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Nonlinear Dynamics and Systems Theory, **17** (1) (2017) 1–4

On the Occasion of the 100th Birthday of Academician Yu. A. Mitropolsky

Yury Alexeevich Mitropolsky was born on January 3, 1917, in the Charnyshs estate located in Kobelyaksky district of Poltava province. During the civil war, in 1918, the Charnyshs residence was completely destroyed which was of common occurrence at that time. Consequently, the Mitropolskys moved to Kiev.

In 1932, Yury Alexeevich finished a 7-year school in Kiev and was employed at a cannery. In 1938, he graduated from a high school with honors, and in the same year he was admitted to Kiev University to the Department of Mathematics and Physics. Upon completion of his third year at Kiev University, when on the day of June 22, 1941, the fascist Germany attacked the Soviet Union, Mitropolsky married his university mate Alexandra Likhacheva to live together happily for more than 60 years.

On July 7, 1941, Mitropolsky joined the Soviet Army and was stationed in an armor division in the town of Chuguyev. In October of 1941, according to the decree issued by the Defense Secretary S. K. Timoshenko, all fourth and fifth year college students were eligible to continue their degrees at the corresponding universities, with forthcoming appointments at military academies. Mitropolsky was sent to the town of Kzyl-Orda in Kazakhstan, where Kiev University was evacuated to. In March of 1942, he successfully passed all the exams and graduated from Kazakh University. Right after that, he was enrolled to Ryazan Artillery Academy in the town of Talgar, which he graduated from in March of 1943 in the rank of a lieutenant. Thereafter, he was sent to the Stepnoy battlefield.

In 1946, after being discharged from the army, Mitropolsky joined the Ukrainian Academy of Sciences in Kiev in the capacity of a junior research fellow. In 1948, Mitropolsky received his Candidate of Science degree (the equivalent of Ph. D. degree). His thesis was titled "The investigation of resonance phenomena in nonlinear systems with variable frequencies". In the same year he joined the Institute of Constructive Mechanics of the Ukrainian Academy of Sciences (now S. P. Timoshenkos Institute of Mechanics of the National Academy of Sciences of Ukraine) in the capacity of a senior research fellow to work under the supervision of N. N. Bogoliubov. In 1951 he received Doctor of Science degree (the equivalent of Habilitation Degree). His thesis was titled "Slow processes in nonlinear oscillatory systems with many degrees of freedom". Earlier he switched over to the Institute of Mathematics of the Ukrainian Academy of Sciences where he took up a position as a senior research fellow. In 1953 Mitropolsky was promoted to the rank of Professor and the Head of Department at the same Institute. In 1956 he became the Associate Provost of Science of this Institute and in 1958 became its Director. He had remained in this capacity up until 1988. Since 1988 he had served as the Honorary Director of the Institute of Mathematics.

In 1958, Mitropolsky was elected the Corresponding Member of the Academy of Sciences of Ukrainian SSR and in 1984 he became the Full Member of the Academy of Sciences of the USSR (now the Russian Academy of Sciences), being at that time the most distinguished academic rank in the USSR.

During the years of his fruitful scientific activity, Mitropolsky had obtained numerous fundamental results in nonlinear mechanics and differential equations. The results of his prolific research were manifested in more than 700 papers and 50 monographs, of which most essential are "Nonstationary Processes in Nonlinear Oscillating Systems" (1955), "Asymptotic Methods in the Theory of Nonlinear Oscillations" (1964), "Averaging Method in Nonlinear Mechanics" (1971), and "Nonlinear Mechanics. Single-Frequency Oscillations" (1997).

The main directions of his science investigations are as follows:

- development of asymptotic methods in nonlinear mechanics;
- development of the single-frequency method;
- contribution to the method of integral manifolds;
- the method of accelerated convergence;
- the averaging method;
- asymptotic methods and averaging method for distributed parameter systems;

- contribution to the theory of systems with delay and small parameter;
- development of the theory of random oscillating processes;
- contribution to the theory of decomposition of systems.

An overview of his most significant works was published in the Journal of Nonlinear Dynamics and Systems Theory **6** (4) (2006) 309–318.

Since 1958, Mitropolsky had focused his attention on the development of the Institute of Mathematics of the Academy of Science of USSR. He initiated new departments setting up to facilitate research in the areas of algebra, probability theory, real and functional analysis and mechanics of special systems.

During that period of time, the post-graduate enrollment was substantially expanded. As the result of Mitropolsky's efforts, the Institute produced about 500 candidates of science and more than 80 doctors of science for their further employments at national universities and research labs in Ukraine, Russia, and other countries. As a consequence of Mitropolsky's colossal scholarly activity, the Institute of Mathematics of the Academy of Science of Ukranian SSR has become the leading scientific center of mathematical research in Ukraine.

Mitropolsky began his pedagogical activity in 1948 at Kiev University to extend it up to 1989. Mitropolsky himself supervised and directed 100 Ph.D. and 25 Habilitation theses in physical and mathematical sciences.

From 1961 to 1992 Mitropolsky had been the Head of the Department of Mathematics, Mechanics and Cybernetics at the Academy of Sciences of Ukrainian SSR. In 1992 Mitropolsky was appointed the Director of the International Mathematical Center of the National Academy of Science of Ukraine and the Counselor of the Presidium of the National Academy of Science of Ukraine. He had held this position until his death.

Mitropolsky had been much involved in editorial work. Since 1967, he had been the Editor-in Chief of the "Ukrainian Mathematical Journal" whose English translation is regularly published in the US. Since 1961, he had been an editorial board member of three Russian and three international journals. Mitropolsky was among main contributors to the 12-volume selected works by N.N. Bogoliubov in the area of mathematics and nonlinear mechanics.

The first international talk by Mitropolsky was given in 1956 at the International Congress of Mathematicians in Bucharest, Romania. Since 1958, he had been an invited speaker to the International Mathematical Congresses held consecutively in Edinburgh, Scotland (1958), Stockholm Sweden (1962), Moscow, Russia (1966), Niece, France (1970), Vancouver, Canada (1974), Warsaw, Poland (1983), Berkeley, USA (1986), and Kyoto, Japan (1990). A series of lectures and talks on particular problems in nonlinear mechanics were delivered by Mitropolsky at various universities in the USA, China, Vietnam, Czechoslovakia, Poland, Mexico, Canada, Italy, and Yugoslavia and at numerous international conferences. Also, his active cooperation over the past two decades with Vietnamese scientists in the area of nonlinear mechanics and theory of differential equations is worth mentioning.

Mitropolsky has been one of the most celebrated scientists who has ever lived in Ukraine and Russia. Consequently, his research, scholarly and pedagogical activities and public service have been highly revered. He was awarded by almost all known highest and most prestigious prizes ever given to a Soviet citizen. Here is the list of some of them: Hero of the Socialist Labor; Honored Activist of Science of UkrSSR; Lenin Prize

Laureate; State Prize Laureate of Ukraine; Federal Prize Laureate of the Soviet Union; Lyapunov Golden Medal; Certificate of the Presidium of Supreme Soviet; Certificate of the Presidium of Supreme Soviet of UkrSSR; Lenin Golden Medal; two Red Star Orders; October Revolution Medal; Labor Red Banner Medal; Second-Degree Great Patriotic War Medal; Fifth Degree Yaroslav Mudryi Order; Bogdan Hmelnitskiy Medal; N. M. Krylov, N. N. Bogoliubov and M. A. Lavrentiev Prizes of the Presidium of the Academy of Sciences of Ukrainian SSR.

As surely as inside his country, Mitropolsky has been treated with a highest honor outside Ukraine and Russia. In 1971, he was elected the foreign member of Bologna Academy of Sciences (Italy). He was also awarded with Silver Medal of the Czechoslovak Academy of Sciences "For Achievements in Science and Deeds for the Mankind". The government of Vietnam awarded him with the Friendship Medals in 1987 and 2001.

On the 14th of June, 2008 the heart of Yury Alexeevich Mitropolsky stopped beating. The inscription on the gravestone at the Baikov cemetery in Kiev says: "The world of mathematics was the world of his life, his religion of purity and perfection" expressing the essence of the life and activity of the great mathematician and mechanical scientist of the 20th century.

Editorial Board of the Journal of Nonlinear Dynamics and Systems Theory respectfully notes his towering stature in Ukrainian and international mechanics and mathematics, his remarkable many-sided talent and outstanding organizational skills and human qualities.

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Approximate Controllability of Non-densely Defined Semilinear Control System with Non Local Conditions

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Abstract: The present paper is devoted to the study of approximate controllability of nondensely defined semilinear control system with nonlocal conditions. The approximate controllability is obtained with nonlinearity satisfying the monotone condition and integral contractor condition. Finally, an example is provided to illustrate the application of the obtained results.

Keywords: *approximate controllability; semilinear systems; nondense domain; nonlocal conditions.*

Mathematics Subject Classification (2010): 93B05, 93C10.

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. Kalman [16] introduced the concept of controllability for finite dimensional deterministic linear control systems. Then Barnett [3] and Curtain [5] introduced the concepts of deterministic control theory in finite and infinite dimensional spaces. Balachandran [2] and Dauer et al. [7] studied the controllability of nonlinear systems in infinite dimensional spaces. The controllability of linear and nonlinear systems in infinite dimensional spaces has been extensively studied by many authors, when the operator A is densely defined, see [2, 7, 15, 19, 21, 23, 26]. On the other hand, we sometimes need to deal with non-densely defined operator. It is a very important case, which occurs in many practical situations. For example, the space C^1 with null values on the boundaries is not dense in the space of continuous functions, see [6]. For more examples and

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details on non-densely defined operators, one can refer [6, 8]. Xianlong [10] considered this case and studied the controllability of the semilinear system with delay, in which the nonlinear function was uniformly bounded. Recently many authors have discussed this case [17, 20].

Moreover, nonlocal conditions have a better effect on the solution and are more precise for physical measurements than classical condition $x(0) = x_0$ alone. Byszewski and Lakshmikantham [4] introduced nonlocal conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since more information was taken into account at the oneset of the experiment, thereby reducing the ill effects incurred by a single initial measurement. Also, in controllability literature, it is common to use fixed point theory to prove the controllability of the system, which makes it necessary to assume certain inequality conditions involving system constants.(For example, see inequality (3) in [9]). In this paper, it is shown that for certain type of nonlinear functions, the non-densely defined semilinear control system with nonlocal conditions is approximately controllable without assuming any inequality conditions on the system constants.

Let U and V be two Banach spaces. $Y = L_2[0, T; U]$ and $Z = L_2[0, T; V]$ be the corresponding function spaces respectively defined on $J = [0, T]$, $0 \leq T < \infty$. Consider the semilinear control system with nonlocal conditions:

$$\left.\begin{array}{rcl} \dfrac{dy(t)}{dt} & = & Ay(t) + Bv(t) + f(t, y(t)) \ \text{ for } t \in (0, T], \\ y(0) & = & y_0 + g(y), \end{array}\right\} \tag{1.1}$$

where the state $y(t)$ takes values in space V and $v : [0, T] \to U$ is the control function. B is a bounded linear operator from U into V. The map $f : [0, T] \times V \to V$ is a purely nonlinear function and $g(y)$ is a continuous function from $C(J, V) \to V$. $A : D(A) \subset V \to V$ is a closed (not necessarily bounded) linear operator whose domain $D(A)$ need not be dense in V. The linear system corresponding to (1.1) is given by

$$\left.\begin{array}{rcl} \dfrac{dy(t)}{dt} & = & Ay(t) + Bu(t) \ \text{ for } t \in (0, T], \\ y(0) & = & y_0 + g(y), \end{array}\right\} \tag{1.2}$$

where $u : [0, T] \to U$ is the control function for the linear system.

2 Preliminaries

We introduce the integrated semigroup.

Definition 2.1 Let V be a Banach space. A one parameter family of bounded linear operators $\{S'(t) : t \geq 0\}$ from V into itself is said to be an integrated semigroup on V if

1. $S'(0) = 0$.
2. $t \to S'(t)$ is strongly continuous.
3. $S'(s)S'(t) = \int_0^s \{S'(t + r) - S'(r)\}dr = S'(t)S'(s)$; for all $t, s \geq 0$.

Definition 2.2 [10] A function $y : [0, T] \to V$ is said to be an integrated solution of the system (1.1) if the following conditions hold

1. y is continuous on $[0, T]$.

2. $\int_0^t y(s)ds \in D(A)$; for all $t \in J$.

3. $y(t) = (y_0 + g(y)) + A\int_0^t y(s)ds + \int_0^t [Bv(s) + f(s, y(s))]ds$.

Definition 2.3 [27] An operator A is called a generator of an integrated semigroup if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ (the resolvent set of A) and there exists a strongly continuous exponentially bounded family $\{S'(t) : t \geq 0\}$ of bounded linear operators such that $S'(0) = 0$ and $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} S'(t)dt$ for all $\lambda > \omega$.

Let $y(T, y_0, v)$ denote the state value of the system (1.1)at time T corresponding to the control $v \in Y$ and the initial value y_0. Now, we introduce the set defined by

$$K_T(f) = \{y(T, y_0, v); v \in Y\}$$

which consists of all the possible final states and is called the reachable set of the system.

Definition 2.4 A control system is said to be approximate controllable on $[0, T]$, if $K_T(f)$ is dense in $\overline{D(A)}$, that is $\overline{K_T(f)} = \overline{D(A)}$.

Throughout this paper, the operator A is assumed to satisfy the following Hille-Yosida condition(without being densely defined), see [25]:
(H_0) there exists a constant $\overline{M} \geq 0$ and $\overline{\omega} \in N$ such that $(\overline{\omega}, \infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \overline{\omega})^n ||R(\lambda, A)^n|| : n \in N \text{ and } \lambda > \overline{\omega}\} \leq \overline{M}, \tag{2.1}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.

It is well known that the above condition is equivalent to the fact that operator A is the generator of a locally Lipschitz integrated semigroup $\{S'(t) : t \geq 0\}$ on V, see [18].

Let A_0 be the part of A defined on the domain

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)} \text{ and } A_0 x = Ax, \text{ for all } x \in D(A_0)\}.$$

Then $\overline{D(A_0)} = \overline{D(A)}$ and the generator A_0 generates a C_0- semigroup $\{T_0(t) : t \geq 0\}$ on $\overline{D(A)}$, see [18]. If the integral solution, as given in Definition 2.2 exists then it is given by

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda \to \infty} \int_0^t T_0(t - s)C(\lambda)[Bv(s) + f(s, y(s))]ds, \tag{2.2}$$

where $C(\lambda) = \lambda R(\lambda, A) = \lambda(\lambda I - A)^{-1}$.

Now, we define the following functions:
$F : Z \to Z$ as

$$(Fy)(t) = f(t, y(t)), \quad y \in Z,$$

and $K : Z \to Z$ as

$$(Ky)(t) = \lim_{\lambda \to \infty} \int_0^t T_0(t - s)C(\lambda)y(s)ds.$$

3 Controllability Results with Monotone Nonlinearity

In this section, we prove the controllability results of the system when the nonlinear function satisfies monotone condition. To prove the approximate controllability of the system (1.1), we assume the following conditions:

(H_1) There exists a constant $\mu > 0$ such that for all $x \in D(A)$

$$< -Ax, x >_V \geq \mu ||x||_V^2.$$

(H_2) Linear system is approximate controllable up to $\overline{D(A)}$.

(H_3) The semigroup $\{T_0(t), t \geq 0\}$ generated by A is compact on $\overline{D(A)}$ and there is a constant $M \geq 0$ such that

$$||T_0(t)|| \leq M, \text{ for all } t \in [0, T].$$

(H_4) f satisfies monotone condition, that is, there is a positive constant β such that

$$< f(t,x) - f(t,y), x - y >_V \leq -\beta ||x - y||_V^2.$$

(H_5) $||Fy||_Z \leq a + b||y||_Z$ where a and b are constants.

(H_6) $R(F) \subseteq \overline{R(B)}$.

(H_7) The function g is a continuous function and there exists a constant M_1 such that

$$||g(y)|| \leq M_1 \text{ for all } y \in D(A).$$

This section has two cases. In subsection 3.1, the controllability is proved for the case when the control operator B is an identity operator and subsection 3.2 contains the general case.

3.1 Controllability of semilinear system when $B = I$

In this subsection, it is proved that the semilinear control system (1.1) in which B is an identity operator is approximate controllable under simple sufficient conditions. Obviously, here $V = U$. In this case, the semilinear control system (1.1) becomes

$$\left.\begin{array}{rcl} y'(t) & = & Ay(t) + Bv(t) + f(t, y(t)); \quad 0 \leq t \leq T, \\ y(0) & = & y_0 + g(y), \end{array}\right\} \tag{3.1}$$

and the system (1.2) becomes

$$\left.\begin{array}{rcl} y'(t) & = & Ay(t) + Bu(t); \quad 0 \leq t \leq T, \\ y(0) & = & y_0 + g(y). \end{array}\right\} \tag{3.2}$$

Before proving the main result, we prove one lemma.

Lemma 3.1 *Under the condition* (H_1)*, the operator* K *which is defined on* Z *satisfies the condition*

$$< Kx, x >_Z \geq \mu ||x||_Z^2 \textit{ for all } x \in Z. \tag{3.3}$$

Proof. Let $x \in Z$. Now, let us define a function ϕ as follows

$$\phi(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)x(s)ds$$

since

$$< Kx, x >_Z = \int_0^T < Kx(t), x(t) >_V dt = \int_0^T < \phi(t), x(t) >_V dt. \tag{3.4}$$

But

$$\begin{aligned} \phi'(t) &= \lim_{\lambda\to\infty} \left[C(\lambda)x(t) + A\int_0^t T_0(t-s)C(\lambda)x(s)ds \right] \\ &= x(t) + A\phi(t) \\ \Rightarrow x(t) &= \phi'(t) - A\phi(t). \end{aligned} \tag{3.5}$$

From the equations (3.4) and (3.5), we get

$$\begin{aligned} < Kx, x >_Z &= \int_0^T < \phi(t), \phi'(t) - A\phi(t) >_V dt \\ &= \int_0^T < \phi(t), \phi'(t) >_V dt + \int_0^T < \phi(t), -A\phi(t) >_V dt. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} \frac{d}{dt} < \phi(t), \phi(t) >_V &= < \phi(t), \phi'(t) >_V + < \phi'(t), \phi(t) >_V dt \\ &= 2 < \phi(t), \phi'(t) >_V \\ \Rightarrow \int_0^T \frac{d}{dt} < \phi(t), \phi(t) >_V dt &= 2\int_0^T < \phi(t), \phi'(t) >_V dt \\ \Rightarrow \int_0^T < \phi(t), \phi'(t) >_V dt &= \frac{1}{2} ||\phi(T)||_V^2 \geq 0 \end{aligned} \tag{3.7}$$

and by the condition (H_1), we have

$$\int_0^T < \phi(t) - A\phi(t) >_V dt \geq \mu ||x||^2. \tag{3.8}$$

Therefore, from the equations (3.6),(3.7) and (3.8), we get

$$< Kx, x >_Z \geq \mu ||Kx||_Z^2.$$

Theorem 3.1 *Under the conditions* $(H_0) - (H_4)$, *the semilinear control system (3.1) is approximate controllable in the time interval* $[0, T]$.

Proof. Let $x(t)$ be the integral solution of the system (3.2) corresponding to the control u. Consider the following semilinear system

$$\left.\begin{aligned} y'(t) &= Ay(t) + u(t) - f(t, x(t)) + f(t, y(t)); \quad 0 \leq t \leq T, \\ y(0) &= y_0 + g(y). \end{aligned}\right\} \tag{3.9}$$

Comparing (3.1) and (3.9), it can be seen that the control function $v(t)$ is chosen that

$$v(t) = u(t) - f(t, x(t)). \tag{3.10}$$

The integral solutions of systems (3.2) and (3.9) can be written as

$$x(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)u(s)ds, \tag{3.11}$$

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[u(s) - f(s, x(s)) + f(s, y(s))]ds. \tag{3.12}$$

Substracting (3.12) from (3.11), we get

$$x(t) - y(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[f(s, x(s)) - f(s, y(s))]ds, \tag{3.13}$$

which in operator theoretic terms can be written as

$$x - y = KFx - KFy.$$

Taking inner product on both sides in Z with $Fx - Fy$, we get

$$< x - y, Fx - Fy >_Z = < KFx - KFy, Fx - Fy >_Z . \tag{3.14}$$

Note, that the left-hand side satisfies the condition (H_4) and it is less than or equal to $-\beta||x-y||_Z^2$ and the right-hand side is nonnegative, from Lemma 3.1. This is possible only when $x = y$ in Z, which implies that $F(x) = F(y)$, where F is Nemytskii operator defined by f. Therefore, from the equation (3.13), it follows that $x(t) = y(t)$ for all $t \in [0, T]$. Thus, any mild solution of the linear system (3.2) is also a mild solution of the semilinear system (3.1), that is, $K_T(f) \supset K_T(0)$, which is dense in $\overline{D(A)}$. Hence, system (3.1) is approximate controllable on $[0, T]$.

3.2 Controllability of semilinear system when $B \neq I$

In this subsection, the approximate controllability of the system (1.1) is proved under some sufficient conditions on the operators A, B and f.

Since $R(F) \subseteq \overline{R(B)}$, (see condition (H_6)), for any given $\epsilon_1 > 0$, there exists a ω in $L_2[0, T; U]$ such that

$$||Fx - Bw||_Z \leq \epsilon_1. \tag{3.15}$$

Before proving the main result, we prove one lemma.

Lemma 3.2 *The solution of the system (1.1) and the corresponding control $v = u - w$ satisfy the following inequality*

$$\begin{aligned} ||y(t)||_V \quad &\leq \quad [M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ &\quad +M\overline{M}aT]e^{M\overline{M}bT}, \end{aligned}$$

where M is a positive constant such that $||T_0(t)|| \leq M$, for each $t \in [0, T]$ and $\overline{M}$ is defined in (2.1).

Proof. Let $y(t)$ be the integral solution of the system (1.1) corresponding to the control $v = u - w$. Then the integral solution of the system (1.1) can be written as

$$\begin{aligned} y(t) &= T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)B(u-w)(s)ds + \\ & \lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds. \end{aligned}$$

Taking V-norm on both sides and using the fact that $\lim_{\lambda\to\infty}||C(\lambda)|| = \overline{M}$, we get

$$\begin{aligned} ||y(t)||_V &\leq M||y_0|| + M||g(y)|| + M\overline{M}\int_0^t ||B(u-w)(s)||_V ds \\ &\quad +M\overline{M}\int_0^t ||f(s,y(s))||_V ds \\ &\leq M||y_0|| + M||g(y)|| + M\overline{M}\sqrt{T}(||Bu||_Z + ||Bw||_Z) \\ &\quad +M\overline{M}\int_0^t (a + b||y(s)||_V)ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + ||Fx||_Z + \epsilon_1) + M\overline{M}aT \\ &\quad +M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + a + b||x||_Z + \epsilon_1) + M\overline{M}aT \\ &\quad +M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}(||Bu||_Z + a + bM||y_0|| + bMM_1 \\ &\quad +M\overline{M}bT||Bu||_Z + \epsilon_1) + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M||y_0|| + MM_1 + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + bM||y_0|| + bMM_1 \\ &\quad +\epsilon_1\} + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq (1 + bM\overline{M}\sqrt{T})M||y_0|| + MM_1 + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a \\ &\quad +bMM_1 + \epsilon_1\} + M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds \\ &\leq M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ &\quad +M\overline{M}aT + M\overline{M}b\int_0^t ||y(s)||_V ds. \end{aligned}$$

Now, Gronwall's inequality implies that

$$\begin{aligned} ||y(t)||_V &\leq [M(1 + bM\overline{M}\sqrt{T})(||y_0|| + M_1) + M\overline{M}\sqrt{T}\{(1 + M\overline{M}bT)||Bu||_Z + a + \epsilon_1\} \\ &\quad +M\overline{M}aT]e^{M\overline{M}bT}, \end{aligned}$$

which completes the proof.

The main result of this chapter is given below.

Theorem 3.2 *Under the conditions $(H_0)-(H_7)$, the semilinear system (1.1) is approximate controllable in the time interval $[0,T]$.*

Proof. Let $x(t)$ be the integral solution of the system (1.2) corresponding to the control u, which can be written as

$$x(t)=T_0(t)(y_0+g(y))+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)(Bu)(s)ds. \tag{3.16}$$

Let $y(t)$ be the integral solution of the system (1.1) corresponding to the control $v=u-w$, which can be written as

$$\begin{aligned} y(t) &= T_0(t)(y_0+g(y))+\lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)B(u-w)(s)ds \\ &\quad +\lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds. \end{aligned} \tag{3.17}$$

From the equations (3.16) and (3.17), we get

$$x(t)-y(t)=\lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)Bw(s)ds-\lim_{\lambda\to\infty}\int_0^t T_0(t-s)C(\lambda)f(s,y(s))ds, \tag{3.18}$$

which in operator theoretic terms can be written as

$$\begin{aligned} x-y &= KBw-KFy \\ &= K(Bw-Fx)+(KFx-KFy). \end{aligned}$$

Taking inner products on both sides in Z with $Fx-Fy$, we get

$$\begin{aligned} <x-y,Fx-Fy>_Z &= <K(Bw-Fx)+(KFx-KFy),Fx-Fy>_Z \\ &= <K(Bw-Fx),Fx-Fy>_Z \\ &\quad +<(KFx-KFy),Fx-Fy>_Z\,. \end{aligned} \tag{3.19}$$

Since, by condition (H_4), the left-hand side of the equation (3.19) is less than or equal to $-\beta||x-y||^2$ and from Lemma 3.1, the second term of the right-hand side is nonnegative, if $<K(Bw-Fx^L),Fx^L-Fx>_Z$ is negligibly small, then from the equation (3.19), it follows that $||x-y||_Z$ is also arbitrary small.

Now, we show that $<K(Bw-Fx^L),Fx^L-Fx>_Z$ is arbitrarily small. For it, by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} |<K(Bw-Fx),Fx-Fy>_Z| &\le ||K(Bw-Fx)||_Z||Fx-Fy||_Z \\ &\le M\overline{M}T||Bw-Fx||_Z\{||Fx||_Z+||Fy||_Z\} \\ &\le M\overline{M}T\epsilon_1\{a+b||x||_Z+a+b||y||_Z\}. \end{aligned} \tag{3.20}$$

From the equation(3.20) and Lemma 3.2, we get

$$|<K(Bw-Fx),Fx-Fy>_Z|\le M\overline{M}CT\epsilon_1, \tag{3.21}$$

where $C=2a+b[M||y_0||+MM_1+M\overline{M}T||Bu||_Z]+b[M(1+bM\overline{M}\sqrt{T})(||y_0||+M_1)+M\overline{M}\sqrt{T}\{(1+M\overline{M}bT)||Bu||_Z+a+\epsilon_1\}+M\overline{M}aT]e^{M\overline{M}bT}$. Thus, for given u and ϵ_1, C is finite. Since ϵ_1 is arbitrarily small, it implies that $< K(Bw - Fx), Fx - Fy >_Z$ is arbitrary small.

Hence, from the equations (3.19), (3.20) and condition (H_4), it follows that $||x-y|| \leq \epsilon_2$ is arbitrary small, for some $\epsilon_2 > 0$.

Further, we prove that $||x(T) - y(T)||_V$ is arbitrary small. Now,

$$\begin{aligned} x(t) - y(t) &= KBw(t) - K(Fy)(t) \\ &= K\{Bw(t) - (Fx)(t)\} + K\{(Fx)(t) - (Fy)(t)\}. \end{aligned} \tag{3.22}$$

Taking norm on both sides in V, we have

$$\begin{aligned} ||x(t) - y(t)||_V &= ||K\{Bw(t) - (Fx)(t)\} + K\{(Fx)(t) - (Fy)(t)\}||_V \\ &\leq ||K\{Bw(t) - (Fx)(t)\}||_V + ||K\{(Fx)(t) - (Fy)(t)\}||_V. \end{aligned}$$

Now,

$$\begin{aligned} ||K\{Bw(t) - (Fx)(t)\}||_V &= \left|\left| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)\{Bw(s) - (Fx)(s)\}ds \right|\right|_V \\ &\leq M\overline{M} \int_0^t ||\{Bw(s) - (Fx)(s)\}ds||_V \\ &\leq M\overline{M}\sqrt{T}||Bw - Fx||_Z \\ &\leq M\overline{M}\sqrt{T}\epsilon_1 \end{aligned}$$

and

$$\begin{aligned} ||K\{(Fx)(t) - (Fy)(t)\}||_V &= \left|\left| \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)\{(Fx)(s) - (Fy)(s)\}ds \right|\right|_V \\ &\leq M\overline{M}\sqrt{T}||Fx - Fy||_Z. \end{aligned} \tag{3.23}$$

The right-hand side of the equation (3.23) can be made arbitrarily small as $||x-y|| \leq \epsilon_2$ and F is continuous on Z. Therefore, it is concluded that for a given ϵ and x, there exists a y such that

$$||x(t) - y(t)|| \leq \epsilon \text{ for all } t \in [0, T]. \tag{3.24}$$

Thus, $||x(t) - y(t)||$ can be made arbitrarily small by choosing suitable ω. It follows that reachable set of the system (1.1) is dense in the reachable set of the system (1.2), which is dense in $\overline{D(A)}$ due to condition (H_3). Hence the theorem is proved.

4 Controllability Results with Integral Contractor Nonlinearity

In this section, approximate controllability of semilinear control system (1.1) is considered when the nonlinear function f has integral contractor.

Let C be the Banach space of all continuous functions from $[0, T]$ to Banach space V with supremum norm. The problem of controllability of infinite dimensional semilinear control systems has been studied widely by many authors, when the nonlinear function is uniformly Lipschitz continuous or monotone, see [11, 21, 22]. In this section, we study the approximate controllability of the system (1.1), when the operator A is not densely defined and the nonlinear function satisfies integral contractor condition, which is a weaker condition in comparison with Lipschitz condition. The concept of contractor was introduced by Altman [1] as a functional analytic tool for solving deterministic operators equations in Banach spaces and subsequently this tool was exploited by many authors for the existence and uniqueness of the solution of nonlinear evolution equations, see [24]. Govindan and Joshi [14] employed this method to investigate optimal control problem and stability problems of nonlinear stochastic control system. George [12] investigated the approximate controllability of semilinear non-autonomous system with nonlinearity satisfying integral contractor condition. Further, George et al. [13] obtained the exact controllability of the third order dispersion equation through the approach of integral contractor. In this section, our aim is to obtain a result similar to that of [28], for the non-densely defined semilinear system (1.1) by replacing Lipschitz condition with integral contractor.

Now, we define the integral contractor.

Definition 4.1 [12] Let $\Gamma : J \times V \to BC(C)$ be a bounded continuous operator and γ be a positive constant such that for any $x, y \in C$, we have

$$\left|\left|f(t, \left(x(t) + y(t) + \int_0^t T_0(t-s)\Gamma(s, x(s))y(s)ds\right) - f(t, x(t)) - \Gamma(t, x(t))y(t)\right|\right| \leq \gamma ||y(t)|| \tag{4.1}$$

Then, we say that f has a bounded integral contractor $\{I + \int T_0\Gamma\}$ with respect to $T_0(t)$. The constant γ will be called the contractor constant.

Remark 4.1 If $\Gamma \equiv 0$, the condition (4.1) reduces to the Lipschitz condition, as we get

$$||f(t, x(t) + y(t)) - f(t, x(t))|| \leq \gamma ||y(t)||. \tag{4.2}$$

Definition 4.2 [12] A bounded integral contractor Γ is said to be regular if the following integral equation

$$\hat{y}(t) = z(t) + \int_0^t T_0(t-s)\Gamma(s, \hat{x}(s))z(s)ds \tag{4.3}$$

has a solution z in C for any $\hat{x}, \hat{y} \in C$.

Let us assume the following conditions:

(H_8) The nonlinear function has bounded integral contractor.

(H_9) $R(F) \subseteq R(B)$.

Theorem 4.1 *Under the conditions (H_2),(H_3), (H_8) and (H_9), the semilinear system (1.1) is approximate controllable on the time interval $[0, T]$.*

Proof. Let $x(t)$ be the integral solution of the linear system (1.2) corresponding to the control u. The integral solution of (1.2) is given by the nonlinear integral equation

$$x(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)Bu(s)ds. \tag{4.4}$$

Let $y(t)$ be the integral solution of the semilinear system (1.2) corresponding to the control v, which satisfies the equation (3.10). Then $y(t)$ can be written as

$$y(t) = T_0(t)(y_0 + g(y)) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[Bu(s) - f(s, x(s)) + f(s, y(s))]ds. \tag{4.5}$$

From (4.4) and (4.5) for all $t \in [0, T]$, we have

$$y(t) - x(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s))]ds. \tag{4.6}$$

By the regularity condition of the integral contractor Γ with $\hat{x} = x$ and $\hat{y} = y - x$, there exists $z \in C$ such that

$$y(t) - x(t) = z(t) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)\Gamma(s, x(s))z(s)ds. \tag{4.7}$$

By the equation (4.6), we have

$$\lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s))]ds = z(t) \tag{4.8}$$
$$+ \lim_{\lambda\to\infty} \int_0^t T_0(t-s)\Gamma(s, x(s))z(s)ds$$
$$\Rightarrow z(t) = \lim_{\lambda\to\infty} \int_0^t T_0(t-s)C(\lambda)[f(s, y(s)) - f(s, x(s)) - \Gamma(s, x(s))z(s)]ds. \tag{4.9}$$

Since

$$||C(\lambda)|| \leq \frac{\lambda\overline{M}}{\lambda - \varpi} \to \overline{M} \text{ as } \lambda \to \infty,$$

taking V-norm on both sides of (4.8), we have

$$||z(t)|| \leq M\overline{M} \int_0^t ||f(s, y(s)) - f(s, x(s)) - \Gamma(s, x(s))z(s)||ds. \tag{4.10}$$

By the condition (H_7), the nonlinear function f has a regular integral contractor, we have

$$\left|\left| f\left(t, x(t) + z(t) + \int_0^t T_0(t-s)\Gamma(s, x(s))z(s)ds\right) - f(t, x(t)) - \Gamma(t, x(t))z(t) \right|\right| \leq \gamma||z(t)||. \tag{4.11}$$

Now, using the equation (4.6), we get

$$||f(t, y(t)) - f(t, x(t)) - \Gamma(t, x(t))z(t)|| \leq \gamma||z(t)||. \tag{4.12}$$

From (4.10) and (4.12), we have

$$||z(t)|| \leq \gamma M\overline{M} \int_0^t ||z(s)||ds.$$

Thus, by using Gronwall's inequality, we get $||z(t)|| = 0$. Therefore by (4.7), we have that $x(t) = y(t)$ for all $t \in [0, T]$. Hence the set of the solutions of the linear system (1.2) is equal to the set of all solutions of the semilinear system (1.1). Here, $K_\tau(f) \supset K_\tau(0)$, which is dense in $\overline{D(A)}$. Therefore, system (1.1) is approximate controllable on $[0, T]$.

5 Example

Consider the following partial differential equation with nonlocal conditions of the form

$$\begin{aligned} \frac{\partial}{\partial t} y(t,x) &= y_{xx}(t,x) + Bv(t,x) + f(t, y(t,x)), \\ y(t,0) &= y(t,\pi) = 0, \\ y(0,x) &= y_0(x) + g(y), \quad 0 \le x \le \pi, 0 \le t \le T. \end{aligned} \tag{5.1}$$

In order to write system (5.1) in the abstract form (1.1), choose $V = C[0,\pi]$(with sup norm) and consider the operator A defined by

$$Aw = w'' = \frac{d^2 w}{dx^2}$$

with the domain

$$D(A) = \{w \in V : w, w' \text{ are absolutely continuous }, w'' \in V, w(0) = w(\pi) = 0\}.$$

Then, $\overline{D(A)} = \{w \in V : w(0) = w(\pi) = 0\}$. It is clear that $\overline{D(A)} \neq V$, and the resolvent set, $\rho(A) \supseteq (0, +\infty)$

$$||(\lambda I - A)^{-1}|| \le \frac{1}{\lambda}$$

for $\lambda > 0$. This implies that A satisfies the Hille-Yosida condition (H_0) on V. It is well known that A generates a C_0 semigroup $T_0(t)$ on $\overline{D(A)}$ for all $t \ge 0$, see [25] and linear system corresponding to (5.1) is approximate controllable in the space $\overline{D(A)}$(condition H_2 is satisfied). Hence, Theorem 4.1 implies the approximate controllability of system (5.1) for any nonlinear function which has integral contractor.

6 Conclusion

In this paper, we introduced a set of sufficient conditions for the approximate controllability of the semilinear control system with nonlocal conditions (1.1) with an important case in which operator A need not be densely defined. Two types of nonlinearity were considered; namely nonlinearity satisfying a monotone condition and nonlinearity having an integral contractor. Some approaches made by earlier authors led to certain inequality conditions involving various system constants. But, in this approach, there is no need of any inequality condition to prove the approximate controllability of system (1.1).

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Integro-differential Equations, Compact Maps, Positive Kernels, and Schaefer's Fixed Point Theorem

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Abstract: Integral equations offer a natural fixed point mapping, while an integro-differential equation

$$x'(t) = -g(t, x(t)) - \int_0^t A(t-s)f(x(s))ds$$

often prompts us to write it as an integral equation. This can be a mistake. We can convert it to an integral equation by a direct fixed point substitution which yields a very fixed point friendly equation. It is a natural sum of a continuous and compact map. Our first contribution here is to note that the direct fixed point process can change the continuous map to a compact map so that we have the sum of two compact maps and it is ready for Schauder's theorem instead of the much more complicated Krasnoselskii theorem which we usually expect to need. Schauder's theorem still requires that we find a self mapping set and that can be difficult. So we continue and combine the direct fixed point process with positive kernel theory so that we have an automatic *a priori* bound on all possible solutions of a homotopy equation. This gives us existence and boundedness of solutions of a wide class of problems from applied mathematics and it solves a classical problem which has been raised several times since 1960. See "Main note" following (3.1). That first contribution is a continuation of an earlier brief note in which we discovered that the direct fixed point process changed a Lipschitz map of an integro-differential equation into a contraction map, while here it changes a continuous map into a compact map.

Keywords: *integro-differential equations; compact maps; positive kernels; Schaefer's theorem; existence; uniqueness; fixed points.*

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1 Introduction

One of the two main goals of this paper is to illustrate a new compactness technique in preparing fixed point mappings for integro-differential equations. It allows us to establish a combination Schauder-Krasnoselskii fixed point theorem especially suited to integro-differential equations. We obtain a fixed point when the mapping is the sum of a compact mapping and a continuous mapping. The result is parallel to one which we presented in [2] in which we showed how to reduce a Lipschitz property to a contraction.

In the classical theory of scalar integro-differential equations

$$y'(t) = -g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds$$

as treated in MacCamy and Wong [13] there is the nagging question of existence and uniqueness of solutions. It is explicitly raised at the bottom of page 6 of their treatment and again by the reviewer in the Mathematical Reviews MR0293355 (45 #2432). Moreover, the question traces back to statements by Levin [11, p. 534]. That paper was the one generating work by Halanay [8] on the use of strongly positive kernels to obtain asymptotic stability properties. In turn, an error in Halanay's paper noted by the reviewer in MR 0176304 (31 #579) generated the work by MacCamy and Wong. This brings us back, then, to Levin's paper with $g(t, x) = 0$, in which he states (but neither proves nor references) that it is possible to use successive approximations, together with a Lipschitz condition on f in every region $|x| \leq X < \infty$ to obtain a unique solution. Our "Main note" following (3.1) will point out that their problem does have a solution on any interval $[0, E]$ and the Lipschitz condition is not needed except for uniqueness.

The papers by Levin, Levin and Nohel, Halanay, and MacCamy and Wong generated a flurry of activity which largely abated by 1990 and as far as we know the existence and uniqueness question was never settled. In any case, we treat the problem of existence as an example of the method we present here as outlined in items 1 through 4 after this paragraph. Much of the work through 1989 is reported in the book by Gripenberg, Londen, and Staffans [7]. As this equation plays a fairly prominent role in some areas of applied mathematics (see for example Levin [11, p. 535] referring to Levin and Nohel [12] on work with reactor dynamics) and as it occupies a position in the general theory of integro-differential equations, it seems to be in order to offer a solution to the question using a fixed point theorem of Schaefer which adds yet another application of the work of MacCamy and Wong. There are four steps and each one seems interesting.

1. Cast the problem in the form of a "direct fixed point mapping", to be explained later.

2. Introduce a Liapunov function yielding an *a priori* bound on all possible solutions including the solution whose existence we prove.

3. Invoke a result that the kernel will generate a compact map.

4. Show that the idea of a direct fixed point mapping allows us to show that continuity of the function $g(t, x)$ generates a compact map. We believe that this idea is new and is a counterpart to a recent result in [2] that if g is Lipschitz then the direct fixed point mapping allows us to show that it generates a contraction mapping.

These properties will fulfill the conditions of Schaefer's fixed point theorem and yield at least one solution on any interval $[0, E]$, $0 < E < \infty$. A review of standard treatments, such as Miller [14, pp. 93-98], shows that investigators generally are willing to accept this as implying that there is a solution on $[0, \infty)$. A look at the figure of Hartman [10, p.

19] raises questions on just how we continue and get a solution on $[0,\infty)$. In fact, there is reason for unease. A close look at (e.g. [9, p. 42]) tells us that this is a deep problem, solved by invocation of Zorn's lemma.

There is a way out if we ask conditions implying uniqueness such as [3, p. 11]. In a sense, this is a return to the old idea of uniqueness implying existence. For with uniqueness we can present a very clear proof of the existence of a unique solution on $[0,\infty)$ and that appears as the last paragraph of this paper.

2 The Mappings and Continuity

We begin with a scalar integro-differential equation

$$y'(t) = -g(t, y(t)) - \int_0^t A(t-s) f(y(s))ds, \quad y(0) = a \in \Re, \quad ' = d/dt, \tag{2.1}$$

in which $f : \Re \to \Re, g : [0,\infty) \times \Re \to \Re$ are both continuous, $A : (0,\infty) \to \Re$ is continuous, and if $\phi : [0,\infty) \to \Re$ is continuous so is $\int_0^t A(t-s)\phi(s)ds$.

Moreover, because we ask no growth condition on either f or g and we seek a global solution it is clear that we must ask sign conditions such as

$$yf(y) \geq 0 \text{ and } yg(t,y) \geq 0. \tag{2.1a}$$

Thus, (2.1) includes both

$$y'(t) = -g(t,y) \text{ and } y'(t) = -\int_0^t A(t-s) f(y(s))ds \tag{2.1b}$$

which are an elementary differential equation and an equation of the type considered by Levin [11]. If we review the conditions of Levin's problem we see that the properties of the kernel which allowed his analysis would be destroyed if we integrate that last equation to get an integral equation which would define a natural fixed point mapping. And that is the case in so many integro-differential equations: they come to us with a very useful kernel and we strive to keep that kernel. In our problems here we start with a positive kernel. If we were to integrate (2.1) and interchange the order of integration then it would no longer be a positive kernel and all the theory of MacCamy and Wong would fail. The following has proven to be a very useful technique.

Under the above conditions, if (2.1) has a continuous solution $y(t)$ then $y'(t)$ is also continuous so we designate that derivative as $\phi(t)$ and write

$$y'(t) = \phi(t), \quad y(t) = y(0) + \int_0^t \phi(s)ds = a + \int_0^t \phi(s)ds. \tag{2.2}$$

Hence, (2.1) is written as

$$\phi(t) = -g\left(t, a + \int_0^t \phi(s)ds\right) - \int_0^t A(t-s) f\left(a + \int_0^s \phi(u)du\right)ds \tag{2.3}$$

which is a standard integral equation.

Given y solving (2.1), then $y(0) = a$ and $y'(t) = \phi(t)$ are both uniquely determined. Letting $y = a + \int_0^t \phi(s)ds$ in (2.1) yields exactly (2.3) so (2.3) is uniquely determined from y and (2.1). On the other hand, if ϕ satisfies (2.3) and if we set $y = a + \int_0^t \phi(s)ds$ in (2.3) we have $y'(t) = \phi(t)$ so (2.1) is uniquely determined.

If $(\mathcal{B}, \|\cdot\|)$ is the Banach space of continuous functions with $\phi \in \mathcal{B}$ if $\phi : [0, E] \to \Re$ with the supremum norm then the natural mapping defined by (2.3) is $\phi \in \mathcal{B}$ implies that

$$(P\phi)(t) = -g\left(t, a + \int_0^t \phi(s))ds\right) - \int_0^t A(t-s)f\left(a + \int_0^s \phi(u)du\right)ds. \tag{2.4}$$

Thus, if $\phi \in \mathcal{B}$ is a fixed point of P, then $\phi : [0, E] \to \Re$ is continuous and satisfies (2.3). The process is reversible and

$$y(t) = a + \int_0^t \phi(s)ds \tag{2.5}$$

is a solution of (2.1).

Continuity A proof on p. 443 of [1] will show that P is continuous when the space is restricted to $[0, E]$.

Remark 2.1 Here is a tour of the paper. In the next section, Theorem 3.1, we show that the first term on the right-hand side of (2.3) defines a compact map. In Section 4 we mention three papers giving conditions ensuring that the integral in (2.3) involving A defines a compact map. Thus, P will define a compact map and is continuous. That much is sufficient to satisfy Schauder's theorem and give a fixed point of P provided that we can find a self mapping set, M. That must be handled in a case-by-case way and it is never simple. But for the vast majority of problems like this occurring in applied mathematics the kernel satisfies a condition which makes the problem simple so we branch at this point and show how that is done, thereby obtaining a bounded solution for a wide class of problems in applied mathematics. The branch starts with the fixed point theorem of Schaefer which requires an *a priori* bound on all possible solutions of a homotopy equation. We use (2.4) to establish continuity and compactness of the mapping. We will need (2.1) to prove boundedness of all possible solutions. The method is important because we do not know how to establish the result without both equations unless we were to ask for a Lipschitz condition on both f and g. $\square$

For reference, here is Schauder's theorem [15, p. 25].

Theorem 2.1 *(Schauder's second theorem) Let $\mathcal{M}$ be a non-empty convex subset of a normed space $\mathcal{B}$. Let T be a continuous mapping of $\mathcal{M}$ into a compact set $\mathcal{K} \subset \mathcal{M}$. Then T has a fixed point.*

3 Compactness of the Maps from g and A

There are three recent papers showing that if $A(t)$ satisfies certain very general conditions then for any bounded set of function $M \subset \mathcal{B}$ the mapping $Q : M \to \mathcal{B}$ defined by $\phi \in M$ implies that

$$(Q\phi)(t) = \int_0^t A(t-s)\phi(s)ds \tag{3.1}$$

maps M into a bounded equicontinuous subset of $\mathcal{B}$ on $[0, E]$ and, hence, into a compact subset of $\mathcal{B}$.

1. The first condition [4, Theorem 5.1] says that both $A(t) = t^{q-1}, 0 < q < 1$, and its resolvent generate the required equicontinuity (Corollary 4.1).

2. There is an interesting condition for equicontinuity given by Dwiggins [6, pp. 48, 51] which asks too many technical details to be safely mentioned here, but we can give a sketch. It is assumed that for decreasing $A(t) > 0$ and

$$W(t) = \int_0^t A(s)ds \quad \& \quad H(t) = \int_0^t A(t-s)|f(s,\phi(s))|ds,$$

then

$$|H(t_2) - H(t_1)| \leq 2KW(t_2 - t_1),$$

where K is the bound on the functions in M. If $W(r)$ is continuous and tends to zero as $r \downarrow 0$ then equicontinuity follows.

3. A necessary and sufficient condition on a nonconvolution case, $C(t,s)$, is found in [5]. The function $C : (0,\infty) \times (0,\infty)$ is continuous for $t > s > 0$. It is of the fading memory type by which we mean

$$0 < s < t_2 < t_1 \implies C(t_2, s) \geq C(t_1, s).$$

For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 \leq t_2 \leq t_1, \quad t_1 - t_2 < \delta \implies \int_{t_2}^{t_1} C(t_1, s)ds < \epsilon.$$

If $\int_0^t C(t,s)ds$ is uniformly continuous on an interval $[0, E]$ then QM of (3.1) is equicontinuous.

Reminder of our original question: The MacCamy-Wong conditions in Theorem 4.1 do satisfy the condition in item 2 above [6] to ensure that A will satisfy the compactness of (3.1). This is the main addition to Theorem 5.2 needed to see that the conclusion of our Theorem 5.3 includes the fact that the MacCamy-Wong problem does have a solution on any interval $[0, E]$ when (1.5) holds.

It is known that integrals smooth so the compactness of (3.1) is not unexpected. But the compactness of g is quite new.

Theorem 3.1 *Let $g : [0, E] \times \Re \to \Re$ be continuous and let M be a closed bounded nonempty subset of $\mathcal{B}$ so that there is a $K > 0$ and if $\phi \in M$ then $\|\phi\| \leq K$. If $L : M \to \mathcal{B}$ is defined by $\phi \in M$ implies*

$$(L\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right),$$

then LM is equicontinuous.

Proof. Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that $\phi \in M$ and $0 \leq t_1 < t_2 \leq E$ with $|t_1 - t_2| < \delta$ then

$$\left|g\left(t_1, a + \int_0^{t_1} \phi(s)ds\right) - g\left(t_2, a + \int_0^{t_2} \phi(s)ds\right)\right| < \epsilon. \tag{3.2}$$

For $\phi \in M$ and $0 \leq t \leq E$ the coordinates in g satisfy $0 \leq t \leq E$ and $\left|a + \int_0^t \phi(s)ds\right| \leq |a| + EK =: J$.

For the given $\epsilon > 0$ by the uniform continuity of g on $[0, E] \times [-J, J]$ there is a $\mu > 0$ such that

$$|t_1 - t_2| < \mu \text{ and } \left| \int_0^{t_1} \phi(s)ds - \int_0^{t_2} \phi(s)ds \right| < \mu$$

imply (3.2).

Now

$$\left| \int_0^{t_1} \phi(s)ds - \int_0^{t_2} \phi(s)ds \right| \leq \int_{t_1}^{t_2} |\phi(s)|ds \leq K|t_2 - t_1| < \mu$$

if $|t_2 - t_1| < \mu/K$ so we choose $\delta = \min[\mu, \mu/K]$ to satisfy (3.2). □

Remark 3.1 Now consider putting the mapping Q of (3.1) and its properties together with the mapping L of Theorem 3.1 into the mapping (2.4). We would have the sum of two compact maps and that is a compact map and it is continuous. Suppose that this sum maps the non-empty closed bounded convex set of continuous functions M on $[0, E]$ into itself. By Schauder's theorem there is a fixed point. We will state that below. But to understand its place in the literature, consider Krasnoselskii's theorem [15, p. 31].

Theorem 3.2 (Krasnoselskii) *Let $\mathcal{M}$ be a closed convex non-empty subset of a Banach space $\mathcal{S}$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{S}$ and that*

(i) $Ax + By \in \mathcal{M}$ for each $x, y \in \mathcal{M}$.

(ii) A is compact and continuous.

(iii) B is a contraction mapping.

Then there exists $y \in \mathcal{M}$ such that $Ay + By = y$.

Notice that in Remark 3.1 we have not required L to be a contraction, but only continuous. Krasnoselskii's theorem is fundamental, but it has proven very hard to satisfy. We would ask only that P be self mapping and would state it as follows invoking Schauder's second theorem [15, p. 25] (stated in Section 2). Recall that continuity of P was mentioned with (2.4).

Theorem 3.3 *Suppose that Q in (3.1) maps bounded subsets of $\mathcal{B}$ on $[0, E]$ into bounded equicontinuous sets. Let the conditions of Theorem 3.1 hold, and suppose that M is a closed bounded convex nonempty set with P defined in (2.4) being continuous and mapping M into M. Then P has a fixed point in M.*

4 The Class of Kernels

Our work branches at this point. We have gotten past the continuous mapping by making it compact and we are going to look at one of many possible ways to proceed from here. It can be challenging to find a self mapping set and now we turn away from Schauder's theorem and, instead, look at Schaefer's theorem which asks for an *a priori* bound on all possible solutions of an equation related to (2.1). When the kernel satisfies conditions (B_1)–(B_4) of Theorem 4.1 then a Liapunov function will give that bound in a very simple way.

On p. 209 Miller [14] defines a large class of kernels which occur frequently in applied mathematics and for which there exist resolvent kernels satisfying fundamental properties given on pp. 212–213. The common kernel $A(t) = t^{q-1}, 0 < q < 1$, satisfies those conditions and it occurs in all fractional differential equations of Riemann-Liouville and

Caputo type as well as many forms of heat problems with $q = 1/2$. Moreover, it allows us to make certain transformations which we have found to be fundamental in papers since 2009. The conditions are

(A1) $A \in C(0, \infty) \cap L^1(0, 1)$,

(A2) $A(t)$ is positive and non-increasing for $t > 0$,

(A3) for each $T > 0$ the function $A(t)/A(t + T)$ is non-increasing in t for $0 < t < \infty$.

We will need a bit more than this and it leads to the origin of this study. Focus on (A2). First, if we ask that $A(t)$ is non-increasing, then we approximate this with the simple requirement that $A'(t) \leq 0$. It is more difficult to specify a condition keeping $A(t)$ positive, but for a start we could ask that $A''(t) \geq 0$. In trying to simplify these three conditions we have unwittingly almost asked for a classical condition that $A(t)$ be a strongly positive kernel. Following MacCamy and Wong [13], $A(t)$ is a positive kernel if whenever $h : [0, \infty) \to \Re$ is continuous then for $T > 0$ we have

$$\int_0^T h(t) \int_0^t A(t-s)h(s)dsdt \geq 0 \tag{4.1}$$

and that will follow under a number of conditions. It is more than sufficient [13, p. 2] if

$$A(t) > 0, A'(t) \leq 0, A''(t) \geq 0, \quad A'(t) \text{ not identically zero,}$$

and we discuss more below.

Remark 4.1 The boundedness which we mentioned in Remark 2.1 will follow from (4.1), but only by using (2.1) or (2.1λ), not the integral equation (2.3). The idea is based on a Liapunov function and works only on an integro-differential equation. Once we get the boundedness of the solution of (2.1) we will transfer it to boundedness of fixed points of the mapping in Schaefer's theorem which requires a λ on the right-hand side. We will denote that equation by (2.4λ) written in Section 5. □

A standard condition for (4.1) to hold is given by a Laplace transform condition [13, p. 2, equation (1.5)], namely its Laplace transform $A^*(s) > 0$ in $\Re s > 0$, which is challenging to verify. Consequently, any claim that it holds needs to be carefully checked. Instead of using that, we would quote a result of MacCamy and Wong [13, p. 16] concerning strongly positive kernels which are positive kernels [13, p. 5, (2.9) and the line below].

In our subsequent theorems we could use the following result, but (4.1) is our basic assumption.

Theorem 4.1 *Let $A(t)$ satisfy the following conditions:*

(B_1) $A(t) \in C^1(0, \infty) \cap L^1(0, 1)$,

(B_2) $A(t) \geq 0, A'(t) \leq 0$,

(B_3) $A'(t)$ *is nondecreasing,*

(B_4) $A(t)$ *is not identically constant.*

Then $A(t)$ defines a strongly positive kernel.

The following corollary provides us examples of strongly positive kernels found widely in applied mathematics. The resolvent mentioned is found in Miller [14, p. 212].

Corollary 4.1 *Both t^{q-1} for $0 < q < 1$ and $R(t)$, the resolvent, define strongly positive kernels as both are completely monotone.*

5 Schaefer's Theorem

The following result by Schaefer is conveniently found in Smart [15, p. 29]

Theorem 5.1 (Schaefer) *Let $(\mathcal{B}, \|\cdot\|)$ be a normed space, P be a continuous mapping of $\mathcal{B}$ into $\mathcal{B}$ which is compact on each bounded subset X of $\mathcal{B}$. Then either*

(i) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

This brings us to (2.4) in which we insert the parameter $\lambda \in (0,1)$ and write

$$\lambda(P\phi)(t) = \lambda\bigg[-g\bigg(t, a + \int_0^t \phi(s))ds\bigg) - \int_0^t A(t-s)f\bigg(a + \int_0^s \phi(u)du\bigg)ds\bigg]. \qquad (2.4\lambda)$$

Here is the array for our work in applying Schaefer's theorem. At the end of the boundedness proof we gather all of this together and state as a formal theorem.

1. The compactness follows from one of our three choices for A in (3.1) which are listed below (3.1) and the compactness of g given in Theorem 3.1.

2. The continuity of P on finite intervals is parallel to that of p. 443 of [1].

3. A fixed point for $\lambda = 1$ is a solution of our beginning equation (2.1). We have established all the conditions of Schaefer's theorem for our mapping P except the *a priori* bound which we will give now as a separate theorem. It contains an interesting pair of steps. To get a clean result we assume that

$$|y| \to \infty \implies \int_0^y f(s)ds \to \infty. \qquad (5.1)$$

There are lengthy arguments yielding boundedness if the integral is infinite for either $y > 0$ or $y < 0$.

Theorem 5.2 *Consider (2.4λ) with f and g continuous, (5.1) satisfied, and let A satisfy (4.1). For each $E > 0$ there is a $K > 0$ such that if ξ is a fixed point of* (2.4λ) *on* $[0, E]$ *then $\|\xi\| \le K$ and for y in (2.5) and (2.1) we have $L > 0$ with $\|y\| \le L$.*

Proof. Equation (2.4λ) is our mapping equation for Schaefer's theorem. We are mapping continuous functions into continuous functions and a fixed point will be a continuous function ϕ. When we do get a fixed point ϕ for $\lambda = 1$ we will return to the paragraph after (2.3) and we will let $y = a + \int_0^t \phi(s)ds$ as a solution of (2.1), finishing this problem. However, that equivalence between (2.1) and (2.3) which we proved in that paragraph does not hold here for $\lambda \ne 1$.

We are applying Schaefer's theorem and we start afresh with ϕ the assumed fixed point of (2.4λ) for some $\lambda \in (0,1)$ so that we are considering

$$\phi(t) = \lambda\bigg[-g(t, a + \int_0^t \phi(s))ds\bigg) - \int_0^t A(t-s)f\bigg(a + \int_0^s \phi(u)du\bigg)ds\bigg]. \qquad (2.4\lambda)$$

The goal is an *a priori* bound on $\phi(t)$ for any $\lambda \in (0,1)$. Working exclusively with this equation we make the substitution $y(t) = a + \int_0^t \phi(s)ds$ so that $y(0) = a$. Thus $y'(t) = \phi(t)$ and we have

$$y'(t) = \lambda\bigg[-g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds\bigg]. \qquad (2.1\lambda)$$

We will get an *a priori* bound on any solution of that equation which is identically (2.4λ) under the substitution $y(t) = a + \int_0^t \phi(s)ds$. A bound on y will be used in (2.1λ) to get a bound on y' which is $\phi(t)$, the assumed fixed point of (2.4λ). That bound will complete Schaefer's requirements giving a solution of (2.4λ) for $\lambda = 1$. We then go back to the paragraph after (2.3) and use that solution of (2.3) to get y in (2.1).

Boundedness of y (but not yet of y') will now follow from the Liapunov function

$$V(y(t)) = \int_0^{y(t)} f(s)ds \tag{5.2}$$

which is positive definite and radially unbounded. In the following differentiation, recall that y' is that continuous function ϕ.

Using the chain rule on (5.2) and (2.1λ) we have

$$\frac{dV(y(t))}{dt} = f(y(t))\frac{dy}{dt} = \lambda f(y(t))\Big[- g(t, y(t)) - \int_0^t A(t-s)f(y(s))ds\Big]$$

so that for any $T \in (0, E]$ we have

$$\begin{aligned} &V(y(T)) - V(y(0)) \\ &= -\lambda\Big[\int_0^T f(y(t))g(t, y(t))dt + \int_0^T f(y(t)) \int_0^t A(t-s)f(y(s))dsdt\Big] \le 0 \end{aligned}$$

by (2.1a), (2.1λ), and (4.1). Hence, $V(y(T)) \le V(y(0))$ or

$$\int_0^{y(T)} f(s)ds \le \int_0^{y(0)} f(s)ds \le \max \int_0^{\pm y(0)} f(s)ds =: L^* \tag{5.3}$$

so by (5.1) there is an $L > 0$ with

$$|y(T)| \le L.$$

This is true also for $\lambda = 1$ which means we have the result for (2.1) *provided the solution does exist which, of course, is the objective of this entire theorem.*

To finish the proof, let

$$G = \max_{0 \le t \le E, |x| \le L} |g(t, x)|$$

and

$$F = \max_{|x| \le L} |f(x)|.$$

Then the bound on y' of any solution of (2.1λ) is given by

$$K = G + \int_0^E A(t)dtF.$$

Let us consider what this proof has given us. We have an *a priori* bound on ϕ. Thus, we now have all the conditions of Schaefer's theorem and a fixed point of P is assured. Taking that back to the paragraph following (2.3) tells us that the existence of $y(t)$ satisfying (2.1) is assured and it is bounded by L. $\Box$

We formally collect all this as the following result.

Theorem 5.3 *Let the conditions with (2.1) hold, together with (2.1a) and the conditions of Theorem 3.1. If, in addition, the conditions of Theorem 5.2 hold, then (2.1) has a solution on any interval* $[0, E]$ *and it is bounded by the L following (5.3).*

If we ask conditions to ensure uniqueness, then we can obtain a solution on $[0, \infty)$ as follows. For each positive integer n construct a solution on $[0, n]$ and continue it to all of $[0, \infty)$ by defining it to be the value at $t = n$. This gives a sequence of functions converging uniformly on compact sets to a continuous function which is a solution on all of $[0, \infty)$ because at any $t > 0$ it agrees with one of the members of the sequence on $[0, t + 1]$. With this we are willing to say that uniqueness implies existence on $[0, \infty)$.

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Analysis of a Set of Trajectories of Generalized Standard Systems: Averaging Technique

Dedicated to the Memory of Yu. A. Mitropolsky

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Abstract: For standard form nonlinear equations with Hukuhara derivative, estimates of deviation of a set of exact solutions from the averaged ones are established and the deviation of a set of trajectories of averaged equations from the equilibrium state is specified in terms of pseudo-linear integral inequalities. Sets of affine systems and problems of approximate integrations and stability over finite interval are considered as applications.

Keywords: *set of standard equations; estimates of set of solutions; set of affine systems; finite-time stability.*

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Introduction

The averaging technique developed in the framework of nonlinear mechanics is a powerful tool widely used for the analysis of nonlinear systems found in various applied investigations (see [1–3] and bibliography therein).

The differential equation with Hukuhara derivative was first considered in [4], where its solution was shown to be a multivalued mapping. In the following papers a lot of authors (see [5, 6] and bibliography therein) established conditions for existence, uniqueness and convergence of successive approximate solutions and many other results. Moreover,

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new estimates of solutions were given and a principle of comparison with the matrix Lyapunov function was presented (see [7, 8]). Also, theorems on stability and boundedness of sets of trajectories of the equations of this type were proved [9].

The present paper deals with the sets of generalized standard systems of equations with Hukuhara derivative (see [6] and bibliography therein) and treats some problems of qualitative behaviour of set of trajectories of averaged equations.

The paper is arranged as follows. In Section 2 the problem for a set of standard systems of differential equations with Hukuhara derivative is stated. In Section 3 an estimate of distance between the set of solutions to the initial and the averaged system is found. In Section 4 the application of Theorem 1 to the analysis of set of solutions to quasilinear equations is considered. In Section 5 an estimate of deviation of the set of solutions to the averaged system from the equilibrium state is obtained. In Section 6 an estimate of deviation of the set of solutions to the affine system from zero is given. In Section 7 concluding remarks are presented and problems for further investigations are discussed.

1 Designations and Definitions

We shall recall designations and definition needed for further presentation of results (see [5] and bibliography therein).

Let E be a real Banach space with the norm $\|\cdot\|$ and 2^E be a totality of all nonempty bounded subsets of the space E with a Hausdorff pseudometric

$$D[A,B] = \max\Big\{ \sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\Big\}, \tag{1}$$

where $A,B \in 2^E$, $d(x,A) = \inf[d(x,y)\colon y \in A]$, $d(x,B) = \inf[d(y,x)\colon x \in B]$.

We designate by $K(E)$, $(K_c(E))$ a totality of all nonempty compact (convex) subsets of E which are considered as subspaces 2^E. Note that on $K(E)$ and $(K_c(E))$ the topology of the space 2^E is induced by the Hausdorff pseudometric (1). Besides, $K(E)$ and $(K_c(E))$ are complete and separable metric spaces if E is separable.

Let $A,B \in K_c(E)$. The set $C \in K_c(E)$ satisfying the relation $A = B + C$ is a Hukuhara difference of the sets A and B. The mapping $F\colon [0,a] \to K_c(E)$ possesses the Hukuhara derivative $D_H F(t_0)$ at the point $t_0 \in J = [0,*]$, $a > 0$, if there exists $D_H F(t_0) \in K_C(E)$ such that the limits

$$\lim\big\{[F(t_0+\tau) - F(t_0)]\tau^{-1}\colon \tau \to 0^+\big\} \text{ and } \lim\big\{[F(t_0) - F(t_0-\tau)]\tau^{-1}\colon \tau \to 0^+\big\} \tag{2}$$

exist in the topology of the space $K_C(E)$ and the both limits are equal to $D_H F(t_0)$.

It is known (see [4]) that if

$$F(t) = X_0 + \int_0^t \Phi(s)\,ds, \quad X_0 \in K_C(E), \tag{3}$$

where $\Phi\colon JK_C(E)$ is an integrable function in the Bohner sense, then $D_H F(t_0)$ exists almost everywhere on J and

$$D_H F(t) = \Phi(t) \text{ almost everywhere on } J. \tag{4}$$

Recall that the Hausdorff pseudometric (1) satisfies the following relations

$$D[U+W, V+W] = D[U,V],$$
$$D[\lambda U, \lambda V] = \lambda D[U,V],$$
$$D[U,V] \le D[U,W] + D[W,V]$$

for all $U, V, W \in K_c(E)$ and $\lambda \in \mathbb{R}_+$. Besides, for $U \in K_c(E)$ we admit that $D[U, \Theta_0] = \|U\| = \sup\{\|u\|;\ \ u \in U\}$, where Θ_0 is a zero element in E.

2 Statement of the Problem

The set of systems of differential equations

$$D_H X = \mu F(t, X), \quad X(t_0) = X_0 \in K_c(E), \tag{5}$$

where $F \in C(\mathbb{R} \times K_C(E), K_c(E))$, and $\mu \in (0,1]$ is a small parameter, is called a generalized standard system. Together with equation (5) we shall consider a partially averaged differential equation (see [6])

$$D_H Y = \mu G(t, Y), \quad Y(t_0) = Y_0 \in K_c(E), \tag{6}$$

for which

$$\lim_{T\to\infty} \frac{1}{T} D\left[\int_0^T F(s,X)ds,\ \int_0^T G(s,Y)ds\right] = 0, \tag{7}$$

for $X, Y \in D^* \subset K_c(E)$.

We assume on the families of equations (5) and (6) as follows.

H_1. There exists a function $M(t,\cdot) > 0$, integrable on J, for all $t \in J$ such that

$$\mu D[G(t,X), G(t,Y)] \le M(t,\mu) D[X,Y]$$

for all $0 < \mu < \mu_1 \in (0,1]$;

H_2. There exist a function $f(t,\cdot) > 0$, integrable on J, $\lim f(t,\mu) = 0$ as $t \to \infty$, and $\alpha > 1$ such that

$$\mu D[F(t,X), G(t,Y)] \le f(t,\mu) D^\alpha[X,Y]$$

for all $(X,Y) \in D^* \subseteq K_c(E)$ and $0 < \mu < \mu_1 \in (0,1]$.

This paper is aimed to obtain estimate of deviation between solutions to the family of equations (5) and (6) and deviation of solutions to averaged equations (6) from the equilibrium state $\Theta_0 \in K_c(E)$.

3 Estimate of the Distance Between Sets of Solutions

We shall estimate deviations between the sets of solutions to the families of equations (5) and (6). Let us show that the following result holds true.

Theorem 1 *In the domain* $Q = \{(t,X)\colon t \ge t_0 \ge 0, X \in D^* \subseteq K_c(E)\}$ *let the following conditions be satisfied:*

(1) *solution* $X(t)$ *of the initial problem for the family of equations* (5) *exists for all* $t \geq t_0$ *and* $0 < \mu < \mu_1,\ \mu_1 \in (0,1]$*;*

(2) *solution* $Y(t)$ *of the family of equations* (6) *with the initial condition* $Y_0 \in D^* \subset D$ *exists for all* $t \geq t_0$ *and* $0 < \mu < \mu_2,\ \mu_2 \in (0,1]$*;*

(3) *limit* (7) *exists uniformly with respect to* $X \in D$*;*

(4) *conditions of hypotheses* H_1 *and* H_2 *are satisfied;*

(5) *for all* $t \in J$ *and* $0 < \mu < \mu_0$ *the inequality*

$$1 - (\alpha-1)D^{\alpha-1}[X_0, Y_0] \int\limits_0^t f(s,\mu) \exp\Big(2(\alpha-1) \int\limits_0^s M(\tau,\mu)d\tau\Big) ds > 0.$$

is valid.

Then the deviation between the sets of solutions to equations (5) and (6) is estimated as

$$D[X(t), Y(t)] \leq$$

$$\leq \frac{D[X_0, Y_0] \exp\Big(\int\limits_0^t M(s,\mu)ds\Big)}{\Big(1 - (\alpha-1)D^{\alpha-1}[X_0, Y_0] \int\limits_0^t f(s,\mu) \exp\Big(2(\alpha-1) \int\limits_0^s M(\tau,\mu)d\tau\Big) ds\Big)^{\frac{1}{\alpha-1}}} \tag{8}$$

for all $t \in J$ *and* $0 < \mu < \mu_0,\ \mu_0 = \min(\mu_1, \mu_2)$.

Proof. We represent the families of equations (5) and (6) in the equivalent form

$$X(t) = X_0 + \mu \int\limits_0^t F(s, X(s))\, ds, \quad X_0 \in D \subset K_c(E),$$

and

$$Y(t) = Y_0 + \mu \int\limits_0^t G(s, Y(s))\, ds, \quad Y_0 \in D^* \subset D,$$

and assume that $D[X_0, Y_0] \neq 0$ for all X_0 and Y_0 in the domain of their values. It is easy to get the following estimates

$$\begin{aligned} D[X(t),Y(t)] &= D\Big[X_0 + \mu \int\limits_0^t F(s, X(s))ds, Y_0 + \mu \int\limits_0^t G(s, Y(s))ds\Big] \\ &= D[X_0, Y_0] + \mu D\Big[\int\limits_0^t F(s, X(s))ds, \int\limits_0^t G(s, Y(s))ds\Big] \\ &\leq D[X_0, Y_0] + \mu D\Big[\int\limits_0^t F(s, X(s))ds, \int\limits_0^t G(s, Y(s))ds\Big] \\ &+ \mu D\Big[\int\limits_0^t G(s, X(s))ds, \int\limits_0^t G(s, Y(s))ds\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq D[X_0, Y_0] + \mu \int\limits_0^t D[F(s, X(s)), G(s, Y(s))] ds \\
&+ \mu \int\limits_0^t D[G(s, X(s)), G(s, Y(s))] ds.
\end{aligned}
\tag{9}
$$

In view of

$$
D[F(t, X), G(t, X)] \leq D[F(t, X), G(t, Y)] + D[G(t, Y), G(t, X)],
$$

under hypotheses H_1 and H_2 we get from estimate (9) that

$$
D[X(t), Y(t)] \leq D[X_0, Y_0] + 2 \int\limits_0^t M(s, \mu) D[X(t), Y(t)]\, ds + \int\limits_0^t f(s, \mu) D^{\alpha}[X(s), Y(s)]\, ds,
\tag{10}
$$

where $0 < \mu < \mu_0$, $\mu_0 = \min(\mu_1, \mu_2)$.

We designate $D[X(t), Y(t)] = m(t)$ and represent inequality (10) as

$$
m(t) \leq m(t_0) + 2 \int\limits_0^t M(s, \mu) m(s)\, ds + \int\limits_0^t f(s, \mu) m^{\alpha}(s)\, ds,
\tag{11}
$$

where $0 < \mu < \mu_0$, $\mu_0 = \min(\mu_1, \mu_2)$, and $t \in J$. Inequality (11) is rewritten in the pseudo-linear form

$$
m(t) \leq m(t_0) + \int\limits_0^t (2M(s, \mu) m(s)\, ds + f(s, \mu) m^{\alpha - 1}(s)) m(s)\, ds
$$

for all $t \in J$. Applying to this inequality the summand estimation technique from [10, 11] we get

$$
m(t) \leq \frac{m(t_0) \exp\left(2 \int\limits_0^t M(s, \mu) ds\right)}{\left(1 - (\alpha - 1) m^{\alpha - 1}(t_0) \int\limits_0^t f(s, \mu) \exp\left(2(\alpha - 1) \int\limits_0^s M(\tau, \mu) d\tau\right) ds\right)^{\frac{1}{\alpha - 1}}}
\tag{12}
$$

under condition (5) of Theorem 1. Estimate (12) yields the assertion of Theorem 1.

Estimate (8) allows one to establish sufficient conditions for the presence of (A, λ)-estimate of approximate integration of the family of equations (5) in the sense of the definition below.

Definition 1 The set of solutions $Y(t)$ of the family of differential equations (6) satisfies (A, λ)-estimate of approximate integration of the family of equations (5) if, given the values λ and A $(0 < \lambda < A)$, the condition $D[X_0, Y_0] < \lambda$ implies that $D[X(t), Y(t)] < A$ for $0 < \mu < \mu_0$ on the common existence interval of solutions $X(t)$ and $Y(t)$.

Corollary 1 Let all conditions of Theorem 1 be satisfied and for given values λ and A the inequality

$$\frac{\exp\Big(2\int\limits_0^t M(s,\mu)ds\Big)}{\Big(1-(\alpha-1)\lambda^{\alpha-1}\int\limits_0^t f(s,\mu)\exp\Big(2(\alpha-1)\int\limits_0^s M(\tau,\mu)d\tau\Big)ds\Big)^{\frac{1}{\alpha-1}}}<\frac{A}{\lambda}$$

holds true for all $0<\mu<\mu_0$ and all $t\in J$.

Then for the set of solutions $X(t)$ of the family of equations (5) the (A,λ)-estimate of approximate integration takes place.

The assertion of Corollary 1 follows immediately from the estimate (8) and Definition 1.

Further we consider the quasilinear equation

$$D_H X = A(t)X+\mu F(t,X),\quad X(0)=X_0\subset D^*, \tag{13}$$

where $A(t)$ is a bounded operator on R_+, $F(t,X)$ is a mapping containing X in power higher than 2.

The solution of problem (13) is the mapping $X(t)=X(t,t_0,X_0)$ satisfying the family of equations (13) almost everywhere on J.

Together with the family of equations (13) we consider a family of averaged equations

$$D_H Y=\overline{A}(t)Y+\mu G(t,Y),\quad Y(t_0)=Y_0\subset D^*, \tag{14}$$

where

$$\overline{A}=\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T A(s)\,ds \tag{15}$$

and $G(t,Y)$ satisfies relation (7). We assume on the family of equations (14) as follows.

H_3. There exists an integrable function $b(t)>0$ for all $t\in J$ such that

$$\|A(t)-\overline{A}\|\le b(t).$$

We find the estimate of deviation of the set of solutions to the averaged equation (14) from the solutions to the initial equations (13).

Theorem 2 *In the domain* $Q=\{(t,X)\colon t\ge 0$ *and* $X\in D\subset K_c(E)\}$ *and let the following conditions be satisfied:*

(1) *there exists a limit* (15) *and the correlation* (7) *holds;*

(2) *the conditions of hypotheses* H_1 *and* H_3 *are satisfied;*

(3) *for* $Y_0\in D^*$ *the solution of averaged equations* (14) *is defined for all* $t\ge 0$ *and* $0<\mu<\mu_0$*;*

(4) *for all* $t\in J$ *and* $0<\mu<\mu_0$ *the inequality*

$$2(\alpha-1)m^{\alpha-1}(t_0)\int\limits_0^t M(s,\mu)\exp\left((\alpha-1)\int\limits_0^s (b(\tau)+f(\tau,\mu))d\tau\right)ds<1$$

is true.

Then the deviation between the sets of solutions to equations (13) *and* (14) *is estimated as*

$$D[X(t), Y(t)] \leq$$
$$\leq \frac{D[X_0, Y_0] \exp\Big(\int\limits_0^t (b(s) + 2M(s,\mu))ds\Big)}{\Big[1 - 2(\alpha-1)m^{\alpha-1}(0)\int\limits_0^t M(s,\mu)\exp\Big((\alpha-1)\int\limits_0^s (b(\tau)+f(\tau,\mu))d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} \tag{16}$$

for all $t \in J$ *and* $0 < \mu < \mu_0$.

Proof. The relations

$$\begin{aligned} X(t) &= X_0 + \int\limits_0^t (A(s)X(s) + \mu F(s, X(s)))\, ds, \\ Y(t) &= Y_0 + \int\limits_0^t (\overline{A}(s)Y(s) + \mu G(s, Y(s)))\, ds \end{aligned} \tag{17}$$

for $D[X_0, Y_0] \neq 0$ yield

$$\begin{aligned} &D[X(t), Y(t)] \\ &\quad = D\Big[X_0 + \int\limits_0^t (A(s)X(s) + \mu F(s, X(s)))ds,\ Y_0 + \int\limits_0^t (\overline{A}(s)Y(s) + \mu G(s, Y(s)))ds\Big] \\ &\quad \leq D[X_0, Y_0] + D\Big[\int\limits_0^t A(s)X(s)ds, \int\limits_0^t \overline{A}(s)Y(s)ds\Big] \\ &\quad + \mu D\Big[\int\limits_0^t F(s, X(s))ds, \int\limits_0^t G(s, Y(s))ds\Big] \\ &\quad \leq D[X_0, Y_0] + \int\limits_0^t (A(s) - \overline{A})D[X(s), Y(s)]ds + \mu \int\limits_0^t D[F(s, X(s)), G(s, X(s))]ds \\ &\quad + \mu \int\limits_0^t D[F(s, X(s)), G(s, Y(s))]ds. \end{aligned}$$

Hence, according to hypotheses H_1 and H_3 it follows that

$$\begin{aligned} D[X(t), Y(t)] \leq D[X_0, Y_0] &+ \int\limits_0^t (b(s) + 2M(s,\mu))D[X(s), Y(s)]\, ds \\ &+ \int\limits_0^t f(s,\mu)D^\alpha[X(s), Y(s)]\, ds \end{aligned} \tag{18}$$

for all $t > t_0$ and $0 < \mu < \mu_0$. Inequality (18) is rewritten as

$$m(t) \le m(t_0) + \int_0^t [(b(s) + 2M(s,\mu))m(s) + f(s,\mu)m^{\alpha}(s)]\,ds$$

and further

$$m(t) \le m(t_0) + \int_0^t [(b(s) + 2M(s,\mu)) + f(s,\mu)m^{\alpha-1}(s)]m(s)\,ds. \tag{19}$$

As in the analysis of inequality (11), we get from estimate (19) the inequality

$$m(t) \le \frac{m(t_0)\exp\Big(\int_0^t (b(s) + 2M(s,\mu))ds\Big)}{\Big[1 - (\alpha-1)(0)\int_0^t f(s,\mu)\exp\Big((\alpha-1)\int_0^s (b(\tau) + 2M(\tau,\mu))d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} \tag{20}$$

provided that condition (4) of Theorem 2 is satisfied for all $t \in J$ and $0 < \mu < \mu_0$. In view of the designation $m(t) = D[X(t), Y(t)]$, for all $t \in J$ estimate (20) completes the proof of Theorem 2.

Corollary 2 Let all conditions of Theorem 2 be satisfied and for the given estimates of the values λ and A the inequality

$$\frac{\exp\Big(\int_0^t (b(s) + 2M(s,\mu))ds\Big)}{1 - 2(\alpha-1)m^{\alpha-1}(0)\int_0^t M(s,\mu)\exp\Big((\alpha-1)\int_0^s (b(\tau) + f(\tau,\mu))d\tau\Big)ds} < \frac{A}{\lambda}$$

holds for all $t > t_0$ and $0 < \mu < \mu_0$. Then for the set of solutions $X(t)$ of the family of equations (13) the (A, λ)-estimate of approximate integration takes place.

The assertion of Corollary 2 follows from estimate (16) and Definition 1.

4 Conditions of (λ, A, J)-stability of Averaged Equation

Further we shall consider a family of averaged equations (13).

Assume that the following conditions are satisfied.

H_4. There exists a constant $a > 0$ such that

$$\|\overline{A}\| < a, \quad a = \text{const} > 0.$$

H_5. There exists a function $N(*, t) > 0$, which is integrable on J, such that

$$\mu D[G(t,Y), \Theta_0] \le N(\mu, t)D^{\alpha}[Y, \Theta_0]$$

for all $t \in J$ and $0 < \mu < \mu_0$ in the domain of values $Y \subset D^*$.

We shall show that the following result is valid.

Theorem 3 *In the domain $Q = \{(t, Y) \colon t \geq 0, Y \in D^* \subset K_c(E)\}$ let the following conditions be satisfied.*

(1) *there exists a solution $Y(t) = Y(t, t_0, Y_0)$ of the averaged equation (14) for all $t \geq 0$ and $Y^* \in D^*$;*

(2) *the conditions of hypotheses H_4 and H_5 be satisfied.*

Then the deviation of the set of solutions $Y(t)$ from the equilibrium state is estimated as

$$D[Y(t), \Theta_0] \leq \frac{n(0) \exp \int\limits_0^t \|\overline{A}\| ds}{\Big[1 - (\alpha - 1) n^{\alpha-1}(0) \int\limits_0^t N(\mu, s) \exp\Big((\alpha - 1) \int\limits_0^s \|\overline{A}\| d\tau\Big) ds\Big]^{\frac{1}{\alpha-1}}} \tag{21}$$

for all $t \in J$ and $0 < \mu < \mu_0$ provided that

$$(\alpha - 1) n^{\alpha-1}(t_0) \int\limits_0^t N(\mu, s) \exp\Big((\alpha - 1) \int\limits_0^s \|\overline{A}\| d\tau\Big) ds < 1$$

for all $t \in J$ and $0 < \mu < \mu_0$.

Proof. For correlation (17) we have

$$\begin{aligned} D[Y(t), \Theta_0] &\leq D[Y_0, \Theta_0] + \int\limits_0^t \overline{A} D[Y(s), \Theta_0]\, ds + \mu \int\limits_0^t D[G(s, Y(s)), \Theta_0]\, ds \\ &\leq D[Y_0, \Theta_0] + \int\limits_0^t \|\overline{A}\| D[Y(s), \Theta_0]\, ds + \int\limits_0^t N(\mu, s) D^\alpha[Y(s), \Theta_0]\, ds. \end{aligned}$$

Hence, for the function $n(t) = D[Y(t), 0]$ estimating the deviation of the set of solutions to the averaged equations from zero in $K_c(E)$, we have the inequality

$$n(t) \leq n(t_0) + \int\limits_0^t \|\overline{A}\| n(s)\, ds + \int\limits_0^t N(\mu, s) n^\alpha(s)\, ds$$

for all $t \in J$ and $0 < \mu < \mu_0$. Applying to this inequality the technique used for the proof of Theorem 1 we arrive at estimate (21). This proves Theorem 3.

Estimate (21) allows one to establish conditions for (λ, A, J)-stability on finite interval of the set of solutions to equations (14).

Definition 2 For given estimates of the values λ, A, J the set of solutions to the averaged equations (14) is (λ, A, J)-stable if $D[Y(t), \Theta_0] < A$ for all $t \in J$, whenever $D[Y_0, \Theta_0] < \lambda$ and $0 < \mu < \mu_0$.

Corollary 3 Let all conditions of Theorem 2 be satisfied and for given estimates of the values λ, A and J the inequality

$$\frac{\exp\Big(\int\limits_0^t \|\overline{A}\| ds\Big)}{\Big[1-(\alpha-1)\lambda^{\alpha-1}\int\limits_0^t N(\mu,s)\exp\Big((\alpha-1)\int\limits_0^s \|\overline{A}\| d\tau\Big)ds\Big]^{\frac{1}{\alpha-1}}} < \frac{A}{\lambda}$$

holds true for all $t \in J$ and $0<\mu<\mu_0$. Then the set of solutions $Y(t)$ of the family of equations (14) is (λ, A, J)-stable.

The assertion of Corollary 3 follows from estimate (21) and Definition 2.

5 Boundedness of Trajectories of Standard Affine Systems

Consider a family of affine systems of the form

$$D_H X(t) = \mu(f(t,X) + g(t,X)U(t)), \tag{22}$$

$$X(t_0) = X_0 \in D \subset K_c(\mathbb{R}^n), \tag{23}$$

where $f(t,X)\colon \mathbb{R}_+ \times D \to K_c(E)$, $g(t,X)$ is an $n\times n$-matrix, $g(t,X)\colon \mathbb{R}_+ \times D^* \to K_c(E)$, $U(t) \in W \subset K_c(E)$ is a control.

Together with the family of equations (22) we consider the averaged equations

$$\begin{aligned} D_H Y(t) &= \mu(\overline{f}(t,Y) + \overline{g}(t,Y)V(t)),\\ Y(t_0) &= Y_0 \in D \subset K_c(\mathbb{R}^n), \end{aligned} \tag{24}$$

where

$$\lim_{T\to\infty} \frac{1}{T} D\left[\int\limits_0^T f(s,X(s))ds, \int\limits_0^T \overline{f}(s,Y(s))ds\right] = 0; \tag{25}$$

$$\lim_{T\to\infty} \frac{1}{T} D\left[\int\limits_0^T g(s,X(s))U(s)ds, \int\limits_0^T \overline{g}(s,Y(s))V(s)ds\right] = 0; \tag{26}$$

for all $(U,V) \in W \subset K_c(E)$.

We shall estimate the deviation of solutions to the family of equations (24) from the state $\Theta_0 \in K_c(E)$. Assume as follows:

H_6. There exists a function $f_1(t,\mu) > 0$, integrable on J and such that

$$D[\overline{f}(t,Y), \Theta_0] \le f_1(t,\mu) D[Y, \Theta_0]$$

for all $Y \in D$ and $0<\mu<\mu_1$.

H_7. There exists a function $f_2(t,\mu) > 0$, integrable on J and such that

$$D[\overline{g}(t,Y)V(t), \Theta_0] \le f_2(t,\mu) D^2[Y, \Theta_0]$$

for all $Y \in D^*$, $V(t) \in W$ and $0<\mu<\mu_1$.

Theorem 4 *In the domain* $Q = \{(t, Y)\colon t \geq 0, Y \in D^* \subset K_c(E)\}$ *for equations* (22) *and* (24) *let*

(1) *there exist limits* (25) *and* (26)*;*

(2) *conditions of hypotheses* H_6 *and* H_7 *be satisfied;*

(3) *for all* $t \geq 0$ *and* $0 < \mu < \mu_0$ *the inequality*

$$1 - D[Y_0, \Theta_0] \int\limits_0^t f_2(s,\mu) \exp\left(\int\limits_0^t f_1(\tau,\mu) d\tau \right) ds > 0 \tag{27}$$

hold true.

Then the deviation of the set of solutions to the family of equations (24) *from zero is estimated as*

$$D[Y(t), \Theta_0] \leq \frac{D[Y_0, \Theta_0] \exp\left(\int\limits_0^t f_1(s,\mu) ds \right)}{1 - D[Y_0, \Theta_0] \int\limits_0^t f_2(s,\mu) \exp\left(\int\limits_0^t f_1(\tau,\mu) d\tau \right) ds} \tag{28}$$

for all $t \geq 0$, $Y \in D^*$ *and* $0 < \mu < \mu_0$.

Proof. Let the limiting relations (25) and (26) be satisfied. From equation (24) we have

$$Y(t) = Y_0 + \mu \int\limits_0^t \left(\overline{f}(s, Y(s)) + \overline{g}(s, Y(s)) V(s) \right) ds$$

and further

$$D[Y(t), Y_0] \leq D[Y_0, \Theta_0] + \mu \int\limits_0^t D[\overline{f}(s, Y(s)), \Theta_0]\, ds + \mu \int\limits_0^t D[\overline{g}(s, Y(s)) V(s), \Theta_0]\, ds. \tag{29}$$

Under conditions of hypotheses H_6 and H_7 we find from inequality (29) that

$$D[Y(t), \Theta_0] \leq D[Y_0, \Theta_0] + \mu \int\limits_0^t f(s, Y(s)) D[Y(s), \Theta_0] + f_2(s,\mu) D^2[Y(s), \Theta_0]\, ds. \tag{30}$$

Designate $n(t) = D[Y(t), \Theta_0]$ and from (30) we get

$$n(t) \leq n(t_0) + \mu \int\limits_0^t (f_1(s,\mu) + f_2(s,\mu) n(s)) n(s)\, ds. \tag{31}$$

Applying Gronwall-Bellman lemma to inequality (31) we arrive at

$$n(t) \leq n(t_0) \exp\left(\int\limits_0^t (f_1(s,\mu) + f_2(s,\mu)) n(s)\, ds \right).$$

Hence

$$-n(t)\exp\int_0^t(-f_2(s,\mu))n(s))\,ds \ge n(t_0)\exp\Bigg(\int_0^t f_1(s,\mu)\,ds\Bigg).$$

Multiplying both sides of this inequality by $f_2(t,\mu) > 0$ we obtain

$$\frac{d}{dt}\Bigg(\exp\Bigg(-\int_0^t f_2(s,\mu)n(s)ds\Bigg)\Bigg) \ge -n(t_0)f_2(t,\mu)\exp\Bigg(\int_0^t f_1(s,\mu)ds\Bigg). \tag{32}$$

Integrating this inequality between t_0 and t we get

$$(n(t))^{-1}\exp\Bigg(\int_0^t f_1(s,\mu)ds\Bigg) \ge 1 - n(t_0)\int_0^t f_2(s,\mu)\exp\Bigg(\int_0^t f_1(\tau,\mu)d\tau\Bigg)ds. \tag{33}$$

Hence follows the estimate of deviation of the family of solutions to equation (24) from the equilibrium state in the (28) form under condition (27). Theorem 4 is proved.

Estimate (28) allows one to establish boundedness conditions for the set of solutions to the averaged affine system (24).

Definition 3 The set of solutions to equations (24) is bounded if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists $\delta(t_0,\varepsilon) > 0$ such that

$$D[Y(t),\Theta_0] < \varepsilon \quad \text{for all} \quad t \ge t_0$$

and $0 < \mu < \mu_0$, whenever $D[Y_0,\Theta_0] < \delta$.

If δ does not depend on t_0, the boundedness of the set of solutions $Y(t)$ is uniform with respect to t_0.

Corollary 4 Let all conditions of Theorem 4 be satisfied and for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that

$$\frac{\exp\Big(\int_0^t f_1(s,\mu)ds\Big)}{1-\delta(\varepsilon)\int_0^t f_2(s,\mu)\exp\Big(\int_0^t f_1(\tau,\mu)d\tau\Big)ds} < \frac{\varepsilon}{\delta(\varepsilon)}$$

for all $t \ge 0$ and $0 < \mu < \mu_0$. Then the set of solutions of equations (24) is uniformly bounded.

The assertion of Corollary 4 follows from estimate (28) and Definition 3.

6 Conclusion

A key element of the approach is the use of nonlinear integral inequalities in the problems of qualitative analysis of the set of trajectories of generalized standard systems. The resulting estimates of evasion of the set of trajectories of the equilibrium state, and the estimate of the distance between the sets of initial and averaged solutions to systems of equations are applicable in many problems of mechanics and applied mathematics in which processes models are the system of equations (5), (13) and (22).

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Robust Output Feedback Stabilization and Optimization of Control Systems

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Abstract: The paper is devoted to the problems of output feedback stabilization, robust stabilization, quadratic optimization and generalized H_∞-control for some classes of linear and nonlinear dynamical systems. Sufficient stability conditions for the zero state are formulated with the joint quadratic Lyapunov function for a family of control systems with uncertain coefficient matrices. The solution of robust stabilization problem and evaluation of the quadratic performance criterion for a family of nonlinear control systems are proposed. Methods for construction of control laws providing a robust stability and specified evaluation of the weighted damping level of input signals and initial perturbations are proposed for a class of linear systems with controllable and observable outputs. The application of the main results reduces to solving the systems of linear matrix inequalities.

Keywords: *pseudolinear system; output feedback; robust stability; linear matrix inequality; quadratic Lyapunov function, H_∞-control.*

Mathematics Subject Classification (2010): Primary: 93C10, 93C35, 93D09, 93D15, 93D21; Secondary: 34D20, 37N35.

1 Introduction

State and output feedback controllers design for dynamic systems with the prescribed and desired properties is a key problem of control theory. At the same time, the properties of control systems such as asymptotic stability, robustness and optimality of the performance indexes are in the foreground. The main problem in H_∞-control theory is connected with suppression of external and initial perturbations (see, e.g., [1–5] as well as review papers [6, 7]).

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It should be noted that the practical applications of many modern methods for synthesis of control systems are based on the construction and solution of linear matrix inequalities (LMI). For this purpose, sufficiently effective computational algorithms and appropriate tools are established in Matlab environment (see [8, 9]).

In this paper, we consider classes of linear and affine control systems for which closed loop systems can be represented in the pseudolinear form

$$\dot{x} = M(x,t)\,x, \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

besides, a matrix function $M(x,t)$ can contain uncertain quantities belonging to certain sets. Intervals, polytopes, affine families of matrices and other objects may serve as the uncertainty sets. To define uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [5, 10, 11]. The applied control laws are of the form of static or dynamic output feedback. It should be noted that at the solution of many control problems the dynamic controllers have great potential as compared with the static controllers.

Our consideration includes the following types of problems:

- output feedback stabilization of control systems (Section 2);
- robust stabilization and optimization of control systems with polyhedral uncertainties (Section 3);
- robust stabilization and weighted perturbation suppression in control systems (Section 4).

Throughout the paper, the following notations are used: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^T > 0$ (≥ 0) is the symmetric positive definite (semidefinite) matrix X; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the inertia of Hermitian matrix $X = X^*$ consisting of the numbers of positive $(\mathrm{i}_+(X))$, negative $(\mathrm{i}_-(X))$ and zero $(\mathrm{i}_0(X))$ eigenvalues (taking into account the multiplicities); $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of A, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix X, respectively; A^+ is the pseudoinverse matrix; W_A is a matrix whose columns make up the bases of the kernel $\mathrm{Ker}\,A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\mathrm{Co}\{A_1, \ldots, A_\nu\}$ stands for a polytope in a matrix space described as a convex full of the set $\{A_1, \ldots, A_\nu\}$, i. e.

$$\mathrm{Co}\{A_1, \ldots, A_\nu\} = \Big\{ \sum_{i=1}^{\nu} \alpha_i A_i : \ \alpha_i \geq 0,\ i = \overline{1,\nu},\ \sum_{i=1}^{\nu} \alpha_i = 1 \Big\}.$$

Note that matrix intervals and affine sets of matrices are described in terms of polytopes.

2 Output Feedback Stabilization of Nonlinear Systems

Consider the following affine nonlinear time-invariant control system

$$\dot{x} = A(x)x + B(x)u, \quad y = C(x)x + D(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is state vector, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x = 0$. We will assume that $\mathrm{rank}\, B(x) \equiv m$ and $\mathrm{rank}\, C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2}$$

where $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^{\perp}$ and $C^{\perp}$ be the orthogonal complements of B and C, respectively, i.e.

$$B^T B^{\perp} = 0, \ \det\left[B, B^{\perp}\right] \neq 0, \ C^{\perp} C^T = 0, \ \det\left[C^T, C^{\perp T}\right] \neq 0.$$

2.1 Static controllers

Formulate stabilizability conditions of the zero state $x = 0$ for systems (1) and (2) through the static output-feedback controller

$$u = Ky, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \{K \in \mathbb{R}^{m \times l} : \det(I_m - KD) \neq 0\}$. Closed loop system (2), (3) has the form

$$\dot{x} = Mx, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K$ is a nonlinear operator with the following properties:
• if $K \in \mathcal{K}_D$, then

$$\mathbf{D}(K) \equiv K(I_l - DK)^{-1}, \quad I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}; \tag{5}$$

• if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1 D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{6}$$

where $\mathbf{D}_1(K_2) = (I_m - K_2 D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;
• if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and

$$\mathbf{D}(K) = K_0. \tag{7}$$

According to (7), to achieve the desired properties and, in particular, to stabilize system (4) it suffices to provide a system with matrix $M_* = A + BKC$ with these properties.

Definition 2.1 System (4) is α-stable if the spectrum $\sigma(M)$ lies in the open left half-plane $\mathbb{C}_\alpha^- = \{\lambda : \operatorname{Re}\lambda + \alpha < 0\}$, where $\alpha \geq 0$.

Theorem 2.1 *The following statements are equivalent:*
1) *There exists static controller* (3) *ensuring α-stability of system* (4).
2) *There exists matrix $X = X^T > 0$ satisfying the relations*

$$B^{\perp T}(AX + XA^T + 2\alpha X)\,B^{\perp} < 0, \tag{8}$$

$$\mathrm{i}(\Delta) = \{l, n, 0\}, \quad \Delta = \begin{bmatrix} AX + XA^T + 2\alpha X & XC^T \\ CX & 0 \end{bmatrix}. \tag{9}$$

3) *There exist mutually inverse matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying the relations* (8) *and*

$$C^{\perp}(A^T Y + YA + 2\alpha Y)\,C^{\perp T} < 0. \tag{10}$$

When one of the statements 2) *or* 3) *is true, then the controller*

$$u = Ky, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{11}$$

where K_0 *is a solution of the LMI*

$$AX + XA^T + 2\alpha X + BK_0CX + XC^TK_0^TB^T < 0, \tag{12}$$

ensures α*-stability of closed loop system* (4)*.*

For the equivalence of the statements 1) and 2) in Theorem 2.1, see [5]. Equivalence of the statements 2) and 3) follows from the correlations (see [10, p. 147])

$$\mathrm{i}_\pm(\Delta) = \mathrm{i}_\pm(\Delta_1) = \mathrm{i}_\pm(C^\perp L_1 C^{\perp T}) + l,$$

where

$$\Delta_1 = R^T\Delta R = \left[\begin{array}{c|cc} C^\perp L_1 C^{\perp T} & 0 & C^\perp L_1 C^+ \\ \hline 0 & 0 & I_l \\ C^{+T} L_1 C^{\perp T} & I_l & C^{+T} L_1 C^+ \end{array}\right],$$

$$L_1 = A^TY + YA + 2\alpha Y, \quad Y = X^{-1}, \quad R = \begin{bmatrix} YC^{\perp T} & 0 & YC^+ \\ 0 & I_l & 0 \end{bmatrix}, \quad \det R \neq 0.$$

For the equivalence of the statements 1) and 3), see also [4].

Theorem 2.2 [12] *Let one of the statements* 2) *or* 3) *of Theorem* 2.1 *hold for system* (2)*. Then relations* (11) *and* (12) *determine static controller ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(x) = x^TYx$ *of nonlinear closed loop system* (1)*,* (11)*.*

2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) consists in finding, if possible, a dynamic control law described by

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \tag{13}$$

where $\xi \in \mathbb{R}^r$ and $r \leq n$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (13) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller

$$\dot{\widehat{x}} = \widehat{A}(\widehat{x})\widehat{x} + \widehat{B}(\widehat{x})\widehat{u}, \quad \widehat{y} = \widehat{C}(\widehat{x})\widehat{x} + \widehat{D}(\widehat{x})\widehat{u}, \quad \widehat{u} = \widehat{K}\widehat{y}, \tag{14}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{y} = \begin{bmatrix} y \\ \xi \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix},$$

$$\widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix}, \quad \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0_{l\times r} \\ 0_{r\times m} & 0_{r\times r} \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (13) has the form

$$\dot{\widehat{x}} = \widehat{M}\,\widehat{x}, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{15}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$, $\widehat{\mathbf{D}}(\widehat{K}) = (I_{m+r} - \widehat{K}\widehat{D})^{-1}\widehat{K}$, and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

Theorem 2.3 *The following statements are equivalent:*

1) *There exists dynamic controller* (13) *of order* $r \le n$ *ensuring* α*-stability of closed loop system* (15)*.*

2) *There exist matrices* X *and* X_0 *satisfying the relations* (8) *and*

$$\mathrm{i}(\Delta_0) = \{l, n, 0\}, \quad X \ge X_0 > 0, \quad \operatorname{rank}(X - X_0) \le r, \tag{16}$$

where

$$\Delta_0 = \begin{bmatrix} AX_0 + X_0A^T + 2\alpha X_0 & X_0C^T \\ CX_0 & 0 \end{bmatrix}.$$

3) *There exist matrices* X *and* Y *satisfying the relations* (8)*,* (10) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0, \quad \operatorname{rank} W \le n + r. \tag{17}$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (15) (see [12]). In [12], a computation algorithm of finding a stabilizing dynamic controller (13) for nonlinear systems (1) has been proposed on the basis of Theorem 2.3.

Remark 2.1 Note, that matrices X and X_0 satisfy statement 2) iff matrices X and $Y = X_0^{-1}$ satisfy statement 3). From (17) it follows that matrices X and Y are positive definite. The rank restriction in (17) always holds in case of full order $r = n$ dynamic regulator.

3 Robust Stabilization and Optimization of Nonlinear Systems

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider a nonlinear operator

$$\mathbf{F}(K) = W + U^T\mathbf{D}(K)V + V^T\mathbf{D}^T(K)U + V^T\mathbf{D}^T(K)\,R\,\mathbf{D}(K)V \tag{18}$$

with $\mathbf{D}(K) = (I_m - KD)^{-1}K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \{K \in \mathbb{R}^{m\times l} :\ K^TPK \le Q\}, \tag{19}$$

where $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T \ge 0$, $W = W^T \le 0$, U, V and D are matrices of suitable sizes. Matrix inequality in (19) is equivalent to the following $KQ^{-1}K^T \le P^{-1}$. Therefore, in case of $m = 1$ the ellipsoid $\mathcal{K}$ is described by a scalar inequality.

Lemma 3.1 [13] *Suppose that the following matrix inequalities hold:*

$$D^TQD + R < P, \quad \Omega = \begin{bmatrix} W & U^T & V^T \\ U & R-P & D^T \\ V & D & -Q^{-1} \end{bmatrix} \leq 0 \ (< 0). \tag{20}$$

Then $\mathbf{F}(K) \leq 0 \ (< 0)$ *for every matrix* $K \in \mathcal{K}$.

Note that Lemma 3.1 is a generalization of the sufficiency statement for a criterion known as Petersen's lemma on matrix uncertainty [14] (see also [15]). In Lemma 3.1 letting $D = 0$, $R = 0$, $P = \varepsilon I_m$ and $Q = \varepsilon I_l$, where $\varepsilon > 0$, we get the sufficiency statement of Petersen's lemma.

Consider a nonlinear control system in the vector-matrix form

$$E(x)\dot{x} = A(x,t)\,x + B(x,t)\,u, \quad y = C(x,t)\,x + D(x,t)\,u, \quad t \geq 0, \tag{21}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$. We construct a set of the static controllers

$$u = K(x,t)\,y, \quad K(x,t) = K_*(x,t) + \widetilde{K}(x,t), \quad \widetilde{K}(x,t) \in \mathcal{K}, \tag{22}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (19). We assume that the matrices E, A, B, C, D, K and K_* depend on x and t continuously and the equilibrium state $x \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leq h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (21), (22) can be represented as

$$E(x)\dot{x} = M(x,t)x, \quad M(x,t) = A + B\mathbf{D}(K)C. \tag{23}$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix K_* in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (23) is Lyapunov asymptotically stable for every matrix $\widetilde{K}(x,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [5,13]).

Theorem 3.1 *Let for some matrix functions* $X(t) = X^T(t)$ *and* $K_*(x,t)$ *at* $x = 0$ *and* $t \geq 0$ *the correlations*

$$\varepsilon_1 I_n \leq X(t) \leq \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leq \varepsilon_2, \tag{24}$$

$$\begin{bmatrix} E^T\dot{X}E + M_*^TXE + E^TXM_* + \varepsilon_0 I_n & E^TXB_* & C_*^T \\ B_*^TXE & -P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{25}$$

hold with $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_*D)^{-1}$. *Then any control* (22) *ensures asymptotic stability of the zero state* $x \equiv 0$ *for system* (23) *and a common Lyapunov function* $v(x,t) = x^TE_0^TX(t)E_0x$, *where* $E_0 = E(0)$.

Consider control system (21) with quadratic quality functional

$$J(u,x_0) = \int_0^\infty \varphi(x,u,t)\,dt, \tag{26}$$

where

$$x_0 = x(0), \quad \varphi(x,u,t) = \left[x^T, u^T\right] \Phi(t) \begin{bmatrix} x \\ u \end{bmatrix},$$

$$\Phi(t) = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix}, \quad R > 0, \quad S \geq NR^{-1}N^T + \eta I_n, \quad \eta > 0, \quad t \geq 0.$$

Theorem 3.2 *Let for some matrix functions $X(t) = X^T(t)$ and $K_*(x,t)$ at $x = 0$ and $t \geq 0$ the correlations* (24) *and*

$$\begin{bmatrix} E^T\dot{X}E + M_*^TXE + E^TXM_* + \Phi_* + \varepsilon_0 I_n & E^TXB_* + N_* + C^TK_*^TR_* & C_*^T \\ B_*^TXE + N_*^T + R_*K_*C & R_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{27}$$

hold with $\varepsilon_0 > 0$, $\Phi_* = L_*^T\Phi L_*$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_*D)^{-1}$, $R_* = (I_m - K_*D)^{-1T}R(I_m - K_*D)^{-1}$, $N_* = N(I_m - K_*D)^{-1}$, $L_*^T = \left[I_n, C^T\mathbf{D}^T(K_*)\right]$. *Then any control* (22) *ensures asymptotic stability of the zero state $x \equiv 0$ for system* (23)*, a common Lyapunov function $v(x,t) = x^TE_0^TX(t)E_0x$, where $E_0 = E(0)$, and a bound on the functional $J(u,x_0) \leq v(x_0, 0)$.*

Corollary 3.1 *Let for some matrix $X = X^T > 0$ and K_* the system of LMI*

$$\begin{bmatrix} M_{*ijk}^TXE_s + E_s^TXM_{*ijk} + L_{*k}^T\Phi L_{*k} & E_s^TXB_{*j} + N_* + C_k^TK_*^TR_* & C_{*k}^T \\ B_{*j}^TXE_s + N_*^T + R_*K_*C_k & R_* - P & D_*^T \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{28}$$

$$i = \overline{1,\alpha}, \quad j = \overline{1,\beta}, \quad k = \overline{1,\gamma}, \quad s = \overline{1,\delta},$$

hold with $M_{*ijk} = A_i + B_j\mathbf{D}(K_*)C_k$, $B_{*j} = B_j(I_m - K_*D)^{-1}$, $C_{*k} = (I_l - DK_*)^{-1}C_k$, $D_* = D(I_m - K_*D)^{-1}$, $R_* = (I_m - K_*D)^{-1T}R(I_m - K_*D)^{-1}$, $N_* = N(I_m - K_*D)^{-1}$, $L_{*k}^T = \left[I_n, C_k^T\mathbf{D}^T(K_*)\right]$. *Then any control* (22) *ensures asymptotic stability of the zero state $x \equiv 0$ for system* (23) *with uncertainties*

$$\begin{aligned} A(0,t) &\in \mathrm{Co}\{A_1, \ldots, A_\alpha\}, \quad B(0,t) \in \mathrm{Co}\{B_1, \ldots, B_\beta\}, \\ C(0,t) &\in \mathrm{Co}\{C_1, \ldots, C_\gamma\}, \quad E(0) \in \mathrm{Co}\{E_1, \ldots, E_\delta\}, \end{aligned} \tag{29}$$

and a bound on the functional $J(u,x_0) \leq \omega = \max\limits_{1 \leq s \leq \delta} x_0^TE_s^TXE_sx_0$.

Note that the proof of Theorems 3.1 and 3.2 follows directly from Lemma 3.1 and Lyapunov theorem on asymptotic stability taking into account representation of derivative of Lyapunov function $v(x,t)$ with respect to system (23) in the form of a quadratic function with matrix of the form (18) and application of formula (6) (see [5,13]).

4 Generalized H_∞-Control

4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$\dot{x} = f(x,w,t), \quad y = g(x,w,t), \quad x(0) = x_0, \quad t \geq 0, \tag{30}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^s$ and $y \in \mathbb{R}^l$ are the state, the norm-limited external perturbations and the output vector, respectively.

Definition 4.1 The dynamical system (30) is called *nonexpansive*, if

$$\int_0^T y(t)^T Q y(t) dt \leq \int_0^T w(t)^T P w(t) dt + x_0^T X_0 x_0$$

for all square-integrable functions $w(t)$ and $T > 0$, where Q, P and X_0 are weight symmetric positive definite matrices.

We introduce the performance criterion of system (30) with respect to output y:

$$J = \sup_{0<\|w\|_P^2 + x_0^T X_0 x_0 < \infty} \varphi(w, x_0), \tag{31}$$

where

$$\varphi(w, x_0) = \frac{\|y\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|y\|_Q^2 = \int_0^\infty y^T Q y \, dt, \quad \|w\|_P^2 = \int_0^\infty w^T P w \, dt.$$

In case of $x_0 = 0$, we denote J by J_0. It is obvious, that $J_0 \leq J$ and $J \leq 1$ for a nonexpansive system. The value J describes the weighted level of external and initial perturbation suppression in system (30). If $P = I_s$, $Q = I_l$ and $X_0 = \rho I_n$, then J and J_0 coincide with known performance criteria of dynamical systems [16]. For the class of linear systems

$$\dot{x} = Ax + Bw, \quad y = Cx + Dw, \quad x(0) = x_0, \tag{32}$$

the value J_0 is equal to H_∞-norm of the transfer function $H(\lambda) = C(\lambda I_n - A)^{-1}B + D$ at $x_0 = 0$ (see, e.g., [3]).

Lemma 4.1 *Let A be a Hurwitz matrix. Then an evaluation $J_0 < \gamma$ for system* (32) *holds iff the LMI*

$$\Phi_\gamma = \begin{bmatrix} A^T X + XA + C^T Q C & XB + C^T Q D \\ B^T X + D^T Q C & D^T Q D - \gamma^2 P \end{bmatrix} < 0 \tag{33}$$

has a solution $X = X^T > 0$. To perform the evaluation $J < \gamma$ it is necessary and sufficient that LMI (33) *has a solution X such that*

$$0 < X < \gamma^2 X_0. \tag{34}$$

Proof. *Sufficiency.* Construct the quadratic Lyapunov function $v(x) = x^T X x$ for system (32) and evaluate the expression

$$\dot{v}(x) + y^T Q y - \gamma^2 w^T P w = [x^T, w^T] \Phi_\gamma \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\dot{v}(x)$ is the derivative of $v(x)$ with respect to system. Integrating given expression and in view of (31) and (33), we have

$$\|y\|_Q^2 \leq \gamma^2 (\|w\|_P^2 + x_0^T X_0 x_0), \quad \varphi(w, x_0) \leq \gamma.$$

The strict matrix inequalities (33) and (34) hold if we replace γ by $\gamma - \varepsilon$ for some small $\varepsilon > 0$. Therefore, $\varphi(w, x_0) \leq \gamma - \varepsilon$ and $J < \gamma$. In particular, in case of $x_0 = 0$ the inequality $J_0 < \gamma$ holds.

Necessity. Use the expansions $Q = \widetilde{Q}^T\widetilde{Q}$, $P = \widetilde{P}^T\widetilde{P}$, $X_0 = \widetilde{X}_0^T\widetilde{X}_0$ and transform system (32):

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{w}, \quad \widetilde{y} = \widetilde{C}\widetilde{x} + \widetilde{D}\widetilde{w}, \quad \widetilde{x}(0) = \widetilde{x}_0,$$

where $\widetilde{x} = \widetilde{X}_0 x$, $\widetilde{y} = \widetilde{Q}y$, $\widetilde{w} = \widetilde{P}w$, $\widetilde{A} = \widetilde{X}_0 A\widetilde{X}_0^{-1}$, $\widetilde{B} = \widetilde{X}_0 B\widetilde{P}^{-1}$, $\widetilde{C} = \widetilde{Q}C\widetilde{X}_0^{-1}$ and $\widetilde{D} = \widetilde{Q}D\widetilde{P}^{-1}$. Then the performance criterion (31) has the form

$$\widetilde{J} = \sup_{0<\|\widetilde{w}\|^2_{I_m}+\widetilde{x}_0^T\widetilde{x}_0<\infty} \frac{\|\widetilde{y}\|_{I_l}}{\sqrt{\|\widetilde{w}\|^2_{I_m} + \widetilde{x}_0^T\widetilde{x}_0}}.$$

If $\widetilde{J} < \gamma$, then for some matrix $\widetilde{X} = \widetilde{X}^T$ (see [16, Theorem 1])

$$0 < \widetilde{X} < \gamma^2 I_n, \quad \widetilde{\Omega} = \begin{bmatrix} \widetilde{A}^T\widetilde{X} + \widetilde{X}\widetilde{A} & \widetilde{X}\widetilde{B} & \widetilde{C}^T \\ \widetilde{B}^T\widetilde{X} & -\gamma^2 I_m & \widetilde{D}^T \\ \widetilde{C} & \widetilde{D} & -I_l \end{bmatrix} < 0$$

or

$$0 < X < \gamma^2 X_0, \quad \Omega = S^T\widetilde{\Omega}S = \begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma^2 P & D^T \\ C & D & -Q^{-1} \end{bmatrix} < 0,$$

where $X = \widetilde{X}_0^T\widetilde{X}\widetilde{X}_0$, $S = \operatorname{diag}\{\widetilde{X}_0, \widetilde{P}, \widetilde{Q}^{-1T}\}$. By Schur's lemma, the last matrix inequality reduces to the form (33) $\square$.

Remark 4.1 If $J_0 < \gamma$, then system (32) with an uncertainty

$$w = \frac{1}{\gamma}\Theta y, \quad \Theta^T P\Theta \leq Q, \tag{35}$$

is robust stable and has a common Lyapunov function $v(x) = x^TXx$. This fact follows from Lemma 4.1 and [13, Theorem 1]. The functional $\varphi(w, x_0)$ on the set of functions (35) accepts the minimum value, if $\Theta^TP\Theta = Q$. In particular, if $k \leq s$ and $x_0 = 0$, then we have $\varphi(w, 0) = \gamma$ for

$$\Theta = (\sqrt{P})^{-1}E\sqrt{Q}, \quad E = \begin{cases} I_k, & k = s, \\ \left[I_k, 0_{k\times s-k}\right]^T, & k < s. \end{cases}$$

It follows from Lemma 4.1 that the performance criteria J and J_0 of system (32) may be computed as the solutions of the corresponding optimization problems:

$$J_0 = \inf\left\{\gamma : \Phi_\gamma < 0,\ X > 0\right\}, \quad J = \inf\left\{\gamma : \Phi_\gamma < 0,\ 0 < X < \gamma^2 X_0\right\}. \tag{36}$$

Consider the affine system with norm-limited external perturbations

$$\dot{x} = A(x)x + B(x)w, \quad y = C(x)x + D(x)w, \quad x(0) = x_0, \tag{37}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in $\mathcal{S}_0$. We can formulate the following lemma (see the proof of sufficiency statement of Lemma 4.1).

Lemma 4.2 *Suppose that there exists a matrix $X = X^T > 0$ satisfying the matrix inequality*

$$\Phi_\gamma(x) = \begin{bmatrix} A^T(x)X + XA(x) + C^T(x)QC(x) & XB(x) + C^T(x)QD(x) \\ B^T(x)X + D^T(x)QC(x) & D^T(x)QD(x) - \gamma^2 P \end{bmatrix} < 0 \quad (38)$$

for all $x \in \mathcal{S}_0$. Then $J_0 \leq \gamma$ and the zero state $x \equiv 0$ of system (37) *with uncertainty* (35) *is robust stable with a common Lyapunov function $v(x) = x^T X x$. In addition, if the restriction $0 < X \leq \gamma^2 X_0$ holds, then $J \leq \gamma$.*

4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria J and J_0 of the form (31). We are interested in control laws that ensure nonexpansivity property of close loop system and minimize J and J_0. A control law is said to be *J-optimal*, if corresponding close loop system has minimum performance criteria J. A J_0-optimal control law is *H_∞-optimal* in case of the identity weight matrices P and Q.

Primarily, we consider the static output-feedback controller

$$u = K_* y + w, \quad (39)$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations and $K_* \in \mathcal{K}_D$ is an unknown matrix. Assuming that $\det \big[I_m - K_* D(x)\big] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding close loop systems in the form

$$\dot{x} = A_*(x)x + B_*(x)w, \quad y = C_*(x)x + D_*(x)w, \quad x(0) = x_0, \quad (40)$$

$$\dot{x} = A_* x + B_* w, \quad y = C_* x + D_* w, \quad x(0) = x_0, \quad (41)$$

where $A_*(x) = A(x) + B(x)\big[I_m - K_* D(x)\big]^{-1} K_* C(x)$, $B_*(x) = B(x)\big[I_m - K_* D(x)\big]^{-1}$, $C_*(x) = \big[I_l - D(x)K_*\big]^{-1} C(x)$, $D_*(x) = \big[I_l - D(x)K_*\big]^{-1} D(x)$, $A_* = A_*(0)$, $B_* = B_*(0)$, $C_* = C_*(0)$, $D_* = D_*(0)$.

Theorem 4.1 [18] *For linear system* (2)*, there exists an output-feedback controller* (39) *such that $J < \gamma$ iff the following correlations are feasible:*

$$W_R^T \begin{bmatrix} A^T X + XA + C^T QC & XB + C^T QD \\ B^T X + D^T QC & D^T QD - \gamma^2 P \end{bmatrix} W_R < 0, \quad (42)$$

$$W_L^T \begin{bmatrix} AY + YA^T + BP^{-1}B^T & YC^T + BP^{-1}D^T \\ CY + DP^{-1}B^T & DP^{-1}D^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \quad (43)$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \quad (44)$$

where $R = [C, D]$, $L = [B^T, D^T]$. Herewith, the zero states $x \equiv 0$ of systems (40) *and* (41) *with uncertainty* (35) *are robust stable with common Lyapunov function $v(x) = x^T X x$.*

Remark 4.2 The gain matrix K_* in Theorem 4.1 may be constructed in the form

$$K_* = K_0(I_l + DK_0)^{-1}, \quad -K_0 \in \mathcal{K}_D, \quad (45)$$

Here K_0 is an arbitrary solution of the LMI

$$L_0^T K_0 R_0 + R_0^T K_0^T L_0 + \Omega < 0, \quad (46)$$

where $R_0 = \left[R, 0_{l\times l}\right]$, $L_0 = \left[L, 0_{m\times m}\right]\widetilde{X}$,

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_m & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -P & D^T \\ C & D & -Q^{-1} \end{bmatrix}.$$

Lemma 4.3 [17] *LMI* (46) *has a solution* K_0 *if and only if*

$$W_{L_0}^T \Omega W_{L_0} < 0, \quad W_{R_0}^T \Omega W_{R_0} < 0, \tag{47}$$

where W_{L_0} (W_{R_0}) *is a matrix whose columns make up the bases of the kernel* $\operatorname{Ker} L_0$ $(\operatorname{Ker} R_0)$.

4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky + w, \quad \xi(0) = 0, \tag{48}$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations, Z, V, U and K are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear close loop system (2), (48) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad y = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{49}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \widehat{M} = \begin{bmatrix} A + BK_0C & BU_0 \\ V_0C & Z_0 \end{bmatrix}, \quad \widehat{N} = \begin{bmatrix} B + BK_0D \\ V_0D \end{bmatrix},$$

$$\widehat{F} = \left[C + DK_0C, DU_0\right], \quad \widehat{G} = D + DK_0D, \ \ K_0 = \mathbf{D}(K),$$

$$U_0 = (I_m - KD)^{-1}U, \ \ V_0 = V(I_l - DK)^{-1}, \ \ Z_0 = Z + VD(I_m - KD)^{-1}U.$$

We give the following auxiliary statement (see also [16] in case of $\gamma = 1$).

Lemma 4.4 *Gain matrices* $X > 0$, $Y > 0$ *and number* $\gamma > 0$, *there are matrices* $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ *and* $Y_2 \in \mathbb{R}^{r\times r}$ *such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{50}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{51}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (49), we get the following result.

Theorem 4.2 [18] *There exists a dynamic controller* (48) *such that the evaluation* $J < \gamma$ *holds for linear system* (49), *iff the LMI system* (34), (42), (43) *and* (51) *is solvable with respect to* $X = X^T > 0$ *and* $Y = Y^T > 0$. *Herewith, a close loop system* (49) *with uncertainty* (35) *is robust stable.*

Remark 4.3 The coefficient matrices of dynamic controller (48) in Theorem 4.2 may be constructed in the form

$$\begin{array}{c} K = (I_m + K_0 D)^{-1} K_0, \quad U = (I_m + K_0 D)^{-1} U_0, \\ V = V_0 (I_l + D K_0)^{-1}, \quad Z = Z_0 - V_0 D (I_m + K_0 D)^{-1} U_0, \end{array} \tag{52}$$

by solving LMI

$$\widehat{L}^T \widehat{K}_0 \widehat{R} + \widehat{R}^T \widehat{K}_0^T \widehat{L} + \widehat{\Omega} < 0, \tag{53}$$

where

$$\widehat{\Omega} = \begin{bmatrix} A^T X + XA & A^T X_1^T & XB & C^T \\ X_1 A & 0 & X_1 B & 0 \\ B^T X & B^T X_1^T & -P & D^T \\ C & 0 & D & -Q^{-1} \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} XB & X_1^T \\ X_1 B & X_2 \\ 0 & 0 \\ D & 0 \end{bmatrix},$$

$$\widehat{R} = \begin{bmatrix} C & 0 & D & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},$$

Here X, X_1 and X_2 are blocks of matrix $\widehat{X}$ in (50).

If $K \in \mathcal{K}_D$, then $\det \big[I_m - KD(x)\big] \neq 0$ for all $x \in \mathcal{S}_0$, where $\mathcal{S}_0$ is some neighbourhood of the point $x = 0$, and nonlinear close loop system (1), (48) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}(\widehat{x})\widehat{x} + \widehat{N}(\widehat{x})w, \quad y = \widehat{F}(\widehat{x})\widehat{x} + \widehat{G}(\widehat{x})w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{54}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (48), (52) ensures robust stability of the zero state $\widehat{x} \equiv 0$ of system (54) with uncertainty (35) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics J_0 and J of system (54), we can apply Lemma 4.2.

4.4 Control systems with controlled and observed outputs

Consider the control system

$$\begin{array}{l} \dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{array} \tag{55}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively. We are interested in static and dynamic control laws that ensure nonexpansivity property of close loop system and minimize the performance criteria J and J_0 with respect to controlled output z of the form (31), where

$$\varphi(w, x_0) = \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|z\|_Q^2 = \int_0^\infty z^T Q z \, dt, \quad \|w\|_P^2 = \int_0^\infty w^T P w \, dt.$$

4.4.1 Static controllers

If we use the static output feedback controller

$$u = Ky, \quad K \in \mathcal{K}_{D_{22}}, \tag{56}$$

then closed loop system (55), (56) has the form

$$\dot{x} = Mx + Nw, \quad z = Fx + Gw, \quad x(0) = x_0, \tag{57}$$

where $M = A+B_2K_0C_2$, $N = B_1+B_2K_0D_{21}$, $F = C_1+D_{12}K_0C_2$, $G = D_{11}+D_{12}K_0D_{21}$, $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^T \begin{bmatrix} A^TX + XA + C_1^TQC_1 & XB_1 + C_1^TQD_{11} \\ B_1^TX + D_{11}^TQC_1 & D_{11}^TQD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{58}$$

$$W_L^T \begin{bmatrix} AY + YA^T + B_1P^{-1}B_1^T & YC_1^T + B_1P^{-1}D_{11}^T \\ C_1Y + D_{11}P^{-1}B_1^T & D_{11}P^{-1}D_{11}^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{59}$$

where $R = \left[\, C_2, D_{21} \,\right]$, $L = \left[\, B_2^T, D_{12}^T \,\right]$.

Theorem 4.3 *For linear system* (55)*, there exists an output feedback controller* (56) *such that* $J < \gamma$ *iff the system of correlations* (44)*,* (58) *and* (59) *is feasible. Herewith, system* (57) *with uncertainty*

$$w = \frac{1}{\gamma}\Theta z, \quad \Theta^T P \Theta \le Q, \tag{60}$$

is robust stable with common Lyapunov function $v(x) = x^TXx$.

If we use a static state feedback $u = Kx$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case the correlations (44) and (58) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad D_{11}^TQD_{11} - \gamma^2 P < 0. \tag{61}$$

Corollary 4.1 *For linear system* (55)*, there exists a state feedback controller* $u = Kx$ *such that* $J < \gamma$ *iff the LMI system* (59) *and* (61) *is solvable for some matrix* $Y = Y^T > 0$. *Herewith, system* (57) *with uncertainty* (60) *is robust stable with common Lyapunov function* $v(x) = \gamma^2 x^T Y^{-1} x$.

Remark 4.4 The gain matrix K in Theorem 4.3 and Corollary 4.1 may be constructed as

$$K = K_0(I_l + D_{22}K_0)^{-1}, \quad -K_0 \in \mathcal{K}_{D_{22}}, \tag{62}$$

where K_0 is an arbitrary solution of LMI:

$$\widehat{L}^T K_0 \widehat{R} + \widehat{R}^T K_0^T \widehat{L} + \Omega < 0,$$

where $\widehat{R} = \left[R, 0_{l\times k}\right]$, $R = \left[\, C_2, D_{21} \,\right]$, $\widehat{L} = \left[L, 0_{m\times s}\right] \widetilde{X}$, $L = \left[\, B_2^T, D_{12}^T \,\right]$,

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^TX + XA & XB_1 & C_1^T \\ B_1^TX & -\gamma^2 P & D_{11}^T \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{63}$$

with $K \in \mathcal{K}_{D_{22}}$, then closed loop system (55), (63) has the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad z = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{64}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \widehat{M} = \begin{bmatrix} A + B_2K_0C_2 & B_2U_0 \\ V_0C_2 & Z_0 \end{bmatrix} = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2,$$

$$\widehat{N} = \begin{bmatrix} B_1 + B_2K_0D_{21} \\ V_0D_{21} \end{bmatrix} = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{F} = \big[C_1 + D_{12}K_0C_2, D_{12}U_0\big] = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2,$$

$$\widehat{G} = D_{11} + D_{12}K_0D_{21} = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{A} = \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \ \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix}, \ \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix},$$

$$\widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \ \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{r\times s} \end{bmatrix}, \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{r\times s} \end{bmatrix},$$

$$\widehat{C}_1 = \big[C_1, 0_{k\times r}\big], \quad \widehat{D}_{12} = \big[D_{12}, 0_{k\times r}\big].$$

Here the blocks of matrix $\widehat{K}_0$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$
$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U,$$

are unknown, and

$$\begin{aligned} K &= (I_m + K_0D_{22})^{-1}K_0, \quad U = (I_m + K_0D_{22})^{-1}U_0, \\ V &= V_0(I_l + D_{22}K_0)^{-1}, \quad Z = Z_0 - V_0D_{22}(I_m + K_0D_{22})^{-1}U_0. \end{aligned} \tag{65}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (64), we get the following result.

Theorem 4.4 *For linear system* (55)*, there exists a dynamic controller* (63) *such that* $J < \gamma$ *iff the system of correlations* (34)*,* (51)*,* (58) *and* (59) *is feasible. Herewith, system* (64) *with uncertainty* (60) *is robust stable.*

Remark 4.5 The coefficient matrices of dynamic controller (63) in Theorem 4.4 may be constructed in the form (65) by solving the LMI

$$\widehat{L}^T\widehat{K}_0\widehat{R} + \widehat{R}^T\widehat{K}_0^T\widehat{L} + \widehat{\Omega} < 0, \tag{66}$$

where

$$\widehat{R} = \big[\widehat{R}_1, 0_{l+r\times k}\big], \ \widehat{R}_1 = \big[\widehat{C}_2, \widehat{D}_{21}\big], \ \widehat{L} = \big[\widehat{L}_1, 0_{m+r\times s}\big]\widetilde{X}, \ \widehat{L}_1 = \big[\widehat{B}_2^T, \widehat{D}_{12}^T\big],$$

$$\widetilde{X}=\begin{bmatrix}\widehat{X}&0&0\\0&0&I_k\\0&I_s&0\end{bmatrix},\quad \widehat{\Omega}=\begin{bmatrix}\widehat{A}^T\widehat{X}+\widehat{X}\widehat{A}&\widehat{X}\widehat{B}_1&\widehat{C}_1^T\\\widehat{B}_1^T\widehat{X}&-\gamma^2P&D_{11}^T\\\widehat{C}_1&D_{11}&-Q^{-1}\end{bmatrix},\quad \widehat{X}=\begin{bmatrix}X&X_1^T\\X_1&X_2\end{bmatrix}.$$

Herewith, system (64) with uncertainty (60) has common Lyapunov function $v(\widehat{x})=\widehat{x}^T\widehat{X}\widehat{x}$.

We give the following algorithm for constructing stabilizing dynamic controller (63) satisfying Theorem 4.4.

Algorithm 4.1 1) calculate the matrices W_R and W_L, where $R=\left[C_2,D_{21}\right]$ and $L=\left[B_2^T,D_{12}^T\right]$;

2) find the matrices $X=X^T>0$ and $Y=Y^T>0$ satisfying (34), (51), (58) and (59);

3) construct the expansion $Z=Y-\gamma^2X^{-1}=V^TV$, $V\in\mathbb{R}^{r\times n}$, $\ker V=\ker Z$ and form the block matrix

$$\widehat{X}=\begin{bmatrix}X&X_1^T\\X_1&X_2\end{bmatrix}>0,\quad X_1=\frac{1}{\gamma}VX,\quad X_2=\frac{1}{\gamma^2}VXV^T+I_r;$$

4) solve the LMI (66) under restriction $\det(I_m+K_0D_{22})\neq 0$;

5) calculate the coefficient matrices of dynamic controller (63) by formula (65).

Static and dynamic output-feedback controllers (56) and (63) with $K\in\mathcal{K}_{D_{22}}$ may be applied to a class of affine systems

$$\begin{aligned}\dot{x}&=A(x)x+B_1(x)w+B_2(x)u,\quad x(0)=x_0,\\z&=C_1(x)x+D_{11}(x)w+D_{12}(x)u,\\y&=C_2(x)x+D_{21}(x)w+D_{22}(x)u.\end{aligned}\tag{67}$$

So, close loop system (63), (67) reduces to the form

$$\dot{\widehat{x}}=\widehat{M}(\widehat{x})\widehat{x}+\widehat{N}(\widehat{x})w,\quad z=\widehat{F}(\widehat{x})\widehat{x}+\widehat{G}(\widehat{x})w,\quad \widehat{x}(0)=\widehat{x}_0.\tag{68}$$

As a result, the dynamic controller (63), (65) ensures robust stability of the zero state $\widehat{x}\equiv 0$ of system (68) with uncertainty (60) and a common Lyapunov function $v(\widehat{x})=\widehat{x}^T\widehat{X}\widehat{x}$. To evaluate characteristics J_0 and J of system (68), we can apply Lemma 4.2.

Remark 4.6 Note, that we have necessary and sufficient conditions for an evaluation $J_0<\gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without usage of additional restriction $X<\gamma^2X_0$. With the use of static state feedback or full order dynamic controllers the problems under consideration are reduced to the solution of LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with the uncertainties

$$A\in\mathrm{Co}\{A^1,\dots,A^{\nu_1}\},\ B_1\in\mathrm{Co}\{B_1^1,\dots,B_1^{\nu_2}\},$$

$$C_1\in\mathrm{Co}\{C_1^1,\dots,C_1^{\nu_3}\},\ D_{11}\in\mathrm{Co}\{D_{11}^1,\dots,D_{11}^{\nu_4}\}.$$

In addition, sufficient statements of these theorems may be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

Example 4.1 Consider a controlled linear damped oscillator described by system (55) with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\delta \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1, 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = D_{22} = 0, \quad x = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}, \quad z = \begin{bmatrix} \varphi \\ u \end{bmatrix}, \quad y = \varphi.$$

Taking into account (36) in the absence of control, we get $J_0 = 1,001$ and $J = 1,289$ assuming that

$$\delta = 0,1, \quad \omega_0 = 1, \quad P = 1, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad X_0 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix},$$

where $q_1 = 0,01$, $q_2 = 0,1$, $\rho_1 = \rho_2 = 0,04$. Figure 1 shows the dependence J of δ and ω_0. The damping level of input signals and initial perturbations of oscillator decreases with the increase of its natural frequency ω_0 and does not change with the increase of the damping factor δ.

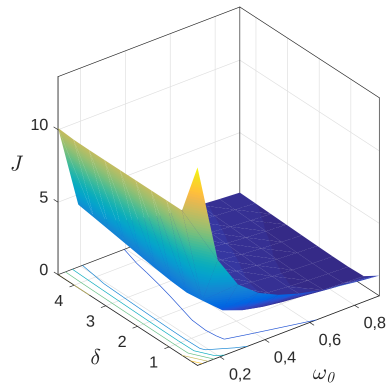

Figure 1: The dependence $J(\delta, \omega_0)$.

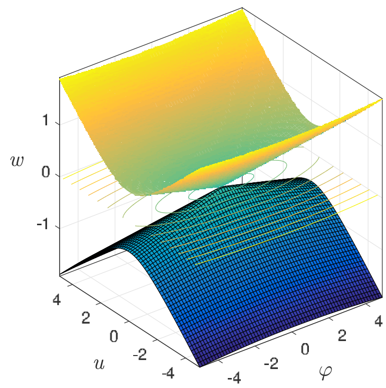

Figure 2: Uncertainty region.

Next, using Algorithm 4.1, we performed minimization of the parameter γ satisfying Theorem 4.4. As a result for $\gamma = 0,865$, we constructed an approximate J-optimal dynamic controller (63) with the coefficient matrices

$$Z = \begin{bmatrix} -0,06612 & -0,09307 \\ 0,23117 & -1,05843 \end{bmatrix}, \quad V = \begin{bmatrix} -0,00037 \\ 0,11011 \end{bmatrix},$$

$$U = \begin{bmatrix} -0,31404 & 3,90247 \end{bmatrix}, \quad K = -0,23776,$$

that provides a robust stability and nonexpansiveness of close loop system. This regulator significantly reduced the damping level of input signals and initial perturbations of oscillator. For example, for the indicated values of parameters we have $J_0 = 0,39062$

and $J = 0,86181 < 1$. The oscillator with constructed regulator preserves asymptotic stability for any perturbation function (see Figure 2)

$$w(t) = \frac{1}{\gamma}\Theta z(t),\ \Theta = \begin{bmatrix}\theta_1, \theta_2\end{bmatrix}, \quad \frac{\theta_1^2}{q_1} + \frac{\theta_2^2}{q_2} \le 1,\ |w| \le \frac{1}{\gamma}\sqrt{q_1\varphi^2 + q_2 u^2}. \tag{69}$$

The dependences $J_0(q_1, q_2)$ and $J(\rho_1, \rho_2)$ for close loop system are shown in Figures 3 and 4, respectively.

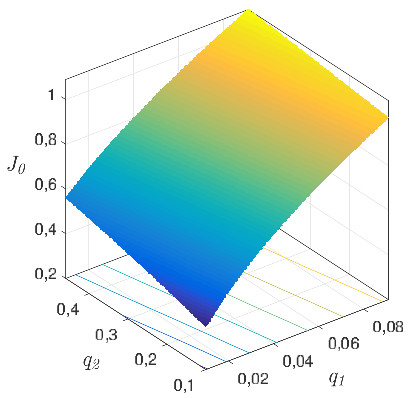

Figure 3: The dependence $J_0(q_1, q_2)$ (close loop system).

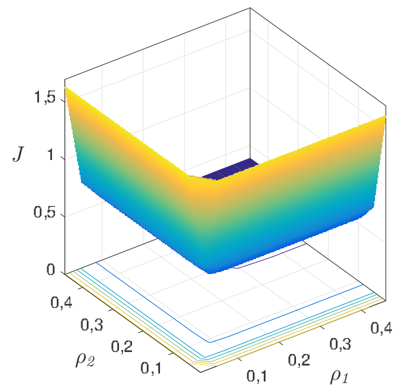

Figure 4: The dependence $J(\rho_1, \rho_2)$ (close loop system).

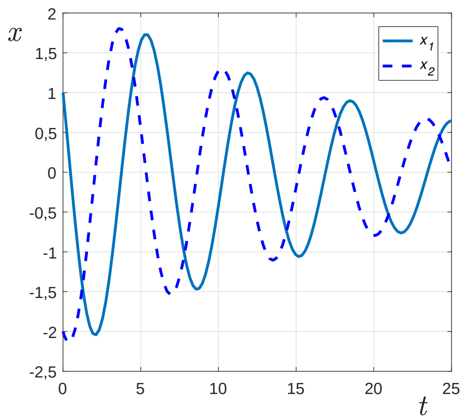

Figure 5: System behavior without control.

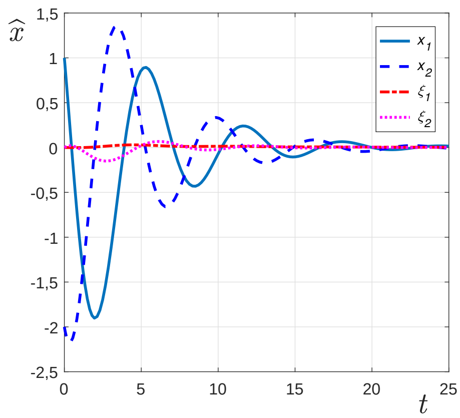

Figure 6: Close loop system behavior.

Figure 5 shows system behavior without control for the initial vector $x_0 = [1, -2]^T$ and Figure 6 shows close loop system behavior for the initial vector $\widehat{x}_0 = [1, -2, 0, 0]^T$, wherein the perturbation function w has the form (69) with $\Theta = \sqrt{P}^{-1} E\sqrt{Q} = \begin{bmatrix}\sqrt{q_1/2}, \sqrt{q_2/2}\end{bmatrix}$, where $E = \begin{bmatrix}1/\sqrt{2}, 1/\sqrt{2}\end{bmatrix}$, $E^T E \le I_2$.

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Relation Between Fuzzy Semigroups and Fuzzy Dynamical Systems

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Abstract: In this work we study a relation between fuzzy semigroups and fuzzy dynamical systems. Some concepts about stability are introduced to evaluate fuzzy semigroups. Several examples are given to illustrate the obtained results.

Keywords: *fuzzy strongly continuous semigroups; fuzzy dynamical systems; stability; Zadehs extension.*

Mathematics Subject Classification (2010): 34A07, 34K36, 35R13.

1 Introduction

Let $\pi(t,.)$ be a flow generated by solutions of autonomous differential equation. M. T. Mizukoshi et al. showed in [13] that the family of applications $\widehat{\pi}(t,.)$, indexed by $\mathbb{R}$, obtained by Zadeh's extension(see [17]) on initial condition of the flow $\pi(t,.)$, satisfies conditions that characterize $\widehat{\pi}(t,.)$ as a dynamical system in the metric space E^n. In [14], authors discuss conditions for existence of equilibrium points for $\widehat{\pi}(t,.)$ and the nature of the stability of such equilibrium points. New results about equilibrium points are presented in [2].

In [1], M. S. Cecconello discusses results obtained in [4] on invariant sets and stability of such fuzzy sets for fuzzy dynamical systems.

The fuzzy dynamical systems we consider here are obtained by Zadeh's extension of dynamical systems defined on subsets of $\mathbb{R}^n$.

In this paper we discuss relationships between fuzzy semigroups and fuzzy dynamical systems and consider results obtained in [1] on invariant sets and stability of such fuzzy sets, but in this case for fuzzy semigroups.

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2 Preliminary Notes

Let $\mathcal{P}_K(\mathbb{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $\mathcal{P}_K(\mathbb{R}^n)$ as usual. Let A and B be two nonempty bounded subsets of $\mathbb{R}^n$. The distance between A and B is defined by the Hausdorf metric

$$d(A,B) = \max\Big(\rho(A,B), \rho(B,A)\Big),$$

where $\rho(A,B) = \sup\limits_{a\in A}\inf\limits_{b\in B} \|a-b\|$ and $\|\ \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Then it is clear that $(\mathcal{P}_K(\mathbb{R}^n), d)$ becomes a complete and separable metric space (see [16]).
Denote

$$E^n = \Big\{ u\ :\ \mathbb{R}^n \longrightarrow [0,1]\ \ |\ \ \text{u satisfies (i)-(iv) below} \Big\},$$

where

(i) u is normal i.e there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{t \in \mathbb{R}^n \,/\, u(t) \geq \alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha \in \mathcal{P}_K(\mathbb{R}^n)$ for all $0 \leq \alpha \leq 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u,\ v \in E^n$, $k \in \mathbb{R}^n$ and $0 \leq \alpha \leq 1$. Define $D : E^n \times E^n \to \mathbb{R}^+$ by the equation

$$D(u,v) = \sup_{0\leq\alpha\leq 1} d\Big([u]^\alpha\ ,\ [v]^\alpha\Big),$$

where d is the Hausdorff metric for non-empty compact sets in $\mathbb{R}^n$.Then it is easy to see that D is a metric in E^n. Using the results in ([16]), we know that
(1) (E^n, D) is a complete metric space;
(2) $D(u+w, v+w) = D(u,v)$ for all $u,\ v,\ x \in E^n$;
(3) $D(k\,u, k\,v) = |k|\ D(u,v)$ for all $u,\ v \in E^n$ and $k \in \mathbb{R}^n$.

On E^n, we can define the substraction $\ominus$, called the H-difference (see [5]) as follows: $u \ominus v$ has sense if there exists $w \in E^n$ such that $u = v + w$.

Nguyens theorem provides an important relationship between α-levels of image of fuzzy subsets and the image of their α-levels by a function $f : X \times Y \longrightarrow Z$. According to [12], if $X \subseteq \mathbb{R}^n$, Y $\subseteq \mathbb{R}^m$ and $f : X \longrightarrow Y$ is continuous, then Zadehs extension $\widehat{f} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ is well defined and

$$\Big[\widehat{f}(u)\Big]^\alpha = f\left([u]^\alpha\right), \quad \forall u \in \mathcal{F}(X), \quad \forall \alpha \in [0,1]. \tag{1}$$

Theorem 2.1 *(see [17]) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function. Then the following conditions are equivalents:*
(i) f is continuous;
(ii) $\widehat{f} : (E^n; D) \longrightarrow (E^n; D)$ is continuous.

2.1 Fuzzy strongly continuous semigroups.

We give here a definition of a fuzzy semigroup.

Definition 2.1 A family $\{T(t), t \geq 0\}$ of operators from E^n into itself is a fuzzy strongly continuous semigroup if

(i) $T(0) = I_{E^n}$, the identity mapping on E^n,

(ii) $T(t+s) = T(t)T(s)$ for all t, $s \geq 0$,

(iii) the function $g : [0, \infty[\rightarrow E^n$, defined by $g(t) = T(t)x$ is continuous at $t = 0$ for all $x \in E^n$ i.e
$$\lim_{t \to 0^+} T(t)x = x,$$

(iv) There exist two constants $M > 0$ and ω such that
$$D\Big(T(t)x, T(t)y\Big) \leq M\, e^{\omega t}\, D(x, y), \quad \text{for}\, t \geq 0, \quad x, \ y \in E^n.$$

In particular if $M = 1$ and $\omega = 0$, we say that $\{T(t), t \geq 0\}$ is a contraction fuzzy semigroup.

Remark 2.1 The condition (iii) implies that the function $t \longrightarrow T(t)(x)$ is continuous on $[0, \infty[$ for all $x \in E^n$.

Definition 2.2 Let $\{T(t), t \geq 0\}$ be a fuzzy strongly continuous semigroup on E^n and $x \in E^n$. If for $h > 0$ sufficiently small, the Hukuhara difference $T(h)x \ominus x$ exits, we define
$$Ax = \lim_{h \to 0^+} \frac{T(h)x \ominus x}{h}$$
whenever this limit exists in the metric space (E^n, D). Then the operator A defined on
$$D(A) = \left\{ x \in E^n : \lim_{h \to 0^+} \frac{T(h)x \ominus x}{h} \,\text{exists} \right\} \subset E^n$$
is called the infinitesimal generator of the fuzzy semigroup $\{T(t), t \geq 0\}$.

Lemma 2.1 *Let A be the generator of a fuzzy semigroup $\{T(t), t \geq 0\}$ on E^n, then for all $x \in E^n$ such that $T(t)x \in D(A)$ for all $t \geq 0$, the mapping $t \to T(t)x$ is differentiable and*
$$\frac{d}{dt}\Big(T(t)\, x\Big) = A\, T(t)\, x, \quad \forall t \geq 0.$$

2.2 Fuzzy dynamical systems.

Definition 2.3 We say that a family of continuous maps, defined on the complete metric space (X, H),
$$\begin{array}{rll} \pi : & \mathbb{R}_+ \times X & \longrightarrow X \\ & (t, x_0) & \longmapsto \pi\,(t, x_0) \end{array}$$
is a dynamical system, or semiflow, if $\pi(t, .)$ satisfies

1. $\pi(0, x_0) = x_0$;
2. $\pi(t, \pi(s, x_0)) = \pi(t + s, x_0)$

for all $t, s \in \mathbb{R}_+$ and $x_0 \in X$.
The set X is called phase space of the dynamical system. In the case of $X \subset \mathbb{R}^2$, X is said to be phase plane.

In dynamical systems context, an orbit (positive) of a point $xo \in X$ is the subset of the phase space defined by

$$\theta(x_0) = \bigcup_{t \in \mathbb{R}_+} \pi(t, x_0) = \{\pi(t, x_0), \quad t \in \mathbb{R}_+\},$$

and for each subset $B \subset X$ we have $\theta(B) = \bigcup_{x_0 \in B} \theta(x_0)$.

The set $\theta(x_0)$ is called periodic orbit if there exists $\tau > 0$ such that $\pi(t + \tau, x_0) = \pi(t, x_0)$. The smallest number $\tau > 0$ for which this property is satisfied is called period of the orbit [6].

The ω-limit of a subset $B \subset X$ is defined as

$$W(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \pi(t, B)}.$$

Remark 2.2 Let $x_0 \in X$, we have

$$W(x_0) = \left\{ y, \quad \exists t_n \longrightarrow +\infty, \quad \lim_{n \to +\infty} \pi(t_n, x_0) = y \right\}.$$

A set $S \subset U$ is called invariant if $\theta(x_0) \subset S$ for all $x_0 \in S$. It follows that S is invariant if and only if $\pi(t, S) = S$ for all $t \in \mathbb{R}_+$.

Example : The orbits and ω-limit are examples of invariant sets.

Definition 2.4 We say that a set $M \subset X$ attracts a set $B \subset \mathbb{R}^n$, by flow $\pi(t, .)$, if $\rho(\pi(t, B), M) \longrightarrow 0$ when $t \to +\infty$. In other words, we say that "a set M attracts a set B" is equivalent to saying that M attracts uniformly all orbits with initial condition in B, that is,

$$\lim_{t \to +\infty} \sup \{\rho(\pi(t, x_0), M) : \quad x_0 \in B\} = 0.$$

The basin of attraction of a set M is the set $A(M)$ defined by

$$A(M) = \{x_0 \in U : \quad \rho(\pi(t, x_0), M) \longrightarrow 0, \quad t \to +\infty\}.$$

The set M is called attractor if there exists an open subset $V \supset M$ such that $V \subset A(M)$. If M is an attractor and attracts compacts subsets of $A(M)$, then M is a uniform attractor.

Definition 2.5 Let $\bar{x} \in X$, $\bar{x}$ is an equilibrium point if $\pi(t, \bar{x}) = \bar{x}$ for all $t \in \mathbb{R}_+$.

Definition of stability for invariant sets is similar to definition of stability for equilibrium points. That is, an invariant set S is stable if for every neighborhood V of S, there exists a neighborhood V' of S such that $\pi(t, V') \subset V$ for all $t \in \mathbb{R}_+$. When S is stable and moreover there exists a neighborhood W such that S attracts points of W then S is an asymptotically stable set.

Theorem 2.2 *([1]) Let M be compact and invariant. Then M is asymptotically stable if and only if M is a uniform attractor.*

Let $\pi(t,.)$ be a dynamical system defined in a subset $U \subset \mathbb{R}^n$, that we will call deterministic dynamical system.

Theorem 2.3 *(see [13]) Let $\pi(t,.) : U \longrightarrow U$ be a deterministic dynamical system. Then $\widehat{\pi}(t,.)$ defined by Zadeh's extension applied in $\pi(t,.)$ has the following properties:*

1. $\widehat{\pi}\,(0, x_0) = x_0$, $\forall x_0 \in \mathcal{F}(U)$;
2. $\widehat{\pi}\,(t+s, x_0) = \widehat{\pi}(t, \widehat{\pi}(s, x_0))$, $\forall x_0 \in \mathcal{F}(U)$, $t, s \geq 0$.

Thus, the Zadeh's extension $\widehat{\pi}(t,.) : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, of deterministic dynamical system $\pi(t,.) : U \longrightarrow U$, is a dynamical system in $\mathcal{F}(U)$ and we will call it fuzzy dynamical system. Then concepts of stability and asymptotic stability for invariant sets in $\mathcal{F}(U)$ follow definitions given previously to general metric spaces.

3 Main Results

3.1 Relation between fuzzy semigroups and fuzzy dynamical systems.

Let $(\pi(t,.))_{t\geq 0}$ be a dynamical system on $\mathbb{R}^n$, i.e for all $t \geq 0$

$$\begin{array}{rll} \pi(t,.) : & \mathbb{R}^n & \longrightarrow \mathbb{R}^n \\ & x & \longmapsto \pi(t,x) \end{array}$$

satisfies

1. $\pi(0,x) = x$, $\forall x \in \mathbb{R}^n$;
2. $\pi(t+s,x) = \pi(t, \pi(s,x))$, $\forall t,s \geq 0, \quad x \in \mathbb{R}^n$;
3. $t \longrightarrow \pi(t,x)$ is continuous for all $x \in \mathbb{R}^n$.

We consider the family $\{T(t),\ t \geq 0\}$ given by

$$\begin{array}{rll} T : & \mathbb{R}_+ \times \mathbb{R}^n & \longrightarrow \mathbb{R}^n \\ & (t,x) & \longmapsto T(t)x = \pi(t,x). \end{array}$$

Then the family of continuous maps $T(t)$ verifies

1. $T(0) = I$;
2. $T(t+s) = T(t)T(s)$, $\forall t,s \geq 0$.

Then the family $\{T(t),\ t \geq 0\}$ defines a strongly continuous semigroup on $\mathbb{R}^n$.

By Zadeh's extension, we can define a fuzzy dynamical system $\widehat{\pi}(t,.)$.

Define a mapping $\left\{\widehat{T}(t), t \geq 0\right\}$ as follows

$$\begin{array}{rll} \widehat{T} : & \mathbb{R}_+ \times \mathbb{R}^n & \longrightarrow \mathbb{R}^n \\ & (t,x) & \longmapsto \widehat{T}(t)x = \widehat{\pi}(t,x). \end{array}$$

From Theorem 2.3, we have

1. $\widehat{T}(0)x = x$, $\forall x \in \mathcal{F}(\mathbb{R}^n)$;
2. $\widehat{T}(t+s)x = \widehat{T}(t) \circ \widehat{T}(s)x$, $\forall x \in \mathcal{F}(\mathbb{R}^n)$, $t, s \geq 0$.

Theorem 3.1 *Let $\{T(t), t \geq 0\}$ be a strongly continuous semigroup on $\mathbb{R}^n$ which satisfies*

$$\|T(t)x\| \leq Me^{wt}\|x\|, \quad \forall x \in \mathbb{R}^n, \quad t \geq 0.$$

Then

$$D\left(\widehat{T}(t)x, \widehat{T}(t)y\right) \leq Me^{wt}D(x,y), \quad \forall x, y \in E^n, \quad t \geq 0.$$

Proof. Let $x, y \in E^n$, $\alpha \in [0,1]$, we have

$$\begin{aligned}
\rho(T(t)[x]^\alpha, T(t)[y]^\alpha) &= \sup_{a \in [x]^\alpha} \inf_{b \in [y]^\alpha} \|T(t)a - T(t)b\| \\
&\leq Me^{wt} \sup_{a \in [x]^\alpha} \inf_{b \in [y]^\alpha} \|a - b\| \\
&\leq Me^{wt}\rho([x]^\alpha, [y]^\alpha) \\
&\leq Me^{wt} \max\left\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\rho(T(t)[y]^\alpha, T(t)[x]^\alpha) &= \sup_{a \in [y]^\alpha} \inf_{b \in [x]^\alpha} \|T(t)a - T(t)b\| \\
&\leq Me^{wt} \sup_{a \in [y]^\alpha} \inf_{b \in [x]^\alpha} \|a - b\| \\
&\leq Me^{wt}\rho([y]^\alpha, [x]^\alpha) \\
&\leq Me^{wt} \max\left\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\right\}.
\end{aligned}$$

This implies

$$\begin{aligned}
d\left([\widehat{T}(t)x]^\alpha, [\widehat{T}(t)y]^\alpha\right) &= d\left(T(t)[x]^\alpha, T(t)[y]^\alpha\right) \\
&= \max\left\{\rho(T(t)[x]^\alpha, T(t)[y]^\alpha), \rho(T(t)[y]^\alpha, T(t)[x]^\alpha)\right\} \\
&\leq Me^{wt} \max\left\{\rho([x]^\alpha, [y]^\alpha), \rho([y]^\alpha, [x]^\alpha)\right\} \\
&= Me^{wt} d\left([x]^\alpha, [y]^\alpha\right).
\end{aligned}$$

Hence, we conclude that

$$D\left(\widehat{T}(t)x, \widehat{T}(t)y\right) \leq Me^{wt}D(x,y).$$

□

Corollary 3.1 $\left\{\widehat{T}(t), t \geq 0\right\}$ *is a fuzzy strongly continous semigroup on E^n.*

Proof. (i) and (ii) are immediate consequences of Theorem 2.3.
Theorem 2.1 ensures (iii).
(iv) follows immediately from Theorem 3.1.

□

Now, we can conclude that from fuzzy dynamical systems, we can define fuzzy strongly continuous semigroups.

Example. We define on $\mathbb{R}$ the family of operator $(\pi(t,.))_{t\geq 0}$ by

$$\pi(t,x) = e^{at}x, \quad a \in \mathbb{R}.$$

$(\pi(t,.))_{t\geq 0}$ is a dynamical system on $\mathbb{R}$. We consider the family $\{T(t), t \geq 0\}$ given by

$$T(t)x = \pi(t,x).$$

$\{T(t), t \geq 0\}$ is a strongly continuous semigroup on $\mathbb{R}$, and the linear operator A defined by $Ax = ax$ is the infinitesimal generator of this semigroup. Then the family of continuous maps $\left\{\widehat{T}(t), t \geq 0\right\}$ defined by $\widehat{T}(t)x = \widehat{\pi}(t,x)$, where $(\widehat{\pi}(t,.)$ is the fuzzy dynamical system obtained by Zadeh's extension applied in $\pi(t,.))$ defines fuzzy strongly continuous semigroups on E^1.

3.2 Invariant and attractor sets for fuzzy strongly continuous semigroups.

In this section, we give the results obtained by M. S. Cecconello, J. Leite , R. C. Bassanezi, A. J. V. Brando (see [1]), but in this case for a fuzzy semigroups.

Let $\{T(t), t \geq 0\}$ be a strongly continuous semigroup on E^n and $\left\{\widehat{T}(t), t \geq 0\right\}$ be the fuzzy strongly continuous semigroup obtained by Zadeh's extension applied in $\{T(t), t \geq 0\}$.

Proposition 3.1 *$\bar{x}$ is an equilibrium point of $T(t)(T(t)\bar{x} = \bar{x})$ if, and only if $\chi_{\{\bar{x}\}}$ is an equilibrium point of $\widehat{T}(t)$, where $\widehat{T}(t)$ is the characteristic function of $\bar{x}$.*

Proof. We have

$$\bar{x} = T(t)\left([\chi_{\{\bar{x}\}}]^\alpha\right) \Leftrightarrow \left[\chi_{\{\bar{x}\}}\right]^\alpha = \left[\widehat{T}(t)(\chi_{\{\bar{x}\}})\right]^\alpha.$$

□

To prove the following results it is suffcient to denote $\widehat{\pi}(t,x) = \widehat{T}(t)x$ in [1].

Theorem 3.2 *Let $S \subset U \subset \mathbb{R}^n$ and consider $S_{\mathcal{F}} \in \mathcal{F}(U)$ defined by*

$$S_{\mathcal{F}} = \{x \in E^n : [x]^\alpha \subset S\}.$$

S is invariant by $T(t)$ if and only if $S_{\mathcal{F}}$ is invariant by $\widehat{T}(t)$.

The ω-limit of a subset $B \subset U$ is defined as

$$\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} T(t)(B)}.$$

Consider the set $\omega_{\mathcal{F}}(B) \subset \mathbb{F}(U)$ defined by

$$\omega_{\mathcal{F}}(B) = \{x \in \mathcal{F}(U) : [x_0]^\alpha \subset \omega(B)\}.$$

Corollary 3.2 *The set $\omega(B)$ is invariant by $T(t)$ if and only if the set $\omega_{\mathcal{F}}(B)$ is invariant by $\widehat{T}(t)$.*

Theorem 3.3 *Let $C \subset U$ be an invariant set by $T(t)$ and consider the set $C_{\mathcal{F}} \subset \mathcal{F}(U)$ defined by*

$$C_{\mathcal{F}} = \left\{x \in \mathcal{F}(U) : [x]^0 \subset C\right\}.$$

Then we have:

1. *C is stable for $T(t)$ if and only if $C_{\mathcal{F}}$ is stable for $\widehat{T}(t)$;*

2. *C is asymptotically stable for $T(t)$ if and only if $C_{\mathcal{F}}$ is asymptotically stable for $\widehat{T}(t)$.*

By the previous theorem we can establish the following result.

Corollary 3.3 *Let $\bar{x} \in U$ be an equilibrium point of $T(t)$. Then*

1. *$\bar{x}$ is stable for $T(t)$ if and only if $\chi\{\bar{x}\}$ is stable for $\widehat{T}(t)$;*

2. *$\bar{x}$ is asymptotically stable for $T(t)$ if and only if $\chi\{\bar{x}\}$ is asymptotically stable for $\widehat{T}(t)$.*

An orbit (positive) of a point x_0 is the subset of the phase space defined by

$$\theta(x_0) = \bigcup_{t \in \mathbb{R}_+} T(t)x_0 = \{T(t)x_0, \quad t \in \mathbb{R}_+\}.$$

Similarly, if θ is a periodic orbit for $T(t)$ then θ is invariant. By Theorem 4.2, the set θ is invariant for $\widehat{T}(t)$ and we have:

Corollary 3.4 *Let θ be a periodic orbit for $T(t)$ with period $\tau > 0$ and $\theta_{\mathcal{F}}$ be the fuzzy periodic set defined by*

$$\theta_{\mathcal{F}} = \left\{x \in \mathcal{F}(U) : [x]^0 \subset \theta\right\}.$$

Then

1. *θ is stable for $T(t)$ if and only if $\theta_{\mathcal{F}}$ is stable for $\widehat{T}(t)$;*

2. *θ is asymptotically stable for $T(t)$ if and only if $\theta_{\mathcal{F}}$ is asymptotically stable for $\widehat{T}(t)$.*

Let A, $B \subset \mathbb{R}^n$ and $A_{\mathcal{F}}, B_{\mathcal{F}} \subset E^n$ be defined, respectively, by

$$A_{\mathcal{F}} = \left\{x \in E^n : [x]^0 \subset A\right\} \quad \text{and} \quad B_{\mathcal{F}} = \left\{x \in E^n : [x]^0 \subset B\right\}.$$

Theorem 3.4 *Set A attracts B by $T(t)$ if and only if $A_{\mathcal{F}}$ attracts $B_{\mathcal{F}}$ by $\widehat{T}(t)$.*

So we have the following result for $\omega(B)$ and the set $\omega_{\mathcal{F}}(B)$:

Corollary 3.5 *The set $\omega(B)$ attracts $B \subset U$ by $T(t)$ if and only if $\omega_{\mathcal{F}}(B)$ attracts $A_{\mathcal{F}} = \left\{x \in \mathcal{F}(U) : [x_0]^0 \subset A\right\}$ by $\widehat{T}(t)$.*

3.3 Some examples

Example: Given the classic initial value problem

$$(2)\begin{cases} x' = -kx, \\ x(0) = x_0, \end{cases}$$

we can verify that its solution is given by $T(t)x_0 = e^{-kt}x_0$ and that the origin is an asymptotically stable equilibrium point for $k > 0$. By Proposition 4.1, we have that $\chi_{\{0\}}$ is an equilibrium point of $\widehat{T}(t)$, since $\bar{x} = 0$ is an equilibrium point of (2).
Moreover, $\chi_{\{0\}}$ is asymptotically stable.

Example: Let us consider the deterministic Verhulst model

$$(3)\begin{cases} x^{'} = ax(1-x), \\ x(0) = x_0, \end{cases}$$

whose solution is given by

$$T(t)x_0 = \frac{x_0}{x_0 + (1-x_0)e^{-at}}.$$

The equilibrium points of (3) are 0 and 1. The first one is unstable while the latter is asymptotically stable.

So, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are equilibrium points of $\widehat{T}(t)$ being the fuzzy strongly semigroup obtained by Zadeh's extension of $T(t)$ and the first one is unstable equilibrium point while the latter is asymptotically stable. Some other details for this example are presented in [1].

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Shunt Active Power Filter Based Harmonics Compensation of a Low-Voltage Network Using Fuzzy Logic System

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Abstract: This paper presents the design of optimal fuzzy logic system, controlled a shunt active power filter (sAPF) for harmonics compensation which is injected by non-linear loads. This method is applied to a sAPF based on a threephase voltage converter at two levels. The main contribution of this paper is the use of P-Q method for reference currents calculation by applying fuzzy logic for better active filter current control accuracy. For pulse generation, we use the PWM strategy. The results reflect clearly the effectiveness of the proposed APF to meet the IEEE-519 standard recommendations on harmonic levels. To validate the theoretical part, work simulations under Matlab-Simulink are provided.

Keywords: *fuzzy logic system; P-Q algorithm method; shunt Active Power Filter; Total Harmonic Distortion (THD).*

Mathematics Subject Classification (2010): 93C42, 03B52, 93E11, 93Cxx.

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1 Introduction

The increasing use of control systems based on power electronics in industry involves more and more disturbance problems in the level of the electrical power supply networks. Non-linear electronic components such as diode/ thyristor rectifiers, switched mode power supplies, arc furnaces, incandescent lighting and motor drives are widely used in industrial and commercial applications. These nonlinear loads create harmonic or distortion current problems in the transmission and distribution network. The harmonics induce malfunctions in sensitive equipment, over voltage by resonance and harmonic voltage drop across the network impedance that affect power quality [1].

The controller is the heart of the active power filter and many studies are being conducted in this area recently. Conventional PI voltage and current controllers have been used to control the harmonic current and the dc voltage of shunt APLC. Recently, fuzzy logic controllers (FLC) are used in power electronic system and drive applications [2].

The power switching devices are driven with specific control strategy to produce current able to compensate harmonic and poor power factor load. In this work, we take the inverter supplied with a continuous source controlled by fuzzy logic system in series with p-q algorithm method.

2 Shunt Active Power Filter

The shunt Active Power Filter is a power electronic device based on the use of power electronic inverters (Figure 1). The shunt active power filter is connected in a common point connection between the source of power system and the load system which present the source of the polluting currents circulating in the power system lines. This insertion is realized via low pass filter such as, L, LC or LCL filters [3].

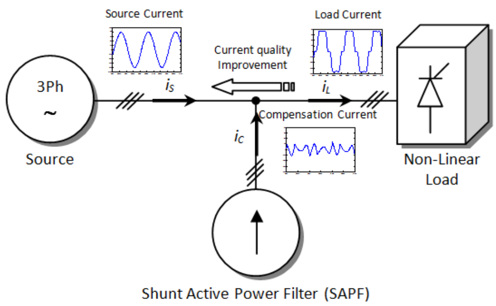

Figure 1: Shunt Active Power Filter principle schematics.

The most important objective of the APF is to compensate the harmonic currents due to the non-linear load. Exactly to sense the load currents and extract the harmonic component of the load current to produce a reference current as shown in Figure 2, The reference current consists of the harmonic components of the load current which the active filter must supply [4,5]. This reference current is fed through a controller and then the switching signal is generated to switch the power switching devices of the inverter, so that the active filter will indeed produce the harmonics required by the load. Finally, the AC supply will only need to provide the load fundamental component, resulting in a low harmonic sinusoidal supply. A shunt active power filter is controlled to supply/extract compensate current to/from the utility Point Common Coupling (PCC).

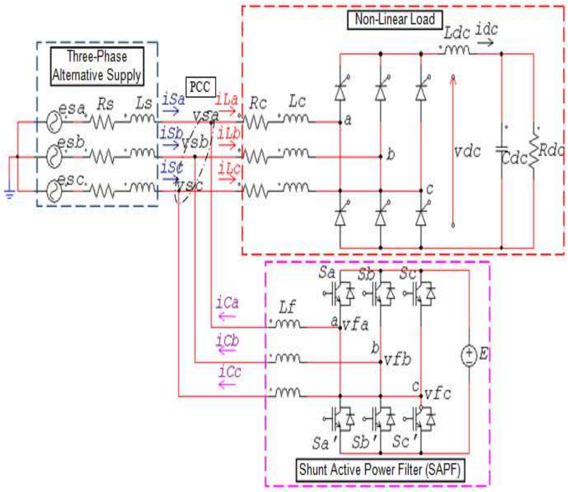

Figure 2: Equivalent schematic of sAPF two levels.

3 Control strategy of the shunt active power filter

The control strategy permits to generate the gating signals to the APF switches. Mainly, we distinguish between two kinds of control techniques. The first one, commonly called the Sigma-Delta Modulation, is based on a hysteresis comparator and is characterized by a non controllable switching frequency. The second one modulates the pulse width and controls the switching frequency: it is called the Pulse width Modulation (PWM). Several PWM techniques exist [6, 7].

Particularly, we cite the carrier-based modulation, the calculated one, and the space vector one. In this paper we applied the carrier-based PWM having the control law described in Figure 3. As shown in Figure 3.a, the pulses (gating signals) are obtained by the intersections of the modulation signal (Δi_C in our case) and one or many carrier signals (generally triangular or saw-toothed signals). This can be realized by comparing the APF error current with the triangular carrier signal (Figure 3.b). After that, the output passes through a hysteresis comparator and is saturated between 0 and 1, corresponding to the two states of the switch. Then, if the saturated output is equal to 1, the leg upper switch is in the 'on' position; else, it is in the 'off' position. Concerning the lower switches, complementarities with the upper ones must be ensured (i.e. Si' = not (Si), i = a, b, c) in order to avoid the opening of voltage sources or the short-cutting of current sources.

In this kind of PWM, the parameters that influence the switching frequency are mainly the modulation index (modulating wave magnitude/carrier signal magnitude) and the carrier frequency.

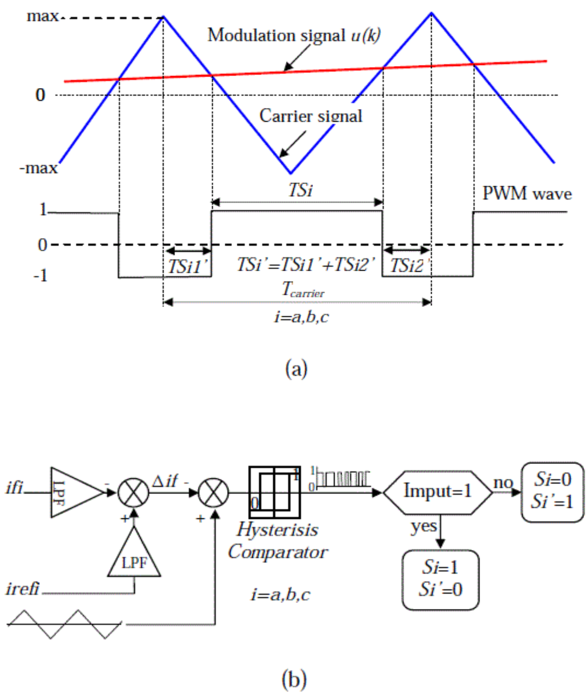

Figure 3: The two-level shunt APF Control law. (a) Carrier-based PWM principle. (b) Pulses generation.

4 Fuzzy Logic Controller

Fuzzy logic serves to represent uncertain and imprecise knowledge of the system, whereas fuzzy control allows taking a decision even if we can not estimate inputs/outputs only from uncertain predicates [8–18]. Figure 4 shows the synoptic scheme of fuzzy controller, which possesses two inputs: the error (e), ($e = i_{ref} - i_C$) and its derivative (de), and one output: the command (cde).

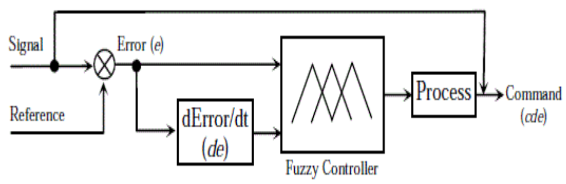

Figure 4: Fuzzy controller synoptic diagram.

Figure 5 illustrates stages of fuzzy control in the considered base of rules and definitions: fuzzification, inference mechanism, and defuzzification.

This step consists of transforming the classical low pass correctors (LPF) on fuzzy ones. The main characteristics of the fuzzy control are:

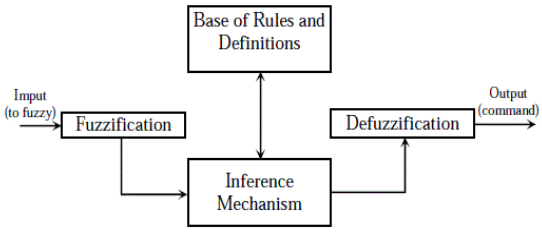

Figure 5: Fuzzy control construction.

- Three fuzzy sets for each of the two inputs (e, de) with Gaussian membership functions.
- Five fuzzy sets for the output with triangular membership functions.
- Implications using the minimum operator, inference mechanism based on fuzzy implication containing five fuzzy rules.
- Defuzzification using the 'centroïd' method.

The establishment of the fuzzy rules is based on the error (e) sign and variation. As explained in Figure 6, and knowing that (e) is increasing if its derivative (de) is positive, constant if (de) is equal to zero, decreasing if (de) is negative, positive if ($i_{Cref} > i_C$), zero if ($i_{Cref} = i_C$), and negative if ($i_{Cref} < i_C$), the command (cde) is:

- zero, if (e) is equal to zero,
- big positive (BP) if (e) is positive both in the increasing and the decreasing cases,
- big negative (BN) if (e) is negative both in the increasing and the decreasing cases,
- negative (N) if (e) is increasing towards zero,
- positive (P) if (e) is decreasing towards zero.

Finally, the fuzzy rules are summarized as follows:

1. If (e) is zero (ZE), then (cde) is zero (ZE).
2. If (e) is positive (P), then (cde) is big positive (BP).
3. If (e) is negative (N), then (cde) is big negative (BN).
4. If (e) is zero (ZE) and (de) is positive (P), then (cde) is negative (N).
5. If (e) is zero (ZE) and (de) is negative (N), then (cde) is positive (P).

The fuzzy inference mechanism used in this work is presented as follows. The fuzzy rules are summarized in Table 1.

$$\mu(u(t)) = \max_{j=1}^{m} [\mu_{A1^j}(e(t)), \mu_{A2^j}(\Delta e(t)), \mu_{B^j}(cde(t))] \tag{1}$$

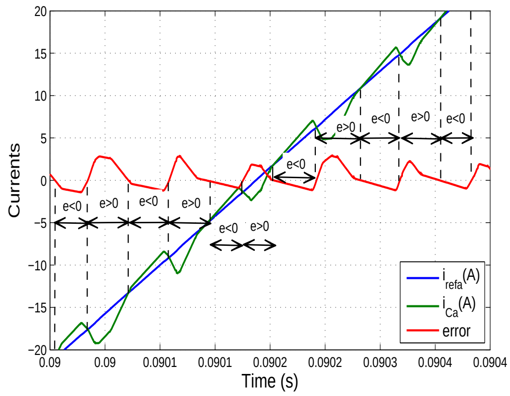

Figure 6: Fuzzy rules establishment.

Fuzzy output $cde(t)$ can be calculated by the center of gravity defuzzification as:

$$cde(t) = \frac{\sum_{i=1}^{m} \mu_B(\mu_i(t))cde_i}{\sum_{i=1}^{m} \mu_B(\mu_i(t))}, \tag{2}$$

where i is the output rule after inferring [19].

		$e(t)$				
	$cde(t)$	**BN**	**SN**	**Z**	**SP**	**BP**
	BN	BN	BN	SN	SN	Z
	SN	BN	SN	SN	Z	SP
$de(t)$	**Z**	SN	SN	Z	SP	SP
	SP	SN	Z	SP	SP	BP
	BP	Z	SP	SP	BP	BP

Table 1: Fuzzy inference rules.

The following Figure 7 to Figure 10 show FIS editor, FIS file viewer, Surface file viewer Fuzzy and the degree of membership for the error and its derivative and the command signal respectively.

5 Mathematical Model of the Instantaneous Power

5.1 Instantaneous active and reactive powers

This method of identification of harmonic currents is simply to eliminate the dc component of instantaneous active and reactive power which is relatively easy to achieve [20]. Respectively denote the vectors of voltages at the connection point $[v_S]$ and load currents

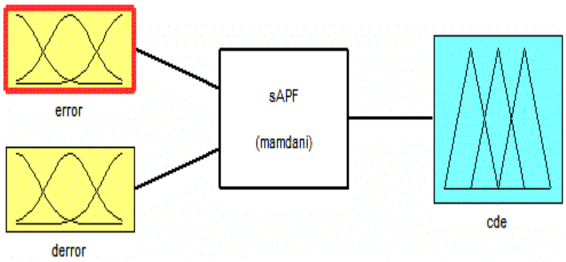

Figure 7: FIS Editor.

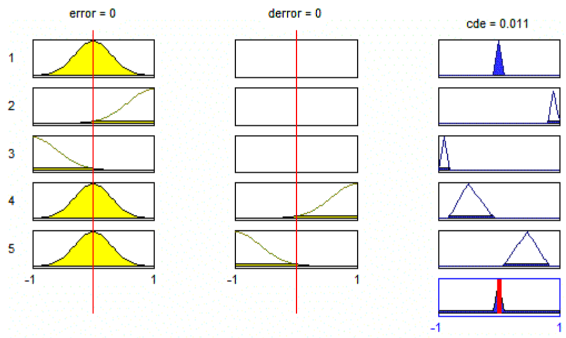

Figure 8: Rule viewer.

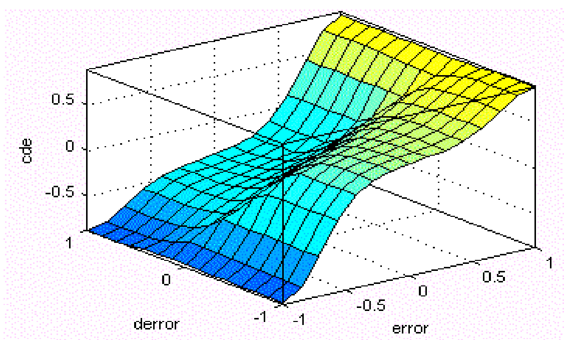

Figure 9: Surface viewer.

$[i_L]$ in a balanced three phase system by:

$$[v_S] = \begin{bmatrix} v_{Sa} \\ v_{Sb} \\ v_{Sc} \end{bmatrix} \tag{3}$$

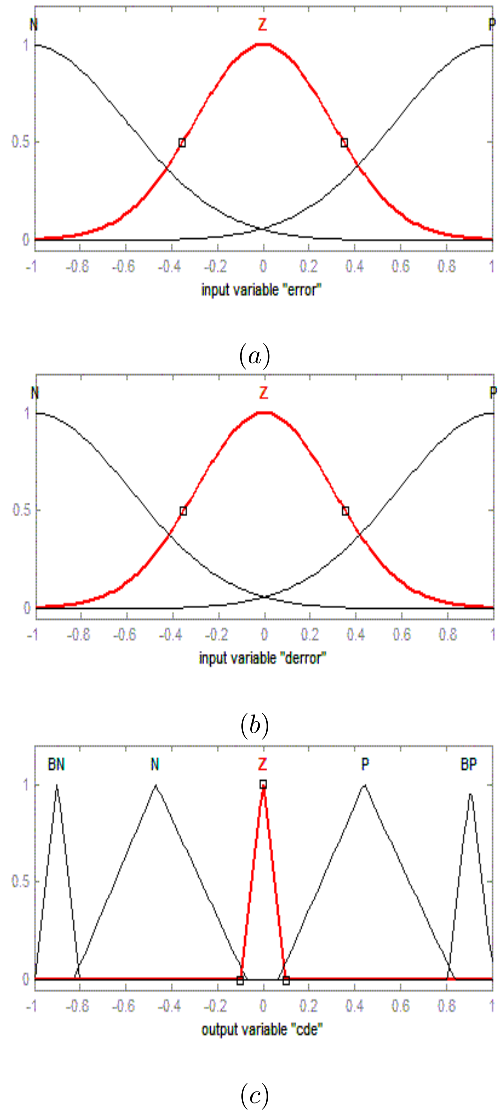

Figure 10: The degree of membership function of fuzzy logic controller for (a) the error, (b) derivative and (c) the command signal.

and

$$[i_L] = \begin{bmatrix} i_{La} \\ i_{Lb} \\ i_{Lc} \end{bmatrix}. \tag{4}$$

The transformation of three-phase instantaneous values of voltage and current in the reference frame of coordinates is given by the following terms:

$$\begin{bmatrix} v_{S\alpha} \\ v_{S\beta} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} v_{Sa} \\ v_{Sb} \\ v_{Sc} \end{bmatrix} \tag{5}$$

and currents :

$$\begin{bmatrix} i_{L\alpha} \\ i_{L\beta} \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} i_{La} \\ i_{Lb} \\ i_{Lc} \end{bmatrix}. \tag{6}$$

The real and imaginary instantaneous power denoted p and q are defined by the following matrix relation:

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} v_{S\alpha} & v_{S\beta} \\ -v_{S\beta} & v_{S\alpha} \end{bmatrix} \begin{bmatrix} i_{L\alpha} \\ i_{L\beta} \end{bmatrix}. \tag{7}$$

By replacing the two-phase voltages and currents by their counterparts phase, we obtain:

$$p = v_{S\alpha} i_{L\alpha} + v_{S\beta} i_{L\beta} = v_{Sa} i_{La} + v_{Sb} i_{Lb} + v_{Sc} i_{Lc}. \tag{8}$$

Similarly, for the imaginary power we have:

$$q = v_{S\alpha} i_{L\beta} - v_{S\beta} i_{L\alpha} = -\frac{1}{\sqrt{3}}[(v_{Sa} - v_{Sb}) i_{Lc} + (v_{Sb} - v_{Sc}) i_{La} + (v_{Sc} - v_{Sa}) i_{Lb}]. \tag{9}$$

From the expression (8), asking:

$$\triangle = v_{S\alpha}^2 + v_{S\beta}^2,$$

we have :

$$\begin{bmatrix} i_{L\alpha} \\ i_{L\beta} \end{bmatrix} = \frac{1}{\triangle} \left\{ \begin{bmatrix} v_{S\alpha} & -v_{S\beta} \\ v_{S\beta} & v_{S\alpha} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \right\} \tag{10}$$

or :

$$\begin{bmatrix} i_{L\alpha} \\ i_{L\beta} \end{bmatrix} = \frac{1}{\triangle} \left\{ \begin{bmatrix} v_{S\alpha} & -v_{S\beta} \\ v_{S\beta} & v_{S\alpha} \end{bmatrix} \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} v_{S\alpha} & -v_{S\beta} \\ v_{S\beta} & v_{S\alpha} \end{bmatrix} \begin{bmatrix} 0 \\ q \end{bmatrix} \right\} = \begin{bmatrix} i_{L\alpha p} \\ i_{L\beta p} \end{bmatrix} + \begin{bmatrix} i_{L\alpha q} \\ i_{L\beta q} \end{bmatrix} \tag{11}$$

with

$$i_{L\alpha p} = \frac{v_{S\alpha}}{\triangle} p, \quad i_{L\alpha q} = -\frac{v_{S\beta}}{\triangle} q, \quad i_{L\beta p} = \frac{v_{S\beta}}{\triangle} p, \quad i_{L\beta q} = \frac{v_{S\alpha}}{\triangle} q.$$

From the expressions (9), we can write:

$$p = p_{\alpha p} + p_{\beta p} + p_{\alpha q} + p_{\beta q} = p_{\alpha p} + p_{\beta p}. \tag{12}$$

The instantaneous powers p and q are expressed as:

$$p = \overline{p} + \widetilde{p}; \quad q = \overline{q} + \widetilde{q} \tag{13}$$

with:

$\overline{p}$ and $\overline{q}$: Continuous power related to the active and reactive fundamental component of the current.

$\widetilde{p}$ and $\widetilde{q}$: Power alternatives related to the sum of harmonic components of current.

$$\begin{bmatrix} i_{Ca}^* \\ i_{Cb}^* \\ i_{Cc}^* \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} i_{C\alpha}^* \\ i_{C\beta}^* \end{bmatrix}. \tag{14}$$

The diagram in Figure 11 shows the steps for obtaining the current harmonic components of nonlinear load [21].

Figure 12 shows the block diagram of the shunt active power filter controlled by fuzzy logic system with p-q algorithm.

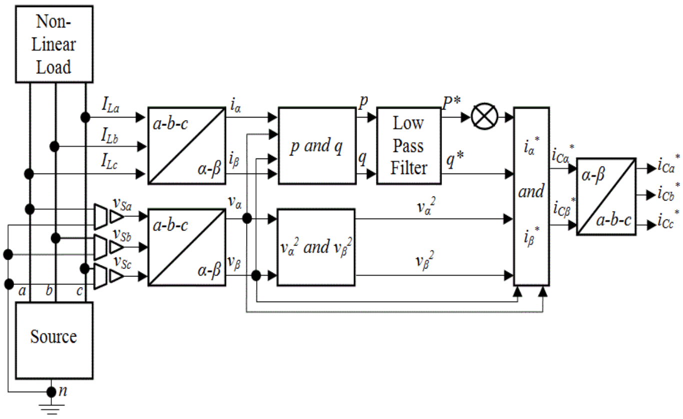

Figure 11: "P-Q" Algorithm extraction of harmonic currents.

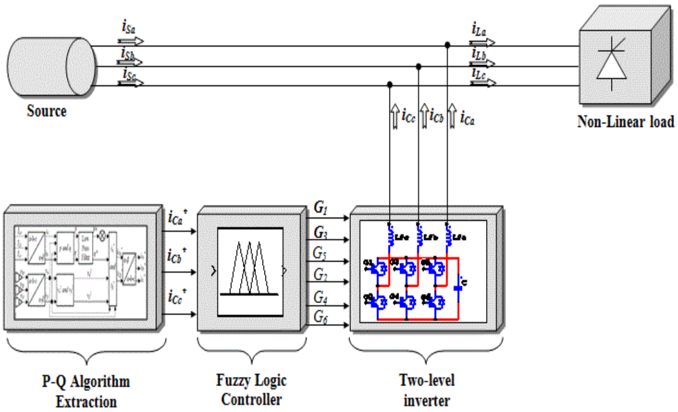

Figure 12: Schematic of a shunt Active Power Filter controlled by a fuzzy logic system.

5.2 Apparent power, reactive power and distortion power

Steady deformed, it must amend the definition of power so that it reflects the current harmonic:

$$S = \sqrt{P^2 + Q^2 + D^2}. \tag{15}$$

Figure 13 shows the vector representation of apparent power.

In single phase, if the instantaneous voltage and current are expressed as:

$$v(t) = \sqrt{2} V_{eff} \sin(\omega t), \tag{16}$$

$$i(t) = \sum_{n=1}^{\infty} \sqrt{2} I_{n,eff} \sin(n\omega t + \phi_n). \tag{17}$$

This is the case for a strong network. Then we have:

$$P = V_{eff} I_{1,eff} \cos(\phi_1), \tag{18}$$

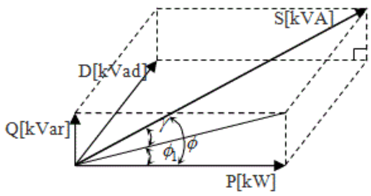

Figure 13: Vector representation of apparent power.

$$Q = V_{eff} I_{1,eff} \sin(\phi_1), \tag{19}$$

$$S = V_{eff} I_{eff}, \tag{20}$$

$$I_{eff} = \sqrt{I_{1,eff}^2 + I_{2,eff}^2 + I_{3,eff}^2 + ... + I_{n,eff}^2}, \tag{21}$$

$$D = V\sqrt{I_{2,eff}^2 + I_{3,eff}^2 + ... + I_{n,eff}^2}. \tag{22}$$

5.3 Total Harmonic Distortion (THD)

Our work focuses on using a parallel active filter, which means we need to calculate the Total Harmonic Distortion of current, as shown in this expression [22]:

$$THD_i = \frac{\sqrt{\sum_{n=2}^{\infty} I_{n(rms)}^2}}{I_{1(rms)}}. \tag{23}$$

6 Simulation Result and Analysis

The SIMULINK toolbox in the MATLAB software is used to model and test the system under steady state and transient conditions before and after using fuzzy logic controller. The system parameters values are summarized in Table 2.

Three-phase network of supply :	
Supply's voltage & frequency,	220 V rms, 50 Hz
Line's inductance Ls & resistance Rs,	19.4 μ H, 0.25 $m\Omega$
Non-linear DC link's :	
InductanceLdc, Resistance Rdc & capacitance Cdc,	20 mH, 6 Ω, 0.01 μF
Inductance L_C & resistance R_C,	0.1 mH, 0.5 Ω
Shunt Active Filter:	
DC supply voltage E, resistance R_f & inductance L_f,	650 V, 4 $m\Omega, 1.5mH$
Control bloc:	
1 st order Low Pass Filter: i_C LPF,	K=1, T= 50 e^{-6}, s
i_{Cref} LPF,	K=1, T = 2 e^{-4}, s
Carrier bipolar saw-toothed, signal magnitude & frequency,	10, 20 kHz
Switching frequency	5 kHz

Table 2: Simulation parameters common to the applications considered.

6.1 Characteristics of the source current before active filtering

The graphs of the source current before application of active filtering are shown in Figures 14, 15 and 16. There is a symmetrical distortion of current i_{La} from the point of half period (Figure 14), which means that the harmonic multiples of 2 and 3 are absent in the spectrum of i_{La} and that only those of rank $(6h \pm 1)$ are present (Figure 15), this is confirmed by the spectrum of i_{La} representing the top 30 most significant harmonics with a THD of 26.37% for an observation period of 0.1 s.

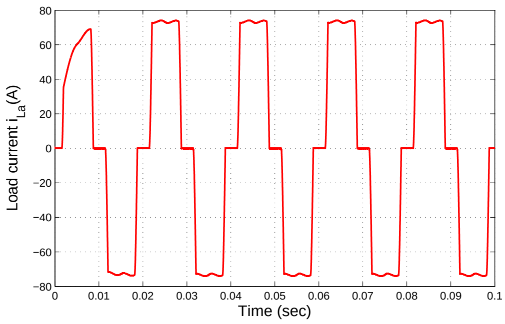

Figure 14: Load current waveform before compensation.

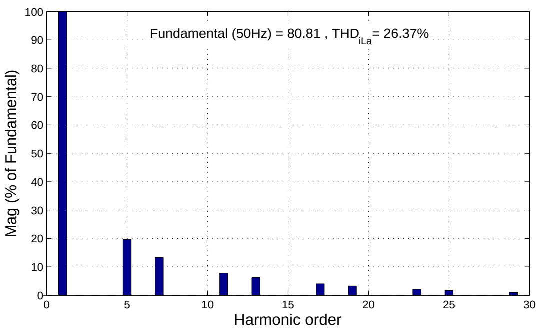

Figure 15: Harmonic spectrum of iLa before compensation.

Harmonic distortion is not the only problem here as Figure 16 shows a degradation in power factor (estimated late 0.0015 s, then $\varphi = 27°$, or $\cos\varphi = 0.891$), so we can expect a change in the reactive energy of the system.

6.2 Characteristics of the source current after active filtering

To improve the waveform i_{Sa}, was inserted an inductance L_C of 0.1 mH and a resistance R_C of 0.5Ω input of the pollutant load with a sAPF, as shown in Figure 2. The result was satisfactory since the distortions have been reduced and it is the same for the THD (I_{Sa}) with a new rate of 2.82%, as shown in Figures 17 and 18.

The source side, the two curves in Figure 19, representing the current and voltage source in phase, despite the presence of a slight delay (delayed i_{Sa} from v_{Sa}) generated

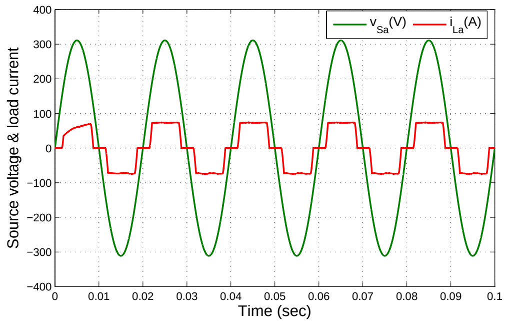

Figure 16: Voltage and current waveforms before compensation.

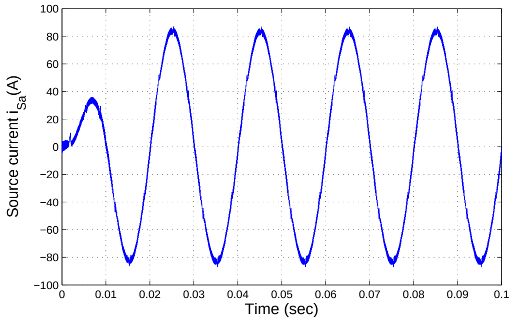

Figure 17: Supply current waveform after compensation.

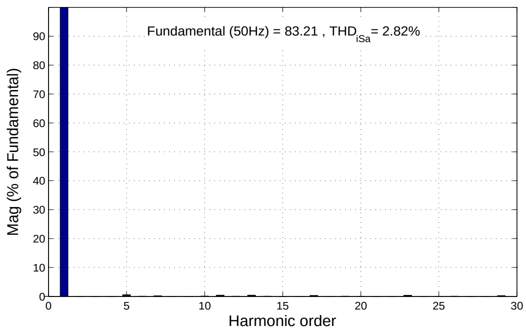

Figure 18: Harmonic spectrum of iSa after compensation.

by L_C and R_C, indicating a power factor corrected, very close to unity. Therefore, so a good compensation of reactive power source.

The deformations in the form of i_{Sa} are in the intersection points nonzero of i_{La} and i_{Ca}, as shown in Figure 20.

The effectiveness of the fuzzy control strategy is illustrated in Figure 21 mentioning the sAPF current pursuing its reference.

Figure 22 gives us an idea about the non-linear DC side current and voltage waveforms

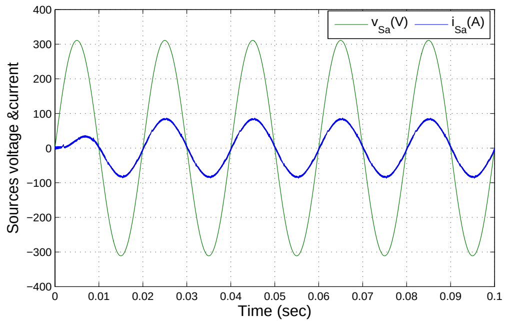

Figure 19: Voltage and current waveforms after compensation.

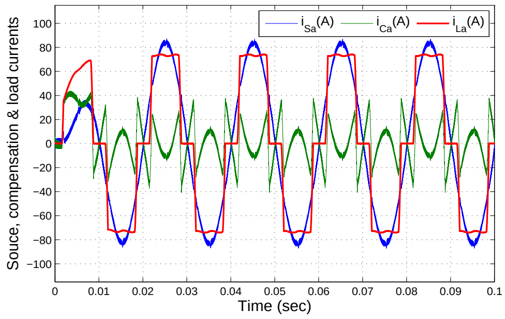

Figure 20: Source current iSa, sAPF filter iCa and non-linear load current iLa.

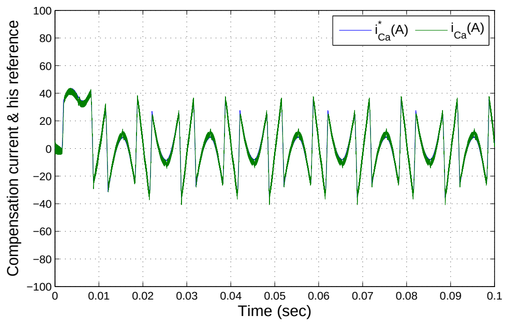

Figure 21: sAPF current and its reference with fuzzy correctors.

with fuzzy controller.

7 Conclusion

The results obtained allow us to visualize the effectiveness of shunt active power filter (sAPF) using a fuzzy controller in series with p-q algorithm method. In fact, the harmonic distortion THD drops from 26.37% to 2.82% after using the active filter. Thus the power factor has been fixed.

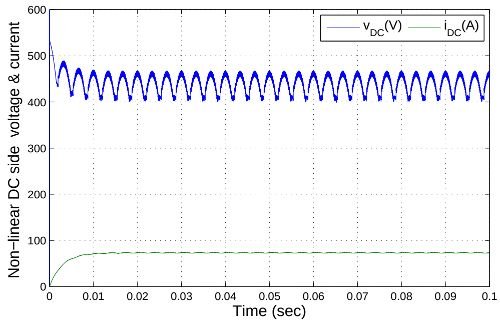

Figure 22: Non-linear DC side current and voltage after compensation.

For future work, we plan to extend our study for other structures by increasing the number of levels of the inverter and compare between them. We also intend to consider a pollution load with more than 26.37% of total harmonic distortion.

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A Simple Approach for Q-S Synchronization of Chaotic Dynamical Systems in Continuous-Time

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Abstract: In this paper, the problem of Q-S synchronization for arbitrary dimensional chaotic dynamical systems in continuous-time is investigated. Based on new control scheme and Lyapunov stability theory, a simple synchronization approach is designed to achieve Q-S synchronization between n-D and m-D continuous-time chaotic systems in arbitrary dimension d. In order to verify the effectiveness of the proposed method, our approach is applied to some typical chaotic systems and numerical simulations are given to validate the derived results.

Keywords: *chaos; Q-S synchronization; continuous-time systems; control scheme; Lyapunov stability.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C10, 93C55.

1 Introduction

Since the discover of synchronization [1, 2], chaos synchronization has played important roles in sciences and enginering, due to its potential applications in secure communication and telecommunications [3–6], control theory [7, 8], biology [9, 10], lasers [11], and so on. Chaos synchronization has received increasing interest and various methods have been proposed for synchronization of chaotic dynamical systems such as adaptive control [12], backstepping design [13], sliding mode control [14], and generalized hamiltonian systems approach [15, 16] etc. Many types of chaos synchronization have been presented such as complete and anti-synchronization [17, 18], hybrid function projective synchronization [19], reduced order function projective combination synchronization [20], etc. Among all types of synchronization, Q-S synchronization is an interesting generalized-type of synchronization which has been extensively considered [21, 22]. In Q-S synchronization,

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different dimensional chaotic systems can be synchronized in arbitrary dimensions due to functional relationships between the states of the master and the slave chaotic systems. Recently, Q-S synchronization has received a great deal of attention and a series of works on Q-S synchronization have been published for chaotic dynamical systems in continuous-time [23–26], and discrete-time [27–29].

The main aim of the present work, is to propose a new general control scheme to study the problem of Q-S synchronization for coupled continuous-time chaotic systems. Based on nonlinear control method, we would like to present a constructive scheme to investigate Q-S synchronization between two different dimensional chaotic systems in arbitrary dimension. The new derived synchronization result is proved using Lyapunov stability theory and numerical examples are used to show the effectiveness of the proposed control method.

The rest of this paper is arranged as follows. In Section 2, the problem of Q-S synchronization in arbitrary dimension is formulated. In Section 3, we present our approach of Q-S synchronization. In Section 4, numerical examples and simulations are used to show the effectiveness of the proposed method. Finally, conclusion is given in Section 5.

2 Problem formulation

Consider the following master chaotic system

$$\dot{X}(t) = F(X(t)), \tag{1}$$

where $X(t) = (x_i(t))_{1\leq i\leq n}$ is the state vector of the master system (1) and $F = (F_i)_{1\leq i\leq n}$ is a differentiable vector function. As a slave system, we consider the following chaotic system

$$\dot{Y}(t) = G(Y(t)) + U, \tag{2}$$

where $Y(t) = (y_i(t))_{1\leq i\leq m}$, is the state vector of the slave system (2), $G = (G_i)_{1\leq i\leq m}$ is a differentiable vector function and $U = (u_i)_{1\leq i\leq m}$ is a vector controller to be determined. The definition of Q-S synchronization for the master system (1) and the slave system (2) is given below.

Definition 2.1 The master system (1) and the slave system (2) are said to be Q-S synchronized, in dimension d, if there exists a controller $U = (u_i)_{1\leq i\leq m}$ and two continuously differentiable vector functions $Q(Y(t)) = (Q_i(Y(t)))_{1\leq i\leq d}$, $S(X(t)) = (S_i(X(t)))_{1\leq i\leq d}$, respectively, such that the synchronization error

$$e(t) = (e_1(t), e_2(t), ..., e_d(t))^T = Q(Y(t)) - S(X(t)), \tag{3}$$

satisfies the condition $\lim_{t\longrightarrow+\infty} \|e(t)\| = 0$.

In order to study Q-S synchronization between the master and the slave systems given in equations (1) and (2), we discuss the asymptotic stability of zero solution of synchronization error system $e(t) = Q(Y(t)) - S(X(t))$ i.e., we find the controllers u_i, $i = 1, 2, \cdots, m$, such that the solutions of the error system $e_i(t) = Q_i(Y(t)) - S_i(X(t))$ go to 0, $i = 1, 2, \cdots, d$, as t goes to $+\infty$.

3 A New Q-S Synchronization Approach

The error system (3), between the master system (1) and the slave system (2), can be derived as

$$\dot{e}\,(t) = DQ\,(Y\,(t)) \times (G(Y\,(t)) + U) - DS\,(X\,(t)) \times F(X\,(t)), \tag{4}$$

where $DQ\,(Y\,(t)) \in \mathbb{R}^{d\times m}$, $DS\,(X\,(t)) \in \mathbb{R}^{d\times n}$ are the Jacobian matrices of the functions Q and S, respectively,

$$DQ\,(Y\,(t)) = \begin{pmatrix} \frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \cdots & \frac{\partial Q_1}{\partial y_m} \\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \cdots & \frac{\partial Q_2}{\partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_d}{\partial y_1} & \frac{\partial Q_d}{\partial y_2} & \cdots & \frac{\partial Q_d}{\partial y_m} \end{pmatrix}, \tag{5}$$

$$DS\,(X\,(t)) = \begin{pmatrix} \frac{\partial S_1}{\partial x_1} & \frac{\partial S_1}{\partial x_2} & \cdots & \frac{\partial S_1}{\partial x_n} \\ \frac{\partial S_2}{\partial x_1} & \frac{\partial S_2}{\partial x_2} & \cdots & \frac{\partial S_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial S_d}{\partial x_1} & \frac{\partial S_d}{\partial x_2} & \cdots & \frac{\partial S_d}{\partial x_n} \end{pmatrix}, \tag{6}$$

and we assume that $d \leq m$. The error system (4) can be described as follows

$$\begin{aligned} \dot{e}_i\,(t) &= \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial y_j} \times (G_j\,(Y\,(t)) + u_j) \right) - \sum_{j=1}^{n} \left(\frac{\partial S_i}{\partial x_j} \times F_j\,(X\,(t)) \right) \\ &= -k_i e_i\,(t) + R_i + \sum_{j=1}^{d} \left(\frac{\partial Q_i}{\partial y_j} \times u_j \right) + \sum_{j=d+1}^{m} \left(\frac{\partial Q_i}{\partial y_j} \times u_j \right),\ 1 \leq i \leq d, \end{aligned} \tag{7}$$

where

$$R_i = k_i\,(Q_i\,(Y\,(t)) - S_i\,(X\,(t))) + \sum_{j=1}^{m} \left(\frac{\partial Q_i}{\partial y_j} \times G_j\,(Y\,(t)) \right) - \sum_{j=1}^{n} \left(\frac{\partial S_i}{\partial x_j} \times F_j\,(X\,(t)) \right), \tag{8}$$

and $k_i \in \mathbb{R}_*^+$, $(1 \leq i \leq d)$ are control constants. To achieve Q-S synchronization between the systems (1) and (2), the vector controller $U = (u_i)_{1\leq i\leq m}$ is chosen as follows

$$U = (u_1, ..., u_d, 0, ..., 0)^T\,, \tag{9}$$

and by using equation (9) into equation (7), the error system (7) can be written as follow:

$$\dot{e}_i\,(t) = -k_i e_i\,(t) + R_i + \sum_{j=1}^{d} \left(\frac{\partial Q_i}{\partial y_j} \times u_j \right), \quad 1 \leq i \leq d, \tag{10}$$

rewriting the error system (10) in the compact form

$$\dot{e}\,(t) = -Ke\,(t) + R + JV, \tag{11}$$

where $e\left(t\right)=\left(e_{i}\left(t\right)\right)_{1\leq i\leq d}$, $K=diag\left(k_{1},...,k_{d}\right)$, $R=\left(R_{i}\right)_{1\leq i\leq d}$,

$$J=\left(\begin{array}{cccc}\frac{\partial Q_1}{\partial y_1} & \frac{\partial Q_1}{\partial y_2} & \cdots & \frac{\partial Q_1}{\partial y_d}\\ \frac{\partial Q_2}{\partial y_1} & \frac{\partial Q_2}{\partial y_2} & \cdots & \frac{\partial Q_2}{\partial y_d}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial Q_n}{\partial y_1} & \frac{\partial Q_n}{\partial y_2} & \cdots & \frac{\partial Q_d}{\partial y_d}\end{array}\right), \tag{12}$$

and $V=\left(u_{1},...,u_{d}\right)^{T}$. Now, we can choose V as follows

$$V=-J^{-1}R, \tag{13}$$

where J^{-1} is the inverse of (12). Substitute equation (13) into equation (11), then the error system can be written as

$$\dot{e}\left(t\right)=-Ke\left(t\right). \tag{14}$$

To study the asymptotic stability of zero solution of the error system (14), we consider the candidate Lyapunov function:

$$V\left(e(t)\right)=\frac{1}{2}e^{T}\left(t\right)e\left(t\right), \tag{15}$$

then the derivative of the function (15) along the solution of the system (14) is given as follows

$$\begin{aligned}\dot{V}\left(e\left(t\right)\right) &= \dot{e}^{T}\left(t\right)e\left(t\right)+e^{T}\left(t\right)\dot{e}\left(t\right)\\ &= -\frac{1}{2}Ke^{T}\left(t\right)e\left(t\right)-\frac{1}{2}Ke^{T}\left(t\right)e\left(t\right)\\ &= -Ke^{T}\left(t\right)e\left(t\right)\\ &= \sum_{i=1}^{d}-k_{i}e_{i}^{2}\left(t\right)<0,\end{aligned}$$

and by Lyapunov stability theory, it is immediate that

$$\lim_{t\to\infty}e_{i}\left(t\right)=0,\quad 1\leq i\leq d, \tag{16}$$

and from the fact

$$\lim_{t\to\infty}\left\Vert e\left(t\right)\right\Vert =0. \tag{17}$$

Hence, we have proved the following result.

Theorem 3.1 *The master system (1) and the slave system (2) are globally* $Q-S$ *synchronized under the control law (9)-(13).*

4 Numerical Examples

In order to show the effectiveness of the presented approach of synchronization, two numerical examples are used to observe Q-S synchronization in 3D between two identical dimensional (3D) chaotic systems and two different dimensional (3D and 4D) chaotic systems , respectively.

4.1 Example 1: Q-S synchronization between Rössler and Sprott-WINDMI systems

In this example, we consider the Rössler system [31] as a master system and the controlled Sprott-WINDMI system [30] as a slave system. The Rössler system and the controlled Sprott-WINDMI system can be described, respectively, as follows

$$\begin{aligned} \dot{x}_1 &= -(x_2+x_3), \\ \dot{x}_2 &= x_1+0.2x_2, \\ \dot{x}_3 &= -5.7x_3+x_1x_3+0.2, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \dot{y}_1 &= y_2+u_1, \\ \dot{y}_2 &= y_3+u_2, \\ \dot{y}_3 &= -y_2-0.7y_3+2.5-e^{y_1}+u_3, \end{aligned} \tag{19}$$

where u_1, u_2 and u_3 are synchronization controllers. In this case, we select the vector functions Q and S respectively as

$$Q(y_1,y_2,y_3) = \left(y_1, \frac{1}{3}y_2^3+y_2, y_3\right)^T, \tag{20}$$

$$S(x_1,x_2,x_3) = (x_1, x_2x_3, x_1+x_3)^T, \tag{21}$$

so

$$DQ(y_1,y_2,y_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_2^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

$$DS(x_1,x_2,x_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3 & x_2 \\ 1 & 0 & 1 \end{pmatrix}. \tag{23}$$

According to our approach presented in Section 3, and by using (20), (21), (22) and (23), the controllers u_1, u_2,and u_3 can be constructed as follows

$$\begin{aligned} u_1 &= -(k_1+1)y_1+k_1x_1-x_2-x_3, \\ u_2 &= \frac{-1}{y_2^2+1}\left[k_2\left(\frac{1}{3}y_2^3+y_2\right)+\left(y_2^2+1\right)y_3-0.2x_2+(5.5-k_2)x_2x_3-x_1x_3-x_1x_2x_3\right], \\ u_3 &= y_2+(0.7-k_3)y_3+e^{y_1}+k_3x_1-x_2+(5.7-k_3)x_3+x_1x_3-2.3, \end{aligned} \tag{24}$$

where the control constants $(k_i)_{1\leq i\leq 3}$ are chosen as $(k_1,k_2,k_3)=(1,2,3)$. The error functions can be written as follows $\dot{e}_1(t) = -e_1(t)$, $\dot{e}_2(t) = -2e_2(t)$ and $\dot{e}_3(t) = -3e_3(t)$. Then, the numerical simulation of the error functions evolution is shown in Figure 2.

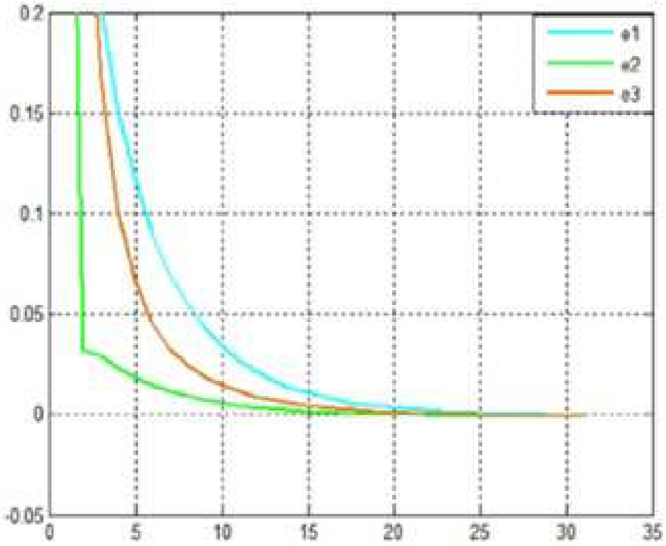

Figure 1: Time evolution of Q-S synchronization errors between the master system (18) and the slave system (19).

4.2 Example 2: Q-S synchronization between Lorenz and hyperchaotic Chen systems

In this example, we consider the Lorenz system [32] as a master system and the controlled hyperchaotic Chen system [33] as a slave system. The Lorenz system and the controlled hyperchaotic Chen system can be described, respectively, as follows

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1), \\ \dot{x}_2 &= 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= -8/3x_3 + x_1x_2, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \dot{y}_1 &= -27.5y_1 + 27.5y_2 + u_1, \\ \dot{y}_2 &= 3y_1 + 19.3y_2 + y_4 - y_1y_3 + u_2, \\ \dot{y}_3 &= -2.9y_4 + u_3, \\ \dot{y}_4 &= -3.3y_1 + y_2^2 + u_4, \end{aligned} \tag{26}$$

where u_1, u_2, u_3 and u_4 are synchronization controllers. In this case, the vector functions Q and S are chosen, respectively, as follows

$$Q\left(y_1, y_2, y_3, y_4\right) = \left(y_1 + y_4, y_2 + y_4, y_4 + y_3\right)^T, \tag{27}$$

$$S\left(x_1, x_2, x_3\right) = \left(x_1x_3, x_2, x_3\right)^T, \tag{28}$$

so

$$DQ\left(y_1,y_2,y_{3,}y_4\right)=\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{pmatrix},\tag{29}$$

$$DS\left(x_1,x_2,x_3\right)=\begin{pmatrix}x_3&0&x_1\\0&1&0\\0&0&1\end{pmatrix}.\tag{30}$$

According to the control law (9)-(13) proposed in Section 3, and by using (27), (28), (29) and (30), the controllers u_1, u_2, u_3 and u_4 can be designed as follows

$$\begin{aligned}u_1 &= (k_1-30.8)y_1+27.5y_2+k_1y_4+y_2^2+(\frac{38}{3}-k_1)x_1x_3 \\ &\quad -10x_3x_2-x_1^2x_2, \\ u_2 &= -0.3y_1+(k_2+19.3)y_2+(k_2+1)y_4-y_1y_3+y_2^2-28x_1 \\ &\quad +(1-k_2)x_2+x_1x_3, \\ u_3 &= -3.3y_1+k_3y_3+(k_3-2.9)\,y_4+y_2^2+(8/3-k_3)-x_1x_2, \\ u_4 &= 0,\end{aligned}\tag{31}$$

where the control constants $(k_i)_{1\le i\le 4}$ are chosen as $(k_1,k_2,k_3,k_4)=(0.1,0.2,0.3,0.4)$. The error functions can be written as follows: $\dot{e}_1\left(t\right)=-0.1e_1\left(t\right)$, $\dot{e}_2\left(t\right)=-0.2e_2\left(t\right)$, $\dot{e}_3\left(t\right)=-0.8e_3\left(t\right)$ and $\dot{e}_3\left(t\right)=-0.4e_3\left(t\right)$. Then, the numerical simulation of the error functions evolution is shown in Figure 2.

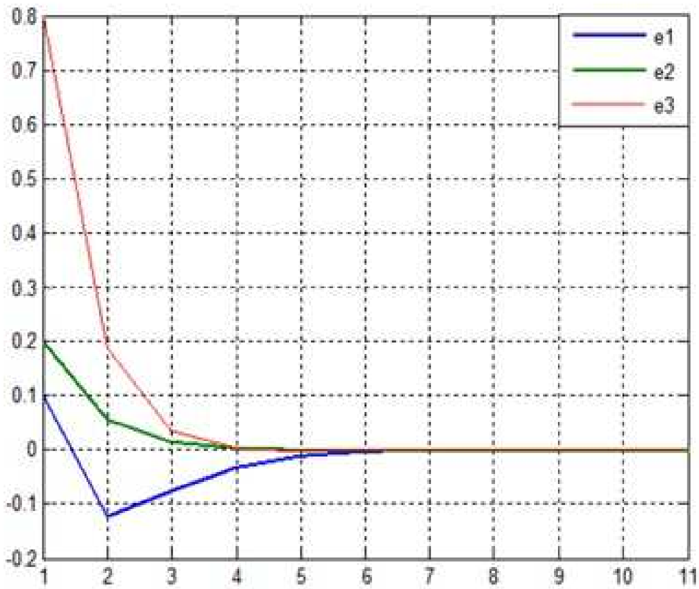

Figure 2: Time evolution of Q-S synchronization errors between the master system (25) and the slave system (26).

5 Conclusion

In this paper, we have developed a new systematic and powerful synchronization scheme, which is used to study Q-S synchronization between two n-D and m-D continuous-time chaotic dynamical systems. Numerical examples are used to verify the effectiveness of the proposed approach.

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Extending the Property of a System to Admit a Family of Oscillations to Coupled Systems

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Abstract: Coupled systems, each one admitting a family of nondegenerate periodic solutions, are considered. The period of oscillations in the family is supposed to depend on a unique parameter. Conditions imposed on weak couplings such that the coupled system admits a family of periodic solutions, which is similar to that of subsystems, are found. Differential equations of general form, as well as reversible mechanical systems are investigated. The existence of resonant orbits in the N-planet problem with one planet in a quasi-circular orbit is proved.

Keywords: *coupled system; differential equation; periodic solution; family; nondegenerate; reversible mechanical system; N-planet; resonant orbits.*

Mathematics Subject Classification (2010): 34A34, 37C27, 37C80, 70F10.

1 Introduction

Investigation of a dynamic model usually implies the consideration of substantial factors. The influence of other (minor) factors is regarded in the frame of the perturbation theory. This influence can either slightly change quantitatively dynamical characteristics of the system, or bring about a new quality. The latter case is usually associated with a bifurcation.

System perturbations result from weak influence of other systems. Taking this into account we consider a new model, which is closed one. The non-regarded influence is modelled by the couplings between the systems to constitute coupled systems. Since the intensity of non-regarded factors is weak, the couplings are expected to be small.

In [1] the closed model containing coupled subsystems (MCCS) is introduced. This model posesses dynamical properties (i.e. run-outs, enregy transfer, etc.) that cannot

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be explained in the frame of perturbation theory. Investigation of this model assumes the two main problem: 1) to find conditions on couplings such that the MCCS inherits dynamical properties of its subsystems; 2) to find qualitatively new effects resulting from couplings between subsystems.

In this paper the first problem is considered for coupled systems. Each system is supposed to admit a family of nondegenerate periodic solutions, where the period depends on a unique parameter. The goal is to find conditons on weak couplings such that the coupled system admits a family of periodic solutions, which is similar to that of subsystems. It is shown that these conditions are always satisfied in the case of symmetric periodic motions of reversible mechanical system. The N-planet problem with one planet in a quasi-circular orbit is considered. The existence of resonant orbits in this problem is proved.

The concept of MCCS arose from the classical perturbation theory. The MCCS describes the dynamics in various problems of classical and celestial mechanics, radioengineering, population dynamics, mechatronics and robotics, biology, medicine, etc. [1, 2]. MCCS can consist of subsystems of diverse nature, the subsystems being described by various type equations. Coupled oscillators (see, for example, [3, 4]) became the classical model that illustrates the complexity of behaviour in coupled systems.

In [1] the formal description of the MCCS is given. Since 2003 systematic investigations concerning the above problems for the MCCS have been carrying out, more than a dozen of papers have been published.

The autonomous models containing families of periodic solutions in subsystems were considered in [2]. In particular, for an MCCS consisting of m subsystems the bifurcation scenario is given. This scenario assumes the bifurcation of the $2m$-family of periodic solutions such that the m-family of periodic solution arises in the MCCS.

Later MCCS with identical subsystems were considered in [5], where the existence of a family of periodic motions such that the period depends on a unique parameter is proven. This paper pushes further the investigations of [2, 5] to extend the results of [5] to MCCS containing different subsystems. It is shown that coupled reversible mechanical systems inherit completely the dynamic property of the subsystems. Thus the problem of extending the dynamic property of subsystems to the MCCS is completely solved.

MCCS belongs to the class of complex systems. Among the characteristic features of the model there are hierarchical and multi-level structure, multi-mode operation, nonlinearity, high order. MCCS is qualified also as a large-scale system. MCCS represents a network. It can be either autonomous or non-autonomous.

Weakly coupled MCCS is a system with a small parameter. Investigations of such systems can apply Yu. A. Mitropolsky's results (cf. [6]), in particular, the single-frequency approach to study nonlinear oscillations in multi-degree of freedom systems [7].

The present paper is dedicated to the 100-year anniversary of Yu.A. Mitropolsky.

2 The Nondegenerate for a Periodic Solution Case

Consider the smooth equation

$$\dot{x} = X(x), \quad x \in R. \tag{1}$$

Denote by $x(x_1^0, \ldots, x_n^0, t)$ the general solution of (1). The necessary and sufficient conditions of the existence of a T-periodic solution are given by

$$f \equiv x(x_1^0, \ldots, x_n^0, T) - x^0 = 0, \tag{2}$$

where $x^0 = (x_1^0, \dots, x_n^0)$ is the initial point at $t = 0$.

Let equation (2) have a solution $x^0 = x^*$, $T = 2\pi$. Calculate the rank Ra of the functional matrix for the function f at the point (x^*, T). Since (1) is autonomous, equation (2) possesses a monoparametric (denote the parameter by γ) family of solutions

$$x^0 = x^*(\gamma), \quad T = 2\pi. \tag{3}$$

Thus we obtain $Ra \le n - 1$.

Definition 2.1 The case of $Ra = n-1$ is referred to as nondegenerate for a periodic solution. The very solution is referred to as nondegenerate.

We use later on the following notion.

Definition 2.2 The isolated periodic solution of an autonomous differential equation is called the cycle.

The following alternative holds [2].

Theorem 2.1 *In the nondegenerate for a periodic solution case the following alternative takes place: the solution is either a cycle or belongs to a family of periodic solutions with the period depending on a unique parameter. If this alternative realizes for equation* (1) *then the nondegenerate for a periodic solution case takes place.*

In this paper the case of family in the alternative is analyzed. According to the law [8, 9] the period on the family depends on a unique parameter and $T = T(h)$. For ordinary points of the family $dT \neq 0$ and for critical points $dT = 0$ [10]. The nondegenerate for a periodic solution case excludes the critical point from consideration. In the case of family the periodic solution is associated with a double zero characteristic exponent (CE) in the Jordan cell [11]. Since $Ra = n - 1$, the remaining CE are nonzero.

Note that there always exists (cf. [11]) a particular solution of the form

$$x_s(t) = e^{\lambda_k t}\varphi_s(t), \quad \varphi_s(t+T) = \varphi_s(t), \quad s = 1, \dots, n,$$

for a T-periodic linear system of the n-th degree. Here λ_k is CE; the total number of CE (regarding their multiplicity) being equal to n.

3 Extending the Dynamic Property

Consider m smooth coupled systems

$$\dot{x}^s = X(x^s) + \varepsilon\tilde{X}^s(\varepsilon, x^1, \dots, x^m), \quad s = 1, \dots, m, \quad x^s \in R^{m_s}. \tag{4}$$

Here ε is a nonnegative numerical parameter such that (4) breaks up into m independent systems at $\varepsilon = 0$. Suppose that the s-th system admits a family of periodic solutions

$$x^s = \varphi^s(h_s, t + \gamma_s), \quad s = 1, \dots, m, \tag{5}$$

which contains two parameters h_s and γ_s. Here the period $T_s = T_s(h_s)$ of (5) depends on h_s and γ_s and represents the shift of the initial point along the trajectory. The s-th system at a fixed $h_s = h_s^*$ admits a periodic solution that depends on γ_s and given $h_s = h_s^*$, $s = 1, \dots, m$, the generating system (i.e. (4) at $\varepsilon = 0$) has an m-family of conditionally periodic solutions with m frequencies. If $T_s(h_s^*) = T^*,\ s = 1,\dots,m$, this family is the family of T^*-periodic solutions with the parameter $\gamma = (\gamma_1,\dots,\gamma_m)$. In view of this the existence conditions of periodic motions for coupled systems are formulated in terms of $\gamma = (\gamma_1,\dots,\gamma_m)$ that corresponds to a chosen $h^* = (h_1^*, ..., h_m^*)$, rather than in terms of h_s, as it is the case in [2] where arbitrary systems are considered.

Let

$$x(\varepsilon, x^0, t) = (x^1(\varepsilon, x^0, t), \dots, x^m(\varepsilon, x^0, t)) \tag{6}$$

be the solution of the Cauchy problem of (4) with the initial point x^0 at $t = 0$. Take the derivative of (6) with respect to ε at $\varepsilon = 0$ when (6) coinsides with the solution given by (5) at $h_s = h_s^*,\ s = 1,\dots,m$. This derivative satisfies the following linear nonhomogenious system with periodic coefficients

$$\frac{d}{dt}\left(\frac{\partial x^s}{\partial \varepsilon}\right) = P^s(h_s^*, t+\gamma_s)\left(\frac{\partial x^s}{\partial \varepsilon}\right) + \tilde{X}^s(0, \varphi^1(h_1^*, t+\gamma_1), \dots, \varphi^m(h_m^*, t+\gamma_m)), \qquad s = 1,\dots,m, \tag{7}$$

where

$$P^s(h_s^*, t+\gamma_s) = \|p_{kj}^s(h_s^*, t+\gamma_s)\|_{k,j=1}^{m_s}, \quad p_{kj}^s(h_s^*, t+\gamma_s) = \left(\frac{\partial X_k^s}{\partial x_j^s}\right)_{x^s=\varphi^s(h_s^*, t+\gamma_s)}, \qquad s = 1,\dots,m.$$

The homogenious part of (7) splits up into m independent systems of m_s-th degree, each one having a unique T^*-periodic solution. The appropriate conjugate system splits up into m subsystems as well. Denote the T^*-periodic solutions of those subsystems by $\psi^s(h_s^*, t+\gamma_s)$ $s = 1,\dots,m$. Consequently, the necessary condition of existence of a T^*-periodic solution for (4) can be written as

$$g_{h^*}(\gamma_1,\dots,\gamma_m) = 0, \tag{8}$$

where the components of g are defined by

$$g_{h^*}^s(\gamma_1,\dots,\gamma_m) = \int_0^{T^*} \sum_{k=1}^{n} \tilde{X}_k^s(0, \varphi^1(h_1^*, t+\gamma_1), \dots, \varphi^m(h_m^*, t+\gamma_m))\psi_k^s(h_s^*, t+\gamma_m)dt, \qquad s = 1,\dots,m.$$

Equation (8) determines the class of couplings that admit the existence of periodic solutions in coupled systems. It will be shown later that (8) turns out to be sufficient under some conditions.

Note that the equation $g = 0$ was used earlier [2] to find h_s^* of the generating family with the parameter γ.

Let us formulate the theorem that establishes sufficient conditions of the existence of periodic solutions in coupled systems.

Theorem 3.1 *Let equation* (8) *have a solution denoted by* γ^*, *i.e.* $g_{h^*}(\gamma^*) = 0$. *Let the rank of the functional matrix of the mapping* $\gamma \to g_{h^*}(\gamma)$ *at* γ^* *be equal to* $m-1$. *Then* (4) *has a periodic solution.*

Proof. The solution $x(\varepsilon, x^0, t+\gamma)$, being T^*-periodic, satisfies equation

$$F \equiv x(\varepsilon, x^0, T^*) - x^0 = 0, \quad F = (F^1, \dots, F^m). \tag{9}$$

Since (4) is autonomous, the solution x^0 of (9) depends on a unique parameter δ and $\gamma = \gamma(\delta)$.

Hence, the problem is to find the root x^0 of (9), which depends on ε such that it satisfies the system

$$F_0^s \equiv x^s(0, x^{s0}, T^*) - x^{s0} = 0, \quad s = 1, \dots, m, \quad x^0 = (x^{10}, \dots, x^{m0}) \tag{10}$$

at $\varepsilon = 0$. System (10) splits up into m subsistems. The s-th subsystem has a family of solutions $x^{s0} = x^{s0}(\gamma_s^*(\delta))$ with the parameter $\gamma_s^*(\delta)$.

At a given ε (9) represents a system of m-th degree in m variables, while its solution $x^0 = x^*(\varepsilon, \gamma(\delta))$ depends on ε.

Rearrange (9) as

$$F_0^s(x^{s0}, T^*) + \varepsilon G^s(\varepsilon, x^0, T^*) = 0, \quad s = 1, \dots, m, \tag{11}$$

where $F_0^s(x^{s*}(0, \gamma^*), T^*) = 0$. For (11) take the increments

$$y_k^s = x_k^{s0} - x_k^{s*}(0, \gamma_s^*), \quad k = 1, \dots, m_s, \quad s = 1, \dots, m. \tag{12}$$

By assumption, the rank Ra^s of the functional matrix of F_0^s at $x^{s0} = x^{s*}(0, \gamma_s^*)$ is equal to $m_s - 1$. Consequently, $m_s - 1$ increments y_k^s of the s-th subsystem can be expressed as functions of the remaining increment, which we denote, to be specific, by y_s^s. Substitute the above functions into the equation for y_s^s, then the s-th subsystem yields a unique equation in y_s^s instead of m_s equations. By repeating this procedure for all m subsystems we obtain a system of m equations

$$\Phi^s(z, T^*) + \varepsilon \Psi^s(\varepsilon, z, T^*) = 0, \quad s = 1, \dots, m, \quad z = (y_1^1, \dots, y_m^m).$$

In this system functions Φ^s do not contain linear terms. According to (12) components of z are of the ε-th order. Equations (12) mean the transition along the trajectory of the generating solution from the point $x^*(0, \gamma^*)$ to the initial point $x^*(0, \gamma)$ for a periodic solution of the system at $\varepsilon \neq 0$. This implies that $\gamma - \gamma^* \sim \varepsilon$, so we obtain the system

$$\Psi^s(0, y_1^1(\gamma), \dots, y_m^m(\gamma), T^*) = 0, \quad s = 1, \dots, m, \tag{13}$$

which coinsides with system of amplitude equations (8).

Let the rank of (13) at the point $\gamma = \gamma^*$ be equal to $m-1$. Then system (9) has a solution, which depends on the parameter δ. This means that there exists a T^*-periodic solution in (4).

Theorem 3.1 can be valid for an isolated point h^*. Such a point can be found by deriving an appropriate amplitude equation and by finding its simple roots [2].

Suppose that $h_s = h_s(\chi)$, $s = 1, \dots, m$ in (5). Then the generating system has an $(m+1)$-family of periodic solutions, and the vector $h^*(\chi)$ can be regarded as a parameter in Theorem 3.1.

So the following theorem holds.

Theorem 3.2 *Let equation* (8) *have a solution* $g_{h^*}(\gamma^*) = 0$. *Let the rank of the functional matrix of the mapping* $\gamma \to g_{h^*}(\gamma)$ *at* γ^* *be equal to* $m-1$. *Then* (4) *has a 2-family of periodic solutions with the period depending on a unique parameter.*

Theorem 3.2 solves the problem of extending the property of having a family of periodic solutions with the period depending on a unique parameter to coupled systems. This result was announced in [12].

The proof of Theorem 3.2 repeats that of Theorem 3.1 with obvious modifications.

4 Coupled Reversible Mechanical Systems

4.1 Symmetric periodic motions

At first consider a separate system

$$\dot{u} = U(u, v), \quad \dot{v} = V(u, v), \tag{14}$$

$$U(u, -v) = -U(u, v), \quad V(u, -v) = V(u, v); u \in R^l, v \in R^n, l \geq n. \tag{15}$$

A series of models in classical and celestial mechanics are described by these equations [13]. Usually u is the vector of generalized coordinates (quasicoordinates) and v is the vector of generalized velocities (quasivelocities). System (14), (15) is the particular case of the reversible dynamical system [14]. It is called the reversible mechanical system.

In what follows the set $M = \{u, v : v = 0\}$, which is called the fixed set, is used.

System (14), (15) always possesses a pair of symmetric with respect to M motions (see Figure 1, a). The solution of (14),(15) that crosses M is called the symmetric motion (Figure 1, b). A symmetric motion can be periodic (symmetric periodic motion, SPM). An SPM crosses M at least twice (Figure 1, c).

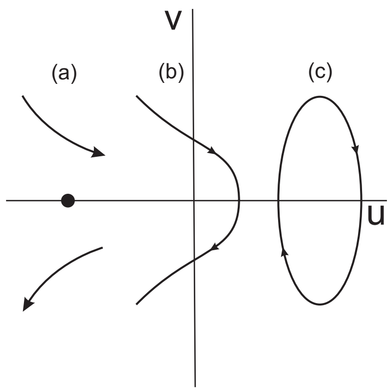

Figure 1: Motion types in a reversible mechanical system. a: a pair of symmetric with respect to M motions; b: symmetric motion; c: symmetric periodic motion.

If system (14), (15) is 2π-periodic in some or all components of v, the appropriate SPM can be either oscillation or rotation. These components take multiple of π values on the fixed set [15].

Denote the symmetric motion by $v(u_1^0, \ldots, u_l^0, t)$, where u^0 is the initial point in M. Then the necessary and sufficient conditions of existence of a T-periodic SPM are given by [15]

$$v_s(u_1^0, \ldots, u_l^0, T/2) = 0, \quad s = 1, \ldots, n. \tag{16}$$

Let system (16) admit a solution

$$u_1^0 = u_1^*, \ldots, u_l^0 = u_l^*, \quad T = T^* = 2\pi. \tag{17}$$

Set up the matrix

$$A = \left\| \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial u_j^0} \right\|,$$

where the partial derivatives are taken at the values (17).

Definition 4.1 The case of $\operatorname{rank} A = n$ is called nondegenerate for a symmetric periodic motion; the very SPM is called nondegenerate.

Note that if $l > n$, a nondegenerate SPM will be degenerate in the sense of Definition 2.1.

A nondegenerate SPM is extended in the phase space over a family of $(l - n + 1)$-th degree. The condition $\operatorname{rank} A = n$ means that the SPM with the initial point (17) is submerged in the family of SPM that depends on arbitrary $l - n$ initial values of the vector u^0 and on the period T (cf. [16]). The law stating that the period depends on a unique parameter is valid over the family of SPM [8, 9]. If system (14) contains a numerical parameter μ and an SPM is nondegenerate at $\mu = 0$ then the property of nondegeneracy is extended for the appropriate SPM over the range $\mu \neq 0$. Given the existence of the SPM family in the system at $\mu = 0$, conditions of the extension of the SPM family over the range $\mu \neq 0$ are found [16]. The appropriate property of the SPM family is called stability with respect to parametric perturbations of the system.

A more general matrix

$$A_1 = \left\| \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial u_j^0} \quad \frac{\partial v_s(u_1^0, ..., u_l^0, T/2)}{\partial t} \right\|$$

can be used instead of A [17]. If $\operatorname{rank} A_1 = n$ (generalized nondegeneracy condition) then the implicit function theorem guarantees the existence of solution of system (16) in the neighborhood of the point (17).

The case of $\operatorname{rank} A = n$ (which implies $\operatorname{rank} A_1 = n$) is described above. If $\operatorname{rank} A = n - 1$ and $\operatorname{rank} A_1 = n$ then system (14) has a $l - n + 1$ family of SPM with the period depending on $l - n + 1$ initial values of u^0.

4.2 SPM families in coupled reversible mechanical systems

Consider the model of coupled reversible mechanical systems [17]. The intensity of coupling is characterized by the small numeric parameter ε such that the model decouples into independent systems of the form (14) at $\varepsilon = 0$. If so, matrices A and A_1 depend on ε. When $\varepsilon = 0$, they are block diagonal with the blocks $A^{(j)}$ $(A_1^{(j)})$) determined by the j-th system. The condition of nondegeneracy of SPM in all systems provides the condition of nondegeneracy of SPM in the coupled model [17]. The nondegeneracy condition for the SPM in all but one systems and the generalized nondegeneracy condition for the SPM in the remaining system yield the generalized nondegeneracy condition for the SPM in the coupled model. So the property of the reversible mechanical system to have SPM can be extended to coupled system in the following way.

Theorem 4.1 *If the nondegeneracy condition for the SPM is satisfied for all but one reversible mechanical systems, while either the nondegeneracy condition for the SPM or the generalized nondegeneracy condition for the SPM is satisfied for the remaining system, then there exists a family of SPM of degree $2 + \sum(l - n)$ in the coupled model of reversible mechanical systems. Here l and n are dimensions of the vectors u and v in the systems, respectively.*

Remark 4.1 If $l = n$ for all systems then the 2-family of SPM of the system is extended to 2-family of SPM of the coupled model.

4.3 Coupled reversible mechanical systems with couplings of general form

Reversible mechanical systems, being coupled, may loose the property of reversibility. This is the case when the couplings are represented by arbitrary functions of u and v such that conditions (15) are not satisfied for coupled systems. As a result, the coupled model is described by differential equations of general form. The following particular case can be distinguished: $l = n$ for all systems, all periodic motions involved are nondegenerate. In this case Theorem 3.2 can be applied to establish the extension of the property from the separate system to coupled systems. Besides, the result holds for both symmetric and non-symmetric periodic motions.

The generalization of Theorem 3.2 turns out to be valid for nondegenerate SPM even if $l \neq n$: the property of having a family of SPM is extended to coupled systems. The accurate statement requires preliminary transformations of coupled systems similar to those represented in [18]. This statement is beyond the scope of the paper.

5 Resonant Orbits in the N-planet Problem

The motion of $N+1$ gravitating bodies with one body (the Sun) being vastly superior in mass to other bodies (the planets) is studied in the frame of the N-planet problem. If the interaction between planets is neglected, the N-planet problem results in N independent two-body problems (the Sun and the planet). The interaction between the planets can be treated as perturbations.

5.1 Parade of planets

In the Solar system the parade of planets phenomenon, where all planets or some of them line up in a straight line, is observed. In the frame of the N-planet problem this phenomenon is associated with the existence of symmetric periodic orbits [19].

The N-planet problem belongs to the class of reversible mechanical systems [17]. Elliptic orbits in the two-body problem are symmetric with respect to the major axis, the radial velocity being zero on the axis. The crossing of the fixed set by the image point means for the N-planet problem that the planets line up in the straight line (parade of planets). Since the parade of planets is periodic, this effect is observed on periodic orbits. Such orbits result in resonances in the planet system.

In the stationary frame of reference the parade of planets is observed on elliptic orbits (orbits of the second type), while in the rotating frame of reference the parade of planets takes place on circular orbits (orbits of the first type).

Orbits of the first type were studied in [19,20], the parade of planets on the orbit of the second type was analyzed in [19]. A simple proof of existence of the orbit of the second type is given in [17]. The parade of planets turns out to occur on the $(N+1)$-parametric family of orbits close to elliptic, the period depending on a unique parameter (namely, on the energy integral). When the generating system comprises a two-body system with a circular orbit and other two-body systems with elliptic orbits, the existence of orbits of the second type has remained an open problem. The interest to this case is due to the fact that the eccentricity of Venus's orbit is 0.007, i.e. the orbit is close to circular. The solution to this problem is given in the paper.

In terms of the classification of oscillation modes in the model containing coupled subsystems [1] the above problem falls under the category of modes with a critical point. According to Theorem 4.1 for reversible mechanical systems the problem of existence of oscillations in this mode has a solution.

Note that the question on the number of periodic orbits of planet systems in the rotating frame of reference was raised in [19, 21, 22].

5.2 Two-body problem. Generalized nondegeneracy condition

Periodic orbits of two-body problems play an important role of the generating orbits in the N-planet problem. Noting that the orbits in the two-body problem are planar, consider only the planar problem. Write the equations in polar coordinates

$$\ddot{\rho} - \rho\dot{\theta}^2 + \frac{k}{\rho^2} = 0, \quad \frac{d}{dt}(\rho^2\dot{\theta}) = 0.$$

Introduce the notation $c = \rho^2\dot{\theta}$. Then

$$\ddot{\rho} - \frac{c^2}{\rho^3} + \frac{k}{\rho^2} = 0, \quad \dot{c} = 0, \quad \dot{\theta} = \frac{c}{\rho^2}. \tag{18}$$

System (18) admits a solution with the fixed $c = c_*$. In this case the first equation represents a conservative one degree of freedom system. This systems admits a family of oscillations with respect to ρ, the period $T(h)$ depending monotonically on the energy integral h. The only exception is the critical point $\rho_* = c_*/k$, which corresponds to the circular orbit; here $dT = 0$ [10]. The last condition implies that $\operatorname{rank} A \leq 1$ for the circular orbits

$$\rho = \rho_*, \quad \dot{\rho} = 0, \quad c = c_*, \quad \theta = (c_*/\rho_*^2)t. \tag{19}$$

Let us prove that $\operatorname{rank} A = 1$.

Derive the equations in variations for the circular orbit:

$$\delta\ddot{\rho} + \frac{k}{\rho_*^3}\delta\rho = 0, \quad \delta\dot{c} = 0, \quad \delta\dot{\theta} = \frac{\delta c}{\rho_*^2} - \frac{2c_*}{\rho_*^3}\delta\rho. \tag{20}$$

It is obvious that the first equation can be integrated independently. This equation has periodic solution $\delta\rho = \cos(k/\rho_*^3)t$, the corresponding Jordan cell in matrix A breaks up so that $\operatorname{rank} A \leq 1$.

The other symmetric solution in system (20) is characterized by the initial point $\delta\rho(0) = 0$, $\delta\dot{\rho}(0) = 0$, $\delta c(0) = 1$, $\delta\theta(0) = 0$, so that $\delta\rho(t) \equiv 0$. At the half-period instant $t = T/2$ we have

$$\delta\theta(T) = (\rho_*^2)^{-1}T/2 \neq 0,$$

consequently, the second Jordan cell remains intact and $\operatorname{rank} A = 1$.

Accoding to (19)

$$\frac{\partial \dot\rho(T/2)}{\partial t} \equiv 0, \quad \frac{\partial \theta(T/2)}{\partial t} = \frac{c}{\rho_*^2} \neq 0$$

on the circular orbits at $t = T/2$, so that the condition $\operatorname{rank} A_1 = 2$ holds. This means that the generalized nondegeneracy condition is satisfied on circular orbits.

5.3 Family of resonant orbits in the N-planet problem

Let us write the equation of motion for the problem in the cylindric coordinates [23, p. 365]

$$\ddot\rho_s - \rho_s \dot\theta_s^2 = \frac{\partial \Omega_s}{\partial \rho_s}, \quad \frac{d}{dt}(\rho_s^2 \dot\theta_s) = \frac{\partial \Omega_s}{\partial \theta_s}, \quad \ddot z_s = \frac{\partial \Omega_s}{\partial z_s}, \quad s = 1, \ldots, N, \tag{21}$$

where

$$\Omega_s = \frac{f(m_0 + m_s)}{\sqrt{\rho_s^2 + z_s^2}} + \Omega_{s1},$$

$$\Omega_{s1} = f \sum_{j=1 (s \neq j)}^{N} m_j \left[\frac{1}{\Delta_{sj}} - \frac{\rho_s \rho_j \cos(\theta_s - \theta_j) + z_s z_j}{(\rho_s^2 + z_s^2)^{3/2}} \right],$$

$$\Delta_{sj}^2 = \rho_s^2 + \rho_j^2 - 2\rho_s \rho_j \cos(\theta_s - \theta_j) + (z_s - z_j)^2,$$

m_0 is the Sun's mass, m_s are the planets' masses, $m_0 \gg m_s$, f is the gravitational constant.

System (21) is invariant with respect to the change of variables

$$\rho \to \rho, \ \theta \to \pm\theta, \ z \to z(-z), \ t \to -t$$

and belongs to the class of reversible mechanical systems. Consider the planar problem ($z \equiv 0$). At $\Omega_{s1} = 0$ ($s = 1, \ldots, N$) the system splits up into N planar two-body problems, which represent the generating system. Suppose that one planet moves along a circular orbit, while other planets move along elliptic orbits. Then the reversible mechanical system for the N-planet problem admits an N-family of SPM. The nondegeneracy condition is satisfied for elliptic orbits of $N-1$ two-body problems, while the generalized nondegeneracy condition is satisfied for the circular orbit of the remaining two-body problem. Perturbations Ω_{s1} depend only on $x_s, \ y_s$, such that (21) remains reversible. Hence, Theorem 4.1 can be applied to establish the extension of the family in the generating system to the N-planet problem.

So we can conclude that in the N-planet problem there exist resonant orbits close to orbits in two-body problems such that the planets line up in a straight line (parade of planets). The parade of planets is observed on the N-family of orbits, where the energy integral h is one of parameters and the period depends only on h.

Let us summarize the above reasoning by

Theorem 5.1 *In the N-planet problem there exists an N-family of planar symmetric resonant periodic orbits close to orbits of the two-body problem, one of the orbits being circular, and the others being elliptic. The planets in such orbits line up (periodically in time) in a straight line (parade of planets).*

6 Conclusion

A great interest to coupled and network systems is being observed at present time. There is a vast variety of the considered models and problem statements. One of the problems to solve is to extend dynamic properties of a separate system to coupled systems. This problem is solved in this paper for the smooth autonomous model containing coupled subsystems. The separate subsystem is supposed to admit a family of periodic solutions with the period depending on a unique parameter. For coupled systems described by ordinary differential equations the problem of extending dynamic properties is solved by finding appropriate couplings. In the case of reversible mechanical systems the dynamic property is completely extended to coupled systems. The obtained results are applied to the N-planet problem with one planet in a circular orbit and the other planets in elliptic orbits. The existence of N-resonant orbits, on which the parade of planets is observed, is established.

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