_n$ is a sequence which converges to $f$ in $W^m L_\varphi$, then there exists a subsequence of $(f_n)_n$ which converge to $f$ almost everywhere.*

***Proof.*** It is an immediate consequence of Theorem 3.2 and Theorem 3.7.

**Theorem 3.8** *Let $\varphi$ be a Musielak–Orlicz function which satisfies the condition $\triangle_2$ and the assumptions of Theorem 2.2. For each $f \in W^m L_\varphi$, there is a $C_\varphi$-quasicontinuous function $g \in W^m L_\varphi$ such that $f = g$ almost everywhere.*

***Proof.*** Let $f \in W^m L_\varphi$. By Theorem 2.3, there exists a sequence $(f_n)$ in $D(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in $W^m L_\varphi$. By Theorem 3.2, there exists a subsequence of $(f_n)$ denoted again by $(f_n)$ such that $f_n \longrightarrow f \;\; C_\varphi - q.u$. The claim follows by Theorem 3.7.

**Remark 3.2** By theorem 2.6 in [7], we have the same result if we replace $W^m L_\varphi$, by $W^m L_\varphi(\Omega)$, where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$.

**Theorem 3.9** *Let $\varphi$ be a Musielak-Orlicz function, uniformly convex which satisfies the condition $\triangle_2$*

*1) If $O$ is an open set of $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ such that $|E| = 0$, then*

$$C_\varphi(O) = C_\varphi(O - E).$$

*2) Let $u$ and $v$ are $C_\varphi$-quasicontinuous functions in $\mathbb{R}^N$, we have*
*i) if $u = v$, almost everywhere in an open set $O \subset \mathbb{R}^N$, then*

$$u = v \ C_\varphi - quasieverywhere \ in \ O,$$

*ii) if , $u \leqslant v$, almost everywhere in an open set $O \subset \mathbb{R}^N$ , then*

$$u \leqslant v \ C_\varphi - quasieverywhere \ in \ O.$$

***Proof.*** 1) It obvious that $C_\varphi(O) \geqslant C_\varphi(O - E)$. Let $u \in A_\varphi(O - E)$ thus $u \geqslant 1$ in an open containing $O - E$. Let the function $f$ define as

$$\begin{cases} f(x) = u(x), & if \ x \in \mathbb{R}^N - E \\ f(x) = 1, & if \ x \in E. \end{cases}$$

We have $f \in A_\varphi(O)$ and $\bar{\rho}_{m,\varphi}(f) = \bar{\rho}_{m,\varphi}(u)$, thus

$$C_\varphi(O) \leqslant \bar{\rho}_{m,\varphi}(u),$$

and by passing to inf we get $C_\varphi(O) \leqslant C_\varphi(O - E)$.

2) Since $C_\varphi$ is an outer capacity we get the results by [16].

**Lemma 3.2** *Let $\Omega$ be a bounded Lipschitz domain in $R^n$, $\varphi$ be a Musielak-Orlicz function which satisfies the condition (9), $\varphi$ and $\varphi^*$ satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $T \in W^{-m}L_{\varphi^*}(\Omega) \cap M(\Omega)$, where $M(\Omega)$ denote the set of Radon measures in $\Omega$. If $X \subset \Omega$ is such that $C_\varphi(X) = 0$, then $X$ is $|T|$ -measurable and $|T|(X) = 0$.*

***Proof.*** It is the same as in [19] and [10].

### 3.3 Theorem of H. Brezis and F. Browder type in Musielak–Orlicz–Sobolev spaces

In this section we generalize the theorem of H. Brezis and F. Browder [10] in the setting of the Musielak–Orlicz–Sobolev spaces $W^m L_\varphi(\Omega)$.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. In this section we study the following question: let $w \in W_0^m L_\varphi\ (\Omega)$ and $T \in W^{-m}L_{\varphi^*}(\Omega)$ such that $T = \mu + h$, where $\mu$ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and h lie $L^1_{loc}(\Omega)$; find sufficient conditions on the data in order for $w$ to belong $L^1(\Omega; d\mu)$, for $hw$ to belong to $L^1(\Omega)$ and finally to have:

$$< T, w >= \int_\Omega w d\mu + \int_\Omega h w dx.$$

This question was solved in [15] in the case of the classical Sobolev spaces, in [5] when $\mu = 0$ in the case of Orlicz–Sobolev spaces and in [1] in the case of Orlicz–Sobolev spaces.

**Theorem 3.10** *Let $\varphi$ be a Musielak–Orlicz function which satisfies the condition (9), $\varphi$ and $\varphi^*$ satisfy the $\triangle_2$ condition and $m \in \mathbb{N}$. Consider $w \in W_0^m L_\varphi(\Omega)$, $w \geqslant 0$ a.e in $\Omega$ and $T \in W^{-m}L_{\varphi^*}(\Omega)$ such that $T = \mu + h$, where $\mu$ lie in $M^+(\Omega)$ (the subset of positive Radon measures) and $h \ \in L^1_{loc}(\Omega)$, assume that:*

$$hw \geqslant -|\Phi| \ \ a.e \ in \ \Omega \ for \ some \ \ \Phi \ in \ L^1(\Omega). \tag{12}$$

*Then:*

$$hw \in L^1(\Omega), w \in L^1(\Omega; d\mu) \ and \ < T, w >= \int_\Omega w d\mu + \int_\Omega h w dx. \tag{13}$$

**Remark 3.3** Note that $\mu(X) = 0$ for all $X \subset \Omega$ such that $C_\varphi(X) = 0$. Indeed by Lemma 3.2

$$|T|(X) = |\mu + h|(X) = 0,$$

but

$$0 \leqslant \mu(X) \leqslant |h|(X) + |\mu + h|(X) = 0.$$

Let prove Theorem 3.10.

***Proof.*** Let $w \in W_0^m L_\varphi(\Omega)$, the Lemma 2.4 of [9] yields the existence of a sequence $w_n$ such that:
**(i)** $w_n \in W_0^m L_\varphi(\Omega) \cap L^\infty(\Omega)$,
**(ii)** supp $w_n$ is compact,
**(iii)** $|w_n| \leqslant |w|$ a.e. in $\Omega$,
**(v)** $w_n \longrightarrow w$ in $W_0^m L_\varphi(\Omega)$.
**(vi)** $w_n w \geqslant 0$ a.e. in $\Omega$.

Following the lines of [15], it is easy to deduce that

$$< \mu + h, w_n >= \int_\Omega w_n d\mu + \int_\Omega h w_n dx. \tag{14}$$

Since $w_n \longrightarrow w$ in $W_0^m L_\varphi(\Omega)$, by using the Theorem 3.2, Lemma3.2 and Remark 3.3 we have

$$w_n \longrightarrow w \quad \mu.a.e \;\; and \;\; a.e. \;\; in \;\; \Omega. \tag{15}$$

We recall that by Theorem 3.9 and Theorem 3.7, for any $v \in W^m L_\varphi(\Omega)$ one has

$$v \geqslant 0 \quad a.e. \;\; in \; \Omega \Leftrightarrow v \geqslant 0 \quad q.e. \;\; in \; \Omega.$$

This equivalence, Remark 3.3 and the fact ($w \geqslant 0$ a.e. in $\Omega$), imply

$$w_n \geqslant 0 \quad a.e. \;\; , \;\; w_n \geqslant 0 \quad \mu.a.e. \quad and \;\; 0 \leqslant w_n \leqslant w \;\; a.e. \;\; in \;\; \Omega. \tag{16}$$

On the other hand from $hw \geqslant -|\Phi|$ and $0 \leqslant w_n \leqslant w$ $a.e.$ $in$ $\Omega$ we have

$$h w_n \geqslant -|\Phi| \; a.e.in \;\; \Omega \tag{17}$$

Since $< \mu + h, w_n >$ is bounded, (14 )and (16) imply $\int_\Omega h w_n dx \leqslant cst$; Similary (14 ) and (17 ) imply $\int_\Omega w_n d\mu \leqslant cst$.

By using (15), (16), (17) and Fatou's lemma we get $hw \in L^1(\Omega)$ and $w \in L^1(\Omega; d\mu)$. Using $0 \leqslant w_n \leqslant w$ $\mu.a.e.$ $in$ $\Omega$ and $|hw_n| \leqslant |hw|$ a.e. in $\Omega$, it is now easy to pass to the limit in (14); we use the convergence of $w_n$ to $w$ in $W_0^m L_\varphi$ $(\Omega)$ for the left hand side and Lebesgue's dominated convergence theorem in each term of the right hand side: we obtain

$$< T, w >= \int_\Omega w d\mu + \int_\Omega h w dx.$$

### 3.4 Application to unilateral problem

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and $m \in \mathbb{N}$. $\varphi$ be a Musielak-Orlicz function which satisfies the condition (9), $\varphi$ and $\varphi^*$ satisfy the $\triangle_2$ condition.

We consider some right hand side $f \in W^{-m}L_{\varphi^*}(\Omega)$ and the convex set

$$K_\Phi = \{v \in W_0^m L_\varphi(\Omega),\ v \geqslant \Phi\ a.e\ in\ \Omega\},$$

where the obstacle $\Phi$ belong to $W_0^m L_\varphi(\Omega) \cap L^\infty(\Omega)$. Let a pseudo-monotone mapping $S$ from $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_{\varphi^*}(\Omega)$. which satisfies the following conditions:
**(1)** $S$ is continuous from each finite-dimensional subspace of $W_0^m L_\varphi(\Omega)$ into $W^{-m}L_{\varphi^*}(\Omega)$ for the weak$^*$ topology.
**(2)** $S$ maps bounded sets into bounded sets.
**(3)** $S$ is coercive, i.e that for some $v_0$ in $K_\Phi \cap L^\infty(\Omega)$

$$\frac{< S(v), v - v_0 >}{||v||_{W_0^m L_\varphi(\Omega)}} \longrightarrow +\infty \quad as \quad ||v||_{W_0^m L_\varphi(\Omega)} \longrightarrow +\infty. \tag{18}$$

Consider finally a carathéodory function $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ witch satisfies :
**(4)** $s.g(x,s) \geqslant 0,\ \forall\ s \in \mathbb{R}\ \ and\ \ a.e\ \ in\ \ \Omega,$
**(5)** $h_t = sup_{|s| \leqslant t}|g(x,s)| \in L^1(\Omega)\ \forall t\ \geqslant 0.$

**Theorem 3.11** *The variational inequality:*

$$u \in K_\Phi,\ g(.,u) \in L^1(\Omega),\ ug(.,u) \in L^1(\Omega)$$

$$< Su, v - u > + \int_\Omega g(.,u)(v-u)dx \geqslant \quad < f, v - u >,\ \forall v \in K_\Phi \cap L^\infty(\Omega)$$

*has at least one solution.*

***Proof.*** **First part** *Approximation and a priori istimates.*
Define $g_n(x,s) = \begin{cases} \chi_n(x) g(x,s) \ \ if \ \ |g(x,s)| \leqslant n, \\ \chi_n(x) n \dfrac{g(x,s)}{|g(x,s)|} \ \ if \ \ |g(x,s)| > n, \end{cases}$
where $\chi_n$ is the characteristic function of the set $\{x \in \Omega : |x| \leqslant n\}$
By by using the proposition 1 of [14] we have the approximate problem

$$\begin{cases} u_n \in K_\Phi, \\ < Su_n, v - u_n > + \displaystyle\int_\Omega g_n(.,u_n)(v-u_n)dx \geqslant \quad < f, v - u_n >,\ \forall v \in K_\Phi \cap L^\infty(\Omega) \end{cases} \tag{19}$$

has at least one solution. Using $v = v_0$ as test function in (19) we get

$$< Su_n, u_n - v_0 > + \int_\Omega g_n(.,u_n)(u_n - v_0)dx \leqslant \ < f, u_n - v_0 > . \tag{20}$$

If $(u_n)$ is not bonded in $W_0^m L_\varphi(\Omega)$ then by the assumptions **(3)** we have

$$(\forall A > 0)(\exists n_0 \in \mathbb{N}) : (\forall n \geqslant n_o)(\frac{< S(u_n), u_n - v_0 >}{||u_n||_{W_0^m L_\varphi(\Omega)}} > A). \tag{21}$$

Let $E_n = \{x \in \Omega : u_n(x) \geqslant 0\}$, by (20) and (21) we have for large $n$ : $A||u_n||_{W_0^m L_\varphi(\Omega)} + \int_{E_n} g_n(.,u_n)(u_n - v_0)dx + \int_{\Omega - E_n} g_n(.,u_n)u_n dx$
$\leqslant \int_{\Omega - E_n} g_n(.,u_n)v_0 dx + ||f||_{W^{-m}L_{\varphi^*}(\Omega)}||u_n||_{W_0^m L_\varphi(\Omega)} + ||f||_{W^{-m}L_{\varphi^*}(\Omega)}||v_0||_{W_0^m L_\varphi(\Omega)}$
Let $G_n = \{x \in \Omega : u_n(x) \geqslant v_o\}$ and $l = sup(|v_0|, |\Phi|)$.

By the assumptions **(4)** and **(5)** we have

$$\int_{E_n \cap G_n} g_n(.,u_n)(u_n - v_0)dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(.,u_n)u_n dx \geqslant 0,$$

$$\int_{E_n \cap G_n^c} g_n(.,u_n)v_0 dx \leqslant \int_\Omega |h_{||l||_{L^\infty(\Omega)}} v_0|,$$

$$\int_{\Omega - E_n} g_n(.,u_n)u_n dx \geqslant 0,$$

$$\int_{\Omega - E_n} g_n(.,u_n)v_0 dx \leqslant \int_\Omega |h_{||\Phi||_{L^\infty(\Omega)}} v_0|.$$

Then we get

$$||u_n||_{W_0^m L_\varphi(\Omega)} \leqslant C_1, \ \forall n \geqslant n_0,$$

which is impossible, thus $(u_n)$ is bounded in $W_0^m L_\varphi(\Omega)$.

It follows that there exists a subsequence, again denoted by $u_n$ such that

$$u_n \rightharpoonup u, \ \ weakly \ \ in \ \ W_0^m L_\varphi(\Omega) \ \ and \ \ a.e. \ \ in \ \ \Omega.$$

Thus

$$g_n(x, u_n(x)) \longrightarrow g(x, u(x)) \ \ a.e. \ \ in \ \ \Omega.$$

From (20) we get

$$\int_\Omega g_n(.,u_n)(u_n - v_0)dx \leqslant C_2. \tag{22}$$

We shall prove

$$\int_\Omega |g_n(.,u_n)(u_n - v_0)|dx \leqslant C_3.$$

Indeed

$$\begin{aligned}
\int_\Omega |g_n(.,u_n)(u_n - v_0)|dx &= \int_{G_n} g_n(.,u_n)(u_n - v_0)dx - \int_{\Omega - G_n} g_n(.,u_n)(u_n - v_0)dx \\
&= -2\int_{\Omega - G_n} g_n(.,u_n)(u_n - v_0)dx + \int_\Omega g_n(.,u_n)(u_n - v_0)dx \\
&\leqslant C_2 + 2\int_{\Omega - G_n} g_n(.,u_n)v_0 dx \\
&\leqslant C_2 + 2\int_\Omega |h_{||b||_{L^\infty}} v_0|dx = C_3,
\end{aligned} \tag{23}$$

where $b = sup(|\Phi|, |v_0|)$.

In order to prove

$$g_n(., u_n) \longrightarrow g(., u) \ \ in \ \ L^1(\Omega), \tag{24}$$

let us observe that, for any $\delta > 0$,

$$|g_n(x, u_n(x))| \leqslant \sup_{|t| \leqslant \delta^{-1} + ||v_0||_{L^\infty}} |g(., t)| + \delta |g_n(x, u_n(x))(u_n(x) - v_0(x))|,$$

and there fore, fore any measurable set $E$ in $\Omega$ we have

$$\int_E |g_n(., u_n)| dx \leqslant \int_E |h_{\frac{1}{\delta} + ||v_0||_{L^\infty}}| + \delta C_3.$$

By Vitali's theorem, we obtain (24).

Furthermore by (22) we have

$$\int_\Omega g_n(., u_n) u_n dx \leqslant C_2 + \int_\Omega g_n(., u_n) v_0 dx.$$

By Fatou's lemma and (24), we get

$$0 \leqslant \int_\Omega g(., u) u dx \leqslant C_2 + \int_\Omega g(., u) v_0 dx.$$

Thus

$$g(., u) u \in L^1(\Omega).$$

**Second part** : *Passing to the limit in (19)*

Let

$$\mu_n = S u_n - f + g_n(., u_n).$$

From (19) it is clear that $\mu_n \in M^+(\Omega)$. Since $S$ maps bounded sets of $W_0^m L_\varphi\ (\Omega)$ in to bounded sets of $W^{-m} L_{\varphi^*}\ (\Omega)$, then we can assume for the same sequence that

$$S u_n \rightharpoonup \chi \ \ weakly \ \ in \ W^{-m} L_{\varphi^*}\ (\Omega),$$

which implies that

$$\mu_n \longrightarrow \mu \ \ in \ \ D^{'}(\Omega),$$

where

$$\mu = \chi - f + g(., u).$$

We put $w = u - \Phi,\ h = -g(., u) \ \ and \ \ T = \mu + h.$

The assumptions of theorem 3.10 are satisfied since $T = \chi - f \in W^{-m} L_{\varphi^*}\ (\Omega)$ and $h \in L^1(\Omega)$. Thus

$$\left\{ \begin{array}{l} u - \Phi \in L^1(\Omega; d\mu), \\ < \chi - f, u - \Phi >= \displaystyle\int_\Omega (u - \Phi) d\mu - \int_\Omega g(., u)(u - \Phi) dx. \end{array} \right. \tag{25}$$

Using $v = \Phi$ as test function in (19 ) we get

$$< S u_n, u_n > \quad \leqslant \quad < S u_n, \Phi > - < f, \Phi - u_n > + \int_\Omega g_n(., u_n)(\Phi - u_n),$$

which gives passing to the limit and then using (25)

$$\begin{cases} limsup_n < Su_n, u_n > \leqslant < \chi, \Phi > - < f, \Phi - u > + \int_\Omega g(., u)(\Phi - u)dx, \\ \qquad\qquad \leqslant < \chi, u > + \int_\Omega (\Phi - u)d\mu \leqslant < \chi, u >; \end{cases} \tag{26}$$

since, by theorem 3.9 we have

$$(\Phi - u) \leqslant 0 \quad \mu.a.e. \quad in \;\; \Omega. \tag{27}$$

Using (26 ) and since $S$ is a pseudo-monotone operator, we obtain

$$\chi = Su \quad and \quad < Su_n, u_n > \longrightarrow < Su, u > .$$

It is now easy to pass to the limit in (19) for any fixed $v \in K_\Phi \cap L^\infty(\Omega)$.

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# Function Projective Dual Synchronization of Chaotic Systems with Uncertain Parameters

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**Abstract:** This paper mainly concerns with the general methods for the function projective dual synchronization of a pair of chaotic systems with unknown parameters. The adaptive control law and the parameter update law are derived to make the states of a pair of chaotic systems asymptotically synchronized up to a desired scaling function by Lyapunov stability theory. The general approach for function projective dual synchronization of Lü system and Lorenz system is provided. Numerical simulation results show that the proposed method is effective and convenient.

**Keywords:** *function projective; dual synchronization; adaptive control; uncertain parameters; Lyapunov stability theory.*

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The essence of studying chaotic systems is to understand their structure and behavior. These systems are deemed important as they reflect and model natural phenomena. One of the main reasons for studying chaotic systems lies in the interest of controlling chaos. Many areas have branched from this study due to practical applications in many fields. The main property of chaotic dynamics is its critical sensitivity to initial conditions which is responsible for initially neighboring trajectories separating from each other exponentially in the course of time. For many years, this feature made chaos undesirable, insofar as the sensitivity to initial conditions of chaotic systems reduces their predictability over long time scales. On the other hand, the capability of chaotic dynamics to amplify small perturbations improves their utility for reaching specific desired states with very high flexibility and low energy cost. In contrast, the process of controlling chaos is directed to improving a desired behavior by making only small time-dependent perturbations in an

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accessible system parameter or dynamical variable. Therefore, understanding the behavior of chaos is crucial in the process of seeking beneficial applications to our lives [1,2,4].

The study of synchronization has been widely explored in a variety of systems including physical, chemical and ecological systems. In the broadest sense, synchronization is often understood as the tendency to undergo resembling evolution in time. Synchronization is an important mechanism for creating order in complex systems. Many nonlinear dynamical systems have been found to show a kind of behavior known as chaos, being characterized as chaotic systems by their extreme sensitivity to initial conditions and having noise-like behaviors. Several types of synchronization behaviors have been demonstrated and identified, such as, complete synchronization [3], phase synchronization [5, 6], anti-phase synchronization [7–10], lag synchronization [11, 12], generalized synchronization [13], projective synchronization [14–21] and so on. Function projective synchronization, which is the generalization of projective synchronization, is one of the important synchronization methods that have been widely investigated to obtain faster communication with its proportional feature. function projective synchronization means that the drive and response systems could be synchronized up to a scaling function. Recently, many authors have investigated the function projective synchronization. It is obvious that the unpredictability of the scaling function in function projective synchronization can additionally improve the security of communication [22–25].

However, the theory of dual synchronization has been intensively reviewed and studied recently. The first study on dual synchronization of chaotic systems has been reported by Tsimring and Sushchik in 1996 in [26]. Later, several dual synchronization methods have been reported, for example, dual synchronization of one dimensional discrete chaotic systems was undertaken in [27], where the authors achieved dual synchronization via specific classes of piecewise-linear maps with conditional linear coupling. In [28], the authors experimentally demonstrated dual synchronization of chaos in two pairs of microchip lasers in a one-way coupling configuration over one transmission channel. In [29], the authors demonstrated that dual synchronization of Lorenz and Rössler systems can be obtained by using the means of Lyapunov stabilization theory. In [30], the authors addressed dual synchronization via output feedback strategy in two different chaotic systems. In [31], the authors achieved dual synchronization of modulated time-delayed system by designing a delay feedback controller. In [32], the author investigated the existence of projective-dual-anticipating, projective-dual, and projective-dual-lag synchronization in a coupled time-delayed systems with modulated delay time using Krasovskii–Lyapunov stability theory. In [33], the authors studied the problem of dual synchronization of two different fractional-order chaotic systems by a linear controller. Finally, in [34–37], the authors investigated dual synchronization and dual anti-dual synchronization using nonlinear and adaptive control. To the best of our knowledge, the function projective dual synchronization of chaotic systems with unknown parameters has not yet been studied by any researcher. Inspired by the previous works, in this paper we propose a new analytic treatment of function projective dual synchronization of chaotic systems using adaptive control method in which a state variable of the drive system dual synchronizes with the state variable of the response system up to a scaling function. Numerical simulations are carried out for adaptive function projective dual synchronization behavior of two chaotic systems with uncertain parameters which are depicted through figures for different particular cases.

## 1 Problem Statement

Consider the following two chaotic systems with uncertain parameters as the drive system:

$$\begin{cases} \dot{x}_1 = f_1(x_1) + F_1(x_1)\alpha, \\ \dot{y}_1 = g_1(y_1) + G_1(y_1)\beta, \end{cases} \tag{1}$$

where $x_1 = (x_{11}, x_{12}, ..., x_{1n})^T \in R^n$ and $y = (y_{11}, y_{12}, ..., y_{1n})^T \in R^n$ are the state vectors of the systems, $f_1 : R^n \to R^n$ and $g_1 : R^n \to R^n$ are two continuous vector functions, $F_1 : R^n \to R^{n\times m}$, $G_1 : R^n \to R^{n\times m}$ are two matrix functions and $\alpha, \beta \in R^m$ are the unknown parameter vectors of the two drive systems. The systems studied in this paper depend linearly on the parameters and many resemble well-known chaotic systems. By a linear combination of the drive systems states, a scalar signal is generated in the form of

$$\varepsilon_d = \sum_{i=1}^{n} (a_i x_1 + b_i y_1) = A^T x_1 + B^T y_1 = C^T x, \tag{2}$$

where $A = (a_1, a_2, ..., a_n)^T$ and $B = (b_1, b_2, ..., b_n)^T$ are known matrices and $C = (A^T \ B^T)^T$ and $x = (x_1^T \ y_1^T)^T$. This generated scalar signal is fed to the response systems which are corresponding to the drive systems. The response systems are

$$\begin{cases} \dot{x}_2 = f_2(x_2) + F_2(x_2)\hat{\alpha} + u_1, \\ \dot{y}_2 = g_2(y_2) + G_2(y_2)\hat{\beta} + u_2, \end{cases} \tag{3}$$

where $x_2 = (x_{21}, x_{22}, ..., x_{2n})^T \in R^n$ and $y_2 = (y_{21}, y_{22}, ..., y_{2n})^T \in R^n$ are the state vectors, $f_2 : R^n \to R^n$ and $g_2 : R^n \to R^n$ are two continuous vector functions, $F_2 : R^n \to R^{n\times m}$, $G_2 : R^n \to R^{n\times m}$ are two matrix functions and $\hat{\alpha}, \hat{\beta} \in R^m$ represent the estimated vectors of unknown parameter vectors $\alpha, \beta$ and $u = (u_1 \ \ u_2)^T \in R^{2n}$ is a controller. By the linear combination of the response systems states a scalar signal is generated in the form of

$$\varepsilon_r = \sum_{i=1}^{n} (a_i x_2 + b_i y_2) = A^T x_2 + B^T y_2 = C^T y. \tag{4}$$

Our goal is to obtain the function projective dual synchronization between the drive and the response systems. Now define the error function between the drive and the response systems as $e_s = \varepsilon_r - h(t)\varepsilon_d = C^T(y - h(t)x)$, where $h(t) = \mathrm{diag}(h_1(t), h_2(t), ..., h_{2n}(t))$ is a scaling matrix. Therefore, for function projective dual synchronization we use adaptive control method to design the control in such a way that the origin becomes asymptotically stable equilibrium point of the error dynamics i.e., $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|x_2 - h(t)x_1\| = 0$, $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y_2 - h(t)y_1\| = 0$, where the scaling function $h(t) \in C^1(0, +\infty)$ and $0 < h(t) < N_h$ for all $t > 0$, ($N_h$ is a positive constant for the function $h(t)$).

### 1.1 Adaptive function projective dual synchronization controller design

System (1) can be rewritten in the following form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} f_1(x_1) \\ g_1(y_1) \end{pmatrix} + \begin{pmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \dot{x} = f(x) + F(x)\Phi, \tag{5}$$

where $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} \in R^{2n}$, $f(x) = \begin{pmatrix} f_1(x_1) \\ g_1(y_1) \end{pmatrix} \in R^{2n}$, $F(x) = \begin{pmatrix} F_1(x_1) & 0 \\ 0 & G_1(y_1) \end{pmatrix} : R^{2n} \to R^{2n\times 2m}$ and $\Phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in R^{2m}$. Similarly, system (3) can be rewritten in the following form:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_2(x_2) \\ g_2(y_2) \end{pmatrix} + \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \dot{y} = g\,(y) + G\,(y)\,\hat{\Phi} + u, \tag{6}$$

where $\dot{y} = \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} \in R^{2n}$, $g(y) = \begin{pmatrix} f_2(x_2) \\ g_2(y_2) \end{pmatrix} \in R^{2n}$, $G(x) = \begin{pmatrix} F_2(x_2) & 0 \\ 0 & G_2(y_2) \end{pmatrix} : R^{2n} \to R^{2n\times 2m}$ and $\hat{\Phi} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \in R^{2m}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in R^{2n}$. Now, define the error vector as

$$e = y - h(t)x. \tag{7}$$

The time derivative of equation (7) is

$$\begin{aligned} \dot{e}(t) &= \dot{y} - h(t)\dot{x} - \dot{h}(t)x \\ &= g(y) + G(y)\hat{\Phi} - h(t)f(x) - h(t)F(x)\Phi - \dot{h}(t)x + u \\ &= h(t)F(x)\tilde{\Phi} + \tilde{F}\hat{\Phi} + \tilde{f} - \dot{h}(t)x + u, \end{aligned} \tag{8}$$

where $\tilde{f} = g(y) - h(t)f(x)$, $\tilde{F} = G(y) - h(t)F(x)$ and $\tilde{\Phi} = \hat{\Phi} - \Phi$. In practical situation, the parameters belonging to the drive and the response systems are always unknown. Therefore, by using adaptive control and the parameters identification techniques, the controller can be designed as:

$$u = -\tilde{f} - \tilde{F}\hat{\Phi} + \dot{h}(t)x - ke - e_s, \tag{9}$$

where

$$e_s = C^T e, \tag{10}$$

denotes the linear coupling of the drive and response systems and the adaptive parameter update laws are chosen as

$$\dot{\hat{\Phi}} = -F^T(x)h(t)e. \tag{11}$$

**Definition 1.1** For the drive system (5) and the response system (6), it is said that the systems (5) and (6) are function projective dual synchronization if there exists a scaling function $h(t)$, such that $\lim\limits_{t\to\infty} \|e(t)\| = 0$.

**Theorem 1.1** *For given synchronization scaling function $h(t)$ and any initial conditions $x(0), y(0)$, the function projective dual synchronization between drive system (5) and response system (6) will occur by the control law (9) and the parameter update law (11).*

***Proof.*** Construct dynamical Lyapunov function candidate in the form of:

$$V = \frac{1}{2}[e^T e + \tilde{\Phi}^T \tilde{\Phi}], \tag{12}$$

with the choice of the controller (9) and the parameter update law (11), the time derivative of $V$ along the trajectories of equation (8) is

$$\dot{V} = e^T \dot{e} + \dot{\tilde{\Phi}}^T \tilde{\Phi} = e^T [h(t)F(x)\tilde{\Phi} - ke - e_s] + [-F(x)^T h(t)e]^T \tilde{\Phi} = -e^T Pe < 0. \tag{13}$$

Suppose we select an appropriate positive definite matrix $P$ such that $\dot{V} < 0$, that is, $\dot{V}$ is negative definite. Then, according to the Lyapunov stability theorem [38], the function projective dual synchronization of the systems (5) and (6) is achieved under the certain chosen controller $u$ and parameters update law. This completes the proof.

## 2 Adaptive Function Projective Dual Synchronization of Chaotic Systems

In this section, we realized the adaptive projective dual synchronization behavior in a pair of chaotic Lorenz and Lü systems, using proposed the technique. Now, define the pair of the drive system equations and the pair of the response system equations as

Drive 1: Lü system [40] is given by

$$\begin{aligned} \dot{x_1} &= \alpha(y_1 - x_1), \\ \dot{y_1} &= -x_1 z_1 + \delta y_1, \\ \dot{z_1} &= x_1 y_1 - \beta z_1. \end{aligned} \tag{14}$$

Drive 2: Lorenz system [41] is given by

$$\begin{aligned} \dot{x_2} &= \sigma(y_2 - x_2), \\ \dot{y_2} &= \rho x_2 - x_2 z_2 - y_2, \\ \dot{z_2} &= x_2 y_2 - \gamma z_2. \end{aligned} \tag{15}$$

So the corresponding response systems are as follows:

Response 1:

$$\begin{aligned} \dot{x_3} &= \hat{\alpha}(y_3 - x_3) + u_1, \\ \dot{y_3} &= -x_3 z_3 + \hat{\delta} y_3 + u_2, \\ \dot{z_3} &= x_3 y_3 - \hat{\beta} z_3 + u_3. \end{aligned} \tag{16}$$

Response 2:

$$\begin{aligned} \dot{x_4} &= \hat{\sigma}(y_4 - x_4) + u_4, \\ \dot{y_4} &= \hat{\rho} x_4 - x_4 z_4 - y_4 + u_5, \\ \dot{z_4} &= x_4 y_4 - \hat{\gamma} z_4 + u_6, \end{aligned} \tag{17}$$

where $\alpha, \delta, \beta, \sigma, \rho, \gamma$, are unknown system parameters, $\hat{\alpha}, \hat{\delta}, \hat{\beta}, \hat{\sigma}, \hat{\rho}, \hat{\gamma}$ are the estimates of $\alpha, \delta, \beta, \sigma, \rho, \gamma$, respectively, and $U = (u_1, u_2, u_3, u_4, u_5, u_6)^T$ is the controller function to be determined. The error dynamical system can be written as

$$\begin{aligned}
\dot{e_1} &= \hat{\alpha}((y_3 - x_3) - h_1(t)(y_1 - x_1)) + h_1(t)\tilde{\alpha}(y_1 - x_1) - \dot{h}_1(t)x_1 + u_1, \qquad (18)\\
\dot{e_2} &= -x_3z_3 + \hat{\delta}(y_3 - h_2(t)y_1) + h_2(t)(x_1z_1 + \tilde{\delta}y_1) - \dot{h}_2(t)y_1 + u_2,\\
\dot{e_3} &= x_3y_3 - \hat{\beta}(z_3 - h_3(t)z_1) - h_3(t)(x_1y_1 + \tilde{\beta}z_1) - \dot{h}_3(t)z_1 + u_3,\\
\dot{e_4} &= \hat{\sigma}((y_4 - x_4) - h_4(t)(y_2 - x_2)) + h_4(t)\tilde{\sigma}(y_2 - x_2) - \dot{h}_4(t)x_2 + u_4,\\
\dot{e_5} &= \hat{\rho}(x_4 - h_5(t)x_2) - x_4z_4 - y_4 + h_5(t)(x_2z_2 + y_2 + \tilde{\rho}x_2) - \dot{h}_5(t)y_2 + u_5,\\
\dot{e_6} &= x_4y_4 - \hat{\gamma}(z_4 - h_6(t)z_2) - h_6(t)(x_2y_2 + \tilde{\gamma}z_2) - \dot{h}_6(t)z_2 + u_6,
\end{aligned}$$

where $e_1 = x_3 - h_1(t)x_1$, $e_2 = y_3 - h_2(t)y_1$, $e_3 = z_3 - h_3(t)z_1$, $e_4 = x_4 - h_4(t)x_2$, $e_5 = y_4 - h_5(t)y_2$, $e_6 = z_4 - h_6(t)z_2$, and $\tilde{\alpha} = \hat{\alpha} - \alpha$, $\tilde{\delta} = \hat{\delta} - \delta$, $\tilde{\beta} = \hat{\beta} - \beta$, $\tilde{\sigma} = \hat{\sigma} - \sigma$, $\tilde{\rho} = \hat{\rho} - \rho$, $\tilde{\gamma} = \hat{\gamma} - \gamma$, respectively. Our goal is to find a suitable adaptive control law and parameter update rule equation so that pair of the two chaotic systems will approach projective dual synchronization for any initial conditions.

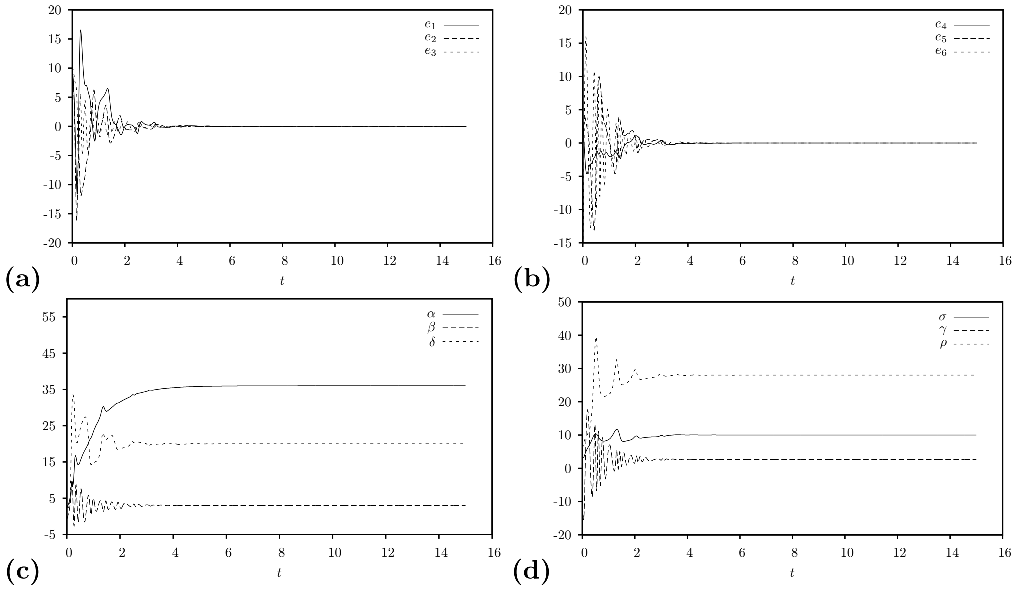


**Figure 1**: (a)–(b): Error signals between drive and response systems for Case I. (c)–(d): Estimated values for unknown parameters for Case I.

**Theorem 2.1** *For given synchronization scaling function matrix* $h(t) = \text{diag}(h_1(t), h_2(t), \ ..., h_6(t))$, *the function projective dual synchronization between the drive systems (14)-(15) and the response systems (16)–(17) will occur if the adaptive*

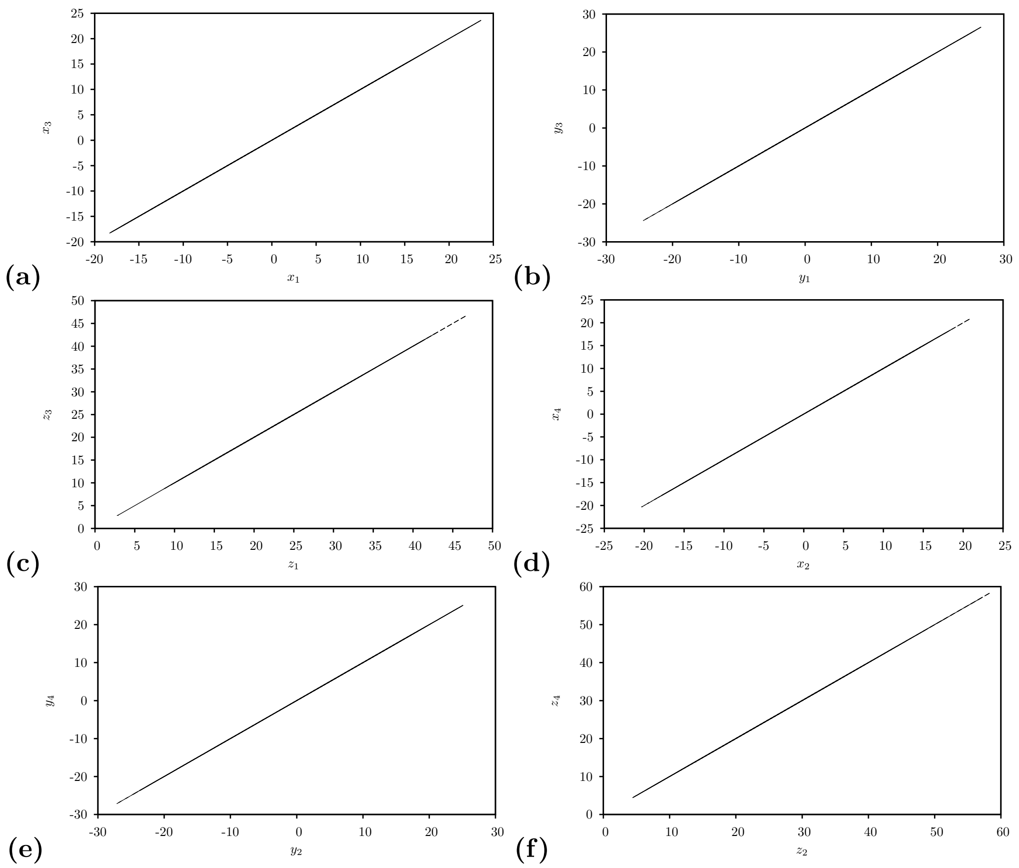


**Figure 2**: Signals $x_1$ versus $x_3$, $y_1$ versus $y_3$, and $z_1$ versus $z_3$ and signals $x_2$ versus $x_4$, $y_4$ versus $y_4$, and $z_2$ versus $z_4$ after dual–synchronization for Case I.

*control law equation is designed as follows*

$$\begin{aligned}
u_1 &= -\hat{\alpha}((y_3 - x_3) - h_1(t)(y_1 - x_1)) + \dot{h}_1(t)x_1 - ke_1 - e_s, \\
u_2 &= x_3 z_3 - h_2(t)x_1 z_1 - \hat{\delta}(y_3 - h_2(t)y_1) + \dot{h}_2(t)y_1 - ke_2 - e_s, \\
u_3 &= h_3(t)x_1 y_1 - x_3 y_3 + \hat{\beta}(z_3 - h_3(t)z_1) + \dot{h}_3(t)z_1 - ke_3 - e_s, \\
u_4 &= -\hat{\sigma}((y_4 - x_4) - h_4(t)(y_2 - x_2)) + \dot{h}_4(t)x_2 - ke_4 - e_s, \\
u_5 &= x_4 z_4 + y_4 - \hat{\rho}(x_4 - h_5(t)x_2) - h_5(t)(x_2 z_2 + y_2) + \dot{h}_5(t)y_2 - ke_5 - e_s, \\
u_6 &= \hat{\gamma}(z_4 - h_6(t)z_2) + h_6(t)x_2 y_2 - x_4 y_4 + \dot{h}_6(t)z_2 - ke_6 - e_s,
\end{aligned} \tag{19}$$

*where*

$$e_s = a_1 e_1 + a_2 e_2 + a_3 e_3 + b_1 e_4 + b_2 e_5 + b_3 e_6 \tag{20}$$

*denotes the linear coupling of the drive and response systems and the adaptive parameter*

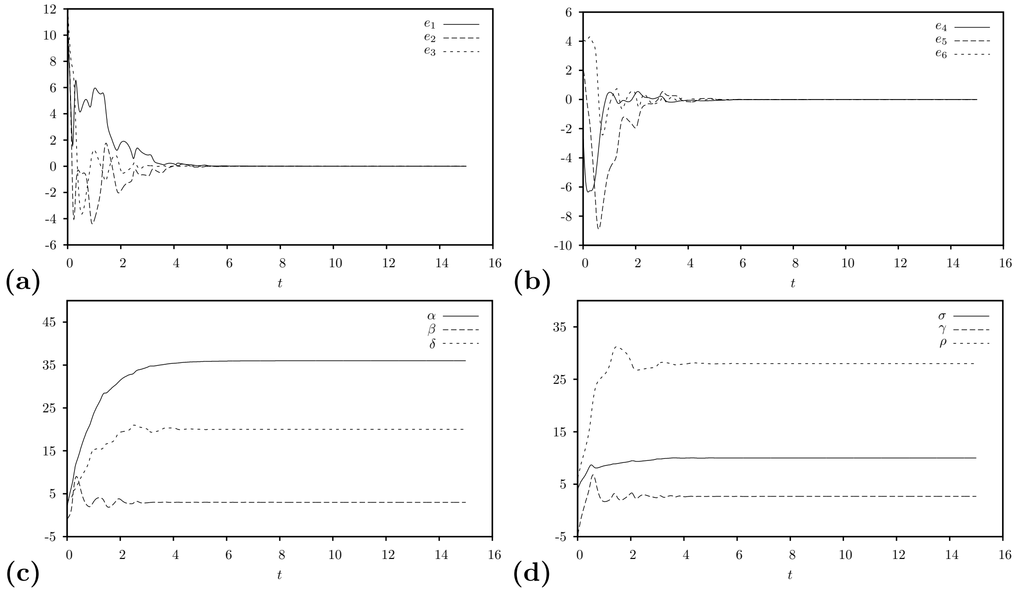


**Figure 3**: (a)–(b): Error signals between drive and response systems for Case II. (c)–(d): Estimated values for unknown parameters for Case II.

*update laws are chosen as*

$$\begin{aligned} \dot{\hat{\alpha}} &= -h_1(t)(y_1-x_1)e_1, \quad \dot{\hat{\delta}} = -h_2(t)y_1e_2, \\ \dot{\hat{\beta}} &= h_3(t)z_1e_3, \quad \dot{\hat{\sigma}} = -h_4(t)(y_2-x_2)e_4, \\ \dot{\hat{\rho}} &= -h_5(t)x_2e_5, \quad \dot{\hat{\gamma}} = h_6(t)z_2e_6. \end{aligned} \tag{21}$$

***Proof.*** Substituting (19) into (18) leads to the following error system

$$\begin{aligned} \dot{e_1} &= h_1(t)\tilde{\alpha}(y_1-x_1) - ke_1 - e_s, \quad \dot{e_2} = h_2(t)\tilde{\delta}y_1 - ke_2 - e_s, \\ \dot{e_3} &= -h_3(t)\tilde{\beta}z_3 - ke_3 - e_s, \quad \dot{e_4} = h_4(t)\tilde{\sigma}(y_2-x_2) - ke_4 - e_s, \\ \dot{e_5} &= h_5(t)\tilde{\rho}x_2 - ke_5 - e_s, \quad \dot{e_6} = -h_6(t)\tilde{\gamma}z_2 - ke_6 - e_s. \end{aligned} \tag{22}$$

Construct a Lyapunov function of the form:

$$V = \frac{1}{2}(e^Te + \tilde{\alpha}^2 + \tilde{\delta}^2 + \tilde{\beta}^2 + \tilde{\sigma}^2 + \tilde{\rho}^2 + \tilde{\gamma}^2). \tag{23}$$

Inserting (20), (21) and (22) into the time derivative of $V$ leads to

$$\begin{aligned} \dot{V} &= e^T\dot{e} + \tilde{\alpha}\dot{\tilde{\alpha}} + \tilde{\delta}\dot{\tilde{\delta}} + \tilde{\beta}\dot{\tilde{\beta}} + \tilde{\sigma}\dot{\tilde{\sigma}} + \tilde{\rho}\dot{\tilde{\rho}} + \tilde{\gamma}\dot{\tilde{\gamma}} \\ &= (h_1(t)\tilde{\alpha}(y_1-x_1)-ke_1-e_s)e_1+(h_2(t)\tilde{\delta}y_1 - ke_2 - e_s)e_2 - (h_3(t)\tilde{\beta}z_3 + e_3 + e_s)e_3 \\ &\quad +(h_4(t)\tilde{\sigma}(y_2-x_2) - ke_4-e_s)e_4+(h_5(t)\tilde{\rho}x_2-ke_5-e_s)e_5-(h_6(t)\tilde{\gamma}z_2+ke_6+e_s)e_6 \\ &\quad -\tilde{\alpha}(h_1(t)(y_1-x_1)e_1) - \tilde{\delta}(h_2(t)y_1e_2) + \tilde{\beta}(h_3(t)z_1e_3) - \tilde{\sigma}(h_4(t)(y_2-x_2)e_4) \\ &\quad -\tilde{\rho}(h_5(t)x_2e_5) + \tilde{\gamma}(h_6(t)z_2e_6) \end{aligned} \tag{24}$$

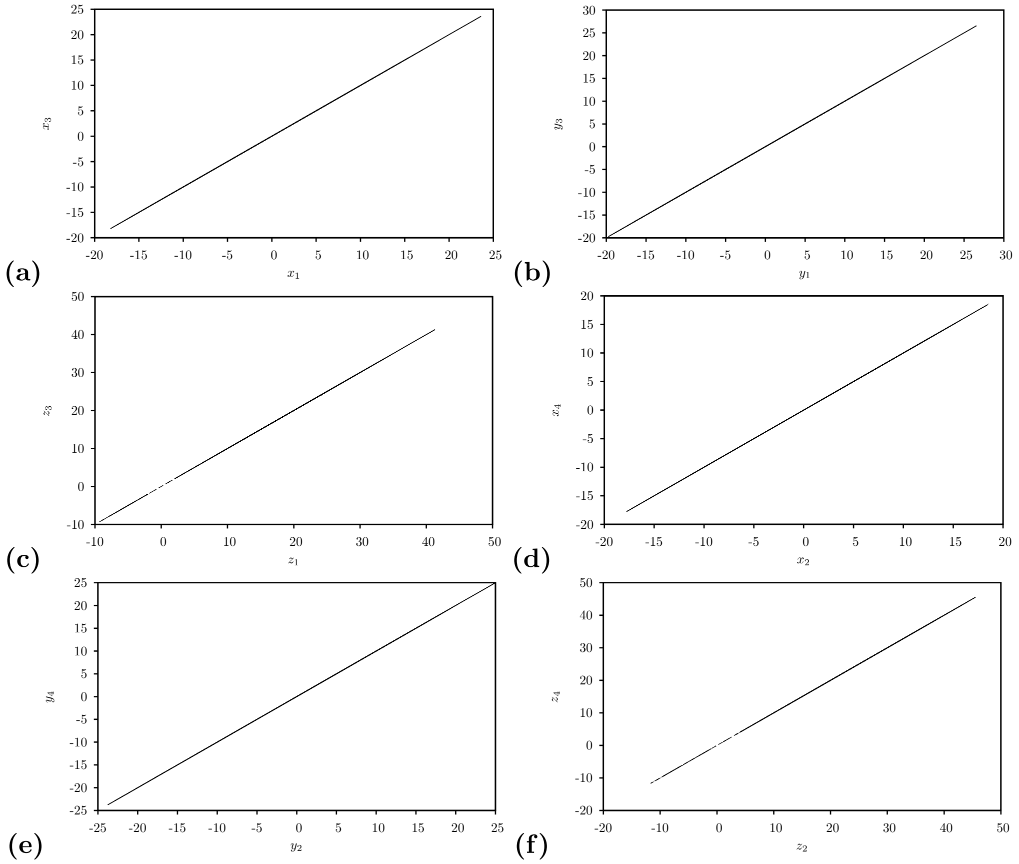


**Figure 4**: Signals $x_1$ versus $x_3$, $y_1$ versus $y_3$, and $z_1$ versus $z_3$ and signals $x_2$ versus $x_4$, $y_4$ versus $y_4$, and $z_2$ versus $z_4$ after dual–synchronization for Case II.

$$\begin{aligned} = \; - & \;[(k+a_1)e_1^2 + (a_1+a_2)e_1e_2 + (a_1+a_3)e_1e_3 + (a_1+b_1)e_1e_4 + (a_1+b_2)e_1e_5 \\ & +(a_1+b_3)e_1e_6 + (k+a_2)e_2^2 + (a_2+a_3)e_2e_3 + (a_2+b_1)e_2e_4 + (a_2+b_2)e_2e_5 \\ & +(a_2+b_3)e_2e_6 + (k+a_3)e_3^2 + (a_3+b_1)e_3e_4 + (a_3+b_2)e_3e_5 + (a_3+b_3)e_3e_6 \\ & +(k+b_1)e_4^2 + (b_1+b_2)e_4e_5 + (b_1+b_3)e_4e_6 + (k+b_2)e_5^2 + (b_2+b_3)e_5e_6 + (k+b_3)e_6^2] \\ = & \; -e^TPe < 0, \end{aligned}$$

where $e = [|e_1|, |e_2|, |e_3|, |e_4|, |e_5|, |e_6|]$ and $P$ is real symmetric matrix. From the Lyapunov theorem of stability [38], it is simple to point out that the zero equilibrium point ($e_i = 0, i = 1, ..., 6$) of the error dynamical system (18) is asymptotically stable if the real symmetric matrix $P$ is positive definite. According to Sylvester's theorem [39], $P$ is positive definite if and only if $\Delta_i > 0, i = 1, 2, ..., 6$, where $\Delta_i$ represents the $i$th order sequential sub determinant of matrix. That is, we should choose the appropriate coupled parameters. Then, we realize the function projective dual synchronization between a pair of Lü systems and a pair of Lorenz systems. This completes the proof.

### 2.1 Numerical simulation and results for function projective dual synchronization

In the present section, the numerical simulations for the function projective dual synchronization of a pair of chaotic systems are studied. The true values of the unknown parameter of the systems ((14)–(15)) are taken as $\alpha = 36, \delta = 20, \beta = 3$, and $\sigma = 10, \rho = 28, \gamma = 8/3$, so both systems exhibit chaotic behavior. The initial values of the estimated unknown parameter vectors of the systems are taken as $\alpha(0) = 2, \delta(0) = -1, \beta(0) = 3$, and $\sigma(0) = 4, \rho(0) = -5, \gamma(0) = 6$. The initial conditions of the drive system (14) and the drive system (15) are taken as $x_1(0) = 1, y_1(0) = 2, z_1(0) = 3, x_2(0) = -9,\ y_2(0) = 5, z_2(0) = 30$, the initial conditions of the response system (16) and the response system (17) are taken as $x_3(0) = 11, y_3(0) = 12, z_3(0) = 13$ and $x_4(0) = -4, y_4(0) = 3, z_4(0) = 10$, respectively. The coupled parameters are valued as $a_i = (1, 1, 1), b_i = (1, 1, 1), i = 1, 2, 3$, for which condition $P$ is positive definite. The real positive constants $k$ is taken as 1.

**Case I.** Let the scaling function be $h_i(t) = 0.9 + \dfrac{t}{1+t^2}, i = (1, .2, ..., 6)$. The simulation results are shown through Fig. 1 (a)–(d), which shows that the dual synchronization errors converge asymptotically to zero and the estimated parameters $\hat{\alpha}, \hat{\delta}, \hat{\beta}$, and $\hat{\sigma}, \hat{\rho}, \hat{\gamma}$ converge to the original parameter $\alpha = 36, \delta = 20, \beta = 3$ , and $\sigma = 10, \rho = 28, \gamma = 8/3$ as $t \to \infty$. Fig. 2 shows the signals after dual synchronization.

**Case II.** Let the scaling function be $h_i(t) = 0.2 + 0.5 \sin\left(\dfrac{\pi t}{10}\right)$. The simulation results are depicted through Fig. 3 (a)–(d), which shows that the dual synchronization errors converge asymptotically to zero and the estimated parameters $\hat{\alpha}, \hat{\delta}, \hat{\beta}$, and $\hat{\sigma}, \hat{\rho}, \hat{\gamma}$ converge to the original parameters $\alpha = 36, \delta = 20, \beta = 3$ , and $\sigma = 10, \rho = 28, \gamma = 8/3$ as $t \to \infty$. Fig. 4 shows the signals after dual synchronization.

## 3 Conclusion

In the present paper, we have successfully demonstrated the function projective dual synchronization between a pair of chaotic systems using adaptive control method with uncertain parameters. The method is applied for the function projective dual synchronization between chaotic Lü and Lorenz systems. This clearly exhibits that the adaptive control method is effective and convenient to achieve the global dual synchronization of a pair of chaotic systems. Eventually some simulation results shown in corresponding figures have illustrated the effectiveness and feasibility of the proposed controller.

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# Global Dynamics of a Cooperative and Supportive Network System with Subnetwork Deactivation

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**Abstract:** In this paper a cooperative and supportive neural network, in which each neuron of main network is supported by a subnetwork of neurons, is considered. The dynamics of supportive subnetwork are subjected to some deactivation with transfer of data to the main network. Results are obtained on influence of this deactivation on global asymptotic behavior of the solutions. Numerical examples are provided to illustrate the results. The results are compared with known results.

**Keywords:** *neural networks; cooperative and supportive systems; deactivation; global asymptotic stability;*

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## 1 Introduction

Neural networks is an exciting area for research with a broad spectrum of applications [1-12, 16, 22-24]. Networks of neurons have taken many shapes as mathematical models basing on the application which they are designed for. Models of Hopfield, Cohen-Grossberg, cellular networks, recurrent networks, cooperative modular networks and spiking neural networks are such popular models to quote [1-9, 13, 20]. Along the cite, a new class of neural networks, designated as co-operative and supportive neural network (CSNN, for short) is introduced in [26]. It consists of a network of neurons called main components each of which is connected to and supported by another network of neurons called subnetwork components. The model is aimed at explaining the dynamics of systems exhibiting hierarchy that takes into account the collective capabilities of components for better performance of the system. Such systems are useful in understanding industrial information management, financial and economic systems which involve distribution and monitoring of various tasks. They are utilized to decompose complex classification tasks into simpler subtasks and puzzle them out. In particular, the network of [26] was utilized for estimation of key parameters in an infectious disease model [25]. For different models of cooperative neural networks and their applications, readers are referred to [9, 14, 15, 17, 21]. It was also claimed that the CSNN model presented in [26] was entirely new and different from all the above neural models in terms of formulation and application. Hereunder, we explain briefly the CSNN model introduced in [26], which we are going to modify and analyze further in the present study.

The model comprises two neuronal fields, say, $F_x$ and $F_y$. Each neuron in $F_x$ is denoted by $x_i$, $i = 1, 2, ..., n$ and is connected to other neurons $x_j$, $j = 1, 2, ...n$ in the same field $F_x$. Also each $x_i$ is connected to $r_i$ number of neurons in the neuronal field $F_y$. These are denoted by $y_{i_k}$, $k = 1, 2, ....r_i$, $1 \leq r_i \leq n$. These $y_{i_k}$'s support $x_i$ in the sense that they coordinate and cooperate with it so that any task assigned to them by $x_i$ will be attended to. The dynamics of the model are described by the following system of equations

$$x_i' = -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, \ i = 1, 2, ..., n,$$
$$y_{i_k}' = -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) + J_{i_k}, \quad k = 1, 2, ..., r_i, \ 1 \leq r_i \leq n. \tag{1}$$

In (1), $a_i$ and $c_{i_k}$ are positive constants known as decay rates, $I_i, J_{i_k}$ are exogenous inputs and $b_{ij}, d_{i_l}$ are the synaptic connection weights which may be real or complex constants. $c_{ii_k}$ is the rate of distribution of information between $x_i$ and $y_{i_k}$. The functions $f_i$, $g_{i_k}$ and $h_{i_k}$ are the neuronal output response functions and are more commonly known as the signal functions.

Besides a study of qualitative behavior of the system, several modifications of the CSNN model (1) are suggested and left as open problems in [26] for enthusiastic readers. Present authors have studied two such modified models [18, 19] of (1) that increase its applicability. Extending this view point, we shall address one more modification of (1) in our present study. Motivation for this stems from the following observations.

The second equation of (1) contains no term that includes $x_i$. That means, the subcomponents $y_{i_k}$ work independently of $x_i$, supply information to $x_i$ and do not bother whether their contribution is fully utilized or are contributing more than what is required. At the same time for the main component $x_i$ there is a need to check this contribution from $y_{i_k}$ either in terms of money or in terms of easing out $y_{i_k}$ from unnecessary production beyond what is required. Thus, there is a need to check the activity of sub-components. Also when $y_{i_k}$ depends mainly on $x_i$ for its survival or activity there should be a term that reflects the interaction between $x_i$ and $y_{i_k}$. Such a term in second equation of (1) represents: (i) physical transfer of subcomponents in case of ancillary manufacturing units; (ii) removal of data/information after transferring it to main component ('cut and paste' instead of 'copy and paste') or (iii) deactivation of subcomponents as soon as the required data is supplied.

Another argument runs as follows. System (1) reflects how $x_i$ receives information from $y_{i_k}$ represented by $g_{i_k}(x_i, y_{i_k})$ but not how it is sent from $y_{i_k}$ - term at receiver's end but not at giver's end. It may also be understood as that $y_{i_k}$ keeps a copy of what ever information/data sent to $x_i$. This may not be possible in all cases. We can not keep copies of physical quantities such as spare parts, components, etc., of the main item in a manufacturing unit. Even in data processing systems, retention of data at too many places may raise security problems. Absence of a term involving $x_i$ may also infer that the requirements of $x_i$ are insignificant when compared to the quantum of work done by $y_{i_k}$ for all its purposes.

In order to incorporate these, we introduce a term which may be utilized for deactivating or resting of $y_{i_k}$ once its task is done. Introduction of such term into the second equation modifies (1) to

$$\begin{aligned} x_i' &= -a_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i, y_{i_k}) + I_i, \\ y_{i_k}' &= -c_{i_k} y_{i_k} + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}) - \bar{c}_{ii_k} \bar{g}_{i_k}(x_i, y_{i_k}) + J_{i_k}. \end{aligned} \tag{2}$$

In (2), the term $\bar{c}_{ii_k} \bar{g}_{i_k}(x_i, y_{i_k})$ denotes the resting or deactivating component for the subsystem. Here each $\bar{c}_{ii_k} > 0$ may be called the rate of de-activation of $y_{i_k}$ by $x_i$. The functional term $\bar{g}_{i_k}(x_i, y_{i_k})$ denotes how the deactivation takes place. System (2) is Model I in [26] which is left open for exploration. Our task in this paper shall be to study the influence of this new term on the dynamics of the system (1). Is this term going to pacify sub-components or influence the entire network will be a question of utmost importance. How to manage its influence using the system parameters may be reasonable task to take up. This we study in terms of stability of equilibrium patterns of the system (2) in the light of existing results on (1).

The paper is organized as follows. In Section 2, we provide conditions for existence and uniqueness of solutions, equilibria for system (2) — basic properties of any such dynamical system. Results on global asymptotic stability of equilibria are obtained in Section 3. The results are compared with earlier results on (1). Examples are provided for illustration of results. Finally a discussion follows in Section 4.

## 2 Basic Properties

In this section, we explain basic properties of (2) such as existence of solutions along with equilibria. This is to be done with appropriate assumptions or restrictions on system parameters and nonlinear functions. To begin with we assume that the response functions $f_j$, $g_{ii_k}$, $h_{i_k}$ and $\overline{g}_{ii_k}$ satisfy local Lipschitz conditions given by

$$\begin{aligned}
\| g_{i_k}(x_i, y_{i_k}) - g_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq M_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + M_{2i_k}|x_i - \overline{x}_i|, &(3)\\
\| \overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(\overline{x}_i, \overline{y}_{i_k}) \| &\leq \overline{M}_{1i_k}|y_{i_k} - \overline{y}_{i_k}| + \overline{M}_{2i_k}|x_i - \overline{x}_i|, &(4)\\
|f_j(x_j) - f_j(\overline{x}_j)| &\leq p_j|x_j - \overline{x}_j|, &(5)\\
|h_{i_l}(y_{i_l}) - h_{i_l}(\overline{y}_{i_l})| &\leq q_{i_l}|y_{i_l} - \overline{y}_{i_l}|, &(6)
\end{aligned}$$

where $M_{1i_k}$, $M_{2i_k}$, $p_j$ and $q_{i_l}$ are positive constants. Then from the theory of differential equations, it is evident that solutions for (2) do exist, are unique and continuable in their maximal intervals of existence.

Since the stability of a system is understood in terms of the stability of its equilibria, we verify whether (2) provides scope for equilibrium patterns to exist. The following result provides one such set of sufficient conditions.

**Theorem 2.1.** *Let $a_i$ and $c_{i_k}$ be positive numbers such that*

$$\begin{aligned}
\frac{1}{a_i}\sum_{j=1}^{n}|b_{ij}|p_j + \frac{1}{a_i}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} + \frac{1}{c_{i_k}}|\overline{c}_{ii_k}|\overline{M}_{2i_k} &< 1, \ i = 1, 2, ..., n.\\
\frac{1}{c_{i_k}}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} + \frac{1}{a_i}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} + \frac{1}{c_{i_k}}|\overline{c}_{ii_k}|\overline{M}_{1i_k} &< 1, \ 1 \leq r_i \leq n. &(7)
\end{aligned}$$

*Then the system (2) has a unique equilibrium solution $(x_i^*, y_{i_k}^*)$ for each $i$, $k$.*

Since several results are available in literature on similar systems, we omit the proof of the above result here and refer the interested readers to [3],[26] for a line of proof based on contraction mapping principle.

Since $(x_i^*, y_{i_k}^*)$ is a constant solution of (2), we have

$$\begin{aligned}
x_i^{*'} = 0 &= -a_i x_i^* + \sum_{j=1}^{n} b_{ij} f_j(x_i^*) + \sum_{k=1}^{r_i} c_{ii_k} g_{i_k}(x_i^*, y_{i_k}^*) + I_i,\\
y_{i_k}^{*'} = 0 &= -c_{i_k} y_{i_k}^* + \sum_{l=1}^{r_i} d_{i_l} h_{i_l}(y_{i_l}^*) - \overline{c}_{ii_k}\overline{g}_{i_k}(x_i^*, y_{i_k}^*) + J_{i_k}. &(8)
\end{aligned}$$

We shall now take up the aspect of stability of equilibrium pattern of (2), assuming its existence tacitly.

## 3 Global Stability Results

In this section we study the influence of deactivation term on the stability of the system. Whether its presence will increase strain on parameters or reduce it when compared to (1) — is the main concern.

Before we present our results, we rearrange system (2) as follows. Utilizing (8) in (2), we get

$$
\begin{aligned}
(x_i - x_i^*)' &= -a_i(x_i - x_i^*) + \sum_{j=1}^{n} b_{ij}[f_j(x_j) - f_j(x_j^*)] + \sum_{k=1}^{r_i} c_{ii_k}[g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)], \\
(y_{i_k} - y_{i_k}^*)' &= -c_{i_k}(y_{i_k} - y_{i_k}^*) + \sum_{l=1}^{r_i} d_{i_l}[h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)] - \overline{c}_{ii_k}[\overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*)].
\end{aligned} \tag{9}
$$

We shall establish our first result now.

**Theorem 3.1.** *Assume that the parameters of the system (2) satisfy the following conditions:*

$$
\begin{aligned}
a_i &> \sum_{j=1}^{n} |b_{ji}| p_i + \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k} + \sum_{k=1}^{r_i} |\overline{c}_{ii_k}| \overline{M}_{2i_k}, \\
c_{i_k} &> \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} + |c_{ii_k}| M_{1i_k} + |\overline{c}_{ii_k}| \overline{M}_{1i_k}.
\end{aligned}
$$

*Assume further that conditions (3) - (6) on response functions hold. Then the equilibrium* $(x_i^*, y_{i_k}^*)$ *is globally asymptotically stable in the sense that all solutions of (2) satisfy the convergence requirement*

$$\lim_{t\to\infty} y_{i_k} \to y_{i_k}^*, \quad \lim_{t\to\infty} x_i \to x_i^*.$$

**Proof.** We consider the functional

$$V(t) = \sum_{i=1}^{n} \Big\{ |x_i - x_i^*| + \sum_{k=1}^{r_i} |y_{i_k} - y_{i_k}^*| \Big\}. \tag{10}$$

The upper right derivative of $V$ along the solutions of (2) utilizing (9) may be given by

$$
\begin{aligned}
D^+V(t) \le \sum_{i=1}^{n} \Big\{ &- a_i|x_i - x_i^*| + \sum_{j=1}^{n} |b_{ij}||f_j(x_j) - f_j(x_j^*)| \\
&+ \sum_{k=1}^{r_i} |c_{ii_k}||g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)| \\
&+ \sum_{k=1}^{r_i} \Big[ - c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i} |d_{i_l}||h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)| \Big] \\
&- \sum_{k=1}^{r_i} |\overline{c}_{ii_k}||\overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*)| \Big\}.
\end{aligned}
$$

Then,

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n}\Big\{ -a_i|x_i - x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j - x_j^*| \\ &\quad + \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k}|x_i - x_i| + \sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k}|y_{i_k} - y_{i_k}^*| \\ &\quad + \sum_{k=1}^{r_i}\Big[ -c_{i_k}|y_{i_k} - y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l} - y_{i_l}^*|\Big] \\ &\quad + \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|[\overline{M}_{1i_k}|y_{i_k} - y_{i_k}^*| + \overline{M}_{2i_k}|x_i - x_i^*|]\Big\}, \end{aligned}$$

using (3) - (6) on the response functions. Thus,

$$\begin{aligned} D^+V(t) &\leq -\sum_{i=1}^{n}\Big\{[a_i - \sum_{j=1}^{n}|b_{ji}|p_i - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}]\ |x_i - x_i^*| \\ &\quad + \sum_{k=1}^{r_i}[c_{i_k} - \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - |c_{ii_k}|M_{1i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}]\ |y_{i_k} - y_{i_k}^*|\Big\} \\ &\leq -\widetilde{A}V < 0, \qquad \text{by hypotheses,} \end{aligned}$$

where $\widetilde{A} = \min\left\{\overline{A}, \overline{B}\right\}$, and

$$\begin{aligned} \overline{A} &= \left\{\min\left[a_i - \sum_{j=1}^{n}|b_{ji}|p_j - \sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} - \sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}\right] > 0,\ 1 \leq i \leq n.\right\} \\ \overline{B} &= \left\{\min\left[c_{i_k} - \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - |c_{ii_k}|M_{1i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}\right] > 0,\ 1 \leq k \leq r_i,\ 1 \leq i \leq n.\right\} \end{aligned}$$

Thus, $D^+V(t) + \widetilde{A}V(t) < 0$. Integrating on both sides with respect to $t$ from 0 to $t$, we have $V(t) < V(0)e^{-\widetilde{A}t} \to 0$ for large $t$. The conclusion follows from the definition of $V$.

We shall present yet another result on global asymptotic stability of equilibrium pattern of (2) using a different Lyapunov functional providing one more set of sufficient conditions on parameters of the system.

**Theorem 3.2.** *Assume that the conditions (3)–(6) on response functions hold. Furthermore the parameters satisfy the following inequalities*

$$\begin{aligned} a_i &> \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|p_j + \frac{1}{2}\sum_{j=1}^{n}|b_{ji}|p_i + \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\ &\quad + \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} + \frac{1}{2}\sum_{k=1}^{r_i}|\overline{c}_{ii_k}|\overline{M}_{2i_k}, \\ c_{i_k} &> \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} + \frac{1}{2}|c_{ii_k}|M_{1i_k} + \frac{1}{2}|\overline{c}_{ii_k}|\overline{M}_{2i_k} + |\overline{c}_{ii_k}|\overline{M}_{1i_k}, \end{aligned} \tag{11}$$

*for all $i$ and $i_k$. Then the equilibrium pattern of (2) is globally asymptotically stable.*

**Proof.** We consider the functional

$$V(t) = \sum_{i=1}^{n} \left\{ \frac{(x_i(t) - x_i^*)^2}{2} + \sum_{k=1}^{r_i} \frac{(y_{i_k} - y_{i_k}^*)^2}{2} \right\}.$$

The derivative of $V$ along the solutions of (1.2), using (3.1), is given by

$$\begin{aligned}
V'(t) &= \sum_{i=1}^{n} \Big[ (x_i(t) - x_i^*)(x_i'(t) - x_i^{*'}) + \sum_{k=1}^{r_i} (y_{i_k}(t) - y_{i_k}^*)(y_{i_k}'(t) - y_{i_k}^{*'}) \Big] \\
&= \sum_{i=1}^{n} \Big[ \Big[ - a_i (x_i(t) - x_i^*)^2 + (x_i(t) - x_i^*) \sum_{j=1}^{n} b_{ij} (f_j(x_j) - f_j(x_j^*)) \\
&\quad + (x_i(t) - x_i^*) \sum_{k=1}^{r_i} c_{ii_k} (g_{i_k}(x_i, y_{i_k}) - g_{i_k}(x_i^*, y_{i_k}^*)) \Big] \\
&\quad + \sum_{k=1}^{r_i} \Big[ - c_{i_k} (y_{i_k}(t) - y_{i_k}^*)^2 + (y_{i_k}(t) - y_{i_k}^*) \sum_{l=1}^{r_i} d_{i_l} [h_{i_l}(y_{i_l}) - h_{i_l}(y_{i_l}^*)] \\
&\quad - (y_{i_k}(t) - y_{i_k}^*) \overline{c}_{ii_k} \Big( \overline{g}_{i_k}(x_i, y_{i_k}) - \overline{g}_{i_k}(x_i^*, y_{i_k}^*) \Big) \Big] \Big] \\
&\leq \sum_{i=1}^{n} \Big[ - a_i (x_i(t) - x_i^*)^2 + |x_i(t) - x_i^*| \sum_{j=1}^{n} |b_{ij}| p_j |x_j(t) - x_j^*| \\
&\quad + |x_i(t) - x_i^*| \sum_{k=1}^{r_i} |c_{ii_k}| \Big[ M_{2i_k} |x_i - x_i^*| + M_{1i_k} |y_{i_k} - y_{i_k}^*| \Big] \\
&\quad + \sum_{k=1}^{r_i} \Big[ - c_{i_k} (y_{i_k}(t) - y_{i_k}^*)^2 + |y_{i_k}(t) - y_{i_k}^*| \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} |y_{i_l} - y_{i_l}^*| \\
&\quad + |y_{i_k} - y_{i_k}^*| |\overline{c}_{ii_k}| \Big( \overline{M}_{2i_k} |x_i - x_i^*| + \overline{M}_{1i_k} |y_{i_k} - y_{i_k}^*| \Big) \Big] \Big],
\end{aligned}$$

utilizing (3)-(6). Employing the inequality $ab \leq \frac{a^2+b^2}{2}$ and rearranging the terms we get

$$\begin{aligned}
V'(t) &\leq \sum_{i=1}^{n} \Big[ - a_i (x_i(t) - x_i^*)^2 + \frac{1}{2} \sum_{j=1}^{n} |b_{ij}| p_j \Big[ (x_i(t) - x_i^*)^2 + (x_j - x_j^*)^2 \Big] \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |c_{ii_k}| M_{2i_k} (x_i - x_i^*)^2 \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |c_{ii_k}| M_{1i_k} \Big[ (y_{i_k} - y_{i_k}^*)^2 + (x_i - x_i^*)^2 \Big] \Big] \\
&\quad - \sum_{k=1}^{r_i} \Big[ c_{i_k} - \frac{1}{2} \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} - \frac{1}{2} \sum_{k=1}^{r_i} |d_{i_k}| q_{i_k} \Big] (y_{i_k} - y_{i_k}^*)^2 \\
&\quad + \frac{1}{2} \sum_{k=1}^{r_i} |\overline{c}_{ii_k}| \overline{M}_{2i_k} (x_i - x_i^*)^2 + \frac{1}{2} |\overline{c}_{ii_k}| \overline{M}_{2i_k} (y_{i_k} - y_{i_k}^*)^2 \\
&\quad + |\overline{c}_{ii_k}| \overline{M}_{1i_k} (y_{i_k} - y_{i_k}^*)^2 \Big].
\end{aligned}$$

Thus,

$$\begin{aligned} V'(t) \quad \leq \quad & -\sum_{i=1}^{n}\Big[a_i - \frac{1}{2}\sum_{j=1}^{n}|b_{ij}|p_j - \frac{1}{2}\sum_{j=1}^{n}|b_{ji}|p_i - \frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{2i_k} \\ & -\frac{1}{2}\sum_{k=1}^{r_i}|c_{ii_k}|M_{1i_k} - \frac{1}{2}\sum_{k=1}^{r_i}\overline{c}_{ii_k}\overline{M}_{2i_k}\Big](x_i - x_i^*)^2 \\ & -\sum_{i=1}^{n}\sum_{k=1}^{r_i}\Big[c_{i_k} - \frac{1}{2}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - \frac{1}{2}\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l} - \frac{1}{2}|c_{ii_k}|M_{1i_k} \\ & -\frac{1}{2}|\overline{c}_{ii_k}|\overline{M}_{2i_k} - |\overline{c}_{ii_k}|\overline{M}_{1i_k}\Big](y_{i_k} - y_{i_k}^*)^2. \end{aligned}$$

Then by assumptions, $V'$ is negative definite, and hence, the conclusion follows employing standard arguments as in earlier case (e.g., [3, 19, 26]).

We shall now provide examples to illustrate these results and establish the criteria provided in these two results are independent.

**Example 3.3.** Consider

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad = \quad & -\begin{pmatrix} 1.49\ x_1 \\ 3.79\ x_2 \end{pmatrix} + \begin{pmatrix} 0.32 & 0.43 \\ 0.18 & 0.24 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ + \quad & \begin{pmatrix} 0.25 & 0.53 \\ 0.85 & 0.95 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}) & g_{2_1}(x_2, y_{2_1}) \\ g_{1_2}(x_1, y_{1_2}) & g_{2_2}(x_2, y_{2_2}) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} y_{1_1}' \\ y_{1_2}' \end{pmatrix}, \quad = \quad & -\begin{pmatrix} 1.25\ y_{1_1} \\ 1.02\ y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.5 & 0.25 \\ 0.3 & 0.1 \end{pmatrix}\begin{pmatrix} h_{1_1}(y_{1_1}) \\ h_{1_2}(y_{1_2}) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix} \\ & -\begin{pmatrix} 0.15\ g_{1_1}(x_1, y_{1_1}) \\ 0.05\ g_{1_2}(x_1, y_{1_2}) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} y_{2_1}' \\ y_{2_2}' \end{pmatrix} \quad = \quad & -\begin{pmatrix} 2.01\ y_{2_1} \\ 1.72\ y_{2_2} \end{pmatrix} + \begin{pmatrix} 0.25 & 0.12 \\ 0.15 & 0.05 \end{pmatrix}\begin{pmatrix} h_{2_1}(y_{2_1}) \\ h_{2_2}(y_{2_2}) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix} \\ & -\begin{pmatrix} 0.75\ g_{2_1}(x_2, y_{2_1}) \\ 0.53\ g_{2_2}(x_2, y_{2_2}) \end{pmatrix}. \end{aligned}$$

Let $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$. Then $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1$. Choose $I_i = 10,\ J_{i_k} = 10,\ i = 1, 2,\ k = 1, 2.$

For the above system, the equilibrium pattern is given by (11.68, 4.33, 6.67, 9.40, 2.69, 3.89). It may be seen that all the conditions of Theorem 3.1 are satisfied, and hence,the equilibrium pattern of the system is globally asymptotically stable by virtue of Theorem 3.1. Also some of the parametric conditions of Theorem 3.2 are violated, it can not be applied here.

**Example 3.4.** Consider

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad = \quad & -\begin{pmatrix} 2.45\ x_1 \\ 3.85\ x_2 \end{pmatrix} + \begin{pmatrix} 0.4 & 0.6 \\ 0.7 & 1.3 \end{pmatrix}\begin{pmatrix} f_1(x_1) \\ f_2(x_2) \end{pmatrix} \\ + \quad & \begin{pmatrix} 0.6 & 0.3 \\ 0.8 & 0.5 \end{pmatrix}\begin{pmatrix} g_{1_1}(x_1, y_{1_1}) & g_{2_1}(x_2, y_{2_1}) \\ g_{1_2}(x_1, y_{1_2}) & g_{2_2}(x_2, y_{2_2}) \end{pmatrix} + \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}\begin{pmatrix} y'_{1_1} \\ y'_{1_2} \end{pmatrix} &= -\begin{pmatrix} 1.4\; y_{1_1} \\ 1.7\; y_{1_2} \end{pmatrix} + \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} h_{1_1}(y_{1_1}) \\ h_{1_2}(y_{1_2}) \end{pmatrix} + \begin{pmatrix} J_{1_1} \\ J_{1_2} \end{pmatrix} \\ &\quad - \begin{pmatrix} 0.3\; g_{1_1}(x_1, y_{1_1}) \\ 0.6\; g_{1_2}(x_1, y_{1_2}) \end{pmatrix}, \\ \begin{pmatrix} y'_{2_1} \\ y'_{2_2} \end{pmatrix} &= -\begin{pmatrix} 1.65\; y_{2_1} \\ 2.5\; y_{2_2} \end{pmatrix} + \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}\begin{pmatrix} h_{2_1}(y_{2_1}) \\ h_{2_2}(y_{2_2}) \end{pmatrix} + \begin{pmatrix} J_{2_1} \\ J_{2_2} \end{pmatrix} \\ &\quad - \begin{pmatrix} 0.4\; g_{2_1}(x_2, y_{2_1}) \\ 0.8\; g_{2_2}(x_2, y_{2_2}) \end{pmatrix}.\end{aligned}$$

Choosing $f_i(x_i) = tanh(x_i), h_{i_k} = tanh(y_{i_k})$ and $g_{i_k}(x_i, y_{i_k}) = x_i + y_{i_k}$, we have $p_j = q_{i_k} = M_{1i_k} = M_{2i_k} = 1$. Let $I_i = 10,\ J_{i_k} = 10,\ i = 1, 2,\ k = 1, 2$.

The equilibrium pattern of the above system is given by ( 6.21, 4.38, 5.51, 4.06, 4.74, 2.18). Clearly, all the conditions of Theorem 3.2 are satisfied here while some of the parametric conditions in Theorem 3.1 are violated. Thus, the unique equilibrium pattern of system is stable by virtue of Theorem 3.2.

It may be concluded from Examples 3.3 and 3.4 that Theorems 3.1 and 3.2 are independent of each other. The examples are simulated using ODE23 of MATLAB and Figures 1 and 2 picturize our theoretical conclusions. We now consider the case where all

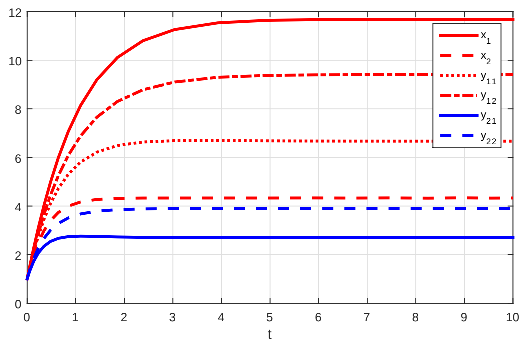


**Figure 1**: Behaviour of solutions in Example 3.3.

contribution of $y_{i_k}$ has been completely received and utilized by $x_i$ as it is. That means, we assume that $\overline{c}_{ii_k}\overline{g}_{i_k}(x_i, y_{i_k}) \equiv c_{ii_k} g_{i_k}(x_i, y_{i_k})$ for all $x_i$ and $y_{i_k}$. Our next result studies the global stability of equilibrium in this case.

**Theorem 3.5.** *Assume that the parameters of the system satisfy the following conditions:*

$$a_i - \sum_{j=1}^{n} |b_{ji}| p_i > 0, \quad c_{i_k} - \sum_{l=1}^{r_i} |d_{i_l}| q_{i_l} > 0,$$

*for all $i$ and $i_k$ and the response functions satisfy (3)-(6). Then the equilibrium pattern of (2) is globally asymptotically stable.*

**Proof.** We employ the same functional as in Theorem 3.1,

$$V(t) \;=\; \sum_{i=1}^{n} \Big[ |x_i - x_i^*| + \sum_{k=1}^{r_i} |y_{i_k} - y_{i_k}^*| \Big]. \tag{12}$$

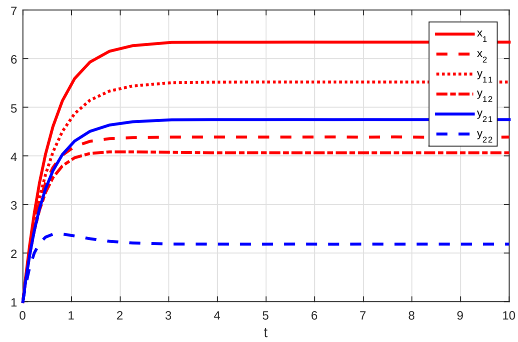


**Figure 2**: Solutions converging to equilibrium values in Example 3.4.

Then we have

$$\begin{aligned}
D^+V(t) \quad &\leq \quad \sum_{i=1}^{n}\Big[-a_i|x_i-x_i^*| + \sum_{j=1}^{n}|b_{ij}||f_j(x_j)-f_j(x_j^*)| \\
&+ \quad \sum_{k=1}^{r_i}|c_{ii_k}||g_{i_k}(x_i,y_{i_k})-g_{i_k}(x_i^*,y_{i_k}^*)| + \sum_{k=1}^{r_i}\Big[-c_{i_k}|y_{i_k}-y_{i_k}^*| \\
&+ \quad \sum_{l=1}^{r_i}|d_{i_l}||h_{i_l}(y_{i_l})-h_{i_l}(y_{i_l}^*)| - \sum_{k=1}^{r_i}|\overline{c}_{ii_k}||\overline{g}_{i_k}(x_i,y_{i_k})-\overline{g}_{i_k}(x_i^*,y_{i_k}^*)|\Big]\Big] \\
&\leq \quad \sum_{i=1}^{n}\Big[-a_i|x_i-x_i^*| + \sum_{j=1}^{n}|b_{ij}|p_j|x_j-x_j^*| \\
&+ \quad \sum_{k=1}^{r_i}\Big[-c_{i_k}|y_{i_k}-y_{i_k}^*| + \sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}|y_{i_l}-y_{i_l}^*|\Big]\Big].
\end{aligned}$$

Therefore,

$$D^+V(t) \leq -\sum_{i=1}^{n}\Big[[a_i-\sum_{j=1}^{n}|b_{ji}|p_i]\ |x_i-x_i^*| + \sum_{k=1}^{r_i}[c_{i_k}-\sum_{l=1}^{r_i}|d_{i_l}|q_{i_l}]\ |y_{i_k}-y_{i_k}^*|\Big]$$

Negative definiteness of $D^+V$ follows from assumptions on parameters. The rest of the argument is similar to that of Theorem 3.1, and thus, omitted.

**Remark 3.6.** Two types of approaches are possible here. For system (1), where the dynamics of subnetwork neurons $y_{i_k}$ (i.e., second equation of (1)) do not include terms of main components $x_i$, the subnetworks are allowed to converge first, $x_i$ waits to receive this contribution and then starts working on its own for a convergence - as worked out in Theorem 4.1 of [26]. Secondly, the case where $x_i$ works together with $y_{i_k}$ and interacts continuously with them for a simultaneous convergence was discussed in Corollary 2.3 of [19]. First situation may be called as a 'serial processing' – elongates the convergence process but the strain on the parameters is considerably less when compared to that in second situation which may be termed as a 'parallel processing'.

It may be noticed from Theorem 3.5 here that the strain on parameters is very less as compared to that of Theorem 4.1 of [26] at the same time allows interactions of $y_{i_k}$'s with $x_i$ as Corollary 2.3 of [19]. Thus, influence of deactivation term $\overline{g}_{i_k}(x_i, y_{i_k})$ in second equation is clear. This also indicates that when the subcomponents contribute exactly what their main components require and the main components receive what they need with a proper interaction with their subcomponents then the system parameters are strained less and thus, paving way for a better performance of the system.

## 4 Discussion

In the present paper, we studied the influence of deactivation dynamics introduced into the supportive subnetwork of a cooperative and supportive network system. We established sufficient conditions for global asymptotic stability of the equilibrium pattern. Examples are provided to establish that the criteria presented are independent of each other. It was assumed in [26] that the subnetworks of the main group always support it. If the subnetwork is an ancillary unit established independently of the main system (but always supports it) and survives on its own (has independent, own dynamics – second equation of system(1)), then main system has no burden. In case if the subnetwork is an ancillary unit that survives only because of main network or is an integral part of the main system which needs to be defunct as soon as the task of main network is finished either to reduce or to avoid unnecessary use of $y_{i_k}$'s, then system (2) comes into play and the study in this paper becomes very relevant and useful. A look at Theorems 3.1 and 3.2 shows that the parameters have to be strained much when the contribution from subnetwork is not utilized as it is or is not known to be the same as that required by main network. On the other hand, the strain on parameters is much less for systems which utilize contributions of its subnetworks completely or equivalently, the subnetworks are contributing exactly what their main group is expecting from them. This is what Theorem 3.5 says. Thus, systems with perfect coordination and cooperation among groups perform well with less strain on constituent components and resources.

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