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# Nonlinear Dynamics and Systems Theory

## An International Journal of Research and Surveys



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# NONLINEAR DYNAMICS AND SYSTEMS THEORY

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# Robust Neural Output Feedback Tracking Control for a Class of Uncertain Nonlinear Systems Without Time-delay

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**Abstract:** This paper investigates the problem of adaptive tracking control by output feedback for a class of uncertain nonlinear systems. These nonlinear systems are subjected to various structured and unstructured uncertainty due essentially to modelling errors, parameter variations and unmodelled dynamics. With the help of error signals generated by the simple linear observer, a radial basis function neural network (RBF NN) is established to approximately compensate on line for these uncertainties. In this note, the neural network operates over system input/output signals without time delay. The stability analysis and tracking performance of the closed-loop system are confirmed through Lyapunov stability theory. The potential of the theoretical results is demonstrated through computer simulations of both nonlinear systems, Van der Pol and tunnel diode circuit.

**Keywords:** *nonlinear systems; feedback control; perturbations; adaptive or robust stabilization; neural nets and related approaches; stability; simulation.*

**Mathematics Subject Classification (2010):** 93C10, 93B52, 93C73, 93D21, 62M45, 70K20, 37M05.

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## 1 Introduction

In practical engineering, a large range of physical systems and devices, such as electromagnetism, mechanical actuators, electronic relay circuits and chaotic systems possess nonlinear and uncertain characteristics [8, 18]. On the other hand, the magnitude of control signal is always limited due to the poorly modelled dynamics of these systems, i.e., for most practical processes, obtaining an exact model is a difficult task or is not possible at all [6]. Therefore, modelling errors, unmodelled dynamics and uncertain parameter variations should be explicitly considered in the control design to enhance robust control performance. If these uncertainties (referred to as inversion errors) are ignored in the control design, the closed-loop control performance will be strongly damaged, and instability may occur. Thus, it is very important to develop powerful robust control techniques for nonlinear systems subjected to high uncertainty.

In recent years, there has been growing attention paid to the control problems of uncertain systems [5,8,26]. As is well known, various adaptive state feedback and output feedback controls have been known as efficient algorithms for designing feedback controllers for a large class of nonlinear systems in the presence of uncertainties [1–3,6,16,20]. These algorithms are expected to exhibit more excellent performance in order to have its outputs track given reference signals. In the same area, [20] discusses backstepping-based approaches to adaptive output feedback control of uncertain systems that are linear with respect to unknown parameters. For systems in which nonlinearities depend only upon the available measurement, [23] and [16] give a solution to the output feedback stabilization problem. In brief, the controller designs and stability analysis of highly uncertain nonlinear dynamic systems have been an important research topic. Unfortunately, the majority of the existing references are deterministic since the exact models are not available and/or their parameters are not precisely known, which prevent the error signals from tending to zero [6].

Recent years have witnessed advances in approximation of high nonlinearity by incorporating neural networks (NNs) and fuzzy logic systems (FLSs) in the control design to achieve excellent tracking performances. Taking advantage of this fact, these intelligent techniques have been widely employed for nonlinear control and identification since they can approximate any nonlinear functions without a priori knowledge of system dynamics [6]. With the help of FLSs and NNs, many approximator based adaptive control approaches were proposed for uncertain nonlinear systems; see, for example, [10, 19, 21, 22, 25, 26] and references therein. In [21, 22, 25], adaptive fuzzy or NN state feedback control schemes for a class of single-input single-output (SISO) nonlinear systems without or with time delays are developed; in [10, 19], adaptive output feedback controllers for SISO nonlinear systems are developed without unmeasured states, while the adaptive fuzzy or NN decentralized output feedback stabilization problem for a class of nonlinear systems is discussed in [26]. [20] proposes a robust adaptive output-feedback controller based on the small-gain theorem in order to overcome the effect of the unmodelled dynamics involved in the considered uncertain systems, whereas a RBF NN augmented backstepping controller for the nonlinear system dynamics is applied in [4] to gain from the approximation ability of NNs and ensure the stability of the closed loop system by an augmented Lyapunov function. Thus, authors in [1, 2, 5] augment adaptive output feedback linearization control using single hidden layer NNs in order to overcome the effect of uncertain parameter and unmodelled dynamics for highly uncertain nonlinear systems, and excellent tracking performances were achieved. With the aid of NN techniques, [27] presents a novel robust adaptive trajectory linearization control (RATLC) method for a class of uncertain nonlinear systems, in which RBF NNs are introduced to approximate the uncertainties online from available measurements. In [3], first, an adaptive neural network (NN) state-feedback controller for a class of nonlinear systems with mismatched uncertainties is proposed. Then, a bound of unknown nonlinear functions is approximated using RBFNNs so that no information about the upper bound of mismatched uncertainties is required.

Moreover, in most real cases, the state variables are unavailable for direct online measurements, and merely input and output of the system are measurable. Therefore, estimating the state variables by observers plays an important role in the control of processes to achieve better performances. During the past several decades, many nonlinear observers have been developed to obtain the estimated states. Thus, [24] and [17] present an output feedback control using a high-gain observer that is applied to estimate the unmeasurable states of the nonlinear systems. A sliding mode observer is proposed in [9] for a class of nonlinear systems to achieve finite time convergence for all error states. Notice that this previous observer makes use of fractional powers to reduce other non-output errors to zero in finite time. For a special class of single-output nonlinear systems, [15] has developed a sliding mode high-gain observer for state and unknown input estimations, so that the disturbance can be estimated from the sliding surface by ensuring the observability of the unknown input with respect to the output. However, these conventional nonlinear observers, such as high-gain observers [17, 24], and sliding mode observers [9, 15] are only applicable to systems with specific model structures.

Recently, observer-based adaptive fuzzy-neural control schemes are proposed for a large class of uncertain nonlinear dynamical systems. [11] proposes an indirect adaptive fuzzy neural network controller with state observer and supervisory controller for a class of uncertain nonlinear dynamic time-delay systems, in which the free parameters of the indirect adaptive fuzzy controller can be tuned on-line by observer based output feedback control law and adaptive laws by means of Lyapunov stability criterion. A novel state and output feedback control law that are developed for the tracking control of a class of multi-input-multi-output (MIMO) continuous time nonlinear systems with unknown dynamics and disturbance input can be found in [23], in which a high-gain observer is utilized to estimate the unmeasurable system states and an output feedback based controller is designed.

In the present paper, we contribute to design only one robust adaptive output feedback controller augmented using a RBF NN to handle uncertainties that exist in two switched SISO nonlinear systems. In the simple strategy followed in this work, first, we involve feedback linearization. Then, we design the adaptive control signal coupled with the robustifying term to compensate adaptively for inversion errors. A vector, that contains a linear combination of the tracking error generated by the linear observer and the compensator states, is exploited in the adaptation laws for the NN parameters. Furthermore, input/output data of the considered systems (without time-delay) is employed as a teaching signal for the NN. Consequently, the obtained robust control scheme not only guarantees the stability of the closed-loop system, but also has strong robustness to uncertainties existing in both nonlinear systems. Computer simulations of switched nonlinear systems, Van der Pol example having fourth-order nonlinear system of relative degree two and tunnel diode circuit model having full relative degree, are used to demonstrate the effectiveness of the proposed approach.

The rest of this paper is organized as follows. First, the system description and control problem are introduced in the next section. Then, the control structure is well detailed in Section 3. Section 4 develops a robust adaptive controller, in which NN augmentation is discussed. In Section 5, faithful stability analysis is elaborated to guarantee the boundedness of the tracking error signals. The efficiency of the proposed control approach is revealed throughout computer simulation in Section 6.

## 2 Problem Formulation

Let the dynamics of an observable uncertain SISO system be given as follows

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \Re^{\mathrm{n}}$ is the state of the plant, $u \in \Re$, and $y \in \Re$ are the control and measurement, respectively.

**Assumption 1.** The functions $f : \Re^{n+1} \longrightarrow \Re^n$ and $h : \Re^n \longrightarrow \Re$ are partially known, and the dynamical system of (1) satisfies the output feedback linearization conditions [14] with relative degree $r$ for all $(x,u) \in \Omega \times \Re$ where $\Omega \subset \Re^n$. Moreover, $n$ need not to be known. Therefore, there exists a mapping that transforms the system in (1) into the so-called normal form [12]:

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1}, \quad i = 1, ..., r-1, \\ \dot{\xi}_r &= h(\xi, u), \\ \xi_1 &= y, \end{aligned} \tag{2}$$

where $h(\xi,u) = L_f^{(r)} h$ are the Lie derivatives, and $\xi = [\xi_1 \quad ... \quad \xi_r]^T$.

The key objective is to design a robust neural output feedback tracking control that utilizes the available measurement $y$, so that $y(t)$ tracks a reference trajectory $y_{ref}(t)$ with bounded error.

## 3 Controller Design

### 3.1 Feedback linearization

Approximate feedback linearization is performed by defining the following control input signal:

$$v = \widehat{h}^{-1}(y,u), \tag{3}$$

where $v$ is a pseudo-control. The function $\widehat{h}(y,u)$ represents the best available approximation of $h(y,u)$. Then, the system dynamics can be formulated as

$$y^{(r)} = v + \vartheta, \tag{4}$$

where

$$\vartheta(\xi, v) = h(\xi_1, \widehat{h}^{-1}(\xi_1, v)) - \widehat{h}(\xi_1, \widehat{h}^{-1}(\xi_1, v)) \tag{5}$$

is the inversion error. Note that the pseudo-control mentioned in (4) is chosen to have the form

$$v = y_{ref}^{(r)} + L_d^c - V_c^s + R_t, \tag{6}$$

where $y_{ref}^{(r)}$ is the $r^{th}$ derivative of the input signal $y_{ref}$ generated by a stable command filter, $L_d^c$ is the output of a linear dynamic compensator, $V_c^s$ and $R_t$, namely adaptive control signal and robustifying term, are designed to overcome $\vartheta$.

With (6), the dynamics in (4) will be expressed as follows

$$y^{(r)} = y_{ref}^{(r)} + L_d^c - V_c^s + R_t + \vartheta. \tag{7}$$

From (5), notice that $\vartheta$ depends on $V_c^s$ and $R_t$ through $v$, whereas $V_c^s - R_t$ has been designed to approximately cancel $\vartheta$.

### 3.2 Linear Dynamic Compensator Design and Tracking Error Dynamics

The output tracking error is defined as $e = y_{ref} - y$. Then the dynamics in (7) can be rewritten as

$$e^{(r)} = -L_d^c + V_c^s - R_t - \vartheta. \tag{8}$$

Note that the adaptive term coupled with the robustifying term $V_c^s - R_t$ are not required when $\vartheta = 0$. Consequently, the error dynamics in (8) reduces to

$$e^{(r)} = -L_d^c. \tag{9}$$

The following linear compensator is introduced to stabilize the dynamics in (9):

$$\begin{cases} \dot{\lambda} = A_q\lambda + b_q e, \\ L_d^c = c_q\lambda + d_q e, \quad \lambda \in \Re^{r-1}. \end{cases} \tag{10}$$

Note that $\lambda$ needs to be at least of dimension $(r-1)$ [7]. This follows from the fact that (9) corresponds to error dynamics that has $r$ poles at the origin. One could elect to design a compensator of dimension $\geq$ r as well. In the future, we will assume that the minimum dimension is chosen.

Returning to (8), notice that the vector $e_r = [e \quad \dot{e} \quad ... \quad e^{(r-1)}]^T$ mutually with the compensator state $\lambda$ will obey the following dynamics, referred to as tracking error dynamics:

$$\begin{cases} \dot{E} = A_k E + b_k[V_c^s - R_t - \vartheta], \\ z = C_k E, \end{cases} \tag{11}$$

where $z$ is the vector of available measurements.

Remind that

$$A_k = \begin{bmatrix} A - d_q bc & -bc_q \\ b_q c & A_q \end{bmatrix}, \; b_k = \begin{bmatrix} b \\ 0 \end{bmatrix}, \; c_k = \begin{bmatrix} c & 0 \\ 0 & I \end{bmatrix} \tag{12}$$

and a new vector

$$E_d = \begin{bmatrix} e_r^T & \lambda^T \end{bmatrix}^T, \tag{13}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T.$$

Note that $A_q, b_q, c_q$ and $d_q$ in (10) should be designed such that $A_k$ is Hurwitz.

### 3.3 Observer Design for the Error Dynamics

Lyapunov-like stability analysis of the error dynamics results in update laws for the adaptive control parameters in terms of $(E)$ for the full-state feedback application [2,5]. In [12] and [13], adaptive state observers are used to provide the necessary estimates in the adaptation terms. In the present paper, we propose a simple linear observer for the tracking error dynamics in (11), and confirm through Lyapunov's direct method that the adaptive part of the control signal coupled with the robustifying term $(V_c^s - R_t)$ cancels successfully the inversion error $(\vartheta)$, if the output of this observer is introduced as an error signal for the adaptive laws. Moreover, a minimal-order observer of dimension $(r-1)$ may be designed for the dynamics in (11).

In what follows, we consider the case of a full-order observer of dimension $(2r-1)$ [12]. To this end, consider the following simple linear observer for the tracking error dynamics in (11):

$$\begin{cases} \dot{\widehat{E}} = A_k \widehat{E} + K(z - \widehat{z}), \\ \widehat{z} = C_k \widehat{E}, \end{cases} \tag{14}$$

where $K$ is a gain matrix, and $z$ that is defined in (11) should be chosen such that $(A_k - KC_k)$ is asymptotically stable.

Let

$$\widetilde{A} = A_k - KC_k, \quad \widetilde{E} = \widehat{E} - E, \quad \widetilde{z} = \widehat{z} - z. \tag{15}$$

Then, the observer error dynamics can be written as

$$\begin{cases} \dot{\widetilde{E}} = \widetilde{A}\widetilde{E} - b_k[V_c^s - R_t - \vartheta], \\ \widetilde{z} = c_k \widetilde{E}. \end{cases} \tag{16}$$

## 4 RBF NN Augmented Controller

### 4.1 NN approximation

Following [12], given a compact set $\mathcal{D} \subset R^{n+1}$ and $\epsilon^* > 0$, the model inversion error $\vartheta(\xi, v)$ can be approximated over $\mathcal{D}$ by a radial basis function neural network (RBF NN)

$$\vartheta(\xi, v) = M^T \phi(\varrho) + \epsilon(d, \varrho), \quad |\epsilon| < \epsilon^*, \tag{17}$$

using the input vector

$$\varrho(t) = [v \quad y]^T \in \mathcal{D}, \quad \|\varrho\| \le \varrho^*, \quad \varrho^* > 0. \tag{18}$$

The adaptive signal is designed as follows

$$V_c^s = \widehat{M}^T \phi(\widehat{\varrho}), \tag{19}$$

where $\widehat{M}$ is the estimate of $M$ that is updated according to the following adaptation law:

$$\dot{\widehat{M}} = -\beta_M[2\phi(\widehat{\varrho})\widehat{E}^T P b_k + \alpha_M(\widehat{M} - M_0)] \tag{20}$$

in which $M_0$ is the initial value of $M$, $P$ is the solution of the Lyapunov equation

$$A_k^T P + P A_k = -Q \tag{21}$$

for some $Q > 0$, $k > 0$, $\beta_M$ is the adaptation gain matrix, and $\widehat{\varrho}$ is an implementable input vector to the NN on the compact set $\Omega_{\widehat{\varrho}}$, defined as $\widehat{\varrho} = [v^T(t) \quad \widehat{y}^T(t)]^T \in \Omega_{\widehat{\varrho}}$, $\widehat{y}_i = \widehat{E}_i + y_{ref}^{(i-1)}, i = 1, ..., r-1$.

Notice that in (19), there is an algebraic loop, since $\widehat{\varrho}$, by definition, depends upon $V_c^s$ through $v$, see (18). However, with bounded squashing functions, this algebraic loop has at least one fixed-point solution as long as $\phi(.)$ is made up of bounded basis functions.

The robustifying term is designed as follows

$$R_t = \widehat{\Psi} sgn(2\widehat{E}^T P b_k), \tag{22}$$

where the adaptive gain $\widehat{\Psi}$ is updated according to the following adaptation law

$$\dot{\widehat{\Psi}} = -\beta_\Psi[2\widehat{E}^T P b_k sgn(2\widehat{E}^T P b_k) + \alpha_\Psi(\widehat{\Psi} - \Psi_0)]. \tag{23}$$

in which $\Psi_0$ is an initial value of $\widehat{\Psi}, \beta_\Psi > 0, \alpha_\Psi > 0$.

Using (17) and (19), we can write the mismatch between the adaptive signal and the real NN as:

$$V_c^s - \vartheta = \widehat{M}^T \phi(\widehat{\varrho}) - M^T \phi(\varrho) - \epsilon = \widetilde{M}^T \widehat{\phi} + M^T \widetilde{\phi} - \epsilon, \tag{24}$$

where $\widetilde{M} = \widehat{M} - M, \widehat{\phi} = \phi(\widehat{\varrho}), \widetilde{\phi} = \phi(\widehat{\varrho}) - \phi(\varrho)$.

Using (24), the error dynamics in (11) and the observer error dynamics in (16) can be reformulated as

$$\dot{E} = A_k E + b_k[\widetilde{M}^T \widehat{\phi} + M^T \widetilde{\phi} - \epsilon - \widehat{\Psi} sgn(2\widehat{E}^T P b_k)], \tag{25}$$

$$\dot{\widetilde{E}} = \widetilde{A}\widetilde{E} + b_k[\widetilde{M}^T \widehat{\phi} + M^T \widetilde{\phi} - \epsilon - \widehat{\Psi} sgn(2\widehat{E}^T P b_k)]. \tag{26}$$

Notice that for radial basis function and many other activation functions that satisfy $|\phi_i| \leq 1, i = 1, ..., N$, there exists an upper bound over the set $\mathcal{D}$

$$\|\phi(\varrho)\| \leq \varpi, \quad \varpi = \max_{\varrho \in \mathcal{D}} \|\phi(\varrho)\|, \tag{27}$$

where $\varpi$ remains of the order one, even if $N$ is large. With this, we have the following upper bound:

$$|M^T \widetilde{\phi}| \leq 2\|M\|\varpi. \tag{28}$$

## 5 Stability Analysis

We confirm through Lyapunov's direct method that if the initial errors of the variables $E^T, \widetilde{E}^T, \widetilde{E}, \widehat{M}^T$ and $\widetilde{\Psi}$ belong to a presented compact set, then the composite error vector $\zeta = \begin{bmatrix} E^T & \widetilde{E}^T & \widehat{M}^T & \widetilde{\Psi} \end{bmatrix}^T$ is ultimately bounded, where $\widetilde{\Psi} = \widehat{\Psi} - \Psi$ and $\Psi = \epsilon^* + 2\varpi\|M\|$. Notice that $\zeta$ can be viewed as a function of the state variables $y, \lambda, \widehat{E}, \widehat{Z}$, the command vector $y_{ref}$, and a constant vector $Z$

$$\zeta = F\big(y, \lambda, \widehat{E}, \widehat{Z}, y_{ref}, Z\big), \tag{29}$$

where $\widehat{Z} = [\widehat{M}^T \quad \widehat{\Psi}]^T$, $Z = [M^T \quad \Psi]^T$. The relation in(29) represents a mapping from the original domains of the arguments to the space of the error variables

$$F : \Omega_y \times \Omega_{y_{ref}} \times \Omega_\lambda \times \Omega_{\widehat{E}} \times \Omega_{\widehat{Z}} \times \Omega_Z \longrightarrow \Omega_\zeta. \tag{30}$$

Recall that (18) introduces the compact set $\mathcal{D}$ over which the NN approximation is valid. From (18), it follows that

$$\varrho \in \mathcal{D} \Longleftrightarrow y \in \Omega_y, \quad v \in \Omega_v. \tag{31}$$

Also, notice that, since the observer in(14) is driven by the output tracking error $e = y_{ref} - y$ and compensator state $\lambda$, having $y \in \Omega_y$, $y_{ref} \in \Omega_{y_{ref}}$, $\lambda \in \Omega_\lambda$, implies that $\widehat{E} \in \Omega_{\widehat{E}}$, the latter being a compact set. According to (6)

$$v = F_v\big(\lambda, \widehat{E}, \widehat{Z}, y_{ref}\big), \tag{32}$$

where $F_v : \Omega_\lambda \times \Omega_{\widehat{E}} \times \Omega_{\widehat{Z}} \times \Omega_{y_{ref}} \longrightarrow \Omega_v$.

Thus, (29), (31) and (32) ensure that $\Omega_\zeta$ is a bound set. Introduce the largest ball, which is included in $\Omega_\zeta$ in the error space

$$L_B = \big\{|\zeta| \|\zeta\| \leq R\big\}, \quad R > 0. \tag{33}$$

For every $\zeta \in L_B$, we have $\varrho \in \mathcal{D}, Z \in \Omega_Z$, where both $\mathcal{D}$ and $\Omega_Z$ are bounded sets.

**Assumption 2.** Assume

$$R > \gamma\sqrt{\frac{T_M}{T_m}} \geq \gamma. \tag{34}$$

where $T_M$ and $T_m$ are the maximum and minimum eigenvalues of the following matrix

$$T = \frac{1}{2}\begin{bmatrix} 2P & 0 & 0 & 0 \\ 0 & 2P & 0 & 0 \\ 0 & 0 & \beta_M^{-1} I & 0 \\ 0 & 0 & 0 & \beta_\Psi^{-1} \end{bmatrix} \tag{35}$$

and

$\gamma = \max(\sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\alpha_{\min}(Q)-2}}, \sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\alpha_{\min}(\widetilde{Q})-2}}, \sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\rho}}$, where $\overline{Z} = \frac{\alpha_M}{2}\|M - M_0\|^2 + \frac{\alpha_\Psi}{2}|\Psi - \Psi_0|^2, \Theta = \|Pb_k\| + \|\widetilde{P}b_k\|$, $\rho = \alpha - \Theta^2(\varpi + 1)^2 > 0$, $\alpha = \frac{1}{2}\min(\alpha_M, \alpha_\Psi)$ and $\widetilde{P}$ satisfies $\widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A} = -\widetilde{Q}$ for some $\widetilde{Q} > 0$ with minimum eigenvalues $\alpha_{min}(\widetilde{Q}) > 2$.

**Theorem 1.** *Let the assumption (1) hold, and let $\alpha_{min}(Q) > 2$ for $Q$ introduced in (21). Then, if the initial errors belong to the set $\Omega_\alpha$ defined in (37), the feedback control laws given by (3) and (6), along with adaptation laws (20) and (23) ensure that the error signals $E, \widetilde{E}$, $\widetilde{M}$ and $\widetilde{\Psi}$ in the closed-loop system are ultimately bounded.*

**Proof.** Take into account the following Lyapunov function:

$$V = E^T PE + \widetilde{E}^T \widetilde{P}\widetilde{E} + \frac{1}{2}\widetilde{M}^T \beta_M^{-1}\widetilde{M} + \frac{1}{2}\widetilde{\Psi}^T \beta_\Psi^{-1}\widetilde{\Psi}. \tag{36}$$

The derivative of $V$ along the tracking error dynamics(25), the observer error dynamics (26), NN weight and adaptive gain adaptation laws (20) and (23) can be formulated as

$$\begin{aligned}\dot{V} = &-E^T PE - \widetilde{E}^T \widetilde{Q}\widetilde{E} - 2\widetilde{E}^T(\widetilde{P} + P)b_k[\widetilde{M}^T\widehat{\phi} + M^T\widetilde{\phi} - \epsilon - \widehat{\Psi}sgn(2\widehat{E}^T Pb_k)]\\ &- 2\widetilde{E}^T Pb_k[\epsilon - M^T\widetilde{\phi} + \Psi sgn(2\widehat{E}^T Pb_k)] - [\alpha_M\widetilde{M}^T(\widehat{M} - M_0)] - \widetilde{\Psi}\alpha_\Psi(\widehat{\Psi} - \Psi_0),\end{aligned}$$

where $\widetilde{E} = \widehat{E} - E, \widehat{\Psi} = \Psi + \widetilde{\Psi}$. Using the following property for vectors $[\widetilde{M}^T(\widehat{M} - M_0)] = \frac{1}{2}\|\widetilde{M}\|^2 + \frac{1}{2}\|\widehat{M} - M_0\|^2 - \frac{1}{2}\|M - M_0\|^2$ , and with (28), the upper bound becomes [13]

$$\dot{V} \le -[\alpha_{min}(Q) - 2]\|\widetilde{E}\|^2 - [\alpha_{min}(\widetilde{Q}) - 2]\|E\|^2 - [\alpha - \Theta^2(\varpi + 1)^2]\|\widetilde{Z}\|^2 + \overline{Z} + 4(\Theta\Psi)^2.$$

Either of the following conditions:
$\|\widetilde{E}\| > \sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\alpha_{\min}(Q)-2}}, \|E\| > \sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\alpha_{\min}(\widetilde{Q})-2}}, \|\widetilde{Z}\| > \sqrt{\frac{4(\Theta\Psi)^2+\overline{Z}}{\rho}}$ will render $\dot{V} < 0$ outside a compact set: $B_\gamma = \{\zeta \in L_B, \|\zeta\| \le \gamma\}$.

Note from (34) that $B_\gamma \subset L_B$. Then, consider the Lyapunov function candidate in (36) and write it as: $V = \zeta^T T\zeta$. Let $\Upsilon$ be the maximum value of the Lyapunov function $V$ on the edge of $B_\gamma$: $\Upsilon = \max_{\|\zeta\|=\gamma} V = \gamma^2 T_M$. Introduce the level set $\Omega_\gamma = \{\zeta V \le \Upsilon\}$. Let $\alpha_v$ be the minimum value of the Lyapunov function $V$ on the edge of $L_B$: $\alpha_v = \min_{\|\zeta\|=R} V = R^2 T_m$. Define the level set

$$\Omega_\alpha = \{\zeta \in L_B, V = \alpha_v\}. \tag{37}$$

Consequently, the condition in (34) guarantees that $\Omega_\gamma \subset \Omega_\alpha$, and thus ultimate boundedness of $\zeta$.

## 6 Application

This paper addresses the design of a robust adaptive controller augmented using a NN to handle the uncertainty of two switched nonlinear systems: Van der Pol model having a fourth-order nonlinear system of relative degree two and the tunnel diode circuit example with full relative degree. This part is devoted to illustrating the performance of the proposed approach. First, we present the dynamics of the considered uncertain systems:

### 6.1 Tunnel diode circuit model

$$\begin{cases} \dot{x}_1 = \dfrac{1}{C}x_2 - \dfrac{1}{C}h(x_1), \\ \dot{x}_2 = -\dfrac{R}{L}x_2 - \dfrac{1}{L}x_1 + \dfrac{u}{L}, \end{cases} \tag{38}$$

where $x_1$ the voltage across the capacitor $C$ and $x_2$ is the current through the inductor $L$. The initial conditions were set as $x_1(0) = 0.1, x_2(0) = 0.0005$, and the element values of the circuit are $R = 1.5k\Omega, L = 1nH$, and $C = 2pF$. Notice that the function $h : \Re \longrightarrow \Re$ represents the characteristic curve of the tunnel diode, $h(x_1) = x_1 + 2x_1^2 + x_1^3 - x_1^4 - 2x_1^5$. We assume that the output $y$ has a full relative degree of $n = r = 2$.

### 6.2 Van der Pol model

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -0.2(x_1^2 - 1)x_2 - 0.2x_3 + \dfrac{u}{\sqrt{|u| + 0.1}}, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -0.2x_4 - x_2 + x_1, \end{cases} \tag{39}$$

with initial conditions $x_1(0) = 0.5, x_2(0) = 1.5, x_3(0) = 0$ and $x_4(0) = 0$. The output $y$ has a relative degree of $r = 2$.

The command signals $y_{ref}$ and $y_{ref}^{(2)}$ are generated through a second -order command filter with natural frequency of $1rad/s$ and damping of 0.7. The following dynamic compensator:

$$\begin{cases} \dot{\lambda} = -6.4\lambda + 4e, \\ L_d^c = -18.2\lambda + 13.04e, \end{cases} \tag{40}$$

places the poles of the closed-loop error dynamics in (9) of both nonlinear systems at $-3.6, -1.4 \pm j$. The observer dynamics in (16) was designed so that its poles are four times faster than those of the error dynamics. A radial basis function NN with five neurons was used in the adaptive control. The functional form for each RBF neuron was defined by

$$\phi_i(\varrho) = e^{-(\varrho-\kappa_{c_i})^T(\varrho-\kappa_{c_i})/\sigma^2}, \quad \sigma = 1, \quad i = \overline{1,6}. \tag{41}$$

The centers $\kappa_{c_i}$, $i = \overline{1,6}$, were arbitrarily selected over a grid of possible values for the vector $\varrho$. The adaptation gains were set to $\beta_M = 1.2$, with sigma modification gain $\alpha_M = 0.001$. The other parameters are : $\alpha_\Psi = 0.012, \beta_\Psi = 0.0015$.

In this paper, we contribute to design one robust adaptive control scheme augmented using a RBF NN in order to make up adaptively for the nonlinearities that exist in both uncertain systems (Van der Pol and tunnel diode circuit model). Therefore, the designed controller forces the system response to track a given reference trajectory with bounded errors. First, set the output $y = x_1$ for each system. Then, we employ feedback linearization, coupled with an on-line NN to handle the inversion errors, according to the equation (7). The dynamic compensator, described in (10) and (40), is designed to stabilize the linearized systems [1,2]. A signal, constituted of a linear combination of the measured tracking error and the compensator states is used to adapt the control laws, such as presented in (20), (22) and (23).

Figure 1 compares the system measurement $y$ without NN augmentation (dashed line) with the reference model output $y_{ref}$ (solid line), clearly demonstrating the almost unstable oscillatory behavior caused by the nonlinear elements ($\vartheta$) in the Van der Pol model in the first half time (0 to 50 seconds) and the nonlinearities of the tunnel diode equation in the last half time (50 to 100 seconds). Meanwhile, with the aid of NN augmentation, Figure 2 shows that the effect of these nonlinearities is successfully eliminated. This is

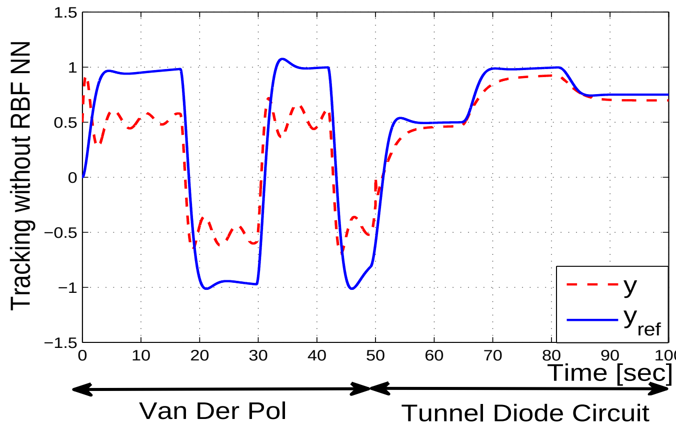


**Figure 1**: Tracking without RBF NN.

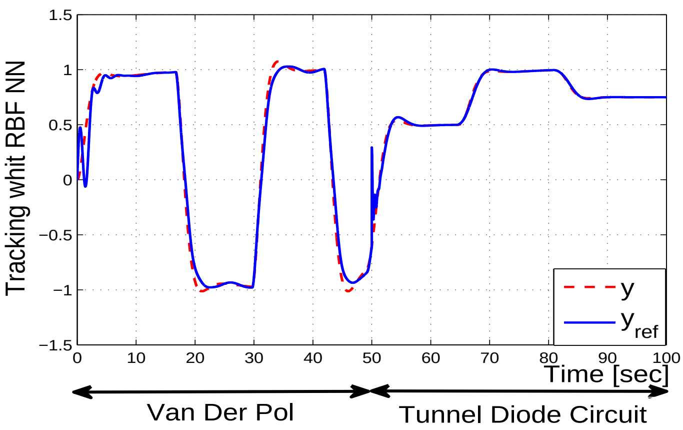


**Figure 2**: Tracking with the aid of RBF NN.

due essentially to the excellent identification of the model inversion error ($\vartheta$) (dashed line) by adaptive control signal and robustifying term ($V_c^s - R_t$) (solid line), which is illustrated in Figure 3.

Figure 4 compares the control efforts ($y_{ref} - y$) without and with adaptation, in which the NN based robust adaptive controller exhibits a steady state tracking error.

As expected, the RBF NN improves the tracking performance due to its ability to "model" nonlinearities. Consequently, simulation results show that the NNs augmented robust adaptive output feedback controller compensates successfully for the uncertainties existing in two different nonlinear systems.

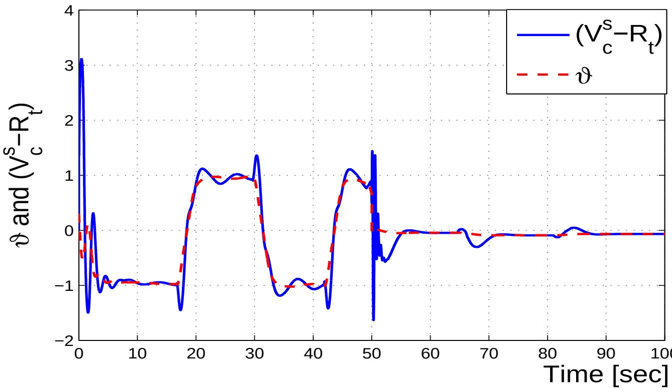


**Figure 3**: Identification of uncertainties ($\vartheta$) by NN ($V_c^s - R_t$).

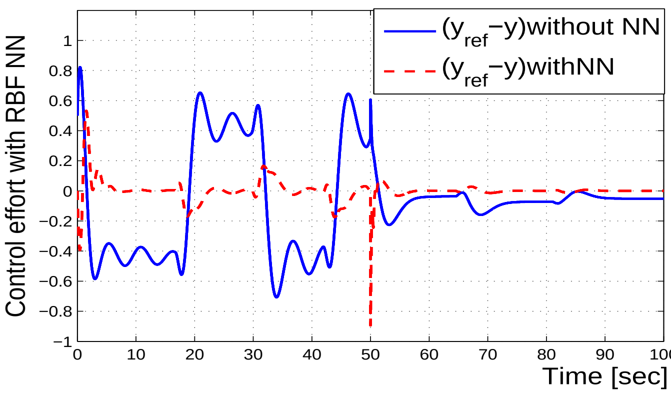


**Figure 4**: Control effort without and with RBF NN.

## 7 Conclusion

In this paper, one robust adaptive output feedback control augmented via RBF NN has been designed to overcome the effect of nonlinearities for both highly uncertain nonlinear systems: Van der Pol and Tunnel Diode Circuit. The derivatives of the tracking error are estimated by the simple linear observer. These estimates are used in the adaptation laws for the NN parameters. Ultimate boundedness of the tracking and observation errors are proven using Lyapunov's direct method. The methodology is applicable for observable and stabilizable systems of unknown but bounded dimension when the relative degree is known. Through Lyapunov-based theoretical analysis and computer simulation, we were able to demonstrate that the proposed RBF NN-based robust adaptive output feedback controller was robust to modeling inaccuracies, and excellent tracking performance was succeeded.

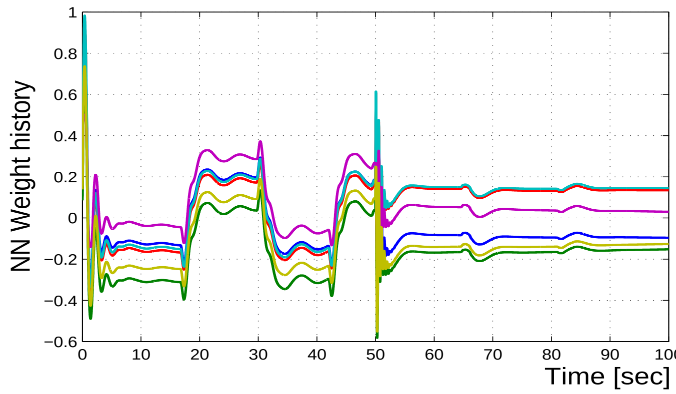


**Figure 5**: NN weights history.

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# Entropy Solutions of a Quasilinear Degenerated Elliptic Unilateral Problems With $L^1$ Data and Without Sign Condition

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**Abstract:** In this paper, we will be concerned with the existence of solutions for strongly nonlinear degenerated elliptic unilateral problems associated with the equation $A(u) + g(x, u, \nabla u) + H(x, \nabla u) = f$, where $A$ is Leray-Lions operator acting from $W_0^{1,p}(\Omega, w)$ to its dual. On the nonlinear term $g(x, s, \xi)$, we assume growth condition on $\xi$ and without assuming the sign condition on $s$, while the function $H(x, \xi)$, which induces a convection term, is only growing at most as $|\xi|^{p-1}$. The right-hand side $f$ belongs to $L^1(\Omega)$.

**Keywords:** *weighted Sobolev spaces; quasilinear degenerated unilateral problems; non-variational inequalities.*

**Mathematics Subject Classification (2010):** 35J15, 35J70, 35J87.

## 1 Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, $1 < p < \infty$ and $w = \{w_i(x), i = 0, \dots, N\}$ be a vector of weight functions on $\Omega$, i.e. each $w_i(x)$ is a measurable strictly positive function on $\Omega$, satisfying some integrability conditions. Let $X = W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector $w$. Consider the following non-linear Dirichlet problem

$$\begin{cases} A(u) + g(x, u, \nabla u) + H(x, \nabla u) = f \quad \text{in } \mathfrak{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega, w), \ g(x, u, \nabla u) \in L^1(\Omega), \ H(x, \nabla u) \in L^1(\Omega), \end{cases} \tag{1}$$

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where $A(u) = -\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $X$ into its dual $X^*$ and $g(x,u,\nabla u)$ is a nonlinear lower-order term that grows at most like $|\nabla u|^p$ satisfying the coercivity condition $|g(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)\,|\xi_i|^p$ for $|s|$ sufficiently large, while the function $H(x,\nabla u)$ is only growing at most as $|\nabla u|^{p-1}$. We study the problem (1) in the non-variational case where the right-hand side $f$ belongs to $L^1(\Omega)$.

Our main goal, in this paper, is to prove an existence result for degenerated unilateral problems associated with (1) in the non-variational case where the source term $f$ belongs to $L^1(\Omega)$ and without assuming the sign condition $g(x,s,\xi)s \geq 0$. More precisely, we prove the existence of solutions for the following nonlinear Dirichlet problem

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega,w), \ u \geq \psi \ \text{ a.e. in } \ \Omega, \\ \langle A(u), T_k(u-v)\rangle + \displaystyle\int_\Omega (g(x,u,\nabla u) + H(x,\nabla u))\, T_k(u-v)\,dx \\ \qquad \leq \ \displaystyle\int_\Omega f\, T_k(u-v)\,dx, \qquad \forall v \in K_\psi \cap L^\infty(\Omega), \ \ \forall k > 0. \end{cases}$$

Note that $\mathcal{T}_0^{1,p}(\Omega,w)$ is the set of measurable functions $u : \Omega \to \mathbb{R}$ such that, for all $k \geq 0$, we have $T_k(u) \in W_0^{1,p}(\Omega,w)$, where $T_k : \mathbb{R} \to \mathbb{R}$ is the truncation at height $k$ defined by $T_k(s) = \max(-k, \min(k,s))$ for all $s \in \mathbb{R}$. $K_\psi$ is the convex set defined by $K_\psi = \{u \in W_0^{1,p}(\Omega,w) : \ u \geq \psi$ almost everywhere (a.e.) in $\Omega\}$ for an obstacle function $\psi : \Omega \to \overline{\mathbb{R}}$ such that $\psi^+ \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$.

For $H \equiv 0$ and in the variational case (i.e. the source term $f$ belongs to $W^{-1,p'}(\Omega,w^*)$), an existence theorem for degenerated unilateral problem related to the equation (1), was proved in [4] where the authors have used the approach based on the strong convergence of the positive part $u_\varepsilon^+$ (respectively negative part $u_\varepsilon^-$). In the non-variational case where $f \in L^1(\Omega)$, the authors of [9] give an existence result for degenerated unilateral problems associated with (1) by another approach based on the strong convergence of truncation. All previous works have used the sign condition for the lower-order nonlinear term $g$, for those who don't use it one can cite that of Porretta [17] and that of Aharouch and Akdim [1] in the classical Sobolev space $W_0^{1,p}(\Omega)$ and that of Aharouch et al. [2] in the weighted case.

When $H$ is not necessarily the null function and in the non weighted case (i.e. $w \equiv 1$), Del Vecchio has solved in [10] the problem (1) where $g$ depends only on $x$ and $u$. If $g$ depends also on $\nabla u$, an existence result for the problem (1) was first proved in [16] by Monetti and Randazzo in the case of equation and secondly in [18] by Youssfi et al. in the case of obstacle problems. Recently in [6], Akdim et al. give an existence result that can be seen as a generalization of [18] in the weighted case.

This paper is organized as follows, Section 2 contains some preliminaries, basic assumptions and some technical lemmas, Section 3 is concerned with main results and their proofs, Section 4 gives an example of equations to which the present result can be applied. Finally, we end with a conclusion and the bibliography adopted in this work.

## 2 Preliminaries

### 2.1 Weighted Sobolev spaces.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and $p$ be a real number such that $1 < p < \infty$. For a measurable function $\gamma$ which is strictly positive a.e. in $\Omega$ we define the weighted space with weight $\gamma$ in $\Omega$ as $L^p(\Omega,\gamma) = \left\{u : \Omega \to \mathbb{R} :\ u\,\gamma^{\frac{1}{p}} \in L^p(\Omega)\right\}$, which is endowed with the norm $\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p\,\gamma(x)\,dx\right)^{\frac{1}{p}}$.

Let $w = \{w_i(x); i = 0, 1, \dots, N\}$ be a vector of weight functions. We suppose in all our considerations that for $0 \le i \le N$, $w_i \in L^1_{loc}(\Omega)$ and $w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega)$. We denote by $W^{1,p}(\Omega, w)$ the weighted Sobolev space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i),\ \forall i = 1, \dots, N$. This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p\,w_0\,dx + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p\,w_i(x)\,dx\right)^{\frac{1}{p}}. \tag{2}$$

To deal with the Dirichlet problem, we use the space $X = W_0^{1,p}(\Omega, w)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \left\{w_i^* = w_i^{1-p'}; i = 0, 1, \dots, N\right\}$ and $p'$ is the conjugate of $p$, that is $p' = p/(p-1)$. For more details we refer the reader to [11–14].

### 2.2 Basic assumptions and some technical lemmas.

We state the following assumptions.

**Assumption** $(\mathcal{H}_1)$:

- The expression
$$|||u|||_X = \left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p\,w_i(x)\,dx\right)^{\frac{1}{p}} \tag{3}$$
is a norm defined on $X$ and it is equivalent to the norm (2).

- There exists a weight function $\sigma$ on $\Omega$ such that $\sigma \in L^1(\Omega)$ and $\sigma^{1-q'} \in L^1_{loc}(\Omega)$ for some parameter $1 < q < p + p'$, ($q' = \frac{q}{q-1}$), such that the Hardy inequality
$$\left(\int_\Omega |u(x)|^q\,\sigma\,dx\right)^{\frac{1}{q}} \le c\left(\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p\,w_i(x)\,dx\right)^{\frac{1}{p}} \tag{4}$$
holds for every $u \in X$ with a constant $c > 0$ independent of $u$. Moreover, the imbedding
$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma), \tag{5}$$
determined by the inequality (4) is compact.

Note that $(X, |||u|||_X)$ is a uniformly convex and thus reflexive Banach space.

**Lemma 2.1** *[3] Let $\varrho \in L^r(\Omega,\gamma)$ and $\varrho_n \in L^r(\Omega,\gamma)$ such that $\|\varrho_n\|_{r,\gamma} \le c$, where $1 < r < \infty$ and $\gamma$ is a weight function on $\Omega$. If $\varrho_n(x) \to \varrho(x)$ a.e. in $\Omega$, then $\varrho_n \rightharpoonup \varrho$ weakly in $L^r(\Omega,\gamma)$.*

**Lemma 2.2** *[3] Assume that $(\mathcal{H}_1)$ holds. Let $u \in W_0^{1,p}(\Omega, w)$, then $T_k(u) \in W_0^{1,p}(\Omega, w)$. Moreover, we have $T_k(u) \to u$ strongly in $W_0^{1,p}(\Omega, w)$.*

**Lemma 2.3** *[5] Assume that $(\mathcal{H}_1)$ holds. Let $(u_n)$ be a sequence of $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$. Then $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, w)$.*

## 3 Main Results

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and $p$ be a real number such that $1 < p < \infty$. Let $A$ be the nonlinear elliptic differential operator in divergence form, defined from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ by $A(u) = -\mathrm{div} a(x, u, \nabla u)$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and all $\xi$, $\xi^* \in \mathbb{R}^N$ $(\xi \neq \xi^*)$, the following assumption.

**Assumption** $(\mathcal{H}_2)$: [(6), (7), (8)]

$$|a_i(x, s, \xi)| \leq \alpha_1 w_i^{\frac{1}{p}}(x)\, [\delta(x) + \sigma^{\frac{1}{p'}}\, |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)\, |\xi_j|^{p-1}], \text{ for } i = 1, \dots, N, \tag{6}$$

$$[a(x, s, \xi) - a(x, s, \xi^*)].[\xi - \xi^*] > 0, \tag{7}$$

$$a(x, s, \xi).\xi \geq \alpha_2 \sum_{i=1}^{N} w_i(x)\, |\xi_i|^p, \tag{8}$$

where $\delta(x)$ is a positive function in $L^{p'}(\Omega)$ and $\alpha_1, \alpha_2$ are positive constants.

**Lemma 3.1** *[3] Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied. Let $(u_n)$ be a sequence of $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)].[\nabla u_n - \nabla u]\, dx \to 0. \text{ Then } \ u_n \to u \text{ in } W_0^{1,p}(\Omega, w).$$

Furthermore, let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be two Carathéodory functions satisfying, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following assumption.

**Assumption** $(\mathcal{H}_3)$: [(9), (10), (11)]

$$|g(x, s, \xi)| \leq c(x) + b(s) \sum_{i=1}^{N} w_i(x)\, |\xi_i|^p, \tag{9}$$

$$|g(x, s, \xi)| \geq \beta \sum_{i=1}^{N} w_i(x)\, |\xi_i|^p \quad \text{for} \quad |s| > \rho, \tag{10}$$

$$|H(x, \xi)| \leq h(x) \sum_{i=1}^{N} w_i^{\frac{1}{p'}}(x)\, |\xi_i|^{p-1}, \tag{11}$$

where $\beta > 0$, $\rho > 0$, $b : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, $c \in L^1(\Omega)$ and $h \in L^r(\Omega)$ with $r > \max(N, p)$.

Finally, we assume that

$$f \in L^1(\Omega). \tag{12}$$

Consider the convex set $K_\psi = \{u \in W_0^{1,p}(\Omega, w) : \ u \geq \psi \ \text{ a.e. in } \Omega\}$ for an obstacle function $\psi : \Omega \to \overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{13}$$

**Definition 3.1** Assume that $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$, (12) and (13) hold true. A function $u$ is an entropy solution of problem (1) if

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega, w), \ u \geq \psi \ \text{ a.e. in } \ \Omega, \\ \langle A(u), T_k(u-v)\rangle + \displaystyle\int_\Omega (g(x,u,\nabla u) + H(x,\nabla u))\, T_k(u-v)\,dx \\ \qquad \leq \ \displaystyle\int_\Omega f\, T_k(u-v)\,dx, \qquad \forall v \in K_\psi \cap L^\infty(\Omega), \ \ \forall k > 0. \end{cases} \tag{14}$$

For the nonlinear Dirichlet boundary value problem (1), we state our main result as follows.

**Theorem 3.1** *Under the assumptions* $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$, (12) *and* (13)*, there exists at least one entropy solution of problem* (1) *(in the sense of Definition 3.1).*

***Proof of Theorem 3.1.***
**Step 1: A priori estimates.** Let $\Omega_n$ be a sequence of compact subsets of $\Omega$ such that $\Omega_n$ is increasing to $\Omega$ as $n \to \infty$. Let us define $H_n(x,\xi) = \frac{H(x,\xi)}{1+\frac{1}{n}|H(x,\xi)|}\,\chi_{\Omega_n}$ where $\chi_{\Omega_n}$ is the characteristic function of $\Omega_n$. Consider the sequence of approximate problems

$$\begin{cases} u_n \in K_\psi, \\ \langle A(u_n), u_n - v\rangle + \displaystyle\int_\Omega (g(x,u_n,\nabla u_n) + H_n(x,\nabla u_n))\,(u_n - v)\,dx \\ \qquad \leq \ \displaystyle\int_\Omega f_n\,(u_n - v)\,dx, \quad \forall v \in K_\psi, \end{cases} \tag{15}$$

where $(f_n)$ is a sequence of smooth functions which converges strongly to $f$ in $L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \leq C_f$. Note that $H_n(x,\xi)$ satisfies the conditions

$$|H_n(x,\xi)| \leq |H(x,\xi)| \ \text{ and } \ |H_n(x,\xi)| \leq n.$$

We define the operator $G_n : W_0^{1,p}(\Omega,w) \to W^{-1,p'}(\Omega,w^*)$ by

$$\langle G_n u, v\rangle = \int_\Omega (g(x,u,\nabla u) + H_n(x,\nabla u))\,v\,dx.$$

Thanks to the classical result, that is Theorem 8.2 of [15], the following lemma which can be proved in the same way as Lemma 4.2 of [4], shows that the problem (15) has at least one solution $u_n$.

**Lemma 3.2** *The operator* $B_n = A + G_n$ *from* $K_\psi$ *into* $W^{-1,p'}(\Omega,w^*)$ *is pseudomonotone. Moreover,* $B_n$ *is coercive in the following sense*

$$\frac{\langle B_n v, v - v_0\rangle}{\|v\|} \to +\infty \ \text{ if } \ \|v\| \to +\infty, \ v \in K_\psi, \ \ \text{where} \ \ v_0 \in K_\psi.$$

Take $v \in K_\psi$ and choose $h \geq \|\psi^+\|_\infty$ so as $\widetilde{v} = T_h(u_n - T_k(u_n - v)) \in K_\psi \cap L^\infty(\Omega)$. We can use in (15) the test function $\widetilde{v}$ and by letting $h \to +\infty$ we obtain

$$\langle A(u_n), T_k(u_n - v)\rangle + \int_\Omega [g(x,u_n,\nabla u_n) + H_n(x,\nabla u_n)]\,T_k(u_n - v)\,dx \\ \leq \int_\Omega f_n\,T_k(u_n - v)\,dx, \quad \text{for all } v \in K_\psi \ \text{ and for all } k > 0. \tag{16}$$

For $k \geq \rho + \|\psi^+\|_\infty$, where $\rho$ is defined in (10), taking $v = \psi^+$ as the test function in (16) we get

$$\begin{aligned}\langle A(u_n), T_k(u_n - \psi^+)\rangle + \int_\Omega [g(x, u_n, \nabla u_n) + H_n(x, \nabla u_n)]\, T_k(u_n - \psi^+)\, dx \\ \leq \int_\Omega f_n\, T_k(u_n - \psi^+)\, dx \end{aligned} \tag{17}$$

which implies by using (11) and Young's inequality

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n - \psi^+)\, dx + \int_\Omega g(x, u_n, \nabla u_n)\, T_k(u_n - \psi^+)\, dx \\
&\quad \leq k\, C_f + k \sum_{i=1}^N \int_\Omega h(x)\, w_i^{\frac{1}{p'}}(x) \left|\frac{\partial u_n}{\partial x_i}\right|^{p-1} dx \\
&\quad \leq k\, C_f + C(k, p, N, \beta) \int_\Omega |h(x)|^p\, dx + \frac{\beta}{k} \sum_{i=1}^N \int_\Omega w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx \\
&\leq C_k + \frac{\beta}{k} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx + \frac{\beta}{k} \sum_{i=1}^N \int_{\{|u_n - \psi^+| > k\}} w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx,
\end{aligned}$$

where $C_k$ is a constant not depending on $n$, which may be different from line to line.

We use (10) and the fact that $|u_n| \geq k - \|\psi^+\|_\infty \geq \rho$ on the set $\{|u_n - \psi^+| > k\}$, then

$$\begin{aligned}
\frac{\beta}{k} \sum_{i=1}^N \int_{\{|u_n - \psi^+| > k\}} w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx \quad &\leq \quad \frac{1}{k} \int_{\{|u_n - \psi^+| > k\}} |g(x, u_n, \nabla u_n)|\, dx \\
&= \quad \frac{1}{k^2} \int_{\{|u_n - \psi^+| > k\}} g(x, u_n, \nabla u_n)\, T_k(u_n - \psi^+)\, dx \\
&\leq \quad \int_\Omega g(x, u_n, \nabla u_n)\, T_k(u_n - \psi^+)\, dx.
\end{aligned}$$

Consequently, we have

$$\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n - \psi^+)\, dx \leq C_k + \frac{\beta}{k} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx$$

which implies that

$$\begin{aligned}
\int_{\{|u_n - \psi^+| \leq k\}} a(x, u_n, \nabla u_n).\nabla u_n\, dx \quad &\leq \quad C_k + \frac{\beta}{k} \sum_{i=1}^N \int_{\{|u_n - \psi^+| \leq k\}} w_i(x) \left|\frac{\partial u_n}{\partial x_i}\right|^p dx \\
&\qquad + \int_{\{|u_n - \psi^+| \leq k\}} \left|a(x, u_n, \nabla u_n).\nabla \psi^+\right| dx
\end{aligned}$$

and by using Young's inequality we obtain for a positive constant $\lambda$

$$\begin{aligned}\int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx &\le C_k+\frac{\beta}{k}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &+\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}}\frac{\lambda^{p'}}{p'}|a_i(x,u_n,\nabla u_n)|^{p'}w_i^{1-p'}(x)\,dx\\ &+\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}}\frac{1}{p\,\lambda^p}w_i(x)\left|\frac{\partial\psi^+}{\partial x_i}\right|^p dx.\end{aligned}$$

By virtue of (6), we get

$$\begin{aligned}\int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx &\le C_k+\frac{\beta}{k}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\int_\Omega\delta^{p'}(x)\,dx+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\int_{\{|u_n-\psi^+|\le k\}}\sigma(x)|u_n|^q\,dx\\ &+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\le C_k+\frac{\beta}{k}\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\int_{\{|u_n|\le k+\|\psi^+\|_\infty\}}\sigma(x)|u_n|^q\,dx\\ &+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\le C_k+\left(\frac{\beta}{k}+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\right)\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx.\end{aligned}$$

Consequently, by using the coercivity condition (8) we obtain

$$\begin{aligned}&\alpha_2\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\qquad\le C_k+\left(\frac{\beta}{k}+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N\right)\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx.\end{aligned}$$

We can choose $\lambda>0$ small enough such that $\alpha_2>\frac{\beta}{k}+\frac{\lambda^{p'}}{p'}\alpha_1^{p'}N$ for $k>\frac{\beta}{\alpha_2}$, then

$$\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\le C_1. \tag{18}$$

On the other hand, from (17) we have

$$\begin{aligned}\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n-\psi^+)\,dx \quad &\le \quad k\,C_f + k\int_\Omega |H_n(x,\nabla u_n)|\,dx\\ &\qquad -\int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla(u_n-\psi^+)\,dx\end{aligned}$$

which implies by using (11), (8) and Young's inequality

$$\begin{aligned}&\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n-\psi^+)\,dx\\ &\quad\le k\,C_f + k\sum_{i=1}^N\int_\Omega h(x)\,w_i^{\frac{1}{p'}}(x)\left|\frac{\partial u_n}{\partial x_i}\right|^{p-1}dx + \int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla\psi^+\,dx\\ &\qquad -\int_{\{|u_n-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx\\ &\quad\le k\,C_f + C(k,p,N,\beta,\lambda)\int_\Omega |h(x)|^p\,dx + \lambda\beta\sum_{i=1}^N\int_\Omega w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\qquad +\int_{\{|u_n-\psi^+|\le k\}} |a(x,u_n,\nabla u_n).\nabla\psi^+|\,dx. \qquad (19)\end{aligned}$$

In view of (18), the last term of the right-hand side of (19) is bounded uniformly in $n$, then

$$\begin{aligned}\int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n-\psi^+)\,dx \quad &\le \quad C_k + \lambda\beta\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\qquad +\lambda\beta\sum_{i=1}^N\int_{\{|u_n-\psi^+|> k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx.\end{aligned}$$

By using (10) we have for $k>\rho+\|\psi^+\|_\infty$

$$\begin{aligned}k\,\beta\sum_{i=1}^N\int_{\{|u_n-\psi^+|>k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx \quad &\le \quad k\int_\Omega |g(x,u_n,\nabla u_n)|\,dx\\ &\le \quad \int_\Omega g(x,u_n,\nabla u_n)\,T_k(u_n-\psi^+)\,dx.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}(k-\lambda)\beta\sum_{i=1}^N\int_{\{|u_n-\psi^+|>k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx \quad &\le \quad C_k+\lambda\beta\sum_{i=1}^N\int_{\{|u_n-\psi^+|\le k\}} w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p dx\\ &\le \quad C_k+\lambda\beta\,C_1.\end{aligned}$$

So that

$$|||u_n|||_X \le C, \qquad (20)$$

where $C$ is a constant not depending on $n$. The boundedness of the sequence $(u_n)$ in $X$ with (5) imply the existence of a function $u$ in $W_0^{1,p}(\Omega,w)$ and a subsequence, still denoted by $(u_n)$, such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega,w),\ \text{strongly in } L^q(\Omega,\sigma) \text{ and a.e. in } \Omega. \qquad (21)$$

**Step 2: Almost everywhere convergence of the gradients.**
We will show successively the following results

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le|u_n|\le j+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx=0, \tag{22}$$

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(u_n)\,(1-\varphi_j(u_n))\,dx=0, \tag{23}$$

$$\begin{aligned}\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega &(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\\ &.(\nabla T_k(u_n)-\nabla T_k(u))\,\varphi_j(u_n)\,dx=0,\end{aligned} \tag{24}$$

and

$$T_k(u_n)\to T_k(u)\ \text{ strongly in }\ W_0^{1,p}(\Omega,w), \tag{25}$$

where the function $\varphi_j$ will be defined below (see (31)).

For (22), consider the function $v=u_n-\eta\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+$, where $\eta$ is a real positive and $B(s)=\int_0^s\frac{b(t)}{\alpha_2}\,dt$ (note that the function $b$ is the one that appeared in (9) and the real positive $\alpha_2$ is the one that appeared in (8)). We have $v\in W_0^{1,p}(\Omega,w)$ and for $j$ large enough and $\eta$ small enough, we can deduce that $v\ge\psi$, thus $v$ is an admissible test function in (15) and we obtain

$$\begin{aligned}\int_\Omega &a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\right)\,dx\\ &+\int_\Omega g(x,u_n,\nabla u_n)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx\\ &+\int_\Omega H_n(x,\nabla u_n)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx\\ &\qquad\le\int_\Omega f_n\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx.\end{aligned} \tag{26}$$

By Lebesgue's theorem the right-hand side goes to zero as $n$ and $j$ tend to infinity. For the last term of the left-hand side, by using (11) we have

$$\begin{aligned}\int_\Omega &\left|H_n(x,\nabla u_n)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\right|\,dx\\ &\le\int_\Omega\left|h(x)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\right|\sum_{i=1}^N w_i^{\frac{1}{p'}}(x)\left|\frac{\partial u_n}{\partial x_i}\right|^{p-1}\,dx\\ &\le\|h\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\|_{L^p(\Omega)}\,\||u_n\||_X^{p-1}.\end{aligned}$$

Therefore, passing to the limit firstly in $j$ and secondly in $n$, we obtain

$$h(x)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\to 0\ \text{ strongly in }\ L^p(\Omega)$$

and from (20) we conclude that

$$\int_\Omega|H_n(x,\nabla u_n)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+|\,dx\to 0,\ \text{ as }\ n\ \text{ and }\ j\ \to\infty. \tag{27}$$

Thus we can write (26) as follows

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\right)\,dx \\ +\int_\Omega g(x,u_n,\nabla u_n)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx\le\varepsilon_1(j,n)$$

where $\varepsilon_i(j,n)$, $(i=1,2,\dots)$, denote various sequences of real numbers which converge to zero when $n$ and $j$ tend to $\infty$. In view of (9) we deduce that

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\,\frac{b(u_n)}{\alpha_2}\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx \\ +\int_\Omega a(x,u_n,\nabla u_n).\nabla T_1(u_n-T_j(u_n))^+\,\exp(B(u_n))\,dx \\ \le\int_\Omega c(x)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx \\ +\int_\Omega b(u_n)\sum_{i=1}^N w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx+\varepsilon_1(j,n),$$

and by using (8) we obtain

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla T_1(u_n-T_j(u_n))^+\,\exp(B(u_n))\,dx \\ \le\int_\Omega c(x)\,\exp(B(u_n))\,T_1(u_n-T_j(u_n))^+\,dx+\varepsilon_1(j,n).$$

We use in the first term of the right-hand side the Lebesgue's theorem and we pass to the limit in $n$ and $j$ to obtain

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le u_n\le j+1\}}a(x,u_n,\nabla u_n).\nabla u_n\,dx=0. \tag{28}$$

On the other hand, the function $v=u_n+\eta\,\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-$ is an admissible test function in the inequality (15) then

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(-\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-\right)\,dx \\ +\int_\Omega g(x,u_n,\nabla u_n)\,(-\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-)\,dx \\ +\int_\Omega H_n(x,\nabla u_n)\,(-\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-)\,dx \\ \le\int_\Omega f_n\,(-\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-)\,dx. \tag{29}$$

Similarly as above, we have

$$\int_\Omega |H_n(x,\nabla u_n)\,(-\exp(-B(u_n))\,T_1(u_n-T_j(u_n))^-)|\,dx\to 0 \qquad \text{and}$$

$$\int_\Omega f_n\,(-\exp(-B(u_n))\,T_1(u_n - T_j(u_n))^-)\,dx \to 0, \quad \text{as} \;\; n \;\; \text{and} \;\; j \;\to \infty.$$

So, (29) yields

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(-\exp(-B(u_n))\,T_1(u_n - T_j(u_n))^-\right)\,dx \\ + \int_\Omega g(x,u_n,\nabla u_n)\,(-\exp(-B(u_n))\,T_1(u_n - T_j(u_n))^-)\,dx \le \varepsilon_2(j,n).$$

As above, we use (9) and then (8) to obtain

$$-\int_\Omega a(x,u_n,\nabla u_n).\nabla T_1(u_n - T_j(u_n))^-\;\exp(-B(u_n))\,dx \\ \le \int_\Omega c(x)\;\exp(-B(u_n))\,T_1(u_n - T_j(u_n))^-\,dx + \varepsilon_2(j,n),$$

which gives as above

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n).\nabla u_n\,dx = 0. \tag{30}$$

Therefore, (22) follows from (28) and (30).

Now, we pass to claim (23). For a nonnegative real parameter $j$ define a function $\varphi_j$ as

$$\begin{cases} \varphi_j(s) = 1, & \text{if} \quad |s| \le j, \\ \varphi_j(s) = 0, & \text{if} \quad |s| \ge j+1, \\ \varphi_j(s) = j+1-|s|, & \text{if} \quad j \le |s| \le j+1. \end{cases} \tag{31}$$

On one hand, the function $v = u_n - \eta\,\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))$ is an admissible test function in the inequality (15), then

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))\right)\,dx \\ + \int_\Omega g(x,u_n,\nabla u_n)\;\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))\,dx \\ + \int_\Omega H_n(x,\nabla u_n)\;\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))\,dx \\ \le \int_\Omega f_n\;\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))\,dx. \tag{32}$$

As in (27) and by Lebesgue's theorem we have

$$\int_\Omega |H_n(x,\nabla u_n)\;\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))|\,dx \to 0 \;\; \text{and}$$

$$\int_\Omega f_n\;\exp(B(u_n))\,T_k(u_n^+ - \psi^+)\,(1-\varphi_j(u_n))\,dx \to 0, \quad \text{as} \;\; n \;\; \text{and} \;\; j \;\to \infty.$$

Then (32) gives

$$\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\right)\,dx$$
$$+\int_\Omega g(x,u_n,\nabla u_n)\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\,dx\le\varepsilon_3(j,n).$$

In view of (9) we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\,\frac{b(u_n)}{\alpha_2}\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\,dx\\ &\quad+\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n^+-\psi^+)\,\exp(B(u_n))\,(1-\varphi_j(u_n))\,dx\\ &\quad+\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,dx\\ &\le\int_\Omega c(x)\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\,dx\\ &\quad+\int_\Omega b(u_n)\sum_{i=1}^N w_i(x)\left|\frac{\partial u_n}{\partial x_i}\right|^p\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\,dx+\varepsilon_3(j,n),\end{aligned}$$

and by using (8) we get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n^+-\psi^+)\,\exp(B(u_n))\,(1-\varphi_j(u_n))\,dx\\ &\quad+\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,dx\\ &\le\int_\Omega c(x)\,\exp(B(u_n))\,T_k(u_n^+-\psi^+)\,(1-\varphi_j(u_n))\,dx+\varepsilon_3(j,n).\end{aligned}\tag{33}$$

In view of (28) and the fact that $\exp(B(u_n))\,T_k(u_n^+-\psi^+)$ is bounded, we conclude that the second integral in the left hand side of the last inequality converges to zero as $n$ and $j$ tend to infinity. The first integral in the right hand side of the same inequality tends to zero when $n$ and $j$ tend to infinity by Lebesgue's theorem. Then we can write the last estimation as follows

$$\begin{aligned}&\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n^+\,\exp(B(u_n))\,(1-\varphi_j(u_n))\,dx\\ &\quad\le\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla\psi^+\,\exp(B(u_n))\,(1-\varphi_j(u_n))\,dx+\varepsilon_4(j,n),\end{aligned}$$

which gives by using the fact that $\exp(B(u_n))$ is bounded

$$\begin{aligned}&\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n^+\,(1-\varphi_j(u_n))\,dx\\ &\qquad\le M\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla\psi^+\,(1-\varphi_j(u_n))\,dx+\varepsilon_4(j,n),\end{aligned}$$

where $M$ is a positive constant. By the growth condition (6) and Young's inequality, the second integral of the last inequality converges to zero as $n$ and $j$ tend to infinity, then we can deduce that

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n^+\,(1-\varphi_j(u_n))\,dx \le \varepsilon_5(j,n).$$

The fact that $\{|u_n^+|\le k\}\subset\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$ implies that

$$\int_{\{|u_n^+|\le k\}} a(x,u_n,\nabla u_n).\nabla u_n\,(1-\varphi_j(u_n))\,dx$$
$$\le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} a(x,u_n,\nabla u_n).\nabla u_n\,(1-\varphi_j(u_n))\,dx \le \varepsilon_5(j,n).$$

Consequently we have for all $k>0$

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\ge 0\}} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(u_n)\,(1-\varphi_j(u_n))\,dx = 0. \tag{34}$$

On the other hand, we can use $v=u_n+\eta\,\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n))$ as the test function in (15) and we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n))\right)\,dx\\
&\quad+\int_\Omega g(x,u_n,\nabla u_n)\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))\,dx\\
&\quad+\int_\Omega H_n(x,\nabla u_n)\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))\,dx\\
&\qquad\le \int_\Omega f_n\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))\,dx.
\end{aligned}\tag{35}$$

As in (27) and by Lebesgue's theorem we have

$$\int_\Omega |H_n(x,\nabla u_n)\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))|\,dx\to 0 \qquad\text{and}$$
$$\int_\Omega f_n\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))\,dx\to 0,\quad\text{as } n \text{ and } j\ \to\infty.$$

Then we can offset the estimation (35) as follows

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n))\right)\,dx\\
&\quad+\int_\Omega g(x,u_n,\nabla u_n)\,(-\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n)))\,dx\le\varepsilon_6(j,n).
\end{aligned}$$

As in (33), we use (9) and then (8) to obtain

$$\begin{aligned}
&\int_{\{u_n\le 0\}} a(x,u_n,\nabla u_n).\nabla T_k(u_n)\,\exp(-B(u_n))\,(1-\varphi_j(u_n))\,dx\\
&\quad+\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n).\nabla u_n\,\exp(-B(u_n))\,T_k(u_n)^-\,dx\\
&\le \int_\Omega c(x)\,\exp(-B(u_n))\,T_k(u_n)^-(1-\varphi_j(u_n))\,dx+\varepsilon_6(j,n).
\end{aligned}\tag{36}$$

By virtue of (30) and the fact that $\exp(-B(u_n))\,T_k(u_n)^-$ is bounded, we conclude that the second integral in the left hand side of (36) converges to zero as $n$ and $j$ tend to infinity. The first term in the right hand side of the same inequality tends to zero when $n$ and $j$ tend to infinity by Lebesgue's theorem. Then (21) implies that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\le 0\}} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla T_k(u_n)\,(1-\varphi_j(u_n))\,dx=0. \tag{37}$$

We arrive at (23) by combining (34) and (37).

Now we will show (24), consider the function $v=u_n-\eta\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)$, we can use it as the test function in (15) for $\eta$ small enough, we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\right)\,dx\\&\qquad+\int_\Omega g(x,u_n,\nabla u_n)\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx\\&\qquad+\int_\Omega H_n(x,\nabla u_n)\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx\\&\qquad\le\int_\Omega f_n\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx.\end{aligned} \tag{38}$$

As in (27) and by Lebesgue's theorem we have

$$\int_\Omega |H_n(x,\nabla u_n)\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)|\,dx\to 0\quad\text{and}$$

$$\int_\Omega f_n\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx\to 0,\quad\text{as}\quad n\quad\text{and}\quad j\ \to\infty.$$

Then (38) yields

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla\left(\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\right)\,dx\\&\qquad+\int_\Omega g(x,u_n,\nabla u_n)\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx\le\varepsilon_7(j,n).\end{aligned}$$

Similarly as above, we use (9) and (8) to get

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n).\nabla(T_k(u_n)-T_k(u))^+\,\exp(B(u_n))\,\varphi_j(u_n)\,dx\\&\qquad-\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n).\nabla u_n\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,dx\\&\qquad\le\int_\Omega c(x)\,\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+\,\varphi_j(u_n)\,dx+\varepsilon_7(j,n)\end{aligned}$$

which gives, by using (28) and the fact that $\exp(B(u_n))\,(T_k(u_n)-T_k(u))^+$ is bounded for the second integral and Lebesgue's theorem for the third integral, the following estimation

$$\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,u_n,\nabla u_n).\nabla(T_k(u_n)-T_k(u))\,\exp(B(u_n))\,\varphi_j(u_n)dx\le\varepsilon_8(j,n),$$

that is

$$\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|\leq k\}} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla(T_k(u_n)-T_k(u)) \exp(B(u_n))\,\varphi_j(u_n)dx$$
$$\leq \int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|>k\}} a(x,u_n,\nabla u_n).\nabla T_k(u) \exp(B(u_n))\,\varphi_j(u_n)\,dx+\varepsilon_8(j,n). \quad (39)$$

By using the fact that $\varphi_j(u_n)=0$ if $|u_n|>j+1$, we have for the second integral of the last inequality

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|>k\}} a(x,u_n,\nabla u_n).\nabla T_k(u) \exp(B(u_n))\,\varphi_j(u_n)\,dx\\
&=\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|>k\}} a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n)).\nabla T_k(u) \exp(B(u_n))\,\varphi_j(u_n)dx\\
&=\varepsilon_9(j,n),
\end{aligned}$$

since

$$\begin{aligned}
\lim_{j\to\infty}\lim_{n\to\infty}&\int_{\{T_k(u_n)-T_k(u)\geq 0,\ |u_n|>k\}} a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n)).\nabla T_k(u)\\
&\exp(B(u_n))\,\varphi_j(u_n)\,dx=\lim_{j\to\infty}\int_{\{|u|>k\}}\Lambda_j.\nabla T_k(u) \exp(B(u))\,\varphi_j(u)\,dx=0,
\end{aligned}$$

where $\Lambda_j$ is the limit of $a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))$ in $\Pi_{i=1}^{N}L^{p'}(\Omega,w_i^*)$ as $n\to\infty$. Therefore (39) becomes by adding $\varepsilon_9(j,n)$ on both sides

$$\begin{aligned}
\lim_{j\to\infty}\lim_{n\to\infty}&\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n)).\nabla(T_k(u_n)-T_k(u))\\
&\times\exp(B(u_n))\,\varphi_j(u_n)\,dx=0. \qquad (40)
\end{aligned}$$

On the other hand, by using $v=u_n+\eta\exp(-B(u_n))\,(T_k(u_n)-T_k(u))^-\,\varphi_j(u_n)$ as the test function in (15) and reasoning as in (40), we obtain

$$\begin{aligned}
\lim_{j\to\infty}\lim_{n\to\infty}&\int_{\{T_k(u_n)-T_k(u)\leq 0\}} (a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))\\
&.\nabla(T_k(u_n)-T_k(u))\,\varphi_j(u_n)\,dx=0. \qquad (41)
\end{aligned}$$

Combining (40) and (41) we arrive at (24).

We pass on to the proof of (25). We have

$$
\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\,dx\\
&= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\,\varphi_j(u_n)\,dx\\
&\qquad + \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\\
&\qquad\qquad \times (1-\varphi_j(u_n))\,dx.\\
&= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\,\varphi_j(u_n)\,dx\\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u_n)\,(1-\varphi_j(u_n))\,dx\\
&\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\,\nabla T_k(u)\,(1-\varphi_j(u_n))\,dx\\
&\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)))\,(\nabla T_k(u_n) - \nabla T_k(u))\,(1-\varphi_j(u_n))\,dx. \qquad (42)
\end{aligned}
$$

The results (24) and (23) respectively give that the first and second terms of the right hand side of the last equality converge to zero as $n$ and $j$ tend to infinity. The third term has the same limit because $(a(x,T_k(u_n),\nabla T_k(u_n)))$ is bounded in $\Pi_{i=1}^N L^{p'}(\Omega,w_i^*)$ uniformly on $n$ from (6) and (20), and $\nabla T_k(u)\,(1-\varphi_j(u_n))$ converges to zero. Finally for this equality, we have $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u))$ weakly in $\Pi_{i=1}^N L^p(\Omega,w_i)$ then the last integral converges to zero. Therefore (42) gives

$$
\lim_{n\to\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))).(\nabla T_k(u_n) - \nabla T_k(u))\,dx = 0
$$

which implies (25) by using Lemma 3.1 and the fact that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega,w)$. So, $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $\Pi_{i=1}^N L^p(\Omega,w_i)$. Consequently, there exists a subsequence still denoted by $(u_n)_n$ such that

$$
\nabla u_n \to \nabla u \ \text{ a.e. in } \ \Omega. \qquad (43)
$$

**Step 3: Equi-integrability of the non-linearities $g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)$.** By using Vitali's theorem we will show that

$$
g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n) \to g(x,u,\nabla u)+H(x,\nabla u) \ \text{ strongly in } \ L^1(\Omega). \qquad (44)
$$

Thanks to (21) and (43) we have $g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n) \to g(x,u,\nabla u)+H(x,\nabla u)$ a.e. in $\Omega$. So it suffices to prove that $g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)$ is uniformly equi-integrable in $\Omega$. For any measurable subset $E$ of $\Omega$ and any $m > 0$ we have

$$\begin{aligned}\int_E |g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)|\,dx &= \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)|dx \\ &\quad + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)|dx. \\ &\le \int_E b(m)\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_m(u_n)}{\partial x_i}\right|^p\right)dx \\ &\quad + \left(\int_E h^p(x)\,dx\right)^{\frac{1}{p}} \sum_{i=1}^N \left(\int_E w_i(x)\left|\frac{\partial T_m(u_n)}{\partial x_i}\right|^p dx\right)^{\frac{1}{p'}} \\ &\quad + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)|dx. \qquad (45)\end{aligned}$$

In view of (25) for any $\varepsilon > 0$ there exists $\mu(\varepsilon, m) > 0$ such that for all $E$ satisfying $|E| < \mu(\varepsilon, m)$ we have

$$\begin{aligned}&\int_E b(m)\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_m(u_n)}{\partial x_i}\right|^p\right)dx \\ &\quad + \left(\int_E h^p(x)\,dx\right)^{\frac{1}{p}} \sum_{i=1}^N \left(\int_E w_i(x)\left|\frac{\partial T_m(u_n)}{\partial x_i}\right|^p dx\right)^{\frac{1}{p'}} < \frac{\varepsilon}{2} \quad \forall n. \qquad (46)\end{aligned}$$

Now let us choose $m$ large enough such that $m \ge 2 + \|\psi^+\|_\infty$, and define a function $\phi_m$ which satisfies

$$\begin{cases} \phi_m(s) = 0, & \text{if } |s| \le m-1, \\ \phi_m'(s) = 1, & \text{if } m-1 \le |s| \le m, \\ \phi_m(s) = \frac{s}{|s|}, & \text{if } |s| \ge m. \end{cases}$$

Note that $u_n - \phi_m(u_n) \in K_\psi$, then by using it as the test function in (16) we get

$$\begin{aligned}\langle A(u_n), T_k(\phi_m(u_n))\rangle + \int_\Omega (g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n))\,T_k(\phi_m(u_n))\,dx \\ \le \int_\Omega f_n\,T_k(\phi_m(u_n))\,dx\end{aligned}$$

which by choosing $k \ge 1$ implies

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n).\nabla u_n\,\phi_m'(u_n)\,dx + \int_\Omega (g(x,u_n,\nabla u_n)+H_n(x,\nabla u_n))\,\phi_m(u_n)\,dx \\ \le \int_\Omega f_n\,\phi_m(u_n)\,dx.\end{aligned}$$

Because of (8) and by using the fact that $\phi_m(u_n)$ and $u_n$ have the same sign we conclude that

$$\int_{\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx \le \int_{\{|u_n|>m-1\}} |H_n(x,\nabla u_n)|\,dx + \int_{\{|u_n|>m-1\}} |f_n|\,dx.$$

The right-hand side of the last inequality converges to 0 uniformly in $n$ when $m$ tends to $\infty$ by using (11), the Hölder inequality, $f_n \to f$ strongly in $L^1(\Omega)$ and the fact that $|\{|u_n| > m\}| \to 0$ uniformly in $n$ when $m \to \infty$. Hence there exists $m(\varepsilon) > 1$ such that

$$\int_{\{|u_n|>m\}} |g(x, u_n, \nabla u_n)|\, dx \le \frac{\varepsilon}{2} \quad \forall n. \tag{47}$$

Finally from (45), (46) and (47) we have

$$\int_E |g(x, u_n, \nabla u_n) + H_n(x, \nabla u_n)|\, dx < \varepsilon \ \ \forall n, \ \text{ if } \ |E| < \mu(\varepsilon) \ \text{ for some } \ \mu(\varepsilon) > 0,$$

which gives the uniform equi-integrability in $\Omega$ of $g(x, u_n, \nabla u_n) + H_n(x, \nabla u_n)$.

**Step 4: Passage to the limit.**

Going back to (16), we have for all $v \in K_\psi \cap L^\infty(\Omega)$ and all $k > 0$

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n - v)\, dx + \int_\Omega (g(x, u_n, \nabla u_n) + H_n(x, \nabla u_n))\, T_k(u_n - v)\, dx \\ \le \int_\Omega f_n\, T_k(u_n - v)\, dx. \end{aligned} \tag{48}$$

From (6) and (20) we have $a(x, u_n, \nabla u_n)$ is bounded in $\Pi_{i=1}^N L^{p'}(\Omega, w_i^*)$, and because of (21) and (43) we have $a(x, u_n, \nabla u_n) \to a(x, u, \nabla u)$ a.e. in $\Omega$. Therefore by Lemma 2.1 we obtain $a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u)$ weakly in $\Pi_{i=1}^N L^{p'}(\Omega, w_i^*)$. For all measurable subsets $E \subset \Omega$ and for $i = 1, \dots, N$ we have

$$\begin{aligned}\int_E w_i^{\frac{1}{p}}(x) \left| \frac{\partial T_k(u_n - v)}{\partial x_i} \right| dx &= \int_E w_i^{\frac{1}{p}}(x) \left| \frac{\partial (u_n - v)}{\partial x_i} \right| \chi_{\{|u_n - v| \le k\}}\, dx \\ &\le \int_E w_i^{\frac{1}{p}}(x) \left( \left| \frac{\partial u_n}{\partial x_i} \right| + \left| \frac{\partial v}{\partial x_i} \right| \right) \chi_{\{|u_n| \le k + \|v\|_\infty\}}\, dx \\ &\le \int_E w_i^{\frac{1}{p}}(x) \left| \frac{\partial v}{\partial x_i} \right| dx + \int_E w_i^{\frac{1}{p}}(x) \left| \frac{\partial T_{k+\|v\|_\infty}(u_n)}{\partial x_i} \right| dx.\end{aligned}$$

By using (21), (25) and the Vitali's theorem we get $\nabla T_k(u_n - v) \to \nabla T_k(u - v)$ strongly in $\Pi_{i=1}^N L^p(\Omega, w_i)$, so that $\int_\Omega a(x, u_n, \nabla u_n).\nabla T_k(u_n - v)\, dx \to \int_\Omega a(x, u, \nabla u).\nabla T_k(u - v)\, dx$ as $n \to \infty$. Finally we use (44) and the fact that $f_n \to f$ strongly in $L^1(\Omega)$ for passing to the limit in (48) and this completes the proof of Theorem 3.1.

## 4 Example

In particular, let us use the special weight functions $w$ and $\sigma$ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \mathrm{dist}(x, \partial\Omega)$ and set $w(x) = d^\lambda(x)$, $\sigma(x) = d^\mu(x)$ (see [3]). As an example of equations to which the result of this paper can be applied, we give the following example. Consider the Carathéodory functions $a_i(x, s, \xi) = w_i\, |\xi_i|^{p-1}\, sgn(\xi_i)$ for $i = 1, \dots, N$, $g(x, s, \xi) = \rho \exp(s^{-2}) \sum_{i=1}^N w_i\, |\xi_i|^p$ with $\rho \in \mathbb{R}$ and $H(x, \xi) = h(x) \sum_{i=1}^N w_i^{\frac{1}{p'}}(x)\, |\xi_i|^{p-1}$, where $h \in L^r(\Omega)$ with $r > \max(N, p)$. We can use the special weight functions $w$ and $\sigma$ already given previously and we shall assume that the weight functions satisfy $w_i(x) = w(x)$ $\forall i = 0, \dots, N$. First, note that $g(x, s, \xi)$

does not satisfy the sign condition. It is easy to show that the Carathéodory functions $a_i(x, s, \xi)$ satisfy the growth condition (6) and the coercivity condition (8). Also the Carathéodory function $g(x, s, \xi)$ satisfies the conditions (9) and (10). Indeed, we have $|g(x, s, \xi)| \leq |\rho| \exp(s^{-2}) \sum_{i=1}^{N} w_i |\xi_i|^p = b(s) \sum_{i=1}^{N} w_i |\xi_i|^p$, where $b(s) = |\rho| \exp(s^{-2})$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and $|g(x, s, \xi)| \geq |\rho| \sum_{i=1}^{N} w_i |\xi_i|^p$, since $\exp(s^{-2}) \geq 1 \ \ \forall s \in \mathbb{R}^*$. For the monotonicity condition, since $w > 0$ a.e. in $\Omega$ we have

$$\sum_{i=1}^{N}(a_i(x,s,\xi)-a_i(x,s,\xi^*))(\xi-\xi^*) = w(x)\sum_{i=1}^{N}(|\xi_i|^{p-1}\, sgn(\xi_i)-|\xi_i^*|^{p-1}\, sgn(\xi_i^*))(\xi-\xi^*) > 0$$

for almost all $x \in \Omega$ and for all $\xi$, $\xi^* \in \mathbb{R}^N$ with $(\xi \neq \xi^*)$.

## 5 Conclusion

Through this result, we tried to answer the question of existence of solutions for some nonlinear elliptic partial differential equations of the form $A(u)+g(x,u,\nabla u)+H(x,\nabla u) = f \in L^1(\Omega)$ in $\Omega$, whose functional framework is the weighted Sobolev spaces. The major difficulty of this work is the sign condition of the first lower order term g that we have eliminated. To overcome this difficulty we have used a technique based on positive and negative parts of some functions in order to choose test functions to show the strong convergence of the truncations.

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# Symmetries Impact in Chaotification of Piecewise Smooth Systems

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**Abstract:** This paper is devoted to a mathematical analysis of a route to chaos for bounded piecewise smooth systems of dimension three subjected to symmetric non-smooth bifurcations. This study is based on period doubling method applied to the associated Poincaré maps. These Poincaré maps are characterized taking into account the symmetry of the transient manifolds. The corresponding Poincaré sections are chosen to be transverse to these transient manifolds, this particular choice takes into account the fact that the system dynamics crosses the intersection of both manifolds. In this case, the dimension of the Poincaré map (defined as discrete map of dimension two) is reduced to dimension one in this particular neighborhood of transient points. This dimension reduction allows us to deal with the famous result "period three implies chaos". The approach is also highlighted by simulation results applied particularly to Chua circuit subjected to symmetric grazing bifurcations.

**Keywords:** *chaotification analysis; period doubling; non-smooth bifurcations; symmetries; Chua circuit.*

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## 1 Introduction

In the literature, hybrid dynamic models can represent systems for which the behavior consists of continuous evolution interspersed by instantaneous jumps in the velocity. More precisely, those systems exhibit non-smoothness or discontinuities in the dynamics and this induces new dynamics phenomena witch are not present in smooth dynamics. However, the field of hybrid systems is not as mature as that of the smooth ones. The corresponding fundamental theoretical concepts have not been so developed. The most known general textbook on hybrid systems is [46] and the book [40] contains qualitative analysis of some classes of hybrid systems. Recently, it was gradually recognized that a particular class of those systems exhibits many interesting phenomena because of the specific complex structure of the state space composed of some different vector fields. In this case, the dynamics of the system can be defined by an ordinary differential equation in each region and the associated Poincaré map is continuous across the border but its derivative is discontinuous. Those systems are called piecewise smooth systems (noted p.w.s systems), they occur naturally in the description of many physical processes as grazing, sliding, switching, friction and so on. This type of dynamics was introduced and studied in many seminal papers [2], [3], [17], [27], [18], [31], [38], [41], [42], [50]. Many books and monographs have been published on this topic. The analysis in [32] generalized several fundamental theories in smooth systems theory to this relevant class of hybrid systems. [12] gave a comprehensive treatment on the theory of p.w.s systems. The reader can also refer to recent survey paper [13] for numerous references therein. Such class of p.w.s systems is common in the literature. Authors in [15], [16], [33] dealt with p.w.s systems from mechanical problems, other applications were performed in control in engineering [3], [48], [37] electromechanical systems [29] or in gene regulatory networks and neurons in computational neuroscience and biology [45]. In those applications, it is often essential to characterize its bifurcations. Those events , known as discontinuity induced bifurcations, occur when an invariant set of the system (as an equilibrium point or a limit cycle) crosses or hits tangentially the switching manifold in the phase space. A pioneering work was carried out by Feigin in [23], [24], [25] who introduced the notion of C-bifurcations and has recently re-evaluated it in [7]. Furthermore, symmetric bifurcations are widespread phenomena, one of the oldest known example is the Lorenz dynamics [47] for the smooth systems and the Chua circuit [21] for the piecewise smooth ones. This kind of symmetric non-smooth transients occurs for example in a multicell chopper coupled with nonlinear load and may generate a chaotic behavior [22] (see [1], [28] for mathematical definitions and characterizations of chaos in dynamical systems). In fact, all those types of bifurcations can give rise to a chaotic behavior. Most notably, p.w.s systems can exhibit robust chaotic behavior that has been conjectured not to exist for smooth systems. This is due to the discontinuous dependence on initial conditions leading to chaotic behavior. Knowing that there exist three main branches of chaotic dynamic systems theory, namely the symbolic dynamics, ergotic theory and bifurcation theory, we focus on the last one in this paper. Those notions can be found in references [28], [30], [43]. Author in [32] generalized several fundamental theories in smooth systems theory including Lyapounov exponents and Conley index of p.w.s systems. Some interesting results in [51] are dedicated to bifurcations and chaos analysis to p.w.s systems. P. Collins gives in [19] an overview of some chaotic hybrid systems. He proposed results on dynamics in switched arrival systems and in systems with periodic forcing.

Hereafter, we propose a mathematical analysis of way to chaos for bounded p.w.s systems of dimension three subjected to symmetric non-smooth bifurcations. We restrict our attention to bimodal p.w.s systems depending on a parameter $\varepsilon$. Such class of p.w.s systems is common in the literature due to its importance in many applications [44], [49]. This work is an extension to symmetric case of the results obtained in [4] and [5] and associated with non-symmetric and non-smooth bifurcations. The suggested procedure is based on four main features: the first one is the Poincaré maps determination associated with p.w.s systems subjected to symmetric non-smooth transitions. It is an extension of the Poincaré Discontinuity Maps (P.D.M.) associated with p.w.s systems subjected to classic non-smooth transitions given in [8], [9], [10]. The Poincaré maps computed here are characterized by a composition of the previous Poincaré maps with some particular maps that take into account the symmetries of the dynamics. The second feature is the special choice of the Poincaré sections relatively to the switching manifolds. Those Poincaré sections are perpendicular to the switching manifolds, this permits to reduce the dimension of the Poincaré maps from two to one, this reduction being available only in a specific neighborhood of the bifurcation points. The third feature is the application of period doubling method based on the famous result of [35] called "period three implies chaos". It is important to mention here that another choice of Poincaré sections will oblige us to be in dimension 2 and thus to use results of Marotto published in 1978 who generalized results of Li and Yorke to discrete systems of dimension greater than one. This result is summarized by "snap-back repealers imply chaos " [39] and was revisited by several authors, see for example [36], [34]. Note that a snap-back repealer is an expanding fixed point such that for very small variations of the bifurcation parameter, the trajectory is repelled and for more larger deviations of this parameter, the process jumps onto the fixed point. As the determination of the snap-back repealer is difficult in general, our purpose is to avoid the corresponding approaches by considering specific choice of Poincaré sections. The fourth feature is the use of a simple and simultaneously powerful mathematical tool that is the implicit function theorem. It guaranties that the expected points for chaotifying the considered system defined on the Poincaré section are close to the bifurcation points and vary continuously with respect to the bifurcation parameter. This is primordial because on the one hand limitedness condition of the trajectories is respected (knowing that if it is not the case, study of chaos has no sense) and on the other hand, the process of period doubling occurs until the dimension of the considered discrete map is reduced to one in the neighborhood of the bifurcation parameter permitting us to use the result "period three implies chaos".

The paper is structured as follows. In Section 2 some preliminaries and statements on the characterization of symmetric non-smooth transitions are provided followed by the determination of the corresponding Poincaré maps. A route to chaos analysis is proposed in Section 3. Section 4 is dedicated to some simulation results: the first one concerns an academic example subjected to symmetric sliding bifurcations and the second one concerns Chua circuit subjected to symmetric grazing bifurcations [20]. The results obtained for both examples highlight the efficiency of the proposed approach. Finally, concluding remarks and some perspectives end the paper.

## 2 Symmetric Non-smooth Transitions and Poincaré Maps Characterization

We propose, in this section, a characterization of symmetric non-smooth transitions and then a determination of the associated Poincaré maps.

### 2.1 Characterization of p.w.s systems subjected to symmetric non smooth transitions

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon), \text{ if } x \in D_1, \\ F_2(x,\varepsilon), \text{ if } x \in D_2, \end{cases} \tag{1}$$

where $x : I \longrightarrow D$, $I \subset R^+$ and $D \supset D_1 \cup D_2$ is an open bounded domain of $R^3$ with

$$D_1 = \{x \in D : |H(x)| < E\}, \quad D_2 = \{x \in D : |H(x)| > E\},$$

where $E$ is a positive fixed real number and $\varepsilon$ is a real parameter defined on a neighborhood of 0 denoted by $V_\varepsilon$, $H : D \to R$ is a continuous function that characterizes the phase space boundary between two regions of smooth dynamics, $H$ defines the two symmetric transient sets:

$$\Pi_1 := \{x \in D : H(x) = E\}, \quad \Pi_2 := \{x \in D : H(x) = -E\},$$

where $\Pi_1$ and $\Pi_2$ are termed the switching manifolds and divide respectively the phase space into the following regions:

$$\Pi_1^+ = \{x \in D : H(x) \geq E\}, \ \Pi_1^- = \{x \in D : H(x) < E\},$$

$$\Pi_2^+ = \{x \in D : H(x) \geq -E\}, \ \Pi_2^- = \{x \in D : H(x) < -E\},$$

$F_1, F_2 : C^1(I,D) \times V_\varepsilon \longrightarrow C^m(I,D)$, $m \geq 4$, where $C^m(I,D)$ is the set of $C^k$ functions defined on $I$ and having values in $R^3$, $C^m(I,D)$ is provided with the following norm: $\|x\| = \sup\limits_{t\in I} \|x(t)\|_e + \sup\limits_{t\in I} \|\dot{x}(t)\|_e + ... + \sup\limits_{t\in I} \left\|x^{(m)}(t)\right\|_e$, $\forall x \in C^m(I,D)$ .

According to [14], $(C^m(I,D), \|.\|)$ is a Banach space.

The vector fields $F_1$ and $F_2$ are defined on both sides of $\Pi_k$, $k = 1, 2$.

Moreover, the system (1) is assumed to depend smoothly on the parameter $\varepsilon$ such that at $\varepsilon = 0$, there exists a periodic orbit $x(.)$ that intersects the switching manifolds $\Pi_1$ and $\Pi_2$ at two points $\overline{x}_1$ and $\overline{x}_2$ corresponding to $\overline{t}$ (where $\overline{t}$ is the period of time associated with the system (1)).

The assumptions given by [11], [8], [10], [13] to characterize the sliding and grazing non-smooth bifurcations are generalized to the symmetric non-smooth cases in the following subsections, notations will be more complicated because all types of grazing and sliding bifurcations are considered here at the same time with the symmetry phenomena.

#### 2.1.1 First case: symmetric sliding bifurcations

Symmetric sliding bifurcations occur on two transient surfaces $\Pi_1$and $\Pi_2$ at two sliding points $\overline{x}_k$, $k = 1, 2$ at time $t_0$ (taken for simplicity to be equal to 0) if the following

For the sake of simplicity, we denote by $x$ the function and also the value of $x$ at time $t$ when the context is without ambiguity.

$x^{(m)}(.)$ denotes the $m^{th}$ derivative of $x(.)$ and $||.||_e$ is a norm defined on $R^3$.

In this paper, indexes $s$ and $g$ are related respectively to sliding and grazing cases.

general sliding conditions are satisfied for each function $H_1 := H - E$ and $H_2 := H + E$:

$C_1^{k,s}$ ) $< \nabla H_k(x(t)), F_2(x(t),0) - F_1(x(t),0) > \in R_+^*$ for all $x(t) \in v_s^k$, where $v_s^k$ is a bifurcation neighborhood in $\Pi_k$.
$C_2^{k,s}$ ) $H_k(\overline{x}_k) = 0$ and $\nabla H_k(\overline{x}_k) \neq 0$.
$C_3^{k,s}$ ) for $i = 1, 2$ and $k = 1, 2 :< \nabla H_k(\overline{x}_k), F_{ki}^0 >= 0$, where $F_{ki}^0 := F_i(\Phi_i(\overline{x}_k, 0), 0), i = 1, 2$, and $\Phi_i$ is the flow associated with $F_i$.

Moreover, each type of the four symmetric sliding bifurcations is characterized by specific assumptions marked as $A_i^{k,s}$ ), $i = 1, 2, 3, 4$ and $k = 1, 2$:
$\mathrm{A}_1^{k,s}$ ) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_1(\bar{x}_k,0)}{\partial x} F_{k1}^0 \right\rangle > 0,$
$\mathrm{A}_2^{k,s}$ ) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_2(\bar{x}_k,0)}{\partial x} F_{k2}^0 \right\rangle > 0,$
$\mathrm{A}_3^{k,s}$ ) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_1(\bar{x}_k,0)}{\partial x} F_{k1}^0 \right\rangle < 0,$
$\mathrm{A}_4^{k,s}$ ) $\left\langle \nabla H_k(\bar{x}_k), (\frac{\partial F_1(\bar{x}_k,0)}{\partial x})^2 F_{k1}^0 \right\rangle < 0.$

### 2.1.2 Second case: symmetric grazing bifurcations

Symmetric grazing bifurcations occur on the two transient surfaces $\Pi_1$and $\Pi_2$ at two grazing points (denoted also for simplicity) $\overline{x}_k$, $k = 1, 2$ at time $t_0 = 0$ if the following general grazing conditions are satisfied on a bifurcation neighborhood $v_s^k$ of $\Pi_k$. for each function $H_1 := H - E$ and $H_2 := H + E$:
$C_1^{k,g}$ ) $H_k(\overline{x}_k) = 0$ and $\nabla H_k(\overline{x}_k) \neq 0$,
$C_2^{k,g}$ ) for $i = 1, 2$ and $k = 1, 2 :< \nabla H_k(\overline{x}_k), F_{ki}^0 >= 0$,
$C_3^{k,g}$ ) for $i = 1, 2.$ and $k = 1, 2$: $\frac{\partial^2 H_k(\bar{x}_k,0)}{\partial x^2} \in R_+^*$,
$C_4^{k,g}$) $(< L_k, F_{k1}^0 >< L_k, F_{k2}^0 >) \in R_+^*$ for each $k = 1, 2$, where $L_k$ is the unit vector perpendicular to $\nabla H(\overline{x}_k)$ at point $\overline{x}_k$.

## 2.2 Determination of Poincaré maps associated with symmetric non smooth transitions

It is assumed that at $\varepsilon = 0$ there exists a periodic orbit $x(.)$ that intersects symmetrically at two points the two symmetric manifolds $\Pi_1$ and $\Pi_2$.It is also requested that this orbit is hyperbolic and hence isolated. This implies that there is no points of sliding (respectively grazing) along the orbit other than $\overline{x}_k$, $k = 1, 2$. Those conditions are defined on an open set such that there exist sufficiently small neighborhoods $V_\varepsilon$ of $\varepsilon = 0$ and $v_{\overline{x}_k}$ of $\overline{x}_k$ such that assumptions $C_j^{k,s}$, $j = 1, 2, 3$, associated with symmetric sliding bifurcations (respectively $C_j^{k,g}$, $j = 1, 2, 3$, associated with symmetric grazing bifurcations) are satisfied.

At this step, in order to compute the corresponding Poincaré maps, let us begin with choosing specially two symmetric Poincaré sections denoted $\Lambda_1$ and $\Lambda_2$ to be perpendicular to $\Pi_1$ and $\Pi_2$ and consider the following diffeomorphism defined by:

$$S : R^2 \times S^1 \to R^2 \times S^1, \quad (x_1, x_2, t) \to S(x_1, x_2, t) = (-x_1, -x_2, t + 2p\pi),$$

where $S^1$ is the unit circle and $p\epsilon Z$ (the set of relative numbers).

The Poincaré maps denoted $P^s$ (for non-symmetric sliding case) and $P^g$ (for the non-symmetric grazing case) are given in details in [8] and [10].

The procedure for computing the Poincaré map is the same for the symmetric sliding and the symmetric grazing case, we directly deal with notation $P^{s,g}$ ,where following the cases, this map corresponds to the sliding or the grazing Poincaré one.

Now, let us consider $P_1^{s,g}$ being the the part of Poincaré map including sliding (respectively grazing) bifurcation on the transient surface $\Pi_1$ going from $\Lambda_1$ to $\Lambda_2$ and consider $P_2^{s,g}$ being the the other part of Poincaré map including sliding (respectively grazing) bifurcation on the transient surface $\Pi_2$ going from $\Lambda_2$ to $\Lambda_1$, then the global Poincaré map of the system subjected to symmetric sliding (respectively symmetric grazing) is given by:

$$P^{s,g} : \Lambda_1 \to \Lambda_2 \quad \text{such that} \quad P^{s,g} = P_2^{s,g} \circ P_1^{s,g}.$$

However, due to the symmetry of the trajectory, maps $P_1^{s,g}$ and $P_2^{s,g}$ are related by the following relation:

$$S \circ P_2^{s,g} \quad = \quad P_1^{s,g} \circ S,$$

this implies that $P^{s,g} = S^{-1} \circ P_1^{s,g} \circ S \circ P_1^{s,g}$.

Taking this fact into account, the Poincaré maps have the following form:

$$P^{s,g}(x,\varepsilon) = \left\{ \begin{array}{ll} S^{-1} \circ P_1^{s,g} \circ S \circ P_1^{s,g}(x,\varepsilon) & \text{if} < \nabla H_1, x > \in R_+ \text{ or } < \nabla H_2, x > \in R_- \\ S^{-1} \circ P_2^{s,g} \circ S \circ P_2^{s,g}(x,\varepsilon) & \text{if} < \nabla H_1, x > \in R_-^* \text{ and } < \nabla H_2, x > \in R_+^* \end{array} \right. \tag{2}$$

In the next section, a rigorous approach of a route to chaos for p.w.s systems subjected to those symmetric non-smooth bifurcations is proposed.

## 3 Analysis of Route to Chaos for P.W.S Systems Subjected to Symmetric Non Smooth Transitions

A mathematical analysis of generated chaos for bounded piecewise smooth systems of dimension 3, subjected to symmetric sliding or grazing bifurcations is now presented. This approach is based on the period doubling method applied to the corresponding Poincaré maps given by (2). Note that these Poincaré maps are discrete maps defined in dimension 2 and thus at this step, the result of Li and Yorke "Period three implies chaos" can not be used because period three does not imply necessarily chaos for continuous flows of dimension three (and so for their corresponding Poincaré maps that are discrete maps of dimension 2). In fact, determinism (non intersection of trajectories) and continuity requirement set constraints on how points of period doubling are defined on the corresponding Poincaré maps and move around the associated orbit. On the other hand, many simulation results show that period doubling can imply chaos for discrete systems of dimension greater than one. This is possible for specific cases when the multidimensional map is described in one direction by a particular map (as the saw-tooth one or the logistic one) while the other directions are characterized by strong contractions or if the process of squeezing and stretching is chosen for particular systems defined in dimension three. Moreover, the process corresponding to a pure rotation does not imply a chaotic attractor but that corresponding to braid implies chaos. In this work, a more general case of dynamic systems is considered and the trick proposed here is to reduce the dimension of the Poincaré map to one in the neighborhood of the transient points. This is possible by choosing a convenient Poincaré map section that is transversal to the switching surface, this neighborhood of $x$ is denoted $v_x^{s,g}$. This main idea is supported by applying the implicit function theorem to $v_x^{s,g}$. It is a simple and a powerful mathematical tool allowing us to generate a "branch" of continuous solutions $x$ with respect to the bifurcation parameter $\varepsilon$ defined in some neighborhood of $\varepsilon = 0$ denoted $v_{\varepsilon=0}^{s,g} \subset V_\varepsilon$ . In this context, the dimension of the discrete map $P^{s,g}$ defined on $v_x^{s,g} \times v_{\varepsilon=0}^{s,g}$ is reduced to 1, without confusion and only for simplicity we denote it also by $P^{s,g}$. Now, the famous result of Li and Yorke can be applied to $P^{s,g}$.

To propose the main result of this paper, we set the following assumptions:
$\mathrm{B}_1^{s,g}$ ) $\frac{\partial P^{s,g}}{\partial x}(0,0) - 1 \neq 0$,
$\mathrm{B}_2^{s,g}$ ) $-\frac{\partial P^{s,g}}{\partial x}(0,0)(\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} + (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} - 1 \neq 0$,
$\mathrm{B}_3^{s,g}$ ) $\frac{\partial P^{s,g}}{\partial x}(\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1}(\frac{\partial P^{s,g}}{\partial x}(0,0)(\frac{\partial P^{s,g}}{\partial x}(0,0) - 1))^{-1} - (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} + 1) - (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1}(\frac{\partial P^{s,g}}{\partial x}(0,0)\ (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1))^{-1} - (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} + 1) - 1 \neq 0$.

**Theorem 3.1**

1. *Symmetric sliding case: Under conditions $C_j^{k,s}$) $j = 1,2,3$, $A_i^{k,s}$), $i = 1,2,3,4$, $k = 1,2$ and $B_i^{s,g}$, $i = 1,2,3$ the bounded p.w.s system (1) admits a chaotic behavior associated with specific type of symmetric sliding transitions.*

2. *Symmetric grazing case: Under conditions $C_j^{k,g}$) $j = 1,2,3,4$, $k = 1,2$ and $B_i^{s,g}$, $i = 1,2,3$ the bounded p.w.s system (1) admits a chaotic behavior associated with symmetric grazing transitions.*

***Proof.*** According to period doubling method, the problem is to determine three distinct points denoted respectively by $x, y$ and $z$ that satisfy: $P^{s,g}(x,\varepsilon) = y$, $P^{s,g}(y,\varepsilon) = z$ and $P^{s,g}(z,\varepsilon) = x$.

So this procedure will be done in three steeps, each step corresponds to the determination of one of the 3 previous searched points.

**First step of the period doubling procedure**: it is performed by the analysis of the following equation:

$$P^{s,g}(x,\varepsilon) = y, \tag{3}$$

$$y := x + \eta, \tag{4}$$

where $\eta$ is a real parameter defined in the neighborhood of $x$.

The equation (3) is equivalent to the following one:

$$\Psi^{s,g}(x,\varepsilon,\eta) := P^{s,g}(x,\varepsilon) - x - \eta = 0. \tag{5}$$

Under assumption $\frac{\partial \Psi^{s,g}}{\partial x}(0,0,0) \neq 0$, (that is equivalent to assumption $\mathrm{B}_1^{s,g}$)), and using the implicit functions theorem, one obtains that $\exists$ a neighborhood of the parameter $\varepsilon$ denoted $\vartheta_{\varepsilon=0}^{s,g} \subset v_{\varepsilon=0}^{s,g}$ in $R$, a neighborhood of the parameter $\eta$ denoted $v_{\eta=0}^{s,g} \subset R$, a neighborhood of $x$ noted $v_{x=0}^{s,g} \subset v_x^{s,g} \subset R$ and a unique application $x^*$: $\vartheta_{\varepsilon=0}^{s,g} \times v_{\eta=0}^{s,g} \longrightarrow v_{x=0}^{s,g}$ solution of $\Psi^{s,g}(x^*(\varepsilon,\eta),\varepsilon,\eta) = 0$ such that $x^*(0,0) = 0$. Furthermore, $x^*$ depends continuously on $\varepsilon$ and $\eta$.

**Second step of the period doubling procedure**: it is equivalent to the analysis of the following equation:

$$P^{s,g}(P^{s,g}(x,\varepsilon),\varepsilon) = z, \tag{6}$$

$$z := y + \mu, \tag{7}$$

where $\mu$ stands for a real parameter defined in the neighborhood of $x$.

Taking into account results of the previous step, the equation (6) becomes equivalent to:

$$\Gamma^{s,g}(\varepsilon,\eta,\mu) := P^{s,g}(x^*(\varepsilon,\eta)+\eta,\varepsilon) - x^*(\varepsilon,\eta) - \eta - \mu = 0 \tag{8}$$

for $(\varepsilon,\eta,\mu) \in \vartheta^{s,g}_{\varepsilon=0} \times \upsilon^{s,g}_{\eta=0} \times R$.

In order to continue the process with the same arguments (i.e. the implicit function theorem applied to $\Gamma^{s,g}$), the following hypothesis is necessary:

$\frac{\partial\Gamma^{s,g}}{\partial\eta}(0,0,0) \neq 0$ that is written in details as $\frac{\partial P^{s,g}}{\partial x^*}(0,0)\frac{\partial x^*}{\partial\eta}(0,0) - \frac{\partial x^*}{\partial\eta}(0,0) - 1 \neq 0$, knowing that $\frac{\partial x^*}{\partial\eta}(0,0) = -(\frac{\partial P^{s,g}}{\partial x^*}(0,0)-1)^{-1}$, this is exactly the stated assumption $\mathrm{B}_2^{s,g}$) and thus, $\exists$ a neighborhood $\upsilon^{s,g}_{\varepsilon=0} \subset \vartheta^{s,g}_{\varepsilon=0}$, a neighborhood $\nu^{s,g}_{\eta=0} \subset \upsilon^{s,g}_{\eta=0}$, a neighborhood of $\mu$ denoted $\nu^{s,g}_{\mu=0} \subset R$ and a unique application $\eta^*{:}\upsilon^{s,g}_{\varepsilon=0} \times \nu^{s,g}_{\mu=0} \longrightarrow \nu^{s,g}_{\eta=0}$ solution of $\Gamma^{s,g}(\varepsilon,\eta^*(\varepsilon,\mu),\mu) = 0$ such that $\eta^*(0,0) = 0$. Furthermore, $\eta^*$ depends continuously on $\varepsilon$ and $\mu$.

**Third step of the period doubling procedure**: the last step of the period doubling is reduced to the analysis of the following equation:

$$P^{s,g}(P^{s,g}(P^{s,g}(x(\varepsilon,\eta),\varepsilon),\varepsilon),\varepsilon) = x. \tag{9}$$

Taking into account the results obtained in the two previous steps, the analysis of this equation (9) becomes equivalent to the analysis of the following one:
for $(\varepsilon,\mu)\epsilon\upsilon^{s,g}_{\varepsilon=0} \times \nu^{s,g}_{\mu=0}$:

$$\Pi^{s,g}(\varepsilon,\mu) := P^{s,g}(x^*(\varepsilon,\eta^*(\varepsilon,\mu)) + \eta^*(\varepsilon,\mu) + \mu,\varepsilon) - x^*(\varepsilon,\eta^*(\varepsilon,\mu)) = 0. \tag{10}$$

In this case, the following hypothesis is required to apply the implicit function theorem to $\Pi^{s,g}$ :

$\frac{\partial\Pi^{s,g}}{\partial\mu}(0,0) \neq 0$ that is equivalent in details to:

$\frac{\partial P^{s,g}}{\partial x^*}\frac{\partial x^*}{\partial\eta}\frac{\partial\eta}{\partial\mu}(0,0) - \frac{\partial x^*}{\partial\eta}\frac{\partial\eta}{\partial\mu}(0,0) - 1 \neq 0$

and as $\frac{\partial\eta}{\partial\mu}(0,0) = -(\frac{\partial\Gamma^{s,g}}{\partial\eta}(0,0,0))^{-1}$, this is exactly the stated assumption $\mathrm{B}_3^{s,g}$).

This permits us to affirm that: $\exists$ a neighborhood $\omega^{s,g}_{\varepsilon=0} \subset \upsilon^{s,g}_{\varepsilon=0}$, a neighborhood $\theta^{s,g}_{\mu=0} \subset \nu^{s,g}_{\mu=0}$ and a unique application $\mu^*$: $\omega^{s,g}_{\varepsilon=0} \longrightarrow \theta^{s,g}_{\mu=0}$ solution of $\Pi^{s,g}(\varepsilon,\mu^*(\varepsilon)) = 0$ such that $\mu^*(0) = 0$. Furthermore, $\mu^*$ depends continuously on $\varepsilon$ .

Thus the period doubling procedure applied to the Poincaré map (2), associated with p.w.s system (1) (reduced to a discrete map of dimension 1 on the neighborhood $\upsilon^{s,g}_x \times \upsilon^{s,g}_{\varepsilon=0}$) is constructed step by step and this system becomes chaotic according to the well-known result "period 3 implies chaos" applied to the discrete map $P^{s,g}$. □

## 4 Simulations Results

### 4.1 Symmetric sliding case

Let us consider an academic model subjected to symmetric sliding bifurcations given by:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) & for\ \ x \in D_1, \\ F_2(x,\varepsilon) & for\ \ x \in D_2, \end{cases} \tag{11}$$

where $D_1 := \left\{x \in R^3 : x_3 - \frac{44}{3}x_1^3 - \frac{41}{2}x_1^2 - 5.3x_1 > 0\right\}$,
$D_2 := \left\{x \in R^3 : x_3 - \frac{44}{3}x_1^3 - \frac{41}{2}x_1^2 - 5.3x_1 \leq 0\right\}$

$$\begin{aligned} F_1(x,\varepsilon) &= \begin{pmatrix} 100 \\ -x_3 \\ -0.7x_1 + x_2 + 0.24x_3 - (\varepsilon x_3)^3 \end{pmatrix}, \\ F_2(x,\varepsilon) &= \begin{pmatrix} -100 \\ -x_3 \\ -0.7x_1 + x_2 + 0.24x_3 - (\varepsilon x_3)^3 \end{pmatrix}, \end{aligned}$$

where $\varepsilon$ is the bifurcation parameter defined near 0.

Applying the procedure presented in Section 2 in order to compute the Poincaré map associated with (11) and the method of chaotification given in Section 3, we obtain the following results:

• For $\varepsilon = 0.4$, there is a limit cycle between the two sides $\Pi_1$ and $\Pi_2$, see Fig. 1.

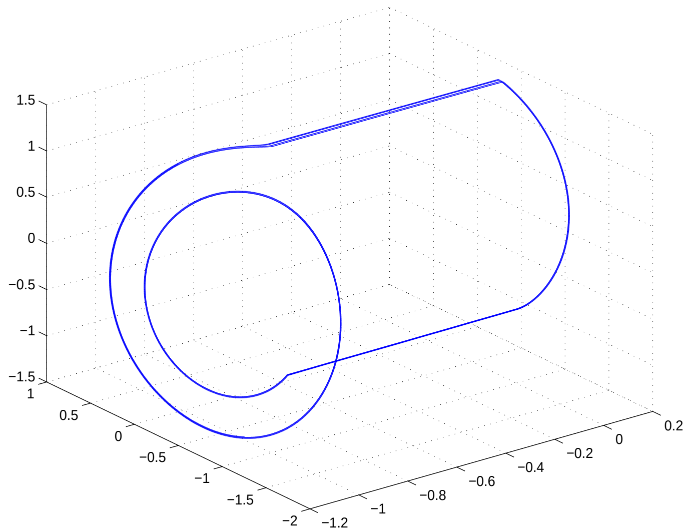


**Figure 1**: Symmetric sliding case: limit cycle for $\varepsilon = 0.4$.

• For $\varepsilon = 0.2$, a symmetric sliding period doubling appears, see Fig. 2.

• For $\varepsilon = -0.05$, a symmetric sliding multi period doubling appears, see Fig. 3.

• For $\varepsilon = -0.23$, a chaotic behavior appears, see Fig. 4.

### 4.2 Symmetric grazing case (Chua circuit)

Let us consider the Chua model subjected to symmetric grazing bifurcations given by:

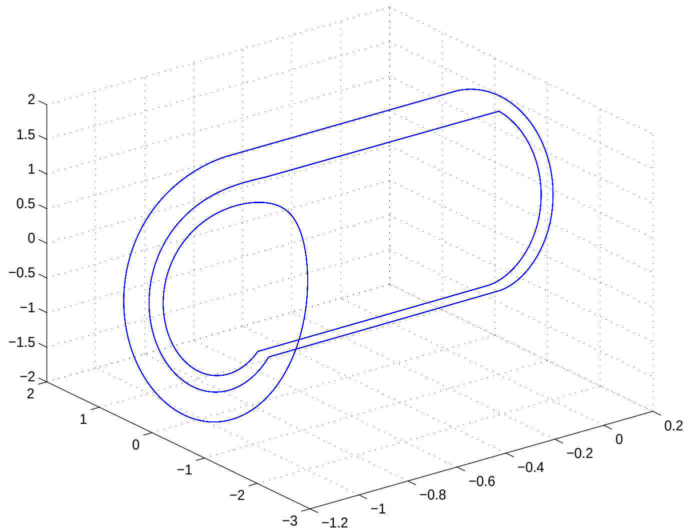


**Figure 2**: Symmetric sliding case: period doubling for $\varepsilon = 0.2$

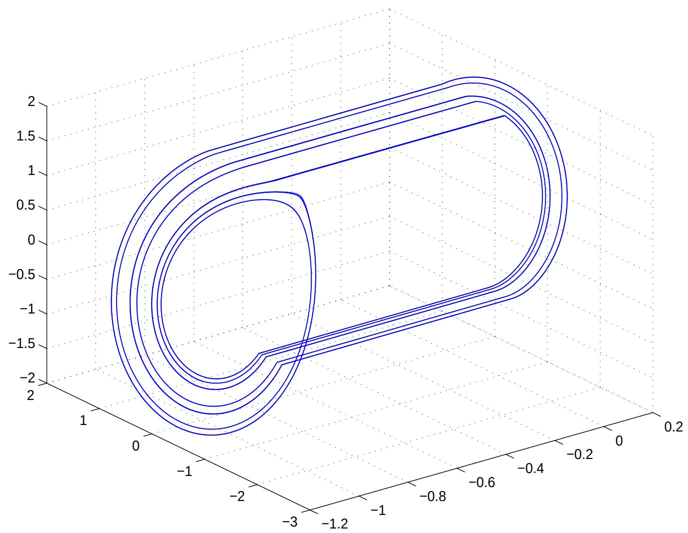


**Figure 3**: Symmetric sliding case: multi period doubling for $\varepsilon = -0.05$.

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \frac{-1}{C_1R}(x_1 - x_2) + \frac{f(x_1,\varepsilon)}{C_1}, \\ \dot{x}_2 & = & \frac{1}{C_2R}(x_1 - x_2) + \frac{x_3}{C_2}, \\ \dot{x}_3 & = & \frac{-x_2}{L}, \end{array} \right. \tag{12}$$

with $f(x_1, \varepsilon) = G_b x_1 + 0.5(G_a(1+\varepsilon) - G_b)(|x_1 + E| - |x_1 - E|$, $R = 2.115K\Omega$, $E = 5.75V$, $C_1 = 10nF$, $C_2 = 100nF$, $G_a(\varepsilon) = \frac{1+\varepsilon}{0.999R}$, $\mathrm{G}_b = \frac{1}{2R}$ and the following initial conditions $(E + 0.3V, 0, -\frac{E}{R})$.

The system (12) can be rewritten according to the general form of systems considered

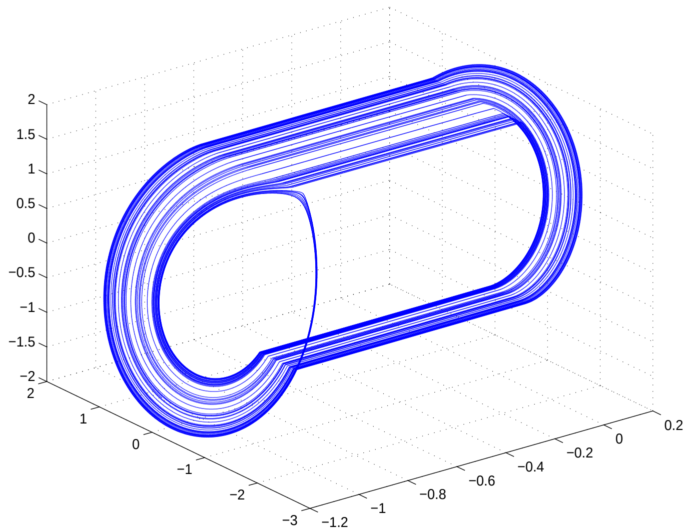


**Figure 4**: Symmetric sliding case: a chaotic behavior for $\varepsilon = -0.23$.

in this paper as:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) & for \;\; x \in D_1, \\ F_2(x,\varepsilon) & for \;\; x \in D_2, \end{cases}$$

with $D_1 = \{x \in R^3 : -E \leq x_1 \leq E\}$, $D_2 = \{x \in R^3 : x_1 > E \text{ or } x_1 < -E\}$,

$$F_1(x,\varepsilon) = \begin{pmatrix} [\alpha_1 + \frac{1}{C_1} G_a(1+\varepsilon)]x_1 - \alpha_1 x_2 \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2, \end{pmatrix},$$

$$F_2(x,\varepsilon) = \begin{cases} F_{2,E}(x,\varepsilon) & \text{for } x_1 > E \;, \\ F_{2,-E}(x,\varepsilon) & \text{for } x_1 < -E, \end{cases}$$

where

$$F_{2,E}(x,\varepsilon) = \begin{pmatrix} [\alpha_1 + \frac{1}{C_1} G_b]x_1 - \alpha_1 x_2 + \frac{1}{C_1}[G_a(1+\varepsilon)G_b]E \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2 \end{pmatrix}$$

and by symmetry

$$F_{2,-E}(x,\varepsilon) = \begin{pmatrix} [\alpha_1 + \frac{1}{C_1} G_b]x_1 - \alpha_1 x_2 + \frac{1}{C_1}[G_a(1+\varepsilon)G_b](-E) \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2 \end{pmatrix},$$

where $\alpha_1 = \frac{-1}{C_1 R}$, $\alpha_2 = \frac{1}{C_2 R}$ and $\alpha_3 = \frac{-1}{L}$, $\varepsilon$ is the parameter bifurcation.

So applying the method presented in Section 2 as for the first example, one determines the Poincaré map associated with this system when a symmetric grazing occurs. The procedure of chaotification given in Section 3 and applied to this Poincaré map gives us the following results:

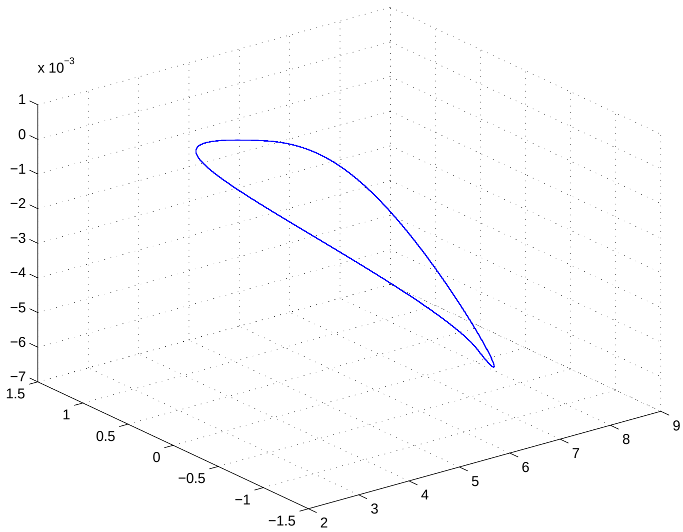


**Figure 5**: Symmetric grazing case (Chua circuit): limit Cycle for $\varepsilon = 0.1$.

• <u>For $\varepsilon = 0.1$</u> (this corresponds to the initial value of $G_a$), there is a limit cycle between the two sides $\Pi_1$ and $\Pi_2$, see Fig. 5.

• <u>For $\varepsilon = 0.2$</u>, a period doubling appears, see Fig. 6.

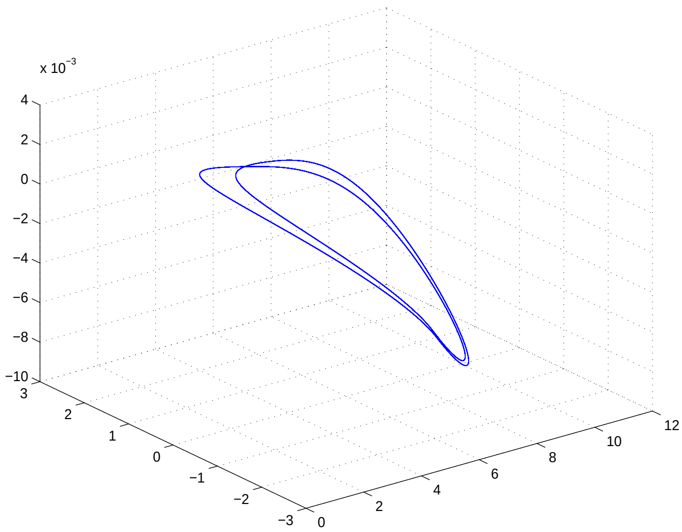


**Figure 6**: Symmetric grazing case (Chua circuit): period doubling for $\varepsilon = 0.2$.

• <u>For $\varepsilon = 0.3$</u>, a Rössler behavior appears, see Fig. 7.

• <u>For $\varepsilon = 0.4$</u>, a double scroll behavior appears, see Fig. 8.

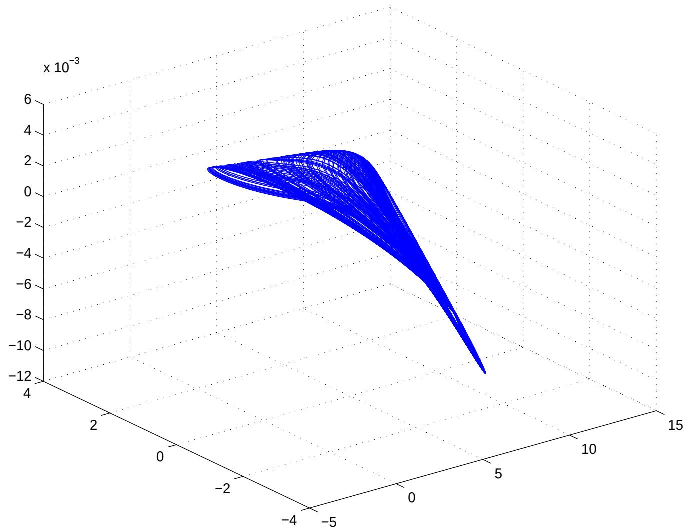


**Figure 7**: Symmetric grazing case (Chua circuit): Rössler attractor for $\varepsilon = 0.3$.

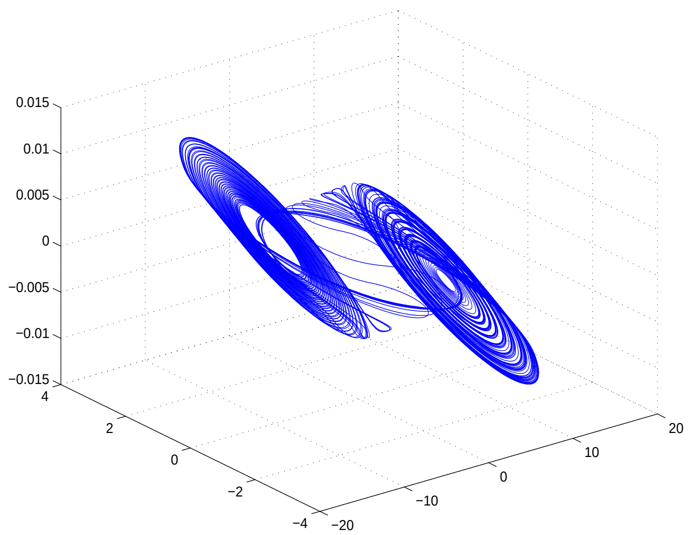


**Figure 8**: Symmetric grazing case (Chua circuit): double scroll attractor for $\varepsilon = 0.4$.

## 5 Conclusion

In this paper, we have proposed a mathematical approach of route to chaos for bounded p.w.s systems of dimension three subjected to symmetric grazing or sliding bifurcations. This approach highlights the fact that it is possible to extend the procedure given in [4,5] to the interesting case of symmetric non-smooth bifurcations. Moreover, simulation results show that it is less complicated to deal with symmetric non-smooth transitions than non-symmetric non-smooth ones. Simulation results were proposed for academic example subjected to symmetric sliding bifurcations and an application of this approach is also done for the well-known Chua circuit where two grazing bifurcations associated with two symmetric transient surfaces appear simultaneously and symmetrically. Many possible perspectives can be investigated such as to generalize the results to other forms of non-smooth transitions, for example corner ones, or to deal with multimodal p.w.s systems.

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# Peculiarities of Wave Fields in Nonlocal Media

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**Abstract:** The paper summarizes the studies of wave fields in structured non-equilibrium media described by means of nonlocal hydrodynamic models. Due to the symmetry properties of models, we derived the invariant wave solutions satisfying autonomous dynamical systems. Using the methods of numerical and qualitative analysis, we have shown that these systems possess periodic, multiperiodic, quasiperiodic, chaotic, and soliton-like solutions. Bifurcation phenomena caused by the variation of nonlinearity and nonlocality degree are investigated as well.

**Keywords:** *nonlocal models of structured media; travelling wave solutions; chaotic attractor; homoclinic curve; invariant tori.*

**Mathematics Subject Classification (2010):** 74D10, 74D30, 37G20, 34A45.

## 1 Introduction

Open thermodynamic systems attract attention of scientists by their synergetic properties, their ability to produce localized nontrivial structures and order. Description of such phenomena requires the creation of new and the refinement of already known mathematical models.

According to [1–3], with the methods of non-equilibrium thermodynamics and the internal variables concept [6], the nonlinear temporally and spatially nonlocal mathematical models have been constructed for non-equilibrium processes in media with structure. In

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this paper, we present the results of investigations of wave processes in such media. To this end, we use the following hydrodynamic type system

$$\begin{gathered}\dot{\rho}+\rho u_x=0, \quad \rho\dot{u}+p_x=\gamma\rho^m,\\ \frac{1}{\rho^2}\frac{\Gamma\varepsilon_r}{\tau_{\mathrm{TP}}}\left\{\left[-\rho_{xx}\left(1+\mathbf{a}\right)+\frac{1}{\rho}\left(\rho_x\right)^2\left(1-\mathbf{a}\Gamma_{\mathrm{V}0}\right)\right]+\left[-\ddot{\rho}\left(1+\mathbf{a}\right)+\right.\right.\\ \left.\left.+\frac{2}{\rho}\dot{\rho}^2\left(1-\frac{\mathbf{a}\left(\Gamma_{\mathrm{V}0}-1\right)}{2}\right)+\frac{1}{\tau_{\mathrm{TP}}}\dot{\rho}\left(1+\mathbf{a}\right)\right]\right\}+\omega_0^2\rho_0^{1-\Gamma_{\mathrm{V}0}}\rho^{\Gamma_{\mathrm{V}0}}\\ -\omega_0^2\rho_0=b\left(p-p_0\right)+b\tau_{\mathrm{TV}}\dot{p}-\frac{\chi_{\mathrm{T}0}}{\chi_{\mathrm{T}\infty}}b\tau_{\mathrm{TV}}^2\ddot{p}-b\Gamma\varepsilon_r\tau_{\mathrm{TV}}\left(p_{xx}+\frac{\rho_x}{\rho}p_x\right),\end{gathered} \tag{1}$$

where

$$\mathbf{a}=T_0\alpha_\infty\Gamma_{\mathrm{V}0}\left(\frac{\rho}{\rho_0}\right)^{\Gamma_{\mathrm{V}0}+1},\ \omega_0^2=\frac{bc_{S0}^2\alpha_0T_0}{\gamma_0},\ b=\frac{\chi_{T0}}{\rho_0\tau_{TP}^2},\ \chi_{\mathrm{T}0}=\rho_0^{-1}c_{\mathrm{T}0}^{-2}=\gamma_\infty\rho_0^{-1}c_{S0}^{-2};$$

$c_{T0}$, $c_{S0}$ are the isothermal and adiabatic frozen velocities of sound; $\gamma_\infty$ is the frozen polytropic index, $\gamma\rho^m$ is the mass force.

Using the characteristic quantities $t_0$, $u_0$, $\rho_0$, let us construct the scale transformation

$$\begin{gathered}t=\bar{t}t_0,\ x=\bar{x}t_0u_0,\ p=\bar{p}\rho_0u_0^2,\ \rho=\bar{\rho}\rho_0,\ u=\bar{u}u_0,\\ \sigma=\frac{\Gamma\varepsilon_r\tau_{TV}}{(t_0u_0)^2},\ \tau_{pT}=\tau_{TV}\frac{\chi_{T0}}{\chi_{T\infty}},\ \tau=\frac{\tau_{TV}}{t_0},\\ h=\frac{\chi_{T0}}{\chi_{T\infty}}\tau^2,\ \kappa=\frac{\omega_0^2}{bu_0^2},\ \chi=\frac{1}{\rho_0u_0^2\chi_{T\infty}},\ a=\delta n\rho^{n+1},\ \delta=T_0\alpha_\infty,\ \Gamma_{V0}=n,\end{gathered} \tag{2}$$

which leads system (1) to the dimensionless form

$$\begin{gathered}\dot{\rho}+\rho u_x=0, \quad \rho\dot{u}+p_x=\gamma\rho^m,\\ \sigma\chi\rho^{-2}\left[-\rho_{xx}(1+a)+\rho_x^2\rho^{-1}(1-an)\right]+h\,\chi\,\rho^{-2}\\ \left[-\ddot{\rho}(1+a)+2\dot{\rho}^2\rho^{-1}(1-0.5a(n-1))+\tau h^{-1}\dot{\rho}(1+a)\right]\\ +\kappa\rho^n=p+\tau\dot{p}-h\ddot{p}-\sigma\left(p_{xx}+\rho_xp_x\rho^{-1}\right).\end{gathered} \tag{3}$$

We would like to emphasize that system (3) can be regarded as a hierarchical set of submodels which are complicated by taking new effects into account. We are thus going to study the chain of nested models and to classify their wave solutions using the methods of qualitative and numerical analysis.

The remainder of the paper is organized as follows. In Section 2 we begin our studies with a simplified version of system (3) keeping the terms with the first temporal derivatives, then attaching the terms with the second temporal or spatial derivatives. The form of wave solutions and the description of techniques for their exploration are presented in detail. Section 3 is devoted to the spatially nonlocal model which is used for investigating the Shilnikov homoclinic structures whose existence and bifurcations are extremely important during chaotic regimes formation. The model incorporating both temporal and spatial nonlocalities is presented in Section 4. Generalizations of the previous models by means of introducing the third temporal derivatives and incorporating physical nonlinearity are given in Section 5 and Section 6, respectively. For all models we derive invariant wave solutions and carry out the qualitative analysis of the corresponding factor-systems.

## 2 Wave Solutions of the Models with Dynamic Equation of State (DES) Incorporating the Second Temporal or Spatial Derivatives

To begin with, let us consider the simplest model with relaxation derived from (3) at $\delta = h = \sigma = 0$, $n = 1$. As has been shown in [5, 6], the system

$$\dot{\rho} + \rho u_x = 0, \quad \rho \dot{u} + p_x = \gamma \rho, \quad \tau(\dot{p} - \chi \dot{\rho}) = \kappa \rho - p, \tag{4}$$

due to its symmetry properties [20], admits the ansatz

$$u = U(\omega) + D, \quad \rho = \rho_0 \exp\left(\xi t + S(\omega)\right), \, p = \rho Z(\omega), \quad \omega = x - Dt, \tag{5}$$

where $D$ is the constant velocity of wave front, $\xi$ determines a slope of the inhomogeneity of the steady solution (5). According to [5], solutions (5) are described by the plane system of ODE which possesses limit cycles and homoclinic trajectories.

If we incorporate the second temporal derivatives in the last equation of system (3), then the previous DES is generalized to the following one:

$$\tau\left(\dot{p} - \chi \dot{\rho}\right) = \kappa \rho - p - h\left\{\ddot{p} + \chi\left(\frac{2}{\rho}\left(\dot{\rho}\right)^2 - \ddot{\rho}\right)\right\}. \tag{6}$$

This model takes into account the dynamics of internal relaxation processes in more detail. As has been shown in [7], wave solutions (5) are described by the system of ODE with three dimensional phase space. This system possesses the limit cycles undergoing the period doubling cascade, and the chaotic attractors.

Consider now the model with relaxation and spatial nonlocality

$$\tau\left(\dot{p} - \chi \dot{\rho}\right) = \kappa \rho - p + \sigma\left\{p_{xx} + \frac{p_x \rho_x}{\rho} - \chi\left(\rho_{xx} - \frac{\rho_x^2}{\rho}\right)\right\}. \tag{7}$$

Solutions (5) satisfy the following dynamical system

$$\begin{aligned} U\frac{dU}{d\omega} &= UW, \; U\frac{dZ}{d\omega} = \gamma U + \xi Z + W(Z - U^2), \\ U\frac{dW}{d\omega} &= \{U^2[\tau(\gamma U + \xi Z - WU^2) + \chi \tau W + Z - \kappa] \\ +\sigma[(\xi + W)(2U(\gamma - UW) &+ \chi W) + (UW)^2]\}\left[\sigma(\chi - U^2)\right]^{-1}. \end{aligned} \tag{8}$$

This system has the fixed point

$$U_0 = -D, \, Z_0 = \frac{\kappa}{1 - 2\sigma(\xi/D)^2}, \, W_0 = 0, \, \gamma = \frac{\xi Z_0}{D} \tag{9}$$

which is the only one lying in the physical parameter range.

We start with analyzing the linearized at the fixed point (9) system (8) with the matrix $\hat{M}$

$$\hat{M} = \begin{pmatrix} 0 & 0 & -D \\ \gamma & \xi & Z_0 - D^2 \\ A & B & C \end{pmatrix},$$

where

$$A = \frac{D\kappa\xi(2\xi\sigma - D^2\tau)}{Q\sigma(2\xi^2\sigma - D^2)}, \, B = \frac{D^2(1 + \xi\tau)}{Q}, \, Q = \sigma(\chi - D^2),$$

$$C = Q^{-1}\left\{\xi\sigma\left(\chi - D^2\right) - \frac{2D^2\kappa\xi\sigma}{D^2 - 2\xi^2\sigma} + D^2\tau\left(\chi - D^2\right)\right\}.$$

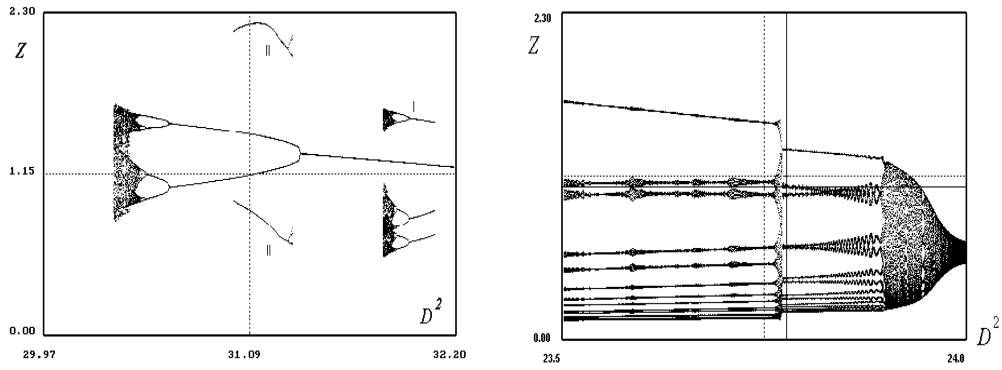


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**Figure 1**: Bifurcation diagrams of system (8) in the plane $(D^2, Z)$ obtained for $\chi = \eta = 50$, $\xi = 1.8$, $\tau = 0.1$, $\sigma = 0.76$ and $\kappa = 14$ (a), $\kappa = 1$ (b).

The well-known Andronov-Hopf bifurcation theorem [21] tells us that periodic solution creation can take place if the spectrum of matrix $\hat{M}$ is $(-\alpha; \pm\Omega i)$. This is so if the following relations hold:

$$\alpha = \xi + C > 0, \tag{10}$$

$$\Omega^2 = AD - B\left(Z_0 - D^2\right) + \xi C > 0, \tag{11}$$

$$\alpha\Omega^2 = \xi\left(AD - Z_0 B\right) > 0. \tag{12}$$

The first two take on the form of inequalities imposing some restrictions on the parameters. The third one determines the neutral stability curve (NSC) in the space $\left(D^2; \kappa\right)$ provided that the remaining parameters are fixed. For $\sigma = 0.76$, $\xi = 1.8$, $\tau = 0.1$, $\chi = 50$, it looks like a parabola with branches directed from left to right, see Figure 2a. Crossing the NSC from right to left, we observe the limit cycle appearance. Development of limit cycle at decreasing $D^2$ is convenient to study by means of the Poincaré section technique [13, 22].

Let us choose the plane $W = 0$ as an intersecting one and find coordinates of intersection points of phase curves which cross-sect the intersecting plane only in one direction. Plotting coordinate $Z$ of the cross-section point along the vertical axis, and the value of the bifurcation parameter $D^2$ along the horizontal one, we will obtain the typical bifurcation diagrams in (Figure 1). From the analysis of diagram Figure 1a we can see that while parameter $D^2$ decreases the development of the limit cycle coincides with the Feigenbaum scenario, followed by the creation of a chaotic attractor. Moreover, in the vicinity of the main limit cycle there are the hidden attractors (designated in Figure 1a by the symbols I and II). These attractors can be visualized by the integrating of system (8) with special initial data only.

In Figure 1b we see the torus development at decreasing $D^2$. According to the diagram, we can distinguish tori with densely wound trajectories and striped tori.

Proceeding in the same way, we get the two-parameter bifurcation diagram (Figure 2) which shows that system (8) possesses the periodic, multiperiodic, quasiperiodic, and chaotic trajectories.

Such a complicated structure of the phase space of the system can be coursed by

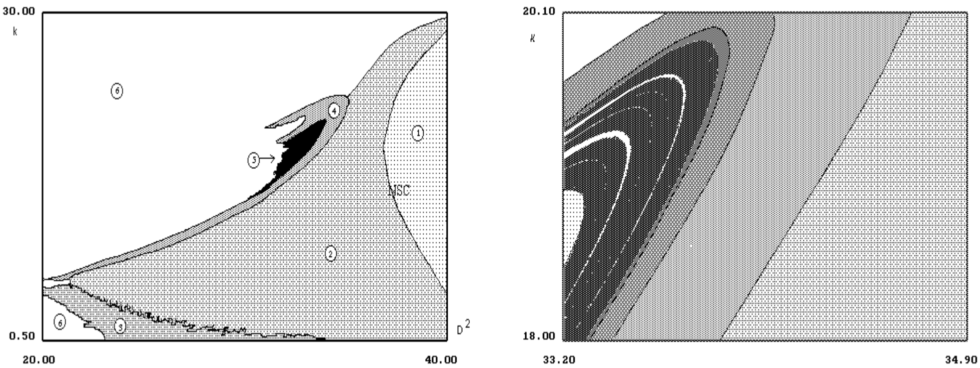


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**Figure 2**: Left: bifurcation diagram of system (8) in parametric space $(D^2, \kappa)$: 1 – stable focus; 2 – $1T$-cycle; 3 – torus; 4 – multiperiodic attractor; 5 – chaotic attractor; 6 – loss of stability. Right: enlargement of part of the left figure: 6 – $3T$-cycle.

homoclinic trajectory existence.

## 3 Homoclinic Loops of Shilnikov Type and Their Bifurcations

It is worth noting that existence of homoclinic trajectories, i.e. loops consisting of the separatrix orbits of hyperbolic fixed point, plays a crucial role [16,19] in the formation of localized regimes (solitary waves) in the phase space of dynamical system. It turned out that the incorporation of spatial nonlocality causes the creation of solitary waves with oscillating tails, whereas the well-known soliton equations have solutions with monotonic asymptotics or compact support (compactons) [17].

For the present, the problem on the existence of homoclinic trajectory of Shilnikov type [18,21] in system (8) has been treated numerically.

We investigate a set of points of parameter space $(D^2,\ \kappa)$ for which the trajectories moving out of the origin along the one-dimensional unstable invariant manifold $W^u$ return to the origin along the two-dimensional stable invariant manifold $W^s$. In practice, for the given values of parameters $\kappa$, $D^2$, we numerically define a distance (the counterpart of split function in [18], p. 198) between the origin and the point $(X^\Gamma(\omega), Y^\Gamma(\omega), W^\Gamma(\omega))$ of the phase trajectory $\Gamma\left(\cdot;\ \kappa,\ D^2\right)$:

$$f^\Gamma\left(\kappa,\ D^2;\ \omega\right) = \sqrt{\left[X^\Gamma(\omega)\right]^2 + \left[Y^\Gamma(\omega)\right]^2 + \left[W^\Gamma(\omega)\right]^2},$$

starting from the fixed Cauchy data $(0, 0, 0.001)$. Next we determine

$$\Phi(\kappa,\ D^2) = \min_{\omega}\{f^\Gamma\} \tag{13}$$

for the part of the trajectory which lies beyond the point at which the distance gains its first local maximum, providing that it still lies inside the ball centered at the origin and having a fixed (sufficiently large) radius (for this case $f^\Gamma(\omega) \leq 5$). The results are presented in Figure 3. The first one is of the most rough scale in this series. Here, white color marks the values of parameters $\kappa$, $D^2$ for which $\Phi > 1.2$, light grey corresponds to

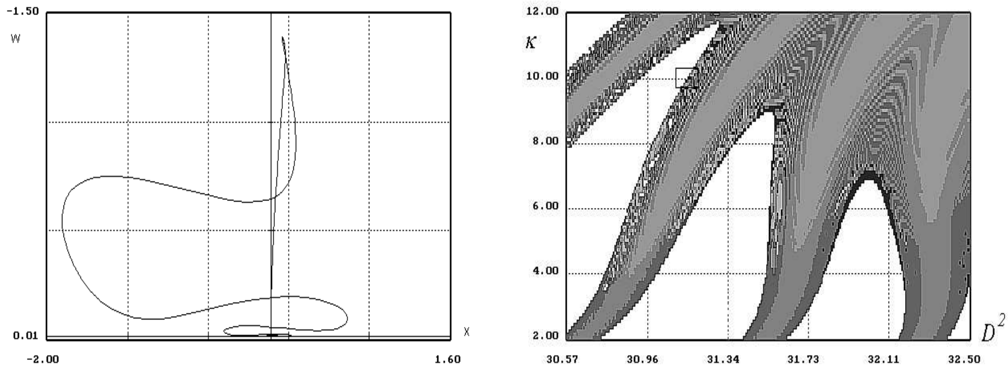


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**Figure 3**: a) Projection of the homoclinic solution of system (8) onto the $(X, W)$ plane. b) A portrait of subset of parameter space $(D^2,\, \kappa)$, corresponding to different intervals of function $f^{\Gamma}_{\min}(D^2,\, \kappa)$ values and the following Cauchy data: $X(0) = Y(0) = 0$, $W(0) = 0.001$: $f^{\Gamma}_{\min} > 1.2$ for white colour; $0.6 < f^{\Gamma}_{\min} \le 1.2$ for light grey; $0.3 < f^{\Gamma}_{\min} \le 0.6$ for grey; $0.01 < f^{\Gamma}_{\min} \le 0.3$ for dark grey; $f^{\Gamma}_{\min} \le 0.01$ for black.

the cases when $0.9 < \Phi < 1.2$ and so on (further explanations are given in the subsequent captions). The black coloured patches correspond to the case when $\Phi < 0.01$. In [11] the structure of the set of points in Figure 3b has been studied in more detail.

## 4 Models with DES taking spatial and temporal nonlocalities into account

Combining the models (6) and (7), we obtain the following spatio-temporal nonlocal model

$$\begin{aligned} \tau\left(\dot{p} - \chi\dot{\rho}\right) &= \kappa\rho - p + \sigma\left\{p_{xx} + \frac{1}{\rho}p_x\rho_x - \eta\left(\rho_{xx} - \frac{\rho_x^2}{\rho}\right)\right\} \\ &\quad -h\left\{\ddot{p} + \eta\left(\frac{2}{\rho}(\dot{\rho})^2 - \ddot{\rho}\right)\right\}. \end{aligned} \tag{14}$$

This model has been studied in [8,14], when the parameters $h$ and $\sigma$ are regarded as small quantities, i.e., equations (6) and (7) are perturbed by the terms with high derivatives. It turned out that the wave localized regimes are saved under perturbations and undergo some smooth changes.

## 5 Models Involving DES with the Third Temporal Derivatives

If we need to describe the relaxing processes in more detail, then we can incorporate the terms with the third temporal derivatives in DES (14). In such case DES has the form [3]

$$\begin{aligned} &\tau\left(\dot{p} - \chi\dot{\rho}\right) = \kappa\rho - p + \sigma\left\{p_{xx} + \frac{1}{\rho}p_x\rho_x - \chi\left(\rho_{xx} - \frac{1}{\rho}\left(\rho_x\right)^2\right)\right\} \\ &-h\left\{\ddot{p} + \chi\left(\frac{2}{\rho}\left(\dot{\rho}\right)^2 - \ddot{\rho}\right)\right\} + \frac{h^2}{\tau}\dddot{p} + \frac{h^2\chi}{\tau}\left\{-\frac{6\dot{\rho}^3}{\rho^2} + \frac{6\dot{\rho}\ddot{\rho}}{\rho} - \dddot{\rho}\right\}. \end{aligned} \tag{15}$$

Solutions (5) satisfy the following dynamical system

$$\begin{aligned}
&U\frac{dU}{d\omega} = UW, \quad U\frac{dZ}{d\omega} = \gamma U + \xi Z + W(Z - U^2), \quad U\frac{dW}{d\omega} = UR,\\
&U\frac{dR}{d\omega} = \left(bU^3\left(\chi - U^2\right)\right)^{-1}\{-\kappa U^2 + \eta\xi\sigma W - 2\xi\sigma U^2 W + \chi\tau U^2 W - h\xi U^4 W\\
&+b\xi^2 U^4 W - \tau U^4 W + \eta\sigma W^2 + (\chi h - \sigma)U^2 W^2 - hU^4 W^2 + b\xi U^4 W^2 - b\chi U^2 W^3\\
&+bU^4 W^3 + \gamma\left(2\xi\sigma U + h\xi U^3 - b\xi^2 U^3 + \tau U^3 + 2\sigma UW\right) + U^2 Z + h\xi^2 U^2 Z\\
&-b\xi^3 U^2 Z + \xi\tau U^2 Z + \left(-\eta\sigma U + U^3\left\{\sigma + \chi h - 4b\chi W - hU^2 + b\xi U^2 + 4bWU^2\right\}\right)R\},
\end{aligned} \tag{16}$$

where $b = h^2/\tau$, and quadrature

$$U\frac{dS}{d\omega} = -(W + \xi).$$

The fixed point of this system has the coordinates

$$U_0 = -D, Z_0 = \frac{\kappa D^2}{D^2 - 2\sigma\xi^2}, W_0 = 0, R_0 = 0. \tag{17}$$

The conditions under which the linearized matrix

$$\hat{M} = \begin{pmatrix} 0 & 0 & a_1 & 0\\ a_2 & a_3 & a_4 & 0\\ 0 & 0 & 0 & a_5\\ a_6 & a_7 & a_8 & a_9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -D & 0\\ \gamma & \xi & Z_0 - D^2 & 0\\ 0 & 0 & 0 & -D\\ a_6 & a_7 & a_8 & a_9 \end{pmatrix}, \tag{18}$$

$$\begin{aligned}
a_6 &= \frac{\kappa\xi\left(-2\xi\sigma + D^2\left(h\xi - b\xi^2 + \tau\right)\right)}{\Delta D\left(2\xi^2\sigma - D^2\right)},\ a_7 = -\frac{1 + h\xi^2 - b\xi^3 + \xi\tau}{\Delta},\\
a_8 &= \frac{\xi\sigma\left(2Z_0 - \eta\right) + D^4\left(h\xi - b\xi^2 + \tau\right) - D^2\left(\chi\tau - 2\xi\sigma\right)}{D^2\Delta},\\
a_9 &= \frac{\chi D^2 h - D^4 h + bD^4\xi + D^2\sigma - \eta\sigma}{D\Delta},\ \Delta = bD\left(\chi - D^2\right)
\end{aligned}$$

admits the spectrum $(\pm\Omega^2 i; -\alpha_1; -\alpha_2)$ have the form

$$B_2 = \frac{B_1}{B_3} + B_0\frac{B_3}{B_1}, \quad B_3^2 - 4B_0\frac{B_3}{B_1} \geq 0, \tag{19}$$

where $B_3 = -a_3 - a_9$, $B_2 = a_3 a_9 - a_5 a_8$, $B_1 = a_5\left(a_3 a_8 - a_1 a_6 - a_4 a_7\right)$, $B_0 = a_1 a_5\left(a_3 a_6 - a_2 a_7\right)$ are the coefficients of characteristic polynomial for the matrix $\hat{M}$.

If we fix the parameters $\chi = \eta = 30$, $\xi = -1.9$, $h = 1$, $\tau = 1$, $b = 1$, $\sigma = 2.7$, then in the plane $(D^2, \kappa)$ equation (19) defines the NSC. Crossing this curve at the point $A(2.2852; 3.7)$, one can observe the appearance of the limit cycle at $D^2 \geq 2.2852$.

In the Poincaré diagram depicted at increasing $D^2$ (Figure 4) we can identify the moments of several period doubling bifurcations leading to the chaotic attractor creation. But the chaotic attractor existing at a short interval of parameter $D^2$ is destroyed. Instead of it in the phase space of system (16) the complicated periodic trajectory in the shape of a loop (Figure 5a) appears.

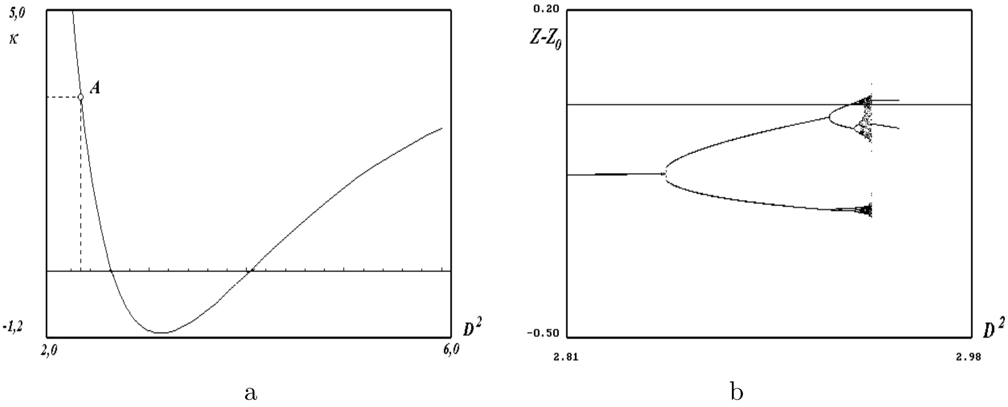


**Figure 4**: a) Neutral stability curve in the plane $(D^2; \kappa)$. b) The bifurcation Poincaré diagram at increasing $D^2$

Consider also the development of oscillating regimes whose basins of attraction are separated from the basin of attraction of the main limit cycle. Integrating dynamical system (16) from initial conditions $(0; 0; 0; 0.01)$ at $D^2 = 2.722$, we see that the phase space of the system, in addition to the main limit cycle, contains the complicated trajectory (Figure 5,a) which can be regarded as a hidden attractor. From the analysis of Poincaré diagram (Figure 6a) it follows that the system weakly responds to the growing of the parameter $D^2$ until $D^2 = 2.7445$. When $D^2 > 2.7445$, the system jumps to another type of oscillations followed by chaotic regime creation.

If we plot the Poincaré diagram at decreasing $D^2$ (Figure 6b) starting from the chaotic attractor, then we observe the periodic trajectory (Figure 5b) that differs from the initial regime (Figure 5a). Note that the periodic trajectory in Figure 5b can be revealed directly by the integration from the initial conditions $(0; 0; 0; 0.1)$.

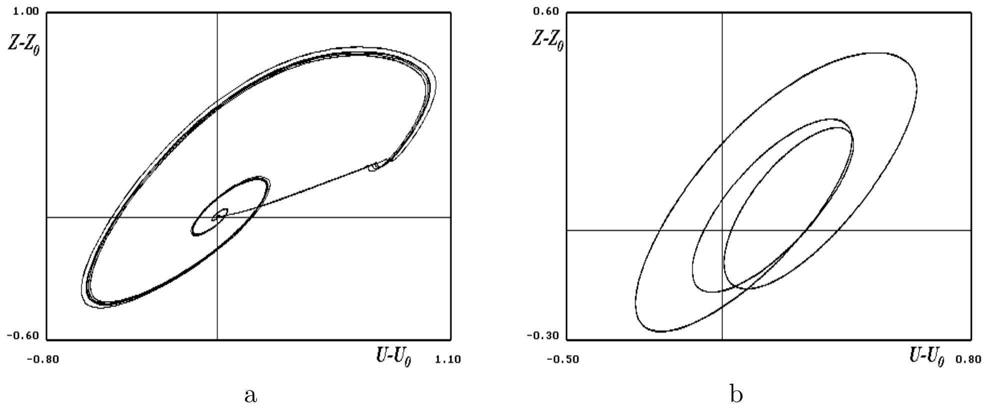


**Figure 5**: Phase portraits of separated trajectories derived at $D^2 = 2.722$, $\kappa = 3.7$, $b = 1$ and under different initial conditions.

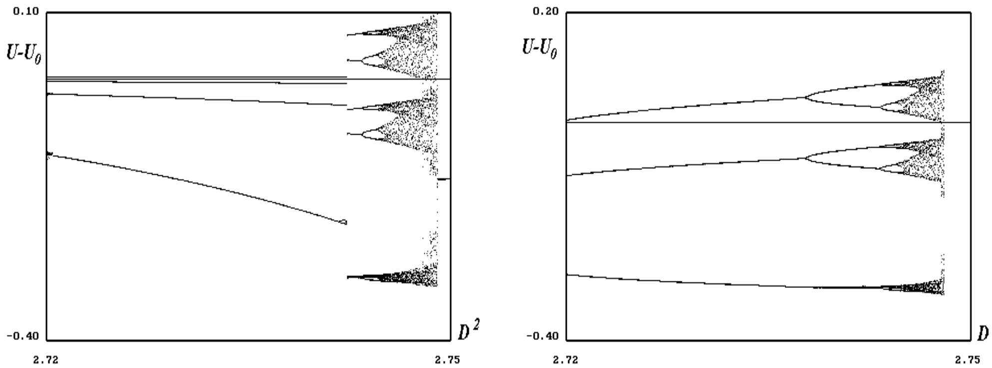


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**Figure 6**: The bifurcation Poincaré diagram of development of separated regime at increasing $D^2$ (a) and decreasing $D^2$. Here $b = 1$.

## 6 DES with Physical Nonlinearity and Second Derivatives

Till now we dealt with the models without physical nonlinearity. Generalizing the previous models in this direction, we obtain the following model [13]

$$\begin{aligned} &\sigma\chi\rho^{-2}\left[-\rho_{xx}\left(1+a\right)+\rho_x^2\rho^{-1}\left(1-na\right)\right] \\ &+h\chi\rho^{-2}[-\ddot{\rho}\left(1+a\right)+2\dot{\rho}^2\rho^{-1}\left(1-0.5a(n-1)\right) \\ &+\tau h^{-1}\dot{\rho}\left(1+a\right)]+\kappa\rho^n=p+\tau\dot{p}-h\ddot{p}-\sigma\left(p_{xx}+\rho_x p_x\rho^{-1}\right), \quad a=\delta n\rho^{n+1}. \end{aligned} \tag{20}$$

Properties of solutions to system (20) can be found out using the symmetry of the system with respect to the Galilei group [20]. One can ascertain by direct verification that system (20) allows the operator

$$\hat{X}=\frac{1}{2\xi}\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}.$$

Let us construct an anzatz with its invariants

$$\rho=R(\omega),\ p=P(\omega),\ u=2\xi t+U(\omega),\ \omega=x-\xi t^2, \tag{21}$$

where parameter $\xi$ is proportional to acceleration of the wave front. Substitution by (21) into the system yields the following quadrature

$$UR=C=\text{const}$$

and the dynamical system

$$\begin{aligned} &R'=W, \quad P'=\gamma R^m-2\xi R+\frac{C^2}{R^2}W, \\ &W'=-(\kappa R^{n+3}-PR^3-P'R^2C\tau-hP'C^2W \\ &+P'R^2\sigma W+\gamma mR^{2+m}\sigma W+\chi L\tau CW+\gamma hmR^mC^2W \\ &+h\chi L(CWR^{-1})^2-2C^2\sigma W^2+\chi M\sigma W^2-2C^4hR^{-2}W^2 \\ &+2h\chi NC^2R^{-2}W^2-2R^3\sigma W\xi-2hRC^2W\xi)\times \\ &((C^2-\chi L)R(\sigma+hC^2R^{-2}))^{-1}, \end{aligned} \tag{22}$$

where $(\cdot)' = \frac{d}{d\omega}(\cdot)$, $L = 1 + a, M = 1 - an, N = 1 - 0.5a(n-1), a = \delta n R^{n+1}$.

The single isolated equilibrium (neglecting the trivial) point has the following coordinates

$$R_0 = \left(\frac{2\xi}{\gamma}\right)^{1/m-1}, \; P_0 = \kappa R_0^n, \; W_0 = 0. \tag{23}$$

At this point the linearized matrix $\hat{M}$ has the form

$$\hat{M} = \begin{pmatrix} 0 & 0 & 1 \\ a_1 & 0 & a_2 \\ a_3 & a_4 & a_5 \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned}
a_1 &= 2\xi(n-1), \quad a_2 = C^2 R_0^{-2}, \quad a_4 = R_0^2 \Delta^{-1}, \\
a_3 &= (2C^3 h \left[C^2 - \chi L\right] \tau \left[\gamma R_0^m - 2\xi R_0\right] R_0^{-2} \\
&\quad + C\chi(n+1)(L-1)\tau\Delta - C\left[C^2 - \chi L\right]\tau\Delta \\
&\quad - \left[C^2 - \chi L\right]\left(C^2 h R_0^{-2} + \sigma\right) \\
&\quad \times \left(\kappa n R_0^{1+n} - C\tau\left(\gamma(2+m)R_0^m - 6\xi R_0\right)\right))/\Delta^2, \\
a_5 &= (C^2\gamma h\left(nR_0^n - R_0^m\right) - C^3\tau + C\chi L\tau \\
&\quad + R_0^2\sigma\left(\gamma\left[R_0^m + nR_0^n\right] - 4R_0\xi\right))/R_0\Delta, \\
\Delta &= \left(C^2 - \chi L\right)\left(C^2 h R_0^{-2} + \sigma\right).
\end{aligned}$$

The NSC for system (22) has the following form

$$G(\xi, \sigma, n, h, \tau, \kappa, \chi) \equiv a_5(a_3 + a_2 a_4) + a_1 a_4 = 0. \tag{25}$$

Let us make the values of parameters fixed as follows:

$$\gamma = 1, \quad \chi = 10, \quad C = -2.8, \quad \sigma = 0.2,$$
$$\tau = 1.1, \quad h = 3.2, \quad \delta = 1.4, \quad n = m = 3.2.$$

Condition (25) allows us to find numerically the value of $\xi_0 = 0.157$ corresponding to birth of the limit cycle.

Let us consider in more detail the influence on the revealed regimes of parameters $n$ and $\delta$ changes, which determine nonlinearity of the medium in the dynamic equation of state. Let us make the value of parameter $\xi = 0.35$ fixed, then there is a limit cycle with period $2T$ in the space of the system, and we construct the bifurcation diagram presented in Figure 7a.

The diagram reveals some peculiarities of system (22) behaviour. In particular, we would like to pay attention to the presence of a "special" point in the parameter plane surrounded by four different types of solutions. One can also see the "windows" of periodicity (area 6) in the chaotic area. To find out the structure of phase space in more detail near area 6, one-parametric Poincaré diagrams were plotted [13].

It turns out that abrupt reconstruction of the chaotic attractor structure can be observed, which is probably caused by the interaction of two (or more) co-existing attractors of the dynamic system. We also reveal that the chaotic trajectory is localized in a more

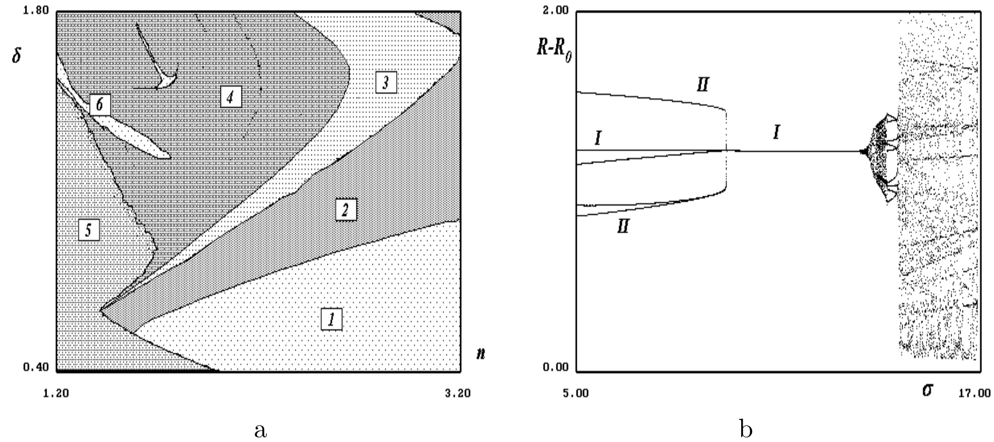


**Figure 7**: a) The two-parametric bifurcation diagram in case of $\gamma = 1$, $\chi = 10$, $C = -2.8$, $\tau = 1.1$, $\sigma = 0.2$, $\kappa = 0.9$, $h = 3.2$, $\xi = 0.35$, $m = 3.2$; b) The Poincaré bifurcation diagram for development of the torus in case of $\delta = 0.4$, $n = 3.2$ (for other values of parameters see Figure 7a) and increasing $\sigma$, where graph I is the basic limit cycle, graph II – complicated periodic trajectory with separated region of attraction.

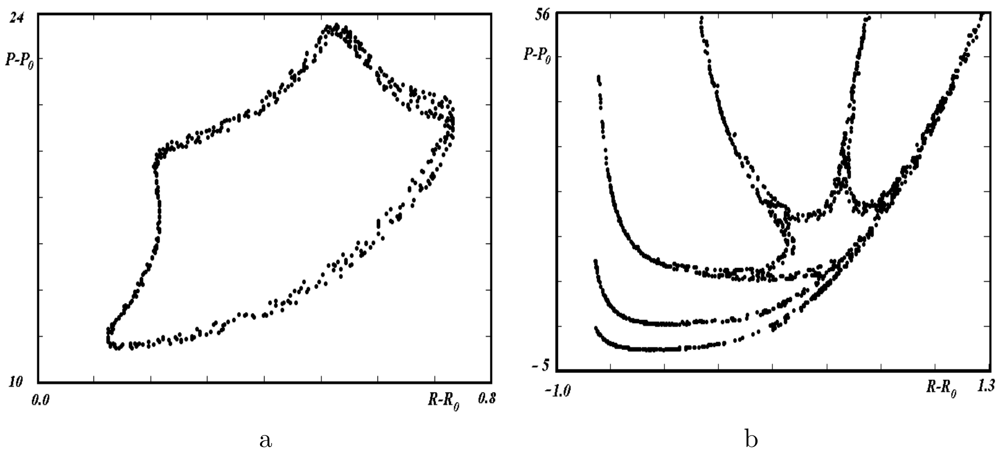


**Figure 8**: a) The Poincaré cross-section of the torus surface in case of $\sigma = 14$ b) The Poincaré cross-section of the chaotic attractor in case of $\sigma = 14.6$. Fixed parameters $\gamma = 1$, $\chi = 10$, $C = -2.8$, $\tau = 1.1$, $\kappa = 0.9$, $h = 3.2$, $\delta = 0.4$, $\xi = 0.35$, $n = m = 3.2$.

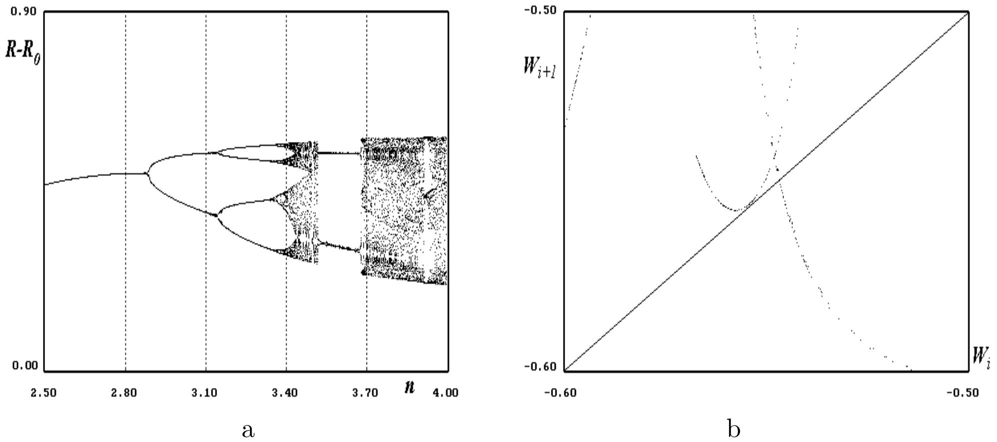


**Figure 9**: a) The bifurcation diagram at increasing $n$. b) The graph of dependence $W_{i+1}$ vs $W_i$ at $n = 4.25$. The fixed values of parameters $\gamma = 1.49$, $\chi = 50$, $C = -1.5$, $\tau = 0.1$, $\kappa = 1.9$, $\sigma = 0.2$, $h = 0.9$, $\xi = 0.18$, $\delta = 0.8$.

narrow area of phase space of system (22), stipulating the appearance of a specific window (area 6) of periodicity with a decrease of $n$. Analysis of two-parametric bifurcation diagrams for $\kappa > 0.9$ shows that the area of existence of chaotic attractors increases and the windows of regular behaviour in case of the increasing $\kappa$ are shifted towards higher values of the nonlinearity parameter $n$.

A crucially different set of bifurcations is observed in case of a change of parameter $\sigma$.

Let us fix the values of parameters $\gamma = 1$, $\chi = 10$, $C = -2.8$, $\tau = 1.1$, $\kappa = 0.9$, $h = 3.2$, $\xi = 0.35$, $n = m = 3.2$ and $\delta = 0.4$. Integrating system (22) with the initial data (0, 0, 0.01) and $\sigma = 5$ within phase space near the equilibrium point, in addition to the limit cycle, other periodic trajectory has been found with a separated pool of attraction (development of this regime with increasing of $\sigma$ is presented in Figure 7b graph II).

The presence of such a regime leads to the assumption on the existence of quasi-periodic regimes. To look for such a regime let us plot a bifurcation diagram of Poincaré for development of basic limit cycle in case of increasing parameter $\sigma$ (Figure 7b graph I).

Another bifurcation, leading to the appearance of the toroidal surface, has been discovered in this system. An intersection of the toroidal attractor with the plane $y_3 = 0$ forms a closed curve, shown in Figure 8a. A further increase of parameter $\sigma$ causes the synchronization of tore frequencies, and finally an abrupt increase of vibrations amplitude, which shows the creation of a crucial new dynamical behavior. To clarify the character of the produced regime, let us analyze the Poincaré section for the case of $\sigma = 14.6$ (Figure 8b). The plotted cross-section is specific for chaotic attractor, which provides reasons for statements on the existence of bifurcation of a quasi-periodic regime with a producing chaotic attractor.

It turned out that system (22) provides another type of chaotic attractor creation, namely, intermittency. Let us fix $\gamma = 1$, $\chi = 50$, $C = -1.5$, $\tau = 0.1$, $\kappa = 1.9$, $\sigma = 0.2$, $h = 0.9$, $\xi = 0.18$.

Plotting the Poincaré bifurcation diagram (Figure 9a), we see that a limit cycle undergoes several period doubling bifurcations resulting in the chaotic attractor creation. But the development of chaotic attractor is interrupted suddenly and new complicated periodic trajectory appears which bifurcates in chaotic attractor as well at increasing $n$. Considering the hereditary sequences (Figure 9b) for chaotic trajectories, we found that the graph of the map $W_{i+1} = f(W_i)$ is close to the bissectrice at $n = 4.25$. As in the case with the Lorentz system, existence of narrow passage leads to the alternation of the chaotic and regular behavior of the system trajectories.

## 7 Conclusions

Finally, we have studied the hierarchical sequences of the mathematical models for non-equilibrium media. Analyzing the wave fields in such media we have shown that the derived models possess wide set of localized wave regimes. In particular, the models with relaxation admit periodic, multiperiodic and chaotic solutions. Spatially nonlocal models have in addition quasiperiodic and solitary wave solutions. All the models demonstrate most bifurcations and scenarios of chaotic regimes creation. The equations of state utilized in this paper are suitable for developing other models of complicated nonequilibrium systems [23].

On the other hand, identifying internal variables with parameters undergoing fluctuations, one can consider these investigations as the problem on the dissipative structures creation under the influence of noise.

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# Stability in Terms of Two Measures for Matrix Differential Equations and Graph Differential Equations

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**Abstract:** In this paper, an attempt has been made to study the qualitative theory of MDEs and its associated GDEs using the Lyapunov function and the concepts of stability in terms of two measures. The theory is well supported with examples. Further, a comparison method wherein the Lyapunov function is used to simplify the complicated MDE is given.

**Keywords:** *matrix differential equations; graph differential equations; stability in two measures.*

**Mathematics Subject Classification (2010):** 65L07, 93D30.

## 1 Introduction

Any natural or manmade systems involve interactions between its constituients, which can be considered as interconnections between them. These interconnections form a network, which can be expressed by a graph [12, 2]. Also, graphs arise naturally when one models organizational structures in social sciences [10]. It has been observed that while many social phenomena change with respect to time, modeling them using static graphs has limited the study. Thus a dynamic graph, a graph that changes with time was introduced [12]. This also led to the concept of a rate of change of a graph with respect to time and a graph differential equation [12]. These concepts were introduced and successfully utilized to study the stability of complex dynamic systems through its associated adjacency matrix [12].

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In [13] the author and her group have utilized the concepts defined in [12] including a graph linear space and its associated matrix linear space. Observing that the study of graph differential equations (GDEs) falls into the realm of differential equations in abstract spaces, the author and her group planned to study GDEs through the associated matrix differential equations (MDEs). This approach appeared to be more reasonable and practical for the study of GDEs. Hence in [13], a weighted directed simple graph was considered as a basic element and existence and uniqueness results were obtained by using monotone iterative technique for the MDE. It is interesting to note that in 2008 a comparison principle for matrix differential equations was developed by Martynyuk [8]. It was realized that simple graphs have no loops and hence in terms of applications a simple graph is not a correct representative of a social structure. This led to the definition of a pseudo simple graph in [14]. Also in [14] a proposition was made that the non linearity of a prey predator model can be preserved using graphs. In [3, 11, 13, 15] many results have been obtained for MDEs and its associated GDEs in terms of iterative techniques and basic theory. With the basic theory well placed the question of studying the qualitative theory of MDEs and its associated GDEs came to the fore. In this direction there is a paper dealing with the stability of dynamic graphs on time scales [2].

The Lyapunov second method, with its advantage of not requiring the knowledge of solutions, has gained increasing significance and gave impetus for developments in the stability theory of differential equations [5]. It is now recognized that the Lyapunov function can be considered to define a generalized distance and can be employed to study various qualitative and quantitative properties of dynamic systems. Further, Lyapunov function serves as a vehicle to transform a given completed differential system into a relatively simpler system and as a result, it is enough to study the properties of solutions of the simpler system.

It was observed that at times a single Lyapunov function might not cater to the needs of a problem and hence a vector Lyapunov function [6] was introduced. In another direction new concepts of stability were defined to be on par with the real world situations. Concepts like partial stability, eventual stability and practical stability were introduced. This posed the question of the possibility of unification of all the definitions. As an answer the concept of stability in terms of two measures [7] was introduced. At this stage, it is appropriate to mention that the study of stability of physical applications is quite appealing. In this context we refer to the following two papers dealing with stability for real world problems [9] and mechanical systems with swiching linear force fields [1].

In this paper, an attempt has been made to study the qualitative theory of MDEs and its associated GDEs using the Lyapunov function and the concept of stability in terms of two measures. The theory is well supported with examples. Further, a comparison method wherein the Lyapunov function is used to simplify the complicated MDE is given.

## 2 Preliminaries

In this section, we introduce all the necessary notation and results that have been developed in earlier works.

**Definition 2.1 Pseudo simple graph:** A simple graph having loops is called a pseudo simple graph.

Let $v_1, v_2, ...v_N$, be $N$ vertice, where $N$ is any positive integer. Let $D_N$ be the set of all weighted directed pseudo simple graphs D=(V, E). Then $(D_N, +, .)$ is a linear space with respect to the operations + and . defined in [12, 13].

Let the set of all matrices be $\mathbb{R}^{N\times N}$. Then $(\mathbb{R}^{N\times N}, +, .)$ is a matrix linear space where '+' denotes matrix addition and '.' denotes multiplication of a matrix by a scalar.

**Definition 2.2 Continuous and differentiable matrix function:**
(1) A matrix function $E : J \to \mathbb{R}^{N\times N}$ defined by $E(t) = (e_{ij}(t))_{N\times N}$ is said to be continuous if and only if each entry $e_{ij}(t)$ is continuous for all $i, j = 1, 2, \ldots, N$ where $e_{ij} : J \to \mathbb{R}$.
(2) A continuous matrix function $E(t)$ is said to be differentiable if and only if each entry $e_{ij}(t)$ is differentiable for all $i, j = 1, 2, \ldots, N$. The derivative of $E(t)$ (if it exists) is denoted by $E'(t)$ and is given by $E'(t) = (e'_{ij}(t))_{N\times N}$.

**Definition 2.3 Continuous and differentiable graph function:** Let $D : J \to D_N$ be a graph function and $E : J \to \mathbb{R}^{N\times N}$ be its associated adjacency matrix function. Then

(1) $D(t)$ is said to be continuous if and only if $E(t)$ is continuous.
(2) $D(t)$ is said to be differentiable if and only if $E(t)$ is differentiable.

Consider the initial value problem

$$D' = G(t, D), \quad D(t_0) = D_0, \tag{2.1}$$

where $G \in C[J \times D_N, D_N]$ and $J = [t_0, T]$. The derivative of a graph function $D$ denoted by $D'$ is the graph function whose edges have weight functions that are derivatives of the weight functions of the corresponding edges of $D$.

The integral of a graph function $D$ denoted by $\int D\ dt$ is the graph function whose edges have weight functions that are integrals of the weight functions of the corresponding edges of $D$. With the above definitions the initial value problem (IVP) of GDE (2.1) can be written as the graph integral equation

$$D(t) = D_0 + \int_{t_0}^{t} G(s, D(s))ds. \tag{2.2}$$

Now using the isomorphism between graphs and matrices we observe that the graph function $G(t, D)$ will be isomorphic to some matrix function $F(t, E)$, and corresponding to (2.1) and (2.2), we can consider the IVP of matrix differential equation

$$E' = F(t, E), \quad E(t_0) = E_0, \tag{2.3}$$

and the matrix integral equation

$$E(t) = E_0 + \int_{t_0}^{t} F(s, E(s))ds, \tag{2.4}$$

where $E_0$ is the adjacency matrix of $D_0$.

In the following sections, we study stability results for the MDE and using the isomorphism that exists between graphs and matrices, we obtain similar results for the corresponding GDE. In order to do so we begin with the following definitions.

**Definition 2.4 Stability:** Consider the differential system

$$E' = F(t, E), \qquad E(t_0) = E_0, \qquad t \geq t_0, \tag{2.5}$$

where $F \in [R_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$. Suppose that the function $F$ is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $E(t) = E(t, t_0, E_0)$ of (2.5). Before proceeding further, we introduce the following classes of functions which are needed in our work

$$K = \{a \in C[R_+, R_+] : a(u) \ is \ \ strictly \ increasing \ in \ u \ and \ a(0) = 0\},$$

$$L = \{\sigma \in C[R_+, R_+] : \sigma(u) \ is \ \ strictly \ decreasing \ in \ u \ and \ \ lim_{u\to\infty}\sigma(u) = 0\},$$

$$KL = \{a \in C[R_+^2, R_+] : a(t, s) \in K \ for \ each \ s \ and \ a(t, s) \in L \ for \ each \ t\},$$

$$CK = \{a \in C[R_+^2, R_+] : a(t, s) \in K \ for \ each \ t\},$$

$$\Gamma = \{h \in C[R_+^2 \times \ \mathbb{R}^{N\times N}, R_+] : inf_{\{t,E\}}h(t, E) = 0\},$$

$$\Gamma_0 = \{h \in \Gamma \ inf \ h(t, E) = 0 \ for \ each \ t \in R_+\}.$$

We are ready to define various stability concepts for the system (2.3) in terms of two measures $h_0, h \in \Gamma$.

**Definition 2.5** The differential system (2.3) is said to be
($S_1$) $(h_0, h)$ -equi-stable if, for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in $t_0$ for each $\epsilon$ such that $h_0(t_0, E_0) < \delta$ implies $h(t, E(t)) < \epsilon$, $t \geq t_0$ where $E(t) = E(t, t_0, E_0)$ is any solution of the system (2.5)
($S_2$) $(h_0, h)$-uniformly stable if the $\delta$ in ($S_1$) is independent of $t_0$;
($S_3$) $(h_0, h)$-equi-attractive-uniformly stable, if for each $\epsilon > 0$ and $t_0 \in R_+$ there exist positive constants $\delta_0 = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that $h_0(t_0, E_0) < \delta_0$ implies that $h(t, E(t)) < \epsilon, t \geq t_0 + T$;
($S_4$) $(h_0, h)$-uniformly attractive, if ($S_3$) holds with $\delta_0$ and T being independent of $t_0$;
($S_5$) $(h_0, h)$-equi-asymptotically stable if ($S_1$) and ($S_3$) hold simultaneously;
($S_6$) $(h_0, h)$-uniformly-asymptotically stable if ($S_2$) and ($S_4$) hold together;
($S_7$) $(h_0, h)$-equi attractive in the large if for each $\epsilon > 0$ and $\alpha > 0$ and $t_0 \in R_+$, there exists a positive number $T = T(t_0, \epsilon, \alpha)$ such that $h_0(t_0, E_0) < \alpha$ implies $h(t, E(t)) < \epsilon, \ t \geq t_0 + T$;
($S_8$) $(h_0, h)$-uniformly attractive in the large if the constant T in ($S_7$) is independent of $t_0$;
($S_9$) $(h_0, h)$-unstable if ($S_1$) fails to hold.

In order to understand the generality of the above stability definitions refer to [ P. 5,6 of [7] ] where examples are given.

Next, we need the following definitions.

**Definition 2.6** Let $h_0, h \in \Gamma$. Then we say that
(i) $h_0$ is finer than $h$ if there exist a $\rho > 0$ and a function $\phi \in CK$ such that $h_0(t, E) < \rho$ implies $h(t, E) \leq \phi(t, h_0(t, E))$;
(ii) $h_0$ is uniformly finer than $h$ if in (i) $\phi$ is independent of $t$;
(iii) $h_0$ is asymptotically finer than $h$ if there exist a $\rho > 0$ and a function $KL$ such that $h_0(t, E) < \rho$ implies $h(t, E) \leq \phi(h_0(t, E), t)$.

**Definition 2.7** Let $V \in C[R_+ \times \ \mathbb{R}^{N\times N}, R_+]$ then V is said to be
(i) $h$-positive definite if there exist a $\rho > 0$ and a function $b \in K$ such that $b(h(t,E)) \leq V(t,E)$ whenever $h(t,E) \leq \rho$;
(ii) $h$- decrescent if there exist a $\rho > 0$ and a function $a \in K$ such that $V(t,E) \leq a(h(t,E))$ whenever $h(t,E) < \rho$;
(iii) $h$-weakly decrescent if there exist a $\rho > 0$ and a function $a \in CK$ such that $V_0(t,E) \leq a(t, h(t,E))$ whenever $h(t,E) < \rho$;
(iv) $h$-asymptotically decrescent if there exist a $\rho > 0$ and a function $a \in KL$ such that $V(t,E) \leq a(h(t,E), t)$ whenever $h(t,E) < \rho$.

For any function $V \in C[R_+ \times \ \mathbb{R}^{N\times N}, R_+]$ we define the function

$$D^+V(t,E) = \lim_{\delta \to 0^+} = \sup \frac{1}{\delta}[V(t+\delta, E + \delta F(t,E)) - V(t,E))] \tag{2.6}$$

for $(t,E) \in R_+ \times \ \mathbb{R}^{N\times N}$.

Let $E(t)$ be a solution of (2.3) existing on $[t_0, \infty)$ and $V(t,E)$ be locally Lipschitzian in E . Then, given $t \geq t_0$, there exists a neighbourhood U of $(t, E(t))$ and an $L > 0$ such that $|V(\tau,\zeta) - V(\tau,\eta)| \leq L||\zeta - \eta||$ for $(\tau,\zeta),(\tau,\eta) \in U$.

## 3 Lyapunov Theorems in Two Measures

In this section we propose to state and prove the theorems due to Lyapunov in terms of two measures for GDEs through its associated MDEs. Though the two theorems of Lyapunov deal with uniform stability and uniform asymptotic stability, we begin with a result on equi stability. We weaken the condition of differentiability of the Lyapunov function by assuming continuity and that it possesses a Dini derivative. We consider the IVP of MDE given by

$$E' = F(t,E), \quad E(t_0) = E_0, \quad t \geq t_0, \tag{3.1}$$

where $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$.

**Theorem 3.1** *Assume that*
*(H1)* $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$, $h \in \Gamma$, $V(t,E)$ *is locally Lipschitzian in E and h-positive definite;*
*(H2)* $D^+V(t,E) \leq 0$, $(t,E) \in S(h,\rho) = \{(t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N},\ h(t,E) < \rho,\ \rho > 0\}$*;*
*(H3)* $h_0 \in \Gamma$, $h_0$ *is finer than* $h$ *and V(t,E) is* $h_0$ *weakly decrescent. Then the system (3.1) is* $(h_0,h)-$ *equi stable.*

**Proof.** From the hypothesis (H1), V is h-positive definite, hence there exist a positive constant $\rho_0 \in (0,\rho)$ and a function $b \in K$ such that

$$b(h(t,E)) \leq V(t,E) \quad whenever \quad h(t,E) \leq \rho_0. \tag{3.2}$$

By hypothesis (H2), V(t,E) is $h_0-$ weakly decrescent, therefore for $t_0 \in \mathbb{R}_+,\ E_0 \in \mathbb{R}^{N\times N}$, there exist a constant $\delta_0 = \delta(t_0) > 0$ and a function $a \in K$ such that $h_0(t_0,E_0) < \delta_0$ implies

$$V(t_0,E_0) \leq a(t_0, h_0(t_0,E_0)). \tag{3.3}$$

Further, the fact that $h_0$ is finer than $h$ implies that there exist a constant $\delta_1 = \delta_1(t_0) > 0$ and a function $\psi \in CK$ such that

$$h(t_0, E_0) \leq \psi(t_0, h(t_0, E_0)) \;\; whenever \;\; h(t_0, E_0) < \delta_1, \tag{3.4}$$

where $\delta_1$ is chosen so that $(t_0, \delta_1) < \rho_0$. Let $\epsilon \in (0, \rho_0)$ and $t_0 \in \psi_+$ be given. Since $a \in CK$, there exists a $\delta_2 = \delta_2(t_0, \epsilon) > 0$ that is continuous in $t_0$ such that

$$a(t_0, \delta_2) < b(\epsilon). \tag{3.5}$$

Choose $\delta(t_0) = min\{\delta_0, \delta_1, \delta_2\}$. Then, using the fact that $h(t_0, E_0) < \delta_0$ and the relations from (3.2) to (3.5) we get

$$b(h(t_0, E_0)) \leq V(t_0, E_0) \leq a(t_0, h_0(t_0, E_0)) < b(\epsilon), \tag{3.6}$$

which in turn yields that $h(t_0, E_0) < \epsilon$. We claim that for every solution $E(t) = E(t, t_0, E_0)$ of (3.1) satisfying $h(t_0, E_0) < \delta$, we have

$$h(t, E(t)) < \epsilon, \quad t \geq t_0. \tag{3.7}$$

If this is not true, there exists a $t_1 > t_0$ such that

$$h(t_1, E(t_1)) = \epsilon \;\; and \;\; h(t, E(t)) < \epsilon, \;\; t \in [t_0, t_1], \tag{3.8}$$

for some solution $E(t, t_0, E_0)$ of (3.1). Set $m(t) = V(t, E(t)), \quad for\ t \in [t_0, t_1]$ and using the fact that V is Lipschitzian in E and the definition of $D^+V(t, E)$ we arrive at

$D^+m(t) \leq 0$, which implies by Lemma 1.1 [4] , that $m(t)$ is nonincreasing in $[t_0, t_1]$, that is $V(t, E(t))$ is nonincreasing in $[t_0, t_1]$, which yields $V(t_1, E(t_1)) \leq V(t_0, E(t_0))$. On combining the relations from (3.5) to (3.8), we obtain

$$b(\epsilon) = V(t_1, E(t_1)) \leq V(t_0, E(t_0)) \leq a(t_0, h_0(t_0, E_0(t_0))) < b(\epsilon) \tag{3.9}$$

which is a contradiction. Hence (3.7) holds, which means that $E(t) < \epsilon$ for all $t \geq t_0$. The proof is complete.

**Theorem 3.2** *Assume that the hypotheses (H1) and (H2) of Theorem 2.1 hold. Further assume that $h_0 \in \Gamma$, $h_0$ is uniformly finer than h, and $V(t, E)$ is $h_0-$ decrescent. Then the system (3.1) is $(h_0, h)-$ uniformly stable.*

**Proof.** Since $h_0$ is uniformly finer than h and $V(t, E)$ is $h_0-$ decrescent, there exist functions $a \in K$ and $\psi \in K$ such that

$$h(t_0, E_0) \leq \psi(h_0(\epsilon)), \tag{3.10}$$

$$V(t_0, E_0) \leq a(h_0(\epsilon)). \tag{3.11}$$

Working along the lines of the proof of Theorem 3.1, the relations (3.2), (3.5), (3.9) together with the relations (3.10) and (3.11) yield the uniform stability of system (3.1). The proof is complete.

**Theorem 3.3** *Assume that*

*(i)* $h_0, \ h \in \Gamma$ *and* $h_0$ *is uniformly finer than h;*

*(ii)* $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N \times N}, \mathbb{R}_+]$, $V(t, E)$ *is locally Lipschitzian in E, h-positive definite,* $h_0-$ *decrescent and*

$$D^+V(t, E) \leq -c(h_0(t, E)), \quad (t, E) \in S(h, \rho), \quad c \in K. \tag{3.12}$$

*Then the system* (3.1) *is* $(h_0, h)$*-uniformly asymptotically stable.*

**Proof.** Since $V(t,E)$ is h-positively definite and $h_0$-decrescent, there exist constants $\rho_0,\ \delta_0$ with $0 \le \rho_0 \le \rho,\ \ \delta_0 > 0$ and functions $a, b \in K$ such that

$$b(h(t,E)) \le V(t,E), \quad (t,E) \in S(h,\rho_0) \tag{3.13}$$

and

$$V(t,E) \le a(h_0(t,\epsilon)), \quad \textit{whenever} \quad h_0(t,E) < \delta_0. \tag{3.14}$$

Since the hypothesis of Theorem 3.2 is satisfied, the system (3.1) is $(h_0,h)$-uniformly stable. Thus setting $\epsilon = \rho_0$, there exists a $\delta_1 = \delta_1(\rho_0) > 0$ such that $h_0(t_0,E_0) < \delta$ implies $h(t,E(t)) < \rho_0,\ \ t \ge t_0$, where $E(t) = E(t,t_0,E_0)$ is any solution of the system (3.1).

Let $0 < \epsilon < \rho_0$. Then the $(h_0,h)$ uniform stability of the system (3.1) yields a $\delta = \delta(\epsilon)$ such that $h_0(t_0,E_0) < \delta$ implies $h(t,E(t)) < \epsilon,\ \ t \ge t_0$. Taking $\overline{\delta} = min\{\delta_0,\delta_1\}$, we assume that $h_0(t_0,E_0) < \overline{\delta}$, and choose $T = T(\epsilon) = a(\overline{\delta})/c(\delta) + 1$.

To show that the system (3.1) is $(h_0,h)$-uniformly stable, it is enough to show that there exists a $t \in [t_0,t_0+T]$ such that $h_0(\overline{t},E(\overline{t})) < \delta$. If the above relation does not hold, then there exists a solution $E(t) = E(t,t_0,E_0)$ of the system (3.1) with $h_0(t_0,E_0) < \overline{\delta}$ such that

$$h(t,E(t)) \ge \delta, \quad t \in [t_0,t_0+T]. \tag{3.15}$$

Let $m(t) = V(t,E(t))$. Then, since $V(t,E)$ is locally Lipschitzian in E, taking Dini derivative we get $D^+m(t) \le D^+V(t,E(t)) \le -c[h_0(t,E(t))],\ \ t \ge t_0$, which yields
$m(t_0+T) - m(t_0) \le -\int_{t_0}^{t_0+T} c(h_0(s,E(s)))ds$. Thus
$\int_{t_0}^{t_0+T}(h_0(s,E(s)))ds \le m(t_0) - m(t_0+T) \le V(t,E(t_0)) \le a(h_0(t_0,E(t_0))) < a(\overline{\delta})$.

On the other hand,
$\int_{t_0}^{t_0+T} c(h_0(s,E(s)))ds \ge c(\delta)T = c(\delta).a(\delta^*)/c(\delta) + 1 = a(\widehat{\delta+1}) > a(\delta^*)$,
which is a contradiction. Thus, the proof of the theorem is complete.

Now we proceed to consider the IVP of GDE given by

$$D' = G(t,E), \qquad D(t_0) = D_0, \tag{3.16}$$

where $G \in C[\mathbb{R}_+ \times D_N, D_N]$. In order to study the stability properties of the system (3.16), we use the existence of an isomorphism between graphs and matrices and state and prove the following theorem.

**Theorem 3.4** *Assume that there exists a function $F(t,E)$ isomorphic to $G(t,D)$ in GDE* (3.16) *such that $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$. Further, assume that there exists a function $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}_+]$ satisfying the hypothesis of Theorem 3.1. Then the system* (3.16) *is equistable.*

**Proof.** Since $F$ is isomorphic to $G$ and the existence of continuous function $F$ is given, we consider the IVP for MDE (3.1). As the hypothesis of Theorem 3.1 is satisfied, we have that the system (3.1) is equistable. Now by virtue of the existence of isomorphism between graphs and matrices, we observe that the Lyapunov function $V$ also caters to the GDE (3.16) and hence the system (3.16) is equistable.

Similar results parallel to Theorem 3.2 and Theorem 3.3 can be established for the IVP of the GDE (3.16).

## 4 Examples

In this section, we proceeed to give examples to each of the theorems in the previous section. We consider a graph differential equation of a system having two vertices and weighted edge functions. Note that we have taken the examples in 7 and extended them suitably to cater to our need.

**Example 4.1** Consider a graph differential equation given by two vertices $V_1$ and $V_2$ and whose derivatives of weighted edges are given by the following equations

$$\begin{cases} e_{11}' = -e_{12}\ e^t, \\ e_{12}' = -\ \dfrac{1}{2}\ e_{12} + e_{11} - e_{21} + \dfrac{1}{2}\ e_{22}, \\ e_{21}' = (e_{11} - e_{21})\ e^t, \\ e_{22}' = -\ \dfrac{1}{2}\ (e_{12} + e_{22})e^t. \end{cases} \tag{4.1}$$

Using the isomorphism between the graphs and the matrices, the fore mentioned graph differential equation can be written as the matrix differential equation given by

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} -e^t x_2 & -\frac{1}{2}\ x_2 + x_1 - x_3 + \frac{1}{2}\ x_4 \\ (x_1 - x_3)e^t & -\frac{1}{2}(x_2 + x_4)e^t \end{bmatrix}, \tag{4.2}$$

where $x_1,\ x_2,\ x_3,\ x_4$ represent the weighted edges $e_{11},\ e_{13},\ e_{13},\ e_{14}$ respectively. Thus

$$E = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}.$$

Now we define the Lyapunov function $V(t, E) = (x_2^2 + x_4^2)e^t + (x_1 - x_3)^2$ and

$$h(t, E) = \sqrt{x_1^2 + x_4^2}, \quad h_0(t, E) = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

Then clearly

$$[h(t, E)]^2 \leq V(t, E) \leq [h_0(t, E)]^2,\ D^+V(t, E) \leq -2(x_1 - x_3)^2\ e^t \leq 0,\ (t, E) \in \mathbb{R}_+ \times \mathbb{R}^{2\times 2}.$$

Hence by Theorem 3.1, the matrix differential equation (4.2) equistable, which in turn yields on using Theorem 3.4, that the graph differential equation (4.1) is also equistable.

**Example 4.2** Consider a graph differential equation associated with two vertices $V_1$ and $V_2$ and weighted edge function $e_{i,j}(t),\quad i, j = 1, 2$ given by the following equations

$$\begin{cases} e_{11}' = -e_{22}, \\ e_{12}' = -\ e_{21} + (1 - e_{12}^2 - e_{21}^2)\ e_{12}\ e^{-t}, \\ e_{21}' = e_{12} + (1 - e_{12}^2 - e_{21}^2)\ e_{21}\ \sin^2 x, \\ e_{22}' = e_{11}. \end{cases} \tag{4.3}$$

Associated with the above graph differential equation (4.3), we can write the matrix differential equation, where $x_1,\ x_2,\ x_3,\ x_4$ represent the $e_{11},\ e_{12},\ e_{21},\ e_{22}$ respectively as

$$E' = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} -x_4 & -x_3 + (1 - x_2^2 - x_3^2)x_2 e^{-t} \\ x_2 + (1 - x^2 - x_3^2)\ x_3 \sin^2 x_2 & x_1 \end{bmatrix}, \tag{4.4}$$

where $E \in \mathbb{R}^{2\times 2}$. Let $V(E) = (x_1^2 + x_4^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2$ and

$$h(E) = \sqrt{(x_2^2 + x_3^2 - 1)^2}, \quad h_0(E) = \sqrt{(x_1^2 + x_4^2 - 1)^2 + (x_2^2 + x_3^2 - 1)^2}.$$

Then clearly $[h(E)]^2 \le V(E) \le [h_0(E)]^2$, $E \in \mathbb{R}^{2\times 2}$ and

$$D^+V(E) = (-4)\ (x_2^2 + x_3^2 - 1)^2\ (x_2^2 e^{-t} + x_3^2 \sin^2 x) \le 0,\ (t, E) \in \mathbb{R}_+ \times \mathbb{R}^{2\times 2}.$$

The $(h_0,\ h)-$uniform stability follows from Theorem 3.2. Observe that

$$\begin{bmatrix} x_1(t) & x_2(t) \\ x_3(t) & x_4(t) \end{bmatrix} = \begin{bmatrix} \sin t & \cos t \\ \sin t & \cos t \end{bmatrix}$$

and has components $(x_1(t),\ x_4(t)) = (\cos t,\ \sin t)$ and $(x_2(t),\ x_3(t)) = (\sin t,\ \cos t)$ which are periodic, hence the system in pairs $(x_1(t),\ x_4(t))$ and $(x_2(t),\ x_3(t))$ is uniformly orbitally stable. It now follows that the considered graph differential equation is also $(h_0 - h)$-uniformly stable.

The following example will illustrate Theorem 3.3.

Consider a graph having two vertices $V_1$ and $V_2$. Suppose a graph differential equation is defined on this graph, where the edges satisfy the relations

$$\begin{cases} e'_{11} = 2e_{12} - e_{11}\ e^t - e_{22}, \\ e'_{12} = -e_{12}(1 + \sin^2\ e_{21}) - 2e_{11}e^{-t} - e_{22}, \\ e'_{21} = -e_{12}\ e^{-t} + e_{11}\ \cos t + e_{21}\ \sin t, \\ e'_{22} = -(e_{12} + e_{11})e^{-t} - e_{22}. \end{cases} \tag{4.5}$$

Then we construct the adjacency matrix by replacing $e_{11},\ e_{12},\ e_{21},\ e_{22}$ by $x_1,\ x_2,\ x_3,\ x_4$ respectively and obtain the matrix differential equation

$$E' = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}' = \begin{bmatrix} 2x_2 - x_1\ e^t - x_4 & -x_2(1 + \sin^2 x_3) - 2x^2 e^{-t} + x_4) \\ -x_2 e^{-t} + x_1 \cos t + x_3 \sin t & -x_2 + x_1 e^{-t} - x_4 \end{bmatrix}, \tag{4.6}$$

where $E \in \mathbb{R}^{2\times 2}$. Define

$$A = \{\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathbb{R}^{2\times 2} : x_1 = x_2 = x_4 = 0\},\quad B = \{\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in \mathbb{R}^{2\times 2} : x_1 = x_4 = 0\}$$

and $V(t, E) = x_1^2 + x_2^2 e^{-t} + x_4^2$. For $E_1 = (c_{ij})_{2\times 2}$ and $E_2 = (d_{ij})_{2\times 2}$, we define

$$d(E_1,\ E_2) = \sqrt{\sum_{i,j=1}^{2} (c_{ij} - d_{ij})^2}$$

and consider $h(t, E) = d(E, B)$ and $h_0(t, E) = d(E, A)$. Then

$$h_0(t, E) = \sqrt{x_1^2 + x_2^2 + x_4^2}, \quad h(t, E) = \sqrt{x_1^2 + x_4^2}$$

which yield $A \subset B$ and

$$[h(t, E)]^2 \le V(t, E) \le [h_0(t, E)]^2.$$

Also

$$D^+V(t, E) \le (-2)[h_0(t, E)]^2.$$

An application of Theorem 3.3 yields that the matrix differential equation (4.6) is $(h_0 - h)$ uniformly asymptotically stable. From which we can make the same conclusion for the graph differential equation (4.5) using the isomorphism between matrices and graphs.

## 5 Comparison Technique

It is well known that a Lyapunov function can be considered as a vehicle to transform a given complicated differential system into a relatively simpler scalar differential equation. Thus using the concept of a Lyapunov function and theory of differential inequalities we obtain a very general comparison principle in terms of two measures. In order to achive our goal we need the following results from [4,15].

Consider the scalar differential equation given by

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \tag{5.1}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $g(t_0) = 0$.

**Definition 5.1** Let $r(t)$ be a solution of (5.1) existing on some interval $I = [t_0, t_0 + \alpha]$, $0 < \alpha < \infty$. Then $r(t)$ is said to be a maximal solution of (5.1) if for every solution $u(t) = u(t, t_0, u_0)$ of (5.1) existing on J, the following inequality holds

$$u(t) \leq r(t), \quad t \in J. \tag{5.2}$$

**Lemma 5.1** *Let $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$ and $r(t) = r(t, t_0, u_0)$ be the maximal solution of* (5.1) *existing on J. Suppose that $m \in C[\mathbb{R}_+, \mathbb{R}_+]$ and $Dm(t) \leq g(t, m(t))$, $t \in J$, where $D$ is any fixed Dini derivative. Then $m(t_0) \leq u_0$ implies $m(t) \leq r(t)$, $t \in J$.*

We now formulate a basic comparison theorem in terms of Lyapunov function V for MDE (3.1).

**Theorem 5.1** *Let $V \in C[R_+ \times \mathbb{R}^{N\times N}, R_+]$ and $V(t, E)$ be locally Lipschitzian in $E$ for each $t \in \mathbb{R}_+$. Assume further that*

$$D^+V(t,E) \leq g(t, V(t,E)), \quad (t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N}, \tag{5.3}$$

*where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of (5.1) existing on J. Then, for any solution $E(t) = E(t, t_0, E_0)$ of (3.1) existing on J, $V(t_0, E_0) \leq u_0$ implies*

$$V(t, E(t)) \leq r(t), \quad t \in J. \tag{5.4}$$

**Proof.** Let $E(t) = E(t, t_0, E_0)$ be a solution of (3.1). Set $m(t) = V(t, E(t))$ such that $V(t_0, E_0) \leq u_0$. Using the fact that $V(t, E)$ is locally Lipschitzian in $E$, the definition of Dini derivative and the relation (5.3) we arrive at the inequality $D^+m(t) \leq g(t, V(t, m(t)))$, $m(t_0) \leq u_0, t \in J$ From Lemma 5.1, we conclude that $V(t, E(t)) \leq r(t), t \in J$, completing the proof.

For the sake of completeness, we define the stability concept for the trivial solution of the comparision equation (5.1). We give here the definition of equistability only.

**Definition 5.2** Let $u(t, t_0, u_0)$ be any solution of (5.1). The trivial solution $u(t) \equiv 0$ of (5.1) is said to be equistable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ that is continuous in $t_o$ for each $\epsilon$ such that $u_0 < \delta$ implies $u(t, t_0, u_0) < \epsilon, t \geq t_0$.

We will now state and prove the following theorem which gives sufficient conditions for the $(h_0, h)$-stability properties of the differential system.

**Theorem 5.2** *Assume that*
(*i*) $h_0,\ h \in \Gamma$ *and* $h_0$ *is uniformly finer than* $h$;
*(ii)* $V \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N},\ \mathbb{R}_+]$, $V(t,E)$ *is locally Lipshitzian in* $E$, $V$ *is* $h-$ *positive definite and* $h_0-$*decrescent;*
*(iii)* $g \in C[\mathbb{R}_+ \times \mathbb{R},\ \mathbb{R}]$ *and* $g(t,0) \equiv 0$;
*(iv)* $D^+V(t,E) \leq g(t,\ V(t,E))$, $(t,E) \in S(h,\rho)$, *where*
$S(h,\rho) = \{(t,E) \in \mathbb{R}_+ \times \mathbb{R}^{N\times N} :\ h(t,E) < \rho,\ \rho > 0\}$.
*Then, the stability properties of the trivial solution of* (4.2) *imply the corresponding* $(h_0,\ h)-$ *stability properties of MDE* (3.1).

***Proof.*** As the proofs of various stability properties are similar, we shall only prove the $(h_0,\ h)-$ equiasymptotic stability property of (3.1). In order to do so, we begin by proving $(h_0,\ h)-$ stability.

Since $V$ is $h-$ positive definite, there exist a $\lambda \in (0,\rho]$ and a $b \in K$ such that

$$b(h(t,E)) \leq V(t,E), \quad (t,E) \in S(h,\ \lambda). \tag{5.5}$$

Let $0 < \epsilon < \lambda$ and $t_0 \in \mathbb{R}_+$ be given and assume that the trivial solution of (5.1) is equistable. Then, given $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta_1 = \delta_1(t_0,\epsilon)$ such that

$$u_0 < \delta \ \ \textit{implies} \ \ u(t,t_0,u_0) < b(\epsilon), \quad t \geq t_0, \tag{5.6}$$

where $u(t,t_0,u_0)$ is any solution of (5.1). Set $u_0 = V(t_0,E_0)$. Using hypotheses (i) and (ii) (i.e., $h_0$ is finer than $h$ and $V$ is $h_0-$ decrescent) we find that there exist a $\lambda_0 > 0$ and a function $a \in K$ such that for $(t_0,E_0) \in S(h_0,\ \lambda_0)$

$$h(t_0,E_0) < \lambda \ \ \textit{and} \ \ V(t_0,E_0) \leq a(h(t_0,E_0)). \tag{5.7}$$

The above relation (5.7) along with the relation (5.5) yields

$$b(h(t_0,E_0)) \leq V(t_0,E_0) \leq a(h_0(t_0,E_0)), \quad (t_0,E_0) \in S(h_0,\ \lambda_0). \tag{5.8}$$

Next choose a positive $\delta = \delta(t_0,\epsilon)$ such that $\delta \in (0,\lambda_0]$, $a(\delta) < \delta_1$ and let $h_0(t_0,E_0) < \delta$. Then from relations (5.8) we get, on using the fact that $\delta_1 < b(\epsilon)$, $h(t_0,E_0) < b(\epsilon)$. Now for any solution $E(t) = E(t,t_0,E_0)$ claim that $h(t,E(t)) < \epsilon$, $t \geq t_0$, whenever $h(t_0,E_0)) < \delta$.

If possible, suppose our claim is incorrect. Then there exist a $t_1 > t_0$ and a solution $E(t)$ of (3.1) such that

$$h(t_1,E(t_1)) = \epsilon \textit{ and } h(t,E(t)) < \epsilon, \quad t_0 \leq t \leq t_1, \tag{5.9}$$

since $h(t_0,E_0) < \epsilon$ whenever $h_0(t_0,E_0) < \delta$. From this we deduce that

$$h(t,E(t)) \in S(h,\ \lambda)$$

for $t_0 \leq t \leq t_1$ and thus by Theorem (5.1), we conclude

$$V(t,E(t)) \leq r(t,t_0,u_0), \quad t_0 \leq t \leq t_1, \tag{5.10}$$

where $r(t,t_0,E_0)$ is the maximal solution of (5.1).

On using the relations (5.5), (5.6), (5.7) and (5.10) we arrive at

$$b(\epsilon) < V(t_1,E(t_1)) \leq r(t,t_0,E_0) < b(\epsilon),$$

which is a contraduction, proving $h_0, h-$equistability of (3.1).

Next, we assume that the trivial solution of (5.1) is equiattractive. Since the equation (5.1) is $(h_0, h)$-stable, we set $\epsilon = \lambda$ which implies that

$$\widehat{\delta_0} = \delta(t_0,\ \lambda).$$

Let $0 < \eta < \lambda$. Then since the equation (5.1) is equiattractive, given $b(\eta) > 0$ and $t_0 \in \mathbb{R}_+$, there exist $\delta_1^* = \delta_1^*(t_0) > 0$ and $T = T(t_0, \eta) > 0$ such that

$$u_0 < \delta_1^* \textit{ implies } u(t, t_0, u_0) < b(\eta), \quad t \geq t_0 + T. \tag{5.11}$$

Choose $u_0 = V(t_0, E_0)$ and working as before, we find a $\delta_0^* = \delta_0^*(t_0) > 0$ such that $\delta_0^* \in (0, \lambda_0]$ and $a(\delta_0^*) < \delta_1^*$. Let $\delta_0 = min(\delta_0^*,\ \widehat{\delta_0})$ and $h(t_0, E(t_0)) < \delta_0$, which implies that $h(t, ...E(t)) < \lambda,\ t \geq t_0$, and hence the relation (5.10) holds for all $t \geq t_0$. Now suppose that the system (5.1) is not $(h_0, h)-$ equialtractive then there exists a sequence $\{t_k\},\ \ t_k \geq t_0 + T,\ t_k \to \infty$ as $k \to \infty$ such that $\eta_k < h(t_k,\ E(t_k))$, where $E(t)$ is any solution of (3.1) such that $h_0(t_0, E_0) < \delta_0$. Then using the above inequality along with relations (5.10) and (5.1), we obtain

$$b(\eta_k) < b(h(t_k, E(t_k))) \leq V(t_k, E(t_k)) < r(t, t_0, E_0) < b(\eta),$$

which is a contradiction. Hence the system (3.1) is $(h_0, h)-$ asymptotically stable and hence the proof.

**Theorem 5.3** *Supose that the function $G \in C[\mathbb{R}_+ \times D_N, D_N]$ in* (3.16) *is isomorphic to a function $F \in C[\mathbb{R}_+ \times \mathbb{R}^{N\times N}, \mathbb{R}^{N\times N}]$. Let $E(t)$ be the solution associated with the system* (3.1) *corresponding to the F obtained above. If the hypothesis of Theorem* 5.2 *is satisfied then the trivial solution or the null graph of GDE* (3.16) *has all the stability properties that the associated MDE possesses.*

**Proof.** Corresponding to the given graph function $G(t, D)$, we construct the matrix function $F(t, E)$. Owing to the isomorphism that exists between graphs and matrices $F(t, E)$ is continuous. Now from hypothesis, $E(t)$ is any solution of MDE (3.1). Also since the hypothesis of Theorem 5.2 is satisfied, we obtain that the zero solution of MDE (3.1) possesses all the stability properties of the comparison equation (5.1). Hence by the isomorphism that exists between graphs and matrices, we have that the zero solution, a null graph function of the GDE (3.16) has all the stability properties that the comparison equation (5.1) possesses. The proof is complete.

## 6 Conclusion

In this paper we have considered a MDE in terms of two measures and studied its stability properties using the basic Lyapunov theorems and the comparison methods. Using the isomorphism that exists between the graphs and matrices, we have extended these results to study the stability properties in terms of two measures, for the GDEs. We have also given examples to verify the stability properties of graph differential equations and its associated matrix differential equations using suitable Lyapunov functions.

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# Boundedness and Square Integrability of Solutions of Nonlinear Fourth Order Differential Equations

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**Abstract:** Sufficient conditions for the boundedness of the solutions to a certain nonlinear fourth order differential equation are given by means of the Lyapunov's second method. We also give criteria for square integrability of solutions and their derivatives. Example is given to illustrate our results.

**Keywords:** *boundedness; stability; Lyapunov function; fourth-order differential equations; $L^2$ solutions; square integrable.*

**Mathematics Subject Classification (2010):** 34D20, 34C11.

## 1 Introduction

Higher-order nonlinear differential equations are frequently encountered in mathematical models of most dynamic processes in electromechanical systems in physics and engineering. The notions of stability and boundedness of solutions are fundamental in the theory and application of differential equations. In this way, both concepts lead to the real world applications. Many results relative to stability, boundedness, square integrability of solutions to differentiel equations have been obtained. See for instance ( [1]– [42]). In discussing stability and boundedness of a nonlinear differential system, Lyapunov's direct method perhaps is the most effective method. Numerous methods have been proposed in the literature to derive suitable Lyapunov functions, but finding a proper Lyapunov's function in general is a big challenge.

The study of fourth order nonlinear differential equations has attracted the interest of many researchers. Many results concerning the stability and boundedness of solutions of fourth order differential equations have been obtained in view of various methods, especially, Lyapunov's method, see, the book of Reissig et al. [28] as a survey and the

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papers of Adesina and Ogundare [2], Cartwright [6], Chukwu [9], Abou-El-Ela and Sadek [1], Ezeilo [12], [14] Ezeilo and Tejumola [15], Harrow [17], Hu [18], Tejumola [30], Tunç [35], [36], [37], [38], Wu and Xiong [42], Vlček [41] and the references cited therein.

In 1956, Cartwright [6] investigated the asymptotic stability of zero solution of various linear and nonlinear fourth order differential equations. In [6], she considered the following differential equations

$$x'''' + a_1 x''' + a_2 x'' + a_3 x' + f(x) = 0, \tag{1}$$

$$x'''' + a_1 x''' + \psi(x') x'' + a_3 x' + a_4 x = 0, \tag{2}$$

$$x'''' + a_1 x''' + a_2 x'' + \psi(x) x' + f(x) = 0. \tag{3}$$

In [22] and [23], Omeike by using the Cauchy formula for the particular solution of nonlinear differential equations, has proved that every solution of the equations

$$x'''' + ax''' + bx'' + cx' + h(x) = p(t), \tag{4}$$

$$x'''' + ax''' + \psi(x'') + g(x') + h(x) = p(t), \tag{5}$$

and its derivatives up to order three are bounded.

In [31], and [39] Tunç established sufficient conditions for the asymptotic stability of the zero solution of the equations and the boundedness of the following equations

$$x'''' + a_1 x''' + \psi(x, x') x'' + a_4 x' + h(x) = 0, \tag{6}$$

$$x'''' + a_1 x''' + \psi(x, x') x'' + g(x') + a_4 x = 0, \tag{7}$$

$$x'''' + ax''' + \psi(x, x', x'') + g(x, x') + h(x) = p(t). \tag{8}$$

The solution which is in $L^2[0,\infty)$ for higher order nonlinear differential equations was also of great interest, but it should be noted that only a few results are related to the fourth order nonlinear differential equations. Namely, in 1989, Andres and Vlček [3], established some sufficient conditions, when all the solutions of (4) are in $L^2[0,\infty)$.

In this paper, we develop the conditions under which all the solutions of the following equation (9) are bounded and are square integrable

$$x'''' + a(t)\Big(p\big(x(t)\big)x''(t)\Big)' + b(t)\Big(q\big(x(t)\big)x'(t)\Big)' + c(t)\,f\big(x(t)\big)x'(t) + d(t)\,h\big(x(t)\big) = e(t), \tag{9}$$

where the primes in (9) denote differentiation with respect to t; the functions $a, b, c, d,$ are continuously differentiable functions. The functions $f, h, p, q,$ and $e$ are continuous functions depending only on the arguments shown. It is also supposed that the derivatives, $p'(x)$, $q'(x)$, $f'(x)$ and $h'(x)$ exist and are continuous.

Equation (9) is equivalent to the system

$$\begin{cases} x' = y \\ y' = z \\ z' = w \\ w' = -a(t)p(x)w - \Big(b(t)q(x) + a(t)\theta_1\Big)z - \Big(b(t)\theta_2 + c(t)f(x)\Big)y - d(t)h(x) + e(t), \end{cases} \tag{10}$$

such that

$$\theta_1(t) = p'(x(t))\, x'(t), \quad \theta_2(t) = q'(x(t))\, x'(t).$$

The continuity of the functions $a, b, c, d, e, p, q, f, p', q', f'$ and $h$ guarantees the existence of the solutions of (9) ( see [11], p. 15). It is assumed that the right hand side of the system (10) satisfies a Lipschitz condition in $x(t), y(t), z(t)$, and $w(t)$. This assumption guarantees the uniqueness of solutions of (9) ( [11], p. 15). The present work was motivated by the papers [3], [23], [31], [39] and the papers mentioned above, where the boundedness and square integrability of solutions for a fourth order nonlinear differential equation was studied. Using Lyapunov's method, we show that every solution $x(t)$ of equation (9) and its derivatives are bounded and square integrable.

## 2 Assumptions and Main Results

First, we state some assumptions on the functions that appeared in (9). Suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, p_0, q_0, a_1, b_1, c_1, d_1, f_1, p_1, q_1, m, M, \delta$, and $\eta_1$, such that the following conditions are satisfied

i) $0 < a_0 \leq a(t) \leq a_1$; $0 < b_0 \leq b(t) \leq b_1$; $0 < c_0 \leq c(t) \leq c_1$; $0 < d_0 \leq d(t) \leq d_1$ for $t \geq 0$.

ii) $0 < f_0 \leq f(x) \leq f_1$; $0 < p_0 \leq p(x) \leq p_1$; $0 < q_0 \leq q(x) \leq q_1$ for $x \in \mathbb{R}$ and $0 < m < \min\{f_0, p_0, 1\}$, $M > \max\{f_1, p_1, 1\}$.

iii) $\dfrac{h(x)}{x} \geq \delta > 0$ ( for $x \neq 0$); $h(0) = 0$.

iv) $\displaystyle\int_0^{+\infty} (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|)\, dt < \eta_1$.

The following lemma will be useful in the proof of the next theorem.

**Lemma 2.1** *[20] Let* $h(0) = 0$, $xh(x) > 0$ $(x \neq 0)$ *and* $\delta(t) - h'(x) \geq 0$ $(\delta(t) > 0)$, *then*

$$2\delta(t) H(x) \geq h^2(x), \quad \text{where} \quad H(x) = \int_0^x h(s) ds.$$

**Theorem 2.1** *In addition to conditions (i)-(iv) being satisfied, suppose that there are positive constants* $h_0, \delta_0, \delta_1, \eta_2$ *and* $\eta_3$ *such that the following conditions hold*

*H1)* $h_0 - \dfrac{a_0 m \delta_0}{d_1} \leq h'(x) \leq \dfrac{h_0}{2}$ *for* $x \in \mathbb{R}$.

*H2)* $\delta_1 = \dfrac{d_1 h_0 a_1 M}{c_0 m} + \dfrac{c_1 M + \delta_0}{a_0 m} < b_0 q_0$.

*H3)* $\displaystyle\int_{-\infty}^{+\infty} (|p'(s)| + |q'(s)| + |f'(s)|)\, ds < \eta_2$.

*H4)* $\displaystyle\int_0^{+\infty} |e(t)|\, dt < \eta_3$.

*Then any solution* $x(t)$ *of (9) and its derivatives* $x'(t), x''(t)$ *and* $x'''(t)$ *are bounded and satisfy*

$$\int_0^{\infty} \big(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\big) ds < \infty.$$

**Remark 2.1** Equation (9) can be rewritten as

$$x''''(t) + a(t)p(x)x''' + \varphi_1(t, x, x')x'' + \varphi_2(t, x, x')x' + d(t)h(x) = e(t),$$

where

$$\varphi_1(t, x, x') = b(t)q(x) + \frac{1}{2}a(t)p'(x)x', \quad and \quad \varphi_2(t, x, x') = b(t)q'(x)x' + c(t)f(x).$$

If we apply Tunç theorem [39] to show that every solution x(t) of (9) is bounded, we must take $\psi(x, x', x'') = \varphi_1(t, x, x')x''$ and $g(x, x') = \varphi_2(t, x, x')x'$ then the boundedness of $\frac{\psi(x,y,z)}{z}$ and $\frac{g(x,y)}{y}$ is needed. However in our theorem this latter condition is not required since we just need to deal with the boundedness of $a(t), b(t), p(x)$, and $q(x)$.

***Proof.*** **Boundedness of solutions**.
First we proof the boundedness of solutions. The proof of this theorem depends on properties of the continuously differentiable function $W = W(t, x, y, z, w)$ defined as

$$W = e^{-\frac{1}{\eta}\int_0^t \gamma(s)\, ds} V, \tag{11}$$

where

$$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|,$$
$$\theta_3(t) = f'(x(t))\, x'(t)$$

and

$$\begin{aligned} 2V \;=\; & 2\beta d(t) H(x) + c(t) f(x) y^2 + \alpha b(t) q(x) z^2 + a(t) p(x) z^2 + 2\beta a(t) p(x) yz \\ & + [\beta b(t) q(x) - \alpha h_0 d(t)] y^2 - \beta z^2 + \alpha w^2 + 2d(t) h(x) y + 2\alpha d(t) h(x) z \\ & + 2\alpha c(t) f(x) yz + 2\beta yw + 2zw, \end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \dfrac{1}{a_0 m} + \epsilon$ , $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon$, $\epsilon$, and $\eta$ are positive constants to be determined later in the proof. We rewrite $2V$ as

$$\begin{aligned} 2V = & a(t) p(x) \left[ \frac{w}{a(t) p(x)} + z + \beta y \right]^2 + c(t) f(x) \left[ \frac{d(t) h(x)}{c(t) f(x)} + y + \alpha z \right]^2 \\ & + \frac{d^2(t) h^2(x)}{c(t) f(x)} + 2\epsilon d(t) H(x) + V_1 + V_2 + V_3, \end{aligned}$$

where

$$\begin{aligned} V_1 &= 2d(t) \int_0^x h(s) \left[ \frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t) f(x)} h'(s) \right] ds, \\ V_2 &= \left[ \alpha b(t) q(x) - \beta - \alpha^2 c(t) f(x) \right] z^2, \\ V_3 &= \left[ \beta b(t) q(x) - \alpha h_0 d(t) - \beta^2 a(t) p(x) \right] y^2 + \left[ \alpha - \frac{1}{a(t) p(x)} \right] w^2. \end{aligned}$$

Now, we will prove that $V$ is positive definite. Take

$$\epsilon < \min \left\{ \frac{1}{a_0 m} \;,\; \frac{d_1 h_0}{c_0 m} \;,\; \frac{b_0 q_0 - \delta_1}{M(a_1 + c_1)} \right\}, \tag{12}$$

then
$$\frac{1}{a_0m}<\alpha<\frac{2}{a_0m},\qquad \frac{d_1h_0}{c_0m}<\beta<2\frac{d_1h_0}{c_0m}. \tag{13}$$
Using conditions (i)-(iii), (H1), (H2) and inequalities (12), (13) we get
$$\begin{aligned}
V_1 &\geq 4d\left(t\right)\frac{d_1}{c_0m}\int_0^x h\left(s\right)\left[\frac{h_0}{2}-h'\left(s\right)\right]ds\geq 0,\\
V_2 &= \Big(\alpha\Big(b\left(t\right)q(x)-\beta a\left(t\right)-\alpha c\left(t\right)f\left(x\right)\Big)+\beta\big(\alpha a\left(t\right)-1\big)\Big)z^2\\
&\geq \alpha\Big(b_0q_0-\frac{d_1h_0a_1}{c_0m}-\frac{c_1M}{a_0m}-\epsilon(a_1+c_1M)\Big)z^2+\beta\big(\frac{1}{m}-1\big)z^2\\
&\geq \alpha\big(b_0q_0-\delta_1-\epsilon M(a_1+c_1)\big)z^2\geq 0,
\end{aligned}$$

and
$$\begin{aligned}
V_3 &\geq \beta\Big(b_0q_0-\frac{\alpha}{\beta}h_0d_1-\beta a_1M\Big)y^2+\Big(\alpha-\frac{1}{a_0m}\Big)w^2\\
&\geq \beta\Big(b_0q_0-\frac{c_0}{a_0}-a_1\frac{d_1h_0M}{c_0m}-\epsilon(c_0m+a_1M)\Big)y^2+\epsilon w^2\\
&\geq \beta\big(b_0q_0-\delta_1-\epsilon M(c_1+a_1)\big)y^2+\epsilon w^2\geq 0.
\end{aligned}$$

Hence, it is evident from the terms contained in the last inequalities, that there exists positive constant $D_0$ such that
$$2V\geq D_0\left(y^2+z^2+w^2+H(x)\right). \tag{14}$$
By Lemma 2.1 and conditions (iii) and (H1) it follows that there is a positive constant $D_1$ such that
$$2V\geq D_1\left(x^2+y^2+z^2+w^2\right). \tag{15}$$
Thus $V$ is positive definite. From (i)-(iii), it is not difficult to see that there is a positive constant $U_1$ such that
$$V\leq U_1\left(x^2+y^2+z^2+w^2\right).$$
By (H3), we have
$$\begin{aligned}
\int_0^t\big(|\theta_1(s)|+|\theta_2(s)|+|\theta_3(s)|\big)ds &= \int_{\alpha_1(t)}^{\alpha_2(t)}\big(|p'(u)|+|q'(u)|+|f'(u)|\big)du\\
&\leq \int_{-\infty}^{+\infty}\big(|p'(u)|+|q'(u)|+|f'(u)|\big)du<\eta_2<\infty,
\end{aligned} \tag{16}$$
where $\alpha_1(t)=\min\{x(0),x(t)\}$, and $\alpha_2(t)=\max\{x(0),x(t)\}$. From inequalities (11), (15), and (16), it follows that
$$W\geq D_2(x^2+y^2+z^2+w^2), \tag{17}$$
where $D_2=\dfrac{D_1}{2}e^{-\frac{\eta_1+\eta_2}{\eta}}$. Also, it is easy to see that there is a positive constant $U_2$ such that
$$W\leq U_2(x^2+y^2+z^2+w^2), \tag{18}$$

for all $x, y, z$ and $w$, and all $t \geq 0$.

Next we show that $\dot{W}$ is negative definite function. The derivative of the function V, along any solution $(x(t), y(t), z(t), w(t))$ of system (10), with respect to t is after simplifying

$$2\dot{V}_{(10)} = -2\epsilon c(t) f(x) y^2 + V_4 + V_5 + V_6 + V_7 + 2(\beta y + z + \alpha w)e(t) + 2\frac{\partial V}{\partial t},$$

where

$$\begin{aligned}
V_4 =& -2\left(\frac{d_1 h_0}{c_0 m} c(t) f(x) - d(t) h'(x)\right) y^2 - 2\alpha d(t)\big(h_0 - h'(x)\big) yz, \\
V_5 =& -2\big(b(t) q(x) - \alpha c(t) f(x) - \beta a(t) p(x)\big) z^2, \\
V_6 =& -2\big(\alpha a(t) p(x) - 1\big) w^2, \\
V_7 =& -a(t)\theta_1\Big(z^2 + 2\alpha z w\Big) - b(t)\theta_2\Big(\alpha z^2 + 2\alpha z w + \beta y^2 + 2yz\Big) \\
&+ c(t)\theta_3\Big(y^2 + 2\alpha y z\Big).
\end{aligned}$$

By conditions (i), (ii), (H1), (H2) and inequality (12), (13) we obtain the following

$$\begin{aligned}
V_4 &\leq -2\left[d(t) h_0 - d(t) h'(x)\right] y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right] yz \\
&\leq -2d(t)\left[h_0 - h'(x)\right] y^2 - 2\alpha d(t)\left[h_0 - h'(x)\right] yz \\
&\leq -2d(t)\left[h_0 - h'(x)\right]\left[\Big(y + \frac{\alpha}{2} z\Big)^2 - \Big(\frac{\alpha}{2} z\Big)^2\right] \\
&\leq \frac{\alpha^2}{2} d(t)\left[h_0 - h'(x)\right] z^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V_4 + V_5 &\leq -2\left[b(t) q(x) - \alpha c(t) f(x) - \beta a(t) p(x) - \frac{\alpha^2}{4} d(t)\left[h_0 - h'(x)\right]\right] z^2 \\
&\leq -2\left[b_0 q_0 - (\frac{1}{a_0 m} + \epsilon) c_1 M - (\frac{d_1 h_0}{c_0 m} + \epsilon) a_1 M - \frac{\alpha^2}{4}\left(a_0 m \delta_0\right)\right] z^2 \\
&\leq -2\left[b_0 q_0 - \frac{M}{a_0 m} c_1 - \frac{d_1 h_0 a_1 M}{c_0 m} - \frac{\delta_0}{a_0 m} - \epsilon M\left(a_1 + c_1\right)\right] z^2 \\
&\leq -2\left[b_0 q_0 - \delta_1 - \epsilon M\left(a_1 + c_1\right)\right] z^2 \leq 0,
\end{aligned}$$

and

$$V_6 \leq -2\left[\alpha a_0 m - 1\right] w^2 = -2\epsilon w^2 \leq 0.$$

Hence, there exists a positive constant $D_3$ such that

$$-2\epsilon c(t) f(x) y^2 + V_4 + V_5 + V_6 \leq -2D_3\left(y^2 + z^2 + w^2\right).$$

From (14), and the Cauchy Schwartz inequality, we get

$$\begin{aligned}
V_7 &\leq a(t)|\theta_1|\Big(z^2 + \alpha(z^2 + w^2)\Big) + b(t)|\theta_2|\Big(\alpha z^2 + \alpha(z^2 + w^2) + \beta y^2 + (y^2 + z^2)\Big) \\
&\quad + c(t)|\theta_3|\Big(y^2 + \alpha(y^2 + z^2)\Big) \\
&\leq \lambda_1(|\theta_1| + |\theta_2| + |\theta_3|)\left(y^2 + z^2 + w^2 + H(x)\right) \\
&\leq 2\frac{\lambda_1}{D_0}\Big(|\theta_1| + |\theta_2| + |\theta_3|\Big) V,
\end{aligned}$$

where $\lambda_1 = \max\big\{a_1(1+\alpha), b_1(1+2\alpha+\beta), c_1(1+\alpha)\big\}$. We get also

$$\begin{aligned} 2\frac{\partial V}{\partial t} &= d'(t)\left[2\beta H(x) - \alpha h_0 y^2 + 2h(x)\,y + 2\alpha h(x)\,z\right] \\ &\quad + c'(t)\left[f(x)y^2 + 2\alpha f(x)yz\right] + b'(t)\left[\alpha q(x)z^2 + \beta q(x)y^2\right] \\ &\quad + a'(t)\left[p(x)z^2 + 2\beta p(x)yz\right]. \end{aligned}$$

Using condition (H1) and Lemma 2.1, we obtain

$$h^2(x) \le h_0 H(x),$$

consequently,

$$\begin{aligned} 2\left|\frac{\partial V}{\partial t}\right| &\le |d'(t)|\left[2\beta H(x) + \alpha h_0 y^2 + \left(h^2(x) + y^2\right) + \alpha\left(h^2(x) + z^2\right)\right] \\ &\quad + |c'(t)|\left[y^2 + \alpha\left(y^2+z^2\right)\right] + |b'(t)|\left[\alpha z^2 + \beta y^2\right] \\ &\quad + |a'(t)|\left[z^2 + 2\beta\left(y^2+z^2\right)\right] \\ &\le \lambda_2\left[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right]\left(y^2+z^2+w^2+H(x)\right) \\ &\le 2\frac{\lambda_2}{D_0}\left[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right]V, \end{aligned}$$

such that $\lambda_2 = \max\big\{2\beta + \alpha h_0 + h_0, \alpha h_0 + 1, \alpha + 1\big\}$. By taking $\dfrac{1}{\eta} = \dfrac{1}{D_0}\max\big\{\lambda_1, \lambda_2\big\}$, we obtain

$$\begin{aligned} \dot V_{(10)} \le &- D_3(y^2+z^2+w^2) + \frac{1}{\eta}\Big(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1| + |\theta_2| + |\theta_3|\Big)V \\ &+ \big(\beta y + z + \alpha w\big)e(t). \end{aligned} \tag{19}$$

From (iv), (H3), (16), (17), (19) and the Cauchy Schwartz inequality, we get

$$\begin{aligned} \dot W_{(10)} &= \left(\dot V_{(10)} - \frac{1}{\eta}\gamma(t)\,V\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le \Big(-D_3\left(y^2+z^2+w^2\right) + \big(\beta y + z + \alpha w\big)e(t)\Big)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \qquad (20) \\ &\le \left(\beta|y| + |z| + \alpha|w|\right)|e(t)| \\ &\le D_4\left(|y| + |z| + |w|\right)|e(t)| \\ &\le D_4\left(3 + y^2 + z^2 + w^2\right)|e(t)| \\ &\le D_4\left(3 + \frac{1}{D_2}W\right)|e(t)| \\ &\le 3D_4|e(t)| + \frac{D_4}{D_2}W|e(t)|, \qquad (21) \end{aligned}$$

where $D_4 = \max\{\alpha, \beta, 1\}$. Integrating (21) from 0 to $t$, and using the condition (H4) and the Gronwall inequality, we obtain

$$\begin{aligned}
W(t,x,y,z,w) &\leq W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3 \\
&\quad +\frac{D_4}{D_2}\int_0^t W\big(s,x(s),y(s),z(s),w(s)\big)|e(s)|ds \\
&\leq \Big(W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds} \\
&\leq \Big(W\big(0,x(0),y(0),z(0),w(0)\big)+3D_4\eta_3\Big)e^{\frac{D_4}{D_2}\eta_3} = K_1<\infty. \quad (22)
\end{aligned}$$

In view of inequalities (17) and (22), we get

$$(x^2+y^2+z^2+w^2)\leq \frac{1}{D_2}W\leq K_2, \quad (23)$$

where $K_2=\dfrac{K_1}{D_2}$. Clearly (23) implies that

$$|x(t)|\leq\sqrt{K_2},\ |y(t)|\leq\sqrt{K_2},\ |z(t)|\leq\sqrt{K_2},\ |w(t)|\leq\sqrt{K_2}\quad \text{for all}\quad t\geq 0.$$

Hence,

$$|x(t)|\leq\sqrt{K_2},\ |x'(t)|\leq\sqrt{K_2},\ |x''(t)|\leq\sqrt{K_2},\ |x'''(t)|\leq\sqrt{K_2}\quad \text{for all}\quad t\geq 0. \quad (24)$$

**Square integrable solutions.**
Now, we proof the square integrability of solutions and their derivatives. We define $F_t=F(t,x(t),y(t),z(t),w(t))$ as

$$F_t=W+\rho\int_0^t\big(y^2(s)+z^2(s)+w^2(s)\big)\,ds,$$

where $\rho>0$. It is easy to see that $F_t$ is positive definite, since $W=W(t,x,y,z,w)$ is already positive definite. Using the following estimate

$$e^{-\frac{\eta_1+\eta_2}{\eta}}\leq e^{-\frac{1}{\eta}\int_0^t\gamma(s)\,ds}\leq 1,$$

by (20) we have the following

$$\begin{aligned}
\dot{F}_{t(10)} &\leq -D_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1+\eta_2}{\eta}} \quad (25)\\
&\quad +D_4\Big(|y(t)|+|z(t)|+|w(t)|\Big)|e(t)| \\
&\quad +\rho\Big(y^2(t)+z^2(t)+w^2(t)\Big).
\end{aligned}$$

By choosing $\rho = D_3 e^{-\dfrac{\eta_1+\eta_2}{\eta}}$ we obtain

$$\begin{aligned}\dot{F}_{t(10)} &\leq D_4\Big(3 + y^2(t) + z^2(t) + w^2(t)\Big)|e(t)| \\ &\leq D_4\Big(3 + \frac{1}{D_2}W\Big)|e(t)| \\ &\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|. \end{aligned} \tag{26}$$

Integrating the last inequality (26) from 0 to $t$, and using again the Gronwall inequality and the condition (H4), we get

$$\begin{aligned}F_t &\leq F_0 + 3D_4\eta_3 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds \\ &\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\int_0^t |e(s)|ds} \\ &\leq \Big(F_0 + 3D_4\eta_3\Big)e^{\dfrac{D_4}{D_2}\eta_3} = K_3 < \infty. \end{aligned} \tag{27}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_3 \quad , \quad \int_0^\infty z^2(s) < K_3 \text{ and } \int_0^\infty w^2(s)ds < K_3,$$

which implies that

$$\int_0^\infty x'^2(s)ds \leq K_3 \quad , \quad \int_0^\infty x''^2(s)ds \leq K_3 \quad , \quad \int_0^\infty x'''^2(s)ds \leq K_3. \tag{28}$$

Next, multiply (9) by $x(t)$ and integrate by parts from 0 to $t$, we obtain

$$\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + L_0, \tag{29}$$

where

$$\begin{aligned}
I_1(t) &= x'(t)x''(t) - x(t)x'''(t) - \int_0^t x''^2(s)ds, \\
I_2(t) &= -a(t)p(x(t))x(t)x''(t) + \int_0^t a'(s)p(x(s))x(s)x''(s)ds \\
&\quad + \int_0^t a(s)p(x(s))x'(s)x''(s)ds, \\
I_3(t) &= -b(t)q(x(t))x(t)x'(t) + \int_0^t b'(s)q(x(s))x(s)x'(s)ds + \int_0^t b(s)q(x(s))x'^2(s)ds, \\
I_4(t) &= -\frac{1}{2}c(t)f(x(t))x^2(t) + \frac{1}{2}\int_0^t c'(s)f(x(s))x^2(s)ds + \frac{1}{2}\int_0^t c(s)f'(x(s))x'(s)x^2(s)ds, \\
I_5(t) &= \int_0^t e(s)x(s)ds,
\end{aligned}$$

and

$$\begin{aligned} L_0 &= x(0)x'''(0) - x'(0)x''(0) + a(0)p(x(0))x(0)x''(0) \\ &\quad + b(0)q(x(0))x(0)x'(0) + \frac{1}{2}c(0)f(x(0))x^2(0). \end{aligned}$$

From (24), (28) and the conditions (i), (ii), (iv), (H3) and (H4), we have

$$\begin{aligned} I_1(t) &\leq 2K_2 + \int_0^t x''^2(s)ds, \\ I_2(t) &\leq a_1MK_2 + MK_2\int_0^t |a'(s)|ds + a_1M\int_0^t x'(s)x''(s)ds, \\ &\leq \frac{3}{2}a_1MK_2 + MK_2\int_0^t |a'(s)|ds, \\ I_3(t) &\leq b_1q_1K_2 + q_1K_2\int_0^t |b'(s)|ds + b_1q_1\int_0^t x'^2(s)ds, \\ I_4(t) &\leq \frac{1}{2}c_1MK_2 + \frac{1}{2}MK_2\int_0^t |c'(s)|ds, + \frac{1}{2}c_1K_2^{\frac{3}{2}}\int_0^t |f'(s)|ds, \\ I_5(t) &\leq \sqrt{K_2}\int_0^t |e(s)|ds. \end{aligned}$$

It follows that

$$\lim_{t\to+\infty} I_1(t) \leq 2K_2 + K_3 = L_1, \quad \lim_{t\to+\infty} I_2(t) \leq \frac{3}{2}a_1MK_2 + MK_2\eta_1 = L_2,$$
$$\lim_{t\to+\infty} I_3(t) \leq b_1q_1K_2 + q_1K_2\eta_1 + b_1q_1K_3 = L_3,$$
$$\lim_{t\to+\infty} I_4(t) \leq \frac{1}{2}c_1MK_2 + \frac{1}{2}MK_2\eta_1 + \frac{1}{2}c_1K_2^{\frac{3}{2}}\eta_2 = L_4, \quad \text{and} \quad \lim_{t\to+\infty} I_5(t) \leq \sqrt{K_2}\eta_3 = L_5.$$

Thus,

$$\lim_{t\to+\infty} \big(I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)\big) \leq \sum_{i=1}^{5} L_i < \infty. \tag{30}$$

Consequently, (29), (30) and condition iii) give

$$\int_0^\infty x^2(s)ds \leq \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds \leq \frac{1}{d_0\delta}\sum_{i=0}^{5} L_i < \infty,$$

which completes the proof of the theorem.

**Remark 2.2** If $e(t) = 0$, similarly to the above proof, the inequality (3.10) becomes

$$\begin{aligned} \dot{W}_{(10)} &= \left(\dot{V}_{(10)} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\leq -D_3\left(y^2 + z^2 + w^2\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\leq -\mu\left(y^2 + z^2 + w^2\right), \end{aligned}$$

where $\mu = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. It can also be observed that the only solution of system (10) for which $\dot{W}_{(10)}(t,x,y,z,w) = 0$ is the solution $x = y = z = w = 0$. The above discussion guarantees that the trivial solution of equation (9) is uniformly asymptotically stable, and the same conclusion as in the proof of Theorem 2.1 can be drawn for square integrability of solutions of equation (9).

## 3 Example

We consider the following fourth order non-autonomous differential equation

$$\begin{aligned}
&x'''' + \left(e^{-t}\sin t + 2\right)\left(\left(\frac{x + 4e^{x} + 4e^{-x}}{4\left(e^{x} + e^{-x}\right)}\right)x''\right)' \\
&+ \left(\frac{\cos t + 7t^2 + 7}{1 + t^2}\right)\left(\left(\frac{\sin x + 6e^{x} + 6e^{-x}}{e^{x} + e^{-x}}\right)x'\right)' \\
&+ \left(e^{-2t}\sin^3 t + 2\right)\left(\frac{x\cos x + 5x^4 + 5}{5\left(1 + x^4\right)}\right)x' \\
&+ \left(\frac{\cos^2 t + t^2 + 1}{10\left(1 + t^2\right)}\right)\left(\frac{x}{x^2 + 1}\right) = \frac{2\sin t}{t^2 + 1},
\end{aligned} \tag{31}$$

by taking
$p(x) = \dfrac{x + 4e^{x} + 4e^{-x}}{4\left(e^{x} + e^{-x}\right)}$ , $q(x) = \dfrac{\sin x + 3e^{x} + 3e^{-x}}{e^{x} + e^{-x}}$ , $f(x) = \dfrac{x\cos x + 5x^4 + 5}{5\left(1 + x^4\right)}$ ,
$h(x) = \dfrac{x}{x^2 + 1}$ , $a(t) = e^{-t}\sin t + 2$ , $b(t) = \dfrac{\cos t + 4t^2 + 4}{1 + t^2}$ ,
$c(t) = e^{-2t}\sin^3 t + 2$ , $d(t) = \dfrac{\cos^2 t + t^2 + 1}{10\left(1 + t^2\right)}$ and $e(t) = \dfrac{2\sin t}{t^2 + 1}$. It follows easily that
$m = \dfrac{9}{10}$ , $M = \dfrac{11}{10}$ , $q_0 = \dfrac{5}{2}$ , $q_1 = \dfrac{7}{2}$ , $h_0 = \dfrac{11}{5}$ , $\delta_0 = \dfrac{3}{2}$ , $a_0 = 1$ , $a_1 = 3$ , $b_0 = 3$ ,
$b_1 = 5$ $c_0 = 1$ , $c_1 = 3$ , $d_0 = \dfrac{1}{10}$ , and $d_1 = \dfrac{1}{5}$. We find $h_0 - \dfrac{a_0 m \delta_0}{d_1} = -4$,
$55 \leq h'(x) \leq \dfrac{h_0}{2} = 1.1$ and $b_0 q_0 = \dfrac{15}{2} > \dfrac{6\,946\,7}{10000} = \dfrac{d_1 h_0 a_1 M}{c_0 m} + \dfrac{c_1 M + \delta_0}{c_0 m} = \delta_1$.

We have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left|p'(x)\right| dx &= \frac{1}{4}\int_{-\infty}^{+\infty} \left|\frac{1}{e^{x} + e^{-x}} + x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right| dx \\
&\leq \frac{1}{4}\int_{-\infty}^{0} \left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right) dx \\
&\quad + \frac{1}{4}\int_{0}^{+\infty} \left(\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{\left(e^{x} + e^{-x}\right)^2}\right) dx = \frac{\pi}{4},
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left|q'(x)\right| dx &= \int_{-\infty}^{+\infty} \left|\frac{\left(e^{x} + e^{-x}\right)\cos x - \left(e^{x} - e^{-x}\right)\sin x}{\left(e^{x} + e^{-x}\right)^2}\right| dx \\
&\leq \int_{-\infty}^{+\infty} \left(\frac{1}{e^{x} + e^{-x}} + \frac{x}{\left(e^{x} + e^{-x}\right)^2}\left(e^{x} - e^{-x}\right)\right) dx = \pi, \text{ and}
\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|f'(x)\right|dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{(\cos x-x\sin x)\left(x^4+1\right)-4x^4\cos x}{\left(x^4+1\right)^2}\right|dx\\ &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\cos x}{x^4+1}-4x^4\frac{\cos x}{\left(x^4+1\right)^2}-x\frac{\sin x}{x^4+1}\right|dx\\ &\leq \frac{1}{5}\int_{-\infty}^{+\infty}\left(\frac{5}{x^4+1}+\frac{x^2}{x^4+1}\right)dx=\frac{6}{5}\sqrt{2}\pi.\end{aligned}$$

Consequently,

$$\int_{-\infty}^{+\infty}\left(\left|p'(s)\right|+\left|q'(s)\right|+\left|f'(s)\right|\right)ds<\infty.$$

A simple computation gives

$$\int_0^{+\infty}|e(t)|\,dt=\int_0^{+\infty}\left|\frac{2\sin t}{t^2+1}\right|dt\leq\int_0^{+\infty}\frac{2}{t^2+1}dt=\pi,$$

$$\begin{aligned}\int_0^{+\infty}\left|a'(t)\right|dt &= \int_0^{+\infty}\left|(\cos t)\,e^{-t}-(\sin t)\,e^{-t}\right|dt\leq\int_0^{+\infty}2e^{-t}dt=2,\\ \int_0^{+\infty}\left|b'(t)\right|dt &= \int_0^{+\infty}\left|-\frac{\sin t}{t^2+1}-2t\frac{\cos t}{\left(t^2+1\right)^2}\right|dt\leq\int_0^{+\infty}\left(\frac{1}{t^2+1}+\frac{2|t|}{\left(t^2+1\right)^2}\right)dt\\ &\leq \int_0^{+\infty}\left(\frac{1}{t^2+1}+\frac{t^2+1}{\left(t^2+1\right)^2}\right)dt=\int_0^{+\infty}\frac{2}{t^2+1}dt=\pi,\\ \int_0^{+\infty}\left|c'(t)\right|dt &= \int_0^{+\infty}\left|3\left(\cos t\sin^2 t\right)e^{-2t}-2\left(\sin^3 t\right)e^{-2t}\right|dt\leq\int_0^{+\infty}5e^{-2t}dt=\frac{5}{2},\end{aligned}$$

and

$$\begin{aligned}\int_0^{+\infty}\left|d'(t)\right|dt &= \int_0^{+\infty}\left|-2(\cos t)\frac{\sin t}{t^2+1}-2t\frac{\cos^2 t}{\left(t^2+1\right)^2}\right|dt\\ &\leq \int_0^{+\infty}\left(\frac{2}{t^2+1}+\frac{2|t|}{\left(t^2+1\right)^2}\right)dt\leq\int_0^{+\infty}\frac{3}{t^2+1}dt=\frac{3\pi}{2}.\end{aligned}$$

Therefore,

$$\int_0^{+\infty}\left(\left|a'(t)\right|+\left|b'(t)\right|+\left|c'(t)\right|+\left|d'(t)\right|\right)dt<+\infty.$$

Thus all the assumptions of Theorem 2.1 hold, so solutions of (31) are bounded and square integrable.

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# Reduced Order Bilinear Time Invariant System by Means of Error Transfer Function Least Upper Bounds

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**Abstract:** The order selection problem of the reduced bilinear time invariant systems is considered in this paper. The r-*th* order reduced bilinear time invariant systems are chosen by using the least upper bound of the difference bilinear system in the proposed $H_2$-norm. The $H_2$-norm of the difference bilinear system is computed by the $H_2$-norm of the error transfer function between the full order and the reduced order of a bilinear time invariant system. The reduced bilinear systems are obtained by using the balanced truncation and the singular perturbation methods. The $H_2$-norm of the difference bilinear systems is a function of controllability gramian or observability gramian of the difference bilinear system. The simulation results in the example confirm the proposed method for obtaining the reduced bilinear system which is similar to the full order bilinear system.

**Keywords:** *bilinear systems; controllability and observability gramians; $H_2$-norm; reduced order bilinear systems; balanced truncation; singular perturbation.*

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## 1 Introduction

In this paper, a criteria for selecting order of a reduced order model of bilinear time invariant systems based on the value alteration of the least upper bounds of a transfer function of difference bilinear systems in the proposed $H_2$-norm is considered. The order selection based on the value alteration of the singular Hankel values, see [3], is not apparent because the decision is influenced by knowledge of the decision makers. The measurement of the model reduction, which is calculated by using the $H_2$-norm is able to characterize the virtue of the reduced order model. The definition of the $H_2$-norm based on transfer function of the bilinear time invariant system which includes the controllability gramian or the observability gramian is then proposed.

The least upper bounds of the error transfer function between the full order and the reduced order model of the bilinear systems in the $H_2$-norm become a tool for modern controller design. The least upper bounds of the error transfer function between full order and reduced order model for the linear systems in the $H_2$-norm have been discussed in [10] and [13]. Therefore, the least upper bounds of the bilinear time invariant systems discussed in [23, 24] are important in model order reduction.

The reduced order bilinear systems are obtained by using the balanced truncation [3] and the singular perturbation methods [22]. Two methods are used because they preserve the dominant state of the original bilinear systems which are based on the controllability or observability gramians. These methods result in the reduced bilinear systems which are nearly optimal for a given least upper bound. The comparison of the least upper bounds of the difference bilinear system using two methods is investigated in the paper. Another method, for example, the moment-matching method is very efficient and numerically robust, but the reduced bilinear systems are not guaranteed as an optimal reduced bilinear system.

In the high order of the bilinear systems, the bottleneck of the balanced truncation and singular perturbation methods can occur in the calculation of controllability or observability gramians. The controllability or observability gramians can be approximated in the frequency domain to reduce the computational cost. Therefore, it is suggested to use the Poor man's truncated balanced realization of the bilinear systems. This approach uses frequency-weighted finite summation to approximate the infinite integration. This method approximates the gramian in the frequency domain without solving the Lyapunov equations [20]. The reduced bilinear systems will be accurate when the bilinear systems have finite bandwidth inputs.

A class of nonlinear system which is linear in inputs and linear in states with a nonlinearity in a product of states and inputs is known as bilinear systems [3]. Mathematical modeling and control design of bilinear systems were discussed in [1] and [8]. The identification of time-invariant bilinear system models in the error-in-variables framework has been discussed in [16]. The error-in-variables framework is dedicated to problem of dynamic system identification in the presence of noise corrupting both input and output measurements. The bilinear control systems have been discussed by using the Lie groups approach in [9] and [19], whereas in [14] it has been discussed how to stabilize the homogeneous bilinear system by sliding mode control. The bilinear systems are naturally found in science and technology problems, for example induction motor drives in [1], paper making machines in [1], quantum mechanics in [19], power systems in [3], suspension systems in [26], circuit electricity in [17], and immunity problems in [18].

The control design problem of a bilinear system is to seek a controller that stabilizes and satisfies a given norm of the closed loop of the bilinear system. Many problems in science and technology are usually formulated in terms of a high order bilinear system. In fact, the order of robust control design is always higher than the order of the system so a reduced-order controller is necessary for application in real problems. Hence, model order reduction and reduced order controller are an important part in the high order control system design.

Model reduction for linear time invariant (LTI) and linear time varying (LTV) systems has been discussed in [2], whereas the model reduction for bilinear systems has been developed by many researchers in [3–7, 12, 15, 21, 22, 25, 27]. Model order reduction methods for nonlinear model have been discussed in [11]. Balanced truncation [3] and singular perturbation [22] methods are used to obtain the reduced order bilinear time invariant systems. In the balanced truncation method, the original bilinear system is transformed to the balanced system. The characterizations of the original bilinear system and the balanced system are the same. In the singular perturbation method, the original bilinear system is transformed into a balanced system which is then divided into two subsystems, i.e. slow and fast mode systems. After that, the reduced bilinear systems are obtained by defining that the velocity of fast mode is zero.

The paper is organized as follows. Section 2 presents the least upper bounds of the transfer function of the bilinear time invariant systems in the $H_2$-norm. Section 3 reviews the balanced truncation and singular perturbation methods for bilinear systems. Section 4 gives the main result that is the least upper bounds of the difference bilinear system. In Section 5, the procedure of selecting the reduced order bilinear system is presented. Section 6 shows the simulation results which illustrate the performance of the proposed algorithm and Section 7 gives conclusions.

## 2 The Least Upper Bounds of Bilinear Systems

Consider a bilinear time invariant system $\mathfrak{B}$ characterized by the following differential equations

$$\mathfrak{B}: \begin{array}{l} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i u_i(t)x(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{array} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the control input, $u_i(t)$ is the $i-$th element of $u(t)$, $y \in \Re^q$ is the output system, $A \in \Re^{nxn}, N_i \in \Re^{nxn}, i = 1, 2, \ldots, m, B \in \Re^{nxm}$, $C \in \Re^{qxn}$, and $D \in \Re^{qxm}$. Suppose the bilinear system (1) is locally stable, $(A, B)$ is controllable, and $(A, C)$ is observable. The bilinear system is called locally stable if the real parts of all eigenvalues of $A$ are negative. The relation of inputs and outputs of the bilinear system (1) can be expressed by the following Volterra series [18]

$$y(t) = \sum_{i=1}^{\infty} \int_{i=0}^{t} \int_{i=0}^{t_1} \cdots \int_{i=0}^{t_{k-1}} \sum_{i1,i2,\ldots,i_k=1}^{m} h_k^{(i_1,i_2,\ldots,i_k)}(t_1, t_2, ..., t_k)$$

$$u_{i1}(t - t_k) \ldots u_{ik}(t - \sum_{k=1}^{i} t_k) dt_1 \ldots dt_k.$$

The regular Volterra kernel $h_k$ can be expressed as [18]

$$h_k^{(i_1,i_2,\ldots,i_k)}(t_1, t_2, ..., t_k) = Ce^{At_k} N_{i1} e^{At_{k-1}} \ldots N_{ik-1} e^{At_1} b_{i_k},$$

where $b_{i_k}$ denotes the $i_k-$th column of $B$ matrix. For the sake of simplicity, $h_k^{(i1,...,ik)}(t_1, t_2, ..., t_k)$ is denoted by $h_k$. The notation $h_k^T$ denotes transpose of the $h_k$.

To deal with the least upper bounds problem, the paper treats the controllability and the observability gramians defined in [3] as follows

**Definition 2.1** The controllability gramian matrix $P$ is defined by

$$P = \sum_{i=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P_i P_i^T dt_1 \dots dt_i,$$

where $P_1(t_1) = e^{At_1}B$, and $P_i(t_1, \dots, t_i) = e^{At_i} \left[ \begin{array}{ccc} N_1 P_{i-1} & \dots & N_m P_{i-1} \end{array} \right], i = 2, 3, \dots$. Analogously, observability gramian matrix $Q$ is defined by

$$Q = \sum_{i=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} Q_i^T Q_i dt_1 \dots dt_i,$$

where $Q_1(t_1) = Ce^{At_1}$, and $Q_i(t_1, \dots, t_i) = \left[ \begin{array}{c} Q_{i-1}N_1 \\ Q_{i-1}N_2 \\ \dots \\ Q_{i-1}N_m \end{array} \right] e^{At_i}, i = 2, 3, \dots$.

The existence and properties of the controllability gramian $P$ and the observability gramian $Q$ which satisfy the generalized Lyapunov equations are presented in [27]. The generalized Lyapunov equations are given by the following equations

$$AP + PA^T + \sum_{i=1}^{m} N_i P N_i^T + BB^T = 0, \tag{2}$$

$$A^T Q + QA + \sum_{i=1}^{m} N_i^T Q N_i + C^T C = 0. \tag{3}$$

If the equation (2) is taken *vec* on two sides then

$$\left( A \otimes I + I \otimes A + \sum_{i=1}^{m} N_i \otimes N_i \right) vec(P) = -vec(BB^T).$$

Therefore, if $A \otimes I + I \otimes A + \sum_{i=1}^{m} N_i \otimes N_i$ is a nonsingular matrix, then a single solution $P$ will be found. If $P$ is a nonnegative matrix then $P$ is called the controllability gramian. The observability gramian $Q$ is obtained by using the similar manner and properties to the equation (3) [27].

Let us introduce a definition of $H_2$-norm of the bilinear system $\mathfrak{B}$ in [23, 24].

**Definition 2.2** Consider the bilinear system (1). The $H_2$-norm of the bilinear system $\mathfrak{B}$ is defined by

$$\|\mathfrak{B}\|_2 = \sqrt{\lambda_{max} \left( \sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \sum_{i1,\dots,ik=1}^{m} h_k h_k^T dt_1 ... dt_k \right)},$$

where $\lambda_{max}(.)$ denotes the maximum of (.) eigenvalues and $h_k$ is the regular Voltera kernel.

Definition 2.2 is an extended form of the Euclidian-induced norm of matrix $M$ which is equivalent to the square root of the maximum eigenvalue of $M^TM$ over a time interval of integration from $t=0$ to $t=\infty$. It is clear that $h_kh_k^T$ is a symmetry and a semi definite positive matrix because $h_k$ is $k-$variate impulse response. The following lemma is obtained from Definition 2.2.

**Lemma 2.1** *[23, 24] Suppose the bilinear system (1) is locally stable. If there exists the controllability gramian $P$ of bilinear system (1) then $\|\mathfrak{B}\|_2=\sqrt{\lambda_{max}(CPC^T)}$. If there exists the observability gramian $Q$ of bilinear system (1) then $\|\mathfrak{B}\|_2=\sqrt{\lambda_{max}(B^TQB)}$.*

***Proof.*** Suppose that

$$J_k^2=\int_0^\infty\int_0^\infty\dots\int_0^\infty\sum_{i1,\dots,ik=1}^{m}h_kh_k^Tdt_1\dots dt_k.$$

When $k=1$ then $J_1^2=\int_0^\infty\sum_{i1=1}^m Ce^{At_1}b_{i1}b_{i1}^Te^{A^Tt_1}C^Tdt_1=C\int_0^\infty P_1P_1^Tdt_1C^T$. When $k=2$ then $J_2^2=\int_0^\infty\int_0^\infty\sum_{i1=1}^m\phi\phi^Tdt_1dt_2=C\int_0^\infty\int_0^\infty P_2P_2^Tdt_1dt_2C^T$, where $\phi=Ce^{At_2}N_1e^{At_1}b_{i1}$, $b_{i1}$ denotes the $i1$-th column of the matrix $B$, and generally $J_k^2=C\int_0^\infty\dots\int_0^\infty P_kP_k^Tdt_1\dots dt_kC^T, i=2,3,\dots$. Therefore, the following result will be obtained by taking the sum from $k=1$ to infinite

$$\sum_{k=1}^\infty J_k^2=C\sum_{k=1}^\infty\int_0^\infty\dots\int_0^\infty P_kP_k^Tdt_1\dots dt_kC^T=CPC^T.$$

Hence, the $H_2$ norm can also be computed by using

$$\|\mathfrak{B}\|_2=\sqrt{\lambda_{max}\left(\sum_{k=1}^\infty J_k^2\right)}=\sqrt{\lambda_{max}(CPC^T)},$$

where $P$ is the controllability gramian of bilinear system (1). Similar reasoning holds for the second case. □

The least upper bounds of $H_2$-norm of the transfer function of the bilinear system are determined as a function of the controllability gramian (the observability gramian) of the bilinear system.

**Lemma 2.2** *[23, 24] Suppose the bilinear system (1) is locally stable. If there exists the controllability gramian $P$ of bilinear system (1) then $\|\mathfrak{B}\|_2<\sqrt{\lambda_{max}(P)}\sqrt{\lambda_{max}(C^TC)}$. If there exists the observability gramian $Q$ of bilinear system (1) then $\|\mathfrak{B}\|_2\le\sqrt{\lambda_{max}(Q)}\sqrt{\lambda_{max}(BB^T)}$.*

***Proof.*** We shall furnish the proof for the controllability gramian $P$, having the same arguments for the observability gramian $Q$. As the controllability gramian $P$ exists, then $P$ is a positive definite matrix. Furthermore, $C^TC$ is a positive semidefinite matrix. According to Lemma 2.1 and properties of the eigenvalues of positive semidefinite matrix, it holds that $\|\mathfrak{B}\|_2=\sqrt{\lambda_{max}\left(CPC^T\right)}\le\sqrt{\lambda_{max}\left(P\right)}\sqrt{\lambda_{max}\left(C^TC\right)}$. □

## 3 Balanced Truncation and Singular Perturbation Methods

According to [3], balanced realization of the bilinear system (1) can be obtained by applying the state space balancing transformation $x_b(t) = T^{-1}x(t)$ to (1). Hence, the new presentation will be obtained as follows

$$\mathfrak{B}_b : \begin{array}{l} \dot{x}_b(t) = A_b x_b(t) + \sum_{i=1}^{m} N_{bi} u_i(t) x_b(t) + B_b u(t), \\ y(t) = C_b x_b(t), \end{array} \tag{4}$$

where $A_b = T^{-1}AT, N_{bi} = T^{-1}N_iT, B_b = T^{-1}B, C_b = CT$. The controllability and the observability gramians of the balanced system are $P_b = T^{-1}PT^{-T}$ and $Q_b = T^TQT$. Furthermore, the system (4) is denoted by $(A_b, B_b, N_{bi}, C_b, D_d)$, $i = 1, ..., m$.

**Definition 3.1** *The system $(A_b, B_b, N_{bi}, C_b, D_b), i = 1, ..., m$ is called the balanced realization of the bilinear system (1) if*

$$P_b = Q_b = \Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n), \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0,$$

*where $P_b$ and $Q_b$ are the controllability gramian and the observability gramian, respectively. Furthermore, $\sigma_k = \sqrt{\lambda_k(P_bQ_b)}, k = 1, ..., n$ is called Hankel singular value of the balanced system, where $\lambda_k(P_bQ_b)$ denotes the k-th eigenvalue of the matrix $P_bQ_b$.*

The balanced system (4) can be partitioned as follows

$$\begin{bmatrix} \dot{x}_{b_1} \\ \dot{x}_{b_2} \end{bmatrix} = \begin{bmatrix} A_{b_{11}} & A_{b_{12}} \\ A_{b_{21}} & A_{b_{22}} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_{b_{11_i}} & N_{b_{12_i}} \\ N_{b_{21_i}} & N_{b_{22_i}} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix} u_i + \begin{bmatrix} B_{b_1} \\ B_{b_2} \end{bmatrix} u,$$

$$y = \begin{bmatrix} C_{b_1} & C_{b_2} \end{bmatrix} \begin{bmatrix} x_{b_1} \\ x_{b_2} \end{bmatrix},$$

where $\dot{x}_{b_1}$ is the velocity of slow mode and $\dot{x}_{b_2}$ is the velocity of fast mode. In the balanced truncation method, the system of the slow mode is selected as the reduced bilinear system. The system which is obtained by the balanced truncation method can preserve the stability, but this method gives high error at low frequencies. Let $\Sigma$ be partitioned as $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$, where $\Sigma_1 = diag[\sigma_1, \sigma_2, ..., \sigma_r]$ and $\Sigma_2 = diag[\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n]$. According to [3], the order selection of the slow mode is based on the ratio of Hankel singular values that is $\frac{\sigma_r}{\sigma_{r+1}} \gg 1$, then, the reduced bilinear system where order $r$ is chosen. Furthermore, the balanced truncation method for bilinear systems has been developed to the singular perturbation method for bilinear systems in [22]. Denote

$$K = A_{b_{12}} + \sum_{i=1}^{m} N_{b_{12_i}} u_i(t), L = A_{b_{22}} + \sum_{i=1}^{m} N_{b_{22_i}} u_i(t), M = A_{b_{21}} + \sum_{i=1}^{m} N_{b_{21_i}} u_i(t),$$

and assume that the velocity of the fast mode is zero, then
$x_{b_2}(t) = -L^{-1}Mx_{b_1}(t) - L^{-1}B_{b_2}u(t)$. Therefore, the reduced bilinear system is given by

$$\dot{x}_{b_1}(t) = (A_{b_{11}} - KL^{-1}M)x_{b_1} + \sum_{i=1}^{m} N_{b_{11_i}} x_{b_1}(t) u_i(t) + (B_{b_1} - KL^{-1})u(t),$$

$$y(t) = (C_{b_1} - C_{b_2} L^{-1} M) x_{b_1}(t).$$

The reduced order bilinear system using the balanced truncation or the singular perturbation methods can be presented by

$$\mathfrak{B}_r : \begin{array}{l} \dot{x}_r(t) = A_r x_r(t) + \sum_{i=1}^{m} N_{ri} u_i(t) x_r(t) + B_r u(t), \\ y_r(t) = C_r x_r(t), \end{array} \tag{5}$$

where $x_r \in \Re^r, r < n, y_r \in \Re^p$, $A_r$ is stable, $(A_r, B_r)$ is controllable and $r$ is order of the reduced bilinear systems.

## 4 The Least Upper Bounds of the Difference Bilinear Systems

Consider the full order model (1) and the reduced order model (5) of the bilinear system. The difference bilinear system is defined as a system in which the transfer function is the difference of transfer function between the full order system (1) and the reduced order system (5) of a bilinear system. The difference of the transfer matrix $k$-variate of the full order model and the reduced order model of the bilinear system is obtained as follows:

$$h_{i_1,\ldots,i_k}(t_1,\ldots,t_k) - h_{ri_1,\ldots,ri_k}(t_1,\ldots,t_k) = \begin{bmatrix} C & -C_r \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_k}$$

$$\begin{bmatrix} N_{i1} & 0 \\ 0 & N_{ri1} \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_{k-1}} \begin{bmatrix} N_{i2} & 0 \\ 0 & N_{ri2} \end{bmatrix}$$

$$\cdots \begin{bmatrix} N_{i_k-1} & 0 \\ 0 & N_{ri_k-1} \end{bmatrix} e^{\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} t_1} \begin{bmatrix} b_{i_k} \\ b_{ri_k} \end{bmatrix}.$$

The difference of the transfer matrix k-variate leads to the difference bilinear system given by

$$\mathfrak{B}_d : \begin{array}{l} \begin{bmatrix} \dot{x} \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} + \sum_{i=1}^{m} \begin{bmatrix} N_i & 0 \\ 0 & N_{r_i} \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix} u_i + \begin{bmatrix} B \\ B_r \end{bmatrix} u, \\ y - y_r = \begin{bmatrix} C & -C_r \end{bmatrix} \begin{bmatrix} x \\ x_r \end{bmatrix}. \end{array} \tag{6}$$

Suppose $\bar{P}$ and $\bar{Q}$ are the controllability gramian and the observability gramian of the difference bilinear system (6), respectively. Therefore, $\bar{P}$ and $\bar{Q}$ are nonnegative matrices and the two following generalized Lyapunov equations are satisfied

$$F\bar{P} + \bar{P}F^T + \sum_{i=1}^{m} H_i \bar{P} H_i^T + S = 0, \tag{7}$$

$$F^T\bar{Q} + \bar{Q}F + \sum_{i=1}^{m} H_i^T \bar{Q} H_i + M = 0, \tag{8}$$

where $F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$, $H_i = \begin{bmatrix} N_i & 0 \\ 0 & N_{r_i} \end{bmatrix}$, $S = \begin{bmatrix} BB^T & BB_r^T \\ B_rB^T & B_rB_r^T \end{bmatrix}$, and
$M = \begin{bmatrix} C^TC & -C^TC_r \\ -C_r^TC & C_r^TC_r \end{bmatrix}$.

Furthermore, the least upper bounds of the error transfer function between the full order (1) and the reduced order (5) of the bilinear time invariant systems in the $H_2$-norm are given by the following theorem.

**Theorem 4.1** *Consider the order of the bilinear system (1) is $n$ and the order of the reduced bilinear system (5) is $r, r = 1, 2, ..., n-1$. Suppose $A$ and $A_r$ are locally stable. If there exists the controllability gramian $\bar{P}$ of the difference bilinear system (6) then*

$$\|\mathfrak{B} - \mathfrak{B}_r\|_2 \le \sqrt{\lambda_{max}(\bar{P})}\sqrt{\lambda_{max}(M)}, \forall r.$$

*If there exist the observability gramian $\bar{Q}$ of the difference bilinear system (6) then*

$$\|\mathfrak{B} - \mathfrak{B}_r\|_2 \le \sqrt{\lambda_{max}(\bar{Q})}\sqrt{\lambda_{max}(S)}, \forall r.$$

**Proof.** Because $A$ and $A_r$ are locally stable then $F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}$ is locally stable. By using Lemma 2.2 and the controllability gramian $\bar{P}$ of the difference bilinear system (6) (the observability gramian $\bar{Q}$ of the difference bilinear system (6)), the least upper bounds as on the right hand side are obtained. □

The results for the linear time invariant systems (LTIS) as a special case of the bilinear time invariant systems when $N_i = 0, \forall i$ is given by the following

**Corollary 4.1** *If $N_i = 0, \forall i$, then (1) will become the linear time invariant system (LTIS). The least upper bound of the transfer function of the LTIS in the $H_2$-norm is*

$$\sqrt{\lambda_{max}(P)}\sqrt{\lambda_{max}(C^TC)},$$

*where $P$ is the controllability gramian of the LTIS. The least upper bound of the $H_2$-norm of the difference of the transfer function for the difference of LTIS is*

$$\sqrt{\lambda_{max}(\bar{P})}\sqrt{\lambda_{max}(M)},$$

*where $\bar{P}$ is the controllability gramian of the difference of LTIS.*

## 5 Procedure to Select the Reduced Order Bilinear System

The following algorithm is used to show that the least upper bounds of the $H_2$-norm of the transfer function of the difference bilinear systems are valid. The algorithm can also be used to choose the reduced order bilinear system which is similar to the full order bilinear system. The input of the algorithm is a bilinear system (1), where $A, B, N_i, C, i = 1, 2, 3, ..., m$ are matrices of suitable dimensions and the order of the bilinear system is $n$.

- Step 1: Choose the method to obtain the reduced order bilinear system.

    1. Reduce the bilinear system (1) by using the balance truncation method.

2. Reduce the bilinear system (1) by using the singular perturbation method.

- Step 2: Calculate the $H_2$-norm and the least upper bounds of the difference bilinear system.

1. Suppose $\beta_{rBT} = \|\mathfrak{B} - \mathfrak{B}_r\|_2$ denotes $H_2$-norm of the transfer function of the difference bilinear systems with the reduced $r$-th order bilinear systems, $r = 1, 2, ..., n-1$ using the balanced truncation method. Calculate $\beta_{rBT}$ by Lemma 2.1 where the gramian matrix $\bar{P}$ satisfies (7). Next, calculate the least upper bounds $\gamma_{rBT}$ by using Theorem 4.1. It is clear that $\beta_{rBT} < \gamma_{rBT}, \forall r = 1, 2, ..., n-1$. The index $BT$ denotes the balanced truncation method.

2. Suppose $\gamma_{rBT}$ denotes the least upper bound of the difference bilinear systems with the reduced $r$-th order bilinear system which is reduced by using the balanced truncation method. Suppose the index $SP$ denotes the singular perturbation method. Calculate $\beta_{rSP}$ by Lemma 2.1, where the gramian matrix $\bar{P}$ satisfies (8). Next, calculate the least upper bounds $\gamma_{rSP}$ by using Theorem 4.1. It is also clear that $\beta_{rSP} < \gamma_{rSP}, \forall r$.

- Step 3: Choose the smallest $r$ of the reduced order bilinear systems $\mathfrak{B}_r$ such that $\frac{\gamma_{(r-1)BT}}{\gamma_{rBT}} \approx 1$, or $\frac{\gamma_{(r-1)SP}}{\gamma_{rSP}} \approx 1$, where $\gamma_{rBT}$ is the least upper bound of the transfer function of the difference of the bilinear systems with the reduced $r$-th order bilinear system using the balanced truncation method, $\gamma_{(r-1)BT}$ for order $r-1$. The index $SP$ is for the singular perturbation method.

## 6 Simulation Results

Consider the circuit bilinear time invariant system as in [17] as follows

$$
\mathfrak{B}: \begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -5 & 2 & 0 & \dots & 0 \\ 2 & -5 & 2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 2 & -5 & 2 \\ 0 & 0 & 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 0 & -3 & 0 & \dots & 0 \\ 3 & 0 & -3 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix} u_1(t)x(t) \\
&+ \begin{bmatrix} 1 & 3 & 0 & \dots & 0 \\ -3 & 1 & 3 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix} u_2(t)x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{bmatrix} x(t).
\end{aligned}
$$

Furthermore, the simulation of circuit bilinear system with order 25 and 15 is presented. The $H_2$-norm and the least upper bounds of the difference bilinear system with order 25 and 15 are obtained by using the proposed algorithm as shown in Figures 1 and 2. It is found that $\beta_{rBT} < \gamma_{rBT}$ for each $r$. When the order of the reduced bilinear system is increased, the value of $\|\mathfrak{B} - \mathfrak{B}_r\|_2$ is decreased and the least upper bounds of the difference of bilinear system are increased. According to Definition 3.1, the Hankel singular values of the circuit bilinear system and the ratio of the Hankel singular values are presented in Table 1. The ratio of the Hankel singular value of each order of the reduced bilinear system from 2 up to 14 is near to 1. Therefore, the order of the reduced bilinear system is not easy to be determined because it depends on knowledge of the decision makers.

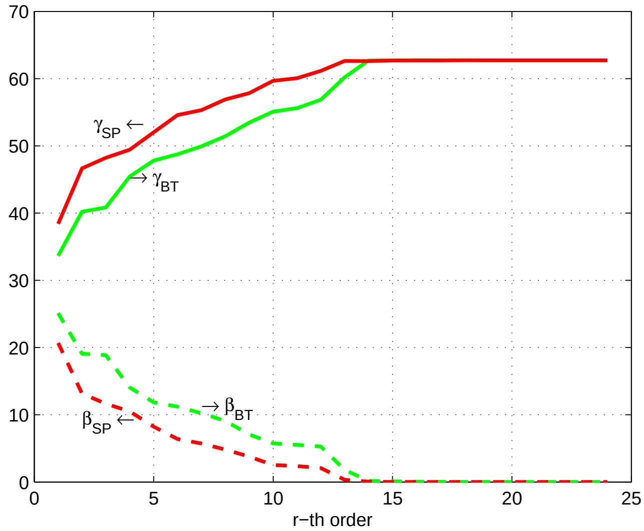


**Figure 1**: The $H_2$-norm $\beta$ and least upper bound $\gamma$ of the difference bilinear system.

For the order of the circuit system is 25, the order of the reduced bilinear systems can be chosen to the 10-th order when the balance truncation method is used to obtain the reduced bilinear system and to the 13-th order when the singular perturbation method is used. The output of the circuit bilinear system is presented in Figures 3 and 4. For the 11-th order reduced bilinear system, the response of the reduced bilinear system is not similar to that of the full order, so it is not recommended as the reduced order model.

The reduced circuit system by using the two methods will have nearly the same response when the order of the reduced bilinear system is 13. For the order of the circuit system is 15, the order of the reduced bilinear systems can be chosen to the 4-th order when the balanced truncation method is used to obtain the reduced bilinear system. The reduced circuit system by using the two methods will have nearly the same response when the order of the reduced bilinear system is 8. The outputs of the circuit bilinear system are shown in Figure 5 for the 4-th order reduced circuit bilinear system.

## 7 Conclusions

The least upper bounds of the difference bilinear time invariant systems were derived by defining the $H_2$-norm of the bilinear systems in terms of the error transfer function. The least upper bounds of the difference bilinear system were presented by the controllability gramian or the observability gramian of the difference bilinear system. The results were

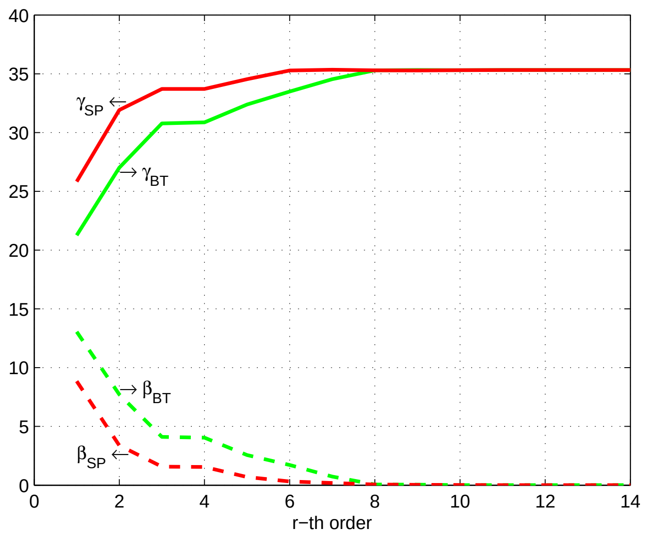


**Figure 2**: The $H_2$-norm $\beta$ and least upper bound $\gamma$ of the difference bilinear system.

| Order | $\sigma_i, i = 1, 2, ..., n = 25$ | $\mathfrak{R}_k, k = 1, 2, ..., 24$ | $\sigma_i, i = 1, 2, ..., n = 15$ | $\mathfrak{R}_k, k = 1, 2, ..., 14$ |
|---|---|---|---|---|
| 1 | 4.5536 | 5.8207 | 3.4232 | 4.3639 |
| 2 | 0.7823 | 1.9669 | 0.7844 | 1.9817 |
| 3 | 0.3977 | 1.0126 | 0.3958 | 1.0977 |
| 4 | 0.3928 | 1.3096 | 0.3606 | 1.1909 |
| 5 | 0.2999 | 1.2173 | 0.3028 | 1.2161 |
| 6 | 0.2464 | 1.3369 | 0.2490 | 1.3118 |
| 7 | 0.1843 | 1.2429 | 0.1898 | 1.2736 |
| 8 | 0.1483 | 1.1669 | 0.1490 | 1.3711 |
| 9 | 0.1271 | 1.3048 | 0.1087 | 1.6485 |
| 10 | 0.0974 | 1.3010 | 0.0659 | 1.8538 |
| 11 | 0.0749 | 1.2387 | 0.0356 | 2.0791 |
| 12 | 0.0604 | 1.1717 | 0.0171 | 2.3844 |
| 13 | 0.0516 | 1.1419 | 0.0072 | 2.8787 |
| 14 | 0.0452 | 1.4572 | 0.0025 | 3.9790 |
| 15 | 0.0310 | 1.6138 | 0.0006 | |
| 16 | 0.0192 | 1.7376 | | |
| 17 | 0.0111 | 1.8585 | | |
| 18 | 0.0059 | 1.9983 | | |
| 19 | 0.0030 | 2.1563 | | |
| 20 | 0.0014 | 2.3551 | | |
| 21 | 0.0006 | 2.6129 | | |
| 22 | 0.0002 | 2.9861 | | |
| 23 | 0.0001 | 3.6031 | | |
| 24 | 0.0000 | 5.0032 | | |
| 25 | 0.0000 | | | |

**Table 1**: Hankel singular value $\sigma_i, i = 1, 2, ..., n$ for the circuit bilinear system and its ratios $\mathfrak{R}_k = \frac{\sigma_k}{\sigma_{k+1}}, k = 1, 2, ..., n-1$.

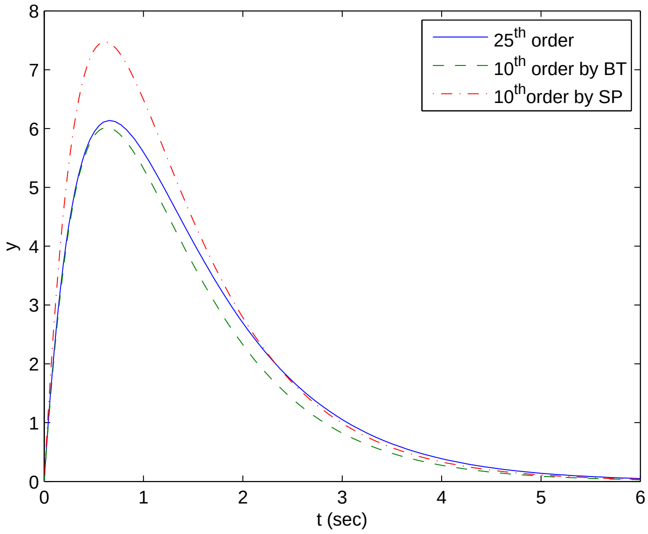


**Figure 3**: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

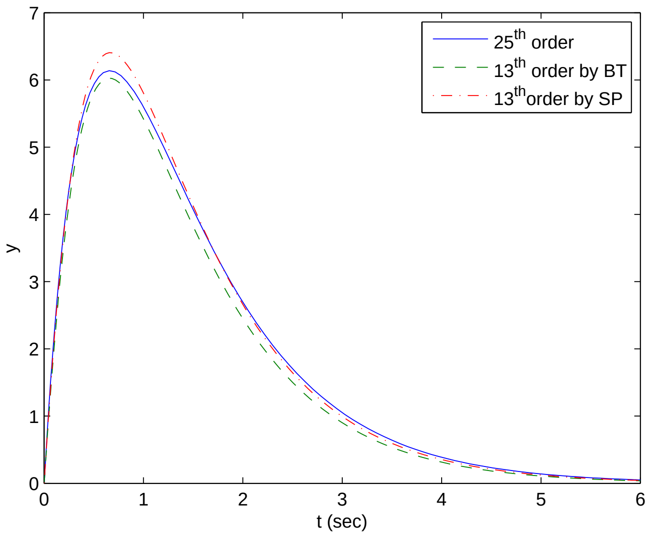


**Figure 4**: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

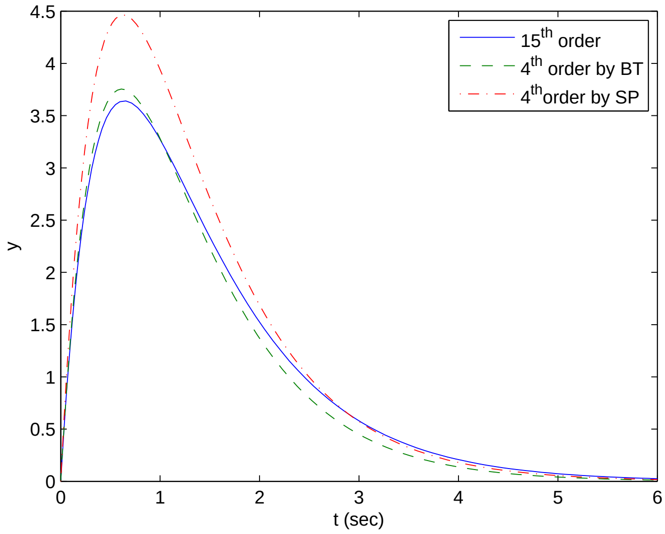


**Figure 5**: The output of the circuit bilinear system, BT: balanced truncation, SP: singular perturbation.

also valid for the linear time invariant systems as a special case. The value of the $\|\mathfrak{B} - \mathfrak{B}_r\|_2$ decreased as the order of the reduced bilinear system was closer to the full order bilinear system.

The order selection of the reduced bilinear system was based on the alteration value of the least upper bounds or the value alteration of $\|\mathfrak{B} - \mathfrak{B}_r\|_2$. The proposed method was easier than using the alteration of the singular Hankel values. The least upper bounds of the transfer function of the bilinear system in $H_2$-norm are a function of the controllability gramian or the observability gramian of the bilinear system. The simulation result showed that the balanced truncation method was better than the singular perturbation method when the system frequency is low and vice versa. Therefore, the order of the reduced bilinear system can be chosen to be smaller when using the balanced truncation method although $H_2$-norm of difference bilinear system was greater when using the singular perturbation method.

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