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Capacity and Non-linear Potential in Musielak-Orlicz Spaces

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Abstract: In this paper we are going to introduce the theory of capacity in Musielak-Orlicz space. We will define the $C_{k,\varphi}$ capacity and the $D_{k,\varphi}$ capacity, prove their main properties, and establish relationship between $C_{k,\varphi}$ and $D_{k,\varphi}$. We shall introduce the theory of non-linear potential and give some of its properties.

Keywords: *Musielak-Orlicz space; Radon measures space; capacity; potential.*

Mathematics Subject Classification (2010): 31C15.

Introduction

The theory of capacity and non-linear potential in the Lebesgue space L^p studied by Maz'ya and Khavin in [10] and Meyers in [11] introduced the concept of capacity and non-linear potential in these spaces and provided very rich applications in functional analysis, harmonic analysis and the theory of partial differential equations. The previous concept was generalised by N. Aissaoui and A. Benkirane in [2] and [3], by replacing L^p by Orlicz space.

The main purpose of this paper is to study the theory of capacity and non-linear potential in Musielak-Orlicz space. Our results generalize those of N. Aissaoui and A. Benkirane in the case of Orlicz spaces [see [3] and [2]]. Let us note that this generalization was touched upon by Fumi-Yuki Maeda, Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura in [9] [see the third paragraph], but we are going to deal with another method.

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The present paper is organized as follows. In the first section, we recall the main results for the Musielak-Orlicz spaces and Radon measure spaces. In the second section, we define the capacity $C_{k,\varphi}$ in the Musielak-Orlicz spaces, give some of its properties, introduce a $D_{k,\varphi}$ capacity in terms of Radon measures and give its relations with $C_{k,\varphi}$. In the third section, we introduce the theory of the non-linear potential and give some of its properties.

1 Preliminaries

1.1 Musielak-Orlicz function

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}^+$ and satisfying the following conditions:
a) $\varphi(x,.)$ is an N-function [convex, increasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0\ \forall t>0$ $\dfrac{\varphi(x,t)}{t} \to 0$ as $t\to 0$, $\dfrac{\varphi(x,t)}{t} \to \infty$ as $t \to \infty$].
b) $\varphi(.,t)$ is a measurable function.

A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. Equivalently, φ admits the representation:
$\varphi(y,t) = \displaystyle\int_0^t a(y,\tau)d\tau$, for all $y\in\Omega$ and $t \geqslant 0$, where $a(y,.) : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, for all $y \in \Omega$: $a(y,0)=0$,
$a(y,t)>0$ for t¿0 and $\lim\limits_{t\to+\infty} a(y,t) = +\infty$.

The function $a(y,.)$ is called the derivative of $\varphi(y,.)$. The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if there exists $K \geqslant 2$ such that

$$\varphi(y,2t) \leqslant K\varphi(y,t), \quad for\ all\ y\in\Omega \quad and\ \ t\geqslant 0.$$

The smallest K is called the Δ_2-constant of φ. When the last inequality holds only for $t \geqslant$ *some* $t_0 > 0$ then φ is said to satisfy the Δ_2-condition near infinity.

1.2 Musielak-Orlicz spaces

Let φ be a Musielak-Orlicz function, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ is a Lebesgue measurable function.

In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability.

The set

$$K_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(u) < \infty\}$$

is called the Musielak-Orlicz class.

The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently:

$$L_\varphi(\Omega) = \{u : \Omega \mapsto \mathbb{R},\ \text{measurable}/\varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < +\infty\,\text{for some}\ \lambda > 0\}.$$

$K\varphi(\Omega)$ is a convex subset of $L\varphi(\Omega)$. If $\Omega = \mathbb{R}^N$ then $L_\varphi(\mathbb{R}^N)$ is denoted by L_φ.

Let

$$\psi(x,s) = sup\{st - \varphi(x,t) \ \ /t \geqslant 0\}.$$

That is, ψ is the Musielak-Orlicz function complementary to $\varphi(x,t)$ in the sense of Young with respect to the variable s. For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the Young inequality [12]

$$t.s \leqslant \varphi(x,t) + \psi(x,s) \text{ for all s, t} \geqslant 0 \ , x \in \Omega. \tag{1}$$

If $s = a(x,t)$ then

$$t.a(x,t) = \varphi(x,t) + \psi(x,a(x,t)) \text{ for all t} \geqslant 0 \ , x \in \Omega. \tag{2}$$

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 : \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) \leqslant 1\}$$

called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_{\psi,\Omega}\leqslant 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [12].

For two complementary Musielak-Orlicz functions φ and ψ let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, we have the *Hölder* inequality [12]

$$|\int_\Omega u(x)v(x)dx| \leqslant ||u||_{\varphi,\Omega} \, |||v|||_{\psi,\Omega} \, . \tag{3}$$

In $L_\varphi(\Omega)$ we have the relation with the norm and the modular:

$$|||u|||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) + 1, \tag{4}$$

$$||u||_{\varphi,\Omega} \leqslant \varrho_{\varphi,\Omega}\,(u) \ \ , \text{if} \, ||u||_{\varphi,\Omega} > 1, \tag{5}$$

$$||u||_{\varphi,\Omega} \geqslant \varrho_{\varphi,\Omega}\,(u) \ \ , \text{if} \, ||u||_{\varphi,\Omega} \leqslant 1. \tag{6}$$

If $\Omega = \mathbb{R}^N$ then two norms $||.||_{\varphi,\mathbb{R}^N}$ and $|||.|||_{\varphi,\mathbb{R}^N}$ are denoted respectively by $||.||_\varphi$. and $|||.|||_\varphi$.

We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant k ¿ 0 such that

$$\lim_{n\to+\infty} \varrho_{\varphi,\Omega}\,(\frac{u_n - u}{k}) = 0.$$

If φ satisfies the $\triangle_2$ condition, then modular convergence coincides with norm convergence.

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_\varphi(\Omega)$ and it is a separable space. The equality $K_\varphi(\Omega) = E_\varphi(\Omega) = L_\varphi(\Omega)$ holds if and only if φ satisfies the $\triangle_2$ condition, for all t or for large t, according to whether Ω has infinite measure or not. The dual of $E_\varphi(\Omega)$ can be identified with $L_\psi(\Omega)$ by means of the pairing $\displaystyle\int_\Omega u(x)v(x)dx$ and the dual norm on $L_\psi(\Omega)$ is equivalent to $||.||_\psi$. The space $L_\varphi(\Omega)$ is reflexive if and only if φ and ψ satisfy the $\triangle_2$ condition, for all t or for large t according to whether Ω has infinite measure or not.

Lemma 1.1 *[8] Let φ be a Musielak-Orlicz function and f_n, f, g be measurable functions.*
(a) If $f_n \longrightarrow f$, almost everywhere , then $\varrho_{\varphi,\Omega}(f) \leqslant \liminf_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(b) If $|f_n| \nearrow |f|$, almost everywhere, then $\varrho_{\varphi,\Omega}(f) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(f_n)$.
(c) If $f_n \longrightarrow f$, almost everywhere, $|f_n| \leqslant |g|$, almost everywhere and $\varrho_{\varphi,\Omega}(\lambda g) < \infty$ for every $\lambda > 0$, then $f_n \to f$ strongly in $L_\varphi(\Omega)$.

Theorem 1.1 *[8] Let φ be a Musielak-Orlicz function.*
(a) $||f||_{\varphi,\Omega} = ||\ |f|\ ||_{\varphi,\Omega}$ for all $f \in L_\varphi(\Omega)$.
(b) If $f \in L_\varphi(\Omega)$, g is a measurable function, and $0 \leqslant |g| \leqslant |f|$ almost everywhere, then:

$$g \in L_\varphi(\Omega) \quad and \quad ||g||_{\varphi,\Omega} \leqslant ||f||_{\varphi,\Omega}\,.$$

(c) If $f_n \to f$ almost everywhere, then: $||f||_{\varphi,\Omega} \leqslant \liminf_{n\to+\infty} ||f_n||_{\varphi,\Omega}$.
(d) If $|f_n| \nearrow |f|$ almost everywhere with $f_n \in L_\varphi(\Omega)$ and $\sup_n ||f_n||_{\varphi,\Omega} < \infty$ then:

$$f \in L_\varphi(\Omega) \quad and \quad ||f_n||_{\varphi,\Omega} \nearrow ||f||_{\varphi,\Omega}\,.$$

Theorem 1.2 *[5] Let φ and ψ be two complementary Musielak-Orlicz functions. Assume that there exists a constant $A > 0$ such that for all $x, y \in \Omega$: $|x - y| \leqslant \frac{1}{2}$ we have:*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leqslant t^{\overline{\frac{A}{log(\frac{1}{|x-y|})}}} \tag{7}$$

for all $t \geq 1$. If $D \subset \Omega$ is a bounded measurable set, then $\int_D \varphi(x,1)dx < \infty$.
ψ satisfies the following condition:

$$\exists C > 0 \ : \ \psi(x,1) \leqslant C, \quad almost\ everywhere\ \ in\ \ \Omega. \tag{8}$$

Under the previous conditions, with $\Omega = \mathbb{R}^N$; $C_0^\infty(\mathbb{R}^N)$ is dense in $L_\varphi(\mathbb{R}^N)$ with respect to the modular topology.

1.3 Measures space

M designates the vector space of Radon measures. M is endowed with the weak topology for which a sequence (μ_n) converges weakly to μ, if for any continuous function f with compact support

$$\lim_{n\to+\infty} \int f d\mu_n = \int f d\mu.$$

M^+ is the cone of positive elements of M.

For all measures $\mu < \infty$, for all $X \subset \mathbb{R}^N$, the variation of μ is defined by:

$$||\mu||(X) = sup\{\sum_1^n |\mu(X_i)| : (X_i)_{i=1\ldots n} \ \ is \ \ an \ \ X \ \ partiton\}.$$

$||\mu||(\mathbb{R}^N) = ||\mu||$ is called the total variation of μ. M_1 designates the Banach space of measures, endowed with the norm total variation. M_1^+ designates the subset of M_1 consisting of positive measures.

Definition 1.1 Let $\mu \in M_1^+$. We say that μ is concentrated on X if $\mu(Y) = 0$ for all $\mu - measurable\ set\ Y$, such that $Y \subset X^c$.

2 Capacity in Musielak-Orlicz Space

2.1 $C_{k,\varphi}$-capacity

Lemma 2.1 *Let Ω be an open set in $\mathbb{R}^N$ and φ be a Musielak-Orlicz function such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \ \forall y \in \Omega \ and \ \ t \geqslant 0.$$

Let $u : \Omega \to \mathbb{R}$ be measurable function and $\alpha > 0$, we define a measurable function $g : \Omega \to \mathbb{R}$ so that

$$g(y) = a(y, \frac{|u(y)|}{2\alpha}), \ \forall y \in \Omega.$$

If $(\frac{u}{\alpha}) \in K_\varphi(\Omega)$ then $g \in K_\psi(\Omega)$, where ψ is the Musielak-Orlicz function complementary to φ.

Proof. For all $y \in \Omega$ and $t \geqslant O$: $\varphi(y,2t) = \displaystyle\int_0^{2t} a(y,\tau)d\tau \ \geqslant \int_t^{2t} a(y,\tau)d\tau.$
Hence $\varphi(y,2t) \geqslant ta(y,t)$, thus for all $y \in \Omega$: $\varphi(y, \frac{|u(y)|}{\alpha}) \geqslant \frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha})$.
On the other hand, we have: $\frac{|u(y)|}{2\alpha} a(y, \frac{|u(y)|}{2\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha}) = \psi(y, a(y, \frac{|u(y)|}{2\alpha}))$.
Therefore, $\psi(y, a(y, \frac{|u(y)|}{2\alpha})) \leqslant \varphi(y, \frac{|u(y)|}{\alpha}) - \varphi(y, \frac{|u(y)|}{2\alpha})$, this implies that

$$\int_\Omega \psi(y, a(y, \frac{|u(y)|}{2\alpha}))dy \leqslant \int_\Omega \varphi(y, \frac{|u(y)|}{\alpha})dy - \int_\Omega \varphi(y, \frac{|u(y)|}{2\alpha})dy,$$

then

$$\varrho_{\psi,\Omega}(g) \leqslant \varrho_{\varphi,\Omega}(\frac{u}{\alpha}) - \varrho_{\varphi,\Omega}(\frac{u}{2\alpha}).$$

Since $\varrho_{\varphi,\Omega}(\frac{u}{2\alpha}) \leqslant \frac{1}{2}\varrho_{\varphi,\Omega}(\frac{u}{\alpha})$ and $\varrho_{\varphi,\Omega}(\frac{u}{\alpha}) < \infty$, the proof is complete.

Lemma 2.2 *If (f_n) is a sequence in $L_\varphi(\Omega)$ such that for all $n \in \mathbb{N}$, $f_n \geqslant 0$, then*

$$||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega}.$$

Proof. Since $0 \leqslant \sup_n f_n \leqslant \sum_n f_n$, thus $||\sup_n f_n||_{\varphi,\Omega} \leqslant ||\sum_n f_n||_{\varphi,\Omega}$.

Let $g_n = \sum_{k=0}^n f_k$ and $f = \sum_n f_n$, we have

$$\frac{g_n}{\sum_n ||f_n||_{\varphi,\Omega}} \nearrow \frac{f}{\sum_n ||f_n||_{\varphi,\Omega}} \ \ \textit{almost everywhere.}$$

By Lemma 1.1, we obtain

$$\varrho_{\varphi,\Omega}(\frac{f}{\sum_n ||f_n||_{\varphi,\Omega}}) = \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{\sum_n ||f_n||_{\varphi,\Omega}}) \leqslant \lim_{n\to+\infty} \varrho_{\varphi,\Omega}(\frac{g_n}{||g_n||_{\varphi,\Omega}}) \leqslant 1.$$

Then

$$||\frac{f}{\sum_n ||f_n||_{\varphi,\Omega}}||_{\varphi,\Omega} \leqslant 1.$$

Therefore,

$$||\sum_n f_n||_{\varphi,\Omega} \leqslant \sum_n ||f_n||_{\varphi,\Omega}\,.$$

Lemma 2.3 *Let φ be a Musielak-Orlicz function, which satisfies the $\triangle_2$ condition, and such that*

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \quad for\ all\ y \in \Omega \ \ and \ \ t \geqslant 0.$$

Let $f \in L_\varphi(\Omega)$, such that $f \geqslant 0$, and $||f||_{\varphi,\Omega} \neq 0$.

We define a measurable function $g : \Omega \to \mathbb{R}$ such that for all $y \in \Omega$; $g(y) = a(y, \frac{f(y)}{||f||_{\varphi,\Omega}})$. Then $\int f(y)g(y)dy = ||f||_{\varphi,\Omega}\, |||g|||_{\psi,\Omega}\,.$

Proof. By Lemma 2.1, we have $g \in L_\psi(\Omega)$ and by the *Hölder* inequality we have

$$\int_\Omega f(x)g(x)dx \leqslant ||f||_{\varphi,\Omega}\, |||g|||_{\psi,\Omega}\,.$$

For the opposite inequality, let $h = \frac{f}{||f||_{\varphi,\Omega}}$, and $v \in L_\varphi(\Omega)$, such that $||v||_{\varphi,\Omega} \leqslant 1$.

For all $y \in \Omega$, we have

$$g(y)h(y) = \varphi(y, h(y)) + \psi(y, g(y))$$

and

$$g(y)v(y) \leqslant \varphi(y, v(y)) + \psi(y, g(y)).$$

Hence for all $y \in \Omega$:

$$g(y)v(y) \leqslant g(y)h(y) - \varphi(y, h(y)) + \varphi(y, v(y)).$$

Then

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \int_\Omega \varphi(y, h(y))dy + \int_\Omega \varphi(y, v(y))dy.$$

Thus,

$$\int_\Omega g(y)v(y)dy \leqslant \int_\Omega g(y)h(y)dy - \varrho_{\varphi,\Omega}(h) + \varrho_{\varphi,\Omega}(v).$$

We have $\varrho_{\varphi,\Omega}(v) \leqslant 1$. On the other hand φ satisfies the $\triangle_2$ condition, then, $\varrho_{\varphi,\Omega}$ is a continuous modular[see [8] Lemma 2.4.3]. We have $||h||_{\varphi,\Omega} = 1$, then $\varrho_{\varphi,\Omega}(h) = 1$ [see [8] Lemma 2.1.14].

Thus,

$$\int g(y)v(y)dy \leqslant \int g(y)h(y)dy$$

implies

$$\sup_{||v||_{\varphi,\Omega} \leqslant 1} \int g(y)v(y)dy \leqslant \int g(y)h(y)dy.$$

Then

$$|||g|||_{\psi,\Omega}\,||f||_{\varphi,\Omega} \leqslant \int f(y)g(y)dy.$$

Definition 2.1 Let T be a class of Borel sets in $\mathbb{R}^N$, and a function $C : T \to [0, +\infty]$.
1) C is called a capacity if the following axioms are satisfied:
i) $C(\emptyset) = 0$.
ii) $X \subset Y \Rightarrow C(X) \leqslant C(Y)$, for all X and Y in T.
iii) For all sequences $(X_n) \subset T$:

$$C(\bigcup_n X_n) \leqslant \sum_n C(X_n).$$

2) C is called an outer capacity if for all $X \in T$:

$$C(X) = \inf\{C(O) : O \supset X, \quad O \ is \ \ open\}.$$

3) C is called an interior capacity if for all $X \subset T$:

$$C(X) = \sup\{C(K) : K \subset X, \quad K \ \ is \ compact\}.$$

4) A property, that holds true except perhaps on a set of zero capacity is said to be true C-quasi-everywhere, (C-q.e).
5) $f \ and \ (f_n)$ are real-valued finite functions C-q.e. We say that (f_n) converges to f in C-capacity if:

$$\forall\varepsilon > 0, \quad \lim_{n\to+\infty} C(\{x : \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

6) $f \ \ and \ \ (f_n)$ are real-valued function finite C-q.e. We say that (f_n) converges to f C-quasi- uniformly, (C-q.u) if

$$(\forall\varepsilon > 0), (\exists\ X \in T) \ : \ C(X) < \varepsilon \ \ and \ (f_n) \ \ converges \ to \ f \ uniformly \ on \ X^c.$$

Remark 2.1 In the following $\Omega = \mathbb{R}^n, \varphi$ is a Musielak-Orlicz function, and $L^+_\varphi = \{\ f \in L_\varphi \ / \ f \geqslant 0\}$.

Theorem 2.1 *Let k be a positive integrable function on $\mathbb{R}^N$. For all $X \subset \mathbb{R}^N$, we put $C_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L_\varphi \quad and \ k * f \geqslant 1 \ on \ \ X\}$, where $k * f$ is the convolution of k and f. $C_{k,\varphi}$ is an outer capacity.*

Remark 2.2 Let $B_{k,\varphi}(X) = \inf\{||f||_\varphi : f \in L^+_\varphi \quad and \quad k * f \geqslant 1 \quad on \ X\}$, then

$$C_{k,\varphi}(X) = B_{k,\varphi}(X).$$

Indeed, it is obvious that $C_{k,\varphi}(X) \leqslant B_{k,\varphi}(X)$. On the other hand, let $f \in L_\varphi$, then $|f| \in L^+_\varphi$ and if $k * f \geqslant 1$ on X, then $k * |f| \geqslant 1$ on X. Thus $B_{k,\varphi}(X) \leqslant ||f||_\varphi$; and therefore $B_{k,\varphi}(X) \leqslant C_{k,\varphi}(X)$.

Proof of Theorem 2.1. It is obvious that $C_{k,\varphi}(\emptyset) = 0$ and $C_{k,\varphi}(X) \leqslant C_{k,\varphi}(Y)$ if $X \subset Y$. Let $(X_n) \subset T$, so that $\sum_i C_{k,\varphi}(X_i) < +\infty$, then $(\forall i \in \mathbb{N})$ $C_{k,\varphi}(X_i) < +\infty$. Thus, $(\forall i \in \mathbb{N})(\forall \varepsilon > 0)$, $(\exists f_i \in L^+_\varphi)$ so that $k*f_i \geqslant 1$ *on* X_i and $||f_i||_\varphi \leqslant C_{k,\varphi}(X_i) + \frac{\varepsilon}{2^i}$.

Let $f = \sup_i f_i$. By Lemma 2.2, we have:

$$||f||_\varphi \leqslant \sum_i ||f_i||_\varphi.$$

We can write

$$||f||_\varphi \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon,$$

which implies that, $f \in L_\varphi$.

Since $k * f \geqslant 1$ *on* $\bigcup_i X_i$,

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i) + \varepsilon, \quad \forall \varepsilon > 0.$$

Hence,

$$C_{k,\varphi}(\bigcup_i X_i) \leqslant \sum_i C_{k,\varphi}(X_i).$$

It remains to show that $C_{k,\varphi}$ is outer. Let $X \subset \mathbb{R}^N$, we have:

$$C_{k,\varphi}(X) \leqslant \inf\{C_{k,\varphi}(O) : O \supset X, \quad O \textit{ is open}\}.$$

For the reverse inequality, if $C_{k,\varphi}(X) = +\infty$ there is nothing to show.
Assume that $C_{k,\varphi}(X) < +\infty$, and let $0 < \varepsilon < 1$, then $\exists\, g \in L^+_\varphi$ so that $k*g \geqslant 1$ *on* X and $||g||_\varphi \leqslant C_{k,\varphi}(X) + \varepsilon$.

Let $g_\varepsilon = \dfrac{g}{1-\varepsilon}$ and $O_\varepsilon = \{x : (k * g_\varepsilon) > 1\}$, thus O_ε is open and

$$\forall x \in X; \quad (k * g_\varepsilon) \geqslant \frac{1}{1-\varepsilon} > 1.$$

Hence, $X \subset O_\varepsilon$. On the other hand, we have $C_{k,\varphi}(O_\varepsilon) \leqslant ||g_\varepsilon||_\varphi$, and we deduce that

$$C_{k,\varphi}(O_\varepsilon) \leqslant \frac{1}{1-\varepsilon}||g||_\varphi \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X) + \varepsilon].$$

Therefore,

$$\inf\{C(O) : O \supset X, \quad O \textit{ isopen}\} \leqslant \frac{1}{1-\varepsilon}[C_{k,\varphi}(X) + \varepsilon], \quad \forall \varepsilon > 0.$$

Thus,

$$\inf\{C(O) : O \supset X, \quad O \textit{ isopen}\} \leqslant C_{k,\varphi}(X).$$

Theorem 2.2 *1) If there exists $f \in L_\varphi$ such that $|k*f| = +\infty$ on X, then $C_{k,\varphi}(X) = 0$.*
2) If $C_{k,\varphi}(X) = 0$ then there exists $f \in L_\varphi^+$ such that $k*f = +\infty$ on X.

Proof. 1) Let $f \in L_\varphi$ such that $|k*f| = +\infty$ on X, then $\forall \alpha > 0,\ |k*f| \geqslant \alpha$ on X. Then $C_{k,\varphi}(X) \leqslant \dfrac{||f||_\varphi}{\alpha},\ \ \forall \alpha > 0$; this means that $C_{k,\varphi}(X) = 0$.
2) If $C_{k,\varphi}(X) = 0$ then $\forall i \in \mathbb{N},\ \exists f_i \in L_\varphi^+ :\ \ k*f_i \geqslant 1$ *on* X *and* $||f_i||_\varphi \leqslant 2^{-i}$.
Let $f = \sum_i f_i$. By Lemma 2.2, $||f||_\varphi \leqslant \sum_i ||f_i||_\varphi$, then $||f||_\varphi < +\infty$.
We deduce that $f \in L_\varphi^+$ and $k*f = +\infty$ *on* X.

Theorem 2.3 *Consider the following propositions :*
i) $f_n \longrightarrow f$ strongly in L_φ.
ii) $k*f_n \longrightarrow k*f$, $C_{k,\varphi}$ $-$capacity.
iii) There is a subsequence $(f_{n_j})_j$ such that : $k*f_{n_j} \longrightarrow k*f$ $C_{k,\varphi}$ $-q.u.$
iv) $k*f_{n_j} \longrightarrow k*f$ in $C_{k,\varphi}$ $-q.e.$
We have

$$i) \Rightarrow\ ii) \Rightarrow\ iii) \Rightarrow iv).$$

Proof. We show $i) \Rightarrow\ ii)$.
By Theorem 2.2, we have $k*f$ *and* $k*f_n$ are finite $C_{k,\varphi}$ $-q.e,\ \ \forall n$.
Let $\varepsilon > 0$; then

$$C_{k,\varphi}(\{x\ : |k*f_n - k*f|(x) > \varepsilon\}) \leqslant \frac{||f_n - f||_\varphi}{\varepsilon}.$$

We show $ii) \Rightarrow\ iii)$.
Let $\varepsilon > 0\ \ \exists\ f_{n_j}$ such that

$$C_{k,\varphi}(\{x\ :\ |k*f_{n_j} - k*f|(x) > 2^{-j}\}) < \varepsilon.2^{-j}.$$

We put

$$E_j = \{x\ :\ |k*f_{n_j} - k*f|(x) > 2^{-j}\} \quad \text{and} \quad G_m = \bigcup_{j \geqslant m} E_j.$$

We have $C_{k,\varphi}(G_m) \leqslant \sum_{j \geqslant m} \varepsilon.2^{-j} < \varepsilon$.
On the other hand :

$$\forall x \in (G_m)^c,\ \forall\ j \geqslant m :\ \ |k*f_{n_j} - k*f|(x) \leqslant 2^{-j}.$$

Thus $k*f_{n_j} \longrightarrow k*f$ $C_{k,\varphi}$ $-q.u.$

We show $iii) \Rightarrow\ iv)$. We have $\forall j \in \mathbb{N}, \exists X_j\ : C_{k,\varphi}(X_j) \leqslant \dfrac{1}{j}$ and $k*f_{n_j} \longrightarrow k*f$ *on* $(X_j)^c$. We put $X = \bigcap_j X_j$, then $C_{k,\varphi}(X) = 0$ *and* $k*f_{n_j} \longrightarrow k*f$ *on* X^c.

Theorem 2.4 *Let φ be a Musielak-Orlicz function that satisfies the $\triangle_2$ condition, and (f_n) be a sequence in L_φ such that $\sum_n |f_n| \in L_\varphi$. Then,*

$$\sum_n (k*f_n) = k*(\sum_n f_n)\ \ C_{k,\varphi}\ -q.e.$$

Proof. **First step**: Assume that $f_n \geqslant 0 \ \forall n \in \mathbb{N}$, and let $g_n = \sum_{i=1}^{n} f_i$ *and* $f = \sum_n f_n$. We have $g_n \to f$ *almost everywhere* and $g_n \leqslant f$. On the other hand, $\varrho_\varphi(\lambda f) < \infty \ \ \forall \lambda > 0$ because $f \in L_\varphi$ and φ satisfies the $\triangle_2$ condition [see [8] paragraph 2.5].

By (c) of Lemma 1.1 we have

$$g_n \to f \ \textit{strongly in} \ L_\varphi.$$

Theorem 2.3 implies that there is a subsequence (g_{n_i}) such that $k * g_{n_i} \to k * f$, $C_{k,\varphi}$-q.e. Since $f_n \geqslant 0, \ \ \forall n \in \mathbb{N} \quad k * g_n \to k * f$, $C_{k,\varphi}$-q.e.

Second step: If f_n has any sign, then $\sum_n f_n^+$ and $\sum_n f_n^-$ are in L_φ because $|\sum_n f_n^+| \leqslant \sum_n |f_n|$, $|\sum_n f_n^-| \leqslant \sum_n |f_n|$ and $\sum_n |f_n| \in L_\varphi$.
By the first step the result follows.

Theorem 2.5 *Let (K_n) be a decreasing sequence of compact and $K = \bigcap_n K_n$. Then* $\lim_{n \to +\infty} C_{k,\varphi}(K_n) = C_{k,\varphi}(K)$.

Proof. First, we observe that $C_{k,\varphi}(K) \leqslant \lim_{n\to+\infty} C_{k,\varphi}(K_n)$. On the other hand, let O be an open set containing K. By the compactness of K, $K_i \subset O$ for all sufficiently large i. Therefore $\lim_{n\to+\infty} C_{k,\varphi}(K_n) \leqslant C_{k,\varphi}(O)$, and since $C_{k,\varphi}$ is an outer capacity, we obtain the claim by taking infimum over open set O containing K.

Theorem 2.6 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. If f_n , $f \in L_\varphi$ such that $f_n \rightharpoonup f$ weakly in L_φ, then:*

$$\liminf(k * f_n) \leqslant (k * f) \leqslant \limsup(k * f_n) \quad C_{k,\varphi} - q.e.$$

Proof. $(L_\varphi \ , \ ||.||)$ is uniformly convex therefore reflexive. By the Banach-Saks theorem, there is a subsequence denoted again by (f_n) such that the sequence $g_n = \frac{1}{n}\sum_{i=1}^{n} f_i$ converges to f strongly in L_φ. By Theorem 2.3, there is a subsequence of (g_n) denoted again by (g_n) such that

$$\lim_{n\to+\infty}(k * g_n) = (k * f) \quad C_{k,\varphi} - q.e.$$

On the other hand,

$$\liminf(k * f_n) \leqslant \lim_{n\to+\infty}(k * g_n) \quad .$$

Therefore,

$$\lim_{n\to+\infty}(k * f_n) \leqslant (k * f) \quad C_{k,\varphi} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

Theorem 2.7 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition, (X_n) be an increasing sequence of sets and $X = \bigcup_n X_n$. Then*

$$\lim_{n\to+\infty} C_{k,\varphi}(X_n) = C_{k,\varphi}(X).$$

Proof. We have $\lim_{n\to+\infty} C_{k,\varphi}(X_n) \leqslant C_{k,\varphi}(X)$. For the reverse inequality, if $C_{k,\varphi}(X) = +\infty$, there is nothing to show.

Assuming that $C_{k,\varphi}(X) < +\infty$, we have

$$\forall n \in \mathbb{N}, \quad \exists f_n \in L_{\varphi}^{+} \; : k * f_n \geqslant 1 \textit{ on } X_n \textit{ and } ||f_n||_{\varphi} \leqslant C_{k,\varphi}(X_n) + \frac{1}{n}.$$

Thus, (f_n) is a bounded sequence in L_{φ}.

On the other hand, L_{φ} is uniformly convex, then it is reflexive because φ is uniformly convex and satisfies the $\triangle_2$ condition, [see [8] Remark 2.4.15]. Hence there exists a subsequence which is denoted again by (f_n), and converges weakly to a function $f \in L_{\varphi}$. Then by Theorem 2.6,

$$\forall n \in \mathbb{N} \; : k * f \geqslant 1 \quad on \quad X_n, \; C_{k,\varphi} - q.e.$$

Therefore,

$$k * f \geqslant 1 \quad on \quad X, \quad C_{k,\varphi} - q.e.$$

Let Y be a subset of X where $k * f \geqslant 1$, then $C_{k,\varphi}(X) = C_{k,\varphi}(Y)$. On the other hand we know that

$$\varphi(y,t) = \int_0^t a(y,\tau)d\tau, \quad \text{for all} \;\; y \in R^N \;\; \textit{and} \; t \geqslant 0.$$

Let the function $g : \mathbb{R}^N \to \mathbb{R}$ be defined by $g(y) = a(y, \frac{|f(y)|}{||f||_{\varphi}})$ for all $y \in \mathbb{R}^N$.
By Lemma 2.1, $g \in L_{\psi}$, and since φ satisfies the $\triangle_2$ condition, we have $L_{\psi} = (L_{\varphi})^*$. Thus,

$$\int f_n(y)g(y)dy \to \int f(y)g(y)dy.$$

By Lemma 2.3, we have

$$\int f(y)g(y)dy = ||f||_{\varphi}|||g|||_{\psi}.$$

By the *Hölder* inequality we have:

$$\int f_n(y)g(y)dy \leq ||f_n||_{\varphi}|||g|||_{\psi}.$$

Therefore,

$$||f||_{\varphi} \leqslant \lim_{n\to+\infty} ||f_n||_{\varphi} \leqslant \lim_{n\to+\infty} (C_{k,\varphi}(X_n) + \frac{1}{n}).$$

Thus,

$$C_{k,\varphi}(X) \leqslant \lim_{n\to+\infty} C_{k,\varphi}(X_n).$$

Corollary 2.1 *Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition. Let $E_n \subset \mathbb{R}^N$, then $C_{k,\varphi}(\liminf E_n) \leqslant \liminf C_{k,\varphi}(E_n)$.*

Proof. Let $E = \liminf E_n$, we have $E = \bigcup_n (\bigcap_{i \geqslant n} E_i)$.
We put $G_n = \bigcap_{i \geqslant n} E_i$. Thus a sequence (G_n) is increasing and by Theorem 2.7, $C_{k,\varphi}(E) = \lim_n C_{k,\varphi}(G_n)$. On the other hand, $C_{k,\varphi}$ is increasing, then $C_{k,\varphi}(G_n) \leqslant C_{k,\varphi}(E_n)$; therefore

$$C_{k,\varphi}(E) \leqslant \liminf C_{k,\varphi}(E_n).$$

Theorem 2.8 *Let φ be a Musielak-Orlicz function which satisfies the assumptions of Theorem 1.2. If φ satisfies the $\triangle_2$ condition, then for each $f \in L_\varphi$, there is a $C_{k,\varphi}$-quasicontinuous function $g \in L_\varphi$ such that $k * f = g \;\; C_{k,\varphi}$ $-q.e.$*

Proof. Let $f \in L_\varphi$, by Theorem 1.2, there exists a sequence (f_n) in $C_0^\infty(\mathbb{R}^N)$ such that $f_n \longrightarrow f$ in L_φ. By Theorem 2.3, there exists a subsequence of (f_n) denoted again by (f_n) such that

$$k * f_n \longrightarrow k * f \quad C_\varphi - q.u.$$

Since k is integrable function and f_n is continuous $\forall n$, then $k * f_n$ is continuous. Thus, the proof is complete.

Definition 2.2 In the terminology of Choquet, C is called a capacity if it satisfies the following four properties:
i) $C(\emptyset) = 0$.
ii) C is increasing.
iii) If (E_n) is an increasing sequence of sets, then $\sup_n C(X_n) = C(\bigcup_n X_n)$.
iv) If (K_n) is a decreasing sequence of compacts, then $\inf_n C(K_n) = C(\bigcap_n K_n)$.

Remark 2.3 Let φ be a Musielak-Orlicz function, uniformly convex, that satisfies the $\triangle_2$ condition. By Theorems 2.1, 2.5 and 2.7 $C_{k,\varphi}$ is a capacity, in the sense of Choquet.

Definition 2.3 Let C be a capacity in the sense of Choquet, and $X \subset \mathbb{R}^N$.
X is called capacitable if

$$C(X) = \sup\{C(K) : K \subset X, \quad K \; iscompact\}.$$

Theorem 2.9 *Let φ be a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition. Then all analytic sets are $C_{k,\varphi}$- capacitable .*

Proof. It is an immediate consequence of Choquet theorem [7].

2.2 Capacity in terms of measure

Theorem 2.10 *Let φ be a Musielak-Orlicz function, k be a positive integrable function on $\mathbb{R}$, and X be a μ-measurable set, for all positive measures μ. We put*
$D_{k,\varphi}(X) = \sup\{||\mu|| \; : \; \mu \in M_1^+, \mu \; is \;\; concentrated \; on \; X \;\; and \; ||k * \mu||_\psi \leqslant 1\}$
where $(k * \mu)(x) = \int k(x-y) d\mu(y)$. Then, $D_{k,\varphi}$ is an interior capacity.

Proof. It is clear that $D_{k,\varphi}(\emptyset) = 0$ and $D_{k,\varphi}(X) \leqslant D_{k,\varphi}(Y)$ if $X \subset Y$.
Let $\mu \in M_1^+$, (X_n) be a sequence of μ-measurable sets and $\mu_n = \mu|_{X_n}$ be defined by

$$\mu_n(Y) = \mu(X_n \cap Y), \quad for\ all\ \ \mu - measurable\ \ set\ \ Y.$$

First we assume that the X_n are pairwise disjoint, then

$$\mu(\bigcup_n X_n) = \sum_n \mu(X_n).$$

If μ is concentrated on $\bigcup_n X_n$ and $||k*\mu||_\psi \leqslant 1$, then $\forall n;\ \ \mu_n \in M_1^+;\ \mu_n$ is concentrated on X_n and $||k*\mu_n||_\psi \leqslant 1$.

On the other hand, we have

$$||\mu|| = \sum_n ||\mu_n|| \leqslant \sum_n D_{k,\varphi}(X_n).$$

Thus,

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

If the X_n are not pairwise disjoint, then by the first case and the fact that $D_{k,\varphi}$ is increasing, we have

$$D_{k,\varphi}(\bigcup_n X_n) \leqslant \sum_n D_{k,\varphi}(X_n).$$

It remains to show that $D_{k,\varphi}$ is interior.

By monotonicity we have

$$\sup\{D_{k,\varphi}(K) : K \subset X,\ K\ \ compact\} \leqslant D_{k,\varphi}(X).$$

Let $\mu \in M_1^+$ and X be a μ-measurable set such that μ is concentrated on X and $||k*\mu||_\psi \leqslant 1$.
Let a compact K be such that $K \subset X$, then $\mu|_K \in M_1^+$, $\mu|_K$ is concentrated on K and $||k * \mu|_K||_\psi \leqslant 1$. Therefore,

$$||\mu|_K||_\psi \leqslant D_{k,\varphi}(K).$$

On the other hand,

$$\sup\{||\mu\setminus_K|| : K \subset X,\ K\ is\ compact\} = ||\mu||.$$

Thus,

$$D_{k,\varphi}(X) \leqslant \sup\{D_{k,\varphi}(K) : K \subset X,\ K\ iscompact\}.$$

Theorem 2.11 *1) $D^*_{k,\varphi}$ is the outer capacity associated with $D_{k,\varphi}$, defined by:*

$$D^*_{k,\varphi}(X) = \inf\{\ D_{k,\varphi}(O)\ :\ O\ isopen\ and\ X \subset O\}.$$

Then,

$$D^*_{k,\varphi}(X) = C_{k,\varphi}(X).$$

2) If φ is a Musielak-Orlicz function, uniformly convex that satisfies the $\triangle_2$ condition, then for all analytic set X we have:

$$D_{k,\varphi}(X) = C_{k,\varphi}(X).$$

Proof. It is the same as that given in [2], Theorem 11.

Theorem 2.12 *Let φ be a Musielak-Orlicz function.*
Let K be a compact of $\mathbb{R}^N$. The following assertions are equivalents.
1)$C_{k,\varphi}(K) = \infty$.
2) $D^*_{k,\varphi}(K) = \infty$.
3) $D_{k,\varphi}(K) = \infty$.
4) There exists $x_0 \in K$ such that $k(x_0 - y) = 0$ almost everywhere.

Proof. It is the same as that given in [3], Theorem 5.

3 Non-linear Potential in Musielak-Orlicz Space

Let φ be a Musielak-Orlicz function. In this section, we propose to study the following variational problem: let X be a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. There exists $f_0 \in L^+_\varphi$ such that $k * f_0 \geqslant 1 \quad C_{k,\varphi} - q.e$ on X, and

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in L^+_\varphi \text{ and } \ k * f \geqslant 1 \ C_{k,\varphi} - q.e \text{ on } X\}.$$

If f_0 exists, it will be called a distribution function of X, and $k * f_0$ is called a potential of X for the $C_{k,\varphi}$ capacity.

Theorem 3.1 *Let φ be a Musielak-Orlicz function and X be a subset of $\mathbb{R}^N$ such that $C_{k,\varphi}(X) < \infty$. $\Omega_X = \{f \in L^+_\varphi : k * f \geqslant 1 \quad C_{k,\varphi} - q.e \text{ on } X\}$, and $Cl^*(\Omega_X)$ is the closure of Ω_X for the topology $\sigma(L_\varphi; E_\psi)$. Then:*
1) There exists a unique $f_0 \in L^+_\varphi$ such that:

$$||f_0||_\varphi = inf\{||f||_\varphi : \ f \in Cl^*(\Omega_X)\}.$$

2) If φ and ψ satisfy the $\triangle_2$ condition, then there exits a unique $f \in L^+_\varphi$ such that:
i) $k * f \geqslant 1$ on X and $||f||_\varphi = C_{k,\varphi}(X)$.
ii) If $C_{k,\varphi}(X) > 0$, then for all $g \in L_\varphi$ such that $k * g \geqslant 0$ on X:

$$\int a(x, \frac{f(x)}{||f||_\varphi})g(x)dx \geqslant 0,$$

where the function a(x,.) is the derivative of the function $\varphi(x, .)$.

Proof. 1) Let the function $\theta : L_\varphi \longrightarrow]-\infty; +\infty]$ be defined by $\theta(f) = ||f||_\varphi \, ; \forall f \in L_\varphi$. θ is lower semi continuous on L_φ, for topology $\sigma(L_\varphi; E_\psi)$ and coercive. Then, there exists a unique $f_0 \in L^+_\varphi$ such that

$$||f_0||_\varphi = inf\{||f||_\varphi : f \in Cl^*(\Omega_X)\}.$$

2) i) Since φ and ψ satisfy the $\triangle_2$ condition, then the space L_φ is reflexive. By Theorem 2.3, Ω_X is strongly closed in L_φ. On the other hand, Ω_X is convex, then there exists a unique $f \in L_\varphi$ such that:

$$||f||_\varphi = inf\{||g||_\varphi : g \in \Omega_X\}.$$

Let Y be a subset of X where $k * f < 1$. Then, $C_{k,\varphi}(X) = C_{k,\varphi}(X - Y)$. Since $k * f \geqslant 1$ on X-Y, $C_{k,\varphi}(X - Y) \leqslant ||f||_\varphi$.

On the other hand, we have $\{g \in L_{\varphi}^{+} : k * g \geqslant 1 \ on \ X\} \subset \Omega_X$; then $||f||_{\varphi} \leqslant C_{k,\varphi}(X)$.
ii) Let $g \in L_{\varphi}$ such that $k * g \geqslant 0$ on X. Then for all $t \geqslant 0$:

$$k * (f + tg) \geqslant 1 \ \ C_{k,\varphi} - q.e \ \ on \ X \ and \ \ (f + tg) \in \ L_{\varphi}.$$

Then,

$$||f + tg||_{\varphi} \geqslant ||f||_{\varphi}.$$

Therefore,

$$||\frac{1}{||f||_{\varphi}}(f + tg)||_{\varphi} \geqslant 1.$$

Thus,

$$\varrho_{\varphi}(\frac{1}{||f||_{\varphi}}(f + tg)) \geqslant 1.$$

On the other hand,

$$\varrho_{\varphi}(\frac{1}{||f||_{\varphi}}f) \leqslant 1.$$

Then, for all $t > 0$

$$\int \frac{1}{t}[\varphi(x, \frac{|f + tg|(x)}{||f||_{\varphi}}) - \varphi(x, \frac{|f(x)|}{||f||_{\varphi}})]dx \geqslant 0.$$

Let $c(x,t) = \varphi(x, \frac{|f + tg|(x)}{||f||_{\varphi}})$. Then, the function $x \longmapsto c(x,t)$ is in L^1 for all $t \in \mathbb{R}$.
On the other hand,

$$\frac{\partial c}{\partial t}(x,t) = a(x, \frac{|f + tg|(x)}{||f||_{\varphi}}).(\frac{g(x)}{||f||_{\varphi}}).sng(f + tg)(x).$$

For $0 < t < 1$ we have:

$$|\frac{\partial c}{\partial t}(x,t)| \leqslant a(x, \frac{|f + g|(x)}{||f||_{\varphi}}).(\frac{g(x)}{||f||_{\varphi}}).$$

By Lemma 2.1, the function: $x \longrightarrow a(x, \frac{|f + g|(x)}{||f||_{\varphi}})$ is in L_{ψ}.
Then the function: $x \mapsto a(x, \frac{|f + g|(x)}{||f||_{\varphi}}).(\frac{g(x)}{||f||_{\varphi}})$ is in L^1.
By Lebesgue's theorem

$$\lim_{t \to 0^+} \int \frac{1}{t}[\varphi(x, \frac{|f + tg|(x)}{||f||_{\varphi}}) - \varphi(x, \frac{|f(x)|}{||f||_{\varphi}})]dx = \frac{1}{||f||_{\varphi}} \int a(x, \frac{|f(x)|}{||f||_{\varphi}})dx \geqslant 0.$$

Remark 3.1 Under the assumptions of Theorem 3.1, 2) ii), if $C_{k,\varphi}(X) > 0$, then for all $g \in L_{\varphi}$ such that $k * g = 0$ on X:

$$\int a(x, \frac{f(x)}{||f||_{\varphi}})g(x)dx = 0.$$

Theorem 3.2 *Let φ be a Musielak-Orlicz function such that φ and ψ satisfy the $\triangle_2$ condition. Let $X \subset \mathbb{R}^N$ such that $0 < C_{k,\varphi}(X) < \infty$ and f be the distribution function of X for the $C_{k,\varphi}$ capacity. For all $g \in L_\varphi$*

$$|\int a(x, \frac{f(x)}{||f||_\varphi})g(x)dx| \leqslant K_\varphi \sup_{x \in X} |(k * g)(x)|.||f||_\varphi,$$

where K_φ is a constant that depends only on φ.

Proof. The inequality is obvious if $\sup_{x \in X} |(k * g)(x)| = +\infty$.
On the other hand, if $\sup_{x \in X} |(k * g)(x)| = 0$, then by Remark 3.1 we have

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx = 0.$$

If $0 < \alpha = \sup_{x \in X} |(k * g)(x)| < +\infty$, then $k * (f - \frac{g}{\alpha})(x) \geqslant 0$ for all $x \in X$.
By Theorem 3.1, we have:

$$\int a(x, \frac{f(x)}{||f||_\varphi}).(f - \frac{g}{\alpha})(x)dx \geqslant 0.$$

Thus,

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx \leqslant \alpha \int a(x, \frac{f(x)}{||f||_\varphi}).f(x)dx.$$

On the other hand, we have for all $x \in \mathbb{R}^N$ and $t \geqslant 0$:

$$\varphi(x, t) = \int_0^t a(x, t)dt \geqslant \int_{\frac{t}{2}}^t a(x, t)dt \geqslant (\frac{t}{2})a(x, \frac{t}{2}).$$

Then,

$$a(x, \frac{f(x)}{||f||_\varphi}).\frac{f(x)}{||f||_\varphi} \leqslant \varphi(x, 2\frac{f(x)}{||f||_\varphi}) \leqslant K'_\varphi \varphi(x, \frac{f(x)}{||f||_\varphi})$$

because φ satisfies the $\triangle_2$ condition.
Therefore,

$$\int a(x, \frac{f(x)}{||f||_\varphi}).g(x)dx \leqslant \alpha.K'_\varphi.\varrho_\varphi(\frac{f}{||f||_\varphi}).$$

Since $\varrho_\varphi(\frac{f}{||f||_\varphi}) \leqslant 1$, the proof is complete.

Theorem 3.3 *Let φ be a Musielak-Orlicz function, uniformly convex which satisfies the $\triangle_2$ condition. Let $(X_i)_i \subset \mathbb{R}^N$. For each i, f_i is the distribution function of X_i for the $C_{k,\varphi}$ capacity. Let $X \subset \mathbb{R}^N$ and f be its distribution function for the $C_{k,\varphi}$ capacity. If $X \subset \liminf X_i$ and $\lim C_{k,\varphi}(X_i) = C_{k,\varphi}(X)$ then, $f_i \longrightarrow f$ in L_φ.*
We have the same result, particularly if $(X_i)_i$ is increasing and $X = \bigcup_i X_i$ or $(X_i)_i$ is a decreasing sequence of compacts and $X = \bigcap_i X_i$.

Proof. $(f_i)_i$ is bounded in L_φ. Since the space L_φ is reflexive, there exists a subsequence denoted again by $(f_i)_i$ which converges weakly in L^+_φ to a function g in L_φ.
By Theorem 2.6, $k * g \geqslant 1$ on X $C_{k,\varphi}\,-q.e.$ Therefore,

$$C_{k,\varphi}\,(X) \leqslant ||g||_\varphi.$$

On the other hand for all $h \in L_\psi$

$$\int f_i(x)h(x)dx \longrightarrow \int g(x)h(x)dx.$$

By*Hölder* inequality, we have:

$$\int f_i(x)h(x)dx \leqslant ||f_i||_\varphi |||h|||_\psi.$$

Thus, $\displaystyle\int g(x)h(x)dx \leqslant \liminf ||f_i||_\varphi |||h|||_\psi \leqslant ||f||_\varphi |||h|||_\psi.$
Let the function $h : x \longrightarrow a(x, \dfrac{g(x)}{||g||_\varphi})$ for all $x \in \mathbb{R}^N$.
By Lemma 2.1, $h \in L_\psi$, and by Lemma 2.3

$$||g||_\varphi |||h|||_\psi = \int g(x)h(x)dx \leqslant ||f||_\varphi |||h|||_\psi.$$

Then,

$$||g||_\varphi \leqslant C_{k,\varphi}\,(X).$$

Thus,

$$||g||_\varphi = C_{k,\varphi}\,(X) \ and\ therefore\ f = g.$$

On the other hand, f is the unique adhesion value of the sequence $(f_i)_i$ for the topology $\sigma(L_\varphi, L_\psi)$. Then, $f_i \longrightarrow f$ weakly in L_φ. Since L_φ is uniformly convex, we have $f_i \longrightarrow f$ strongly in L_φ.

Theorem 3.4 *Let φ be a Musielak-Orlicz function. Let F be a closed subset of $\mathbb{R}^N$ such that $D_{k,\varphi}\,(F) < \infty$. For all $r \in \mathbb{R}^*_+$: $F_r = F \cap \{x \in \mathbb{R}^N : |x| > r\}$.
If $\lim\limits_{r\to+\infty} D_{k,\varphi}\,(F_r) = 0$ then there exists a measure $\mu \in M^+_1$ such that μ is concentrated on F; $||k * \mu||_\psi \leqslant 1$ and $D_{k,\varphi}\,(F) = ||\mu||$, where μ is called a distribution measure of F for $D_{k,\varphi}$. Particularly, if K is a compact such that $D_{k,\varphi}\,(K) < \infty$ then K possesses a distribution measure for $D_{k,\varphi}$.*

Proof. It is the same as that given in [3], Theorem 4.

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