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Minima of Some Integral Functional: Existence and Regularity

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Abstract: We prove the existence and the regularity of minima for functional whose prototype is:

$$J(u) = \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega F.\nabla u\, dx, \qquad u \in\ W_0^{1,p}(\Omega),$$

where Ω is a bounded domain of $I\!R^N$, $p > 1$ and $\alpha > 0$. The function F belongs to some Lebesgue space.

Keywords: *non-linear elliptic equations; degenerate coercive truncations; calculus of variations.*

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1 Introduction and Statement of Results

In this paper, we deal with the study of minima for functional whose prototype is:

$$J(u) = \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega F.\nabla u\, dx, \qquad u \in\ W_0^{1,p}(\Omega), \tag{1.1}$$

where Ω is a bounded open subset of $I\!R^N$, $N \geq 2, \alpha > 0$, and and $1 < p < N$. The datum F belongs to the space $(L^r(\Omega))^N$ for some $r \geq 1$.

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The search of sufficient condition to secure that the functional $J(u) = \int_\Omega a(x,u,\nabla u)\,dx$ attained an extreme value has a long history (see B. Dacorogna [8]). R. Tahraoui, A. Cellina and S. Perrotta in [6,12] prove that the functional J admits a unique minimum, without any assumptions on a, except for the lower semi-continuity and the growth condition. Landes in [10] has shown that if J is weakly lower semi-continuous at one fixed level set, then this level set is an extreme value of J or the defining a is convex in the gradient.

The functional J (see (1.1)) is defined on $W_0^{1,p}(\Omega)$, when $r \geq p'$, but it may not be coercive on the same space as u becomes large (see Example 3.3 of [3]). Thus even if J is lower semi-continuous on $W_0^{1,p}(\Omega)$ as a consequence of the De Giorgi theorem, the lack of coerciveness implies that J may not attain its minimum on $W_0^{1,p}(\Omega)$ even in the case in which J is bounded from below (see Example 3.2 of [3]). To overcome this difficulty we will reason (as in [3]) by extending the functional J to $W_0^{1,q}(\Omega)$ for some $q < p$ depending on α. Thus functional attains its minimum on this larger space when $r \geq q'$. In the same way we cite the recent works of Boccardo and Orsina [1,2].

In this paper, we will prove several results of existence and regularity of minima (depending on the summability of the datum F) for functional J.

Let us give the precise assumptions on the problem that we will study. Let Ω be a bounded open subset of $I\!R^N, N \geq 2$. Let $1 < p < N$, and let $a : \Omega \times I\!R^N \to I\!R$ be a Caratheodory function (that is, $a(.,t)$ is measurable on Ω for every $t \in I\!R$, and $a(x,.)$ is continuous on $I\!R$ for almost every x in Ω), such that the following assumption

$$\frac{\beta_0}{(1+|t|)^{\alpha p}} \leq a(x,t) \leq \beta_1 \tag{1.2}$$

for almost every x in Ω, for every t in $I\!R$ where α, β_0 and β_1 are positive constants. We furthermore suppose that:

$$0 < \alpha < \frac{1}{p'}. \tag{1.3}$$

The function F is such that:

$$|F| \in \quad L^r(\Omega) \quad \text{for some} \quad r \geq p'. \tag{1.4}$$

Example of the function a that satisfies (1.2) is:

$$a(x,t) = \frac{\beta_0}{(b(x)+|t|)^{\alpha p}},$$

where b is a measurable function on Ω such that:

$$0 < \beta_2 \leq b(x) \leq \beta_3 \quad \text{for almost everywhere in} \quad \Omega, \tag{1.5}$$

where β_2 and β_3 are two positive constants.

Similar problems have been considered in [3], more precisely the authors have studied the existence and the regularity of minima for functional:

$$I(u) = \int_\Omega a(x,u)|\nabla u|^p\,dx - \int_\Omega f.u\,dx, \qquad u \in \;\; \alpha > 0, \tag{1.6}$$

where f belongs to $L^r(\Omega)$ for some $r \geq p'$. The following regularity was proved in [3] in light of various summability of the source term

$$\begin{array}{ll} r > \dfrac{r}{p} & \Rightarrow \ u \in \ W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \\ \left(\dfrac{p^*}{1+\alpha p}\right)' \leq r < \dfrac{r}{p} & \Rightarrow \ u \in \ W_0^{1,p}(\Omega) \cap L^s(\Omega), \\ (p^*(1-\alpha))' \leq r < \left(\dfrac{p^*}{1+\alpha p}\right)' & \Rightarrow \ u \in \ W_0^{1,\rho}(\Omega) \cap L^s(\Omega), \end{array}$$

where

$$s = \frac{Nr(p(1-\alpha)-1)}{N-rp}, \qquad \rho = \frac{Nr(p(1-\alpha)-1)}{N-r(1+\alpha p)}.$$

Following this way, in this paper, we are interested in the existence and the regularity of minima for functional $J(v)$.

Notations :

In the sequel we will use the following functions of a real variable depending on a parameter $k > 0$:

$$T_k(s) = \max(-k, \min(k, s)), \qquad G_k(s) = s - T_k(s). \tag{1.7}$$

Furthermore, we will denote by c or $c_1, c_2, ...$, various constants which may depend on the data of the problem, whose value may vary from line to line.

If $1 < \sigma < N$, we denote by $\sigma^* = \frac{N\sigma}{N-\sigma}$ the Sobolev embedding exponent for the space $W_0^{1,\sigma}(\Omega)$.

If $u \ : \ \Omega \to I\!R$ is a Lebesgue measurable function, we define, for all $k \geq 0$

$$A_k = \{x \in \Omega \ \ : \ \ |u(x)| \geq k\}, \quad B_k = \{x \in \Omega \ \ : \ \ k \leq |u(x)| \leq k+1\}. \tag{1.8}$$

If E is a Lebesgue measurable subset of $I\!R^N$, we denote by $|E|$ its N-dimensional Lebesgue measure.

We extend the definition of J to a larger space, namely $W_0^{1,q}(\Omega)$, with $q = \frac{Np(1-\alpha)}{N-\alpha p} < p$, in the following way:

$$I(v) = \begin{cases} J(v), & \text{if } \displaystyle\int_\Omega a(x,v)|\nabla v|^p \, dx < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.9}$$

For the sake of simplicity, in the following we suppose that:

$$a(x,t) = \frac{1}{(1+|t|)^{\alpha p}}. \tag{1.10}$$

Our results are the following:

Theorem 1.1 *Let* $q = \dfrac{Np(1-\alpha)}{N-\alpha p}$*, and let* F *be a function such that* $|F| \in L^r(\Omega)$ *with* $r \geq q'$*. Then there exists a minimum* u *of* I *on* $W_0^{1,q}(\Omega)$*.*

The second result considers the case where $|F|$ has a high summability.

Theorem 1.2 *Let F be such that $|F| \in L^r(\Omega)$ with $r > \frac{N}{p-1}$. Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$; thus J attains its minimum on $W_0^{1,p}(\Omega)$.*

Remark 1.1 Note that the condition $0 < \alpha < \dfrac{1}{p'}$ implies that $\dfrac{N}{p-1} > q'$.

Remark 1.2 Observe that the condition on r does not depend on α, and the result also does not depend on α. The main tool of the proof will be an $L^\infty(\Omega)$ estimate, which then implies the $W_0^{1,p}(\Omega)$ estimate.

Theorem 1.3 *Let F be such that $|F| \in L^r(\Omega)$ with*

$$\frac{Np'}{N - \alpha p'(N-p)} \leq r < \frac{N}{p-1}.$$

Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,p}(\Omega) \cap L^s(\Omega)$; thus J attains its minimum on $W_0^{1,p}(\Omega)$, where

$$s = \frac{Nr\,(p(1-\alpha)-1)}{N - r(p-1)}.$$

Remark 1.3 Since $0 < \alpha < \dfrac{1}{p'}$ we have:

$$\frac{Np'}{N - \alpha p'(N-p)} < \frac{N}{p-1}.$$

Remark 1.4 Observe that if the minima are not bounded, we still have that they belong to $W_0^{1,p}(\Omega)$. The $W_0^{1,p}(\Omega)$ regularity result will be proveded combining the information that u belongs to $L^s(\Omega)$ with the fact that u is minimum.

Remark 1.5 As a consequence of the previous theorem, if $r = \dfrac{N}{p-1}$, we have that any minimum u belongs to $W_0^{1,p}(\Omega)$ and to $L^s(\Omega)$, for every $s < +\infty$.

If we decrease the summability of F, we find minima of I which do not in general belong any more to $W_0^{1,p}(\Omega)$.

Theorem 1.4 *Let F be such that $|F| \in L^r(\Omega)$ with*

$$q' \leq r < \frac{Np'}{N - \alpha p'(N-p)}.$$

Then any minimum u of I on $W_0^{1,q}(\Omega)$ belongs to $W_0^{1,\rho}(\Omega) \cap L^s(\Omega)$; thus J attains its minimum on $W_0^{1,\rho}(\Omega)$, where

$$\rho = \frac{Nr\,(p(1-\alpha)-1)}{N - \alpha pr}.$$

Remark 1.6 Note that the condition $0 < \alpha < \dfrac{1}{p'}$ implies that:

$$q' < \frac{Np'}{N - \alpha p'(N-p)}.$$

Remark 1.7 If α tends to $\dfrac{1}{p'}$ both $\dfrac{Np'}{N-\alpha p'(N-p)}$ and q' converge to $\dfrac{N}{p-1}$, so that Theorems 1.3 and 1.4 cannot be applied if $\alpha = \dfrac{1}{p'}$.

The paper is organized as follows. In the next section we prove the existence of minima for J, in the third section we give the proof of Theorem 1.2 (proof of bounded minima), while the fourth section is devoted to the proof of Theorems 1.3 and 1.4.

2 Existence of Minima

In order to prove that there exists a minimum of I on $W_0^{1,q}(\Omega)$, we are going to prove that I is both coercive and weakly lower semicontinuous on $W_0^{1,q}(\Omega)$.

Theorem 2.1 *Let F be such that: $|F| \in L^r(\Omega)$ with $r \geq q'$. Then I is coercive and weakly lower semi-continuous on $W_0^{1,q}(\Omega)$.*

Proof. The weak lower semi-continuity is a consequence of a theorem by De Giorgi (see [9]). As far as the coerciveness is concerned, it is enough to consider v in $W_0^{1,q}(\Omega)$ such that $I(v)$ is finite.

We have

$$\int_\Omega |\nabla v|^q \, dx = \int_\Omega \frac{|\nabla v|^q}{(1+|v|)^{\alpha q}} (1+|v|)^{\alpha q} \, dx,$$

therefore, by the Hölder inequality we get:

$$\int_\Omega |\nabla v|^q \, dx \leq c \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(\int_\Omega (1+|v|)^{\frac{\alpha pq}{p-q}} \, dx \right)^{1-\frac{q}{p}}.$$

By the fact that $q^* = \frac{\alpha pq}{p-q}$ and Sobolev embedding theorem we obtain:

$$\int_\Omega |\nabla v|^q \, dx \leq c \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(1 + \left(\int_\Omega |\nabla v|^q \, dx \right)^{\frac{q}{q}} \right)^{1-\frac{q}{p}},$$

which implies that if $R = \|v\|_{W_0^{1,q}(\Omega)}$

$$R^p \leq \left(\int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}} \, dx \right)^{\frac{q}{p}} \left(1 + R^{q^*} \right)^{1-\frac{q}{p}}. \tag{2.1}$$

On the other hand we have:

$$\begin{aligned} \left| \int_\Omega F.\nabla v \, dx \right| &\leq c \left(\int_\Omega |F|^{q'} \, dx \right)^{\frac{1}{q'}} \left(\int_\Omega |\nabla v|^q \, dx \right)^{\frac{1}{q}}. \\ &\leq cR. \end{aligned} \tag{2.2}$$

Thus, by (2.1) and (2.2) we obtain:

$$I(v) \geq c \frac{R^p}{(1+R^{q^*})^{\frac{p}{q}-1}} - cR.$$

Using the definition of q, it is easy to check that:

$$p - q^* \left(\frac{p}{q} - 1 \right) > 1,$$

so that

$$\lim_{R \to +\infty} I(v) = +\infty.$$

That is I is coercive on $W_0^{1,q}(\Omega)$.

By standard results, we deduce that there exists the minimum of I on $W_0^{1,q}(\Omega)$ and then Theorem 1.1 is proved.

3 Bounded Minima

By Theorem 2.1 there exists u in $W_0^{1,q}(\Omega)$ such that

$$I(u) = \min \left\{ I(u), \quad v \in W_0^{1,q}(\Omega) \right\},$$

i.e.

$$I(u) \leq I(v) \quad \text{for all} \quad v \in W_0^{1,q}(\Omega). \tag{3.1}$$

3.1 Some lemmas

To prove the bounded minima, we need the following lemmas.

Lemma 3.1 *[4] Let w be a function in $W_0^{1,\sigma}(\Omega)$ such that, for k greater than some k_0*

$$\int_{A_k} |\nabla w|^\sigma \, dx \leq c k^{\theta\sigma} |A_k|^{\frac{\sigma}{\sigma^*} + \varepsilon},$$

where $\varepsilon > 0$, $0 \leq \theta < 1$. Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant which depends on $c, \theta, \sigma, N, \varepsilon, k_0$.

The proof of this lemma can be found in the Appendix of [4], its proof is based on the lemma according to Stampacchia [11].

Lemma 3.2 *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then*

$$\int_{A_k} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \leq \int_{A_k} F.\nabla G_k(u) \, dx, \quad \forall \; k > 0, \tag{3.2}$$

where A_k is as in (1.8) *and G_k is the function defined in* (1.7).

Proof. We have, $I(u) \leq I(0) = 0$, then

$$\int_\Omega \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \leq \int_\Omega F.\nabla u \, dx < +\infty.$$

On the other hand, we have for all $k > 0$

$$\int_\Omega \frac{|\nabla T_k(u)|^p}{(1 + |T_k(u)|)^{\alpha p}} \, dx = \int_{\{|u| \leq k\}} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \leq \int_\Omega \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx < +\infty.$$

We take $v = T_k(u)$ as a test function in (3.1) to obtain:

$$\int_{A_k} \frac{|\nabla u|^p}{(1 + |u|)^{\alpha p}} \, dx \leq \int_{A_k} F.\nabla G_k(u) \, dx, \quad \forall \; k > 0.$$

3.2 Proof of Theorem 1.2

Let σ be such that $r' < \sigma < q < p$, this implies that $\dfrac{1}{r}+\dfrac{1}{\sigma} < 1$, then by Hölder inequality, we have:

$$\begin{aligned}\int_{A_k} |F.\nabla G_k(u)|\, dx &\leq \|F\|_{L^r}\left[\int_{A_k} |\nabla G_k(u)|^\sigma\, dx\right]^{\frac{1}{\sigma}} .|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}\\ &\leq c\left[\int_{A_k} |\nabla G_k(u)|^\sigma\, dx\right]^{\frac{1}{\sigma}} .|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}\end{aligned}$$

and by Lemma 3.2, we deduce that:

$$\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq c\left[\int_{A_k} |\nabla G_k(u)|^\sigma\, dx\right]^{\frac{1}{\sigma}} .|A_k|^{1-\frac{1}{\sigma}-\frac{1}{r}}. \tag{3.3}$$

Moreover, by the Hölder inequality, we obtain:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx &= \int_{A_k} \frac{|\nabla u|^\sigma}{(1+|u|)^{\alpha\sigma}}(1+|u|)^{\alpha\sigma}\, dx\\ &\leq \left[\int_{A_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx\right]^{\frac{\sigma}{p}}\left[\int_{A_k}(1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\, dx\right]^{1-\frac{\sigma}{p}},\end{aligned}$$

therefore, by (3.3), we have:

$$\int_{A_k} |\nabla u|^\sigma\, dx \leq c|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}}\left[\int_{A_k}(1+|u|)^{\frac{\alpha\sigma p}{p-\sigma}}\, dx\right]^{\frac{p-\sigma}{p-1}}. \tag{3.4}$$

Since if $k\geq 1$, one has on A_k that $1+|u| \leq 2(k+|G_k(u)|)$, we can write:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx \leq\ & c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}+\frac{p-\sigma}{p-1}}\\ &+|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}}\left[\int_{A_k}|G_k(u)|^{\frac{\alpha\sigma p}{p-\sigma}}\, dx\right]^{\frac{p-\sigma}{p-1}}\Big\}.\end{aligned}$$

Now, we choose σ such that $\dfrac{\alpha\sigma p}{p-\sigma} < \sigma^*$, and therefore, using Hölder's and Sobolev's inequalities one obtains:

$$\begin{aligned}\int_{A_k} |\nabla u|^\sigma\, dx \leq\ & c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}+\frac{p-\sigma}{p-1}}\\ &+|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}-\frac{\alpha p}{p-1}\cdot\frac{\sigma}{\sigma^*}}\left[\int_{A_k}|\nabla u|^\sigma\, dx\right]^{\frac{\alpha p}{p-1}}\Big\}.\end{aligned}$$

Using the Young's inequality with exponents $\frac{1}{\alpha p'}$ and $\frac{1}{1-\alpha p'}$, on the second term on the right side, we get:

$$\begin{aligned}&|A_k|^{\left(1-\frac{1}{\sigma}-\frac{1}{r}\right)\frac{\sigma}{p-1}-\frac{\alpha p}{p-1}\cdot\frac{\sigma}{\sigma^*}}\left[\int_{A_k}|\nabla u|^\sigma\, dx\right]^{\frac{\alpha p}{p-1}}\\ &\qquad\leq \frac{1}{2}\int_{A_k}|\nabla u|^\sigma\, dx + c|A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}}\end{aligned}$$

so that we have:

$$\int_{A_k} |\nabla u|^\sigma \, dx \leq c\Big\{k^{\frac{\alpha\sigma p}{p-1}}|A_k|^{1-\frac{\sigma}{r(p-1)}} + |A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}}\Big\}. \tag{3.5}$$

As can be seen by means of straightforward calculations, the assumptions on r and α, imply that:

$$1-\frac{\sigma}{r(p-1)} < \left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}.$$

Moreover, since u belongs to $W_0^{1,q}(\Omega)$, we have that $|A_k|$ tends to zero as k tends to infinity, thus there exists k_0 such that if $k \geq k_0$, we have:

$$|A_k|^{\left(p-1-\frac{\sigma}{r}-\alpha p\frac{\sigma}{\sigma^*}\right)\frac{1}{(p-1)(1-\alpha p')}} < |A_k|^{1-\frac{\sigma}{r(p-1)}}$$

and so (3.5) implies that:

$$\int_{A_k} |\nabla u|^\sigma \, dx \leq ck^{\frac{\alpha\sigma p}{p-1}}|A_k|^{1-\frac{\sigma}{r(p-1)}} \quad \forall\, k \geq k_0.$$

It is easy to see that $1-\dfrac{\sigma}{r(p-1)}-\dfrac{\sigma}{\sigma^*}>0$ since $r>\dfrac{N}{p-1}$ and $\dfrac{\alpha p}{p-1}$ belongs to $(0,1)$ since $0<\alpha<\dfrac{1}{p'}$.

Thus, by Lemma 3.1 u belongs to $L^\infty(\Omega)$. On the other hand,

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq \int_\Omega F.\nabla u \, dx < +\infty.$$

The $L^\infty(\Omega)$ estimate implies that:

$$\frac{1}{(1+\|u\|_{L^\infty(\Omega)})^{\alpha p}}\int_\Omega |\nabla u|^p \, dx \leq \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq c$$

and so u belongs to $W_0^{1,p}(\Omega)$.

Theorem 1.3 is proved.

Remark 3.1 Observe that the condition $\dfrac{\alpha\sigma p}{p-\sigma}<\sigma^*$ is equivalent to $\sigma<q$.

4 Summability of Unbounded Minima

This section will be devoted to the proof of Theorems 1.3 and 1.4. We begin with technical results, which will be used later.

4.1 Preliminary lemmas

Lemma 4.1. *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then for all $k \in I\!N$, we have:*

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq \frac{c}{1+k}\int_{A_k} F.\nabla u \, dx + \int_{B_k} F.\nabla u \, dx, \tag{4.1}$$

where A_k and B_k are as in (1.8).

Proof.

- If $k = 0$, the result is trivial since u is minimum of I.
- Let $k > 0$, we take $v = u - T_1(u - T_k(u))$ as test function in (3.1), we obtain:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega F.\nabla u\, dx \leq \int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}}\, dx - \int_\Omega F.\nabla v\, dx$$

which implies that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_\Omega \frac{|\nabla v|^p}{(1+|v|)^{\alpha p}}\, dx \leq \int_{B_k} F.\nabla u\, dx$$

and by definition of v, we deduce that:

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx + \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx - \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|v|)^{\alpha p}}\, dx \leq \int_{B_k} F.\nabla u\, dx$$

and then

$$\begin{aligned} \int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx &\leq \int_{A_{k+1}} |\nabla u|^p \left\{ \frac{1}{(1+|v|)^{\alpha p}} - \frac{1}{(1+|u|)^{\alpha p}} \right\} dx \\ &\quad + \int_{B_k} F.\nabla u\, dx \\ &\leq \int_{A_{k+1}} |\nabla u|^p \frac{(1+|u|)^{\alpha p} - (1+|v|)^{\alpha p}}{(1+|v|)^{\alpha p}(1+|u|)^{\alpha p}}\, dx \\ &\quad + \int_{B_k} F.\nabla u\, dx. \end{aligned} \tag{4.2}$$

Since $|v| = |u| - 1$ on A_{k+1}, we easily obtain that there exists a positive constant c such that

$$(1+|u|)^{\alpha p} - (1+|v|)^{\alpha p} \leq c(1+|v|)^{\alpha p-1}.$$

Thus (4.2) becomes

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq c \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|v|)(1+|u|)^{\alpha p}}\, dx + \int_{B_k} F.\nabla u\, dx.$$

Since $|v| \geq k$ on A_{k+1}, we have:

$$\int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx \leq \frac{c}{1+k} \int_{A_{k+1}} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\, dx + \int_{B_k} F.\nabla u\, dx.$$

Using (3.2) we thus obtain (4.1).

Lemma 4.2 *Let u be the minima of I in $W_0^{1,q}(\Omega)$, then for all $\gamma \geq 1$, we have:*

$$\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\, dx \leq c_1 + c_2 \int_\Omega |F|^{p'} |u|^{p(\alpha p' + \gamma - 1)}\, dx, \tag{4.3}$$

where c_1 and c_2 are two positive constants.

Proof. Let $\gamma \geq 1$, we have:

$$\begin{aligned}
\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\, dx &= \sum_{k=0}^{+\infty} \int_{B_k} |\nabla u|^p |u|^{p(\gamma-1)}\, dx \\
&\leq \sum_{k=0}^{+\infty} \int_{B_k} |\nabla u|^p (1+k)^{p(\gamma-1)}\, dx \\
&\leq c \sum_{k=0}^{+\infty} \int_{B_k} \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} (1+k)^{p(\gamma-1)+\alpha p}\, dx.
\end{aligned} \tag{4.4}$$

Thus, by (4.1) we obtain:

$$\begin{aligned}
\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\, dx \leq\ & c \sum_{k=0}^{+\infty} (1+k)^{p(\gamma-1)+\alpha p-1} \int_{A_k} |F|.|\nabla u|\, dx \\
&+ c \sum_{k=0}^{+\infty} (1+k)^{p(\gamma-1)+\alpha p} \int_{B_k} |F|.|\nabla u|\, dx.
\end{aligned} \tag{4.5}$$

Observe that, for $k \in I\!N$, we have:

$$\int_{A_k} |F|.|\nabla u|\, dx = \sum_{h=k}^{+\infty} \int_{B_h} |F|.|\nabla u|\, dx. \tag{4.6}$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{+\infty} (1+k)^{p(\gamma-1)+\alpha p-1} \int_{A_k} |F|.|\nabla u|\, dx \\
&\qquad = \sum_{k=0}^{+\infty} (1+k)^{p(\gamma-1)+\alpha p-1} \sum_{h=k}^{+\infty} \int_{B_h} |F|.|\nabla u|\, dx.
\end{aligned} \tag{4.7}$$

Therefore, changing the order of summation, and recalling that:

$$\sum_{k=0}^{h} k^l \leq c(1+h)^{l+1} \tag{4.8}$$

with $c = c(l)$, we have:

$$\begin{aligned}
&\sum_{k=0}^{+\infty} (1+k)^{p(\gamma-1)+\alpha p-1} \int_{A_k} |F|.|\nabla u|\, dx \\
&\qquad = \sum_{h=0}^{+\infty} \sum_{k=0}^{h} (1+k)^{p(\gamma-1)+\alpha p-1} \int_{B_h} |F|.|\nabla u|\, dx \\
&\qquad = \sum_{h=0}^{+\infty} (1+h)^{p(\gamma-1)+\alpha p} \int_{B_h} |F|.|\nabla u|\, dx.
\end{aligned} \tag{4.9}$$

We obtain from (4.5) that:

$$\begin{aligned}\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)}\,dx &\le c\sum_{k=0}^{+\infty}(1+k)^{p(\gamma-1)+\alpha p}\int_{B_k}|F|.|\nabla u|\,dx\\ &\le c\sum_{k=0}^{+\infty}\int_{B_k}|F|.|\nabla u|(1+|u|)^{p(\gamma-1)+\alpha p}\,dx\\ &\le c\int_\Omega |F|.|\nabla u|\,dx + c\int_\Omega |F|.|\nabla u||u|^{p(\gamma-1)+\alpha p}\,dx.\end{aligned}$$

By Young's inequality and the fact that $\int_\Omega |F|.|\nabla u|\,dx < +\infty$, we deduce (4.3).

Lemma 4.3 *Let $\lambda > 0$ and let $u \in W_0^{1,q}(\Omega)$ be the minimum of I, then we have:*

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx \le c\int_\Omega |F|^{p'}(1+|u|)^{\alpha p p'-\lambda}\,dx. \tag{4.10}$$

Proof. Let $\lambda > 0$ and let $u \in W_0^{1,q}(\Omega)$ be the minimum of I, we have:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &= \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}(1+|u|)^{\alpha p-\lambda}\,dx\,dx\\ &\le c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}\frac{|\nabla u|^p}{(1+|u|)^{\alpha p}}\,dx\end{aligned} \tag{4.11}$$

and this implies, by (4.1) that:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &\le c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\quad +c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx.\end{aligned} \tag{4.12}$$

Using (4.6) one has

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\qquad = \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\sum_{h=k}^{+\infty}\int_{B_h}|F|.|\nabla u|\,dx.\end{aligned}$$

Changing the order of summation and using (4.8), we have:

$$\begin{aligned}&\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda-1}\int_{A_k}|F|.|\nabla u|\,dx\\ &\qquad \le \sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx.\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13), we get:

$$\begin{aligned}\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx &\leq c\sum_{k=0}^{+\infty}(1+k)^{\alpha p-\lambda}\int_{B_k}|F|.|\nabla u|\,dx\\ &\leq c\sum_{k=0}^{+\infty}\int_{B_k}|F|.|\nabla u|(1+|u|)^{\alpha p-\lambda}\,dx\\ &\leq c\int_\Omega |F|.|\nabla u|(1+|u|)^{\alpha p-\lambda}\,dx.\end{aligned}$$

Now, the Young's inequality implies that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx \leq c\int_\Omega |F|^{p'}(1+|u|)^{\alpha pp'-\lambda}\,dx.$$

4.2 Proof of Theorem 1.3

We begin with the following technical lemma.

Lemma 4.4 *Let* $\gamma = \dfrac{(1-\alpha p')(r(p-1))^*}{p^*}$, *we have*

i) $s = \gamma p^* = \dfrac{pr(\alpha p'+\gamma-1)}{r-p'}$

ii) $\gamma \geq 1$ *if and only if* $r \geq \dfrac{Np'}{N-\alpha p'(N-p)}$

iii) $1-\dfrac{p'}{r} < \dfrac{p}{p^*}$ *if and only if* $r < \dfrac{N}{p-1}$.

Theorem 4.1 *Under the hypotheses of Theorem* 1.3, *we have the following estimations*

i) $\displaystyle\int_\Omega |u|^s\,dx \leq c_3$,

ii) $\displaystyle\int_\Omega |\nabla u|^p\,dx \leq c_4$,

where c_3 *and* c_4 *are two positive constants.*

Proof.

i) We have, by Lemmas 4.2, 4.4 and Sobolev embedding

$$\begin{aligned}\left(\int_\Omega |u|^s\,dx\right)^{\frac{p}{p^*}} &= \left(\int_\Omega |u|^{\gamma p^*}\,dx\right)^{\frac{p}{p^*}} \leq \int_\Omega |\nabla u|^p|u|^{p(\gamma-1)}\,dx\\ &\leq c+\int_\Omega |F|^{p'}|u|^{p(\alpha p'+\gamma-1)}\,dx.\end{aligned} \tag{4.14}$$

Applying the Holder inequality, we obtain:

$$\left(\int_\Omega |u|^s\,dx\right)^{\frac{p}{p^*}} \leq \left(\int_\Omega |F|^r\,dx\right)^{\frac{p'}{r}}\left(\int_\Omega |u|^{\frac{pr(\alpha p'+\gamma-1)}{r-p'}}\,dx\right)^{1-\frac{p'}{r}}.$$

Then by $i), iii)$ of Lemma 4.4 and Young's inequality, we deduce

$$\int_\Omega |u|^s \, dx \leq c_3.$$

ii) We have:

$$\begin{aligned}\int_\Omega |\nabla u|^p \, dx &= \int_{\{|u|\leq 1\}} |\nabla u|^p \, dx + \int_{\{|u|\geq 1\}} |\nabla u|^p \, dx \\ &\leq c \int_\Omega \frac{|\nabla u|^p}{(1+|u|)^{\alpha p}} \, dx + \int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx \\ &\leq c \int_\Omega |F||\nabla u| \, dx + \int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx\end{aligned}$$

and from (4.14), we get:

$$\int_\Omega |\nabla u|^p |u|^{p(\gamma-1)} \, dx \leq c_4,$$

which implies that:

$$\int_\Omega |\nabla u|^p \, dx \leq c_5.$$

4.3 Proof of Theorem 1.4

We begin with the following technical lemma.

Lemma 4.5 *Let* $\lambda = \dfrac{pN - r(p-1)(N - \alpha p'(N-p))}{N - r(p-1)}$*, we have the following properties :*

i) $s = \dfrac{\lambda\rho}{p-\rho} = \dfrac{r(\alpha p p' - \lambda)}{r - p'}$,

ii) $\lambda > 0$ *if and only if* $r < \dfrac{Np'}{N - \alpha p'(N-p)}$,

iii) $(1 - \dfrac{p'}{r})\dfrac{\rho}{p} + 1 - \dfrac{\rho}{p} < \dfrac{\rho}{s}$.

Theorem 4.2 *Under the hypotheses of Theorem* 1.4*, we have the following estimations:*

i) $\displaystyle\int_\Omega |u|^s \, dx \leq c_6$,

ii) $\displaystyle\int_\Omega |\nabla u|^\rho \, dx \leq c_7$,

where c_6 *and* c_7 *are two positive constants.*

Proof. Since $\rho^* = s$, we have by Sobolev embedding:

$$\begin{aligned}\left(\int_\Omega |u|^s\,dx\right)^{\frac{\rho}{s}} &= \left(\int_\Omega |u|^{\rho^*}\,dx\right)^{\frac{\rho}{\rho^*}} \le c\int_\Omega |\nabla u|^\rho\,dx \\ &= c\int_\Omega \frac{|\nabla u|^\rho}{(1+|u|)^{\frac{\lambda\rho}{p}}}(1+|u|)^{\frac{\lambda\rho}{p}}\,dx.\end{aligned} \tag{4.15}$$

Applying Hölder inequality, we have:

$$\left(\int_\Omega |u|^s\,dx\right)^{\frac{\rho}{s}} \le c\left[\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx\right]^{\frac{\rho}{p}}\left[\int_\Omega (1+|u|)^{\frac{\lambda\rho}{p-\rho}}\,dx\right]^{1-\frac{\rho}{p}}. \tag{4.16}$$

On the other hand by Lemma 4.2 and Hölder inequality, we deduce that:

$$\int_\Omega \frac{|\nabla u|^p}{(1+|u|)^\lambda}\,dx \le \left[\int_\Omega (1+|u|)^{\frac{r(\alpha pp'-\lambda)}{r-p'}}\,dx\right]^{1-\frac{p'}{r}}. \tag{4.17}$$

From (4.15), (4.16) and (4.17), we get:

$$\begin{aligned}\left(\int_\Omega |u|^s\,dx\right)^{\frac{\rho}{s}} &\le c\int_\Omega |\nabla u|^\rho\,dx \\ &\le \left[\int_\Omega (1+|u|)^{\frac{r(\alpha pp'-\lambda)}{r-p'}}\,dx\right]^{(1-\frac{p'}{r})\frac{\rho}{p}} \\ &\quad\times\left[\int_\Omega (1+|u|)^{\frac{\lambda\rho}{p-\rho}}\,dx\right]^{1-\frac{\rho}{p}}\end{aligned} \tag{4.18}$$

which implies, by using Lemma 4.5

$$\left(\int_\Omega |u|^s\,dx\right)^{\frac{\rho}{s}} \le \left[\int_\Omega (1+|u|)^s\,dx\right]^{(1-\frac{p'}{r})\frac{\rho}{p}+1-\frac{\rho}{p}}.$$

Finally, by the lemma 4.5 and Hölder inequality, we deduce that:

$$\int_\Omega |u|^s\,dx \le c_6.$$

Therefore by (4.18), we also have :

$$\int_\Omega |\nabla u|^\rho\,dx \le c_7.$$

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Hybrid Projective Synchronization of Fractional Order Chaotic Systems with Fractional Order in the Interval (1,2)

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Abstract: A hybrid projective synchronization scheme for two identical fractional-order chaotic systems with fractional order $1 < q < 2$ has been discussed in this paper. Based on the stability theory of fractional-order systems, a controller for the synchronization of two identical fractional-order chaotic systems is designed. To illustrate the effectiveness of the proposed scheme, we discuss two examples: (i) the fractional-order Lorenz chaotic system with fractional-order $q = 1.17$, (ii) the fractional-order Lu chaotic system with fractional-order q = 1.13. The numerical simulations exhibit the validity and feasibility of the proposed scheme.

Keywords: *fractional order in the interval (1,2); chaotic systems; hybrid projective synchronization.*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N30, 37N35, 70K99.

1 Introduction

The theory of derivatives of fractional order, i.e., non-integer order, goes back to Leibniz's note in his list to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed. Fractional calculus is a 300 year old mathematical topic. Although it has a long history, the applications of fractional calculus to physics and engineering are just a recent focus of interest [1] and [2]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [3], dielectric polarization [4], electrode–electrolyte polarization [5], electromagnetic waves [6], quantitative finance [7], and quantum evolution of complex systems [8]. In recent years, chaotic phenomenon has been found in many

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fractional-order nonlinear systems, such as the fractional-order Lorenz chaotic system [9], [10], Chuas fractional-order chaotic circuit system [10], the fractional-order modified Duffing chaotic system [11], the fractional-order Rossler chaotic system [12], [13], the fractional-order Chen chaotic system [10]- [12], the fractional-order memristor chaotic system [14], and so on.

In 1999, projective synchronization was first proposed by Mainieri and Rehacek [15], where the drive and response systems were synchronized up to a scaling factor. Its proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication [16]. Both complete synchronization and anti-phase synchronization are special cases of projective synchronization. Recently, various kinds of projective synchronization for fractional order chaotic systems without time-delay have been studied, such as hybrid projective synchronization [17], generalized projective synchronization [18], function projective synchronization [19], lag projective synchronization [20] and modified projective synchronization [21].

However, many previous synchronization methods [22]- [25], [26]- [29] for fractional-order chaotic systems only focused on the fractional-order $0 < q < 1$, while in fact, there are many fractional-order systems with fractional-order $1 < q < 2$ in the real world. For example, the time fractional heat conduction equation [30], the fractional telegraph equation [31], the time fractional reaction-diffusion systems [31], the fractional diffusion-wave equation [32], the space-time fractional diffusion equation [33], the super-diffusion systems [34], etc., but the chaos phenomenon was not considered in [30]- [35]. Meanwhile, based on numerical simulation, Ge and Jhuang [31] reported some results on synchronization of the fractional order rotational mechanical system with fractional-order $q = 1.1$. Up to now, there seem to be no results on chaotic synchronization for fractional-order chaotic systems with $1 < q < 2$ through precise theorization. So, how to achieve the chaotic synchronization for fractional-order nonlinear systems with $1 < q < 2$ through precise theorization is an interesting and open question of academic significance as well as practical importance.

Motivated by the above mentioned discussion, in this paper we propose a hybrid projective synchronization approach for a class of fractional-order chaotic systems with fractional-order $1 < q < 2$ through precise theorization. To show the effectiveness of the proposed scheme, the hybrid projective synchronization for a fractional-order Lorenz chaotic system with fractional-order $q = 1.17$ and Lu fractional-order chaotic system with fractional-order $q = 1.13$ are discussed, respectively. The numerical simulations have indicated the validity and feasibility of our scheme.

2 The Review and the Approximation of a Fractional Operator

The differintegral operator, denoted by ${}_aD_t^q$, is a combined differentiation-integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral in a single expression and is defined by:

$$
{}_aD_t^q = \begin{cases} \frac{d^q}{dt^q}, & q > 0, \\ 0, & q = 0, \\ \int_a^t (d\tau)^{-q}, & q < 0. \end{cases} \tag{1}
$$

There are some definitions for fractional derivatives [1]. The commonly used definitions are Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. The Grunwald-Letnikov definition is given by:

$$
\begin{aligned}
{}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\
&= \lim[\frac{t-a}{N}]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f\Big(t - j\Big[\frac{t-a}{N}\Big]\Big).
\end{aligned} \tag{2}
$$

The Riemann-Liouville definition is the simplest and easiest definition to use. This definition is given by:

$$
\begin{aligned}
{}_aD_t^q f(t) &= \frac{d^q f(t)}{d(t-a)^q} \\
&= \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d(\tau),
\end{aligned} \tag{3}
$$

where n is the first integer which is not less than q, i.e., $n-1 \leq q < n$ and Γ is the Gamma function defined as:

$$
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{4}
$$

For functions $f(t)$ having n continuous derivatives for $t \geq 0$ where $n-1 \leq q < n$, the Grunwald–Letnikov and the Riemann–Liouville definitions are equivalent. The Laplace transforms of the Riemann–Liouville fractional integral and derivative are given as follows:

$$
L\{{}_0D_t^q f(t)\} = S^q F(s), q \leq 0, \tag{5}
$$

$$
L\{{}_0D_t^q f(t)\} = S^q F(s) - \sum_{k=0}^{n-1} S_0^k D_t^q f(0), n-1 < q \leq n \in N. \tag{6}
$$

Unfortunately, the Riemann-Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that it requires the knowledge of the non-integer order derivatives of the function at $t = 0$. This problem does not exist in the Caputo definition that is sometimes referred to as smooth fractional derivative in literature. This definition of derivative is defined by

$$
{}_0D_t^q = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m-1 < q < m, \\ \frac{d^m f(t)}{dt^m}, & q = m, \end{cases} \tag{7}
$$

where m is the first integer larger than q. It is found that the equations with Riemann–Liouville operators are equivalent to those with Caputo operators by homogeneous initial conditions assumption [1].

3 Stability of Fractional Order Systems

Stability of fractional systems has been thoroughly investigated where necessary and sufficient conditions have been derived in [39]. The stability region of a linear set of fractional order equations, each of order q, such that $1 < q < 2$ is shown in Figure 1. An autonomous system is asymptotically stable iff $\mid arg\lambda \mid > \frac{q\pi}{2}$ is satisfied for all eigenvalues λ of matrix A. Also this system is stable iff $\mid arg\lambda \mid \geq \frac{q\pi}{2}$ is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy $\mid arg\lambda \mid > \frac{q\pi}{2}$, and have geometric multiplicity one [38].

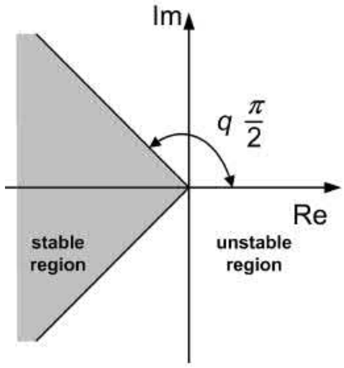

Figure 1: Stability of fractional order systems such that $1 < q < 2$.

4 System Description Problem Formulation for Hps Between Fractional Order Systems

In this section we put a glimpse of methodology and problem formulation for hybrid projective synchronization between fractional order chaotic systems via tracking control. The fractional order chaotic drive and response systems can be described as follows:

$$\frac{d^q(x)}{dt^q} = f(x) \tag{8}$$

and

$$\frac{d^q(y)}{dt^q} = g(y) + \phi(x,y), \tag{9}$$

where $x \in R^n, y \in R^m$ are state vectors of the drive system (8), and the response system (9) and $f, g : R^n \to R^n$ are continuous vector functions, respectively, $\phi(x,y)$ is a vector controller to be designed.

Definition 4.1 For the drive system (8) and the response system (9), the Hybrid Projective Synchronization (HPS) is achieved if there exists an $n \times n$ invertible matrix A such that

$$\lim_{t\to\infty} \|e(t)\| = \|Ay - x\| = 0$$

where $\| \bullet \|$ is an Euclidean norm.

Remark 4.1 If $A = \sigma I; \sigma \in R$, the HPS problem will reduce to Projective Synchronization(PS) where I is an $n \times n$ matrix with proper dimensions. In particular, if $\sigma = 1$ and $\sigma = -1$ the problem is further simplified to complete synchronization and anti-phase synchronization, respectively. If $A = diag(a_1, a_2, ..., a_n)$, where $a_1, a_2, ..., a_n$ are not all zeros and $a_i \neq a_j$ for some i and j, then the modified projective synchronization will appear. Therefore CS, AS, PS, and MPS are the special cases of hybrid projective synchronization.

In order to obtain the HPS for the fractional order chaotic system we consider that for fractional order chaotic system (8) as drive system, and construct a response system as follows

$$\frac{d^q}{dt^q}(y) = A^{-1}\Big[f(Ay) + \phi(x,y)\Big], \tag{10}$$

where A^{-1} is the inverse of the invertible matrix A, $y \in R^n$ are state vectors of the response system (10) and $\phi(x,y)$ is a controller which may be designed.

Define the HPS error between the response system (10) and the drive system (8) as

$$e = Ay - x,$$

$$e = (e_1, e_2, ..., e_n),$$

$$e_i = \Big(\sum_{j=i}^{n} a_{ij} y_j\Big) - x_i \qquad (i, j = 1, 2, ..., n).$$

Let

$$f(Ay) - f(x) = E(x,e). \tag{11}$$

Now we assume that the error vector e can be subdivided into two vectors $e_\alpha = (e_{n1}, e_{n2}, ..., e_{nk})$ and $e_\beta = (e_{n(k+1)}, e_{n(k+2)}, ..., e_{nl})$, so that $E(x,y)$ has the following form:

$$E(x,e) = \begin{pmatrix} B_\alpha e_\alpha + h_1(x, e_\alpha, e_\beta) \\ B_\beta e_\beta + h_{21}(x, e_\alpha, e_\beta) + h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}, \tag{12}$$

where $h_1(x, e_\alpha, e_\beta) \in R^m, h_{21}(x, e_\alpha, e_\beta) \in R^{n-m}, h_{22}(x, e_\alpha, e_\beta) \in R^{n-m}$ and $\lim_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$, respectively. $B_\alpha \in R^{n\times m}$ and $B_\beta \in R^{(n-m)\times(n-m)}$ are constant matrices.

Rewrite the controller $\phi(x,y)$ as follows

$$\phi(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix}, \tag{13}$$

where $\mu_\alpha(x,e) \in R^m$ and $\mu_\beta(x,e) \in R^{n-m}$, respectively.

Now the following theorem is based on the stability of fractional order chaotic systems, which gives the final destination to problem formulation.

Theorem 4.1 *If the controller $\phi(x,y)$ in the response system* (10) *can be chosen as*

$$\phi(x,y) = \mu(x,e) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix},$$

where $Q_\alpha \in R^{m\times m}$ and $Q_\beta \in R^{(n-m)\times(n-m)}$ are suitable constant matrices respectively. If all the eigenvalues of $B_\alpha + Q_\alpha$ satisfy $| arg\lambda_i | > \frac{q\pi}{2}, (i = 1, 2, ..., m)$ and all the eigenvalues of $B_\beta + Q_\beta$ satisfy $| arg\lambda_i | > \frac{q\pi}{2}, (i = 1, 2, ..., n-m)$, then HPS between drive and response system can be achieved.

Remark 4.2 In order to use the stability theory of linear fractional-order systems [37], the controller $\phi(x,y)$ or $\mu(x,y)$ is chosen as $\begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}$. Moreover, the nonlinear term $h_{21}(x, e_\alpha, e_\beta) \in R^{n-m}$ in the error dynamic system (12) or response system (10) is preserved.

5 Illustrative Examples

In this section, to show the effectiveness of the hybrid projective synchronization approach, we apply the hybrid projective synchronization scheme for the fractional-order Lorenz chaotic system with fractional-order $1 < q < 2$ and the fractional order Lu system with fractional order $1 < q < 2$ respectively.

5.1 HPS for fractional order Lorenz chaotic system with $1 < q < 2$.

The fractional order Lorenz system is a system of three ordinary differential equations displaying a chaotic behaviour for certain values of parameters (σ, β, γ), and fractional orders $(q_1 = q_2 = q_3 = q)$. The fractional order Lorenz system with parameter values $(\sigma, \beta, \gamma) = (10, 28, 8/3)$ and fractional order q with $1 < q < 2$ is given by

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= 10(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} &= \frac{8}{3}x_1 - x_2 - x_1x_3, \\ \frac{d^q x_3}{dt^q} &= x_1x_2 - 28x_3. \end{aligned} \tag{14}$$

The chaotic attractor of fractional order Lorenz system for different values of q, $1 < q < 2$ is depicted in Figures 2-7.

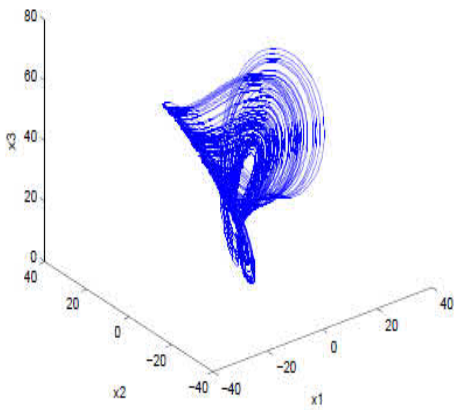

Figure 2: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.15$.

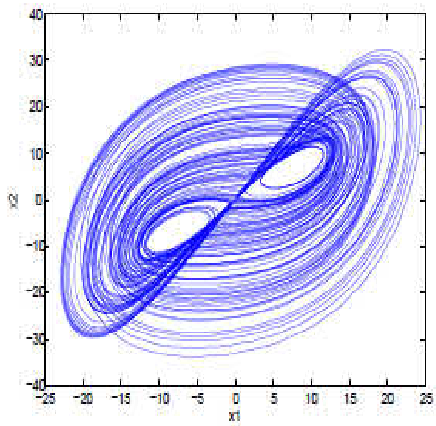

Figure 3: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.15$.

According to the HPS scheme presented in the above section, the response system is described by

$$\begin{pmatrix} \frac{d^q y_1}{dt^q} \\ \\ \frac{d^q y_2}{dt^q} \\ \\ \frac{d^q y_3}{dt^q} \end{pmatrix} = A^{-1} \begin{pmatrix} 10(\sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j) \\ 8/3(\sum_{j=1}^{3} a_{1j}y_j) - \sum_{j=1}^{3} a_{2j}y_j - \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{3j}y_j \\ \sum_{j=1}^{3} a_{1j}y_j \sum_{j=1}^{3} a_{2j}y_j - 28 \sum_{j=1}^{3} a_{3j}y_j \end{pmatrix} + A^{-1}\phi(x, y), \tag{15}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a reversible matrix and A^{-1} is its reverse matrix.

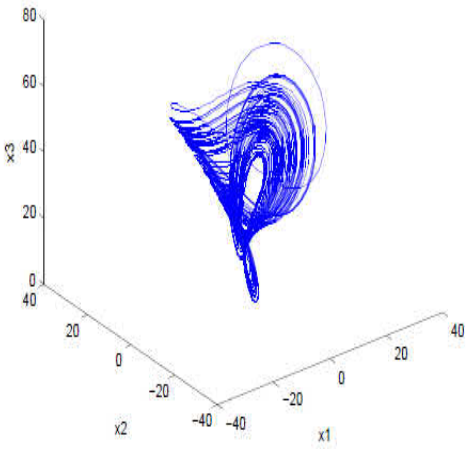

Figure 4: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.16$.

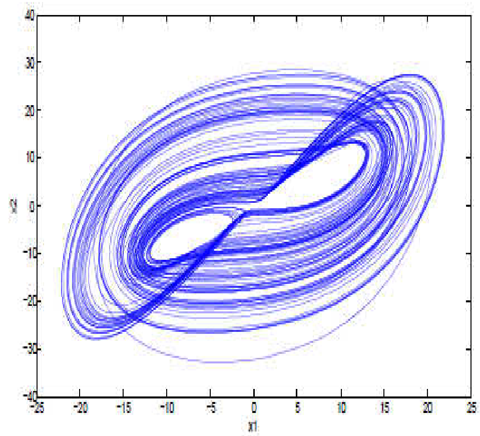

Figure 5: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.16$.

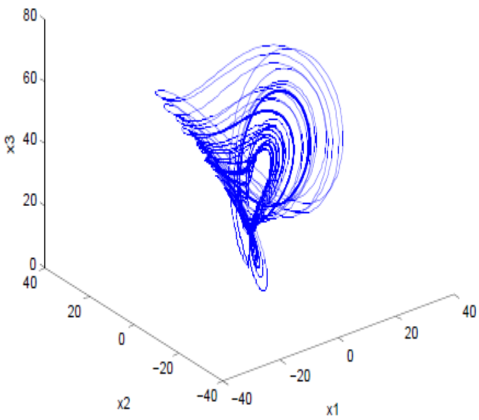

Figure 6: 3D chaotic attractor of the Lorenz system with $q_1 = q_2 = q_3 = 1.17$.

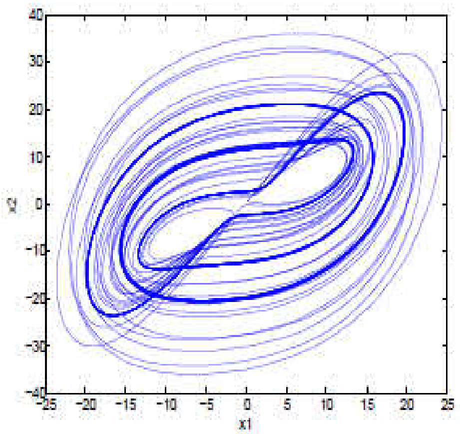

Figure 7: 2D projection of the Lorenz system with $q_1 = q_2 = q_3 = 1.17$.

According to definition of HPS error dynamics, we have

$$\begin{aligned} \frac{d^q e}{dt^q} &= A\frac{d^q y}{dt^q} - \frac{d^q x}{dt^q} \\ &= f(Ay) - f(x) + \phi(x,y). \end{aligned} \tag{16}$$

Let

$$f(Ay) - f(x) = E(x,e). \tag{17}$$

Therefore, from (16) we have

$$\frac{d^q e}{dt^q} = E(x,e) + \phi(x,y). \tag{18}$$

Our goal is to find $E(x,e)$ and design a controller to achieve HPS.

Now from equation (17) we have

$$E(x,e)=\begin{pmatrix} 10(\sum\limits_{j=1}^{3} a_{2j}y_j-\sum\limits_{j=1}^{3} a_{1j}y_j) \\ 8/3(\sum\limits_{j=1}^{3} a_{1j}y_j)-\sum\limits_{j=1}^{3} a_{2j}y_j-\sum\limits_{j=1}^{3} a_{1j}y_j\sum\limits_{j=1}^{3} a_{3j}y_j \\ \sum\limits_{j=1}^{3} a_{1j}y_j\sum\limits_{j=1}^{3} a_{2j}y_j-28\sum\limits_{j=1}^{3} a_{3j}y_j \end{pmatrix}-\begin{pmatrix} 10(x_2-x_1) \\ \frac{8}{3}x_1-x_2-x_1x_3 \\ x_1x_2-28x_3 \end{pmatrix} \quad (19)$$

which gives

$$E(x,e)=\begin{pmatrix} 10e_2-10e_1 \\ \frac{8}{3}e_1-e_2-e_1x_3-e_3x_1-e_1e_3 \\ e_1e_2+x_1e_2+x_2e_1-28e_3 \end{pmatrix}. \quad (20)$$

We choose the following:
$e_\alpha=e_1; e_\beta=(e_2,e_3)^T; B_\alpha=-10; h_1(x,e_\alpha,e_\beta)=10e_2, B_\beta=\begin{pmatrix} -1 & 0 \\ 0 & -28 \end{pmatrix}; h_{21}(x,e_\alpha,e_\beta)=\begin{pmatrix} \frac{8}{3}e_1-e_1x_3--e_1e_3 \\ e_1e_2+x_2e_1 \end{pmatrix}$; and $h_{21}(x,e_\alpha,e_\beta)=\begin{pmatrix} -e_3x_1 \\ e_2x_1 \end{pmatrix}$. Clearly $\lim\limits_{e_\alpha\to 0} h_{21}(x,e_\alpha,e_\beta)=0$.

According to Theorem 4.1, the controller $\phi(x,y)$ is now defined as

$$\phi(x,y)=\begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix}=\begin{pmatrix} Q_\alpha e_\alpha-h_1(x,e_\alpha,e_\beta) \\ Q_\beta e_\beta-h_{22}(x,e_\alpha,e_\beta) \end{pmatrix}. \quad (21)$$

So, from equations (20) and (21) error dynamics can be rewritten as:

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q}&=(B_\alpha+Q_\alpha)e_\alpha, \\ \frac{d^q e_\beta}{dt^q}&=(B_\beta+Q_\beta)e_\beta+h_{21}(x,e_\alpha,e_\beta). \end{aligned} \quad (22)$$

Therefore, choose suitable matrices $Q_\alpha\in R^1$ and $Q_\beta\in R^{2\times 2}$ such that all the eigenvalues of $(B_\alpha+Q_\alpha)$ satisfy $\mid arg\lambda_i\mid>\frac{q\pi}{2}(i=1)$ and all the eigenvalues of $(B_\beta+Q_\beta)$ satisfy $\mid arg\lambda_i\mid>\frac{q\pi}{2}\ (i=1,2)$.

Since equation (22) is asymptotically stable with equilibrium points $e_\alpha=0, e_\beta=0$. Obviously $\lim\limits_{e_\alpha\to 0} h_{21}(x,e_\alpha,e_\beta)=0$. This implies that the HPS between drive system and response system can be achieved.

5.2 Numerical simulations

Parameters of the fractional order Lorenz system are $(\sigma,\beta,\gamma)=(10,8/3,28)$ and fractional order is taken to be $q=1.17$, for which the system displays a chaotic behaviour. In equation (22), we choose $Q_\alpha=8$ and $Q_\beta=\begin{pmatrix} -1 & 0 \\ 0 & 24 \end{pmatrix}$, which gives that the stability condition of the above Theorem 4.1 is satisfied, as eigenvalue of $(B_\alpha+Q_\alpha)$ is -2 and eigenvalues of $(B_\beta+Q_\beta)$ are -2 and -4 and for all eigenvalues condition of Theorem 4.1 has been satisfied as $\mid arg\lambda_i\mid\geq\frac{q\pi}{2}$, where $q=1.17$. The initial conditions for the master and slave systems are $(x_1(0),x_2(0),x_3(0))=(7,9,6)$ and $(y_1(0),y_2(0),y_3(0))=(6,8,5)$, respectively and $A=\begin{pmatrix} 1 & 0 & 0.83 \\ 1 & 0 & -0.03 \\ 1 & -1 & 0.16 \end{pmatrix}$. Then for $(e_1(0), e_2(0), e_3(0)) = (0.50, -0.80, -1)$ and $T_{sim} = 20$, diagram of convergence of errors (Figures 9-11) is the witness for achieving hybrid projective synchronization between the drive and response systems.

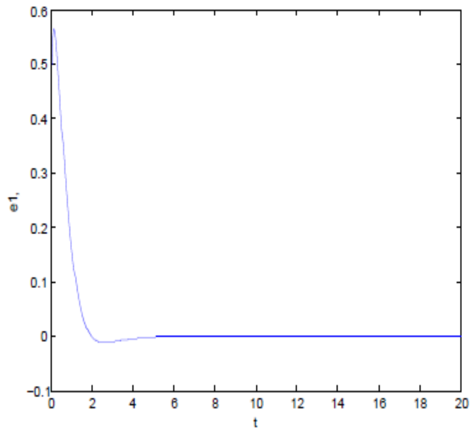

Figure 8: The synchronization error signal $e_1(t)$.

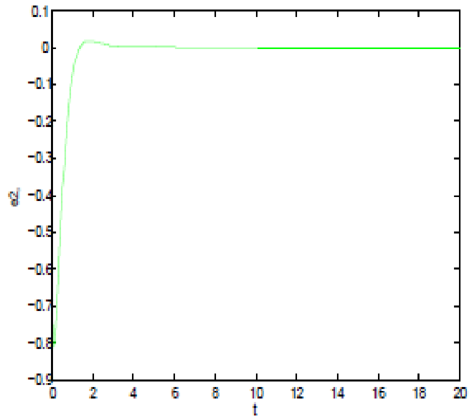

Figure 9: The synchronization error signal $e_2(t)$.

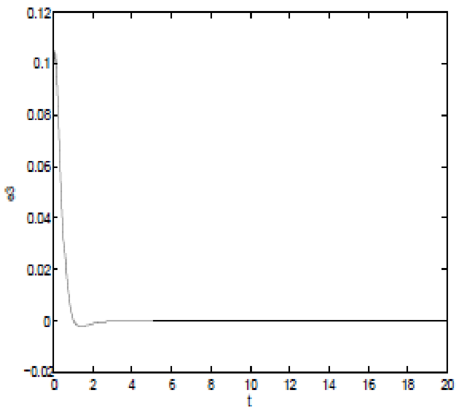

Figure 10: The synchronization error signal $e_3(t)$.

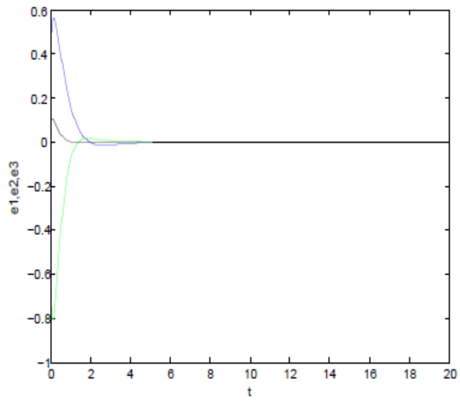

Figure 11: Error convergence diagram for HPS.

5.3 HPS for fractional order Lu chaotic system with with fractional order $1 < q < 2$

The fractional order Lu system is a system of three fractional order differential equation exhibiting chaotic behaviour for certain values of parameters. The equation of the system is:

$$\begin{aligned} \frac{d^q x_1}{dt^q} &= 36(x_2 - x_1), \\ \frac{d^q x_2}{dt^q} &= 20x_2 - x_1 x_3, \\ \frac{d^q x_3}{dt^q} &= x_1 x_2 - 3x_3. \end{aligned} \tag{23}$$

The chaotic attractor of fractional order Lu system for different values of q, $1 < q < 2$ is depicted in Figures 12-17.

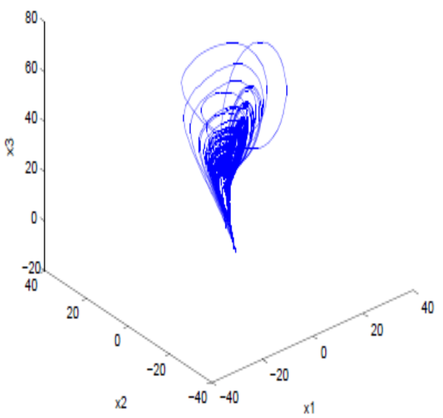

Figure 12: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.11$.

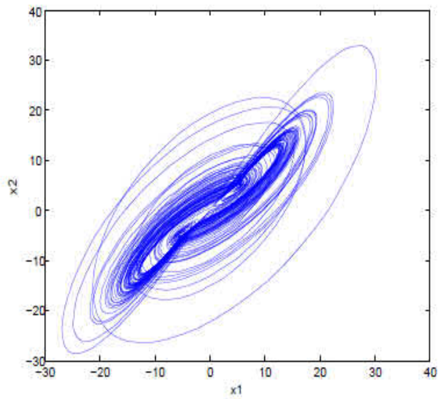

Figure 13: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.11$.

According to the HPS scheme presented in the above section, the response system is described by

$$\begin{pmatrix} \frac{d^q y_1}{dt^q} \\ \frac{d^q y_2}{dt^q} \\ \frac{d^q y_3}{dt^q} \end{pmatrix} = A^{-1} \begin{pmatrix} 36(\sum_{j=1}^{3} a_{2j} y_j - \sum_{j=1}^{3} a_{1j} y_j) \\ 20(\sum_{j=1}^{3} a_{2j} y_j) - \sum_{j=1}^{3} a_{1j} y_j \sum_{j=1}^{3} a_{3j} y_j \\ \sum_{j=1}^{3} a_{1j} y_j \sum_{j=1}^{3} a_{2j} y_j - 3 \sum_{j=1}^{3} a_{3j} y_j \end{pmatrix} + A^{-1} \phi(x, y), \tag{24}$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a reversible matrix and A^{-1} is its reverse matrix.

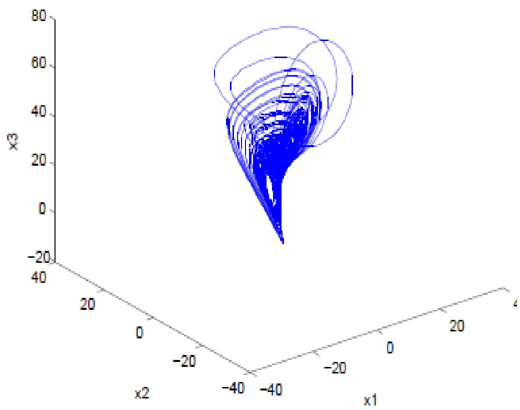

Figure 14: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.12$.

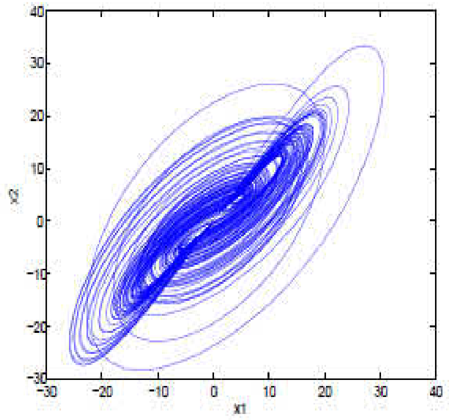

Figure 15: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.12$.

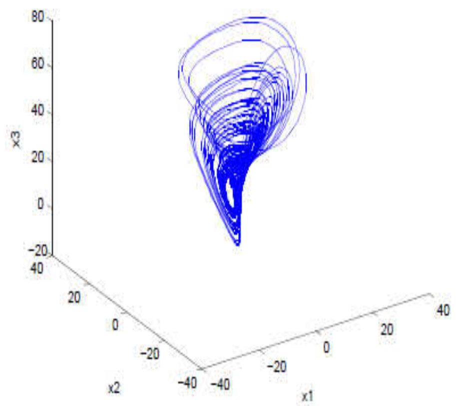

Figure 16: 3D chaotic attractor of the Lu system with $q_1 = q_2 = q_3 = 1.13$.

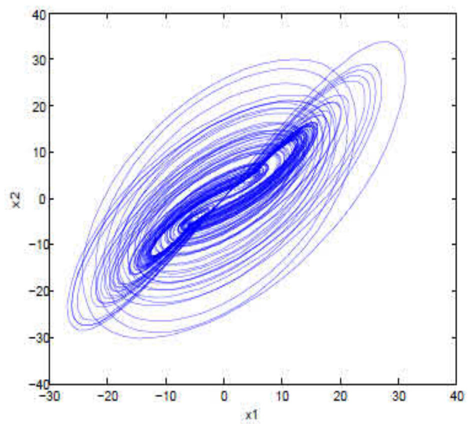

Figure 17: 2D projection of the Lu system with $q_1 = q_2 = q_3 = 1.13$.

Now, according to definition of HPS error dynamics, we have

$$\begin{aligned}\frac{d^q e}{dt^q} &= A\frac{d^q y}{dt^q} - \frac{d^q x}{dt^q} \\ &= f(Ay) - f(x) + \phi(x,y). \end{aligned} \tag{25}$$

Let

$$f(Ay) - f(x) = E(x,e). \tag{26}$$

Therefore, (25) implies that

$$\frac{d^q e}{dt^q} = E(x,e) + \phi(x,y). \tag{27}$$

Our goal is to find $E(x,e)$ and design a controller to achieve HPS. Equation (10), gives

$$E(x,e) = \begin{pmatrix} 36(\sum\limits_{j=1}^{3} a_{2j}y_j - \sum\limits_{j=1}^{3} a_{1j}y_j) \\ 20(\sum\limits_{j=1}^{3} a_{2j}y_j) - \sum\limits_{j=1}^{3} a_{1j}y_j \sum\limits_{j=1}^{3} a_{3j}y_j \\ \sum\limits_{j=1}^{3} a_{1j}y_j \sum\limits_{j=1}^{3} a_{2j}y_j - 3\sum\limits_{j=1}^{3} a_{3j}y_j \end{pmatrix} - \begin{pmatrix} 36(x_2 - x_1) \\ 20x_1 - x_2 - x_1x_3 \\ x_1x_2 - 3x_3 \end{pmatrix} \tag{28}$$

which gives

$$E(x,e) = \begin{pmatrix} 36e_2 - 36e_1 \\ 20e_2 - e_1x_3 - e_3x_1 - e_1e_3 \\ e_1e_2 + x_1e_2 + x_2e_1 - 3e_3 \end{pmatrix}. \tag{29}$$

We choose

$$e_\alpha = e_1,\ e_\beta = (e_2, e_3)^T;\ B_\alpha = -36;\ h_1(x, e_\alpha,\ e_\beta) = 36e_2,$$

$$B_\beta = \begin{pmatrix} 20 & 0 \\ 0 & -3 \end{pmatrix};\ h_{21}(x, e_\alpha,\ e_\beta) = \begin{pmatrix} -e_1x_3 - e_1e_3 \\ e_1e_2 + x_2e_1 \end{pmatrix};\ h_{22}(x, e_\alpha, e_\beta) = \begin{pmatrix} -e_3x_1 \\ e_2x_1 \end{pmatrix}.$$

Clearly, $\lim\limits_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$.

According to Theorem 4.1, the controller $\phi(x,y)$ is now defined as

$$\phi(x,y) = \begin{pmatrix} \mu_\alpha(x,e) \\ \mu_\beta(x,e) \end{pmatrix} = \begin{pmatrix} Q_\alpha e_\alpha - h_1(x, e_\alpha, e_\beta) \\ Q_\beta e_\beta - h_{22}(x, e_\alpha, e_\beta) \end{pmatrix}. \tag{30}$$

So from equation (29) and (30) error dynamical system can be rewritten as:

$$\begin{aligned} \frac{d^q e_\alpha}{dt^q} &= (B_\alpha + Q_\alpha)e_\alpha, \\ \frac{d^q e_\beta}{dt^q} &= (B_\beta + Q_\beta)e_\beta + h_{21}(x, e_\alpha, e_\beta). \end{aligned} \tag{31}$$

Therefore, choose suitable matrices $Q_\alpha \in R^1$ and $Q_\beta \in R^{2\times 2}$ such that all the eigenvalues of $(B_\alpha + Q_\alpha)$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2}$ $(i = 1)$ and all the eigenvalues of $(B_\beta + Q_\beta)$ satisfy $\mid arg\lambda_i \mid > \frac{q\pi}{2} (i = 1,2)$

The equilibrium points $e_\alpha = 0, e_\beta = 0$ of system (31) is asymptotically stable.

Obviously, $\lim\limits_{e_\alpha \to 0} h_{21}(x, e_\alpha, e_\beta) = 0$. This implies that the HPS between drive system and response system can be achieved.

5.4 Numerical Simulations

Parameters of the fractional order Lu system are $(a,b,c) = (36,3,20)$, and fractional order is taken $q = 1.13$ for which the system displays a chaotic behaviour. In equation (31), we choose

$$Q_\alpha = 34, \quad Q_\beta = \begin{pmatrix} -23 & 0 \\ 0 & -2 \end{pmatrix}.$$

This implies that the stability conditions of Theorem 4.1 are satisfied, as eigenvalue of $(B_\alpha + Q_\alpha)$ is -2 and eigenvalues of $(B_\beta + Q_\beta)$ are -3 and -5, and for all eigenvalues condition of Theorem 4.1 are satisfied as $\mid arg\lambda_i \mid \geq \frac{q\pi}{2}$, where $q = 1.13$. The initial conditions for

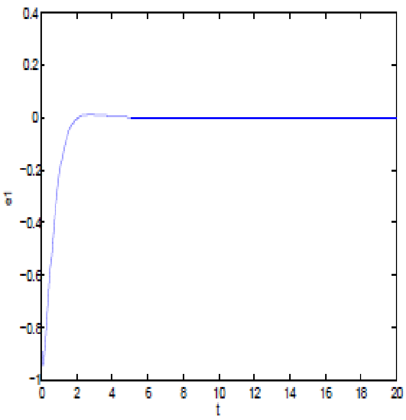

Figure 18: The synchronization error signal $e_1(t)$.

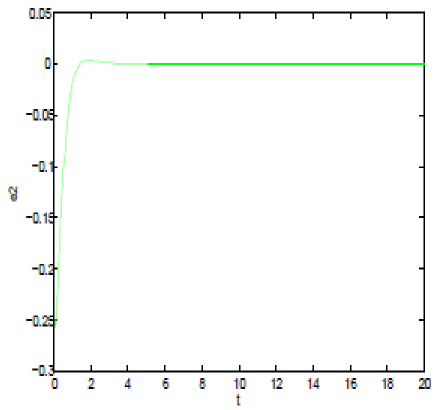

Figure 19: The synchronization error signal $e_2(t)$.

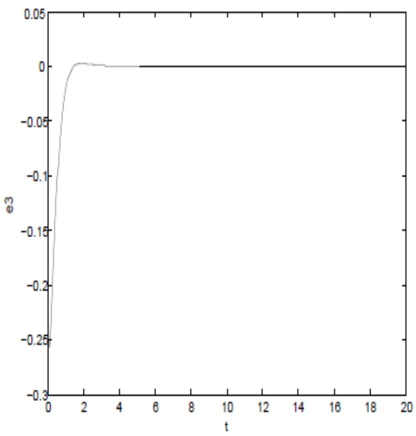

Figure 20: The synchronization error signal $e_3(t)$.

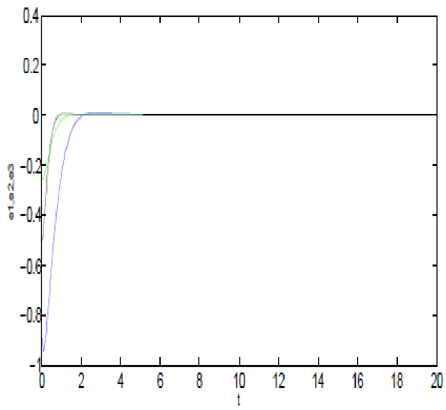

Figure 21: Error convergence diagram for HPS.

the master and slave systems are $(x_1(0), x_2(0), x_3(0)) = (2,3,6)$, $(y_1(0), y_2(0), y_3(0)) = (4,5,8)$, respectively, and

$$A = \begin{pmatrix} 1 & 0 & -0.18 \\ 0 & -0.42 & 1 \\ 1 & 1.83 & 0 \end{pmatrix}.$$

Then for $(e_1(0), e_2(0), e_3(0)) = (-0.89, -0.25, -0.50)$ and $T_{sim} = 20$, diagram of convergence of errors (Figures 18-21) is the witness of achieving hybrid projective synchronization between the drive and response systems.

6 Conclusion

In this paper, we have investigated a new synchronization scheme to achieve hybrid projective synchronization for two identical fractional order chaotic systems with fractional order q such that $1 < q < 2$ via tracking control method and stability of fractional order system. Hybrid projective synchronization (HPS) is a more general definition of projective synchronization, in which the drive system and response system could be synchronized up to a vector function factor. HPS is different from the PS and more beneficial to enhance security of communication than any other synchronization because it is obvious that the unpredictability of the vector function factor in HPS is more than that of the same scaling factor in PS. The numerical simulations exhibit the validity and feasibility of the proposed scheme. Numerical and computational results are in excellent agreement.

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Existence and Uniqueness Results by Progressive Contractions for Integro-Differential Equations

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Abstract: In this brief note we present a simple proof of global existence and uniqueness of a solution of an integro-differential equation

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds,$$

where f and g satisfy a Lipschitz condition with constant $K = K(t)$ where $K(t)$ is allowed to tend to infinity with t. The proof employs the idea of progressive contractions. It is a general fixed point theorem for differential equations.

Keywords: *fixed points; existence; uniqueness; progressive contractions; integro-differential equations.*

Mathematics Subject Classification (2010): 45J05, 37C25, 47H09.

1 Introduction

This is the third in a series of very short notes which we are constructing to illustrate the power, flexibility, and simplicity of a technique which we call *progressive contractions* to obtain a unique global solution of various kinds of differential and integral equations. We have applied the method to integral equations [4], fractional differential equations [6] of the type considered in [2], and integral equations of the Krasnoselskii type featuring a sum of two operators [5]. Each of the problems is of an essentially different type and the title of each note is chosen to allow interested readers to detect which subject is being treated.

In most of the existing literature investigators prove existence and uniqueness of solutions of differential equations by writing them as integral equations and applying

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some type of fixed point theorem which can be tedious and challenging, often patching together solutions on short intervals after making complicated translations. Here, we make three simple short steps, two of which are actually the same. Moreover, we treat the equation directly without changing into an integral equation and we use a method which we introduced earlier and called *direct fixed point mappings*. Each of the three steps is an elementary contraction mapping on a short interval.

Examples of direct fixed point mappings can be seen in [1, 3, 7, 8]. In each case there are excellent reasons for not first converting to an integral equation. In this note there are two reasons. First, while one can prove that there is an inversion because of the fundamental properties of contractions, we see no way to actually achieve it in a workable form. The second reason is accidental. We had begun by asking a contraction condition on g which had been necessary in earlier work with integral equations, but noticed that the integral in the mapping allowed us to ask only a Lipschitz condition. The result is still true when f is identically zero and that means there is a simple proof of global existence in case of an ordinary differential equation with only a (possibly growing) Lipschitz condition.

The equation we treat is the scalar equation

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds, \quad ' = \frac{d}{dt}, \quad x(0) = a \in \Re, \tag{D}$$

although a vector system is handled in the same way. In that case, x, g, f are vectors and A is an $n \times n$ matrix. As we are obtaining solutions on $[0, \infty)$ and asking no sign conditions, it is clear that we will need some growth restrictions. As we are asking for uniqueness it is also clear that we will need something of a Lipschitz condition. In fact, we will ask for a Lipschitz condition on f and g, but the Lipschitz "constant" can grow to infinity as t tends to infinity.

In order to obtain an integral equation for mapping, we write the direct fixed point equation as

$$\xi(t) = g\left(t, a + \int_0^t \xi(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \xi(u)du\right)ds \tag{1.1}$$

so that if we obtain a continuous solution of (1.1), then

$$x(t) = a + \int_0^t \xi(s)ds$$

will be a continuously differentiable solution of the original equation (D).

Specifically, we ask that

$$f, g : [0, \infty) \times \Re \to \Re \text{ are continuous,} \tag{1.2}$$

and for each $E > 0$ there is a $K = K(E) > 0$ such that

$$0 \le t \le E, \quad x, y \in \Re \implies |g(t, x) - g(t, y)| \le K|x - y|, \tag{1.3}$$

$$0 \le t \le E, \quad x, y \in \Re \implies |f(t, x) - f(t, y)| \le K|x - y|. \tag{1.4}$$

Finally, we ask that

$$A : (0, \infty) \to \Re \quad \text{be continuous,} \tag{1.5}$$

that if $\phi : [0,\infty) \to \Re$ is continuous then

$$\int_0^t A(t-s)\phi(s)ds \text{ be continuous,} \tag{1.6}$$

and that

$$\int_0^t |A(s)|ds \text{ be continuous and converge to zero as } t \downarrow 0. \tag{1.7}$$

For the E and K pick $\alpha \in (0,1)$ and then choose a positive $T^* < 1$ with $KT^* < \alpha$. Finally, select $T = T(K, T^*) > 0$ with $T < T^* < 1$ so that, collecting:

$$K\int_0^T |A(s)|ds < \frac{1-\alpha}{2}, \quad T^*K < \alpha, \quad 0 < T < T^* < 1. \tag{1.8}$$

We begin with a solution to (1.1) on $[0, E]$ and parlay it to $[0,\infty)$.

2 Existence and Uniqueness

Theorem 2.1 *If conditions (1.2) –(1.8) hold then for each* $E > 0$ *and each* $a \in \Re$ *there is a unique solution* $\xi(t)$ *of (1.1) on* $[0, E]$.

Proof. For the given $E > 0$ find $K > 0$ satisfying (1.3) and(1.4), while T satisfies (1.8) with

$$0 < T < T^* < 1, \quad KT^* < \alpha < 1. \tag{2.1}$$

Divide $[0, E]$ into n pieces of length $S < T$ and with end points $0 = T_0, T_1, ..., T_n = E$ so that

$$S = T_i - T_{i-1} < T < 1. \tag{2.2}$$

We will take two steps leading to an induction which generalizes the second step. The first step takes place in a Banach space, but the subsequent step is in a complete metric space.

Step 1. Let $(\mathcal{M}_1, |\cdot|_1)$ be the Banach space of continuous functions $\phi : [0, T_1] \to \Re$ with the supremum norm. Define $P_1 : \mathcal{M}_1 \to \mathcal{M}_1$ by $\phi \in \mathcal{M}_1$ which implies that

$$(P_1\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right) + \int_0^t A(t-s)f\left(s, a + \int_0^s \phi(u)du\right)ds. \tag{2.3}$$

Notice that if P_1 has a fixed point ξ_1, then

$$\frac{d}{dt}\left[a + \int_0^t \xi_1(u)du\right] = \xi_1(t)$$

and

$$x(t) = a + \int_0^t \xi_1(s)ds$$

satisfies (D) with $x(0) = a$.

Let us see that we have a contraction. If $\phi, \psi \in \mathcal{M}_1$ then by (1.8)

$$\int_0^t |\phi(s) - \psi(s)|ds \le T^*|\phi - \psi|_1 \le |\phi - \psi|_1, \quad KT^* < \alpha$$

so

$$
\begin{aligned}
|(P_1\phi)(t) - (P_1\psi)(t)| &\le K\left|a + \int_0^t \phi(s)ds - a - \int_0^t \psi(s)ds\right| \\
&+ \int_0^t |A(t-s)|K \int_0^s |\phi(u) - \psi(u)|duds \\
&\le \alpha|\phi - \psi|_1 + |\phi - \psi|_1 K \int_0^t |A(s)|ds \\
&\le |\phi - \psi|_1 \left[\alpha + \frac{1-\alpha}{2}\right] = \frac{1+\alpha}{2}|\phi - \psi|_1,
\end{aligned}
$$

a contraction with unique fixed point ξ_1 solving (2.3) on $[0, T_1]$.

Step 2. Let $(\mathcal{M}_2, |\cdot|_2)$ be the *complete metric space* of continuous functions $\phi : [T_0, T_2] \to \Re$ with the supremum metric and $\phi(t) = \xi_1(t)$ for $T_0 \le t \le T_1$. Define $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ which implies

$$
(P_2\phi)(t) = g\left(t, a + \int_0^t \phi(s)ds\right) + \int_0^t A(t-s) f\left(s, a + \int_0^s \phi(u)du\right)ds. \tag{2.4}
$$

As ξ_1 is a fixed point of P_1 on $[T_0, T_1]$ for $0 \le t \le T_1$ we have for any $\phi \in M_2$ that

$$
\begin{aligned}
(P_2\phi)(t) &= g\left(t, a + \int_0^t \xi_1(s)ds\right) + \int_0^t A(t-s) f\left(s, a + \int_0^s \xi_1(u)du\right)ds \\
&= \xi_1(t)
\end{aligned} \tag{2.5}
$$

and so P_2 does map $\mathcal{M}_2 \to \mathcal{M}_2$.

Let us see that P_2 is a contraction. If $\phi, \psi \in \mathcal{M}_2$ then

$$
\begin{aligned}
|(P_2\phi)(t) - (P_2\psi)(t)| &\le K\left|\int_0^t [\phi(s) - \psi(s)]ds\right| \\
&+ \int_0^t |A(t-s)|K\left|\int_0^s [\phi(u) - \psi(u)]du\right|ds.
\end{aligned}
$$

Let $T_1 \le t \le T_2$ and fix s at any value $0 \le s \le T_1$. Then examine the last integral above. As $s \le T_1$, then $0 \le u \le T_1$ and so $\phi(u) = \psi(u)$ and that last integral is zero. This is true for every value of s with $0 \le s \le T_1$. If $|\phi|^{[T_1,T_2]}$ denotes the sup then as $S = T_2 - T_1 < T^*$

$$
\int_{T_1}^{T_2} |\phi(s) - \psi(s)|ds \le T^*|\phi - \psi|^{[T_1,T_2]} \le |\phi - \psi|^{[T_1,T_2]} = |\phi - \psi|_2. \tag{2.6}
$$

Hence we may continue the above display as

$$
\begin{aligned}
&= K\left|\int_{T_1}^t [\phi(s) - \psi(s)]ds\right| + \int_{T_1}^t |A(t-s)|K \int_{T_1}^s |\phi(u) - \psi(u)|duds \\
&\le KT^*|\phi - \psi|^{[T_1,T_2]} + \int_{T_1}^t |A(t-s)|K|\phi - \psi|^{[T_1,T_2]}ds \\
&(\text{ by a change of variable and } |\phi - \psi|_2 = |\phi - \psi|^{[T_1,T_2]}) \\
&\le |\phi - \psi|_2\left[\alpha + \frac{1-\alpha}{2}\right] = \frac{1+\alpha}{2}|\phi - \psi|_2
\end{aligned}
$$

a contraction with unique fixed point ξ_2 on $[0,T_2]$. Note that $\xi_1=\xi_2$ on $[0,T_1]$ because both are unique and the definition of the space demands it.

This Step 2 is the first step in the induction since it has the first complete metric space with the function ξ_1. We pattern the induction on $\mathcal{M}_2$ which uses ξ_1 from Step 1, the mapping P_2 which truncates the integrals using the ξ_1, and the fixed point ξ_2 which is the final product of Step 2 and upon which Step 3 relies.

Inductive hypothesis. Assume that we have a solution $\xi_{i-1}(t)$ satisfying (1.1) for $0\le t\le T_{i-1}$.

From this and the assumptions (1.2)–(1.8) we will obtain a solution $\xi_i(t)$ satisfying (1.1) for $0\le t\le T_i$. That will complete the induction for we can then reach E with the solution ξ_n satisfying (1.1) on $[0,E]$. The proof will then be complete.

Let ξ_{i-1} satisfy (1.1) on $[0,T_{i-1}]$ for $i-1\ge 1$. Let $(\mathcal{M}_i,|\cdot|_i)$ be the complete metric space of continuous functions $\phi:[0,T_i]\to\Re$ with the supremum metric and for $0\le t\le T_{i-1}$ every function satisfies $\phi(t)=\xi_{i-1}(t)$. Next, we define $P_i:\mathcal{M}_i\to\mathcal{M}_i$ by $\phi\in\mathcal{M}_i$ which implies that

$$(P_i\phi)(t)=g\left(t,a+\int_0^t\phi(s)ds\right)+\int_0^t A(t-s)f\left(s,a+\int_0^s\phi(u)du\right)ds.$$

Because ξ_{i-1} is a solution on $[0,T_{i-1}]$ if $0\le t\le T_{i-1}$ then $(P_i\xi_{i-1})(t)=\xi_{i-1}(t)$ and so the mapping is into $\mathcal{M}_i$.

We now show that P_i is a contraction. If $\phi,\psi\in\mathcal{M}_i$ then

$$\begin{aligned}
&|(P_i\phi)(t)-(P_i\psi)(t)|\le K\left|\int_0^t[\phi(s)-\psi(s)]ds\right|\\
&+\int_0^t|A(t-s)|K\left|\int_0^s[\phi(u)-\psi(u)]du\right|ds\\
&\text{(as in Step 2 at this same point in the display and now } T_{i-1}\le t\le T_i\text{)}\\
&=K\left|\int_{T_{i-1}}^t[\phi(s)-\psi(s)]ds\right|+\int_{T_{i-1}}^t|A(t-s)|K\int_{T_{i-1}}^s|\phi(u)-\psi(u)|duds\\
&\le KT^*|\phi-\psi|^{[T_{i-1},T_i]}+\int_{T_{i-1}}^t|A(t-s)|K|\phi-\psi|^{[T_{i-1},T_i]}ds\\
&\text{(by a change of variable and } |\phi-\psi|_i=|\phi-\psi|^{[T_{i-1},T_i]})\\
&\le|\phi-\psi|_i\left[\alpha+\frac{1-\alpha}{2}\right]=\frac{1+\alpha}{2}|\phi-\psi|_i,
\end{aligned}$$

a contraction with unique fixed point ξ_i on $[0,T_i]$. Note that $\xi_{i-1}=\xi_i$ on $[0,T_{i-1}]$ because both are unique and the definition of the space demands it. □

Theorem 2.2 *Under the conditions of Theorem 2.1 there is a unique solution ξ of (1.1) on* $[0,\infty)$.

Proof. Using Theorem 2.1 we construct a unique solution ξ_n on every interval $[0,n]$ for every positive integer n. Extend each of those solutions to the interval $[0,\infty)$ by defining ξ_n past n by the function $\xi_n^*=\xi_n(n)$ for $t>n$. Thus we have a sequence of uniformly continuous functions on $[0,\infty)$ which converge uniformly on compact sets to a continuous function ξ which is a solution of (1.1) because at every value of t the function on $[0,t]$ coincides with any ξ_n for $n>t$. □

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New State Space Modelling Approach and Unknown Input Observer Design for the Assessment of Temperature Polarization Phenomenon in Direct Contact Membrane Distillation

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Abstract: The objective of this paper is three fold. Firstly, a new modeling approach for direct contact membrane distillation (DCMD) is developed. Based on dynamic bi-dimensional configuration, an uncertain non linear state space model that takes into account all the uncertainties generated by discretization errors and plant parameters variation is derived. It is worth noticing that most of the MD configuration processes have been modeled as steady-state one-dimensional systems. Stationary two-dimensional MD models have been considered only in very few studies. The obtained bi-dimensional state space model of DCMD process is also implemented using Matlab and compared with data published in the literature. Secondly, it is theoretically demonstrated that, by measuring only the inlet and outlet temperatures of the DCMD process, one can recover the temperature profile inside the DCMD process using observers. This is an important point, since most of the existing literatures compute the temperature profile by empirical methods without taking into account disceretization errors and uncertainties. Thirdly, a new unknown input observer is developed to estimate temperature polarization inside the membrane. The convergence of the temperature estimation error to zero is theoretically proved and verified by simulation. Of particular interest, the designed observer can be used for the assessment of temperature polarization phenomena and hence preventing some fouling problems.

Keywords: *direct contact membrane distillation; dynamic modeling; heat and mass transfer; unknown input observer; polarization coefficient; fouling.*

Mathematics Subject Classification (2010): 35Q35, 35Q79, 93A30, 93B07, 93C10, 93C20, 93D30.

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1 Introduction

Membrane distillation (MD) process is an emerging technology for water treatment. The driving force of the MD process results from the pressure difference of vapor formed by a difference in solution's temperature on both sides of a hydrophobic membrane [1]. The advantages of DCMD lie in its simplicity, the need of only small temperature differences and nearly 100% rejection of dissolved solids [1]. Furthermore, thanks to their low energy demand, DCMD processes can be equipped with renewable energy equipment such as solar collectors [2] and solar distillers [3].

Most of researches on DCMD focus on modeling the heat and mass transfer phenomenon inside the membrane, and most of the MD configuration processes have been modeled as steady-state one-dimensional systems using empirical heat and mass transfer equations [4]. Only few publications use stationary one or two-dimensional heat-transfer equation to simulate a particular configuration more accurately. Although many semi-empirical models have been developed, a detailed model for temperature polarization on flat-plate MD processes is still lacking. [5]. In [4] theoretical modeling and experimental analysis of direct contact membrane distillation has been done in steady-state. In [6] a dynamic modeling of direct contact membrane distillation processes has been presented. In [7] performance investigation of a solar-assisted direct contact membrane distillation system is conducted.

This paper presents a different approach using a new bi-dimensional dynamic model to predict the membrane temperature and the pure water flux. It proposes to derive an uncertain state model based on the finite element approximation of the temperature partial differential equations (PDE) and then to build an observer to estimate all temperatures and temperature dependant parameters inside the process from the only measurable data which are inlet and outlet temperatures.

Because temperature distribution inside the membrane is not accessible for measurement this observer is very useful and can be considered as a software sensor to estimate it. The observer developed in this paper is designed in a cascade structure and is specific to the presented DCMD model. It is useful as a means to monitor inner temperature evolution in order to prevent and avoid severe or irreversible fouling situations by predicting their occurrence with a good timing and launching the predefined appropriate maintenance routine [8].

The paper is organized as follows: in the next section, the theoretical equations describing heat and mass transfer in DCMD are introduced and followed by a brief description of fouling phenomenon and its effect on polarization coefficient. In Section 3, a new bi-dimensional state model for DCMD process is developed and simulated. After that, the observability of the whole set of internal dynamic variables is demonstrated, and the new unknown input observer that predicts inner temperature profiles is presented in Section 4. Simulations are conducted to show the efficiency of the proposed observer.

2 DCMD Theoretical Modeling

In Direct Contact Membrane Distillation (DCMD) both sides of the membrane are in direct contact with a liquid stream. On the upper side of the membrane shown in Figure 1 the hot liquid (i.e. hot seawater) flows in the evaporator channel, whilst on the bottom side, a cold liquid (i.e. cooled permeate or distillate) is circulated. Heat and mass transfer occurs from the hotter to the colder side. The liquid in the evaporator channel is constantly refilled and reheated, whilst the volume of the liquid in the permeate channel increases and heats up. One of the main features of DCMD is that the gas gap between the membrane surface and the condensate stream is very narrow and only exists due to the hydrophobic nature of the membrane. This causes the temperature of the membrane surface in contact with the condensate to be very close to that of the condensate stream itself, thus allowing high temperature drops across the membrane, i.e. high driving forces for mass transfer. Conversely, the direct contact configuration causes a relatively high heat loss as the membrane is the only barrier for the transfer of sensible heat [2].

Mathematical equations describing those phenomena are given in the following paragraphs.

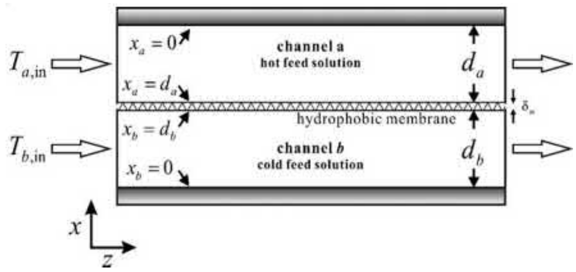

Figure 1: Schematic diagram of DC membrane distillation process [4].

2.1 Mass transfer

The mass transfer driving force across the membrane is the difference in saturated pressure components on both membrane surfaces due to the temperature gradient. The general mass flux form can be expressed as follows:

$$J = c_m \Delta P^{sat} = c_m \left(P_a^{sat} - P_b^{sat}\right), \tag{1}$$

where P_a^{sat}, P_b^{sat} are the saturated pressure of water on the hot and the cold feed membrane surfaces respectively and c_m is the membrane coefficient.

For non-ideal binary mixtures [9], [10], the flux can be determined by:

$$J = c_m \left[(1 - x_{NaCl}) \left(1 - 0.5x_{NaCl} - 10x_{NaCl}^2\right) P_a^{sat} - P_b^{sat}\right], \tag{2}$$

where x_{NaCl} is the mole fraction of $NaCl$ in saline solution.

In the following, the index "s" stands for "side". I.e. $s = a$ for the hot side and $s = b$ for the cold side.

Saturated pressures can be determined by the Antoine equation where T_s is the temperature in $^\circ C$, $s = a, b$:

$$P_s^{sat} = 133.32 \times 10^{\left(8.10765 - \left(\frac{1450.286}{T_s + 235}\right)\right)}. \tag{3}$$

The membrane coefficient c_m in (1) can be estimated by a weighted sum via parameters $\alpha(T)$ and $\beta(T)$ of the Knudsen diffusion and the Poiseuille (viscous) flow models [11]:

$$c_m = c_k + c_p,$$

$$c_m = 1.064\ \alpha\,(T)\,\frac{\varepsilon r}{\tau\delta_m}\sqrt{\frac{M_w}{RT_m}} + 0.125\beta\,(T)\,\frac{\varepsilon r^2}{\tau\delta_m}\ \frac{M_w P_m}{\eta_v R T_m}\ , \tag{4}$$

where $\alpha\,(T)$ and $\beta\,(T)$ are the Knudsen diffusion model and Poiseuille flow model contributions, respectively, M_w is the molecular weight of water, P_m is the mean saturated pressure in membrane, R is the gas constant, r is the pore radius, T_m is the mean temperature in membrane, δ_m is the thickness of membrane, ε is the porosity of membrane, η_v is the gas viscosity and τ is the tortuosity factor. The tortuosity of a porous hydrophobic membrane was estimated by [12].

2.2 Heat transfer

For a laminar and symmetrical flow, symmetrical temperature distribution and without internal energy generation; the temperature propagation in DCMD process is described by the following equation [10]:

$$\rho C_p\left(\frac{\partial T}{\partial t} + \underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial z}}_{convection}\right) = k\left(\underbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}}_{conduction}\right). \tag{5}$$

Considering that conduction effect is along x axis and that convection effect is along z axis, we obtain the basic equation used in DCMD modeling [4]:

$$\rho C_p\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z}\right) = k\frac{\partial^2 T}{\partial x^2}\ . \tag{6}$$

Velocity along z axis is given by

$$v(x) = 6\overline{v}_s\left(\frac{x}{d_s} - \frac{x^2}{d_s^2}\right), \tag{7}$$

where $\overline{v}_s = \overline{v} = \frac{Q}{d_s W}$ is the mean velocity, Q the volumetric flow, W is the channel width and d_s is its height. Here $d_a = d_b = d$.

We rewrite (5) as follows:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial x^2} - v\frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial x^2} - v\frac{\partial T}{\partial z}\ , \tag{8}$$

$$\alpha = \frac{k}{\rho C_p}\ . \tag{9}$$

"α" or convective heat transfer coefficient is a time/temperature varying parameter [13] since it depends on thermal conductivity (k), specific heat (C_p) and the density of the seawater (ρ). One can consider variation of α using empirical relations found in specific literature such as those proposed in [4].

2.3 Boundary conditions

The boundary conditions for modelling the DCMD process are given in [4]:

$$\begin{cases} T_s\left(x,0\right) = T_{s,in}, \\ \frac{dT_s(0,z)}{dx} = 0, \\ k_a \frac{\partial T_a(d,z)}{\partial x} = -\left[\lambda J + \frac{k_m}{\delta_m}\left(T_a\left(d,z\right) - T_b\left(d,z\right)\right)\right], \\ k_b \frac{\partial T_b(d,z)}{\partial x} = \left[\lambda J + \frac{k_m}{\delta_m}\left(T_a\left(d,z\right) - T_b\left(d,z\right)\right)\right]. \end{cases} \tag{10}$$

2.4 Fouling and polarization coefficient in DCMD

2.4.1 Fouling

Fouling in general is the accumulation of unwanted deposits (foulants) on the surface or inside the pores of the membrane that degrade its permeation flux and salt rejection performances (see [8] and references therein such as [14] and [15]). It is one of the major problems in membrane-based processes that reduce the temperature difference across the membrane or increase in temperature polarization leading to lesser driving force [16] (Figure 2).

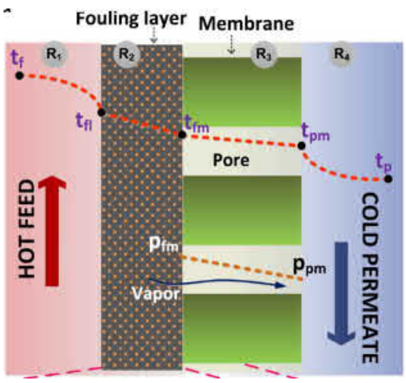

Figure 2: Fouling layer on membrane [8].

The foulants found in membrane technology can be divided into three broad groups according to the fouling material [17]. (a) Inorganic fouling or the deposition of inorganic particles such as calcium carbonate, calcium sulfate, NaCl, ferric oxide, aluminum oxide, etc; (b) organic fouling or the deposition of organic matters such as humic acid, fulvic acid, protein, polysaccharides, and polyacrylic polymers and (c) biological fouling caused by microorganisms such as bacteria and fungi, sludge, algae, yeast, etc. In most cases, a single fouling mechanism does not occur in real MD processes, but a combination of different fouling materials and mechanisms that makes it more complicated to deal with.

Fouling occurs as an external surface fouling referring to the build-up of deposits or gel-like layers on the outer surface of the feed-side of the membrane. Two types of fouling layers are observed [18] both of which decrease the permeate flux: the porous that provides additional heat resistance, thus decreasing the permeate flux and the non-porous deposit layers which reduce the transport of water vapor across the membrane. It also occurs as pore blocking fouling which happens when scales or foulants are formed inside the pores of the membrane causing a partial blocking or gradual narrowing of the pore, or a complete pore blocking (Figure 3) [19].

Figure 3: Surface (external) and pore-blocking (internal) fouling [8].

External surface fouling is usually reversible and can be eliminated by chemical cleaning, while internal fouling or pore blocking is in most cases, irreversible leading to damage of the membrane due to compaction of foulants [20].

Fouling is affected by different factors such as [21] (a) foulant characteristics (concentration, molecular size, solubility, diffusivity, hydrophobicity, charge,etc.); (b) membrane properties (hydrophobicity, surface roughness, pore size, surface charge, and surface functional groups); (c) operational conditions (flux, solution temperature, and flow velocity), and (d) feed water characteristics (solution chemistry, pH, ionic strength, and presence of organic/ inorganic matters).

2.4.2 Temperature polarization coefficient

In most MD fouling investigations, membrane fouling is represented by the permeate flux decline [22]. Although membrane fouling is generally interpreted by flux decline, this approach is inadequate for characterizing fouling development in MD, especially due to the effect of temperature in the operation [23] , [24]. Characterizing the foulant on the MD membrane would provide valuable guidance to the effective application of MD operation such as membrane cleaning as well as deciding the necessity for a pretreatment [25]. It is then important to investigate fouling situations taking into account the temperature distribution characteristics such as Temperature Polarization Coefficient (TPC).

The temperatures at the boundary layers of both the feed (hot side) and permeate (cold side) T_{am} and T_{bm} respectively are different from those at the bulk temperatures T_a and T_b due to temperature polarization. Changes in the driving force (i.e., difference in partial water vapor pressure caused by temperature difference) are usually evaluated through TPC given by $TPC = \frac{T_{am}-T_{bm}}{T_a-T_b}$. It indicates the thermal efficiency of the MD system, wherein a value nearing unity suggests good thermal efficiency, and values nearing zero means otherwise [26].

TPC was found to decrease with the decrease of the pore diameter of the fouling layer and also with the decrease of the membrane resistance with respect to the external resistance (see [8] for more information about fouling effects on TPC and methods for fouling monitoring and cleaning).

3 State Space Model Developpement

3.1 Formulation

Since the temperature has a bidimensional space distribution $T = T(x, z)$, we first consider $(M + 1)$ columns separated by constant distance Δz along the z axis with indexes $j = 0, \ldots M$ that divide each side of the process into (M) subsystems $\Sigma^s{}_{j=1,..,M}$. In both sides of the process, we consider $(N + 2)$ lines separated by constant distance Δx along the x axis with indexes $i = 0, \ldots N + 1$. Let ${}^jT_{s,i}$ be the temperature of the point (i, j) defined by column j and line i in the side s as depicted in Figure 4 bellow.

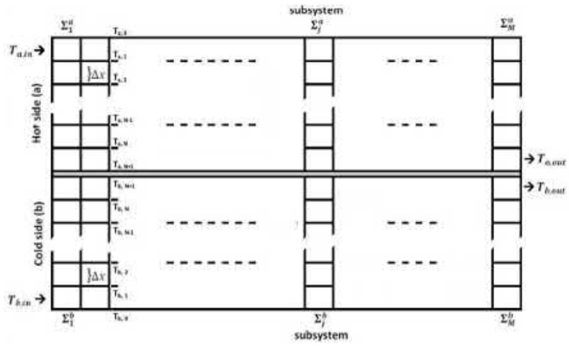

Figure 4: System subdivision.

3.1.1 Derivative terms approximation

Most papers simplify the partial differential equations into an ordinary differential equations system by using the finite difference techniques derived from Taylor's formula with first or second order accuracy [4], [13]

$$\begin{cases} f^{''}(x) = \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right], \\ f^{'}(x) = \frac{1}{h}\left[f(x+h) - f(x)\right] \;\; or \\ f^{'}(x) = \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right], \end{cases} \tag{11}$$

so that for a given point (i, j), conduction term along x axis can be approximated by

$$\frac{\partial^2 \left({}^jT_i\right)}{\partial x^2} = \frac{1}{\Delta x^2}\left({}^jT_{i+1} - 2{}^jT_i + {}^jT_{i-1}\right) \tag{12}$$

and the convection term along z axis by

$$\frac{\partial \left({}^jT_i\right)}{\partial z} = \frac{1}{\Delta z}\left({}^{j+1}T_i - {}^jT_i\right). \tag{13}$$

For our modeling purpose we consider that temperature propagation along z axis is low and can be approximated by a perturbed linear function, therefore we use the following expression for $\frac{\partial\left({}^jT_i\right)}{\partial z}$ where ${}^j\beta_i$ is a bounded perturbation term resulting from modeling approximation

$$\frac{\partial \left({}^jT_i\right)}{\partial z} = \frac{1}{\Delta z}\left({}^jT_i - {}^{j-1}T_i\right) + {}^j\beta_i \,. \tag{14}$$

The velocity profile for the considered point (i, j) is the same for all columns and is given by

$$\begin{cases} v_i = 6\overline{v}\left(\frac{x_i}{d} - \frac{x_i{}^2}{d^2}\right), \\ x_0 = 0,\ x_i = i.\Delta x,\ x_{N+1} = d. \end{cases} \tag{15}$$

Writing (8) for a given point (i, j) in "s" side and substituting ((12),(14)) in it, gives:

$$\frac{\partial\left({}^{j}T_{s,i}\right)}{\partial t} = {}^{j}\alpha_{s,i}\,\frac{\partial^2\left({}^{j}T_{s,i}\right)}{\partial x^2} - v_i\frac{\partial\left({}^{j}T_{s,i}\right)}{\partial z}, \tag{16}$$

$$\begin{aligned}\frac{\partial\left({}^{j}T_{s,i}\right)}{\partial t} = &\left[{}^{j}\alpha_{s,i}\,\frac{1}{\Delta x^2}\left({}^{j}T_{s,i+1} - 2{}^{j}T_{s,i} + {}^{j}T_{s,i-1}\right) - \frac{v_{s,i}}{\Delta z}{}^{j}T_{s,i}\right] + \\ &+\frac{v_{s,i}}{\Delta z}{}^{j-1}T_{s,i} + v_{s,i}{}^{j}\beta_{s,i}\ .\end{aligned} \tag{17}$$

The sign of $\left(v_i{}^{j}\beta_{s,i}\right)$ does not matter because the perturbation term ${}^{j}\beta_{s,i}$ is unknown and parameter α for a given subsystem Σ_j^s is (see [4] for $k_{s,i}$, $\rho_{s,i}$ expressions)

$${}^{j}\alpha_{s,i} = \frac{k_{s,i}}{\rho_{s,i}C_{ps}}\ . \tag{18}$$

3.2 Notations and boundary conditions

3.2.1 State variables, output, and input

For a given subsystem Σ_j^s, consider lines with indexes $i = 1, \dots N$ and build a state vector where each state variable reflects the temperature of (i, j) point

$${}^{j}x_s = \left[{}^{j}x_{s,1} \dots {}^{j}x_{s,N}\right]^T = \left[{}^{j}T_{s,1} \dots {}^{j}T_{s,N}\right]^T. \tag{19}$$

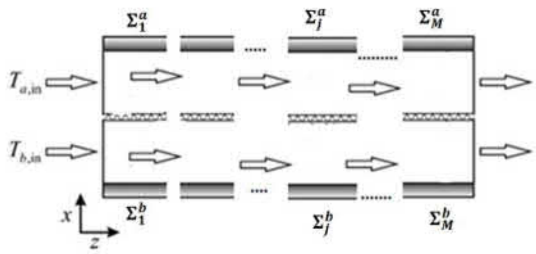

Figure 5: Subsystems in cascade.

Since the flow is laminar one can consider that the output of each subsystem is its own entire state vector. In addition, due to boundary conditions the measurable output temperature of the whole DCMD process given by the last subsystem $(j = M)$ is the same at all lines. That means:

$${}^{j}y_s = {}^{j}C_s{}^{j}x_s = {}^{j}x_s \quad \forall\ j. \tag{20}$$

With the choices made above for ${}^{j}x_s$ and ${}^{j}y_s$, it is easy to see from (17) that the input of each subsystem is the output of the previous one, i. e.

$$ {}^{j}u_s = {}^{j-1}y_s. \tag{21} $$

3.2.2 Boundary conditions

Application of boundary conditions (10) gives:

At the first (resp. last) column: $j = 0$ (resp. $j = M$) that correspond to the first and last inner vertical wall of the DCMD process for both sides, the temperature is the same at all lines and is equal to the inlet (resp. outlet) temperature:

$$ \begin{cases} {}^{0}T_{s,i} = T_{s,in} & \forall\ i, \\ {}^{M}T_{s,i} = T_{s,out} & \forall\ i. \end{cases} \tag{22} $$

At the first line $i = 0$ (corresponds to the first inner horizontal wall of the DCMD process)

$$ {}^{j}T_{s,0} = \frac{4{}^{j}T_{s,1} - {}^{j}T_{s,2}}{3}\ . \tag{23} $$

At the last line $i = N+1$ (corresponds to the boundary layer with the membrane) [4].

For the hot side

$$ {}^{j}T_{a,N+1} = \frac{1}{3}\left[4{}^{j}T_{a,N} - {}^{j}T_{a,N-1} - \frac{2\Delta x}{k_{a,N}}\left(\lambda J + \frac{k_m}{\delta_m}\left({}^{j}T_{a,N} - {}^{j}T_{b,N}\right)\right)\right]\ . \tag{24} $$

And for the cold side

$$ {}^{j}T_{b,N+1} = \frac{1}{3}\left[4{}^{j}T_{b,N} - {}^{j}T_{b,N-1} + \frac{2\Delta x}{k_{b,N}}\left(\lambda J + \frac{k_m}{\delta_m}\left({}^{j}T_{a,N} - {}^{j}T_{b,N}\right)\right)\right]\ . \tag{25} $$

3.3 Parameter variation and modelling approximation

Considering for both sides that parameter ${}^{j}\alpha_{s,i}$ has small unknown but bounded variations around a nominal well-known constant value α_{sn} gives (index n means nominal value) :

$$ {}^{j}\alpha_{s,i} = \alpha_{sn} + \Delta^{j}\alpha_{s,i}\ , s = \{a, b\}\,, \tag{26} $$

$$ \begin{cases} \alpha_{sn} = \frac{k_{an}}{\rho_{sn}C_{ps}}, \\ \left|\Delta^{j}\alpha_{s,i}\right| \leqslant \sigma_{\alpha_s} \quad ; \quad \sigma_{\alpha_s} > 0. \end{cases} \tag{27} $$

In addition, the bounded perturbation term ${}^{j}\beta_{s,i}$ introduced in (14) is such that:

$$ \left|{}^{j}\beta_{s,i}\right| \leqslant \sigma_{\beta_s} \quad ; \quad \sigma_{\beta_s} > 0\ . \tag{28} $$

Then, gathering all variations $\Delta^j\alpha_{s,i}$ in one vector ${}^j\theta_{\alpha s}$ and all perturbation terms ${}^j\beta_{s,i}$ in one vector ${}^j\theta_{\beta s}$ gives :

$$ {}^j\theta_{\alpha s} = \left[{}^j\theta_{\alpha s,1} \dots {}^j\theta_{\alpha s,N}\ \right]^T = \left[\Delta^j\alpha_{s,1}\ \dots\ \Delta^j\alpha_{s,N}\ \right]^T, \tag{29} $$

$$ {}^j\theta_{\beta s} = \left[{}^j\theta_{\beta s,1} \dots {}^j\theta_{\beta s,N}\ \right]^T = \left[{}^j\beta_{s,1} \dots {}^j\beta_{s,N}\ \right]^T. \tag{30} $$

3.4 Equations for state model

The previous states, inputs, and outputs choices, with parameter variation and perturbation terms (17), give a state model of temperature variation at each point (i, j) of the whole process as follows:

$$ \begin{aligned} {}^j\dot{x}_{s,i} = &\left[\alpha_{sn}\frac{1}{\Delta x^2}\left({}^jx_{s,i+1} - 2^jx_{s,i} + {}^jx_{s,i-1}\right) - \frac{v;_{s,i}}{\Delta z}{}^jx_{s,i}\right] + \frac{v_{s,i}}{\Delta z}{}^ju_{s,i} + \\ &+ \frac{1}{\Delta x^2}\left({}^jx_{s,i+1} - 2^jx_{s,i} + {}^jx_{s,i-1}\right){}^j\theta_{\alpha s,i} + v_{s,i}{}^j\theta_{\beta s,i}\ . \end{aligned} \tag{31} $$

Equation (31) needs to be detailed for indexes $i = 1$ and $i = N$ in order to include boundary conditions.

For $i = 1$, ${}^jx_{s,0}$ is obtained from (23), and then (31) gives

$$ \begin{aligned} {}^j\dot{x}_{s,1} = &\left[\left(-\frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2} - \frac{v_{s,1}}{\Delta z}\right){}^jx_{s,1} + \frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2}{}^jx_{s,2}\right] + \frac{v_{s,1}}{\Delta z}{}^ju_{s,1} + \\ &+ \frac{1}{\Delta x^2}\left(-\frac{2}{3}{}^jx_{s,1} + \frac{2}{3}{}^jx_{s,2}\right){}^j\theta_{\alpha s,1} + v_{s,1}{}^j\theta_{\beta s,1}\ . \end{aligned} \tag{32} $$

For $1 < i < N$, (31) gives:

$$ \begin{aligned} {}^j\dot{x}_{s,i} = &\left[\frac{\alpha_{sn}}{\Delta x^2}{}^jx_{s,i-1} - \left(2\frac{\alpha_{sn}}{\Delta x^2} + \frac{v_{s,i}}{\Delta z}\right){}^jx_{s,i} + \frac{\alpha_{sn}}{\Delta x^2}{}^jx_{s,i+1}\right] + \frac{v_{s,i}}{\Delta z}{}^ju_{s,i} + \\ &+ \frac{1}{\Delta x^2}\left({}^jx_{s,i-1} - 2^jx_{s,i} + {}^jx_{s,i+1}\right){}^j\theta_{\alpha s,i} + v_{s,i}{}^j\theta_{\beta s,i}\ . \end{aligned} \tag{33} $$

For $i = N$, getting ${}^jx_{a,N+1}$ and ${}^jx_{b,N+1}$ from (24-25) and then considering the following coupling term between Σ_j^a and Σ_j^b

$$ {}^jx_{ab} = 2\Delta x\left(\lambda J + \frac{k_m}{\delta_m}\left({}^jx_{a,N} - {}^jx_{b,N}\right)\right) \tag{34} $$

gives

$$ \left.\begin{aligned} {}^jx_{a,N+1} &= \tfrac{1}{3}\left[4^jx_{a,N} - {}^jx_{a,N-1} - \tfrac{{}^jx_{ab}}{k_{a,N}}\right] \\ {}^jx_{b,N+1} &= \tfrac{1}{3}\left[4^jx_{b,N} - {}^jx_{b,N-1} + \tfrac{{}^jx_{ab}}{k_{b,N}}\right] \end{aligned}\right\}. \tag{35} $$

This has a compact form as we introduce the variable $\overline{s}$

$$\overline{s} = \frac{2s-a-b}{b-a} = \begin{cases} -1, \ if \ s=a, \\ +1, \ if \ s=b. \end{cases} \tag{36}$$

Equations (35) become:

$$^{j}x_{s,N+1} = \frac{1}{3}\left[4^{j}x_{s,N} - {}^{j}x_{s,N-1} + \overline{s}\frac{^{j}x_{ab}}{k_{s,N}}\right]. \tag{37}$$

Thus (31) gives for $i = \ N$

$$\begin{aligned} {}^{j}\dot{x}_{s,N} = \left[\frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2}{}^{j}x_{s,N-1} - \left(\frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2} + \frac{v_{s,i}}{\Delta z}\right){}^{j}x_{s,N}\right] + \frac{v_{s,N}}{\Delta z}{}^{j}u_{s,N} + \\ + \frac{\alpha_{sn}}{\Delta x^2}\overline{s}\frac{^{j}x_{ab}}{3k_{s,N}} \ + \frac{1}{\Delta x^2}\left(\frac{2}{3}{}^{j}x_{s,N-1} - \frac{2}{3}{}^{j}x_{s,N} + \frac{\overline{s}}{3k_{s,N}}{}^{j}x_{ab}\right){}^{j}\theta_{\alpha s,N} + v_{s,N}{}^{j}\theta_{\beta s,N} \ . \end{aligned} \tag{38}$$

Now, before presenting our first proposition about the new in cascade state model for the DCMD, in particular the state model of a given subsystem Σ_j^s, let us introduce the following matrices, derived from (32), (33) and (38).

- dynamic matrices

$$^{j}A_{s} = \begin{bmatrix} ^{j}A_{s1} \\ \vdots \\ ^{j}A_{sN} \end{bmatrix}. \tag{39}$$

The lines of $^{j}A_{s}$ and their elements are

$$\left\{ \begin{array}{c} ^{j}A_{s1} = [\ a_{s1,1} \quad a_{s1,2} \quad 0 \quad 0\ldots 0]\,, \\ \\ ^{j}A_{si} = [0\ldots 0 \quad a_{si,i-1} \quad a_{si,i} \quad a_{si,i+1} \ 0\ldots 0]\,, \\ \\ ^{j}A_{sN} = [0\ldots 0 \quad a_{sN,N-1} \quad a_{sN,N}]\,, \end{array} \right\} \tag{40}$$

$$\left\{ \begin{array}{c} a_{s1,1} = -\left(\frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2} + \frac{v_{s,1}}{\Delta z}\right) \quad , \quad a_{s1,2} = \ \frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2}, \\ \\ a_{si,i-1} = \frac{\alpha_{sn}}{\Delta x^2}, \ a_{si,i} = -\left(2\frac{\alpha_{sn}}{\Delta x^2} + \frac{v_{s,i}}{\Delta z}\right), \ a_{si,i+1} = \frac{\alpha_{sn}}{\Delta x^2}, \\ \\ a_{sN,N-1} = \frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2} \quad , \quad a_{sN,N} = -\left(\frac{2}{3}\frac{\alpha_{sn}}{\Delta x^2} + \frac{v_{s,N}}{\Delta z}\right), \end{array} \right\} \tag{41}$$

- input and output matrices

$$\left. \begin{array}{c} ^{j}B_{s} = diag\left(^{j}B_{si}\right), \ ^{j}B_{si} = \frac{v_{s,i}}{\Delta z} \quad \forall \ i, \\ \\ ^{j}C_{s} = I_N \ , \qquad \forall \ j, \end{array} \right\} \tag{42}$$

- perturbation term.

Let ${}^j\theta_s$ be the vector of all unknown bounded uncertainties due to parameter variation and modeling approximation and ${}^j\Psi_s\left({}^jx_s, {}^j\theta_s\right)$ be the vector containing all the resulting perturbation terms. It follows:

$$ {}^j\theta_s = \begin{bmatrix} {}^j\theta_{\alpha s} \\ {}^j\theta_{\beta s} \end{bmatrix}, \tag{43} $$

$$ {}^j\Psi_s\left({}^jx_s, {}^j\theta_s\right) = \begin{bmatrix} {}^j\Psi_{s1} \\ \vdots \\ {}^j\Psi_{sN} \end{bmatrix}. \tag{44} $$

For $i = 1$

$$ {}^j\Psi_{s1} = \frac{1}{\Delta x^2}\left(-\frac{2}{3}{}^jx_{s,1} + \frac{2}{3}{}^jx_{s,2}\right){}^j\theta_{\alpha s,1} + v_{s,1}{}^j\theta_{\beta s,1}\ . \tag{45} $$

For $1 < i < N$

$$ {}^j\Psi_{si} = \frac{1}{\Delta x^2}\left({}^jx_{s,i-1} - 2{}^jx_{s,i} + {}^jx_{s,i+1}\right){}^j\theta_{\alpha s,i} + v_{s,i}{}^j\theta_{\beta s,i}\ . \tag{46} $$

For $i = N$:

$$ {}^j\Psi_{sN} = \frac{\alpha_{sn}}{\Delta x^2}\overline{s}\frac{{}^jx_{ab}}{3k_{s,N}} + \frac{1}{\Delta x^2}\left(\frac{2}{3}{}^jx_{s,N-1} - \frac{2}{3}{}^jx_{s,N} + \frac{\overline{s}}{3k_{s,N}}{}^jx_{ab}\right){}^j\theta_{\alpha s,N} + v_{s,N}{}^j\theta_{\beta s,N}\ , $$

$$ \begin{aligned} {}^j\Psi_{sN} = &\left[\frac{\alpha_{sn}}{\Delta x^2}\overline{s}\frac{{}^jx_{ab}}{3k_{s,N}}\frac{1}{{}^j\theta_{\alpha s,N}} + \frac{1}{\Delta x^2}\left(\frac{2}{3}{}^jx_{s,N-1} - \frac{2}{3}{}^jx_{s,N} + \frac{\overline{s}}{3k_{s,N}}{}^jx_{ab}\right)\right]{}^j\theta_{\alpha s,N} \\ &+ v_{s,N}{}^j\theta_{\beta s,N}\ . \end{aligned} \tag{47} $$

A more compact expression of ${}^j\Psi_s\left({}^jx_s, {}^j\theta_s\right)$ would be :

$$ {}^j\Psi_s\left({}^jx_s, {}^j\theta_s\right) = {}^j\Psi_{s\alpha}{}^j\theta_{\alpha s} + {}^j\Psi_{s\beta}{}^j\theta_{\beta s} = {}^j\Psi_s\left({}^jx_S\right){}^j\theta_s \tag{48} $$

such that

$$ \left.\begin{aligned} {}^j\Psi_{s\beta} = diag\left({}^j\Psi_{s\beta i}\right)\ ,\ {}^j\Psi_{s\alpha} = diag\left({}^j\Psi_{s\alpha i}\right), \\ {}^j\Psi_s\left({}^jx_S\right) = \left[{}^j\Psi_{s\alpha}\ \ {}^j\Psi_{s\beta}\right], \end{aligned}\right\}, \tag{49} $$

${}^j\Psi_{s\alpha i}$ and ${}^j\Psi_{s\beta i}$ are the coefficients of ${}^j\theta_{\alpha s,i}$ and ${}^j\theta_{\beta s,i}$ in relations (45) to (47) and ${}^j\theta_s$ is introduced in (43).

In the following, we give the statement of the uncertain bi-dimensional cascade state model for DCMD process.

Proposition 3.1 *Consider the DCMD process theoretically modeled in Section 2 with the above mentioned matrices and vectors ${}^{j}A_s$, ${}^{j}B_s$, ${}^{j}C_s$, ${}^{j}x_s$, ${}^{j}u_s$, ${}^{j}y_s$, ${}^{j}\Psi_s$ and ${}^{j}\theta_s$. Then, the inner temperature profile can be predicted using the following set of state space models defined for subsystems Σ_j^s (Figure 5)*

$$\Sigma_j^s : \begin{cases} {}^{j}\dot{x}_s = {}^{j}A_s{}^{j}x_s + {}^{j}B_s{}^{j}u_s + {}^{j}\Psi_s\left({}^{j}x_S\right){}^{j}\theta_s, \\ {}^{j}y_s = {}^{j}C_s{}^{j}x_s \quad j = 1, \dots M. \end{cases} \tag{50}$$

Proof. Direct consequence of the above developments and relations. □

Remark 3.1

- The model is built in cascade as represented in Figure 5 where each subsystem Σ_j^s is supplied by the previous one (Σ_{j-1}^s) and acts on the next one (Σ_{j+1}^s) .
- This form of bi-dimensional state model of DCMD process is introduced for the first time to the best of our knowledge [5] and gives a complete description of the process behavior. It is appropriate for observer based control/monitoring approaches as we will demonstrate in next sections.
- On the basis of this model, we will build an unknown input observer which gives access to suitable information such as polarization ratio and polarization coefficient since it permits to estimate all (i, j) points' temperatures.
- The aim of the work is to give a means to monitor inner temperature evolution in order to prevent and avoid severe or irreversible fouling situations by predicting their occurrence with a good timing and launch the predefined appropriate maintenance routine.
- ${}^{j}\Psi_s\left({}^{j}x_S\right){}^{j}\theta_s$ behaves as a perturbation term and contains errors due to approximation and parameter variation.
- Simulations were conducted to compare model results with some literature data.

3.5 Model simulation

Simulation of the developed state model showed steady state results comparable to [4] using the same data such as geometry, physical properties and operating conditions. The bi-dimensional simulation depicted in Figure 6 shows that temperature in the hot side decreases along x and z axes of the membrane, in the same way the cold side temperature increases along the x and the z axes, which is in agreement with the polarization phenomenon. Figure 7 to Figure 10 show vertical and longitudinal temperature distribution as well as the variation of mass flux densities and velocity effect on pure water production and membrane temperature.

4 Prediction of Temperature Profiles Using Observers

4.1 States and inputs observing

In practice, only the inlet and outlet temperatures are measurable. The profile and longitudinal temperature distributions are not accessible but are very important because they

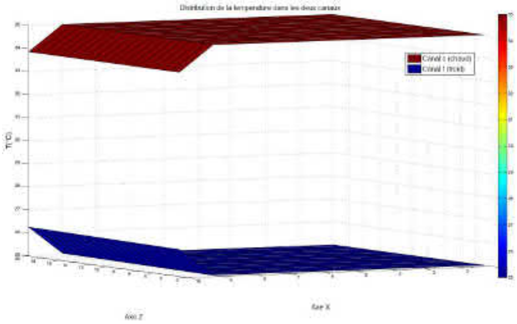

Figure 6: Temperature distribution in two dimensions (up: hot, down: cold).

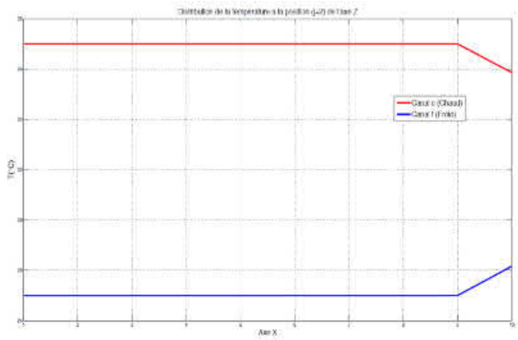

Figure 7: Temperature along x axis for a given z.

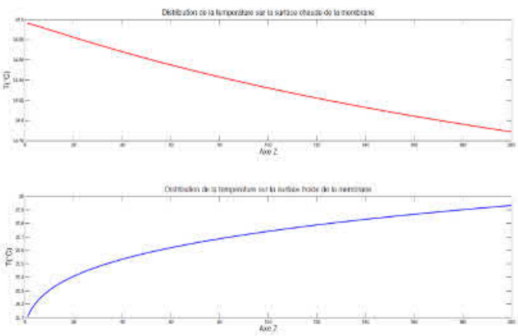

Figure 8: Temperature distribution along the membrane.

describe the polarization phenomenon which is the major driving force for pure water production. The need of an observer arises. The observer should estimate all temperatures inside the process and from those temperatures one could estimate temperature-variable parameters such as polarization coefficient, polarization ratio, and pure water flux.

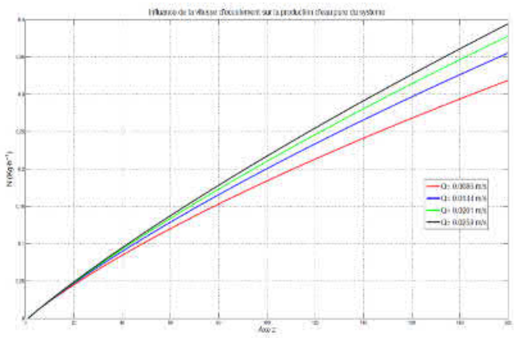

Figure 9: Velocity effect on water production.

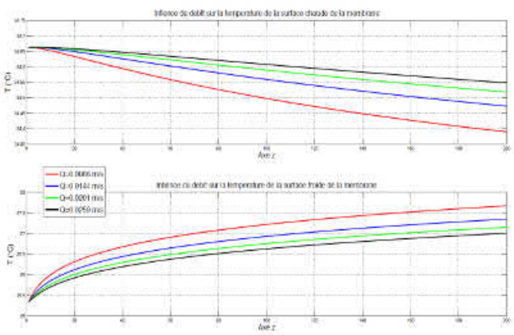

Figure 10: Velocity effect on membrane temperature.

It was stated in Section 3, that the measurable outlet temperature is the same at all lines of the last subsystem Σ_M^s which gives a measurement of the entire output vector. This is due to boundary condition, laminar flow and because generally channel depth (d) is small in DCMD.

On the other hand, outputs, states and inputs have equivalent roles: knowing the state of a subsystem, gives its output and the input of the next one. Conversely, the input informs about the output and the state of the previous one. This motivates the need to build an unknown input observer (UIO) starting from the known (measurable) output of the last subsystem Σ_M^s. The proposed global UIO is built in cascade form (like the state model) as shown bellow in Figure 11 for one side of the process.

The known output ${}^M\hat{y}_s$ (so the state) of the last subsystem Σ_M^s is used with the UIO to estimate its unknown input vector which is the output (and the state) of the previous subsystem Σ_{M-1}^s (${}^M\hat{u}_s = {}^{M-1}\hat{y}_s$). The obtained output (${}^{M-1}\hat{y}_s$) is then used with the UIO to estimate the input of the subsystem Σ_{M-1}^s. This principle is applied to ascend to the first subsystem which has a known input (inlet temperature) and so doing one can have access to all temperatures inside the process.

This structure has a lot of advantages; increasing the accuracy of the model by increasing the number of subsystems; when estimating temperatures inside the process, it is possible to estimate the flux in each part of the membrane, the total flux, and different

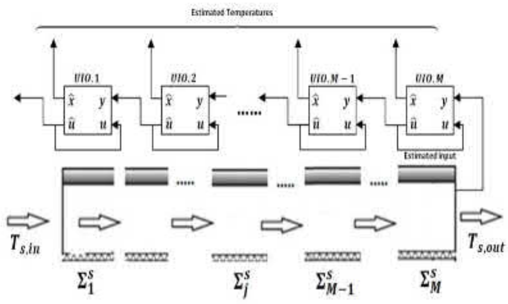

Figure 11: Diagram of the unknown input observer (UIO).

parameters such as polarization ratio and polarization coefficient.

Proposition 4.1 *The states and inputs of models defined in equation (50) are fully observable.*

Proof. Considering the model of Σ_j^s and keeping in mind that for $1 < j < M$, the state vector is the output of the subsystem and the input of the next one, we gather all state vectors in one global vector and write the global unperturbed state model that includes all subsystems. Then, we prove the global state observabilty by showing that global observabilty matrix has a full rank. State and input observablity of each subsystem Σ_j^s follows from the global state observabilty as they are parts of the global state vector.

The unperturbed model of Σ_j^s (without ${}^j\Psi_s\left({}^jx_S\right){}^j\theta_s$) is :

$$\begin{pmatrix} {}^jC_s = I_N \\ {}^ju_s = {}^{j-1}y_s = {}^{j-1}C_s{}^{j-1}x_s = {}^{j-1}x_s \end{pmatrix} \Longrightarrow {}^j\dot{x}_s = {}^jA_s{}^jx_s + {}^jB_s{}^{j-1}x_s \ , \tag{51}$$

which gives for $j = M, \ldots, 1$

$$\begin{cases} {}^M\dot{x}_s = {}^MA_s{}^Mx_s + {}^MB_s{}^{M-1}x_s, \\ \qquad\vdots \\ {}^1\dot{x}_s = {}^1A_s{}^1x_s + {}^1B_s{}^0x_s, \end{cases} \tag{52}$$

with the compact writing

$$\begin{cases} \dot{X}_g = A_gX_g + B_gU_g, \\ Y_g = C_gX_g, \end{cases} \tag{53}$$

where X_g is the global $[(N \cdot M) \times 1]$ state vector, $U_g = {}^0x_s = {}^1u_s$ and $Y_g = {}^My_s$ are both external measurable input and output temperatures.

$$X_g = \begin{bmatrix} {}^Mx_s{}^T & {}^{M-1}x_s{}^T \ldots & {}^1x_s{}^T \end{bmatrix}^T, \quad U_g = {}^1u_s, \ and \ Y_g = {}^My_s = {}^MC_s{}^Mx_s = {}^Mx_s \ , \tag{54}$$

where A_g is a square $[(N \cdot M) \times (N \cdot M)]$ matrix formed by $(N \times N)$ sized zero matrices except for the main diagonal blocks formed by matrices ${}^jA_s$, $j = M, \ldots, 1$ and upper next diagonal blocks formed by matrices ${}^{j}B_{s}$, $j = M, \ldots, 2$,

$$\left(\begin{array}{c} A_g = \begin{bmatrix} A_{g1} \\ \vdots \\ A_{gM} \end{bmatrix}, \begin{cases} A_{g1} = \left[\begin{bmatrix} {}^{M}A_{s} \end{bmatrix} \begin{bmatrix} {}^{M}B_{s} \end{bmatrix} [0] \ldots [0] \right], \\ A_{gj} = \left[[0] \ldots [0] \begin{bmatrix} {}^{j}A_{s} \end{bmatrix} \begin{bmatrix} {}^{j}B_{s} \end{bmatrix} [0] \ldots [0] \right], \\ A_{gM} = \left[[0] \ldots [0] \begin{bmatrix} {}^{1}A_{s} \end{bmatrix} \right], \end{cases} \\ \hline B_g = \begin{bmatrix} [0] \\ \vdots \\ [0] \\ \begin{bmatrix} {}^{1}B_{s} \end{bmatrix} \end{bmatrix}, C_g = \left[\begin{bmatrix} {}^{M}C_s \end{bmatrix} [0] \ldots [0] \right] = \left[[I_N] [0] \ldots [0] \right]. \end{array} \right) \tag{55}$$

Observability matrix is calculated as follows

$$\mathcal{O}(A_g, C_g) = \begin{bmatrix} \mathcal{O}_1 \\ \vdots \\ \mathcal{O}_M \end{bmatrix} = \begin{bmatrix} C_g \\ C_g A_g \\ \vdots \\ C_g A_g^{M-1} \end{bmatrix}, \tag{56}$$

where $\mathcal{O}(A_g, C_g)$ is triangular due to the particular form of C_g and A_g

$$\begin{cases} \mathcal{O}_1 = C_g = [[I_N] [0] \ldots \ [0]], \\ \mathcal{O}_2 = C_g A_g = \left[\begin{bmatrix} {}^{M}A_{s} \end{bmatrix} \begin{bmatrix} {}^{M}B_{s} \end{bmatrix} [0] \ldots \ [0] \right], \\ \mathcal{O}_3 = \mathcal{O}_2 A_g = \left[[\mathcal{O}_{31}] [\mathcal{O}_{32}] \begin{bmatrix} {}^{M}B_{s} {}^{M-1}B_{s} \end{bmatrix} \ [0] \ldots \ [0] \right], \\ \mathcal{O}_4 = \mathcal{O}_3 A_g = \left[[\mathcal{O}_{41}] [\mathcal{O}_{42}] [\mathcal{O}_{43}] \begin{bmatrix} {}^{M}B_{s} {}^{M-1}B_{s} {}^{M-2}B_{s} \end{bmatrix} \ [0] \ldots \ [0] \right], \\ \vdots \\ \mathcal{O}_r = [[\mathcal{O}_{r1}] \ldots [\mathcal{O}_{rr-1}] [\mathcal{O}_{rr}] \ \ [0] \ldots \ [0]], \end{cases} \tag{57}$$

with diagonal blocks given by

$$\begin{cases} \mathcal{O}_{11} = I_N, \\ \mathcal{O}_{rr} = \prod_{k=0}^{k=r-2} {}^{M-k}B_{s}. \end{cases} \tag{58}$$

This yields the simple expression of its determinant

$$|\mathcal{O}(A_g, C_g)| = \prod_r |\mathcal{O}_{rr}| \ . \tag{59}$$

Due to regularity of all ${}^{j}B_{s}$ matrices ($\left|{}^{j}B_{s}\right| \neq 0$) it follows that $\mathcal{O}(A_g, C_g)$ has a full $(N \cdot M)$ rank and thus the global state X_g is fully observable. The state and input of all subsystems Σ_j^s are observable since they are parts of the global state vector. □

4.2 Observer design

The state-models obtained above have the same form for all subsystems in both sides. In order to avoid useless notations, we built the observer (without loss of generality) on the basis of the following state form where vectors u, x, y and θ and matrices A, B, C and Ψ respectively have the same form and role as in (50)

$$\begin{cases} \dot{x} = Ax + Bu + \Psi(x)\,\theta, \\ y = Cx. \end{cases} \tag{60}$$

Based on the "known" output of system (60) the design aim is to ensure that observer state and input $(\hat{x}, \hat{u})$ converge to the system state and input (x, u) even with the effect of the unknown perturbation term $\Psi(x)\,\theta$. We deal with the worst case by considering the maximum possible deviation of θ since we don't need a precise estimation for it.

Proposition 4.2 *Consider the perturbation term $\Psi(x, \theta)$ introduced in (44), and a vector θ_m, such that*

$$\max\left(\|\theta\|, \left\|\hat{\theta}\right\|\right) \leqslant \|\theta_m\| \leqslant \sigma_\theta \ , \tag{61}$$

where $\hat{\theta}$ is the estimate of θ and σ_θ is a positive scalar. Then, Ψ has the following properties:

1) $\Psi(x, \theta)$ is bounded on θ i.e.

$$\begin{cases} \|\Psi(x,\theta)\| \leqslant \|\Psi(x,\theta_m)\| \leqslant \sigma_\theta \|\Psi(x)\|, \\ \left\|\Psi\left(x,\hat{\theta}\right)\right\| \leqslant \|\Psi(x,\theta_m)\| \leqslant \sigma_\theta \|\Psi(x)\|. \end{cases} \tag{62}$$

2) $\Psi(x, \theta)$ is Lipchitz on x i.e.

$$\exists\ \sigma_\Psi > 0 \mid \ \|\Psi(x,\theta) - \Psi(\hat{x},\theta)\| \leqslant \sigma_\Psi \|x - \hat{x}\| \leqslant \sigma_\Psi \tilde{x} \tag{63}$$

and

$$\left\|\Psi(x)\,\theta - \Psi(\hat{x})\,\hat{\theta}\right\| = \|\Delta\Psi\| \leqslant \sigma_\Psi \sigma_\theta \tilde{x} \ . \tag{64}$$

Proof. Property 1 results from the multiplicative form of $\Psi(x, \theta)$ given in (48). Properties 2 of Ψ are proved in appendix. □

These properties are used in the following.

Proposition 4.3 *Consider the state space model (60) and the following unknown input observer:*

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B\hat{u} + \Psi(\hat{x})\,\hat{\theta} + L(y - \hat{y}), \\ \hat{y} = C\hat{x}, \\ \dot{\hat{u}} = \ \eta(y - \hat{y}). \end{cases} \tag{65}$$

Then, estimation errors $\tilde{x} = x - \hat{x}$ and $\tilde{u} = u - \hat{u}$ converge asymptotically to zero if we find symmetric positive-definite matrices P, R and gains η, L with appropriate dimensions that fulfill the following LMI condition:

$$\begin{bmatrix} \left[(A - LC)^T P + P(A - LC) + 2\sigma_\Psi \sigma_\theta P\right] & \left[PB - C^T \eta^T R\right] \\ \left[B^T P - R\eta C\right] & 0 \end{bmatrix} < 0. \tag{66}$$

Proof. State estimation error dynamic is:

$$\dot{\tilde{x}} = (A - LC)\,\tilde{x} + B\tilde{u} + \left(\Psi\,(x)\,\theta - \Psi\,(\hat{x})\,\hat{\theta}\right),$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x} + B\tilde{u} + \Delta\Psi \ . \tag{67}$$

Dynamics of u are negligible with respect to $\hat{u}$ due to earlier supposed almost linear temperature propagation along z axis. Thus :

$$\dot{\tilde{u}} = \dot{u} - \dot{\hat{u}} = \dot{u} - \eta C\tilde{x} = -\eta C\tilde{x} \ . \tag{68}$$

Now consider the Lyapunov function ([27], [28]) with symmetric positive-definite matrices P, R:

$$V = \tilde{x}^T P\tilde{x} + \tilde{u}^T R\tilde{u}, \tag{69}$$

$$\dot{V} = \dot{\tilde{x}}^T P\tilde{x} + \tilde{x}^T P\dot{\tilde{x}} + \tilde{u}^T R\dot{\tilde{u}} + \dot{\tilde{u}}^T R\tilde{u},$$

$$\dot{V} = \dot{\tilde{x}}^T P\tilde{x} + \tilde{x}^T P\dot{\tilde{x}} - \tilde{u}^T R\eta C\tilde{x} - \tilde{x}^T C^T\eta^T R\tilde{u},$$

$$\begin{array}{rl} \dot{V} & = \tilde{x}^T\left[(A - LC)^T P + P(A - LC)\right]\tilde{x} + \tilde{u}^T B^T P\tilde{x} \\ & +\tilde{x}^T PB\tilde{u} + 2\tilde{x}^T P\Delta\Psi - \tilde{u}^T R\eta C\tilde{x} - \tilde{x}^T C^T\eta^T R\tilde{u} \end{array},$$

$$\begin{array}{rl} \dot{V} & = \tilde{x}^T\left[(A - LC)^T P + P(A - LC)\right]\tilde{x} + \tilde{u}^T\left[B^T P - R\eta C\right]\tilde{x} \\ & +\tilde{x}^T\left[PB - C^T\eta^T R\right]\tilde{u} + 2\tilde{x}^T P\Delta\Psi \ . \end{array} \tag{70}$$

From (64):

$$2\tilde{x}^T P\Delta\Psi \leq 2\sigma_\Psi\sigma_\theta\tilde{x}^T P\tilde{x} \ . \tag{71}$$

Therefore,

$$\begin{array}{rl} \dot{V} \leq & \tilde{x}^T\left[(A - LC)^T P + P\,(A - LC) + 2\sigma_\Psi\sigma_\theta P\right]\tilde{x} + \tilde{u}^T\left[B^T P - R\eta C\right]\tilde{x} \\ & +\tilde{x}^T\left[PB - C^T\eta^T R\right]\tilde{u} \end{array} \tag{72}$$

or also $\dot{V} \leq [\tilde{x} \ \ \tilde{u}]^T M_V \begin{bmatrix} \tilde{x} \\ \tilde{u} \end{bmatrix}$.

M_V is a matrix given by

$$M_V = \begin{bmatrix} \left[(A - LC)^T P + P\,(A - LC) + 2\sigma_\Psi\sigma_\theta P\right] & \left[PB - C^T\eta^T R\right] \\ \left[B^T P - R\eta C\right] & 0 \end{bmatrix}. \tag{73}$$

The estimation errors $(\tilde{x}, \tilde{u})$ asymptotically converge to zero if we find matrices η, L, P and R that give a negative-definite $\dot{V}$ ($\dot{V} < 0$). This condition can be announced in the LMI form of Proposition 4.

$$\dot{V} < 0 \Leftrightarrow M_V < 0 \Leftrightarrow$$

$$\begin{bmatrix} \left[(A - LC)^T P + P\,(A - LC) + 2\sigma_\Psi\sigma_\theta P\right] & \left[PB - C^T\eta^T R\right] \\ \left[B^T P - R\eta C\right] & 0 \end{bmatrix} < 0 \ . \tag{74}$$

□

Remark 4.1

- The LMI given in (66) can be solved using the LMI toolbox of MATLAB.
- The proposed observer is similar in spirit the adaptive observers developed in [29], [30], [31], [32].

4.3 Observer simulation

Observer simulation generated a distribution of internal temperatures comparable to those obtained by the model as shown in Figure 12. Figures 13 and 14 show convergence of state estimation errors to zero respectively for hot and cold stream.

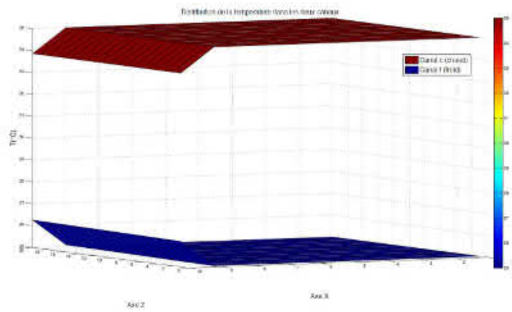

Figure 12: Temperature distribution obtained by the observer.

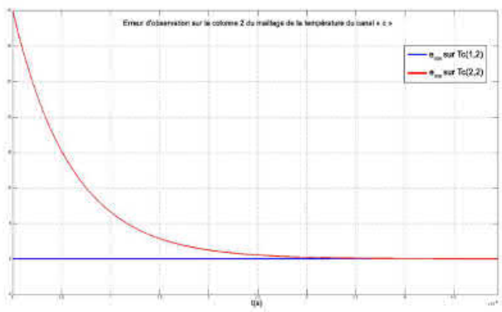

Figure 13: State error convergence in hot side.

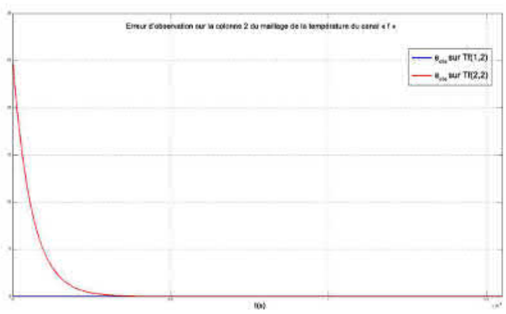

Figure 14: State error convergence in cold side.

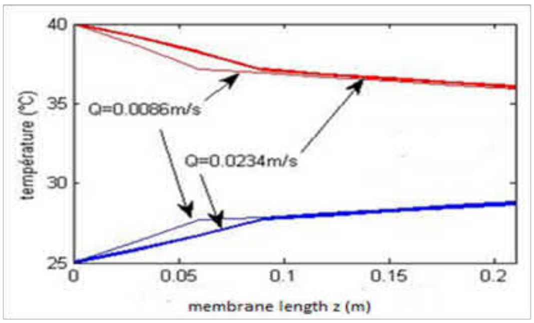

Figure 15: Temperature distribution along z axes.

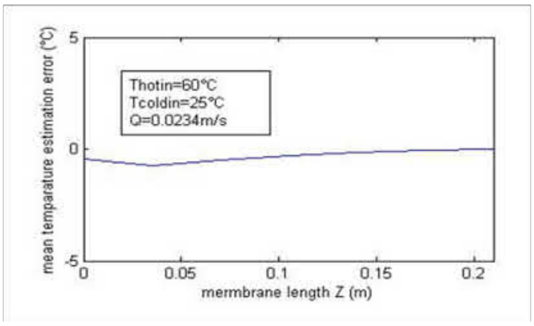

Figure 16: Temperature estimating error along z axes.

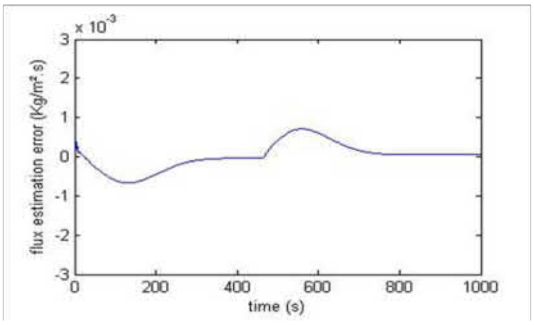

Figure 17: Pure water flux production estimation error.

Other simulations have been made using a different set of parameters for the observer. Figures 15 and 16 show a good estimation of longitudinal temperature distribution compared to those obtained using the model, while Figure 17 shows the ability of the observer to estimate pure water production under varying working conditions (inlet temperature decreases at t=400s) and Figure 18 shows the evolution of temperature polarization coefficient estimating error for a given longitudinal position.

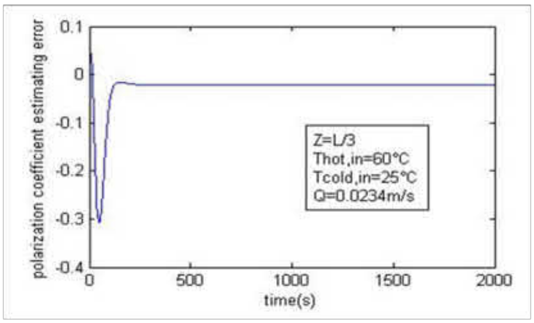

Figure 18: Polarization coefficient estimating error inside DCMD.

Having a good TPC estimation can be very helpful when investigating fouling situations on the basis of pure water decrease information. Thus, the observer-based approach would serve as a means to make further studies about fouling characterization considering in the same time temperature polarization effect on pure water production.

5 Conclusion

In this paper, an observer-based approach is proposed to estimate the temperature profiles inside DCMD unit. This allows predicting the polarization coefficient of the latter and hence can be used to monitor fouling situations. Of particular importance, the convergence of the observation error is proved using Lyapunov direct method and LMI constraints. The performed simulations show the effectiveness of the proposed approach which can be generalized to others types of membrane distillation processes.

Appendix: 2^{nd} Property of $\Psi(x,\theta)$

The objective is to verify that $\Psi(x,\theta)$ is Lipchitz on x i.e.

$$\exists\, \sigma_\Psi > 0,\ \|\Psi(x,\theta) - \Psi(\hat{x},\theta)\| \le \sigma_\Psi \|x-\hat{x}\| \le \sigma_\Psi \tilde{x}\ . \tag{75}$$

For more simplicity, relations (43) to (49) describing the vector ${}^j\Psi_s\left({}^jx_s, {}^j\theta_s\right)$ of all perturbation terms, are used without indexes s and j. Therefore, given that Ψ_β in (49) does not depend on x, then for the same constant vector $\theta = [\theta_\alpha\ \theta_\beta\ ]^T$, (48) gives:

$$\Psi(x,\theta) - \Psi(\hat{x},\theta) = [\Psi_\alpha(x) - \Psi_\alpha(\hat{x})]\,\theta_\alpha\ . \tag{76}$$

It is then sufficient having (61) to verify that $\Psi_\alpha(x)$ is Lipchitz on x

$$\begin{aligned}&\|\Psi(x,\theta) - \Psi(\hat{x},\theta)\| \le \|\Psi_\alpha(x) - \Psi_\alpha(\hat{x})\|\,\|\theta_\alpha\| \le \\ &\|\Psi_\alpha(x) - \Psi_\alpha(\hat{x})\|\,\|\theta_m\| \le \sigma_\theta \|\Psi_\alpha(x) - \Psi_\alpha(\hat{x})\|\ .\end{aligned} \tag{77}$$

Consider the vector $\Delta\Psi_\alpha = \Psi_\alpha(x) - \Psi_\alpha(\hat{x}) = [\Delta\Psi_{\alpha,i}]\ ,\ i = 1,\ldots,N$ such that

$$\|\Psi(x) - \Psi(\hat{x})\|^2 = \|\Delta\Psi_\alpha\|^2 = \sum_{i=1}^{N} (\Delta\Psi_{\alpha,i})^2 \tag{78}$$

in the following we will make use of this simple property :

$$(p \pm d)^2 \geq 0 \ \Leftrightarrow p^2{+}d^2 \geq \mp 2pd\,. \tag{79}$$

We have for $i = 1$

$$\Delta\,\Psi_{\alpha,1} = \frac{2}{3}\frac{1}{\Delta x^2}\left(-\tilde{x}_1 + \tilde{x}_2\right),$$

$$\left(\Delta\,\Psi_{\alpha,1}\right)^2 = \frac{4}{9}\frac{1}{\Delta x^4}\left(\tilde{x}_1^2 + \tilde{x}_2^2 - 2\tilde{x}_1\tilde{x}_2\right).$$

Using (79) gives : $-2\tilde{x}_1\tilde{x}_2 \leq \tilde{x}_1^2 + \tilde{x}_2^2$. Then

$$\left(\Delta\,\Psi_{\alpha,1}\right)^2 \leq \frac{8}{9}\frac{1}{\Delta x^4}\ \left(\tilde{x}_1^2 + \tilde{x}_2^2\right) \leq \frac{1}{\Delta x^4}\ \left(\tilde{x}_1^2 + \tilde{x}_2^2\right)\ . \tag{80}$$

For $1 < i < N$

$$\Delta\,\Psi_{\alpha,i} = \frac{1}{\Delta x^2}\left(\tilde{x}_{i-1} - 2\tilde{x}_i + \tilde{x}_{i+1}\right),$$

$$\left(\Delta\,\Psi_{\alpha,i}\right)^2 = \frac{1}{\Delta x^4}\left(\tilde{x}_{i+1}^2 + 4\tilde{x}_i^2 + \tilde{x}_{i-1}^2 - 4\tilde{x}_{i+1}\tilde{x}_i - 4\tilde{x}_i\tilde{x}_{i-1} + 2\tilde{x}_{i+1}\tilde{x}_{i-1}\right)\ .$$

Using (79) gives $2\tilde{x}_{i+1}\tilde{x}_{i-1} \leq \tilde{x}_{i+1}^2 + \tilde{x}_{i-1}^2$, $-4\tilde{x}_{i+1}\tilde{x}_i \leq 4\tilde{x}_{i+1}^2 + \tilde{x}_i^2$, and $-4\tilde{x}_i\tilde{x}_{i-1} \leq 4\tilde{x}_{i-1}^2 + \tilde{x}_i^2$. Thus

$$\left(\Delta\,\Psi_{\alpha,i}\right)^2 \leq \frac{6}{\Delta x^4}\left(\tilde{x}_{i+1}^2 + \tilde{x}_i^2 + \tilde{x}_{i-1}^2\right)\ . \tag{81}$$

For $i = N$:

$$\Delta\,\Psi_{\alpha,N} = \frac{1}{\Delta x^2}\left[\frac{2}{3}\left(\tilde{x}_{N-1} - \tilde{x}_N\right) + \left(\frac{\alpha_n}{\theta_{\alpha,N}} + 1\right)\tilde{x}_{ab}\right] \leq \frac{1}{\Delta x^2}\left[\frac{2}{3}\left(\tilde{x}_{N-1} - \tilde{x}_N\right) + \sigma_1\tilde{x}_{ab}\right]\ ,$$

where σ_1 is a majoration of $\left(\frac{\alpha_n}{\theta_{\alpha,N}} + 1\right)$ and $\tilde{x}_{ab}$ obtained from (34) verifies :

$\tilde{x}_{ab} = \frac{\Delta 2xk_m}{\delta_m}\left(\tilde{x}_{a,N} - \tilde{x}_{b,N}\right) \Rightarrow \tilde{x}_{ab}^2 = \left(\frac{2\Delta xk_m}{\delta_m}\right)^2\left(\tilde{x}_{a,N} - \tilde{x}_{b,N}\right)^2 \leq \sigma_2\tilde{x}_N^2$ where σ_2 is an appropriate majoration.

It follows:

$$\left(\Delta\,\Psi_{\alpha,N}\right)^2 \leq \frac{1}{\Delta x^4}\left[\left(\frac{2}{3}\left(\tilde{x}_{N-1} - \tilde{x}_N\right)\right)^2 + 2\frac{2\sigma_1}{3}\tilde{x}_{ab}\left(\tilde{x}_{N-1} - \tilde{x}_N\right) + \sigma_1^2\tilde{x}_{ab}^2\right]. \tag{82}$$

As for $\Delta\,\Psi_{\alpha,1}$ we obtain $\left(\frac{2}{3}\left(\tilde{x}_{N-1} - \tilde{x}_N\right)\right)^2 \leq \tilde{x}_N^2 + \tilde{x}_{N-1}^2$ and (79) gives: $2\tilde{x}_{ab}\left(\tilde{x}_{N-1} - \tilde{x}_N\right) \leq \tilde{x}_{ab}^2 + \left(\tilde{x}_{N-1} - \tilde{x}_N\right)^2 \leq \sigma_2\tilde{x}_N^2 + 2\tilde{x}_N^2 + 2\tilde{x}_{N-1}^2$.

Gathering the terms and taking $\sigma_N = \left(1 + \sigma_2\frac{2\sigma_1}{3} + 2\frac{2\sigma_1}{3} + \sigma_1^2\sigma_2\right)$, one gets:

$$\left(\Delta\,\Psi_{\alpha,N}\right)^2 \leq \frac{1}{\Delta x^4}\left[\tilde{x}_N^2\left(1 + \sigma_2\frac{2\sigma_1}{3} + 2\frac{2\sigma_1}{3} + \sigma_1^2\sigma_2\right) + \tilde{x}_{N-1}^2\left(1 + 2\frac{2\sigma_1}{3}\right)\right]$$

$$\left(\Delta\,\Psi_{\alpha,N}\right)^2 \leq \frac{\sigma_N}{\Delta x^4}\left[\tilde{x}_N^2 + \tilde{x}_{N-1}^2\right]\ . \tag{83}$$

Finally from (80), (81)-(83)

$$\sum_{i=1}^{N} (\Delta \Psi_{\alpha,i})^2 = (\Delta \Psi_{\alpha,1})^2 + \sum_{i=2}^{N-1} (\Delta \Psi_{\alpha,i})^2 + (\Delta \Psi_{\alpha,N})^2$$

$$\leq \frac{1}{\Delta x^4} \left[\tilde{x}_1^2 + \tilde{x}_2^2 + 6 \sum_{i=2}^{N-1} \left(\tilde{x}_{i+1}^2 + \tilde{x}_i^2 + \tilde{x}_{i-1}^2 \right) \ + \sigma_N \left[\tilde{x}_N^2 + \tilde{x}_{N-1}^2 \right] \right]$$

$$\leq \frac{\max(6, \sigma_N)}{\Delta x^4} \left[\tilde{x}_1^2 + \tilde{x}_2^2 + \sum_{i=3}^{N} \tilde{x}_i^2 + \sum_{i=2}^{N-1} \tilde{x}_i^2 + \sum_{i=1}^{N-2} \tilde{x}_i^2 \ + \tilde{x}_N^2 + \tilde{x}_{N-1}^2 + \tilde{x}_1^2 + \tilde{x}_N^2 \right]$$

$$\sum_{i=1}^{N} (\Delta \Psi_{\alpha,i})^2 \leq \frac{3\max(6, \sigma_N)}{\Delta x^4} \sum_{i=1}^{N} \tilde{x}_i^2 \tag{84}$$

and

$$\| \Psi_\alpha (x) - \Psi_\alpha (\hat{x}) \| \leq \sqrt{\frac{3\max(6, \sigma_N)}{\Delta x^4}} \sqrt{\sum_{1}^{N} \tilde{x}_i^2}. \tag{85}$$

There exists $\sigma_\Psi > 0$ such that

$$\| \Psi (x, \theta) - \Psi (\hat{x}, \theta) \| \leq \sigma_\Psi \| x - \hat{x} \| \leq \sigma_\Psi \tilde{x} \ ,$$

i.e. $\Psi (x, \theta)$ is Lipschitz on x.

From the above inequality and relation (61), one has

$$\left\| \Psi (x) \, \theta - \Psi (\hat{x}) \, \hat{\theta} \right\| = \| \Delta \Psi \| \leq \sigma_\Psi \sigma_\theta \tilde{x} \ .$$

□

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Searching Functional Exponents for Generalized Fourier Series and Construction of Oscillatory Functions Spaces

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Abstract: This paper is intended to provide a framework for further developments of the theory of generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \ t \in \ R, \tag{1}$$

where $a_k \in \mathcal{C}, \ k \geq 1, \ \lambda_k : R \to R, \ k \geq 1$. Series of the form (1) will be called, in this paper, *series representing oscillatory functions*, by the last term understanding the sum of any series of the form (1), when convergent in some sense, classical or generalized, such as summability procedure or, in respect to a certain norm on the space of series, or in the associated function space of sums or generalized sums. A basic idea we follow is to start from linear spaces of series like (1), then to organize them by introducing a norm or a kind of convergence. The connection between a space of generalized trigonometric series of the form (1) and the space of functions resulting from introducing a topology/norm is our main objective. It is also emphasized that the preceding stages of Fourier analysis, i.e., the classical trigonometric series (the first stage) or the almost periodic functions (the second stage) are also parts of the *third stage* in the development of Fourier analysis. This study is based on classical theory of Fourier Analysis and on the theory of almost periodicity, as developed since 1920's to present. It is also based on methods and results of functional analysis.

Keywords: *generalized Fourier series; oscillatory functions; trigonometric series; almost periodic functions.*

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1 Introduction

Series of the form (1) and construction of spaces of oscillatory functions, consisting of the sum or generalized 'sum', have been investigated by researches during the last 20-25 years. We shall further provide the references, adequate to the subject. It has to be emphasized that both engineering and mathematical literature contain results related to this topic, generated by the applied problems. Mathematicians have started a theory related to the series of the form (1) and their attached oscillatory function spaces. The method used consists in completing certain spaces of generalized trigonometric polynomials, with respect to uniform convergence as basic tool, or the convergence in the mean (of order 2).

Since we take the *series* as primary element in the construction of function spaces of oscillatory functions, we need to proceed with the investigation of spaces whose elements are series of the form (1), to organize them algebraically and then topologically to obtain the series spaces. After the construction of series spaces, we shall be able to obtain the function spaces, consisting of oscillatory functions.

First, let us briefly present the examples already existing in the literature, due to Osipov [15] and Zhang [17]– [20]. These mathematical constructions have been preceded by contributions coming from the engineering literature, due to several researchers, and mentioned in the references to Zhang's papers quoted above. Such applied sources have appeared, particularly, in the IEEE publications, during the last two decades, sporadically, in other journals.

It is interesting to mention the fact that the first stage of development of Fourier analysis (in its main goal of establishing the connection between series and functions), besides many other aspects, started in the 18-th century with names like Euler and continued its vigorous development in the 19-th century, when a great number of mathematicians brought very important contributions, starting with Fourier.

An example connected to the advancement of the first stage is the proof of a conjecture due to Luzin (from 1915), about the convergence almost everywhere, of the Fourier series of any functions $f : [0, 2\pi] \to R$, $f \in L^2([0, 2\pi], R)$. In such case,

$$f \simeq \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \sin jt + b_j \cos jt),$$

where a_j, b_j are given by the classical Euler formulae. The sign $\simeq$ above can be substituted by the sign $=$, excepting a subset of $[0, 2\pi]$ of Lebesgue measure zero. This result is due to Carleson (1966).

The books by Bary [1] and Zygmund [22] are almost of encyclopedic type for the Fourier Analysis, the first stage, since its inception until the mid of the 20-th century. Needless to say that the first stage is not yet quitting the scene and new contributions are abundant.

In the 1920's, the second stage is appearing with Bohr, followed by Stepanov, Bochner and Besicovitch, to mention only a few of the great contributors to the theory of almost periodicity, a kind of oscillatory motion, more complex than periodicity.

The current mathematical literature, dedicated to the case of almost periodic functions is quite rich, the following quotations providing a rather complete source for this subject: Bohr [3], Besicovitch [2], Favard [11], Levitan [13], Fink [12], Levitan [13] and Zhikov [14], Corduneanu [4, 5].

The development of Science and Technology, especially in the 20-th century, lead to the new form, a generalized one, for Fourier series (trigonometric, when not generated by a function). This new type of series is of the form presented in formula (1) above, with the functional exponents $\lambda_k(t) : R \to R$, $k \geq 1$, subject to conditions further specified. In Zhang's papers quoted above, several spaces of oscillatory functions are constructed, starting with a class of generalized Fourier exponents, of the form

$$\lambda(t) = \sum_{k=1}^{m} c_k \exp[iq_k(t)],\ t \in R,\ c_k \in \mathcal{C}, \tag{2}$$

where $q_k(t)$ are defined by formulae like

$$q(t) = \begin{cases} \sum\limits_{i=1}^{m} \lambda_j t^{\alpha_j}, & t \geq 0, \\ -\sum\limits_{j=1}^{m} \lambda_j (-t)^{\alpha_j}, & t < 0, \end{cases} \tag{3}$$

with $\lambda_j \in R,\ j = 1, 2, ..., m$ and $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. The class of generalized exponents in (2), (3), is denoted by $Q(R, R)$ and, according to Zhang [21], it has been considered by Gelfand in another context.

The first space of oscillatory function, defined by Zhang [19], has been called the space of *strong limit power* functions and denoted by $SLP(R, R)$, is obtained by completing the linear space of all generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{n} c_k \exp[iq_k(t)],\ t \in R, \tag{4}$$

with $q_k(t)$ as in (3) and $c_k \in \mathcal{C}$, $k = 1, 2, ..., n = n(P)$, the norm being the supremum, on R, of the polynomial $P(t)$ in (4). Of course, the topology induced by this norm is that of uniform convergence on R. Consequently, the construction of the space $SLP(R, \mathcal{C})$ is achieved by the method of completion of linear vector spaces, in this case, the norm being the $\sup\limits_{R} |\cdot|$.

Therefore, the space $SLP(R, \mathcal{C})$ is a Banach space over $\mathcal{C}$, which is also a subspace of the richer Banach space $BC(R, \mathcal{C})$, of continuous and bounded maps from R into $\mathcal{C}$, with the uniform convergence on R.

Taking the space $SLP(R, \mathcal{C})$ as a base space, new oscillatory function spaces have been constructed by Zhang [20], namely the Besicovitch type spaces, similar to the spaces $B_1(R, \mathcal{C})$, or $B_2(R, \mathcal{C})$. For the first case, one has to complete $SLP(R, \mathcal{C})$ with respect to the norm $f \to M(|f|)$, while in the second case, of the space $B_2(R, \mathcal{C})$, the norm chosen for the completion procedure will be $f \to \{M(|f|^2)\}^{1/2}$.

The interested reader can find the details in Zhang's papers, quoted above, or in the book by Corduneanu et al. [10]. Many properties are known for classical almost periodic functions, in which case the functional exponents are linear functions, of the form λt, $t \in R$, $\lambda \in R$.

In summarizing the discussion above, about the oscillatory function spaces constructed by Zhang, one can notice the following steps which are necessary in the procedure: first, one needs a set (possibly with an algebraic structure) of generalized functional exponents, say $\{f(t)\}$, such that $\exp[if(t)]$ has the following property:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[if(t)] dt$$

exists (as a finite complex number); second, for the completion property of the space of oscillatory functions, one needs to choose a topology, or a norm, based on which we obtain the completed (or Banach) space. In the case we use a seminorm, instead of a norm, the need to work with a factor space is required. See, for instance, Corduneanu [5].

To briefly summarize the connection between the series and its sum, let us denote this connection by

$$f(t) \simeq \sum_{k=1}^{\infty} c_k \exp[iq_k(t)] \tag{5}$$

and provide the formulae ($k \geq 1$)

$$c_k = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-iq_k(t)]dt. \tag{6}$$

As proved in Zhang's quoted papers, the Parseval equation

$$\sum_{k=1}^{\infty} |c_k|^2 = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \tag{7}$$

also holds. It has many implications, among them we mention the uniqueness of the generalized Fourier series attached to a function $f \in SLP(R, \mathcal{C})$. Or, the one to one correspondence between the elements of the space $SLP(R, \mathcal{C})$ and those of the space ℓ^2.

For more details on these matters, the reader is invited to consult the Appendix to the book of Corduneanu et al. [10]. See also the paper by Zhang [19], for the construction of the spaces of Besicovitch type, $B_1(R, \mathcal{C})$ and $B_2(R, \mathcal{C})$, by using the completion method, as specified above, by using the norms $M(|f|)$ and $\{M(|f|^2)^{1/2}$, with respect to which the space $SLP(R, \mathcal{C})$ is not complete. In the paper of Corduneanu [8], the space $B^2_\lambda(R, \mathcal{C})$ is constructed by this method, for an arbitrary set $\lambda = \{\lambda_\alpha,\ \alpha \in$ an arbitrary set of generalized Fourier exponents$\}$.

The remaining part of the Introduction will be concerned with the space constructed by Osipov [15], also pertaining to the third stage in the development of Fourier Analysis.

The Osipov space is known under the name of *Bohr-Fresnel almost periodic functions space*. Actually, these functions are oscillatory in the sense of adopted definition and a result of Osipov states: Let $f(t) : R \to \mathcal{C}$ be a Bohr-Fresnel almost periodic function. Then, there exists a Bohr almost periodic function $F(t, x) : R \times R \to \mathcal{C}$, such that $f(t) = F(t, t^2)$, $t \in R$. Of course, the result shows the close relationship between Bohr and Bohr-Fresnel almost periodic functions, but the theory of the later is much more complex, as it appears in the book of Osipov, quoted above.

Following our procedure in constructing new spaces of oscillatory functions, we shall start from the set of all formal trigonometric series, of the form

$$\sum_{k=1}^{\infty} c_k \exp(i\alpha t^2 + 2i\lambda_k t), \tag{8}$$

where $c_k \in \mathcal{C}$, $\alpha, \lambda_k \in R$, $k \geq 1$. One usually assumes that λ_k's are distinct.

It follows from Zhang's case discussed above that each term in (8) has a finite limit (Poincaré) on the whole real axis. Moreover, if the series in (8) is absolutely convergent and denotes the sum by $f(t)$, then the connections between f and series (8) are given by the formulae for coefficients, in terms of $f(t)$:

$$c_k = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)\exp(-i\alpha t^2 - 2i\lambda_k t)dt. \tag{9}$$

Let us note that α is a real number which is determined by the function $f(t)$. Also, the formula (9) is valid in cases when the series (8) is not necessarily absolute (hence, also uniform) convergent. The right hand side of (9) makes sense in more general situations, as we shall see. It is, again, the Poincaré mean value on R.

If one assumes the condition

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty, \tag{10}$$

which is less restrictive than the condition of absolute convergence, we obtain a larger space of oscillatory functions, which is in a slighter modified form – the space of Osipov [15], consisting of oscillatory functions.

We shall list now some properties of the space of Bohr–Fresnel almost periodic functions, presented in detail in Osipov's book quoted above.

We point out the fact that the Parseval type equation

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = \sum_{k=1}^{\infty} |c_k|^2, \tag{11}$$

where the c_k's are given by (9), holds true for every B^2-almost periodic function.

Another property, following from (1) and some extra arguments, is the *uniqueness* of the generalized Fourier series, associated to a function in the Bohr-Fresnel space.

As shown in Section 2 below, to each sequence $\{c_k;\ k \geq 1\}$ satisfying (10), there corresponds a unique Bohr-Fresnel function. The *approximation* property is also valid, in the following format (different than in Osipov's text): Any function $f(t)$, in the class of Bohr-Fresnel almost periodic functions, can be approximated with any degree of accuracy by polynomials in this class, with frequencies belonging to the set of frequencies in its generalized Fourier series. Using the norm derived from the Poincaré mean value

$$M(f) = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)dt, \tag{12}$$

the approximation property can be stated: for each $\varepsilon > 0$, there exists $n \in N$, such that

$$M\left\{ |f(t)| - \sum_{k=1}^{n} c_k \exp(i\alpha t^2 + 2i\lambda_k t)|^2 \right\} < \varepsilon^2, \tag{13}$$

with $c_k,\ k = 1, 2, ..., n$, given by (9). We notice that, unlike in the case of Zhang's space $SLP(R, \mathcal{C})$, the 'measure' of length used is based on the Poincaré mean value, inducing a convergence in the mean (of order 2), instead of the uniform convergence, achieved by the sup-norm.

We shall conclude this introductory remarks related to the oscillatory function spaces constructed by Osipov and Zhang, mentioning the fact that, in the paper [21] by Zhang et al. the case of generalized Fourier exponents having the form of a quadratic polynomial, with real valued coefficients, has been thoroughly investigated, all possible cases (for constructing a space of oscillatory functions) being emphasized.

2 Finding Generalized Functional Fourier Exponents

From the form of formula (1), we realize that in order to attach a function to the series which we would like to represent an oscillatory function, with some basic properties encountered for classical Fourier series or the ones characterizing various types of almost periodic functions, two necessary conditions have to be satisfied:

First, we must find the generalized Fourier exponents, denoted by $\lambda_k(t),\ k \geq 1$; more precisely, we need to identify sets we shall represent by Λ, containing sequences of functions $R \to \mathcal{C}$, at least locally integrable on R. Since each sequence of $\lambda_k(t)$'s must contain distinct terms, it is obvious that Λ has to be at least countable. Moreover, in case we want to represent certain functions $R \to \mathcal{C}$ by such series, which means we have to determine the coefficients of the series like (1), we realize that, each sequence involved, must be formed from mutually 'orthogonal' elements. This condition will be imposed in the form suggested by Poincaré mean value, namely

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp[i(\lambda_j(t)-\lambda_k(t))]dt = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases} \tag{14}$$

This condition is also suggested by the theory of Hilbert (rather pre-Hilbert) spaces, but we are not getting into details here.

Second, one needs to make precise the kind of attaching to a given series of the type (1), a function that could be reasonably called a generalized sum. Of course, the most natural way is to have a condition assuring the convergence of the series with respect to a certain norm. Since this is a rather restrictive condition (if, for instance, we keep in mind the fact that the classical Fourier series of a continuous function is only summable to the generating function, using Euler's formulae for coefficients), we may use, when adequately, instead of a norm, a seminorm. This feature will lead to further problems when constructing the spaces of oscillatory functions, but it serves well our purpose, as we see below, in this paper.

We can obtain sequences $\{\lambda_k(t);\ k \geq 1\}$, such that (14) is satisfied, if we can construct distinct solutions of the equation/relation

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp[i\lambda(t)]dt = \begin{cases} 0, & \lambda(t) \not\equiv 0, \\ 1, & \lambda(t) \equiv 0. \end{cases} \tag{15}$$

Indeed, if $\lambda_k(t),\ k \geq 1$, are distinct solutions of (15), then the sequence $\{\lambda_k(t); k \geq 1\}$ satisfies obviously the relationship (14).

Let us determine solutions of the equation/relation (15), choosing a simple procedure based on Cauchy's integral theorem.

Namely, limiting our considerations to those $\lambda(t) : R \to R$, which constitute restrictions of entire functions $\lambda = \lambda(z),\ z \in \mathcal{C}$ and applying Cauchy's theorem for a closed contour, consisting of the interval of the real axis $(-\ell, \ell)$ and the semicircle C_ℓ having $(-\ell, \ell)$ as diameter, situated in the half-plane $\operatorname{Im} z \geq 0$, one obtains for $\ell > 0$

$$\int_{-\ell}^{\ell}\exp[i\lambda(t)]dt + \int_{C_\ell}\exp[i\lambda(z)]dz = 0, \tag{16}$$

on C_ℓ being from ℓ to $-\ell$. Let $\Lambda(z)$ be a primitive of $e^{i\lambda(z)}$, which is also an entire function. Then (16) yields for $\ell > 0$,

$$\int_{-\ell}^{\ell} \exp[i\lambda(t)]dt = \Lambda(-\ell) - \Lambda(\ell). \tag{17}$$

From (15) one derives now the condition for λ:

$$\ell^{-1}[\Lambda(\ell) - \Lambda(-\ell)] = o(1),\ \ \ell \to \infty. \tag{18}$$

Consequently, the equation/relation (18) provides a source for obtaining $\lambda(z)$, such that $\Lambda'(z) = \exp(i\lambda(z))$ and, taking a sequence of distinct solutions of (18), we have the possibility of constructing series of the form (1).

Let us notice that the second case in (15) is obviously verified, i.e., when $\lambda(z) \equiv 0$. If one chooses $\lambda(z) = \lambda z$, $\lambda \in R$, $\lambda \neq 0$, $z \in \mathcal{C}$, then we derive from above

$$\lim (i\lambda\ell)^{-1}[e^{i\lambda\ell} - e^{-i\lambda\ell}] = 0,\ \text{ as } \ell \to \infty. \tag{19}$$

Since the bracket is bounded as $\ell \to \infty$, there results the validity of (18).

Hence, the series resulting from the above considerations, namely

$$\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t),\ \ t \in R, \tag{20}$$

with λ_k being arbitrary real numbers, are series for oscillatory functions.

But we recognize in (20) the Fourier series corresponding with the almost periodic functions. Depending on the nature of their convergence of summability, we obtain the classical Bohr almost periodic functions and its multiple generalizations (Stepanov, Besicovitch, the AP_r-almost periodic functions).

Remark 2.1 From the formula (17), we can draw the following conclusion. If $\Lambda(z)$ is a function satisfying the condition $\Lambda(\ell) = \Lambda(-\ell)$, i.e., is an even function, then (17) is verified. This is a rather special case and we invite the readers to find other solutions to the equation/relation (18), in the class of entire functions.

We shall deal now, with another condition imposed, to the function $\lambda(t)$, namely

$$\lambda(-t) = -\lambda(t),\ \ t \in R. \tag{21}$$

Finding generalized Fourier exponents, in the class of odd functions on R, leads to another relation/equation similar to (18). This restriction was also imposed by Zhang, when constructing the space $SLP(R, \mathcal{C})$.

Let us notice that the left hand side in (17) can be rewritten as

$$\begin{aligned}\int_{-\ell}^{\ell} \exp[i\lambda(t)]dt &= \int_{-\ell}^{0} \exp[i\lambda(t)]dt + \int_{0}^{\ell} \exp[i\lambda(t)]dt \\ &= 2\int_{0}^{\ell} \cos\lambda(t)dt,\ \ \ell > 0,\end{aligned} \tag{22}$$

if we take into account (21) and change t for $-t$ in the first integral. Therefore, in order to satisfy the first condition of (15) it is necessary and sufficient to satisfy the equation/relation

$$\int_0^\ell \cos\lambda(t)dt = o(\ell), \quad \text{as } \ell \to \infty. \tag{23}$$

Only odd solutions $\lambda(t)$ at least locally integrable are candidates for functional exponents in series representing oscillatory functions.

In what follows, we shall deal with finding nontrivial solutions to the equation/relation (23), as well as (18).

The relation/equation (23) has, indeed, nontrivial solutions. We notice that any function of the form $\lambda(t) = \mu t$, with $\mu = \text{const.} \in R$ and $t \in R$, is an odd function which satisfies both (18) – as seen above, and (23). Hence, we reobtain the functional exponents that characterize various classes of almost periodic functions. This remark is a confirmation of the fact that the oscillatory functions contain the classical cases of periodic and almost periodic functions. More comments on these matters will be made in forthcoming text.

Of course, it is interesting to emphasize classes of generalized exponents, using the equation/relation (23). And let us examine the case of oscillatory functions of Osipov [15] type.

Still remaining in the classical field, let us remind the Fresnel integrals, related to his theory in Optics: for $\alpha > 0$, one has

$$\int_0^\infty \cos(\alpha t^2)dt = (2\alpha)^{-1}\sqrt{\frac{\pi}{2}}\,\cdot \tag{24}$$

Taking (24) into account, we find out that the relation/equation (23) is verified by any function $\alpha t^2,\ \alpha > 0,\ t \geq 0$. In order to obtain the odd function satisfying (21), one has to consider (on R) $\lambda(t) = \alpha t^2$ for $t \geq 0$ and $\lambda(t) = -\alpha t^2$ for $t < 0$. Then we rely on Zhang's et al. results in [21] to find that $\lambda(t)$ defined above can be used to construct generalized trigonometric polynomials, based on quadratic algebraic polynomials. This means, generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{m} a_k \exp[i(\alpha t^2 + \beta_k t)], \tag{25}$$

with $\alpha, \beta_k \in R,\ 1 \leq k \leq m$. There is no free term at the exponent, because it is absorbed by a_k. This approach, used by Zhang and his collaborators, does not lead to the original space constructed by Osipov. The method used by Osipov [15] requires that polynomials of the form (25), with $\lambda(t) = \alpha t^2 + \beta_k t$, $t \in R$, be used to construct the functions "sum" on the whole R. More precisely, (24) can be used only on R_+, or on the whole R. In such a way, we actually obtain two spaces of oscillatory functions, based on second degree algebraic polynomials as functional exponents. In the introduction, we have sketched the construction of the original Osipov space. The details are given in Osipov's book [15], besides a short presentation of Bohr's theory, to serve for the parallelism between two concepts of almost periodicity (actually, Bohr-Fresnel functions constitute an example of oscillatory functions, even though they can be represented by the classical Bohr almost periodic functions). Their Fourier series is representative for the third stage of Fourier Analysis.

Concerning Zhang's $SLP(R, \mathcal{C})$ functions, one sees from their construction that they are odd functions. The fact of possessing a finite Poincaré mean value is proven in the paper by Zhang [19].

Let us now consider an example corresponding to $\lambda(z) = \sin \lambda z,\ a \in R,\ z \in \mathcal{C}$. Obviously, $\lambda(z)$ is an odd function. But this $\lambda(z)$ is not a solution of (23). The associated generalized Fourier series is

$$\sum_{k=1}^{\infty} a_k \exp[i \sin \lambda_k t],\ \ t \in R, \tag{26}$$

which is characteristic for the third stage of Fourier Analysis. If we admit the condition $\{a_k;\ k \geq 1\} \in \ell^1(N, \mathcal{C})$, then (26) is absolutely and uniformly convergent on R. Since every term is a Bohr almost periodic function, the series is convergent to a function $f \in AP(R, \mathcal{C})$. In other words, this case is an example of a series whose construction is not based on the use of equation (18) or (23), but the sum is an oscillatory function, even of classical type.

Of course, if instead of the condition imposed above, $\{a_k;\ k \geq 1\} \subset \ell^1(N, \mathcal{C})$, one chooses another similar one, the result may lead to other classical spaces of almost periodic or oscillatory functions. It is also clear that the same oscillatory function can be represented by different types of generalized Fourier series. An in depth study of this fact would be welcome.

One can find many other sequences of generalized Fourier exponents, just relying on above considerations. An example, also resulting from Zhang's constructions, is given by a sequence of odd degree polynomials, say like $\mu(z) = a_1 z + a_3 z^3 + \cdots + a_{2k+1}^{2k+1}$. Indeed, these polynomials and their linear combinations are satisfying the request appearing in Zhang's construction of generalized Fourier series [19]. These exponents satisfy, starting with $k = 1$, requirements coming from applications.

We shall prove now a lemma, showing how one can get more complex generalized exponents, relying on some already found.

Lemma 2.1 *Let us assume we are given a set of generalized exponents, say* $\Lambda = \{\lambda_\alpha(t) : \alpha \in A\}$*, where* A *is a set of indices, at least countable. If* $\varphi : R \to R$ *is a locally integrable map, such that* $\lim \exp]i\varphi(t)]$ *exists when* $t \to \infty$*, while* $\{\lambda_j(t);\ j \geq 1\} \in \Lambda$ *and form an orthogonal system as shown in* (14)*, then the sequence* $\{\varphi(t) + \lambda_j(t);\ j \geq 1\} \subset \Lambda$ *is also orthogonal in the sense shown by* (14).

The proof is immediate if we notice that $[\varphi(t) + \lambda_j(t)] - [\varphi(t) + \lambda_k(t)] = \lambda_j(t) - \lambda_k(t)$, and take (14) into account.

In this way, we have obtained in case of Osipov's kind of generalized Fourier series, i.e., the Bohr-Fresnel case of almost periodic functions: $\alpha t^2 + \beta_k t,\ k \geq 1$, representing the exponents of terms in the series for Osipov's oscillatory functions.

We invite the reader to investigate solutions of the form $\lambda(t) = t^\alpha$, $\alpha \in R_+$, for the equation (23). Also, for the relation/equation (18). In particular, the odd polynomials mentioned above, justified by Zhang's argument.

In concluding this section, we shall make two brief remarks/suggestions, which may be helpful in the search of new classes of generalized Fourier exponents.

First one is related to the use of the general formula for residues, instead of Cauchy's integral theorem. This formula has the form, with notations similar to those in (16),

$$\int_{-\ell}^{\ell} e^{i\lambda(t)} dt + \int_{c_\ell} e^{i\lambda(z)} dz = 2\pi i \sum \operatorname{res}(e^{i\lambda(z)}), \tag{27}$$

the Σ being extended at the poles of $\exp[i\lambda(z)]$, within the interior of the semidisc formed by c_ℓ and $(-\ell,\ell)$. The function $\exp(i\lambda(z))$ must be meromorphic, with zeros at $(z_1,z_2,...,z_n)\in\mathcal{C}^n$, so that, for large enough ℓ, one can take the limit of both sides in (21), as $\ell\to\infty$. Apparently, this is not an easy task, but in the affirmative case it will provide other solutions for determining generalized Fourier exponents.

Second remark relates to the notation Λ for the set of generalized exponents. It is obvious that, from algebraic point of view, this set of real valued functions must form at least an additive group. This can be seen, for instance, from the formulas providing the coefficients of a generalized Fourier series, such as (6), (9), or the orthogonality conditions.

Zhang [19] required more algebraic conditions, for instance the ring structure for Λ, a necessity imposed by the fact that the product of two function in Λ, must be in Λ.

3 Construction of a Space of Oscillatory Functions

In Section 1, we have summarily presented the construction of the oscillatory function spaces, following the two authors who have brought significant contributions to the development of the third stage of Fourier Analysis. We shall present, in this section, the construction of a space of oscillatory functions, denoted by '$AP_1(R,\mathcal{C};\Lambda)$, the AP just reminding us of the case of almost periodic functions, which functions are also *oscillatory* type (see the definition in the Abstract of the paper). It is the corresponding, more general, case of the space $AP_1(R,\mathcal{C})$, see Corduneanu [6, 7], the name of Poincaré being properly attached, since he has provided the first example of an almost periodic function (Bohr), in a rather important case: when the Fourier series attached is absolutely and uniformly convergent on R.

The first step in the construction consists in specifying the set/class of generalized Fourier/trigonometric series, of the form (1), which will be the elements of $AP_1(R,\mathcal{C};\Lambda)$. Namely, to obtain the space $AP_1(R,\mathcal{C};\Lambda)$, we shall assume that all series of the form (1), for which

$$\sum_{k=1}^{\infty}|a_k|<\infty, \tag{28}$$

will be the elements of $AP_1(R,\mathcal{C};\Lambda)$, and only them.

Since the series satisfying (28) imply the absolute convergence, due to the fact $|a_k\exp[i\lambda_k(t)]|\le|a_k|$, $k\ge1$, $t\in R$, $\lambda_k\in\Lambda$, the norm on this space appears naturally to be the one given in (28), i.e.,

$$\left|\sum_{k=1}^{\infty}a_k\exp[i\lambda_k(t)]\right|_{AP_1}=\sum_{k=1}^{\infty}|a_k|. \tag{29}$$

Hence, the set $AP_1(R,\mathcal{C};\Lambda)$ is a linear normed space on $\mathcal{C}$. Moreover, this space is a Banach space, i.e., complete as a linear metric space, a statement which is implied by the completeness of the space $\ell^1(R,\mathcal{C})$.

We shall try now to derive some properties of this space, particularly looking at its connections with function spaces on R. The natural approach seems to be in attaching to the series (1), the function representing its sum. This means the correspondence/map is given by

$$\sum_{k=1}^{\infty}a_k\exp[i\lambda_k(t)]\to\sum_{k=1}^{\infty}a_k\exp[i\lambda_k(t)],\ \ t\in R, \tag{30}$$

with the left hand side in (39) regarded as the formal series, while the right hand side is the sum of the series, i.e., a function $f : R \to \mathcal{C}$.

It is obvious that $f = f(t)$, $t \in R$, is a complex valued function, defined on R and taking values which are uniformly bounded, by the right side in (29). It is also a continuous and bounded map from R into $\mathcal{C}$, which tells us that $AP_1(R, \mathcal{C}; \Lambda) \subset BC(R, \mathcal{C})$ = the space of bounded and continuous maps from R into $\mathcal{C}$. We have admitted that Λ consists of continuous functions. When this condition does not hold for the elements of $AP_1(R, \mathcal{C}; \Lambda)$, we can obtain spaces of measurable functions (for instance), more general than $BC(R, \mathcal{C})$.

Let us summarize now the discussion above regarding the space $AP_1(R, \mathcal{C}; \Lambda)$ and its Banach space structure, over the field $\mathcal{C}$. We need to keep in mind that $AP_1(R, \mathcal{C}; \Lambda)$ can be regarded either as a series space or a function space. Their isomorphism is the motivation for using the same notation for both of them. We shall write now the formula which represents the space $AP_1(R, \mathcal{C}; \Lambda)$:

$$AP_1(R, \mathcal{C}; \Lambda) = \Bigg\{ f : R \to \mathcal{C}, f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \\ \sum_{k=1}^{\infty} |a_k| < \infty,\ \lambda_k(t) \in \Lambda,\ k \geq 1 \Bigg\}. \tag{31}$$

The norm is given by formula (29). The completion of the space $AP_1(R, \mathcal{C}; \Lambda)$ follows easily from the following argument. Indeed, from our assumption (25), there follows that $AP_1(R, \mathcal{C}; \Lambda)$ is the closure of the subset of generalized trigonometric polynomials of the form $\sum_{k=1}^{n} a_k \exp[i\lambda_k(t)]$, with a_k and $\lambda_k(t)$, $k \geq 1$, as considered above.

Since the completion of a linear normed space is the minimal complete Banach space, containing the given linear normed space, while any element of $AP_1(R, \mathcal{C}; \Lambda)$ can be regarded as the limit in the sense of the norm, we obtain a contradiction if we assume that there exists a complete linear space, larger than $AP_1(R, \mathcal{C}; \Lambda)$, i.e., containing at least one element outside $AP_1(R, \mathcal{C}; \Lambda)$, which can be reached by the limit process with terms from the space of trigonometric polynomials of the above shown form (sections of the series in the space $AP_1(R, \mathcal{C}; \Lambda)$).

Theorem 3.1 *The space of oscillatory functions $AP_1(R, \mathcal{C}; \Lambda)$ is constructed in the following steps:*

1) *One chooses a set Λ, at least countable, consisting of continuous functions $R \to R$, such that any sequence $\{\lambda_k(t);\ k \geq 1\} \subset \Lambda$ is orthogonal in the sense of Poincaré's mean value on R, as shown in formula* (14).

2) *See Section* 2 *for details in obtaining such a set Λ.*

3) *One considers the set of all generalized Fourier series of the form* (1), *with $\{\lambda_k(t),\ k \geq 1\} \subset \Lambda$, which can be routinely organized as a linear space over $\mathcal{C}$.*

4) *In order to introduce a topology/convergence on this linear space, we have denoted it by $AP_1(R, \mathcal{C}; \Lambda)$, we consider on it the norm defined by* (29).

5) *One derives, as shown above, that the space $AP_1(R, \mathcal{C}; \Lambda)$ is a Banach space, by proving its completeness in the norm* (29).

Remark 3.1 The isomorphism of the series space $AP_1(R, \mathcal{C}; \Lambda)$ and the function spaces of the sums of its series, in other words, the one to one correspondence between the series and functions-sums, will follow easily when we are able to prove the uniqueness theorem for Fourier generalized series in $AP_1(R, \mathcal{C}; \Lambda)$, based on Parseval's formula

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{32}$$

to be established in the sequel. There is an alternative approach, based on the formula for the coefficients, in terms of the sum of the series

$$a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_k(t)] dt. \tag{33}$$

Both approaches will be substantiated in the presentation to follow.

Remark 3.2 Since we shall deal with product of elements/series of $AP_1(R, \mathcal{C})$, we notice that this operation (Cauchy's rule of multiplication can be performed only in case when Λ is an additive group of real valued functions $\lambda = \lambda(t) : R \to R$, which we shall use to form the generalized Fourier series.

Now, let us prove the formula (32), which establishes the connection between the function $f(t) : R \to \mathcal{C}$, and its generalized Fourier series in (30). One obtains, by multiplying both sides by $\exp[-i\lambda_j(t)] \neq 0$, the following relation:

$$f(t) \exp[i\lambda_j(t)] = \sum_{k=1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t)], \tag{34}$$

which we can integrate from $-\ell$ to ℓ, both sides, the second, term by term. This follows from the condition $\{a_k;\ k \geq 1\} \subset \ell^1(N, \mathcal{C})$, taking also into account the fact that each exponential has module equal to 1. This leads to the equation

$$\begin{aligned} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_j(t)] dt &= \int_{-\ell}^{\ell} \sum_{k=1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t))] dt \\ &= \int_{-\ell}^{\ell} \sum_{k=1}^{n} a_k \exp[i(\lambda_k(t) - \lambda_j(t)] dt \\ &+ \int_{-\ell}^{\ell} \sum_{k=n+1}^{\infty} a_k \exp[i(\lambda_k(t) - \lambda_j(t)] dt, \end{aligned} \tag{35}$$

assuming $n > j$. Both sides of this equation must be multiplied by $(2\ell)^{-1}$ and then take the limit as $\ell \to \infty$. Taking into account the equations (14), one obtains from above, since

$$\left| (2\ell)^{-1} \int_{-\ell}^{\ell} \sum_{k=n+1}^{\infty} a_k \exp[i\lambda_k(t) - \lambda_j(t)] dt \right| \leq \sum_{k=n+1}^{\infty} |a_k| < \varepsilon,$$

provided $n > N(\varepsilon) \subset N$, and what remains from (33) when $\ell \to \infty$ is:

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)[-i\lambda_j(t)] dt = a_k,$$

i.e., the formula (32) for the coefficients of the function $f(t)$ = the sum of the associated Fourier series, with generalized exponents from Λ.

We can now proceed to prove the validity of the Parseval formula (32), for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. Indeed, we have

$$\int_{-\ell}^{\ell} |f(t)|^2 dt = \int_{-\ell}^{\ell} f(t)\bar{f}(t) dt = \int_{-\ell}^{\ell} \Sigma\, \overline{\Sigma} dt, \tag{36}$$

with Σ from (29)-(31); but, for large n, we can also write

$$\begin{aligned} f(t)\bar{f}(t) &= \sum_{k=1}^{n} |a_k|^2 + \sum_{\substack{k,j=1 \\ k\neq j}}^{n} a_k \bar{a}_j e^{i[\lambda_k(t) - \bar{\lambda}_j(t)]} \\ &+ \left[\sum_{k=n+1}^{\infty} a_k e^{i\lambda_k t}\right] \bar{r}_n(t) + \left[\sum_{k=n+1}^{\infty} \bar{a}_k e^{-i\lambda_k(t)}\right] r_n(t) + |r_n(t)|^2, \end{aligned} \tag{37}$$

with

$$r_n(t) = \sum_{k=n+1}^{\infty} a_k e^{i\lambda_k t}.$$

Let us integrate both sides of the last equation (37) above, from $-\ell$ to ℓ, and multiply both sides by $(2\ell)^{-1}$. If one takes into account the relationships (14), n is sufficiently large, such that $|r_n(t)| < \varepsilon < 1$ for $n \geq N(\varepsilon)$, then, integrating leads to the inequality (38) below, as $\ell \to \infty$:

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt - \sum_{k=1}^{n} |a_k|^2 \leq (2M+1)\varepsilon, \tag{38}$$

where $M = \sum_{k=1}^{\infty} |a_k| < \infty$, because each of the last two terms in (37) is dominated in modulus by M, while $|r_n(t)|^2 < \varepsilon^2 < \varepsilon$. From (38) one obtains the Bessel inequality, which easily leads to Parseval (32). See our book [5], for instance.

Therefore, we conclude that Parseval's formula (32) is valid for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. We shall see, in the sequel, that its validity takes place in richer spaces of generalized Fourier series, containing $AP_1(R, \mathcal{C}; \Lambda)$.

To continue with the properties of the elements/functions of the space $AP_1(R, \mathcal{C}; \Lambda)$, we shall remark first that the *boundedness* on R, of each $f \in AP_1(R, \mathcal{C}; \Lambda)$, with Λ consisting of continuous generalized exponents, is a direct consequence of the norm definition in formula (31). Let us point out the fact that this property remains valid in more general spaces than $\mathcal{C}$, for example when $\mathcal{C}$ is substituted by a complex Banach space.

Another important fact following from the Parseval formula (32) is the existence of the Poincaré mean value of the square of any $f \in AP_1(R, \mathcal{C}; \Lambda)$. This property will be taken in constructing a richer space of oscillatory functions, denoted by $AP_2(R, \mathcal{C}; \Lambda)$.

We notice the property of *continuity* of the functions in $AP_1(R, \mathcal{C}; \Lambda)$, fact easily derived if we admit the continuity of elements in Λ (the generalized Fourier exponents) and we rely on the absolute and uniform convergence of the series constituting the space $AP_1(R, \mathcal{C}; \Lambda)$.

Concerning the property of uniform *continuity* of functions in $AP_1(R, \mathcal{C}; \Lambda)$, known to be valid for the special case when $\Lambda = \{\lambda t;\ \lambda, t \in R\}$, we notice that we should look closer at the set Λ of generalized exponents, the answer to the problem being certainly determined by the properties of the elements of Λ.

Let us consider the formula from (31), namely

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)],\ \ t \in R, \tag{31$'$}$$

and estimate the difference $f(t+h) - f(t)$, $h > 0$. One finds, based on the absolute convergence of the series involved,

$$f(t+h) - f(t) = \sum_{k=1}^{\infty} a_k[\exp i\lambda_k(t+h) - \exp i\lambda_k(t)],\ \ t \in R, h > 0, \tag{39}$$

with help from the classical formula

$$\exp i\alpha = \cos\alpha + i \sin\alpha,\ \ \alpha \in R, \tag{40}$$

one easily derive the Lipschitz type inequality for $t \in R$, $h > 0$, $\varepsilon \geq 1$:

$$|\exp i[\lambda_k(t+h)] - \exp[i\lambda_k(t)]| \leq 2|\lambda_k(t+h) - \lambda_k(t)|.$$

Therefore, one obtains from (39)

$$|f(t+h) - f(t)| \leq 2\sum_{k=1}^{\infty} |a_k|\, |\lambda_k(t+h) - \lambda_k(t)|, \tag{41}$$

an inequality which can be discussed in regard to the properties of the set Λ of generalized exponents.

The most direct answer seems to be the following:

The sequence $\{\lambda_k(t),\ k \geq 1\} \subset \Lambda$ admits a continuity module on R, say $\omega(h)$, with $h \to 0$ implying $\omega(h) \to 0$. In other words, one obtains from (41), $f(t+h) \to f(t)$ as $h \to 0$, uniformly with respect to $t \in R$. A more stringent condition would be to have ω as a continuity module for all $\lambda(t) \in \Lambda$. This answer, in the weak form, is suggested by the case when $\Lambda = \{\lambda t;\ \lambda \in R,\ t \in R\}$, i.e., the almost periodic case for the space $AP_1(R, \mathcal{C})$ of Poincaré. In this case, with $\lambda_k(t) = \lambda_k t$, $\lambda_k \in R - \{0\}$, $t \in R$, the continuity module is $\omega_k(h) = |\lambda_k| h$.

Another formulation related to the concept of module of continuity could be phrased in terms of equicontinuity of functions in the set Λ, or some of its parts; for instance, the sequence of exponents $\{\lambda_k(t);\ k \geq 1\}$ is equicontinuous if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $|\lambda_k(t) - \lambda_k(s)| < \varepsilon$, for any $t, s \in R$, such that $|t - s| < \delta$. In particular, any sequence $\{\lambda_k(t);\ k \geq 1\} \subset \Lambda$, which is uniformly convergent on R, satisfies the conditions of equicontinuity. Also, a compact subset, countable or not, of Λ, which is compact in respect to the uniform convergence (for instance, a compact subset of the space $BC(R, \mathcal{C})$.

Obviously, from the discussion above, we can infer that the problem of uniform continuity of functions in $AP_1(R, \mathcal{C}; \Lambda)$ has more than one answer. We invite the reader to consider other cases when the uniform continuity is assured.

In the last part of this section, we will consider an example of a space in the same category as the space $AP_1(R, \mathcal{C}; \Lambda)$, which presents a particular flavour and allows the illustration of several kinds of convergence. Also, this example will display a sort of classical type of space.

Namely, we shall assume that the set of generalized exponents is given by $\Lambda = AP(R, R)$, i.e., the space of real valued almost periodic functions in the sense of Bohr.

In this case, the series of the real parts of the terms in $\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, which has the form $\alpha_0 + \sum_{k=1}^{\infty} [\alpha_k \cos \lambda_k(t) + \beta_k \sin \lambda_k(t)]$, appears to belong to the third stage of generalized Fourier Analysis.

Let us notice that each term in the series above reminding us of the classical form of Fourier series is in $AP_r(R)$, which means that a third stage in Fourier Analysis can produce spaces of oscillatory functions also belonging to the classical heritage. Of course, the main problem in constructing spaces of oscillatory functions consists in obtaining new spaces, not pertaining to the classical category. The kind of convergence we associate with the linear space of formal series, like (1), may or may not lead to the space $AP(R, \mathcal{C})$, or to a subspace of the latter in the case $\Lambda = AP(R, R)$.

With these considerations, we end the problems/properties related to the space $AP_1(R, \mathcal{C}; \Lambda)$, moving to another space of oscillatory functions, constructed in a similar manner as above and relying on the construction and the consequences for the space $AP_1(R, \mathcal{C}; \Lambda)$.

4 Construction of the Space $AP_2(R, \mathcal{C}; \Lambda)$

In constructing the space of oscillatory functions, denoted by $AP_2(R, \mathcal{C}; \Lambda)$, we can associate the names of Besicovitch and Zhang to this type of space. In case of classical spaces of almost periodic functions, the space $AP_2(R, \mathcal{C})$ represents the Besicovitch space. In case of oscillatory functions spaces, the first examples are those described in Section 1 (Introduction) of this paper, when $\Lambda = Q(R, R)$. See formulae (3) and (4) for details. This type of space, with a special choice of Λ, is due to Zhang, who was the first to express the need of getting more comprehensive spaces of oscillatory functions, than the spaces of almost periodic functions. This need is motivated by the applications of Fourier Analysis, found in engineering literature and pertinent references are included in Zhang's papers. His pseudo almost periodic functions (1992, Ph.D. thesis), which have generated a vast literature in the last 20 years constitute a convincing example that shows the necessity of constructing new spaces of oscillatory functions. Moreover, the pseudo almost periodic functions appear as "perturbations" of the classical almost periodic functions, while their theory has many points of contact with the old theory.

In order to construct the space of oscillatory functions $AP_2(R, \mathcal{C}; \Lambda)$, we will introduce in the linear (algebraic) space of generalized trigonometric series, with Λ as in the case of the space $AP_1(R, \mathcal{C}; \Lambda)$ already described, the norm

$$\left| \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \right|_{AP_2} = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \tag{42}$$

i.e., the norm of the classical space $\ell^2 = \left\{ a_k,\ k \geq 1, \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\}$ of Hilbert.

In the space of sum functions, associated to the series space $AP_2(R, \mathcal{C}; \Lambda)$, we shall use the seminorm, compatible with (42), which looks

$$|f|_{AP_2} = \left[\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right]^{1/2}, \tag{43}$$

which is derived from Poincaré mean value on R and has been used by Besicovitch in the space $B^2(R, \mathcal{C})$ of his almost periodic functions (a natural generalization of Bohr's theory).

The compatibility will result from the validity of Parseval's formula (32), whose validity has been already established in AP_1. In order to obtain Parseval's formula in case $f \in AP_2(R, \mathcal{C}; \Lambda)$, we can proceed in the same way as in case of the space $AP_1(R, \mathcal{C}; \Lambda)$.

But we need, first, to look closely to the relationship/correspondence between series in AP_2 and sum-function attached. We shall show, first, that to each series in AP_2 one can attach a function belonging to the space $L^2_{\text{loc}}(R, \mathcal{C})$. Indeed, for such a series of the form $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, with $\{a_k;\ k \geq 1\} \subset \ell^2$ and $\lambda_k : R \to R$, we can write for $n, p \in N$,

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k(t)) \right|^2 dt = \sum_{k=n+1}^{n+p} |a_k|^2, \tag{44}$$

taking into account the orthogonality of the sequence of $\lambda_k(t)$'s and the relationship $|u|^2 = u\bar{u}$. From (44) and our assumption, we have included in defining the $AP_2(R, \mathcal{C}; \Lambda)$,

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{45}$$

we conclude that the series of $AP_2(R, \mathcal{C}; \Lambda)$ are convergent with respect to the seminorm chosen for this space. Moreover, the convergence in $AP_2(R, \mathcal{C}; \Lambda)$ is implying the convergence in $L^2_{\text{loc}}(R, \mathcal{C})$. This property of Fourier series is proven in our paper [8], in the special case $\Lambda = \{\lambda t;\ \lambda \in R,\ t \in R\}$. It remains valid in the general case, when $\lambda_k(t)$, $k \geq 1$, are more general functions than in the case $\lambda_k(t) = \lambda_k t$, $k \geq 1$, $\lambda \in R$, corresponding to the almost periodic functions of all known types.

We shall write, as usual in the theory of oscillatory functions, including the classical types, in the traditional form

$$f(t) \simeq \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)],\ t \in R, \tag{46}$$

the fact that the function $f(t)$ is constructed by means of the series in the right hand side of (46). The manner of determining the coefficients a_k, $k \geq 1$, in terms of $f(t)$, will be discussed in this section. The formulae providing the a_k's are

$$a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp[-i\lambda_k(t)] dt, \tag{47}$$

i.e., formally, the same as (32), valid for $f \in AP_1(R, \mathcal{C}; \Lambda)$.

In order to derive (47) for $f \in AP_2(R, \mathcal{C}; \Lambda)$, we shall mention the fact that the space $AP_1(R, \mathcal{C}; \Lambda)$ is everywhere dense in the space $AP_2(R, \mathcal{C}; \Lambda)$. This property follows from the fact that, taking into account the definitions of the norm/seminorm in the spaces AP_1 and AP_2, the generalized trigonometric polynomials of the form

$$P(t) = \sum_{k=1}^{n} a_k \exp[i\lambda_k(t)], \ t \in R, \tag{48}$$

constitute everywhere dense sets in both spaces AP_1 and AP_2. Of course, the exponents $\lambda_k(t)$, $1 \leq k \leq n$, are chosen from Λ, for either space.

Let us notice that (49) is elementary in case of $f(t)$ being polynomial of the form (48). We have proven its validity, above in this section, for any $f \in AP_1(R, \mathcal{C}; \Lambda)$. Since $AP_1 \subset AP_2$, due to the inclusion $\ell^1 \subset \ell^2$, we can regard the whole operations as taking place in the space $AP_2(R, \mathcal{C}; \Lambda)$.

As observed above, for each $f \in AP_2(R, \mathcal{C}; \Lambda)$, there exists a sequence in $AP_1(R, \mathcal{C}; \Lambda)$, such that for each $f \in AP_2(R, \mathcal{C}; \Lambda)$ one has $f^{(j)} \to f$ in $AP_2(R, \mathcal{C}; \Lambda)$, as $j \to \infty$. But the convergence of a sequence in either space AP_1 or AP_2, is uniform on coordinates. That means that from

$$f^{(j)} \to f \text{ in } AP_2(R, \mathcal{C}; \Lambda), \tag{49}$$

there follow the convergence relations

$$a_k^j \to a_k \text{ as } j \to \infty, \ k \geq 1, \text{ uniformly}. \tag{50}$$

There remains to prove that a_k, $k \geq 1$, are indeed the coefficients of $f \in AP_2(R, \mathcal{C}; \Lambda)$.

It is useful to remark the following: If one deals with a countable set of series like the set of series for $f^{(j)}$, $j \geq 1$, there is no loss of generality if we assume that all series have the same generalized Fourier exponents. This is achieved by adding terms, with zero coefficients, after having the set of all exponents, forming a sequence, hence a countable set. This operation does not influence the conditions of convergence (31) and (45).

There remains to prove that the limits a_k, $k \geq 1$, are given by the formulae (47), i.e.,

$$a_k^{(j)} - a_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} [f^{(j)}(t) - f(t)] \exp[-i\lambda_k(t)] dt, \tag{51}$$

tends to zero as $j \to \infty$, $k \geq 1$.

The following estimates are routine in a calculus course. Indeed, one has

$$\begin{aligned} &\left| (2\ell)^{-1} \int_{-\ell}^{\ell} [f^{(j)}(t) - f(t)] \exp[-i\lambda_k(t)] dt \right| \\ &\qquad \leq (2\ell)^{-1} \int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)| dt \\ &\qquad \leq (2\ell)^{-1} \left[\int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)|^2 dt \right]^{1/2} (2\ell)^{1/2} \\ &\qquad = \left[(2\ell)^{-1} \int_{-\ell}^{\ell} |f^{(j)}(t) - f(t)|^2 dt \right]^{1/2}. \end{aligned} \tag{52}$$

The last term in (52) is as $\ell \to \infty$, exactly the norm of $f^{(j)}(t) - f(t) \in AP_2(R, \mathcal{C}; \Lambda)$, which implies it tends to zero as $j \to \infty$, by the choice of the approximating sequence $\{f^{(j)}(t);\ j \geq 1\} \subset AP_2(R, \mathcal{C}; \Lambda)$. Taking into account (51) and (52), one obtains what is required to derive that (49) is correct, it representing the connection between the Fourier series and its generalized sum, in $AP_2(R, \mathcal{C}; \Lambda)$.

Based on facts easily obtained in case of the space $AP_1(R, \mathcal{C}; \Lambda)$, which is dense in the space $AP_2(R, \mathcal{C}; \Lambda)$, we can extend results from $AP_1(R, \mathcal{C}; \Lambda)$ to the richer space $AP_2(R, \mathcal{C}; \Lambda)$, using the procedure above, when getting the formulae for the coefficients of the generalized Fourier series.

For instance, the Parseval equality (32), valid for $f \in AP_1(R, \mathcal{C})$, can be extended as proceeded above for $f \in AP_2(R, \mathcal{C}; \Lambda)$. It will look exactly as (32), which in the geometry of the Hilbert space $\ell^2 = \ell^2(N, \mathcal{C})$ means that the "length" of the limit of a convergent sequence is the limit of the sequence of lengths of the terms in the sequence. We leave to the reader the task of carrying out the details of the proof of (32), for $f \in AP_2(R, \mathcal{C}; \Lambda)$. Of course, Λ has to be the same set of generalized Fourier exponents, in AP_1 and AP_2.

Another proof of the Parseval formula (32) can be obtained based on the model we inherited from the classical period of almost periodicity. The details can be found in the author's book [5], as well as in many other sources. Instead of the exponents $\lambda_k t$, for almost periodic functions, one can substitute the general exponents $\lambda_k(t) \in \Lambda$, for oscillatory functions.

Further properties of the space $AP_2(R, \mathcal{C}; \Lambda)$ can be derived, taking into account its structure of a Banach space, whose elements are generalized Fourier series of the form (1).

We want to define the identity of two series of the form $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k(t)]$, with the usual significance of the data involved: one must have $\lambda_k(t) \in \Lambda =$ the set of generalized exponents, the same in both formal series and with equal coefficients for the same exponent.

When a norm or a seminorm is defined, usually implying a kind of convergence, we obtain a linear normed space which requires the completeness in order to become a Banach space. Another type of condition can be imposed, to help organizing the space of series (for instance, a kind of summability).

Once found a way of organizing the space of series like a linear metric space, the next step is to move from the series space to a function space, the series playing the role of a vehicle, or an intermediate stage, in the construction of the function space. We have illustrated this in constructing the spaces $AP_1(R, \mathcal{C}; \Lambda)$ and $AP_2(R, \mathcal{C}; \Lambda)$. In the literature, see particularly the quotation in the bibliography to this paper under the names of Osipov and Zhang, cases which we have summarily presented in the Introduction. Basically, one obtains such spaces of oscillatory functions by using the procedure of completion with respect to various norms or seminorms of simpler spaces (usually, classical ones).

5 More Spaces of Oscillatory Functions

It is clear from the preceding sections of this paper, including the Introduction, that once we succeed to find a set Λ of generalized Fourier exponents, one can construct several types of spaces of oscillatory functions. So far, we've got acquainted, to some extent, with the spaces built up by Osipov, Zhang and those in Sections 3 and 4, denoted by $AP_1(R, \mathcal{C}; \Lambda)$ and $AP_2(R, \mathcal{C}; \Lambda)$. In case of spaces $AP_1(R, \mathcal{C}; \Lambda)$ and $AP_2(R, \mathcal{C}; \Lambda)$, the set of generalized Fourier exponents Λ does not possess an algebraic structure, necessarily. The operations of multiplication of elements will imply the necessity of having the set Λ organized as an additive group of real-valued functions on R. The classical examples, periodic and almost periodic, illustrate the need and the involved groups: in the periodic case, the set Λ is given by $\Lambda = \{\lambda t;\ \lambda = k\omega,\ k \in \mathcal{Z},\ \omega > 0,\ \text{constant}\}$ = any closed subgroup of the topological group R; in the almost periodic case, $\Lambda = \{\lambda t;\ \lambda, t \in R\}$. In the Introduction, in case of the examples due to Osipov and Zhang, the generalized exponents for the Osipov type oscillatory functions have the form $\Lambda = \{at^2 + b_k t;\ a \in R,\ b_k \in R\}$, while in case of Zhang constructions, the generalized Fourier exponents belong to the set $Q(R; R)$; see formulae (3), (4), where the definition of the set $Q(R; R)$ is provided.

Section 2 is attempting to provide some tools in finding generalized exponents for series forming oscillatory spaces. The problem of finding such exponents must be investigated further. Some suggestions must come from the applicative problems. One can construct already many spaces of oscillatory functions, but their significance is depending of their area of applications.

In this closing section, we shall briefly list and describe some other spaces of oscillatory functions, constructed in several ways, always starting with a set of formal series characteristic for oscillatory functions and giving some comments on possible developments of the theory, formulating also some open problems. Of course, these ideas are directed toward the theoretic, but also deeply practical aim, to have in the future a developed theory of the spaces of oscillatory functions. This development, if achieved, will certainly constitute the third stage in the Fourier theory of vibrations and waves.

We shall start with the definition of the oscillatory function spaces we shall denote by $AP_r(R, \mathcal{C}; \Lambda)$, $1 < r < 2$, the cases $r = 1, 2$ being treated in the preceding section. Taking the example from existing literature, namely Shubin [16] and Corduneanu [5], the series spaces $AP_r(R, \mathcal{C}; \Lambda)$ will be formed from the generalized Fourier series like (1), i.e., $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda(t)]$, with $a_k \in \mathcal{C}$ and $\lambda_k \in C(R, R)$, $\lambda_k \in \Lambda$, $k \geq 1$, with the following property:

$$\sum_{k=1}^{\infty} |a_k|^r < \infty. \tag{53}$$

We introduce the norm, in the linear space (over $\mathcal{C}$), of the set of formal series of the form (1), by the formula

$$\left| \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \right|_{AP_1} = \left(\sum_{k=1}^{\infty} |a_k|^r \right)^{1/r}. \tag{54}$$

These norms are known as Minkowski's norms and the related inequalities are making easier the proof of the norm properties in linear normed spaces. The completion of the space of series satisfying (53) follows from the simple remark that the polynomials associated to such series (sections like $\sum\limits_{k=1}^{n} a_k \exp[i\lambda_k(t)]$), form an everywhere dense set in $AP(R, \mathcal{C}; \Lambda)$. Hence, the spaces $AP_r(R, \mathcal{C}; \Lambda)$ can be organized as Banach spaces.

These spaces, with $1 < r < 2$, enjoy many properties that can be derived from the inclusions

$$AP_1 \subset AP_r \subset AP_s \subset AP_2,\ 1 < r < s < 2, \tag{55}$$

which show that they are part of AP_2, the space we have constructed above. In particular, being also AP_2-series, they have Fourier type series (generalized) of the form (1).

For a more detailed discussion of one space in the categories of AP_r-spaces, one can consult the author's paper [6]. Several applications are provided for several types of functional differential equations, including integral equations, convolution and mixed types of functional equations (integro-differential, convolutions, delay type).

Like in the special case when $\Lambda = \{\lambda t;\ \lambda, t \in R\}$, i.e., the almost periodic type of functions, the series spaces $AP_r(R, \mathcal{C}; \Lambda)$, with the same Λ, they form a scale of oscillatory functions when we regard their elements as parts of $AP_2(R, \mathcal{C}; \Lambda)$, for which space we have more accessible information (they are modeled on the Hilbert space $\ell^2(N, \mathcal{C})$). The stronger type of convergence we find in $AP_1(R, \mathcal{C}; \Lambda)$, while the weaker one corresponds to $AP_2(R, \mathcal{C}; \Lambda)$.

We point out the fact that spaces of this scale have been seldom in attention of researchers. Many problems, like convergence of their series in different meanings (say, pointwise to uniform or a.e.) still wait for detailed investigation. Also, the problem of compactness for sets in such spaces is still unsolved, excepting in case of Zhang's space $SLP(R, \mathcal{C}; \Lambda)$, for $\Lambda = Q(R; R)$. See Zhang [19] and the Appendix in Corduneanu's et al. book [10].

With regard to the space $SLP(R, \mathcal{C}; \Lambda)$, in more general cases than a specific Λ has been considered, it is worth getting in some details of the construction. This type of space is different from those in the scale $AP_r(R, \mathcal{C}; \Lambda)$, $1 \leq r \leq 2$, in the fact that, instead of conditions on the coefficients only, like (28), (45), from the beginning one imposes the type of convergence. Namely, the space $SLP(R, \mathcal{C}; \Lambda)$ is the function space whose elements/functions can be *uniformly* approximated on R by means of generalized trigonometric polynomials of the form (4).

Since Zhang wanted to organize the space as an algebra, which idea brought some advantages, the special type of Λ has been used. A question: are there other choices for Λ, in order to achieve new spaces in the family of $SLP(R, Q)$?

Zhang [19] relied on this space ($\Lambda = Q$) to construct two new spaces of generalized Fourier type (oscillatory functions spaces).

These new spaces generalize the Besicovitch type of almost periodicity. In very brief format, the first of these spaces is obtained by completing the space $SLP(R, \mathcal{C}; Q)$ with respect to the norm

$$f \to \{M(|f|^2)\}^{1/2}, \tag{56}$$

while the second is the completion of $SLP(R, \mathcal{C}; \Lambda)$ with respect to the norm

$$f \to M(|f|), \tag{57}$$

where M stands for the Poincaré mean value on R. It turns out that the normed spaces with either norm (56) or (57) are not complete, in general. For instance, the norm which is given by (57), derives from

$$M(g) = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} g(t) dt,$$

and satisfies the inequality $|M(g)| \leq |g|$, where $|g|$ represents the supremum norm (as used by Zhang in constructing $SLP(R, \mathcal{C}; Q)$). But, in the case of almost periodic functions space $AP(R,\mathcal{C})$, the space itself generates the Besicovitch space $B(R,\mathcal{C})$, or $B = B^1$, which is not complete. See an example in Corduneanu et al. [10], the Appendix, or in [8]. Examples for oscillatory functions spaces await their apparition. That's depending on the possibility of getting an adequate Λ.

In the author's paper [7], one finds a reconstruction of the Bohr space $AP(R,\mathcal{C})$, starting from the set of all formal trigonometric series of the form $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda t]$, $t \in R$, with $a_k \in \mathcal{C}$ and $\lambda_k \in R$, $k \geq 1$.

The condition which allows us to detach those that characterize those of Bohr almost periodic functions is somewhat of a different nature than conditions (28) and (45), utilized above. Instead of imposing conditions on coefficients, of a quantitative nature, we shall require that the series be summable by a linear method (for instance, the Cesaro-Fejer-Bochner method), with respect to the uniform convergence on R. The set of exponents, apparently, does not play a direct role, such method being also based on the coefficients.

Indeed, it is known, from the theory of Bohr almost periodic functions, that their series are summable by the Cesaro-Fejer-Bochner method with respect to the uniform convergence on R. Then, the "sum" is Bohr almost periodic. In other words, a trigonometric series like $\sum\limits_{k=1}^{\infty} a_k \exp[i\lambda_k t]$, $a_k \in \mathcal{C}$, $\lambda \in R$, $k \geq 1$, is characterizing an almost periodic function in Bohr space $AP(R,\mathcal{C})$, iff it is summable with respect to the uniform convergence on RF. As we know, the uniform convergence is induced by the supremum norm.

In concluding this paper, we emphasize again the need of investigation of these spaces of series, like (1), defining the third stage of development in generalized Fourier Analysis. Of course, the Fourier Analysis has many other chapters, inspired by the investigation of classical series and the extension of such aspects appears as a future task for researchers.

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Approximate Controllability of Nonlocal Impulsive Fractional Order Semilinear Time Varying Delay Systems

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Abstract: This paper concerns with approximate (exact) controllability of nonlocal impulsive fractional order semilinear control system with time varying delay. Simple sufficient conditions for the controllability are derived by assuming that the corresponding linear control system is controllable. The results are established under the Lipschitz continuity of nonlinear function. In particular, compactness of the semigroup and uniform boundedness of nonlinear function both are dropped. Finally, some examples are given to illustrate the developed theory.

Keywords: *fractional order semilinear systems; time varying delay; reachable set; approximate controllability.*

Mathematics Subject Classification (2010): 26A33, 34K37, 34K45, 93B05.

1 Introduction

During the last three decades, various problems on fractional order systems have been investigated. Fractional order semilinear equations arise in the modeling of the problems in engineering, physics, medicine, finance, control and many other fields. Particularly, fractional order equations frequently appear in diffusion process, electrical science, electrochemistry, control science and several more. For more details see [1–6] and the references cited therein.

Controllability is the qualitative property of dynamical systems and is of particular importance in mathematical control theory. In literature various controllability problems for different types of semilinear dynamical systems have been studied [7–19] using several

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methods. Among these methods, the fixed point approach is frequently used to show the controllability of the system, in which the authors converted the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse in a function space [20–24]. In this approach, an inequality condition is always required that involves various system parameters and sometimes this condition is difficult to verify in applications.

A large number of physical dynamic systems and biological processes include time varying delay. The delays in engineering systems such as electric systems are often time-varying and sometimes vary violently with time. It is however not necessary that a system containing either time-invariant or time-varying delays is controllable. Thus the study of various types of controllability is important for application points of view. Tomar and Kumar [25] proved the approximate controllability of first order semilinear system with time varying delay. In [26] Muthukumar et al. showed the approximate controllability of nonlinear stochastic evolution time-varying delay systems. The approximate controllability of semilinear system in which the nonlinear term contains fixed delay in the state has been addressed in [14, 27]. The approximate controllability of semilinear fractional control systems, where the control function depends on multi-delay arguments and the nonlocal condition is fractional, is discussed by Debbouche and Torres [28]. Recently, Ji [29] studied the approximate controllability of fractional order control system without the compactness conditions or Lipschitz conditions for the nonlocal function.

The dynamics of many processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. Short term perturbations from continuous and smooth dynamics are involved in these phenomena and the duration of these perturbations is negligible in comparison with the duration of an entire evolution. Impulsive equations have been developed in important fields of science and technology such as modeling of impulsive problems in physics, population dynamics, ecology, biotechnology, etc. and hence the study of such systems is important. The existence and uniqueness of the mild solution of fractional order impulsive semilinear system is discussed in [30, 31]. Using Krasnoselskii's fixed point theorem Tai and Wang [32] studied the controllability of fractional order impulsive neutral functional integrodifferential systems in Banach space. Sufficient conditions for the controllability of the impulsive fractional evolution integrodifferential equations in Banach spaces are established using Banach's fixed point theorem [33]. Kumar and Sukavanam [34] proved approximate controllability of fractional order semilinear delayed systems under the Lipschitz continuity of nonlinear function and extended the results for impulsive systems also. Using Darbo-Sadovskii's fixed point theorem, sufficient conditions for approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space are derived by Balasubramaniam et al. [35]. However, it should be stressed here that there is no paper on approximate controllability of impulsive nonlocal fractional order system so far in which the nonlinear term contains time varying delay. This is the motivation of the present paper.

The main objective of this paper is to provide simple sufficient conditions for approximate controllability of semilinear systems (2). In this approach, uniform boundedness of nonlinear function, compactness of C_0-semigroup and inequality condition involving system parameters are not required. Hence the results are more general and applied to a large number of class of semilinear systems. To establish the results a relation between the reachable set of semilinear system and that of the corresponding linear system is shown. Finally, sufficient conditions for the controllability of fractional order impulsive system (1) are obtained. The nonlinear term and nonlocal condition make the paper different from [34].

The paper is organized as follows: in Section 2, the problem formulation is presented. We give some basic definitions and lemma in Section 3. Sufficient conditions for approximate controllability are obtained in Section 4. To illustrate the theory some examples are provided in Section 5.

2 Problem Formulation

Let V, $\hat{V}$ be Banach spaces and $Z = L_2([0,\tau];V)$, $Y = L_2([0,\tau];\hat{V})$ be the corresponding function spaces. Further, let $\mathcal{C}_t := C([-r,t];V)$, $r > 0$, $0 \le t \le \tau < \infty$ be a Banach space of all continuous functions from $[-r,t]$ into V and the norm on $\mathcal{C}_t$ be defined by

$$\|\varphi\|_{\mathcal{C}_t} = \sup_{-r\le\eta\le t} \|\varphi(\eta)\|_V.$$

Let $0 < t_1 < t_2 < ... < t_m < \tau$. Consider the following fractional order nonlocal impulsive system with time varying delay

$$\left.\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(\sigma(t))), && t \in]0,\tau]; \\ h(x) &= \varphi, && \text{on } [-r,0]; \\ \Delta x|_{t=t_k} &= I_k(x(t_k)), && k = 1,2,...,m, \end{aligned}\right\} \tag{1}$$

where ${}^cD_t^\alpha$ is the Caputo fractional derivative of order α; $1/2 < \alpha < 1$. The state $x(\cdot)$ takes values in Banach space V; the control function $u(\cdot)$ takes values in Y; $A : D(A) \subseteq V \to V$ is a linear operator with dense domain $D(A)$ generating a C_0-semigroup $T(t)$; B is a bounded linear operator from $\hat{V}$ to V; the function $f : [0,\tau] \times V \to V$ is nonlinear; $\sigma : [0,\tau] \to [-r,\tau]$ is a nondecreasing, non-expensive map such that it satisfies delay property i.e. $\sigma(t) \le t$, $\forall\ t \in [0,\tau]$; $h : \mathcal{C}_0 \to \mathcal{C}_0$ and there exists a function $\chi \in \mathcal{C}_0$ such that $h(\chi) = \varphi$. For some examples of h one can see [36]. Here I_k, $k = 1,2,...,m$ are appropriate functions and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Let $PC([-r,\tau],V) = \{x : [-r,\tau] \to V : x(t)$ be continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$. It is easy to see that $PC([-r,\tau],V)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| :\ t \in [0,\tau]\}.$$

To establish sufficient conditions for controllability of system (1), we first discuss controllability of the following nonlocal fractional order semilinear control system with time varying delay

$$\left.\begin{aligned} {}^cD_t^\alpha x(t) &= Ax(t) + Bu(t) + f(t, x(\sigma(t))), && t \in]0,\tau]; \\ h(x) &= \varphi, && \text{on } [-r,0]. \end{aligned}\right\} \tag{2}$$

3 Preliminaries

In this section some basic definitions and lemma, which are useful for further developments, are given.

Definition 3.1 A real function $f(t)$ is said to be in the space C_α, $\alpha \in \mathbb{R}$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty[$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 3.2 The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

where Γ is the Euler gamma function.

Definition 3.3 If the function $f \in C_{-1}^m$ and m is a positive integer then we can define the fractional derivative of $f(t)$ in the Caputo sense as

$${}^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \text{ where } m-1 \leq \alpha < m.$$

Definition 3.4 [37] A function $x \in \mathcal{C}_\tau$ is said to be the mild solution of (2) if it satisfies

$$\begin{aligned} x(t) &= S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x(\sigma(s)))] ds, \ t \in [0, \tau]; \\ x(t) &= \chi(t), \quad t \in [-r, 0], \end{aligned}$$

where

$$\begin{aligned} S_\alpha(t)x &= \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta, \\ T_\alpha(t)x &= \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) x d\theta. \end{aligned}$$

Here $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha})$ is the probability density function defined on $(0, \infty)$, that is $\phi_\alpha(\theta) \geq 0$, and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$. We define $\psi_\alpha(\theta)$ as $\psi_\alpha(\theta) = \frac{1}{\pi}\Sigma_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$, $\theta \in (0, \infty)$.

Definition 3.5 Let $x(\tau)$ be the state value of system (2) at time τ corresponding to the control u. The system (2) is said to be approximately controllable in time interval $[0, \tau]$, if for every desired final state ξ and $\epsilon > 0$ there exists a control function $u \in Y$ such that the solution of (2) satisfies

$$\|x(\tau) - \xi\| \leq \epsilon.$$

The above definition gives exact controllability of system (2) iff $\epsilon = 0$.

The set $K_\tau(f) = \{x(\tau) \in V : x(\cdot)$, is the mild solution of (2)$\}$ and is called the reachable set of the system (2). If $f \equiv 0$, then the system (2) is known as the corresponding linear system and denoted by $(2)^*$. In this case, $K_\tau(0)$ denotes the reachable set of the linear system $(2)^*$.

Definition 3.6 The system (2) is said to be approximately (exactly) controllable on $[0, \tau]$ if $\overline{K_\tau(f)} = V$ $(K_\tau(f) = V)$, where $\overline{K_\tau(f)}$ denotes the closure of $K_\tau(f)$. Clearly, the corresponding linear system $(2)^*$ is approximately controllable if $\overline{K_\tau(0)} = V$.

Lemma 3.1 *For any fixed* $t \geq 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are linear and bounded operators, that is, for any* $x \in V$, $\|S_\alpha(t)x\| \leq M\|x\|$ *and* $\|T_\alpha(t)x\| \leq \frac{M\alpha}{\Gamma(\alpha+1)}\|x\|$, *where* M *is a constant such that* $\|T(t)\| \leq M$, *for all* $t \in [0,\tau]$ *(see Lemma 3.2 [37]).*

We now define the operator $F : Z \to Z$ as

$$[Fx](t) = f(t, x(\sigma(t))); \; x \in Z.$$

The following conditions are required to establish the results:

$[H1]$ The nonlinear function satisfies the Lipschitz continuity, i.e. there exists some positive constant l such that

$$\|f(t,x) - f(t,y)\|_V \leq l\|x-y\|_{\mathcal{C}_\tau}, \text{ for all } x, y \in V.$$

Remark 3.1 Under assumption $[H1]$ one can easily verify that the mild solution of system (2) exists and is unique.

$[H2]$ Range of function F is a subset of closure of range of B, i. e.

$$R(F) \subseteq \overline{R(B)}.$$

Remark 3.2 To support this condition an example is given. Also if $B = I$ the range condition is trivially true. In several real life problems the above condition is also satisfied [38].

$[H3]$ The linear system $(2)^*$ is approximately controllable.

4 Main Results

4.1 Controllability of semilinear system

Theorem 4.1 *Under the assumptions* $[H1]$-$[H3]$ *the fractional order semilinear control system (2) is approximately controllable.*

Proof. To prove the result, we will show that $K_\tau(0) \subset K_\tau(f)$. For this, we assume that $x(\cdot)$ is the mild solution of $(2)^*$ corresponding to a control $u \in Y$ which is given by

$$\left.\begin{array}{ll} x(t) = S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)Bu(s)ds, & t \in [0,\tau]; \\ x(t) = \chi(t), & t \in [-r, 0]. \end{array}\right\} \tag{3}$$

Since $Fx \in \overline{R(B)}$ (by $[H2]$), for a given $\epsilon > 0$ there exists a control function $w \in Y$ such that

$$\|Fx - Bw\|_Z \leq \epsilon. \tag{4}$$

We now assume that $y(t)$ is the mild solution of (2) corresponding to the control $(u - w)$ in Y then

$$\left.\begin{array}{ll} y(t) = S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\{B(u-w) + [Fy]\}(s)ds, & t \in [0,\tau]; \\ y(t) = \chi(t), & t \in [-r, 0]. \end{array}\right\} \tag{5}$$

If $t \in [0, \tau]$ then from (3) and (5), we have

$$\begin{aligned} x(t) - y(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bw - Fy](s)ds \\ &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bw - Fx](s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Fx - Fy](s)ds. \end{aligned}$$

Taking norm on both sides and using (4), we get

$$\begin{aligned} \|x(t) - y(t)\|_V &\le \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|Bw(s) - Fx(s)\|_V ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|[Fx](s) - [Fy](s)\| ds \\ &\le \frac{M\alpha}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} \times \\ &\quad \Big(\int_0^t \|Bw(s) - Fx(s)\|^2 ds \Big)^{1/2} \\ &\quad + \frac{M\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|[Fx](s) - [Fy](s)\|_V ds \\ &\le \frac{M\alpha}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} \Big(\|Fx - Bw\|_Z \Big) \\ &\quad + \frac{M\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \|[Fx](s) - [Fy](s)\|_V ds \\ &\le \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \Big(\int_0^t (t-s)^{2\alpha-2} ds \Big)^{1/2} + \frac{M\alpha}{\Gamma(\alpha+1)} \times \\ &\quad \int_0^t (t-s)^{\alpha-1} \|f(s, x(\sigma(s))) - f(s, y(\sigma(s)))\|_V ds \\ &\le \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \\ &\quad + \frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^\tau (\tau-s)^{\alpha-1} \|x - y\|_{\mathcal{C}_\tau} ds. \end{aligned}$$

For all values of $t \in [-r, \tau]$, we have

$$\|x(t) - y(t)\|_V \le \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} + \frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^\tau (\tau-s)^{\alpha-1} \|x - y\|_{\mathcal{C}_\tau} ds.$$

Using Gronwall's inequality, we get

$$\begin{aligned} \||x - y\|_{\mathcal{C}_\tau} &\le \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \exp\left(\frac{Ml\alpha}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} ds \right) \\ &\le \frac{M\alpha\epsilon}{\Gamma(\alpha+1)} \sqrt{\frac{\tau^{2\alpha-1}}{2\alpha-1}} \exp\left(\frac{Ml\tau^\alpha}{\Gamma(\alpha+1)} \right). \end{aligned}$$

Since the right hand side of above inequality depends on $\epsilon > 0$ and ϵ is arbitrary, it is clear that $\|x - y\|_{\mathcal{C}_\tau}$ can be made arbitrary small by choosing suitable value of control function w. It now follows that the reachable set of system (2) is dense in the reachable set of system $(2)^*$, which is dense in V due to condition $[H3]$. Hence the approximate controllability of $(2)^*$ implies that of the semilinear control system (2). This completes the proof.

4.2 Controllability of Impulsive System

We now prove the approximate controllability of the system (1).

Definition 4.1 [30, 31] The mild solution of the system (1) is a function $x \in PC([-r, \tau]; V)$ such that it satisfy the following integral equation

$$x(t) = \begin{cases} S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) + f(s, x(\sigma(s)))]ds, & t \in]0, t_1]; \\ S_\alpha(t-t_1)[x(t_1^-) + I(x(t_1^-))] + \int_{t_1}^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) & \\ +f(s, x(\sigma(s)))]ds, & t \in]t_1, t_2]; \\ \cdots & \\ S_\alpha(t-t_m)[x(t_m^-) + I(x(t_m^-))] + \int_{t_m}^t (t-s)^{\alpha-1} T_\alpha(t-s)[Bu(s) & \\ +f(s, x(\sigma(s)))]ds, & t \in]t_m, \tau]; \\ \chi(t), & t \in [-r, 0]. \end{cases}$$

To establish the result we need one more hypothesis on the impulsive function as follows: $[H4]$ The functions I_k, $k = 1, 2, \cdots, m$ are continuous and uniformly bounded.

Theorem 4.2 *Under the assumptions [H1]–[H4] the fractional order semilinear control system (1) is approximately controllable.*

Proof. Let $y(t)$ be the mild solution of (1) corresponding to the control $(u - w)$ then

$$y(t) = \begin{cases} S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]0, t_1]; \\ S_\alpha(t-t_1)[y(t_1^-) + I(y(t_1^-))] + \int_{t_1}^t (t-s)^{\alpha-1} T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]t_1, t_2]; \\ \cdots \\ S_\alpha(t-t_m)[y(t_m^-) + I(y(t_m^-))] + \int_{t_m}^t (t-s)^{\alpha-1} T_\alpha(t-s)[B(u-w)(s) \\ +f(s, y(\sigma(s)))]ds, \quad t \in]t_m, \tau]; \\ \chi(t), \quad t \in [-r, 0]. \end{cases}$$

The mild solution $x(t)$ of $(2)^*$ corresponding to a control u is given by

$$\begin{aligned} x(t) &= S_\alpha(t)\chi(0) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s) ds, \quad t \in]0, \tau]; \\ x(t) &= \chi(t), \quad t \in [-r, 0]. \end{aligned}$$

To show the approximate controllability of semilinear system (1), we divide the interval $[-r, \tau]$ into subintervals $[-r, 0]$, $]0, t_1]$, $]t_1, t_2]$, $\cdots$, $]t_m, \tau]$. Now if $t \in]-r, t_1]$ the approximate controllability of the system follows from Theorem 4.1. If $t \in]t_1, t_2]$, since both $y(t_1^-)$ and $I(y(t_1^-))$ are bounded, we are able to prove the approximate controllability in the interval $t \in]t_1, t_2]$ as shown in Theorem 4.1. Similarly, we can show the approximate controllability in subsequent intervals. This completes the proof of the theorem.

5 Examples

In this section, we give examples to show the effectiveness of the developed theory.

Example 5.1 Let $V = L_2(0,\pi)$ and $A \equiv \frac{d^2}{dx^2}$ with $D(A)$ consisting of all $y \in V$ with $\frac{d^2y}{dx^2}$ and $y(0) = 0 = y(\pi)$. Put

$$\phi_n(x) = (\frac{2}{\pi})^{1/2}\sin(nx); 0 \le x \le \pi,\ n = 1, 2, \cdots,$$

then $\{\phi_n\}$ is an orthonormal base for V and ϕ_n is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. Then the C_0-semigroup $T(t)$ generated by A has $\exp(\lambda_n t)$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions, see [39].

Define an infinite-dimensional space $\hat{V}$ by

$$\hat{V} = \left\{ u \mid u = \sum_{n=2}^{\infty} u_n\phi_n,\ \text{with}\ \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.$$

The norm in $\hat{V}$ is defined by

$$\|u\|_{\hat{V}} = \left(\sum_{n=2}^{\infty} u_n^2\right)^{1/2}.$$

Define a continuous linear map B from $\hat{V}$ to V as

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \text{ for } u = \sum_{n=2}^{\infty} u_n\phi_n \in \hat{V}.$$

Let us consider the following fractional order semilinear control system of the form

$$\begin{aligned} {}^cD_t^\alpha y(t,x) &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t, y(\sigma(t)));\ t \in [0,\tau], 0 < x < \pi \\ y(t,0) &= y(t,\pi) = 0;\ t > 0 \\ y_0(x) &= \frac{1}{r}\int_{-r}^{0} \exp(2s) y(s,x)ds. \end{aligned} \tag{6}$$

Let $\sigma(t) = \frac{t^2}{t^2+1} - r$ be time varying aftereffect such that $\sigma(t) \le t$ for all $t \in [0,\tau]$. If we take $h(y)(t) = g(y)$ for $y \in \mathcal{C}_0$, $t \in [-r, 0]$; $\varphi = y_0$, where $g : \mathcal{C}_0 \to V$ is such that $g(y)(x) = \frac{1}{r}\int_{-r}^{0} \exp(2s)y(s,x)ds$. Thus we are able to define a function $\chi \in \mathcal{C}_0$ such that $\chi(t) = \frac{1}{k}y_0$ on $[-r, 0]$ with $k = \frac{1}{r}\int_{-r}^{0} \exp(2s)ds \ne 0$ and

$$h(\chi)(t) = \frac{1}{r}\int_{-r}^{0} \exp(2s)\left(\frac{1}{k}y_0\right) ds = y_0 = \varphi(t).$$

Thus the system (6) can be written in the abstract form given by (2). If the conditions $[H1]$–$[H3]$ are satisfied, then the approximate controllability of system (6) follows from Theorem 4.1. For example, if we consider the function f as

$$f(t,z) = l\|z\|(\phi_3(z) + \phi_4(z)),$$

where l is a positive constant. Here it is clear that f satisfies $[H3]$ with Lipschitz constant l and $R(f) \subset R(B)$. However, it should be noted that the nonlinear term is not uniformly bounded.

Example 5.2 Let us consider the following fractional order impulsive system with finite delay

$$
\begin{aligned}
{}^cD_t^\alpha y(t,x) &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t, y(t-r,x)); \ t \in [0,\tau], 0 < x < \pi, \\
y(t,0) &= y(t,\pi) = 0; \ t > 0, \\
y(t,x) &= \varphi; \ t \in [-r, 0], \\
y(t_k^+, x) &- y(t_k^-, x) = I_k(y(t_k^-, x)); \ k = 0, 1, 2, \cdots, \qquad (7)
\end{aligned}
$$

where $I_k > 0, \ k = 1, 2, \cdots, m$ and $\varphi \in \mathcal{D} = \{\nu : [-r, \tau] \to V : \nu(t)$ is continuous everywhere except for some t_k at which $\nu(t_k^-)$ and $\nu(t_k^+)$ exist and $\nu(t_k^-) = \nu(t_k)\}$.

The system (7) can be reformulated in the abstract form given by (1). The approximate controllability of the system (7) follows from Theorem 4.2 if the conditions [H1]–[H4] are satisfied.

Conclusion

The approximate controllability of nonlocal impulsive fractional order semilinear time varying delay systems is proved. In literature, fixed-point theory has been used to establish the approximate controllability of semilinear control systems. This approach needs certain inequality conditions involving various system parameters which are sometimes difficult to be verified. Here, the approximate controllability of nonlocal impulsive fractional order semilinear control system has been proved for a certain class of nonlinear functions under simple sufficient conditions.

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Deterministic Random Dynamics Generated by Non-linear Non-invertible Transformations of Oscillating Functions

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Abstract: We investigate the generation of highly complex dynamics within non-invertible transformations of specific sets of continuous-time variables. We show that the time series complexity indices depend on the previous values emerging from the initial variables, through analytical complexity models for Fourier spectra, Lyapunov exponents and correlation functions. In some cases, these systems can produce completely unpredictable dynamics in a deterministic way. A comparison of the theory with standard numerical complexity estimators is presented.

Keywords: *strange attractors, chaotic dynamics; complex behavior, chaotic systems; random number generation.*

Mathematics Subject Classification (2010): 37D45, 34C28, 65C10.

1 Introduction

Polynomial functions used as non-linear, non-invertible transformations can be found in a large number of systems that exhibit chaos and hyperchaos, among which we can highlight a modified version of Chua's circuit [1], and the Ikeda system implemented in an electro-optical feedback oscillator with time delay [2–4], whose nonlinear transfer characteristic is a sine function. In discrete-time one-dimensional systems, the non-invertibility is essential for the existence of chaos [5], for example, the Logistic, Tent and Bernoulli maps use this kind of nonlinearity. These maps have been used for the development of the *deterministic*

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randomness theory, which includes random maps that generate unpredictable sequences [6–9], demonstrating that autonomous dynamical systems, containing nonlinear terms described by periodic functions of the variables, can generate random dynamics.

Complexity theory has developed a set of measures that can be used according to the system under study [19–21], and the information we need to extract for an application or theory. In that sense, the complexity indices we consider to characterize the dynamics are the correlation function and the Lyapunov exponent, providing us information about the dependence between different values of the data at different times, and the level of sensitive dependence on the set of initial values, respectively. In addition, the Fourier spectrum is an index that we choose in order to determine if there exist phase coherences [22] in the spectral structures under consideration. These three indices fulfil the role of highly complex dynamics indicators. Analytical complexity indices for chaotic and random maps have also been developed to characterize chaotic dynamics accurately, therefore important mathematical models have emerged to study the statistical properties of the Ulam [23], Bernoulli [24] and piecewise linear maps [25].

In applications and experiments, continuous-time hyperchaos has been generated by non-linear, non-invertible static transformations of low-dimensional signals in electronic [14] and electro-optical systems [15], prompting important developments of chaotic communications schemes [16] and as high-quality random number generators [17, 18]. These developments inspired us to investigate the properties that allow non-invertible transformations of oscillating functions to control the generation of deterministic randomness without the need for any external random input.

In the present paper, we show that there exists a deterministic method to generate unpredictable dynamics, in the sense that x_{n+1} can not be determined by any sequence $x_n, x_{n-1}, \ldots, x_0$ of previous values, controlled by the topological properties of the nonlinear function. We discuss how those complexity indices of the dynamics depend on the kind of nonlinearity.

2 Preliminaries and Models

The well-known discrete-time solution of the equation that satisfies the Chebyshev polynomial, applied k-times onto itself, is

$$x_n = \cos[k^n \arccos x_0]. \tag{1}$$

Under the initial condition $x_0 = \cos(\pi q/p)$, this function can generate aperiodic orbits [10]

$$x_n = \cos\left[k^n \left(\frac{\pi q}{p}\right)\right], \tag{2}$$

where p and q are real numbers.

In previous approaches, the parameter k can be set as a rational number greater than one, obtaining highly complex maps [11]. In fact, depending on the particular value of k, these functions can produce truly random numbers. For example, if k is chosen irrational, the numbers generated by equation (2) are statistically independent [12, 13].

Generally, equation (2) has been studied by the representation $x_n = h(f(n))$, with $h(\cdot)$ as a non-invertible function and $f(n)$ as an oscillating dynamics, able to generate unpredictable time series. Consequently, if initially we consider $f(n)$ as a string of values $\{z_i\}$, it is possible to rewrite equation (1) for the initial condition $x_0 = \cos[\pi(z_1, z_2, \ldots, z_n)]$ as follows,

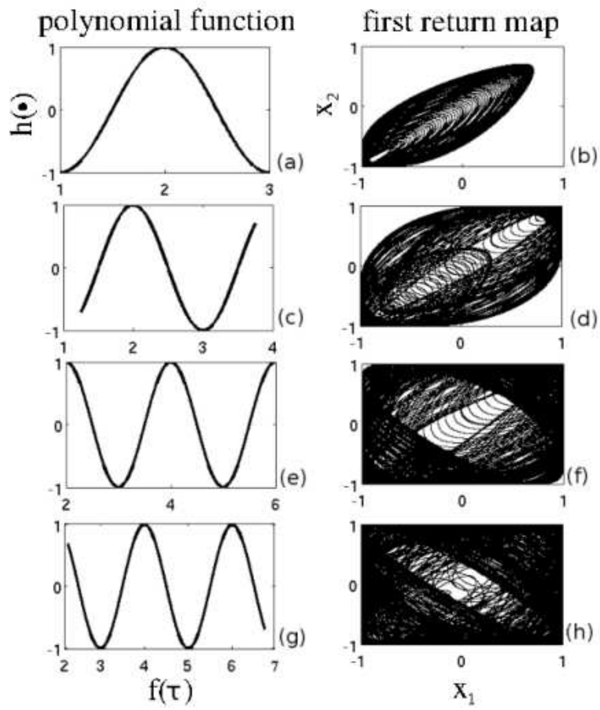

Figure 1: Polynomial functions with the variation of k and first return maps of the dynamics produced. The function to be transformed is a time series $z(t)$ from the Rössler attractor.

$$x_n = \cos[k^n \pi(z_1, z_2, \ldots, z_m)], \tag{3}$$

where $k \in \mathbb{N} - \{1\}$ is the iteration parameter of the map, n is the discrete time of the sinusoidal map and m is the number of values of the sequence $\{z_i\}$. From a simple inspection of this equation we notice that the iteration variable k acts as an amplifying parameter or a gain, that can be used to control the number of maxima and minima of the static sinusoidal function used to transform the oscillating dynamics.

The sequence $\{z_i\}$ is a known time series that can be defined by the following oscillating functions $z(t)$: `a)` periodic, `b)` 2-quasiperiodic, `c)` Rössler [26] and `d)` Lorenz [27] chaotic attractors. These oscillating functions have been chosen to analyze the generation of unpredictable time series starting from time series $z(t)$ with different levels of complexity. Therefore, equation (3) is represented as follows

$$x_n^{(t)} = \cos(k^n \pi z(t)) \equiv h(f(n,t)). \tag{4}$$

This discrete-time expression means that for each n it is possible to set a relevant continuous-time string of values $z(t)$. The use of continuous time series $z(t)$ is relevant for practical purposes. In deed, some of this signals can be implemented and solved experimentally by Analog Computing Techniques, allowing us to obtain deterministic random number generators for scientific and engineering tasks.

The chaotic time series from the Rössler attractor [26] is used as $z(t)$ in Figure 1, where we show the first-return maps. The parameter k is modified in $h(\cdot)$ as follows:

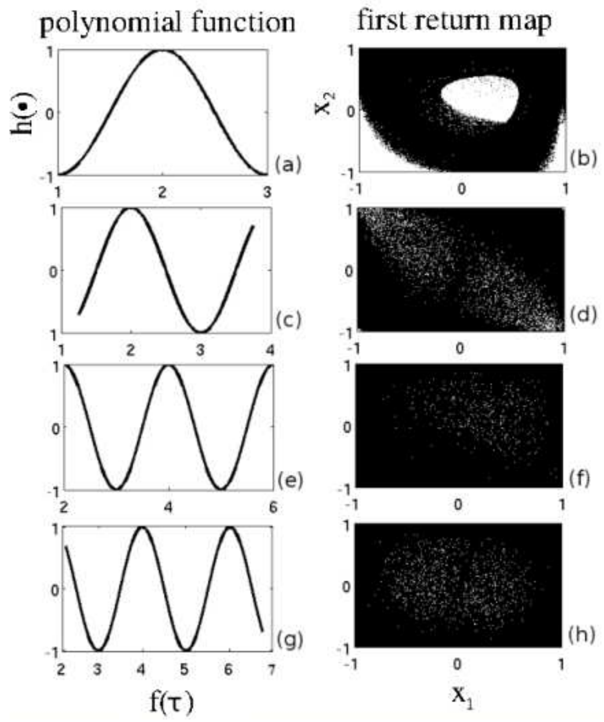

Figure 2: Polynomial functions with the variation of k and first return maps of the dynamics produced. The function to be transformed is a time series $z(t)$ from the Lorenz attractor.

(a) $k = 2$, (c) $k = 3$, (e) $k = 4$ and (g) $k = 5$. The stretch-folding mechanism [5] is helpful to explain qualitatively the action of the function $h(\cdot)$ on the quasiperiodic signal $z(t)$: Figure 1.b. shows that in the stretch step the object is elongated to twice its original length, causing the well-known exponential divergence between near orbits. In the folding step the interval is folded back around its center, with the orbits bounded to the interval $[-1, 1]$. When we increase k further and make available more peaks and valleys on $h(\cdot)$, the process of stretch-folding is accumulated.

In the case of the Lorenz system, we emphasize the fact that the first return map generated with $z(t)$, see Figure 2.b, has lost its initial topological shape when $k = 2$, i.e., the dynamics changed from a 2-scroll to a single-scroll chaotic attractor. This indicates that the initial topology of an object can be changed by the stretch-folding process provided by the non-invertible transformation function. In Figures 1.h and 2.h there occurs a high-dimensional mixture of orbits such that it is not longer possible to infer the structure of the object before the transformation.

3 Complexity Indices

We begin the Fourier spectrum analysis from the simplest oscillating functions, i.e. periodic time series. The transformed function is

$$h(f(t)) = \cos[G \sin(\omega t)], \tag{5}$$

with $\omega = 2\pi 50$ and $G = k^n\pi$. By Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ we express $h(\cdot)$ as follows:

$$h(f(t)) = \mathbb{R}e\{e^{jG\sin(\omega t)}\} = \mathbb{R}e\{y(t)\}, \tag{6}$$

with $y(t) = e^{jG\sin(\omega t)}$. This function is written as the exponential Fourier series,

$$y(t) = e^{jG\sin(\omega t)} = \sum_{p=-\infty}^{\infty} y_p e^{jp\omega t}, \tag{7}$$

where y_p is the exponential Fourier coefficient,

$$y_p = \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-j\omega t}dt = J_p(G). \tag{8}$$

Substituting equation (8) in equation (6), we obtain the time series in terms of the Bessel function of first kind,

$$h(f(t)) = \sum_{p=-\infty}^{\infty} J_p(G)\cos(p\omega t). \tag{9}$$

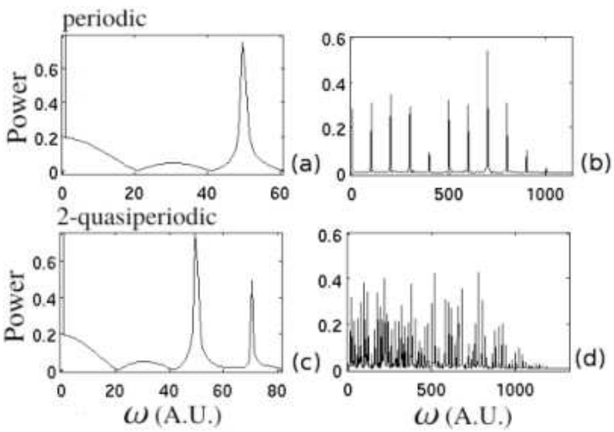

Figure 3: Spectral characteristics of the time series, initially (a) and (c), and when $G = 5\pi$ (b) and (d). For periodic (upper) and 2-quasiperiodic (lower) functions.

Next we consider the transformed function from the 2-quasiperiodic time series:

$$h(f(t)) = \cos[G(\sin(\omega_1 t) + 0.5\sin(\omega_2 t))], \tag{10}$$

where $\omega_1 = 2\pi 50$ and $\omega_2 = 2\pi 50\sqrt{2}$. Following the above procedure, the transformed time series in terms of Bessel functions of the first kind is

$$h(f(t)) = \sum_{p_1,\, p_2=-\infty}^{\infty} J_{p_1}(G)J_{p_2}(G)\cos(p_1\omega_1 + p_2\omega_2)t. \tag{11}$$

The Fourier spectrum $h(f(\omega))$ is obtained from the Fourier transform. As expected, as $G \to \infty$, the number of represented frequencies will be greater, because these depend on the p values of the Bessel function $J_p(G)$, i.e., if G is large, the number p of spectral peaks will be bigger [28]. In Figures 3.a and 3.c we show periodic and 2-quasiperiodic functions, respectively. Amplifying the periodic time series with $G = 5\pi$ just in the case where $n = 1$, see Figures 3.b and 3.d, we found that the spectra of the transformed functions have been filled due to the increment of G with respect to the initial spectra.

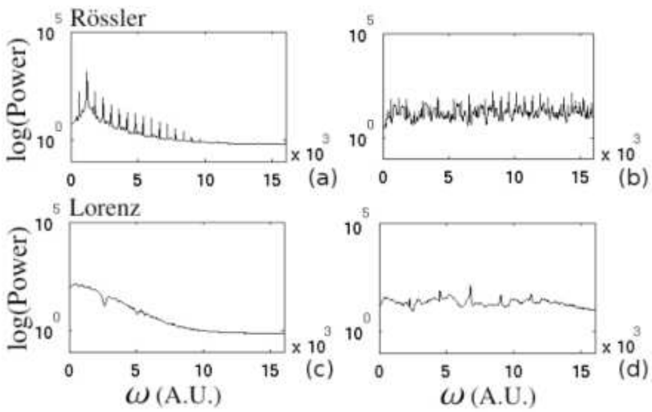

Figure 4: Spectral characteristic of the time series, initially (a) and (c), and when $G = 5\pi$ (b) and (d). For Rössler (upper) and Lorenz (lower) chaotic functions.

Using chaotic functions, we assume in the Fourier analysis that the chaotic time series can be expressed as a sum of m harmonic functions, therefore it is correct to write the time series as follows

$$h(f(t)) = \sum_{p_1=-\infty}^{\infty} \cdots \sum_{p_m=-\infty}^{\infty} J_{p_1}(G) \cdots J_{p_m}(G) \times \cos(p_1\omega_1 + \cdots + p_m\omega_m)t. \tag{12}$$

The Rössler and Lorenz spectra are shown before (Figures 4.a,c) and after the transformation (Figures 4.b,d). These transformed broadband spectra are qualitatively comparable with that of white noise, over the range of frequencies that we have considered initially. They are composed by a very large number of Bessel functions $J_n(G)$. Interestingly, the phase coherences that existed in the spectra before the transformation have disappeared, especially in the most coherent one which is the Rössler system.

As yet we have characterized the maps by their qualitative complexity. In the following, we will add statistical analysis to strengthen the qualitative perception about those complex maps.

The correlation function is determined via the Frobenius-Perron operator applied to the Chebyshev polynomials of the first kind. We found the topological conjugate of the equation (1) to the piecewise map with the application of $x \to x = \cos[\pi z(t)]$, then $(z(t))_{n+1} = \pm[k(z(t))_n - s]$ in $(z(t))_n : [0,1] \to [0,1]$, for all t

$$(z(t))_{n+1} = \begin{cases} k(z(t))_n, & 0 \leq (z(t))_n \leq 1/k, \\ -k(z(t))_n + 1, & 1/k < (z(t))_n \leq 1. \end{cases} \tag{13}$$

This equation indicates that the nth state constitutes a domain of values depending on $z(t)$ initially chosen.

The Frobenius-Perron operator [29] acting on an arbitrary function $B(z(t))$, from the equation (13),

$$\mathcal{H}B(z(t)) = \frac{1}{k}B\left(\frac{z(t)}{k}\right) - \frac{1}{k}B\left[\frac{1}{k}\left(1 - z(t)\right)\right], \tag{14}$$

allows us to determine the normalized correlation function of orbits belonging to an attractor Ω given by

$$C_n = \frac{\langle \delta z(t)|\mathcal{H}^n \delta z(t)\rangle}{\langle \delta z(t)|\delta z(t)\rangle}, \tag{15}$$

where $\delta z(t) = z(t) - \langle z(t)\rangle$, and consider the linear operator

$$\mathcal{H}\psi_l(z(t)) = \epsilon_l \psi_l(z(t)), \tag{16}$$

where ϵ_l is an eigenvalue and $\psi_l(z(t))$ is an eigenfunction of $\mathcal{H}$. We find ϵ_l by constructing an orthonormal system [30], with $\epsilon_l = \langle \mathcal{H}\psi_l|\psi_m\rangle$,

$$\epsilon_l = \frac{1}{k^{l+1}}[1 - (-1)^l], \tag{17}$$

where

$$|\psi_m\rangle = \left|(-1)^m \delta^{(m)}(z(t))/m!\right\rangle,\ \langle\psi_l| = \langle z(t)^l|,\ B(z) = z^l,\ l, m = 0, 1, 2, \ldots.$$

If $l = 1$, associated to the first degenerate eigenvalue from $\mathcal{H}$, and approximating $\delta z(t) = b_1\psi_1$ in equation (15), we obtain $C_n = C_0\epsilon_1^n$, with $\epsilon_1 = 2k^{-2}$

$$C_n(k) = C_0\left(\frac{2}{k^2}\right)^n, \tag{18}$$

where C_0 is the correlation function of the attractor initially.

Moreover, the largest Lyapunov exponent can be determined by [31]

$$\lambda_{max} = \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{\delta(z(t))_n}{\delta(z(0))_0}\right|, \tag{19}$$

with $|(\delta(z(0))_0| \to 0$, indicating that the rate of divergence of nearby trajectories depends on the Lyapunov exponent of the initial attractor. Now let us apply the transformation $x \to x = \cos[\pi z(t)]$ in the equation (3), obtaining $(z(t))_n = \pm[k^t(z(t))_0 - s]$, and substituting in equation (19),

$$\lambda_{max} = \lambda_0 + n\ln|k|, \tag{20}$$

where $\lambda_0 = \ln|\delta(z(t))_0/\delta(z(0))_0|$. Similarly, as in the correlation function expression, the Lyapunov exponent depends on its initial exponent value, evaluated for the attractor used as initial condition.

The analytical expressions just determined are exact expressions that evaluate the complexity of the generated maps. We need to corroborate their behavior by comparing the numerical estimations of the autocorrelation coefficients and the largest Lyapunov exponents using the software package TISEAN [32].

Figures 5 and 6 show that the analytical approaches in equations (18) and (20) are accurate, because the Lyapunov exponent $\lambda(k)$ increases and the correlation function $C(k)$ decreases when k is increased, after setting $n = 1$. Here, the graphics have been separated into functions that can be expressed as polynomials of even and odd degree:

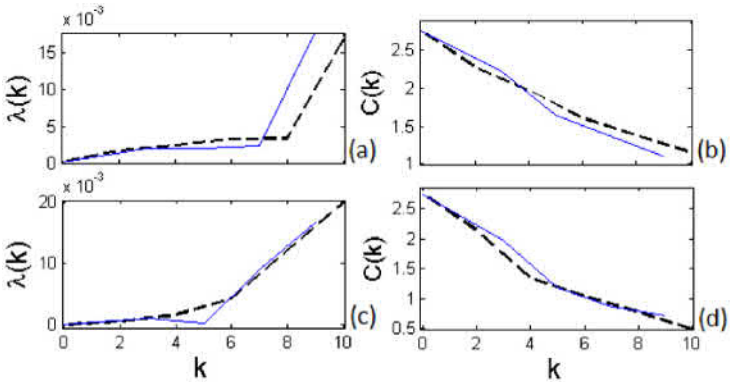

Figure 5: Complexity indices: largest Lyapunov exponent $\lambda(k)$ and autocorrelation coefficient $C(k)$ of periodic (a,b) and 2-quasiperiodic (c,d) functions when the gain k is modified.

the solid curve corresponds to the odd polynomial functions and the dotted curve corresponds to the even ones. This separation is established in order to visualize that the behavior of odd degree polynomials generate first return maps qualitatively more entangled. The statistical analysis shows that the polynomials of even and odd degree produce approximately the same evolution of the complexity index. We highlight that the rate of mixing increases when the gain parameter k is increased, it means that no large time intervals are needed to reach close to zero values for correlation decay.

The Lyapunov exponent has an increasing behavior when k is incremented. This result agrees with equation (20) and indicates the intrinsic relationship between the Lyapunov exponent and the correlation function: as the exponential divergence of nearby trajectories increases due to the polynomial function, the correlation function decreases, i.e., the time series becomes more unpredictable when $k \to \infty$. Furthermore, it was found that the greater the complexity of the time series $z(t)$, the greater the Lyapunov exponents will be, and the autocorrelation coefficients will approach zero for values of k increasingly large. All the correlations of the time series analyzed decay to zero, but the more complex ones decay faster due to the dependence on C_0.

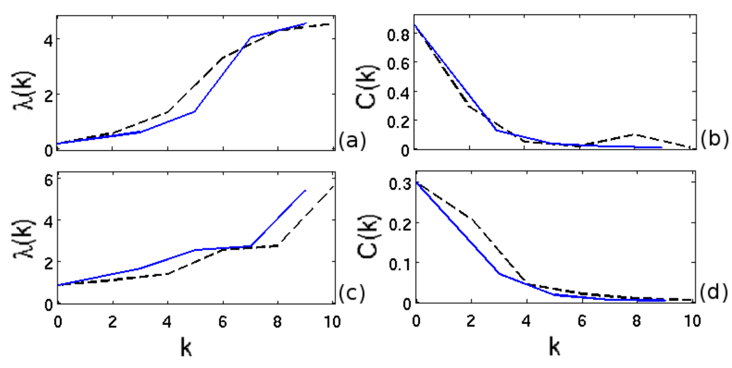

Figure 6: Complexity indices: largest Lyapunov exponent $\lambda(k)$ and autocorrelation coefficient $C(k)$ of Rössler (a,b) and Lorenz (c,d) functions when the gain k is varied.

4 Conclusion

Complex dynamics has been generated by non-linear, non-invertible transformations of oscillating functions. These transformations made the time series become unpredictable as measured by correlations and Lyapunov exponents. Objects in the phase space exhibited abrupt qualitative changes in their properties when k is increased, indicating that these systems are able to modify the initial topology of the object. Deep characterization of the polynomial functions showed that the non-invertibility is associated with the generation of complexity, as shown by the decay to zero of the autocorrelation coefficients and increased Lyapunov exponents, both analytically and numerically calculated. The Fourier analysis showed that as the number of peaks and valleys were increased in the transformation function, the time series exhibited spectra with increasing bandwidth, losing the structure present initially and rendering the signals white-noise like. This loss of structure in the spectra speaks of linear correlations being removed by the transformation, revealing a controlled increase of complexity by this process.

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