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Extremal Mild Solutions for Finite Delay Differential Equations of Fractional Order in Banach Spaces

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Abstract: In this paper, we study the existence and uniqueness of extremal mild solutions for finite delay differential equations of fractional order in Banach spaces with the help of the monotone iterative technique based on lower and upper solutions. This technique uses the iterative procedure starting from a pair of ordered lower and upper solutions to obtain the extremal mild solutions. We also use the theory of fractional calculus, semigroup theory and measures of noncompactness to obtain the results. An example is presented to illustrate the main result.

Keywords: *fractional delay differential equations; semigroup theory; monotone iterative technique; Kuratowskii measures of noncompactness.*

Mathematics Subject Classification (2010): 34A08, 34G20, 34K30.

1 Introduction

In this paper, our aim is to study the existence of extremal mild solutions for the following finite delay differential equations of fractional order in an ordered Banach space X of the form:

$$\begin{cases} {}^cD^\alpha x(t) &= Ax(t) + f(t, x_t), \quad t \in J = [0, b], \\ x_0(\nu) &= \phi(\nu), \quad \nu \in [-a, 0], \end{cases} \tag{1}$$

where state $x(.)$ takes value in the Banach space X endowed with norm $\|.\|$; ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$; $A : D(A) \subset X \to X$ is a closed linear densely defined operator; A is an infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. The function $f : J \times D \to X$ is given nonlinear function, here $D = C([-a, 0], X)$. If $x : [-a, b] \to X$ is a continuous function, then x_t denotes the

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function in D defined as $x_t(\nu) = x(t+\nu)$ for $\nu \in [-a, 0]$, here $x_t(.)$ represents the time history of the state from the time $t-a$ up to the present time t, and $\phi(.) \in D$.

Fractional calculus is generalization of ordinary differential equations and integration to arbitrary non integer orders. The subject is as old as differential calculus when it was invented by Newton and Leibnitz in the seventieth century. It has proved a valuable tool to describe many phenomena, arising in Engineering, Physics, Economics and Science. Indeed, we can find numerous applications in electrochemistry, control, porous media, electromagnetic, etc. (see [1–8]). Hence, in recent years, the researchers have paid more attention to fractional differential equations. In [9–19], the authors have discussed the existence of solutions of delay differential equations with or without fractional order.

This work is motivated by works [24, 26]. In this paper, we study the existence of extremal mild solutions of delay system (1) by using the monotone iterative technique. In the recent years, the monotonic iterative technique is also used to deal with fractional differential equations (see, for instance, [20–26] and references therein). The monotone iterative technique based on lower and upper solutions helps us to solve the differential equation with various kinds of boundary conditions. This technique uses the iterative procedure starting from a pair of ordered lower and upper solutions. The sequences of iterations uniformly converge to the extremal mild solutions between the lower and upper solutions. Further we prove the uniqueness of the solutions of the system. We also use the theory of fractional calculus, semigroup theory and measures of noncompactness to obtain the results. To the best of our knowledge, up to now, no work has been reported on finite delay differential equations of fractional order by using the monotone iterative technique.

The rest of paper is organized as follows. In the next Section we give some basic definitions and notations. In Section 3, we study the existence of extremal mild solution of delay system (1) and uniqueness of solutions of the system. Finally, in Section 4, we present an example to illustrate our results.

2 Preliminaries

In this section, we introduce some basic definitions and notations which are used throughout this paper. We denote by X a Banach space with the norm $\|.\|$ and $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. This means that there exists $M \geq 1$ such that $\sup_{t\in J} \|T(t)\| \leq M$.

Definition 2.1 (see [8]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function f is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad t > 0,$$

where Γ is the gamma function, and $f \in L^1([0, b], X)$.

Definition 2.2 (see [8]) The fractional derivative of order $0 \leq n-1 < \alpha < n$ in the Caputo sense is defined as

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where f is an n-times continuous differentiable function and Γ is a gamma function.

If f is an abstract function with values in a Banach space X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Let $P = \{y \in X : y \geq \theta\}$ (θ is a zero element of X) be positive cone in X which defines a partial ordering in X by $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$ we write $x < y$. The cone P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$ and P is said to be fully regular if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots, \quad \sup_n \|x_n\| < \infty$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in X$. Clearly full regularity of P implies the normality of P.

Since $C([-a,b],X)$ is the Banach space of all continuous X-valued functions on interval $[-a,b]$ with norm $\|.\|_C = \sup_{t\in[-a,b]} \|x(t)\|$. Then $C([-a,b],X)$ is an ordered Banach space whose partial ordering $\leq$ reduced by positive cone $P_C = \{x \in C([-a,b],X) \mid x(t) \geq \theta,\ t \in [-a,b]\}$. Similarly D is also an ordered Banach space with norm $\|.\|_D = \sup_{t\in[-a,0]} \|x(t)\|$ and partial ordering $\leq$ reduced by $P_D = \{x \in C([-a,0],X) \mid x(t) \geq \theta, t \in [-a,0]\}$. P_C and P_D are also normal cones with the same normal constant N. For $x, y \in C(I,X)$ with $x \leq y$, denote the ordered interval $[x,y] = \{z \in C(I,X),\ x \leq z \leq y\}$ in $C(I,X)$, and $[x(t),y(t)] = \{u \in X | x(t) \leq u \leq y(t)\}$ $(t \in I)$ in X, here $I = [-a,b]$ or $I = [-a,0]$.

Let $C^\alpha([-a,b],X) = \{u \in C([-a,b],X) :\ {}^cD^\alpha u$ exists on $[0,b]$, ${}^cD^\alpha u|_{[0,b]} \in C([0,b],X)$ and $u(t) \in D(A)$ for $t \geq 0\}$. An abstract function $u \in C^\alpha([-a,b],X)$ is called a solution of (1) if $u(t)$ satisfies equation (1).

Definition 2.3 (see [26]) The function $y \in C^\alpha([-a,b],X)$ is called a lower solution of the problem (1) if it satisfies the following inequalities

$$\begin{cases} {}^cD^\alpha y(t) \leq Ay(t) + f(t,y_t), & t \in I = [0,b], \\ y_0(\nu) \leq \phi(\nu), & \nu \in [-a,0]. \end{cases} \tag{2}$$

If all inequalities of (2) are reversed, we call $y(\cdot)$ an upper solution of the problem (1).

Lemma 2.1 *If h satisfies a uniform Hölder condition, with exponent $\beta \in (0,1]$, then the unique solution of the linear initial value problem*

$$\begin{cases} {}^cD^\alpha x(t) = Ax(t) + h(t), & t \in J, \\ x(0) = x_0 \in X, \end{cases} \tag{3}$$

is given by

$$x(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s))ds, \quad t \in J, \tag{4}$$

where

$$U(t) = \int_0^\infty \psi_\alpha(\vartheta)T(t^\alpha\vartheta)d\vartheta, \quad V(t) = \alpha\int_0^\infty \vartheta\psi_\alpha(\vartheta)T(t^\alpha\vartheta)d\vartheta, \tag{5}$$

$$\psi_\alpha(\vartheta) = \frac{1}{\alpha}\vartheta^{-1-1/\alpha}\rho_\alpha(\vartheta^{-1/\alpha}).$$

Note that $\psi_\alpha(\vartheta)$ satisfies the condition of a probability density function defined on $(0,\infty)$, that is $\psi_\alpha(\vartheta) \geq 0$, $\int_0^\infty \psi_\alpha(\vartheta)d\vartheta = 1$ and $\int_0^\infty \vartheta\psi_\alpha(\vartheta) = \frac{1}{\Gamma(1+\alpha)}$. Also the term $\rho_\alpha(\vartheta)$ is defined as

$$\rho_\alpha(\vartheta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\vartheta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \vartheta \in (0,\infty).$$

Definition 2.4 A function $x(.) \in C([-a,b],X)$ is said to be a mild solution of the system (1) if $x(t)=\phi(t)$ on $[-a,0]$ and the following integral equation is satisfied:

$$x(t) = U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s,x_s)ds, \quad t \in J, \tag{6}$$

where $U(t)$ and $V(t)$ are defined by (5).

Lemma 2.2 *The following properties are valid:*

(i) for fixed $t \geq 0$ *and any* $x \in X$*, we have*

$$\|U(t)x\| \leq M\|x\|, \quad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|.$$

(ii) The operators are $U(t)$ *and* $V(t)$ *are strongly continuous for all* $t \geq 0$*.*

(iii) If $S(t)(t > 0)$ *is a compact semigroup in* X*, then* $U(t)$ *and* $V(t)$ *are norm-continuous in* X *for* $t > 0$*.*

(iv) If $S(t)(t > 0)$ *is a compact semigroup in* X*, then* $U(t)$ *and* $V(t)$ *are compact operators in* X *for* $t > 0$*.*

Definition 2.5 A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup, if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Now we recall the definition of Kuratowski's measure of noncompactness, which is used in the next section to study the existence of extremal mild solutions for finite delay differential equation of fractional order.

Definition 2.6 (see [27,28]) Let X be a Banach space and $\mathcal{B}(X)$ be family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0 \ : S \text{ admits a finite cover by sets of diameter } \leq \delta \ \},$$

where $S \in \mathcal{B}(X)$, is called the Kuratowski measure of noncompactness.
Clearly $0 \leq \mu(S) < \infty$.

We need to use the following basic properties of the μ measure.

Lemma 2.3 (see [27,28]) *Let* S*,* S_1 *and* S_2 *be bounded sets of a Banach space* X*. Then:*

(i) $\mu(S) = 0$ *if and only if* S *is relatively compact set in* X*;*

(ii) $\mu(S_1) \leq \mu(S_2)$ *if* $S_1 \subset S_2$*;*

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$*;*

(iv) $\mu(\lambda S) \leq |\lambda|\mu(S)$ *for any* $\lambda \in \mathbb{R}$*.*

Lemma 2.4 (see [27,28]) *If* $W \subset C([a,b],X)$ *is bounded and equicontinuous on* $[a,b]$*, then* $\mu(W(t))$ *is continuous for* $t \in [a,b]$ *and*

$$\mu(W) = \sup\{\mu(W(t)), t \in [a,b]\}, \text{ where } W(t) = \{x(t) : x \in W\} \subseteq X.$$

Remark 2.1 (see [27,28]) If B is a bounded set in $C([a,b],X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \leq \mu(B)$.

Lemma 2.5 (see [27,28]) *Let $B = \{u_n\} \subset C(I,X)(n = 1,2,\ldots)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integrable on I, and*

$$\mu\left(\left\{\int_I u_n(t)dt \mid n = 1,2,\ldots\right\}\right) \leq 2\int_I \mu(B(t))dt, \text{ here } I = [a,b]. \tag{7}$$

3 Main Result

In this section, we prove the existence of extremal mild solutions of the problem (1) and then prove the uniqueness in the next theorem.

Theorem 3.1 *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N and $T(t)(t \geq 0)$ be a positive operator. Also assume that the Cauchy delay problem (1) has a lower solution $x^{(0)} \in C([-a,b],X)$ and an upper solution $y^{(0)} \in C([-a,b],X)$ with $x^{(0)} \leq y^{(0)}$. The system (1) has minimal and maximal mild solutions between $x^{(0)}$ and $y^{(0)}$ if the following assumptions (H1)-(H4) are satisfied:*

(H1) The function $f : J \times D \to X$ is such that for $t \in J$, the function $f(t,.) : D \times X \to X$ is continuous and for all $\varphi \in D$, the function $f(.,\varphi)$ is strongly measurable.

(H2) For any $t \in [0,b]$, the function $f(t,.) : D \to X$ satisfies the following

$$f(t,\varphi_1) \leq f(t,\varphi_2),$$

where $\varphi_1, \varphi_2 \in D$ with $x_t^0 \leq \varphi_1 \leq \varphi_2 \leq y_t^0$.

(H3) There exists a constant $L \geq 0$ such that

$$\mu(f(t,E)) \leq L\left[\sup_{-a\leq\nu\leq 0} \mu(E(\nu))\right],$$

for a.e. $t \in J$ and $E \subset D$, where $E(\nu) = \{\varphi(\nu) : \varphi \in E\}$.

(H4) $K = \frac{2MLb^\alpha}{\Gamma(\alpha+1)} < 1$,

Proof. Let $B = [x^{(0)}, y^{(0)}] = \{x \in C([-a,b],X) \mid x^{(0)} \leq x \leq y^{(0)}\}$. We define a map $Q : B \to C([-a,b],X)$ by

$$Qx(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1}V(t-s)f(s,x_s)ds, & t \in [0,b], \\ \phi(t), & t \in [-a,0]. \end{cases} \tag{8}$$

By (H2) and for any $x \in B$, we have that

$$f(t,x_t^{(0)}) \leq f(t,x_t) \leq f(t,y_t^{(0)}).$$

By the normality of the positive cone P, there exists a constant $k > 0$ such that

$$\|f(t,x_t)\| \leq k, \quad x \in B. \tag{9}$$

Clearly $Q : B \to C([-a,b],X)$ is continuous. Let $x,y \in B$ and $x \le y$, then $x(t) \le y(t),\ t \in [-a,b]$. Therefore, for any $t \in [0,b]$, $x_t \le y_t$ in the ordered Banach space D. Now by positivity of operators $U(t)$ and $V(t)$, (H2), we have

$$Qx \le Qy. \tag{10}$$

For showing $x^{(0)} \le Qx^{(0)}$ and $Qy^{(0)} \le y^{(0)}$, we let ${}^cD^\alpha x^{(0)}(t) = Ax^{(0)}(t) + \xi(t),\ \ t \in J$, then by Definition 2.3, Lemma 2.1 and the positivity of $U(t)$ and $V(t)$ for $t \in J$, we get that

$$\begin{aligned} x^{(0)}(t) =& U(t)x^{(0)}(0) + \int_0^t (t-s)^{\alpha-1} V(t-s)\xi(s)ds \\ &\le U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, x_s^{(0)})ds, \quad t \in J \end{aligned}$$

and also $x^{(0)}(t) \le \phi(t) = Qx^{(0)}(t),\ t \in [-a,0]$. Thus $x^{(0)}(t) \le Qx^{(0)}(t),\ t \in [-a,b]$. Similarly we can prove that $Qy^{(0)}(t) \le y^{(0)}(t),\ t \in [-a,b]$. Thus $Q : B \to B$ is an increasing monotonic operator. Now we define the sequences as

$$x^{(n)} = Qx^{(n-1)} \text{ and } y^{(n)} = Qy^{(n-1)}, \quad n = 1,2,\ldots, \tag{11}$$

and from (10), we have

$$x^{(0)} \le x^{(1)} \le \ldots x^{(n)} \le \ldots \le y^{(n)} \le \ldots \le y^{(1)} \le y^{(0)}. \tag{12}$$

Now we show that Q is equicontinuous on $[-a,b]$. For this, we let any $x \in B$ and $t_1, t_2 \in [-a,b]$ with $t_1 \le t_2$. First we take $t_1,\ t_2 \in [-a,0]$, then $\|Qx(t_2) - Qx(t_1)\| = \|\phi(t_2) - \phi(t_1)\| \to 0$ as $\phi(.)$ is continuous and $t_1 \to t_2$ independent of $x \in B$. Further, if $t_1, t_2 \in J$ with $t_1 \le t_2$ and by (9), then we have that

$$\begin{aligned} \|Qx(t_2) - Qx(t_1)\| \le& \|U(t_2)\phi(0) - U(t_1)\phi(0)\| \\ &+ \| \int_0^{t_1} (t_2-s)^{\alpha-1} \left[V(t_2-s) - V(t_1-s)\right] f(s,x_s)ds\| \\ &+ \| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] V(t_1-s) f(s,x_s)ds\| \\ &+ \int_{t_1}^{t_2} (t-s)^{\alpha-1} V(t_2-s) f(s,x_s)ds \\ \le& \|U(t_2)\phi(0) - U(t_1)\phi(0)\| \\ &+ k\int_0^{t_1} (t_2-s)^{\alpha-1} \|V(t_2-s) - V(t_1-s)\| ds \\ &+ \frac{Mk}{\Gamma(\alpha)} \int_0^{t_1} |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| ds \\ &+ \frac{Mk}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t-s)^{\alpha-1} ds \\ =& I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{13}$$

where

$$\begin{aligned}
I_1 =& \|U(t_2)\phi(0) - U(t_1)\phi(0)\|, \\
I_2 =& k\int_0^{t_1} (t_2 - s)^{\alpha-1}\|V(t_2 - s) - V(t_1 - s)\|\ ds, \\
I_3 =& \frac{Mk}{\Gamma(\alpha)}\int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\ ds, \\
I_4 =& \frac{Mk}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t - s)^{\alpha-1} ds.
\end{aligned}$$

For any $\epsilon \in (0, t_1)$, we have

$$\begin{aligned}
I_2 \leq& k\int_0^{t_1-\epsilon} (t_2 - s)^{\alpha-1}\|V(t_2 - s) - V(t_1 - s)\| ds \\
&+ k\int_{t_1-\epsilon}^{t_1} (t_2 - s)^{\alpha-1}\|V(t_2 - s) - V(t_1 - s)\| ds \\
\leq& k\int_0^{t_1-\epsilon} (t_2 - s)^{\alpha-1} ds. \sup_{s\in[0,t_1-\epsilon]} \|V(t_2 - s) - V(t_1 - s)\| \\
&+ \frac{2Mk}{\Gamma(\alpha)}\int_{t_1-\epsilon}^{t_1} (t_2 - s)^{\alpha-1} ds. \qquad (14)
\end{aligned}$$

By Lemma 2.2, we get that $I_2 \to 0$ as $t_1 \to t_2$ and $\epsilon \to 0$ independent of $x \in B$. From expression of I_1, I_3 and I_4, we can easily show that $I_2 \to 0$, $I_3 \to 0$ and $I_4 \to 0$ as $t_2 \to t_1$ independent of $x \in B$. Therefore $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ independent of $x \in B$. Thus for $t_1, t_2 \in [-a, b]$ with $t_1 \leq t_2$, we have that $\|Qx(t_2) - Qx(t_1)\| \to 0$ as $t_1 \to t_2$ independent of $x \in B$. Therefore $Q(B)$ is equicontinuous on $[-a, b]$.

From (8), we must have $x^{(n)}(t) = y^{(n)}(t) = \phi(t)$, $n = 1, 2, \ldots,$ $t \in [-a, 0]$. So $x^{(n)} \to \phi$ and $y^{(n)} \to \phi$ on $[-a, 0]$. Let $S = \{x^{(n)}\}_{n=1}^{\infty}$. The normality of positive cone P and (12) imply that S is bounded. Note that $\mu(S(t)) = 0$, for any $t \in [-a, 0]$. Since $S(t) = \{x^{(1)}(t)\} \cup \{Q(S)(t)\}$ for any $t \in J$, then $\mu(S(t)) = \mu(Q(S)(t))$ for any $t \in J$. By using (H3), (8), (11) and for $t \in J$, we have that

$$\begin{aligned}
\mu(S(t)) =& \mu\left(\left\{U(t)\phi(0) + \int_0^t (t - s)^{\alpha-1}V(t - s)f(s, x_s^{(n)})ds\right\}_{n=1}^{\infty}\right) \\
\leq& \mu\left(\left\{\int_0^t (t - s)^{\alpha-1}V(t - s)f(s, x_s^{(n)})ds\right\}_{n=1}^{\infty}\right) \\
\leq& \frac{2M}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\mu\left(\left\{f(s, x_s^{(n)})\right\}_{n=1}^{\infty}\right) ds \\
\leq& \frac{2M}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} L \sup_{-a\leq\nu\leq 0} \mu\left(\left\{x^{(n)}(s + \nu)\right\}_{n=1}^{\infty}\right) ds
\end{aligned}$$

$$\leq \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{0\leq\tau\leq s} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right) ds$$
$$\leq \frac{2ML}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds. \sup_{0\leq\tau\leq b} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right)$$
$$\leq \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)} \sup_{0\leq\tau\leq b} \mu\left(\left\{x^{(n)}(\tau)\right\}_{n=1}^{\infty}\right). \tag{15}$$

Since $\left\{Qx^{(n)}\right\}_{n=0}^{\infty}$, i.e. $\left\{x^{(n)}\right\}_{n=1}^{\infty}$, are equicontinuous on $[-a,b]$ and $\mu(S(t))=0$, for any $t\in[-a,0]$, then Lemma 2.4 and inequality (15) imply that

$$\mu(S) \leq \frac{2MLb^{\alpha}}{\Gamma(\alpha+1)} \mu\left(\left\{x^{(n)}\right\}_{n=1}^{\infty}\right) = K\mu(S). \tag{16}$$

Since $K<1$ as given in (H4), this implies that $\mu(S)=0$, i.e. $\mu(\{x^{(n)}\}_{n=1}^{\infty})=0$. Thus the set $\{x^{(n)}: n\geq 1\}$ is relatively compact in B. So we have that the sequence $\{x^{(n)}\}$ has a convergent subsequence in B. In view of (12), we can easily show that $\{x^{(n)}\}$ itself is convergent in B. So there exists $\underline{x}\in B$ such that $x^{(n)}\to\underline{x}$ as $n\to\infty$. By (8)and (11), we have that

$$x^{(n)}(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, x_s^{(n-1)}) ds, & t\in[0,b], \\ \phi(t), & t\in[-a,0]. \end{cases} \tag{17}$$

Taking $n\to\infty$ and Lebesgue dominated convergence theorem, we have that

$$\underline{x}(t) = \begin{cases} U(t)\phi(0) + \int_0^t (t-s)^{\alpha-1} V(t-s) f(s, \underline{x}_s) ds, & t\in[0,b], \\ \phi(t), & t\in[-a,0]. \end{cases} \tag{18}$$

Then $\underline{x}\in C([-a,b],X)$ and $\underline{x}=Q\underline{x}$. Thus $\underline{x}$ is a fixed point of Q, hence $\underline{x}$ becomes a mild solution of (1). Similarly we can prove that there exists $\overline{x}\in C([-a,b],X)$ such that $y^{(n)}\to\overline{x}$ as $n\to\infty$ and $\overline{x}=Q\overline{x}$. Let $x\in B$ be any fixed point of Q, then by (10), $x^{(1)}=Qx^{(0)}\leq Qx=x\leq Qy^{(0)}=y^{(1)}$. By induction, $x^{(n)}\leq x\leq y^{(n)}$. Using (12) and taking the limit as $n\to\infty$ we conclude that $x^{(0)}\leq\underline{x}\leq x\leq\overline{x}\leq y^{(0)}$. Hence $\underline{x}$, $\overline{x}$ are the minimal and maximal mild solutions of the finite delay differential equations of fractional order (1) on $[x^{(0)}, y^{(0)}]$ respectively. □

In the next theorem, we shall prove the uniqueness of the solution of system (1) by using monotone iterative procedure. For this we make the the following additional assumption:

(H5) $f: J\times D\to X$ is a continuous function and there exists a constant $\eta\geq 0$ such that

$$f(t,\varphi_2) - f(t,\varphi_1) \leq \eta(\varphi_2(\nu) - \varphi_1(\nu)), \quad \text{for some } \nu\in[-a,0]$$

for any $t\in J$ and $x_t^{(0)}\leq\varphi_1\leq\varphi_2\leq y_t^{(0)}$.

Theorem 3.2 *Let X be an ordered Banach space, whose positive cone P is normal with normal constant N and $T(t)(t\geq 0)$ be a positive operator. Also assume that the Cauchy delay problem (1) has a lower solution $x^{(0)}\in C([-a,b],X)$ and an upper solution $y^{(0)}\in C([-a,b],X)$ with $x^{(0)}\leq y^{(0)}$. If the assumptions (H2) and (H5) hold and $K=\frac{2MN\eta b^{\alpha}}{\Gamma(\alpha+1)}<1$, then the Cauchy delay problem (1) has a unique mild solution between $x^{(0)}$ and $y^{(0)}$.*

Proof. Let $\{x^{(n)}\} \subset B$ be monotone increasing sequence. For any $m, n = 1, 2, \ldots$, with $m > n$, by H(4) and H(6), we have that

$$\theta \leq f(t, x_t^{(m)}) - f(t, x_t^{(n)}) \leq \eta(x_t^{(m)}(\nu) - x_t^{(n)}(\nu)).$$

Using the normality of the positive cone P, we get

$$\|f(t, x_t^{(m)}) - f(t, x_t^{(n)})\| \leq N\eta\|x_t^{(m)}(\nu) - x_t^{(n)}(\nu)\|. \tag{19}$$

From the definition of measure of noncompactness and (19), we get

$$\mu\left(\left\{f\left(s, x_t^{(n)}\right)\right\}\right) \leq N\eta \sup_{-a\leq\nu\leq 0} \mu\left(\left\{x_t^{(n)}(\nu)\right\}\right). \tag{20}$$

From (19), f is a Lipschitz continuous for second variable. So f satisfies the assumptions (H1) and (H3) with $L = N\eta$. Thus all the conditions of Theorem 3.1 are satisfied, the Cauchy delay problem (1) has maximal and minimal solutions on the ordered interval $B = [x^{(0)}, y^{(0)}]$.

Let $\underline{x}(t)$ and $\overline{x}(t)$ be the minimal solution and maximal solution of Cauchy delay problem (1) respectively on the ordered interval $B = [x^{(0)}, y^{(0)}]$. Since $\underline{x}(t) \equiv \overline{x}(t)$ for $t \in [-a, 0]$, then we have to prove that $\overline{x}(t) \equiv \underline{x}(t)$ on J for the uniqueness. By (8), (H5) and the positivity of operator $U(t)$ and $V(t)$ and take $t \in J$, we get

$$\begin{aligned}
\theta &\leq \overline{x}(t) - \underline{x}(t) = Q\overline{x}(t) - Q\underline{x}(t) \\
&= \int_0^t (t-s)^{\alpha-1} V(t-s)\left[f(s, \overline{x}_s) - f(s, \underline{x}_s)\right] ds \\
&\leq \eta \int_0^t (t-s)^{\alpha-1} V(t-s)(\overline{x}_s(\nu) - \underline{x}_s(\nu)) ds, \qquad \text{for some } \nu \in [-a, 0].
\end{aligned}$$

By applying the normality of the positive cone P, we get

$$\begin{aligned}
\|\overline{x}(t) - \underline{x}(t)\| &\leq N\eta\| \int_0^t (t-s)^{\alpha-1} V(t-s)(\overline{x}_s(\nu) - \underline{x}_s(\nu)) ds\| \\
&\leq \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x}_s(\nu) - \underline{x}_s(\nu)\| ds \\
&= \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x}(s+\nu) - \underline{x}(s+\nu)\| ds \\
&\leq \frac{MN\eta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\overline{x} - \underline{x}\| ds \\
&\leq \frac{MN\eta b^\alpha}{\Gamma(\alpha+1)} \|\overline{x} - \underline{x}\|.
\end{aligned} \tag{21}$$

Inequality implies that $\|\overline{x} - \underline{x}\| \leq K\|\overline{x} - \underline{x}\|$. Since $K < \frac{1}{2}$, then $\|\overline{x} - \underline{x}\| = 0$, i.e. $\overline{x} = \underline{x}$ on $[-a, b]$. Hence $\overline{x} = \underline{x}$ is the unique mild solution of the Cauchy delay problem (1) between $x^{(0)}$ and $y^{(0)}$. □

4 Example

Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following finite delay patial differential equation of fractional order:

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} z(t,y) = \frac{\partial^2}{\partial y^2} z(t,y) + 2\eta \sin(\frac{z(t-1,y)}{2}), \quad (t,y) \in [0,\frac{\pi}{2}] \times [0,\pi], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,\frac{\pi}{2}], \\ z(\nu,y) = \phi(\nu,y) \quad \nu \in [-1,0], \end{cases} \tag{22}$$

where $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}$ is the Caputo fractional partial derivative, $0 \leq \eta \leq \min\{\frac{2}{\sqrt{\pi}}, \frac{\sqrt{\pi}}{4M}\}$, $f : J \times D \to X$ is a nonlinear functions, here $D = C([-1,0] \times [0,\pi], X)$ and $\phi(\nu,y) \in D$.

Let $P = \{\phi \in X | \phi(y) \geq 0 \text{ a.e. } y \in [0,\pi]\}$. Then P is a normal cone in Banach space X and its normal constant is 1, i.e. $N = 1$. We define an operator $A : X \to X$ by $Av = v''$ with domain

$$D(A) = \{v \in X : v, v' \text{ is absolutely continuous } v'' \in X, v(0) = v(\pi) = 0\}.$$

It is well known that A is an infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $\{T(t), t \geq 0\}$ in X. Now we define $x(t)(y) = z(t,y)$, ${}^cD_t^{\frac{1}{2}} x(t)(y) = \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} z(t,y)$, $f(t,x_t)(y) = 2\eta \sin(\frac{z(t-1,y)}{2})$, $x(\nu)(y) = \phi(\nu)(y) = \phi(\nu,y)$. Therefore, the above impulsive fractional differential equation (22) can be written as the abstract form (1).

The continuous function ϕ is such that $0 \leq \phi(\nu,y) \leq -\nu y(\pi - y)$, $(\nu,y) \in [-1,0] \times [0,\pi]$. Let $v(t,y) = 0$, $(t,y) \in [-1,\frac{\pi}{2}] \times [0,\pi]$. Then $f(t, v_t(\nu,y)) = 0$ for $t \in [0,\frac{\pi}{2}]$ and $\phi(\nu,y) \geq v(\nu,y)$ for $\nu \in [-1,0]$. Thus v becomes a lower solution of the problem (1). Now we take $w(t,y)$ such that

$$w(t,x) = \begin{cases} ty(\pi - y), \quad (t,y) \in [0,\frac{\pi}{2}] \times [0,\pi], \\ -ty(\pi - y), \quad (t,y) \in [-1,0] \times [0,\pi]. \end{cases}$$

Note that $\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} w(t,y) = \frac{2t^{\frac{1}{2}} y(\pi - y)}{\sqrt{\pi}}$ and $\frac{\partial^2}{\partial y^2} w(t,y) = -2t$. Since $\frac{t^{\frac{1}{2}} y(\pi-y)}{2} \geq \frac{ty(\pi-y)}{2}$ for $0 \leq t \leq 1$, the function sin(.) is increasing for interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\frac{4}{\sqrt{\pi}} \geq 2\eta$, these imply that

$$\frac{2t^{\frac{1}{2}} y(\pi - y)}{\sqrt{\pi}} \geq 2\eta \sin(\frac{ty(\pi - y)}{2}) \geq 2\eta \sin(\frac{(t-1)y(\pi - y)}{2}).$$

Thus

$$\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} w(t,y) \geq \frac{\partial^2}{\partial y^2} w(t,y) + 2\eta \sin(\frac{w(t-1,y)}{2}),$$

and $w(\nu,y) \geq \phi(\nu,y)$ for $\nu \in [-1,0]$. So w is an upper solution of the problem (1). Clearly the function $f(t,\varphi)$ is increasing in φ for $v \leq \varphi \leq w$, so the assumptions (H2) is satisfied. Since the function sin(.) is Lipschitz function and is increasing for interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. So the function f satisfies the following condition:

$$0 \leq f(t, z^{(2)}(t-1,y)) - f(t, z^{(1)}(t-1,y)) \leq \eta(z^{(2)}(t-1,y) - z^{(1)}(t-1,y)), \quad \nu \in [-1,0]$$

for any $v(t,y) \le z^{(1)}(t,y) \le z^{(2)}(t,y) \le w(t,y)$, $(t,y) \in [-1, \frac{\pi}{2}] \times [0,\pi]$. This means

$$\theta(y) \le f(t, x_t^{(2)})(y) - f(t, x_t^{(1)})(y) \le \eta(x_t^{(2)}(-1)(y) - x_t^{(1)}(-1)(y))$$

for any $v \le x^{(1)} \le x^{(2)} \le w$. Thus the assumption (H5) is also satisfied. At last $K = \frac{2MN\eta}{\Gamma(1+\frac{1}{2})} = \frac{4M\eta}{\sqrt{\pi}} < 1$. All the conditions of the Theorem 3.2 are satisfied, hence the system (22) has a unique solution. □

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