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Cubic Operators Corresponding to Graphs

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Abstract: We introduce a notion of a cubic stochastic operator corresponding to graph. We prove that each such operator has a unique fixed point. Besides, it is shown that any trajectory of such cubic stochastic operator exponentially rapidly converges to this fixed point.

Keywords: *quadratic stochastic operator; cubic stochastic operator; Volterra and non-Volterra operators.*

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1 Introduction

The history of quadratic stochastic operator (QSO) can be traced to Bernshtein's work [1]. Since then the theory of QSOs has been further developed motivated by their frequent occurrence in several problems of physical, economical and biological systems, where QSOs serve as a tool for the study of dynamical properties and modeling, see [2,4–12,15, 19–23]. While they were originally introduced as "evolutionary operators" to describe the dynamics of gene frequencies for given laws of heredity in mathematical population genetics, QSOs and the dynamical systems they describe have become interesting objects of study in their own right from a purely mathematical point of view. For a recent review on the theory of quadratic operators see [7].

In modern scientific investigations non-linear operators of higher order arise. Nowadays another class of nonlinear operators which are different from QSOs arises. In particular, *cubic stochastic operator* (CSO) can be obtained in gene engineering and free population with ternary production. In paper [17] the concept of *cubic stochastic operator* was introduced.

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One such subclass that arises naturally in the biological context is given by the additional restriction

$$p_{ijk,l} = 0, \quad \text{if} \quad l \notin \{i, j, k\} \quad \text{for all} \quad i, j, k, l. \tag{1}$$

These CSOs describe a reproductory behaviour where the offspring is a genetic copy of one of its parents and are called *Volterra operators*. The asymptotic behaviour of trajectories of this kind of CSOs for some particular cases were analysed in [13, 14, 17, 18].

However, in the non-Volterra case (i.e. when condition (1) is violated), many questions remain open and there seems to be no general theory available.

In all of the above-mentioned references the authors investigated trajectories of a CSO on finite dimensional unit simplex. However, it seems natural to consider the problem for an infinite dimensional CSO. This can be done, e.g., by using a method of infinite dimensional Volterra quadratic stochastic operator considered in [16].

The paper is organised as follows. In Section 2 we recall definitions and well known results from the theory of Volterra and non-Volterra CSOs . In Section 3 we define a new class of non-Volterra CSOs and show that a CSO from this class has a unique fixed point. Moreover, we prove that the trajectory of such operators has the regularity property and consequently the ergodic hypothesis is verified.

2 Preliminaries and Known Results

Let $[m] = \{1, 2, ..., m\}$. By the $(m-1)-$ simplex we mean the set

$$S^{m-1} = \{\mathbf{x} = (x_1, ..., x_m) \in R^m : x_i \geq 0, \quad \sum_{i=1}^{m} x_i = 1\}.$$

Each element $\mathbf{x} \in S^{m-1}$ is a probability measure on $[m]$ and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

A cubic stochastic operator (CSO) $V : S^{m-1} \mapsto S^{m-1}$ has the form

$$V : x'_l = \sum_{i,j,k=1}^{m} p_{ijk,l} x_i x_j x_k, \quad (l = 1, \ldots, m), \tag{2}$$

where $p_{ijk,l}$ is a coefficient of heredity and

$$p_{ijk,l} \geq 0, \quad \sum_{l=1}^{m} p_{ijk,l} = 1, \quad (i, j, k, l = 1, ..., m). \tag{3}$$

More precisely $p_{ijk,l}$ is the conditional probability $P(l|i, j, k)$ with which the ith, jth and kth species interbreed successfully, when they produce an individual l. We assume that there is no difference whatever the "next" is, and in any generation the "parents" i, j, k are independent, that is $P(i, j, k) = P(i)P(j)P(k) = x_i x_j x_k$, i.e. we consider models of free population.

For a given $\mathbf{x}^{(0)} \in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\}$, $n = 0, 1, 2, \ldots$ of an initial point $\mathbf{x}^{(0)}$ under the action of CSO (2) with (3) is defined by $\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)})$, where $n = 0, 1, 2, \ldots$

A point $\mathbf{x} \in S^{m-1}$ is called a fixed point of V if $V(\mathbf{x}) = \mathbf{x}$. A CSO V on S^{m-1} is called *regular* if for any initial point $\mathbf{x} \in S^{m-1}$ the limit $\lim_{n\to\infty} V^n(\mathbf{x})$ exists. The biological interpretation of the regularity of a CSO is rather clear: in the long run the distribution of species in the next generation coincides with the distribution of species in the previous one, i.e., it is stable.

For a nonlinear dynamical system, Ulam [21] suggested an analogue of a measure-theoretic ergodicity in the form of the following ergodic hypothesis: a QSO V is said to be *ergodic* if the limit $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=0}^{n-1}V^k(\mathbf{x})$ exists for any $\mathbf{x}\in S^{m-1}$.

On the basis of numerical calculations, Ulam [21] conjectured that for any QSO the ergodic hypothesis holds. In [22], Zakharevich proved that this conjecture is false in general. In [17], the authors proved that a class of Volterra CSOs has the ergodic property. The biological interpretation of non-ergodicity of a CSO is the following: in the long run the behavior of the distributions of species is unpredictable.

Evidently, any regular CSO and, more generally, any CSO for which every trajectory converges to a (not necessarily strict) periodic orbit is ergodic, but the converse is not true.

In [18] a construction of a cubic stochastic operator is given. This construction depends on a probability measure μ which is initially given on a fixed graph G. Using the construction of CSO for μ defined as product of measures given on components of G a wide class of non-Volterra CSOs is described. It is proved that the non-Volterra CSOs can be reduced to N number of Volterra CSOs defined on the components, where N is the number of components.

In [3] a class of non-Volterra cubic operators is given and the dynamical systems generated by these CSOs are studied.

3 Asymptotic Behaviour of CSOCGs

Recall the notion of infinite dimensional simplex following [16]. Denote by S the following set:

$$S=\{\mathbf{x}=(x_i): x_i\geq 0, i\in\mathbb{N},\ \ \sum_{i=1}^{\infty}x_i=1\}.$$

Clearly, S is the closed convex hull of vectors of the form $\mathbf{e}_k=(0,0,...,1,0,0,...)$, where the unite is the $k-$th position, and precisely these vectors are the extreme elements of S.

We define an operator $V:S\mapsto S$ as follows

$$(V(\mathbf{x}))_l=\sum_{i,j,k=1}^{\infty}p_{ijk,l}x_ix_jx_k,\quad l\in\mathbb{N},\quad \mathbf{x}=(x_i)\in S, \tag{4}$$

where

$$p_{ijk,l}\geq 0,\quad \sum_{l=1}^{\infty}p_{ijk,l}=1,\quad i,j,k,l\in\mathbb{N}. \tag{5}$$

and the values $p_{ijk,l}$ do not change for any permutation of i,j, and k.

Definition 3.1 An operator defined by conditions (4) and (5) is called an infinite dimensional cubic stochastic operator.

Let $G = (\Lambda, L)$ be a graph without multiple edges, where Λ is the set of vertices which is at most a countable set, L is the set of edges of the graph G. Enumerate the vertices of the graph G by elements of $[m]_0 = \{0\} \cup \mathbb{N}$. For the vertices $i, j \in \Lambda$ define

$$\delta_{ij} := \begin{cases} 1, & \text{if } \{i,j\} \subset L; \\ 0, & \text{otherwise}; \end{cases}$$

and we denote $\langle i,j,k \rangle$ if $\delta_{ij} + \delta_{jk} + \delta_{ki} > 1$ and by $\rangle i,j,k \langle$ we denote the case $\delta_{ij} + \delta_{jk} + \delta_{ki} \le 1$.

We define the coefficients of heredity as follows:

$$p_{ijk,l} := \begin{cases} 1, & \text{if } l = 0, \ \ \rangle i,j,k\langle, \ \ i,j,k \in [m]_0 \ \text{ or } \ \langle i,j,k \rangle, \ \ 0 \in \{i,j,k\}; \\ 0, & \text{if } l \neq 0, \ \ \rangle i,j,k\langle, \ \ i,j,k \in [m]_0 \ \text{ or } \ \langle i,j,k \rangle, \ \ 0 \in \{i,j,k\}; \\ \ge 0, & \text{if } \langle i,j,k \rangle, \ \ i,j,k \in \mathbb{N}. \end{cases} \tag{6}$$

The biological interpretation of the coefficients (6) is obvious: the individuals i, j and k might produce the offspring $l \neq 0$ if they are neighboring points of a graph.

Definition 3.2 For any fixed graph G, CSO satisfying conditions (4), (5) and (6) is called the cubic stochastic operator corresponding to the graph (CSOCG).

Remark 3.1 Any CSOCG is non-Volterra, because $p_{ijk,0} \neq 0$ if $\rangle i,j,k\langle$ and $ijk \neq 0$.

Arbitrary CSOCG has the form

$$V: \begin{cases} x_0' = \sum\limits_{i \in [m]_0} x_i^3 + 3x_0^2 \sum\limits_{i \in \mathbb{N}} x_i + 6x_0 \sum\limits_{i,j \in \mathbb{N}} x_i x_j + 6 \sum\limits_{\substack{i,j,k \in \mathbb{N}: \\ \rangle i,j,k\langle}} x_i x_j x_k + 6 \sum\limits_{\substack{i,j,k \in \mathbb{N}: \\ \langle i,j,k \rangle}} p_{ijk,0} x_i x_j x_k \\ x_l' = 6 \sum\limits_{\substack{i,j,k \in \mathbb{N}: \\ \langle i,j,k \rangle}} p_{ijk,l} x_i x_j x_k, \quad l \in \mathbb{N}. \end{cases} \tag{7}$$

Denote $\text{int}S = \{\mathbf{x} \in S : x_i > 0, \ \ i \in \mathbb{N}\}$. Let $\omega(\mathbf{x}^0)$ be the set of limit points of a trajectory $\{V^k(\mathbf{x}^0) \in S : k = 0, 1, 2, \dots\}$. Using Lyapunov functions, one can handle the set of limit points. Recall the definition of a Lyapunov function.

Definition 3.3 A continuous function $\varphi : intS \to \mathbb{R}$ is called a Lyapunov function for the operator (4) if $\varphi(V(\mathbf{x})) \ge \varphi(\mathbf{x})$ for all $\mathbf{x}$ (or $\varphi(V(\mathbf{x})) \le \varphi(\mathbf{x})$ for all $\mathbf{x}$).

Theorem 3.1 *Any CSOCG* (7) *has a unique fixed point (1,0,0,...). Moreover for an initial* $\mathbf{x}^{(0)} \in S$, *the trajectory of operator* (7) *tends to this fixed point exponentially rapidly.*

Proof. It is easy to verify that $\mathbf{e}_0 = (1, 0, 0, ...)$ is a fixed point. We consider the function

$$\varphi(\mathbf{x}) = \sum_{k \in \mathbb{N}} x_k. \tag{8}$$

The function (8) will be a Lyapunov function for the operator (7). Indeed,

$$\begin{aligned} \varphi(V(\mathbf{x})) &= \sum_{l \in \mathbb{N}} x_l' = \sum_{l \in \mathbb{N}} \sum_{\substack{i,j,k \in \mathbb{N}: \\ \langle i,j,k \rangle}} p_{ijk,l} x_i x_j x_k = \sum_{\substack{i,j,k \in \mathbb{N}: \\ \langle i,j,k \rangle}} \sum_{l \in \mathbb{N}} p_{ijk,l} x_i x_j x_k \\ &\le \sum_{\substack{i,j,k \in \mathbb{N}: \\ \langle i,j,k \rangle}} x_i x_j x_k \le \left(\sum_{l \in \mathbb{N}} x_l \right)^3 \le \sum_{l \in \mathbb{N}} x_l = \varphi(\mathbf{x}). \end{aligned} \tag{9}$$

It is evident, that $\varphi(\mathbf{x}^{(n+1)}) \leq \varphi(\mathbf{x}^{(n)}), n = 0, 1, \dots$ implies that $\varphi(\mathbf{x})$ is a Lyapunov function, that is $\{\varphi(\mathbf{x}^{(n)})\}_{n=0}^{\infty}$ is a decreasing sequence and converges to some limit ξ. We claim that $\xi = 0$. Indeed, from (9) one has

$$\varphi(\mathbf{x}^{(n+1)}) \leq (\varphi(\mathbf{x}^{(n)}))^3 \leq \left(\varphi\big(\mathbf{x}^{(0)}\big)\right)^{3^n}. \tag{10}$$

If $x_0^{(0)} \neq 0$, then from (10) using $\varphi\big(\mathbf{x}^{(0)}\big) = \sum\limits_{k\in\mathbb{N}} x_k^{(0)} = 1 - x_0^{(0)}$ we obtain

$$\lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 0. \tag{11}$$

If for an initial point it holds that $x_0^{(0)} = 0$, then from (7) it is easy to see that

$$V(\mathbf{x}^{(0)}) \in \mathrm{int}S = \{\mathbf{x} \in S : x_i > 0, \sum_{k\in\mathbb{N}} x_k = 1\}, \tag{12}$$

that is $x_0' \neq 0$.

Thus from (11) and (12) it should be

$$\lim_{n\to\infty} x_k^{(n)} = 0, \quad \text{for} \quad \text{any} \quad k \in \mathbb{N},$$

consequently

$$\lim_{n\to\infty} \mathbf{x}^{(n)} = \mathbf{e}_0, \quad \text{for} \quad \text{any} \quad \mathbf{x}^{(0)} \in S.$$

Since the limit is obtained for any $\mathbf{x}^{(0)} \in S$, we conclude that $(1, 0, 0, ...)$ is unique fixed point. This completes the proof.

If an operator has the regularity property then it satisfies the ergodic hypothesis. By Theorem 3.1, a CSOCG is a regular transformation, so as a corollary we have the following theorem.

Theorem 3.2 *Any CSOCG* (7) *is an ergodic transformation.*

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