

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

EDITOR-IN-CHIEF A.A.MARTYNYUK

The S.P.Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine,
Nesterov Str. 3, 03680 MSP, Kiev-57, UKRAINE / e-mail: anmart@stability.kiev.ua
e-mail: amartynyuk@voliacable.com

HONORARY EDITORS

T.A.BURTON, Port Angeles, WA, USA
S.N.VASSILYEV, Moscow, Russia

MANAGING EDITOR I.P.STAVROULAKIS

Department of Mathematics, University of Ioannina
451 10 Ioannina, HELLAS (GREECE) / e-mail: ipstav@cc.uoi.gr

REGIONAL EDITORS

P.BORNE (France), e-mail: Pierre.Borne@ec-lille.fr
C.CORDUNEANU (USA), e-mail: concord@uta.edu
C. CRUZ-HERNANDEZ (Mexico), e-mail: ccruz@cicese.mx
P.SHI (United Kingdom), e-mail: pshi@glam.ac.uk
K.L.TEO (Australia), e-mail: K.L.Teo@curtin.edu.au
H.I.FREEDMAN (Canada), e-mail: hfreedma@math.ualberta.ca

EDITORIAL BOARD

Aleksandrov, A.Yu. (Russia)	Kloeden, P. (Germany)
Artstein, Z. (Israel)	Kokologiannaki, C. (Greece)
Bajodah, A.H. (Saudi Arabia)	Lazar, M. (The Netherlands)
Bohner, M. (USA)	Leonov, G.A. (Russia)
Braiek, N.B. (Tunisia)	Limarchenko, O.S. (Ukraine)
Chang M.-H. (USA)	Lopez-Gutierrez, R.M. (Mexico)
Chen Ye-Hwa (USA)	Nguang Sing Kiong (New Zealand)
D'Anna, A. (Italy)	Peterson, A. (USA)
Dauphin-Tanguy, G. (France)	Rasmussen, M. (United Kingdom)
Dshalalow, J.H. (USA)	Rushchitsky, J.J. (Ukraine)
Eke, F.O. (USA)	Shi Yan (Japan)
Enciso G. (USA)	Siljak, D.D. (USA)
Fabrizio, M. (Italy)	Sira-Ramirez, H. (Mexico)
Georgiou, G. (Cyprus)	Sree Hari Rao, V. (India)
Guang-Ren Duan (China)	Stavrakakis, N.M. (Greece)
Izobov, N.A. (Belarussia)	Vatsala, A. (USA)
Kalauch, A. (Germany)	Wang Hao (Canada)
Khusainov, D.Ya. (Ukraine)	Wuyi Yue (Japan)

ADVISORY EDITOR

A.G.MAZKO, Kiev, Ukraine
e-mail: mazko@imath.kiev.ua

ADVISORY COMPUTER SCIENCE EDITORS

A.N.CHERNIENKO and L.N.CHERNETSKAYA, Kiev, Ukraine

ADVISORY LINGUISTIC EDITOR

S.N.RASSHYVALOVA, Kiev, Ukraine

© 2013, InforMath Publishing Group, ISSN 1562-8353 print, ISSN 1813-7385 online, Printed in Ukraine
No part of this Journal may be reproduced or transmitted in any form or by any means without
permission from InforMath Publishing Group.

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. Nonlinear Dynamics and Systems Theory (ND&ST) is an international journal devoted to publishing peer-refereed, high quality, original papers, brief notes and review articles focusing on nonlinear dynamics and systems theory and their practical applications in engineering, physical and life sciences. Submission of a manuscript is a representation that the submission has been approved by all of the authors and by the institution where the work was carried out. It also represents that the manuscript has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that the authors have agreed that the copyright in the article shall be assigned exclusively to InforMath Publishing Group by signing a transfer of copyright form. Before submission, the authors should visit the website:

<http://www.e-ndst.kiev.ua>

for information on the preparation of accepted manuscripts. Please download the archive Sample_NDST.zip containing example of article file (you can edit only the file Samplefilename.tex).

(2) Manuscript and Correspondence. Manuscripts should be in English and must meet common standards of usage and grammar. To submit a paper, send by e-mail a file in PDF format directly to

Professor A.A. Martynyuk, Institute of Mechanics,
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine
e-mail: anmart@stability.kiev.ua; center@inmech.kiev.ua

or to one of the Regional Editors or to a member of the Editorial Board. Final version of the manuscript must typeset using LaTeX program which is prepared in accordance with the style file of the Journal. Manuscript texts should contain the title of the article, name(s) of the author(s) and complete affiliations. Each article requires an abstract not exceeding 150 words. Formulas and citations should not be included in the abstract. AMS subject classifications and key words must be included in all accepted papers. Each article requires a running head (abbreviated form of the title) of no more than 30 characters. The sizes for regular papers, survey articles, brief notes, letters to editors and book reviews are: (i) 10-14 pages for regular papers, (ii) up to 24 pages for survey articles, and (iii) 2-3 pages for brief notes, letters to the editor and book reviews.

(3) Tables, Graphs and Illustrations. Each figure must be of a quality suitable for direct reproduction and must include a caption. Drawings should include all relevant details and should be drawn professionally in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach the electronic file of the artwork (preferably in PCX format).

(4) References. References should be listed alphabetically and numbered, typed and punctuated according to the following examples. Each entry must be cited in the text in form of author(s) together with the number of the referred article or in the form of the number of the referred article alone.

Journal: [1] Poincare, H. Title of the article. *Title of the Journal* Vol. 1 (No.1), Year, Pages. [Language]

Book: [2] Liapunov, A.M. *Title of the book*. Name of the Publishers, Town, Year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book*. (Eds.). Name of the Publishers, Town, Year, Pages. [Language]

(5) Proofs and Sample Copy. Proofs sent to authors should be returned to the Editorial Office with corrections within three days after receipt. The corresponding author will receive a sample copy of the issue of the Journal for which his/her paper is published.

(6) Editorial Policy. Every submission will undergo a stringent peer review process. An editor will be assigned to handle the review process of the paper. He/she will secure at least two reviewers' reports. The decision on acceptance, rejection or acceptance subject to revision will be made based on these reviewers' reports and the editor's own reading of the paper.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2011: SNIP – 0.84, RIP – 0.77

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

AIMS AND SCOPE

Nonlinear Dynamics and Systems Theory is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed. This journal and the individual contributions published therein are protected under the copyright by International InforMath Publishing Group.

PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 4 issues in 2013, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher (<mailto:anmart@stability.kiev.ua>), SWETS Information Services B.V. (<mailto:Operation-Academic@nl.swets.com>), EBSCO Information Services (<mailto:journals@ebSCO.com>), or website of the Journal: <http://e-ndst.kiev.ua>. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.



Asymptotic Estimates Related to an Integro Differential Equation

Monica De Angelis

*University of Naples "Federico II", Dep. Math. Appl. "R.Caccioppoli",
Via Claudio n.21, 80125, Naples, Italy.*

Received: May 22, 2013; Revised: July 9, 2013

Abstract: The paper deals with an integrodifferential operator which models numerous phenomena in superconductivity, in biology and in viscoelasticity. Initial-boundary value problems with Neumann, Dirichlet and mixed boundary conditions are analyzed. An asymptotic analysis is achieved proving that for large t , the influences of the initial data vanish, while the effects of boundary disturbances are everywhere bounded.

Keywords: *initial-boundary problems for higher order parabolic equations; Laplace transform; superconductivity; FitzHugh-Nagumo model.*

Mathematics Subject Classification (2010): 44A10, 35K57, 35A08, 35K35.

1 Introduction

If $u = u(x, t)$, let us consider the following integrodifferential equation

$$\mathcal{L}u \equiv u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau = F(x, t, u), \quad (1)$$

where ε, a, b, β are positive constants, x denotes the direction of propagation and t is the time. According to the meaning of $F(x, t, u)$, equation (1) describes the evolution of several linear or non linear physical models. For instance, when $F = f(x, t)$, (1) is related to the following linear phenomena:

- motions of viscoelastic fluids or solids [1–4];

* Corresponding author: <mailto:modeange@unina.it>

- heat conduction at low temperature [5–7],
- sound propagation in viscous gases [8].

When $F = F(x, t, u)$, some non linear phenomena involve equation (1) both in superconductivity and biology.

• *Superconductivity* – Let u be the difference between the wave functions phases of two superconductors in a Josephson junction. The equation describing tunnel effects is the following one:

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u - \gamma, \quad (2)$$

where constant γ is a forcing term proportional to a bias current, while the $\varepsilon - term$ and the $\alpha - term$ account for the dissipative normal electron current flow, respectively along and across the junction [9, 10].

Equation (2) can be obtained by (1) as soon as one assumes

$$a = \alpha - \frac{1}{\varepsilon}, \quad b = -\frac{a}{\varepsilon}, \quad \beta = \frac{1}{\varepsilon}, \quad (3)$$

and F is such that

$$F(x, t, u) = - \int_0^t e^{-\frac{1}{\varepsilon}(t-\tau)} [\sin u(x, \tau) - \gamma] d\tau. \quad (4)$$

Besides, when the case of an exponentially shaped Josephson junction (ESJJ) is considered, the evolution of the phase inside this junction is described by the third order equation:

$$(\partial_{xx} - \lambda \partial_x) (\varepsilon u_t + u) - \partial_t(u_t + \alpha u) = \sin u - \gamma, \quad (5)$$

where λ is a positive constant generally less than one and the terms λu_{xt} and λu_x represent the current due to the tapering junction. In particular λu_x corresponds to a geometrical force driving the fluxons from the wide edge to the narrow edge. [10–12] An (ESJJ) provides several advantages with respect to a rectangular junction ([14] and reference therein). For instance, in [11] it has been proved that it is possible to obtain a voltage which is not chaotic anymore, but rather periodic excluding, in this way, some among the possible causes of large spectral width. It is also proved that the problem of trapped flux can be avoided. Numerous applications and devices involve Josephson junctions, for example SQUIDS which are very versatile and can be used in a lot of fields. (see f.i. [15] and references therein).

Moreover, if $u = e^{\lambda x/2} \bar{u}$, (5) turns into an equation like (2) and hence into (1).

• *Biology* – Let us consider the FitzHugh-Nagumo system (FHN) which models the propagation of nerve impulses. [16]:

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - v + f(u), \\ \frac{\partial v}{\partial t} = b u - \beta v. \end{cases} \quad (6)$$

Here, $u(x, t)$ models the transmembrane voltage of a nerve axon at a distance x and time t , while $v(x, t)$ is an auxiliary variable acting as a recovery variable. Besides, the function $f(u)$ has the qualitative form of a cubic polynomial

$$f(u) = -a u + \varphi(u) \quad \text{with} \quad \varphi = u^2 (a + 1 - u), \tag{7}$$

while ε, b, β are non negative and the parameter a , representing the threshold constant, is generally $0 < a < 1$. (see f.i. [17] and references therein)

Denoting by v_0 the initial value of v , system (6) (7) can be given the form of the integrodifferential equation (1) as soon as one puts:

$$F(x, t, u) = \varphi(u) - v_0(x) e^{-\beta t}. \tag{8}$$

In this paper, initial value problems with Neumann, Dirichlet and mixed boundary conditions for (1) are considered. By means of properties of the fundamental solution $K_0(x, t)$ of the operator \mathcal{L} , appropriate estimates are obtained. The function $K_0(x, t)$ has already been determined and analyzed in [18] and an analysis related to a Neumann boundary problem has been conducted in [19]. The aim of this paper is an asymptotic analysis for the initial boundary value problem both with Dirichlet conditions and with mixed conditions. These cases involve x -derivative of theta functions $\theta(x, t)$ and $\theta^*(x, t)$ which are determined in Section (3). So, effects of boundary perturbations can be evaluated by means of a well known theorem on asymptotic behavior of convolutions. As an example, according to the equivalence between operator \mathcal{L} and the FHN system, an estimate of the solution related to the reaction-diffusion system (6) is obtained proving that, for large t , effects determined by boundary disturbance are bounded.

2 Some Models of Superconductivity and Biology

Let T be an arbitrary positive constant and

$$\Omega_T \equiv \{ (x, t) : 0 \leq x \leq L ; 0 < t \leq T. \}$$

(I) A first example is related to *Neumann* boundary conditions (NBC)

$$\left\{ \begin{array}{ll} \mathcal{L} u = F(x, t, u), & (x, t) \in \Omega_T, \\ u(x, 0) = u_0(x), & x \in [0, L], \\ u_x(0, t) = \psi_1(t), \quad u_x(L, t) = \psi_2(t), & 0 < t \leq T. \end{array} \right. \tag{9}$$

In superconductivity, this problem occurs when the magnetic field, proportional to the phase gradient, is assigned [20,21]. In mathematical biology, it can refer to a two-species reaction diffusion system subjected to flux boundary conditions [16]. The same conditions are present in case of pacemakers [22] and are applied also to study distributed (FHN) systems [23] or to solve FHN systems by means of numerical calculations [24].

(II) Another example concerns *Dirichlet* boundary conditions (DBC)

$$\left\{ \begin{array}{ll} \mathcal{L} u = F(x, t, u), & (x, t) \in \Omega_T, \\ u(x, 0) = u_0(x), & x \in [0, L], \\ u(0, t) = g_1(t), \quad u(L, t) = g_2(t), & 0 < t \leq T. \end{array} \right. \tag{10}$$

In superconductivity, $(10)_3$ refer to the phase boundary specifications [12–14]. In excitable systems these conditions occur when the behavior of a single dendrite has to be determined and the voltage level is fixed [22] or when the pulse propagation in a continuum of heart cells is studied [22, 25]. Besides, the Dirichlet problem is also considered to determine universal attractors both for Hodgkin-Huxley equations and for FHN systems, [26] and for stability analysis and asymptotic behavior of reaction-diffusion systems solutions, [27–31], or in hyperbolic diffusion [32].

(III) At last, *mixed* boundary conditions (MBC) as

$$\begin{cases} \mathcal{L}u = F(x, t, u), & (x, t) \in \Omega_T, \\ u(x, 0) = u_0(x), & x \in [0, L], \\ u(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), & 0 < t \leq T, \end{cases} \quad (11)$$

occur in many physical examples both in superconductivity (see, f.i. [33] and references therein) and in biology, as shown in [16, 22]. In particular, in [34], mixed boundary conditions are considered in order to give qualitative information concerning both the threshold problem and the asymptotic behavior of large solutions for the FHN system.

When $F = f(x, t)$ is a linear function, problems (9)-(11) can be solved by Laplace transformation with respect to t . Let $z(x, t)$ be an arbitrary function admitting Laplace transform $\hat{z}(x, s)$

$$\hat{z}(x, s) = \int_0^\infty e^{-st} z(x, t) dt = \mathcal{L}_t z. \quad (12)$$

Referring to the parameters a, β, b, ε of the operator \mathcal{L} , if

$$\sigma^2 = s + a + \frac{b}{s + \beta}, \quad \tilde{\sigma}^2 = \sigma^2 / \varepsilon, \quad (13)$$

we denote by $\theta(x, s)$ and $\theta^*(x, s)$ the following Laplace transforms:

$$\hat{\theta}(y, \tilde{\sigma}) = \frac{\cosh[\tilde{\sigma}(L - y)]}{2 \varepsilon \tilde{\sigma} \sinh(\tilde{\sigma} L)} = \quad (14)$$

$$= \frac{1}{2 \sqrt{\varepsilon} \sigma} \left\{ e^{-\frac{y}{\sqrt{\varepsilon}} \sigma} + \sum_{n=1}^{\infty} \left[e^{-\frac{2nL+y}{\sqrt{\varepsilon}} \sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}} \sigma} \right] \right\},$$

$$\hat{\theta}^*(y, \tilde{\sigma}) = \frac{\sinh[\tilde{\sigma}(L - y)]}{2 \varepsilon \tilde{\sigma} \cosh(\tilde{\sigma} L)} = \quad (15)$$

$$= \frac{1}{2 \sqrt{\varepsilon} \sigma} \left\{ e^{-\frac{y}{\sqrt{\varepsilon}} \sigma} + 2 \sum_{n=1}^{\infty} \left(e^{-\frac{4nL+y}{\sqrt{\varepsilon}} \sigma} + e^{-\frac{4nL-y}{\sqrt{\varepsilon}} \sigma} \right) - \sum_{n=1}^{\infty} \left(e^{-\frac{2nL+y}{\sqrt{\varepsilon}} \sigma} + e^{-\frac{2nL-y}{\sqrt{\varepsilon}} \sigma} \right) \right\}.$$

Then the Laplace transform solutions of the linear problems (9)-(11) can be obtained by means of standard techniques and it results:

- Formal solution for initial boundary problem with (NBC)

$$\begin{aligned} \hat{u}(x, s) &= \int_0^L [\hat{\theta}(|x - \xi|, s) + \hat{\theta}(|x + \xi|, s)] [u_0(\xi) + \hat{f}(\xi, s)] d\xi \\ &\quad - 2 \varepsilon \hat{\psi}_1(s) \hat{\theta}(x, s) + 2 \varepsilon \hat{\psi}_2(s) \hat{\theta}(x - L, s). \end{aligned} \quad (16)$$

- Formal solution for (DBC)

$$\hat{u}(x, s) = \int_0^L [\hat{\theta}(|x - \xi|, s) - \hat{\theta}(x + \xi, s)] [u_0(\xi) + \hat{f}(\xi, s)] d\xi - 2 \varepsilon \hat{g}_1(s) \hat{\theta}_x(x, s) + 2 \varepsilon \hat{g}_2(s) \hat{\theta}_x(x - L, s). \tag{17}$$

- Formal solution for (MBC)

$$\hat{u}(x, s) = \int_0^L [\hat{\theta}^*(x + \xi, s) - \hat{\theta}^*(|x - \xi|, s)] [u_0(\xi) + \hat{f}(\xi, s)] d\xi + 2 \varepsilon \hat{h}_1(s) \hat{\theta}_x^*(x, s) + 2 \varepsilon \hat{h}_2(s) \hat{\theta}_x^*(L - x, s). \tag{18}$$

3 $K_0(x, t)$ and $\theta(x, t)$ Properties

The Neumann boundary value problem has already been solved in [19]. Let us consider now cases (II) and (III).

Let $K_0(x, t)$ be the fundamental solution of the linear operator \mathcal{L} defined in (1). It has already been determined in [18] and one has:

$$K_0(r, t) = \frac{1}{2\sqrt{\pi\varepsilon}} \left[\frac{e^{-\frac{r^2}{4t} - at}}{\sqrt{t}} - \sqrt{b} \int_0^t \frac{e^{-\frac{r^2}{4y} - ay}}{\sqrt{t-y}} e^{-\beta(t-y)} J_1(2\sqrt{by(t-y)}) dy \right], \tag{19}$$

where $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ is the Bessel function of first kind. Function K_0 has the same basic properties of the fundamental solution of the heat equation, and in the half-plane $\Re s > \max(-a, -\beta)$ it results:

$$\mathcal{L}_t K_0 \equiv \int_0^\infty e^{-st} K_0(r, t) dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon}\sigma}, \tag{20}$$

where σ is defined in (13)₁.

Among other properties, in [18] the following estimates have been proved:

$$\int_{\mathbb{R}} |K_0(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{b} \pi t e^{-\omega t} \int_0^t d\tau \int_{\mathbb{R}} |K_0(x - \xi, \tau)| d\xi \leq \beta_0, \tag{21}$$

$$|K_0| \leq \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}} [e^{-at} + bt E(t)], \tag{22}$$

where constants ω , β_0 and $E(t)$ are given by:

$$\omega = \min(a, \beta), \quad \beta_0 = \frac{1}{a} + \pi\sqrt{b} \frac{a + \beta}{2(a\beta)^{3/2}}, \tag{23}$$

$$E(t) = \frac{e^{-\beta t} - e^{-at}}{a - \beta} > 0.$$

Moreover, denoting by

$$K_i(r, t) = \int_0^t e^{-\beta(t-\tau)} K_{i-1}(x, \tau) d\tau \quad (i = 1, 2) \tag{24}$$

kernels $K_1(x, t)$ and $K_2(x, t)$ have the same properties of $K_0(x, t)$. Hence, the following theorem holds [18]:

Theorem 3.1 For all the positive constants a, b, ε, β it results:

$$\int_{\mathfrak{R}} |K_1| d\xi \leq E(t); \quad \int_0^t d\tau \int_{\mathfrak{R}} |K_1| d\xi \leq \beta_1, \quad (25)$$

$$\int_{\mathfrak{R}} |K_2(x - \xi, t)| d\xi \leq tE(t), \quad (26)$$

where $\beta_1 = (a\beta)^{-1}$.

In order to obtain inverse formulae of (17) and (18), let us apply (20) to (14)(15). Then, one deduces the following functions which are similar to *theta functions*:

$$\begin{aligned} \theta(x, t) &= K_0(x, t) + \sum_{n=1}^{\infty} [K_0(x + 2nL, t) + K_0(x - 2nL, t)] \\ &= \sum_{n=-\infty}^{\infty} K_0(x + 2nL, t), \end{aligned} \quad (27)$$

$$\theta^*(x, t) = 2 \sum_{n=-\infty}^{\infty} K_0(x + 4nL, t) - \sum_{n=-\infty}^{\infty} K_0(x + 2nL, t). \quad (28)$$

Some of the properties of function $\theta(x, t)$ have already been evaluated in [19]. Precisely, denoting by $C = 2\varepsilon \pi^2 / (6eL^2)$ and letting

$$C_0 = \frac{1}{2\sqrt{\varepsilon\omega}} + \frac{b\omega^{-3/2}}{4\sqrt{\varepsilon}|a-\beta|} \left[1 + \frac{C}{b}|a-\beta| + \frac{3C}{2\omega} \right], \quad (29)$$

the $\theta(x, t)$ function, defined in (27), satisfies the following inequalities:

$$\int_0^L |\theta(|x - \xi|, t)| d\xi \leq (1 + \sqrt{b}\pi t) e^{-\omega t}, \quad (30)$$

$$\int_0^t d\tau \int_0^L |\theta(|x - \xi|, t)| d\xi \leq \beta_0; \quad \int_0^{\infty} |\theta(x, \tau)| d\tau \leq C_0, \quad (31)$$

and, it results:

$$\lim_{t \rightarrow \infty} \theta(x, t) = 0; \quad \lim_{t \rightarrow \infty} \int_0^t \theta(x, \tau) d\tau = \frac{1}{2\varepsilon\sigma_0} \frac{\cosh \sigma_0 (L - x)}{\sinh(\sigma_0 L)}, \quad (32)$$

where $\sigma_0 = \sqrt{\left(a + \frac{b}{\beta}\right) \frac{1}{\varepsilon}}$.

Furthermore, as for $\frac{\partial \theta}{\partial x}$, from (19), it is well-rendered that the x derivative of the integral term vanishes for $x \rightarrow 0$, while the first term represents the derivative with respect to x of the fundamental solution related to the heat equation. So, by means of classic theorems (see, f.i. [35] p. 60), conditions (10)₃ are surely satisfied.

Moreover, one has:

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t \theta_x(x, \tau) \, d\tau &= \frac{1}{2\varepsilon} \frac{\sinh \sigma_0 (x - L)}{\sinh(\sigma_0 L)}, \\ \lim_{t \rightarrow \infty} \int_0^t \theta_x^*(x, \tau) \, d\tau &= -\frac{1}{2\varepsilon} \frac{\cosh \sigma_0 (L - x)}{\cosh(\sigma_0 L)}. \end{aligned} \tag{33}$$

4 Asymptotic Behaviours

When the source term $F = f(x, t)$ is a prefixed function depending only on x and t , then, initial boundary value problems (10) (11) are linear and can be solved explicitly. Moreover, when $F = F(x, t, u)$ depends also on the unknown function $u(x, t)$, then these problems admit integral differential formulations and one has:

- Integro differential equation for problem (10) (DBC):

$$\begin{aligned} u(x, t) &= \int_0^L [\theta(|x - \xi|, t) - \theta(x + \xi, t)] u_0(\xi) \, d\xi - \\ &2\varepsilon \int_0^t \theta_x(x, t - \tau) g_1(\tau) \, d\tau + 2\varepsilon \int_0^t \theta_x(x - L, t - \tau) g_2(\tau) \, d\tau \\ &+ \int_0^t d\tau \int_0^L [\theta(|x - \xi|, t - \tau) - \theta(x + \xi, t - \tau)] F(\xi, \tau, u(x, \tau)) \, d\xi. \end{aligned} \tag{34}$$

- Integro differential equation for (11) (MBC):

$$\begin{aligned} u(x, t) &= \int_0^L [\theta^*(|x - \xi|, t) - \theta^*(x + \xi, t)] u_0(\xi) \, d\xi - \\ &2\varepsilon \int_0^t \theta_x^*(x, t - \tau) h_1(\tau) \, d\tau + 2\varepsilon \int_0^t \theta_x^*(L - x, t - \tau) h_2(\tau) \, d\tau \\ &+ \int_0^t d\tau \int_0^L [\theta^*(|x - \xi|, t - \tau) - \theta^*(x + \xi, t - \tau)] F(\xi, \tau, u(x, \tau)) \, d\xi. \end{aligned} \tag{35}$$

Now, if \mathcal{B}_T denotes the Banach space

$$\mathcal{B}_T \equiv \left\{ z(x, t) : z \in C(\Omega_T), \|z\| = \sup_{\Omega_T} |z(x, t)|, < \infty \right\} \tag{36}$$

and D is the following set:

$$D \equiv \{(x, t, u) : (x, t) \in \Omega_T, -\infty < u < \infty,$$

then, let us assume the source term $F(x, t, u)$ be defined and continuous on D and uniformly Lipschitz continuous in (x, t, u) for each compact subset of Ω_T . Besides, let F be a bounded function for bounded u and there exists a constant C such that:

$$|F(x, t, u_1) - F(x, t, u_2)| \leq C |u_1 - u_2|.$$

So, by means of standard methods related to integral equations and owing to basic properties of K_0 , it is possible to prove that the mappings defined by (34) (35) are a contraction of \mathcal{B}_T in \mathcal{B}_T and so they admit a unique fixed point $u(x, t) \in \mathcal{B}_T$. [35, 36]

In order to enable a quicker reading, attention will be paid only to the initial boundary value problem with Dirichlet conditions. However, all the following analysis can be applied to the mixed problem, too.

At first, let us consider $g_i = 0$ ($i = 1, 2$) and let

$$\|u_0\| = \sup_{0 \leq x \leq L} |u_0(x)|, \quad \|F\| = \sup_{\Omega_T} |F(x, t, u)|.$$

In [18] the following theorem has been proved:

Theorem 4.1 *When $g_i = 0$ ($i = 1, 2$), solution (34), for large t , verifies the following estimate:*

$$|u(x, t)| \leq 2 [\|F\| \beta_0 + \|u_0\| (1 + \sqrt{b} \pi t) e^{-\omega t}], \quad (37)$$

where $\omega = \min(a, \beta)$ and β_0 is defined by (23)₂.

As for contributes of boundary data, the well known theorem will be considered [37]:

Theorem 4.2 *Let $h(t)$ and $\chi(t)$ be two continuous functions on $[0, \infty[$. If they satisfy the following hypotheses*

$$\exists \lim_{t \rightarrow \infty} \chi(t) = \chi(\infty), \quad \exists \lim_{t \rightarrow \infty} h(t) = h(\infty), \quad (38)$$

$$\dot{h}(t) \in L_1[0, \infty), \quad (39)$$

then, it results:

$$\lim_{t \rightarrow \infty} \int_0^t \chi(t - \tau) \dot{h}(\tau) d\tau = \chi(\infty) [h(\infty) - h(0)]. \quad (40)$$

According to this, it is possible to state:

Theorem 4.3 *Let g_i ($i = 1, 2$) be two continuous functions converging for $t \rightarrow \infty$. In this case one has:*

$$\lim_{t \rightarrow \infty} \int_0^t \theta_x(x, \tau) g_i(t - \tau) d\tau = g_{i, \infty} \frac{1}{2\varepsilon} \frac{\sinh \sigma_0 (x - L)}{\sinh \sigma_0 L}, \quad (41)$$

where $\sigma_0 = \sqrt{\left(a + \frac{b}{\beta}\right) \frac{1}{\varepsilon}}$.

Proof. Let us apply (40) with $h = \int_0^t \theta_x(x, \tau) d\tau$ and $\chi = g_i$ ($i = 1, 2$). Then, (41) follows by (33)₁.

5 An Example: Estimate for the FitzHugh Nagumo System

When $u(x, t)$ is determined, by means of (6), the $v(x, t)$ component is given by

$$v(x, t) = v_0 e^{-\beta t} + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau. \tag{42}$$

To achieve the expression of the solution (u, v) , let us denote by $f_1 * f_2$ the convolution

$$f_1(\cdot, t) * f_2(\cdot, t) = \int_0^t f_1(\cdot, t) f_2(\cdot, t - \tau) d\tau.$$

So that, referring to Dirichlet conditions, if

$$G(x, \xi, t) = \theta(|x - \xi|, t) - \theta(x + \xi, t),$$

and denoting by $N(x, t)$ the following known function depending on the data (u_0, v_0, g_1, g_2) :

$$\begin{aligned} N(x, t) = & -2\varepsilon g_1(t) * \theta_x(x, t) + \\ & + 2\varepsilon g_2(t) * \theta_x(x - L, t) + \int_0^L u_0(\xi) G(x, \xi, t) d\xi - e^{-\beta t} * \int_0^L v_0(\xi) G(x, \xi, t) d\xi, \end{aligned} \tag{43}$$

it results:

$$\begin{aligned} v(x, t) = & v_0 e^{-\beta t} + b e^{-\beta t} * N(x, t) \\ & + b e^{-\beta t} * \int_0^L G(x, \xi, t - \tau) * \varphi[\xi, \tau, u(\xi, \tau)] d\xi. \end{aligned} \tag{44}$$

So, the asymptotic effects due to initial disturbances are vanishing, while the effects of the source terms are bounded. Indeed, letting

$$\|u_0\| = \sup_{0 \leq x \leq L} |u_0(x)|, \quad \|v_0\| = \sup_{0 \leq x \leq L} |v_0(x)|,$$

and

$$\|\varphi\| = \sup_{\Omega_T} |\varphi(x, t, u)|,$$

by means of (8) (34) and (44) and owing to the estimates (21)₁, (25), (26), the following theorem holds:

Theorem 5.1 *For regular solution (u, v) of the (FHN) model, when $g_1 = g_2 = 0$, the following estimates hold:*

$$\begin{cases} |u| \leq 2[\|u_0\| (1 + \pi\sqrt{b}t) e^{-\omega t} + \|v_0\| E(t) + \beta_0 \|\varphi\|], \\ |v| \leq \|v_0\| e^{-\beta t} + 2[b(\|u_0\| + t \|v_0\|) E(t) + b\beta_1 \|\varphi\|]. \end{cases} \tag{45}$$

As for the asymptotic effects of boundary perturbations g_1, g_2 by means of (41), when $u_0 = 0$ and $F = 0$, one has

$$\begin{cases} u = g_{1,\infty} \frac{\sinh \sigma_0 (L-x)}{\sinh \sigma_0 L} + g_{2,\infty} \frac{\sinh \sigma_0 x}{\sinh \sigma_0 L}, \\ v = \frac{b}{\beta} \left[g_{1,\infty} \frac{\sinh \sigma_0 (L-x)}{\sinh \sigma_0 L} + g_{2,\infty} \frac{\sinh \sigma_0 (x)}{\sinh \sigma_0 L} \right]. \end{cases} \quad (46)$$

6 Remarks

- The paper is concerned with the nonlinear integral equation (1) whose kernel is a Green function with numerous basic properties typical of the diffusion equation.
- Neumann, Dirichlet and mixed boundary conditions are considered, and integro differential formulations of *non linear* problems are obtained.
- The asymptotic behavior for initial boundary value problem with Dirichlet conditions is evaluated, showing that effects due to initial disturbances vanish, while the influences of the source term and boundary perturbations are everywhere bounded.
- The analysis related to Dirichlet conditions can be applied to mixed problem, too. Indeed, like $\theta(x, t)$, also the Green function $\theta^*(x, t)$ defined in (15) depends on the fundamental solution K_0 .
- The equivalence among equation (1) and numerous models allow us to apply asymptotic theorems to many other problems related to various physical fields.

Acknowledgment

This work has been performed under the auspices of G.N.F.M. of I.N.d.A.M. and of Programma F.A.R.O. (Finanziamenti per l' Avvio di Ricerche Originali, III tornata) "Controllo e stabilita' di processi diffusivi nell'ambiente", Polo delle Scienze e Tecnologie, Universita' degli Studi di Napoli Federico II (2012).

References

- [1] Bini, D., Cherubini, C. and Filippi, S. Viscoelastic Fizhugh-Nagumo models. *Physical Review E* 041929 (2005).
- [2] Renno, P. and De Angelis, M. Diffusion and wave behavior in linear Voigt model. *C. R. Mecanique* **330** (2002) 21–26.
- [3] De Angelis, M., Monte, A. M. and Renno, P. On fast and slow times in models with diffusion. *Mathematical Models and Methods in Applied Sciences* **12** (2002) 1741–1749.
- [4] Morrison, J. A. Wave propagations in rods of Voigt material and visco-elastic materials with three-parameters models. *Quart. Appl. Math.* **14** (1956) 153–169.
- [5] Flavin, J. N. and Rionero, S. *Qualitative Estimates for Partial Differential Equations: An introduction*. Boca Raton, Florida: CRC Press, 1996.
- [6] Morro, A., Payne, L. E. and Straughan, B. Decay, growth, continuous dependence and uniqueness results of generalized heat theories. *Appl. Anal.* **38** (1990) 231–243.
- [7] Straughan, B. Heat Waves. In: *Springer Series in Applied Mathematical Sciences*, Vol. 177, 2011.
- [8] Lamb, H. *Hydrodynamics*. Cambridge University Press, 1971.

- [9] Barone, A. and Paternò, G. *Physics and Applications of the Josephson Effect*. Wiley-Interscience, New-York, 1982
- [10] De Angelis, M. and Fiore, G. Existence and uniqueness of solutions of a class of third order dissipative problems with various boundary conditions describing the Josephson effect. *J. Math. Anal. Appl.* **404**(2) (2013) 477–490.
- [11] Benabdallah, A., Caputo, J. G. and Scott, A. C. Laminar phase flow for an exponentially tapered Josephson oscillator. *Appl. Phys.* **88**(6) (2000) 3527–3540.
- [12] De Angelis, M. and Fiore, G. Diffusion effects in a superconductive model. *Communications on Pure and Applied Analysis* **13** (1) (January 2014) DOI10.3934/cpaa.2014.13.
- [13] De Angelis, M. On a non linear third - order parabolic equation. *Applied Mathematics Letters* **14** (2001) 425–430.
- [14] De Angelis, M. On exponentially shaped Josephson junctions. *Acta Appl. Math* **122** (I) (2012) 179–189.
- [15] Cybart, S. A. et al. Dynes Series array of incommensurate superconducting quantum interference devices. *Appl. Phys Lett.* **93** (2008) 182502 1–3.
- [16] Murray, J. D. *Mathematical Biology. I and II*. New York: Springer-Verlag, 2002.
- [17] De Angelis, M. and Renno, P. On the Fitzhugh Nagumo model. In: *14 Conferences on Waves and stability in Continuous Media*. Word Sci. Publ. Hackensack, N.Y., 2008, 193–198.
- [18] Renno, P. and De Angelis, M. Existence, uniqueness and a priori estimates for a non linear integro - differential equation. *Ricerche di Mat.* **57** (2008) 95–109.
- [19] Renno, P. and De Angelis, M. Asymptotic effects of boundary perturbations in excitable systems. Submitted to *Discrete and Continuous Dynamical Systems - Series B*, <http://arxiv.org/pdf/1304.3891v1.pdf>.
- [20] D’Anna, A., De Angelis, M. and Fiore, G. Existence and uniqueness for some 3rd order dissipative problems with various boundary conditions. *Acta Appl. Math.* **122** (2012) 255–267.
- [21] Jaworski, M. Fluxon dynamics in an exponentially shaped Josephson junction. *Physical Review B* **71** (2005) 214515, 6 pages.
- [22] Keener, J. P. and Sneyd, J. *Mathematical Physiology*. New York: Springer-Verlag, 1998.
- [23] Nekhamkina, O. and Sheintuch, M. Boundary-induced spatiotemporal complex patterns in excitable systems. *Phys. Rev. E* **73** (2006) 066224 1–4.
- [24] Dikansky, A. Fitzhugh-Nagumo equations in a nonhomogeneous medium. *Discrete and continuous dynamical systems*. Supplement Volume **2005** 216–224.
- [25] Artebrant, R. Bifurcating solutions to the monodomain model equipped with FitzHugh-Nagumo kinetics. *J. Appl. Math.* doi:10.1155/2009/292183.
- [26] Marion, M. Finite dimensional attractors associated to partly dissipative reaction diffusion equations. *SIAM J Math Anal.* **20**(4) (1989) 816–844.
- [27] De Angelis, M. On a parabolic operator of dissipative system. <http://arxiv.org/abs/1307.1887>.
- [28] Torricollo, I. On the Dynamics of the nonlinear duopoly game. *International Journal of Non-Linear Mechanics* (2013) doi: 10.1016/j.ijnonlinmec. 2013.06.011.
- [29] Capone, F., De Cataldis, V. and De Luca, R. On the nonlinear stability of an epidemic SEIR reaction-diffusion model. *Ricerche mat.* (2013) (62) 161–181.
- [30] De Angelis, M. A priori estimates for excitable models. *Meccanica* (2013) doi: 10.1007/s11012-013-9763-2.

- [31] Raheem, A. Existence and uniqueness of a solution of Fisher-KKP type reaction diffusion equation. *Nonlinear Dynamics and Systems Theory* **13**(2) (2013) 193–202.
- [32] Gentile, M. and Straughan, B. Hyperbolic diffusion with Christov - Morro theory. *Mathematics and Computers in Simulation* (2012) doi.org/10.1016/j.matcom.2012.07.010.
- [33] Bergeret F. S., Volkov, A. F. and Efetov, K. V. *Exotic Proximity Effects in Superconductor*. Science Series II: Mathematics, Physics and Chemistry, Vol. **154**. Berlin: Springer, 2004.
- [34] Rauch, J, and Smoller, J. Qualitative theory of the FitzHugh-Nagumo equations. *Advances in mathematics* **27** (1978) 12–44.
- [35] Cannon, J. R. *The one-dimensional heat equation* . New York: Addison-Wesley Publishing Company, 1984.
- [36] De Angelis, M., Maio, A. and Mazziotti, E. Existence and uniqueness results for a class of non linear models. In: *Mathematical Physics models and engineering sciences* (Ed. Liguori, Italy), 2008, 191–202.
- [37] Berg, L. *Introduction to the Operational Calculus*. Amsterdam: North Holland Publ., 1967.



Possibilistic Modeling of Dynamic Uncertain Processes

O.S. Bychkov* and Ie.V. Ivanov

Taras Shevchenko National University of Kyiv, 64 Volodymyrska St., 01601 Kyiv, Ukraine

Received: June 30, 2010; Revised: July 15, 2013

Abstract: In the paper a new class of uncertain differential equations based on the possibility theory is introduced. It is argued that this class is well-suited for modeling uncertain dynamic processes when the uncertainty has a non-probabilistic nature, or when the available statistical information is not sufficient for constructing a reliable stochastic model. The problems of existence and uniqueness of solutions of the proposed equations are studied and a numerical method for their solution is provided.

Keywords: *possibility theory; dynamical system; possibilistic walk process; Cauchy problem.*

Mathematics Subject Classification (2010): 34A07, 34A12.

1 Introduction

The methods of (quantitative) possibility theory [7, 10, 11, 20] allow one to estimate the level of possibility of some event with respect to possibilities of other events on the basis of subjective opinions of experts. These methods are useful for reasoning about uncertain processes and phenomena in cases when the lack of statistical information does not allow one to apply probabilistic methods, or when uncertainty has a non-probabilistic nature. The applications such as prognostication of social-economic phenomena, medical diagnostics, modeling of human-machine systems, etc. often require differential equations with uncertainty in the structure and/or parameters. However, in these applications the available statistical information is often rather limited or unreliable (because of absence of repetitions of the studied phenomena under the same conditions). Therefore, it is reasonable to apply non-probabilistic uncertainty theories (e.g. possibility theory) in such cases [4, 20].

* Corresponding author: <mailto:bos.knu@gmail.com>

However, to the best of our knowledge, in the context of possibility theory a theory of uncertain differential equations has not been developed in the literature. In contrast, in the context of L. Zadeh's fuzzy set theory, fuzzy differential equations were studied extensively [2, 3, 8, 13–16, 19]. Such studies often consider either ordinary differential equations with fuzzy parameters [15], or equations of the evolution of a membership function [2, 12, 14, 15]. Although these approaches sometimes provide an alternative to stochastic modeling, they have some drawbacks. Differential equations with fuzzy parameters do not allow one to describe uncertain dynamic changes in the law of evolution (right-hand side of equation), because fuzzy parameters do not depend on time. The equations of evolution of membership function are not direct generalizations of ordinary differential equations. In most applications differential equations describe an evolution of the state of a system, but it is not obvious how to convert a state equation into an equation describing evolution of a membership function.

In this paper we propose a different approach to modeling of uncertain dynamics, which is based on possibility theory. We argue that it addresses the disadvantages of fuzzy differential equations described above.

Our class of possibilistic differential equations is based on the notion of a possibilistic walk process. Such equations can be considered as possibilistic analogs of stochastic Ito equations which have a wide range of applications in stochastic modeling. We will study the problems of existence and uniqueness of solutions of these equations and provide a numerical method for their solution.

2 Preliminaries

We will use the following framework of (quantitative) possibility theory [4, 7]. Let X be a non-empty set of elementary events and (X, \mathbf{A}) , $\mathbf{A} \subseteq 2^X$ be a measurable space. The elements of \mathbf{A} are called (compound) events.

Definition 2.1 A possibility measure is a function $P : \mathbf{A} \rightarrow [0, 1]$ such that

$$P\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} P(A_i)$$

for any collection $(A_i)_{i \in I}$ of elements of \mathbf{A} such that $\bigcup_{i \in I} A_i \in \mathbf{A}$.

Definition 2.2 A necessity measure is a function $N : \mathbf{A} \rightarrow [0, 1]$ such that

$$N\left(\bigcap_{i \in I} A_i\right) = \inf_{i \in I} N(A_i) \tag{1}$$

for any collection $(A_i)_{i \in I}$ of elements of \mathbf{A} such that $\bigcap_{i \in I} A_i \in \mathbf{A}$.

Definition 2.3 A possibility space is a tuple (X, \mathbf{A}, P, N) , where P and N are respectively a possibility and necessity measure on the measurable space (X, \mathbf{A}) .

Definition 2.4 A possibility space (X, \mathbf{A}, P, N) is called regular, if $P(X) = 1$, $N(X) = 1$, and $N(A) = 1 - P(\neg A)$ for all $A \subseteq X$ (where $\neg A$ denotes the complement of a set $A \subseteq X$).

Definition 2.5 A possibility space (X, \mathbf{A}, P, N) is called complete, if $\mathbf{A} = 2^X$ (the power set of X).

The assumptions of the regular possibility space are rather standard and are used in many works on possibility theory [11, 20]. It was shown in the work [5] that a regular possibility space (X, \mathbf{A}, P, N) can be embedded in some complete regular possibility space $(X, 2^X, P', N')$, where P' and N' are extensions of P and N . For this reason, in this article we will consider only complete regular possibility spaces.

Let us fix a complete regular possibility space $(X, 2^X, P, N)$ and denote

$$X_\alpha = \{x \in X \mid P(\{x\}) > \alpha\}$$

for each $\alpha \in [0, 1]$. In particular, X_0 is the set of elementary events which have non-zero possibility.

Let $\mathbb{R}_+ = [0, +\infty)$ and T be a finite or infinite interval in \mathbb{R}_+ . Under our assumption of completeness of the possibility space we will use the following terminology:

- A *possibilistic variable* is a (total) function $\xi : X \rightarrow Y$; if $Y = \mathbb{R}$, then ξ is called a scalar possibilistic variable; if $Y = \mathbb{R}^d$ (where d is a natural number), then ξ is called a vector possibilistic variable.
- The *distribution* of a possibilistic variable $\xi : X \rightarrow Y$ is a mapping $\mu_\xi : Y \rightarrow [0, 1]$ such that $\mu_\xi(y) = P\{x \in X \mid \xi(x) = y\}$.
- Possibilistic variables $\xi_k : X \rightarrow Y_k$, $k = 1, 2, \dots, m$ are called *non-interactive* (independent), if the distribution $\mu_{\xi_1, \xi_2, \dots, \xi_m}$ of the vector possibilistic variable $(\xi_1, \xi_2, \dots, \xi_m)$ satisfies the condition

$$\mu_{\xi_1, \xi_2, \dots, \xi_m}(u_1, u_2, \dots, u_m) = \min\{\mu_{\xi_1}(u_1), \mu_{\xi_2}(u_2), \dots, \mu_{\xi_m}(u_m)\}$$

for all $u_1 \in Y_1, u_2 \in Y_2, \dots, u_m \in Y_m$.

- A *possibilistic process* is a (total) function $p : T \times X \rightarrow Y$; if $Y = \mathbb{R}$, then p is called a scalar process; if $Y = \mathbb{R}^d$, then p is called a vector process.
- A *trajectory* of a possibilistic process $p : T \times X \rightarrow Y$ is a mapping $t \mapsto p(t, x)$ for a fixed $x \in X$.
- the *distribution of a process* $p : T \times X \rightarrow Y$ is a function $F_p : 2^{T \times Y} \rightarrow [0, 1]$, where

$$F_p(q) = P(\{x \in X \mid \forall t \in T p(t, x) = q(t)\})$$

for each function $q : T \rightarrow Y$, i.e. $F_p(q)$ is a possibility of the event "q is a trajectory of p".

- An α -*trajectory* of p (where $\alpha \in [0, 1)$) is a function $q : T \rightarrow Y$ such that $F_p(q) > \alpha$, i.e. q is a trajectory of p with a possibility level greater than α .

We will abbreviate $P(\{x \in X \mid pred(\xi(x))\})$ as $P\{pred(\xi)\}$, where $pred$ is some predicate. For example, $P\{\xi = y\}$ will denote $P(\{x \in X \mid \xi(x) = y\})$. Also, we will usually omit the second argument (elementary event) of a possibilistic process. For example, $P\{p(t) = 1\}$ will denote $P(\{x \in X \mid p(t, x) = 1\})$.

We will denote by $\|\cdot\|$ the Euclidean norm on \mathbb{R}^d .

Definition 2.6 [4]. A possibilistic variable $\xi : X \rightarrow \mathbb{R}^d$ is called normal, if its distribution has a form

$$\mu_\xi(y) = \varphi\left(\left\|\Xi^{-1/2}(y - y_0)\right\|^2\right),$$

where $\varphi : \mathbb{R}_+ \rightarrow [0, 1]$ is a monotonically decreasing function such that $\lim_{u \rightarrow +\infty} \varphi(u) = 0$ and $\varphi(0) = 1$, y_0 is a constant vector (mean value), Ξ is a positive-definite matrix (covariance-like matrix).

Definition 2.7 [4,20]. A possibilistic walk process $w : \mathbb{R}_+ \times X \rightarrow \mathbb{R}^d$ is a possibilistic process such that:

1. w has non-interactive increments, i.e. for any time moments $0 \leq t_1 < t_2 < \dots < t_{n+1}$, the possibilistic variables $w(t_{i+1}) - w(t_i)$, $i = 1, 2, \dots, n$ are non-interactive.
2. For each $t_0 \geq 0$, $t > t_0$, $y, y_0 \in \mathbb{R}^d$ the transition possibility has a form

$$P\{w(t) = y, w(t_0) = y_0\} = \varphi\left(\frac{\left\|\Xi^{-1/2}(y - y_0)\right\|^2}{t - t_0}\right),$$

where Ξ (a covariance-like matrix of w) is a positive-definite matrix, and $\varphi : \mathbb{R}_+ \rightarrow [0, 1]$ (a distribution function of w) is a monotonically decreasing function such that $\lim_{u \rightarrow +\infty} \varphi(u) = 0$ and $\varphi(0) = 1$.

3. $w(0, x) = 0$.

Possibilistic walk processes can be considered as analogs of stochastic Wiener processes. The existence of a possibilistic walk process was established in [6], where it was proved that for any φ such that $\lim_{u \rightarrow +\infty} \varphi(u) = 0$, $\varphi(0) = 1$ and for any positive-definite matrix Ξ there exists a possibility space and a possibilistic walk process w such that φ is a distribution function of w and Ξ is a covariance-like matrix of w .

3 Main Result

Let w be a scalar possibilistic walk process with $\Xi = 1$ and a distribution function $\varphi : \mathbb{R}_+ \rightarrow [0, 1]$. Let D be a domain in \mathbb{R}^d (where $d \geq 1$), and $a : \mathbb{R}_+ \times D \rightarrow \mathbb{R}^d$ and $b : \mathbb{R}_+ \times D \rightarrow \mathbb{R}^d$ be continuous mappings. Let $(t_0, y_0) \in \mathbb{R}_+ \times D$.

We will use the following lemma to construct our class of possibilistic differential equations:

Lemma 3.1 [6]. For each $\alpha \in [0, 1]$, $t \in \mathbb{R}_+$, and $x \in X_\alpha$, the trajectory $t \mapsto w(t, x)$ is locally absolutely continuous and satisfies the following inequality almost everywhere on \mathbb{R}_+ (with respect to Lebesgue measure):

$$\left|\frac{\partial w(t, x)}{\partial t}\right| \leq \sqrt{\varphi^{-1}(\alpha)}.$$

Consider the following initial-value problem with parameter $x \in X$:

$$dy(t, x) = a(t, y(t, x))dt + b(t, y(t, x))dw(t, x), \quad (2)$$

$$y(t_0, x) = y_0, \quad (3)$$

or the same problem in the integral form:

$$y(t, x) = y_0 + \int_{t_0}^t a(s, y(s, x))ds + \int_{t_0}^t b(s, y(s, x))dw(s, x). \tag{4}$$

Definition 3.1 An α -solution (where $\alpha \in [0, 1)$) of the problem (2)-(3) (or the problem (4)) on an interval $I \subseteq \mathbb{R}_+$ is a possibilistic process $y : I \times X \rightarrow D$ such that for each $x \in X_\alpha$ the trajectory $t \mapsto y(t, x)$ is locally absolutely continuous and satisfies (2)-(3) almost everywhere on I (in the sense of Lebesgue measure).

A solution of the problem (2)-(3) is a 0-solution of this problem.

We will use a special notion of uniqueness of solutions:

Definition 3.2 The problem (2)-(3) (or the problem (4)) has a unique α -solution on I , if it has some α -solution, and each two α -solutions of (2)-(3) on I are equal on the set $I \times X_\alpha$. The problem (2)-(3) has a unique solution, if it has a unique 0-solution.

Let us denote by $B(y_0, r) = \{y \in \mathbb{R}^d \mid \|y - y_0\| \leq r\}$ a closed ball in \mathbb{R}^d .

Theorem 3.1 (About existence and uniqueness of α -solution) *Assume that the functions $a(t, y)$ and $b(t, y)$ are continuous on the set $C = I \times B(y_0, r)$, where $I = [t_0, t_0 + \Delta t]$, $\Delta t > 0$, $r > 0$, and satisfy Lipschitz condition with respect to y , i.e.*

$$\|a(t, y) - a(t, z)\| \leq L\|y - z\|, \quad \|b(t, y) - b(t, z)\| \leq L\|y - z\|,$$

for some constant $L > 0$ and all $t \in I$, $y, z \in B(y_0, r)$.

Let $\alpha \in (0, 1)$, $M_a = \max_{(t,y) \in C} \|a(t, y)\|$, $M_b = \max_{(t,y) \in C} \|b(t, y)\|$. Then the problem (2)-(3) has a unique α -solution on $[t_0, t_0 + h)$, where

$$h = \min \left\{ \frac{1}{2L}, \frac{1}{\sqrt{2L} \sqrt[4]{\varphi^{-1}(\alpha)}}, \frac{r}{M_a + \sqrt{\varphi^{-1}(\alpha)}M_b}, \Delta t \right\}.$$

Proof. Let us fix a number $\epsilon \in (0, 1)$ and denote $I_\epsilon = [t_0, t_0 + \epsilon h]$. Consider the space F_ϵ of all continuous functions $f : I_\epsilon \rightarrow B(y_0, r)$ such that $f(t_0) = y_0$. Let us define a uniform metric on this space:

$$\rho_\epsilon(f, g) = \max_{t \in I_\epsilon} \|f(t) - g(t)\|.$$

Because $B_\epsilon(y_0, r)$ is closed, it is a complete subspace of \mathbb{R}^d . Then the space of all continuous (and bounded) functions $f : I_\epsilon \rightarrow \mathbb{R}^d$ with metric ρ_ϵ is complete. Thus F_ϵ is a (non-empty) complete metric space.

Let us fix an elementary event $x_0 \in X_\alpha$ and consider the mapping $\Phi_\epsilon : F_\epsilon \rightarrow (I_\epsilon \rightarrow \mathbb{R}^d)$ such that

$$\Phi_\epsilon(f)(t) = y_0 + \int_{t_0}^t a(s, f(s))ds + \int_{t_0}^t b(s, g(s))dw(s, x_0).$$

For each $f \in F_\epsilon$ the function $t \mapsto \Phi_\epsilon(f)(t)$ is defined and continuous I_ϵ , because $h \leq \Delta t$, $s \mapsto a(s, f(s))$ and $s \mapsto b(s, f(s))$ are continuous on I_ϵ , and the trajectory $s \mapsto w(s, x_0)$ is absolutely continuous on I_ϵ .

Also, we have $\Phi_\epsilon(f)(t_0) = y_0$ and for each $t \in I_\epsilon$,

$$\begin{aligned} \|\Phi_\epsilon(f)(t) - y_0\| &\leq \sup_{t \in I_\epsilon} \left(\int_{t_0}^t \|a(s, f(s))\| ds + \int_{t_0}^t \|b(s, f(s))\| \left| \frac{\partial w(s, x_0)}{\partial s} \right| ds \right) \leq \\ &\leq \epsilon h \left(\max_{(s,y) \in C} \|a(s, y)\| + \sqrt{\varphi^{-1}(\alpha)} \max_{(s,y) \in C} \|b(s, y)\| \right) = \\ &= \epsilon h (M_a + \sqrt{\varphi^{-1}(\alpha)} M_b) \leq \epsilon r < r \end{aligned}$$

by Lemma 3.1. Thus Φ_ϵ maps F_ϵ to itself. Let us prove that Φ_ϵ is a contracting mapping. The Lipschitz condition implies that

$$\begin{aligned} \rho_\epsilon(\Phi_\epsilon(f), \Phi_\epsilon(g)) &\leq \max_{t \in I_\epsilon} \left(\int_{t_0}^t \|a(s, f(s)) - a(s, g(s))\| ds + \right. \\ &\quad \left. \int_{t_0}^t \|b(s, f(s)) - b(s, g(s))\| \left| \frac{\partial w(s, x_0)}{\partial s} \right| ds \right) \leq \\ &\leq (L + L\epsilon h \sqrt{\varphi^{-1}(\alpha)}) \epsilon h \rho(f, g) \leq \\ &\leq 2L \max \left\{ \epsilon h, \sqrt{\varphi^{-1}(\alpha)} (\epsilon h)^2 \right\} \rho(f, g) \leq \max \{ \epsilon, \epsilon^2 \} \rho(f, g), \end{aligned}$$

because $h \leq \min \left(\frac{1}{2L}, \frac{1}{\sqrt{2L} \sqrt{\varphi^{-1}(\alpha)}} \right)$. Then the mapping Φ_ϵ is contracting, because $\epsilon \in (0, 1)$. By the Banach fixed point theorem, Φ has a unique fixed point. Obviously, this fixed point is absolutely continuous and satisfies (2)-(3) almost everywhere on I_ϵ . On the other hand, it is easy to see that every absolutely continuous function which satisfies (2)-(3) almost everywhere on I_ϵ is a fixed point of Φ_ϵ . Then because $\epsilon \in (0, 1)$ and $x_0 \in X_\alpha$ are arbitrary, it is straightforward to show that the problem (2)-(3) has a unique α -solution on $[t_0, t_0 + h)$ in sense of Definition 3.2. \square

Theorem 3.2 (About global existence and uniqueness of solution)

Assume that the functions $a(t, y)$ and $b(t, y)$ are continuous on the set $C = [t_0, +\infty) \times \mathbb{R}^d$ and satisfy a local Lipschitz condition with respect to y : there exists a continuous function $L : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\|a(t, y) - a(t, z)\| \leq L(r) \|y - z\|,$$

$$\|b(t, y) - b(t, z)\| \leq L(r) \|y - z\|,$$

for all $t \geq 0$, $r > 0$, and $y, z \in B(y_0, r)$. Assume that the functions $a(t, y)$, $b(t, y)$ satisfy the following growth conditions for some constant $K > 0$:

$$\|a(t, y)\|^2 \leq K (1 + \|y\|^2),$$

$$\|b(t, y)\|^2 \leq K (1 + \|y\|^2).$$

Then the problem (2)-(3) has a unique solution on $[t_0; +\infty)$.

Proof. Let us choose an arbitrary $x_0 \in X_0$. Then $x_0 \in X_\alpha$ for some $\alpha \in (0, 1)$.

Assume that a function $t \mapsto y(t, x_0)$ is defined (and continuous) on some segment $I = [t_0, t_0 + h]$, $h > 0$, and satisfies (4) on I . Then

$$\|y(t, x_0) - y_0\| \leq \int_{t_0}^t \|a(s, y(s, x_0))\| ds + \int_{t_0}^t \|b(s, y(s, x_0))\| \left| \frac{\partial w(s, x_0)}{\partial s} \right| ds.$$

This inequality, the growth conditions, and Lemma 3.1 imply that

$$\|y(t, x_0) - y_0\| \leq \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) \int_{t_0}^t \left(1 + K\|y(s, x_0)\|^2\right)^{1/2} ds. \tag{5}$$

Then for all $t \in I$,

$$\|y(t, x_0)\| \leq R(t),$$

where a scalar function $R(t)$ satisfies the following Cauchy problem:

$$R(t) = \|y_0\| + \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) \int_{t_0}^t \left(1 + KR(t)^2\right)^{1/2} ds. \tag{6}$$

It is easy to check that (6) has the following solution defined for all $t \geq t_0$:

$$R(t) = \frac{1}{\sqrt{K}} \sinh \left(\sqrt{K} \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) (t - t_0) + \sinh^{-1}(\sqrt{K}\|y_0\|) \right).$$

Thus any extension of $t \mapsto y(t, x_0)$ from $I = [t_0, t_0 + h]$ to $[t_0, t_0 + h']$, $h' > h$ which satisfies (4) has a norm bounded from above by the function $R(t)$.

For each $r > 0$ let us denote

$$h(r) = \min \left\{ \frac{1}{2L(r)}, \frac{1}{\sqrt{2L(r)} \sqrt[4]{\varphi^{-1}(\alpha)}}, \frac{r}{M(r) + \sqrt{\varphi^{-1}(\alpha)}M(r)} \right\},$$

$$M(r) = \sqrt{K(1 + (r + \|y_0\|)^2)}.$$

The growth conditions imply that

$$\max_{t \geq 0, y \in B(y_0, r)} \|a(t, y)\| \leq M(r), \quad \max_{t \geq 0, y \in B(y_0, r)} \|b(t, y)\| \leq M(r).$$

Then from Theorem 3.1 we have that for each $t'_0 \geq t_0$, $r' > 0$ and $y'_0 \in \mathbb{R}^d$ such that $B(y'_0, r') \subseteq B(y_0, r)$ the problem (2) together with initial condition $y(t'_0) = y'_0$ has a unique α -solution $y_{r', t'_0, y'_0}(t, x)$ on $[t'_0, t'_0 + h(r')]$ (because we can choose an arbitrary $\Delta t > 0$ in the statement of Theorem 3.1).

Let us fix an arbitrary $\tau > t_0$. Let us construct a finite or infinite sequences of trajectories $y_1(t), y_2(t), \dots$, positive numbers r_0, r_1, r_2, \dots and time moments t_1, t_2, \dots (t_0 is defined as in the statement of this theorem) such that

- $r_0 = R(\tau) - \|y_0\| + 1$;
- if $n \geq 0$ and $t_n < \tau$, then

- $t_{n+1} = t_n + h(r_n)/2$,
- $y_{n+1}(t) = y_{r_n, t_n, y_n(t_n)}(t, x_0)$ for all $t \in [t_n, t_{n+1}]$ (here $y_0(\cdot) = y_0$),
- $r_{n+1} = r_0 - (R(t_{n+1}) - \|y_0\|)$.

The sequence (t_n) is increasing and (r_n) is decreasing (because the function R is strictly monotone). These sequences may be finite, if (t_n) reaches or becomes greater than τ . It is easy to check by induction on n that the functions y_1, y_2, \dots, y_n are indeed correctly defined and their concatenation is a trajectory which satisfies (2)-(3) on $[t_0, t_n]$ using the inclusion $B(y_n(t_n), r_n) \subseteq B(y_0, r_0)$ which follows from (5) and (6).

If we assume that the sequence (t_n) is infinite, then it is bounded from above (by τ) and the equation $t_{n+1} = t_n + h(r_0 - R(t_n) + \|y_0\|)/2$ holds for all $n \geq 1$. Then because of continuity of the functions h and R , we have $h(r_0 - R(\lim_{n \rightarrow \infty} t_n) + \|y_0\|) = h(1 + R(\tau) - R(\lim_{n \rightarrow \infty} t_n)) = 0$. But this is impossible when $\lim_{n \rightarrow \infty} t_n < \tau$. Thus the sequence (t_n) is finite or its elements tend to τ . This implies that there exists a trajectory $t \mapsto y(t, x_0)$ which satisfies (2)-(3) on $[t_0, \tau)$.

Because $\tau > t_0$, $\alpha \in (0, 1)$ and $x_0 \in X_\alpha$ are arbitrary, we conclude that the problem (2)-(3) has a unique solution on $[t_0; +\infty)$ in the sense of Definition 3.1. Uniqueness of this solution (in sense of Definition 3.2) easily follows from Theorem 3.1. \square

4 Numerical Solution

Definition 4.1 The (t, α) -cut of a solution of the equation (4) is the set

$$Y(t, \alpha) = \{y(t, x) \mid x \in X_\alpha\},$$

where $t \in \mathbb{R}_+$, $\alpha \in [0, 1)$.

The (t, α) -cut contains all points which can be reached by α -trajectories of a solution at time t . The family of all cuts of the solution gives a complete description of its distribution.

Definition 4.2 An estimate of (t, α) -cut of the solution of the equation (4) is a set $\hat{Y}(t, \alpha) \subseteq \mathbb{R}^n$ such that the (t, α) -cut $Y(t, \alpha)$ is a dense subset of $\hat{Y}(t, \alpha)$.

By a *numerical solution* of the equation (4) we mean some numerical representation of a family of estimates of (t_i, α_i) -cuts for a finite set of pairs $\{(t_i, \alpha_i) \mid i \in I\}$. The numerical solution gives information about sets which can be reached by the solution of (4) with a given level of possibility.

Let us associate with the equation (4) the following dynamical system with scalar input control $u(t)$:

$$dz(t) = a(t, z(t))dt + b(t, z(t))u(t), \quad (7)$$

$$z(t_0) = y_0. \quad (8)$$

Let us denote by $BU(r)$ the set of all bounded measurable controls $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\sup_t |u(t)| \leq r$.

Theorem 4.1 [6]. The set $U(t, \alpha) \subseteq \mathbb{R}^n$ of all points which can be reached by the system (7), (8) at time t by means of controls $u \in BU(\sqrt{\varphi^{-1}(\alpha)})$ is an estimate of (t, α) -cut of the solution of equation (4).

This theorem reduces the problem of finding numerical solution of the equation (4) to the problem of finding reachable sets of the controlled system (7), (8). The problem of finding reachable sets is well studied [1] and can be solved numerically using existing tools such as dynamic programming.

5 Numeric Example

Let us consider how the results obtained above can be applied to the problem of modeling dynamics of epidemics. We start with a simple Ross epidemic model [18]. In this model the population of N individuals is divided into two groups:

- susceptible individuals, S ;
- infective individuals, I .

It is assumed that the following statements hold:

- (1) the population is homogeneous, there are no births, deaths, immigrations and emigrations;
- (2) there is no latent period of the infection, recoveries from illness are not taken into account;
- (3) the infection rate is proportional to the fraction of infectives.

The model is described by the following equations:

$$S(t) = N - I(t), \quad (9)$$

$$\frac{dI}{dt} = aI(t)(N - I(t)), \quad (10)$$

where $S(t)$ and $I(t)$ are the numbers of susceptible and infective individuals at time t , N is the total number of individuals (constant), a is a positive constant.

The model (9)-(10) can be improved by taking into account recovery and transmission of disease from external source as described in [9]. Let us denote by $y(t) = I(t)/N$ the *fraction* of infected individuals. The improved model has the form

$$y'(t) = ay(t)(1 - y(t)) - by(t) + c(1 - y(t)), \quad (11)$$

where

- $a > 0$ is the rate of transmission from individual to individual;
- $b > 0$ is the rate of recovery;
- $c > 0$ is the rate of transmission from external source.

Although the model (11) is more accurate than (9)-(10), it is still a rather rough simplification of the real dynamics of epidemics. To take into account inaccuracy of (9), following [9] let us add a dynamic uncertainty to this model:

$$dy(t, x) = ay(t, x)(1 - y(t, x))dt - by(t, x)dt + c(1 - y(t, x))dt + \sigma(y(t))dw(t, x), \quad (12)$$

where $\sigma(y)$ is a function of the form $\delta y(1 - y)$, $\delta > 0$. This equation differs from a stochastic epidemic model proposed in [9] in that the uncertainty is modeled by a possibilistic walk process $w(t, x)$ instead of the Wiener process. This allows one to estimate the influence of uncertainties on propagation of epidemics on the basis of expert opinions [11, 20] instead of statistical data (the latter may be very limited for new or unfamiliar types of infections).

Then we obtain the final possibilistic epidemic model:

$$\begin{aligned} dy(t, x) &= ay(t, x)(1 - y(t, x))dt - by(t, x)dt + \\ &+ c(1 - y(t, x))dt + \delta y(t)(1 - y(t, x))dw(t, x), \\ y(0, x) &= y_0. \end{aligned} \quad (13)$$

It is not difficult to check that this problem has a unique solution (to simplify this task it is sufficient to assume that y always takes values in $[0, 1]$, because it represents a fraction of infected individuals).

The solution $y(t, x)$ is a possibilistic process. Let us find an estimate of α -cut of $y(t, x)$. Let us apply the system (7)-(8) to the equation (13):

$$\begin{aligned} z'(t) &= az(t)(1 - z(t)) - bz(t) + \\ &+ c(1 - z(t)) + \delta z(t)(1 - z(t))u(t), \\ z(0) &= y_0. \end{aligned} \quad (14)$$

Let us define $y_1(t)$, $y_2(t)$ as solutions of the following equations:

$$\begin{aligned} y_1'(t) &= ay_1(t)(1 - y_1(t)) - by_1(t) + c(1 - y_1(t)) - \\ &- \sqrt{\varphi^{-1}(\alpha)}|\delta y_1(t)(1 - y_1(t))|, \quad y_1(0) = y_0, \end{aligned} \quad (15)$$

$$\begin{aligned} y_2'(t) &= ay_2(t)(1 - y_2(t)) - by_2(t) + c(1 - y_2(t)) + \\ &+ \sqrt{\varphi^{-1}(\alpha)}|\delta y_2(t)(1 - y_2(t))|, \quad y_2(0) = y_0. \end{aligned} \quad (16)$$

It is easy to verify that the segment $[y_1(t), y_2(t)]$ is a reachable set at time t for the system (14) with controls $u \in BU(\sqrt{\varphi^{-1}(\alpha)})$. So the set $[y_1(t), y_2(t)]$ is an estimate of α -cut of the solution of (14) by Theorem [6].

Assume that $\alpha > \varphi(a^2/\delta^2)$. Then non-negative stationary solutions of the equations (15), (16) are given by the following expressions:

$$\begin{aligned} \hat{y}_1(\alpha) &= \frac{a - b - c + \delta C_\alpha + \sqrt{(a - b + c + \delta C_\alpha)^2 + 4bc}}{2(a + \delta C_\alpha)}, \\ \hat{y}_2(\alpha) &= \frac{a - b - c - \delta C_\alpha + \sqrt{(a - b + c - \delta C_\alpha)^2 + 4bc}}{2(a - \delta C_\alpha)}, \end{aligned}$$

where $C_\alpha = \sqrt{\varphi^{-1}(\alpha)}$.

Thus we can accept that for large t , the fraction of infected individuals belongs to the segment $[\hat{y}_1(\alpha), \hat{y}_2(\alpha)]$ with the level of possibility α .

Let us consider a numerical example.

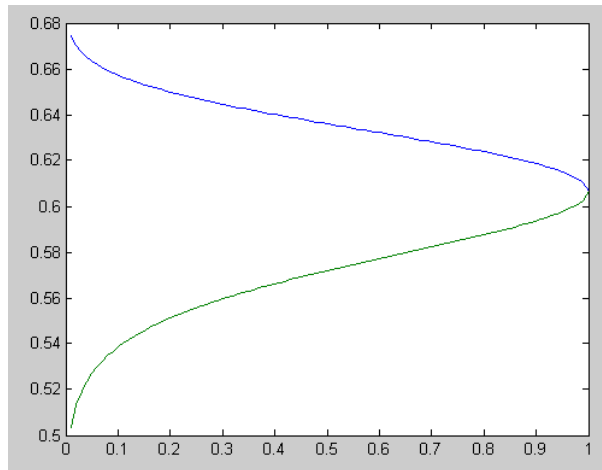


Figure 1: The lower and upper bound for the fraction of infected individuals when $a = 1$. The horizontal axis represents the possibility level α .

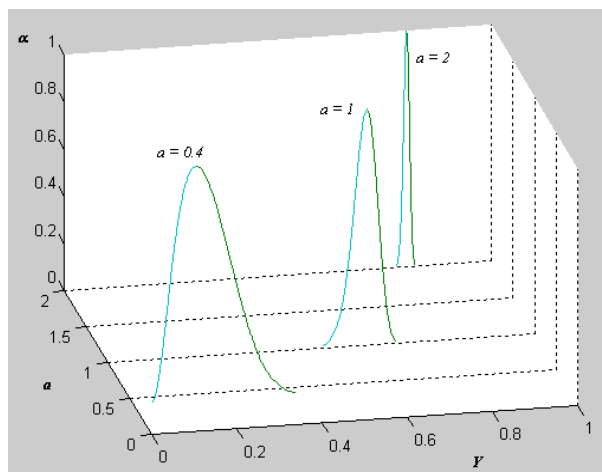


Figure 2: The lower and upper bounds for the fraction of infected individuals for different values of parameter a .

Example 5.1 Let $\varphi(x) = \exp(-x)$, $a = 1$, $b = 0.4$, $c = 0.01$, $\delta = 0.1$. Figure 1 shows the curves \hat{y}_1 (lower bound) and \hat{y}_2 (upper bound). The horizontal axis represents the possibility level α . Figure 2 shows the similar curves for different values of a (but the possibility level α is represented on vertical axis).

Figures 1 and 2 were produced by the following MATLAB [17] program:

```
function r = f(alpha,a,s)
    b = 0.4; c = 0.01;
    dC = s * 0.1 * sqrt(-log(alpha));
```

```

r = (a-b-c+dC + sqrt((a-b+c+dC).^2+4*b*c))./(2*(a+dC));

I = 0.01:0.01:1; nul = zeros(length(I));
plot3(f(I,0.4,1),nul+0.4,I, f(I,0.4,-1),nul+0.4,I); hold on
plot3(f(I,1,1),nul+1,I, f(I,1,-1),nul+1,I);
plot3(f(I,2,1),nul+2,I, f(I,2,-1),nul+2,I);

```

6 Conclusion

In the paper we have studied the problem of modeling of uncertain dynamics using the methods of possibility theory. We have constructed a new class of uncertain differential equation with respect to a possibilistic walk process. We have studied the problems of existence and uniqueness of solutions of these equations and proposed a method which can be used to solve them numerically.

The obtained results can be used for modeling social-economic and ecological phenomena, medical diagnostic tasks, and other uncertain processes or phenomena in which available statistical information is not sufficient for constructing a reliable stochastic model, or the uncertainty has a non-probabilistic nature.

References

- [1] Aubin, J.P. Fuzzy differential inclusions. *Problems of Control and Information Theory* **19** (1990) 55–67.
- [2] Buckley, J.J. and Feuring, T. Fuzzy differential equations. *Fuzzy Sets and Systems* **110** (2000) 43–54.
- [3] Buckley, J.J. and Qu, Y. Solving fuzzy equations: a new solution concept. *Fuzzy Sets and Systems* **50** (1992) 1–14.
- [4] Bychkov, A.S. Modelling and studying fuzzy dynamical systems. *Journal of Mathematical Sciences* **3** (2009) 466–479.
- [5] Bychkov, O.S., Ivanov, Ie.V. Existence of a coherent κ -additive extension of measures in possibility theory. *J. Control Systems and Computers* **6** (2009) 35–41 [Russian]
- [6] Bychkov, O.S. and Ivanov, Ie.V. Numerical solution of a fuzzy differential equation. *J. Mathematical machines and systems* **1** (2009) 31–39. [Ukrainian]
- [7] Bychkov, O.S. About possibility theory and its application. *Reports of National Academy of Sciences of Ukraine* **5** (2007) 7–12. [Ukrainian]
- [8] Chen, X.W. and Liu, B. Existence and uniqueness theorem for uncertain differential equations. *Fuzzy Optimization and Decision Making* **1** (2010) 69–81.
- [9] Cobb, L. and Thrall, R.M. Mathematical frontiers of the social and policy sciences. In: *Proc. AAAS Selected Symposium*. Boulder, Colorado, Westview Press, Inc., 1981.
- [10] Cooman, G. Possibility theory. Part I: Measure- and integral-theoretic groundwork. *International Journal of General Systems* **25** (1997) 291–371.
- [11] Dubois, D. and Prade, H. *Possibility theory*. Plenum Press, New York, 1988.
- [12] Friedmanan, Y. and Sandler, U. A new approach to fuzzy dynamics. In: *Proc. of the 12th IAPR International Conf.* Jerusalem, Israel, 1994.
- [13] Kaleva, O. The Cauchy problem for fuzzy differential equations. *Fuzzy Sets and Systems* **35** (1990) 389–396.
- [14] Kloeden, P. E. Fuzzy dynamical systems. *Fuzzy Sets and Systems* **7** (1982) 275–296.

- [15] Lakshmikantham, V. and Mohapatra, R. *Theory of Fuzzy Differential Equations and Inclusions*. Taylor and Francis, London, 2003.
- [16] Leland, R.P. Fuzzy differential systems and Malliavin calculus. *Fuzzy Sets and Systems* **70** (1995) 59–73.
- [17] MathWorks – MATLAB and Simulink for Technical Computing. Online:
<http://www.mathworks.com>
- [18] Medlock, J. Mathematical analysis of epidemiology models. Online:
http://people.oregonstate.edu/~medlockj/other/mathEpiIntro_I.pdf
- [19] Park, J.Y. and Han, H.K. Existence and uniqueness theorem for a solution of fuzzy differential equations. *Internat. J. Math. & Math. Sci.* **22** (1999) 271–279.
- [20] Pytiev, Yu.P. *Possibility. Elements of Theory and Applications*. Moscow: URSS, 2000. [Russian]
- [21] Seikkala, S. On the fuzzy initial value problem. *Fuzzy Sets and Systems* **24** (1987) 319–330.



On Solutions to a Nonautonomous Neutral Differential Equation with Deviating Arguments

Rajib Haloi

*Department of Mathematical Sciences
Tezpur University, Assam, India 784028.*

Received: July 10, 2012; Revised: July 23, 2013

Abstract: The main objective of this paper is to study solutions of a non-autonomous neutral differential equation of parabolic type with a deviating argument in an arbitrary Banach space. The main results are obtained by the Sobolevskii–Tanabe theory of parabolic equations and the Banach fixed point theorem.

Keywords: *analytic semigroup; neutral differential equation; Banach fixed point theorem.*

Mathematics Subject Classification (2010): 34G20, 34K30, 34K40, 47N20.

1 Introduction

A natural way of generalizing differential equations is allowing the unknown function to appear with different values of the argument. Thus, differential equations with a deviating argument are differential equations in which the unknown function and its derivative appear in different places of the argument. This type of equations arise in many fields such as the theory of automatic control, the theory of self-oscillating systems, the problems of long-term planning in economics, the study of problems related with combustion in rocket motion, a series of biological problems, and many other areas [2]. One of the important examples is the process in fuel injection system for high-speed diesel engines which can be modeled as differential equations with a deviating argument of neutral type (see [2]).

The purpose of this work is to study solutions of the following type of neutral equation in a Banach space $(X, \|\cdot\|)$:

$$\left. \begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A(t)u(t) &= f(t, u(t), u(h(u(t), t))), \quad t > 0; \\ u(0) &= u_0, \end{aligned} \right\} \quad (1)$$

* Corresponding author: <mailto:rajib.haloi@gmail.com>

where $u : \mathbb{R}_+ \rightarrow X$. Here, we assume that $-A(t)$, for each $t \geq 0$, generates an analytic semigroup of bounded linear operators on X . The continuous functions f, g and h satisfy suitable conditions in their arguments and the function $a : [0, T] \rightarrow [0, T]$ satisfies the delay properties.

The plentiful applications motivate the development of the theory of differential equation with deviating arguments (see e.g. [1, 6–9, 13, 14, 17–19] and references cited therein).

Fu and Liu [6] have considered the following abstract neutral functional equation with infinite delay:

$$\begin{aligned} \frac{d}{dt}[u(t) + f(t, u_t)] + A(t)u(t) &= g(t, u_t), \quad t \in (0, T], \\ u_0 &= \phi \in \mathcal{C}_0. \end{aligned}$$

Here $u(t)$ takes values in a Banach space X , the family $\{A(t) : t \in [0, T], T \in [0, \infty)\}$ of unbounded linear operators generates a bounded linear evolution operators on X , the function $f : [0, T] \times \mathcal{C}_0 \rightarrow X$ is uniformly Lipschitz continuous in both variables, the function $g : [0, T] \times \mathcal{C}_0 \rightarrow X$ satisfies suitable conditions (here \mathcal{C}_0 is a phase space defined appropriately). The existence of a solution has been obtained by the Sadovskii fixed point principle.

In [8], Haloi *et. al.* have studied the existence of solutions to the following differential equation

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A(t)[u(t) + g(t, u(a(t)))] &= f(t, u(t), u(h(u(t), t))), \quad t > 0; \\ u(0) &= u_0. \end{aligned}$$

The main results are obtained by the Banach fixed point theorem without any regularity assumption on the function g .

Using the Banach fixed point theorem and the Sobolevskii–Tanabe theory of parabolic equations, we prove the existence, uniqueness and asymptotic stability of a solution to Problem (1). The main results generalize some results of [7], [9], [14] and [19]. The work is organized as follows. In Section 2, we provide preliminaries, assumptions and lemmas that will be needed for proving the main results. In Section 3, we prove the main results. Finally, we discuss an example as an application of the abstract results.

2 Preliminaries and Assumptions

This section deals with basic assumptions, preliminaries and lemmas necessary for proving the main results. For more details, we refer to [4, 12, 15, 16].

Let $(X, \|\cdot\|)$ be a complex Banach space. Let $\{A(t) : 0 \leq t \leq T, 0 \leq T < \infty\}$ be a family of linear operators on the Banach space X . We use the following assumptions.

- (H₁) For each $t \in [0, T]$, $A(t)$ is closed linear operator with domain $D(A)$ of $A(t)$ independent of t and dense in X .
- (H₂) For each $t \in [0, T]$, the resolvent $R(\lambda; A(t))$ exists for all $\text{Re } \lambda \leq 0$ and there is a constant $C > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \leq \frac{C}{|\lambda| + 1}, \quad \text{Re } \lambda \leq 0, \quad t \in [0, T].$$

(H₃) For each fixed $s \in [0, T]$, there are constants $C > 0$ and $\rho \in (0, 1]$ such that

$$\|[A(t) - A(\tau)]A(s)^{-1}\| \leq C|t - \tau|^\rho$$

for any $t, \tau \in [0, T]$. Here C and ρ are independent of t, τ and s .

It is well known that the assumption (H₂) implies that for each $s \in [0, T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \geq 0\}$ in $\mathfrak{L}(X)$, where $\mathfrak{L}(X)$ denotes the Banach algebra of all bounded linear operators on X . Then there exist positive constants C and δ such that

$$\|e^{-tA(s)}\| \leq Ce^{-\delta t}, \quad t \geq 0; \quad (2)$$

$$\|A(s)e^{-tA(s)}\| \leq \frac{Ce^{-\delta t}}{t}, \quad t > 0, \quad (3)$$

for all $s \in [0, T]$ [4]. In the remainder of this work, C will denote a constant independent of s, t .

Theorem 2.1 [4, 15] *If the assumptions (H₁)–(H₃) hold, then there exists a unique fundamental solution $\{U(t, s) : 0 \leq s \leq t \leq T\}$ to homogeneous Cauchy problem.*

Now consider the following inhomogeneous Cauchy problem

$$\frac{d}{dt}u(t) + A(t)u(t) = h(t), \quad t > t_0 \geq 0, \quad u(t_0) = u_0. \quad (4)$$

Let $C^\beta([t_0, T]; X)$ denote the space of all X -valued functions $h(t)$, that are uniformly Hölder continuous on $[t_0, T]$ with exponent β , where $0 < \beta \leq 1$. Then $C^\beta([t_0, T]; X)$ is a Banach space endowed with the norm

$$\|h\|_{C^\beta([t_0, T]; X)} = \sup_{t_0 \leq t \leq T} \|h(t)\| + \sup_{t, s \in [t_0, T], t \neq s} \frac{\|h(t) - h(s)\|}{|t - s|^\beta}.$$

Then we have the following theorem.

Theorem 2.2 [4, 15] *Let the assumptions (H₁)–(H₃) hold. If $h \in C^\beta([t_0, T]; X)$, then there exists a unique solution to Problem (4). Furthermore, the solution is given by*

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^t U(t, s)h(s)ds, \quad t_0 \leq t \leq T,$$

and $u : [t_0, T] \rightarrow X$ is a strongly continuously differentiable on $(t_0, T]$.

It follows from the assumptions (H₂) that the negative fractional powers of the operator $A(t)$ is well defined and defined as

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\tau A(t)} \tau^{\alpha-1} d\tau$$

for $\alpha > 0$. Then $A(t)^{-\alpha}$ is a one-to-one and bounded linear operator on X [4]. We define the positive fractional powers of $A(t)$ by $A(t)^\alpha \equiv [A(t)^{-\alpha}]^{-1}$. It can be seen that $A(t)^\alpha$

is closed linear operator with domain $D(A(t)^\alpha)$ dense in X and $D(A(t)^\alpha) \subset D(A(t)^\beta)$ if $\alpha > \beta$. For $0 < \alpha \leq 1$, let $X_\alpha = D(A(0)^\alpha)$ and equip the space X_α with the graph norm

$$\|x\|_\alpha = \|A(0)^\alpha x\|.$$

Then $(X_\alpha, \|\cdot\|_\alpha)$ is a Banach space. If $0 < \alpha \leq 1$, the embeddings $X_1 \hookrightarrow X_\alpha \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, we define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α , and endow the space $X_{-\alpha}$ with the natural norm

$$\|x\|_{-\alpha} = \|A(0)^{-\alpha} x\|.$$

Then $(X_{-\alpha}, \|\cdot\|_{-\alpha})$ is a Banach space. The following assumptions are necessary for proving the main results. For $0 < \alpha \leq 1$, let V_α and $V_{\alpha-1}$ be open sets in X_α and $X_{\alpha-1}$ respectively. For each $u \in V_\alpha$ and $u_1 \in V_{\alpha-1}$, there are closed balls such that $B_\alpha \equiv B_\alpha(u, r) \subset V_\alpha$ and $B_{\alpha-1} \equiv B_{\alpha-1}(u_1, r_1) \subset V_{\alpha-1}$ for $r > 0$ and $r_1 > 0$.

(H4) There exist constants $L_f \equiv L_f(t, u, u_1, r, r_1) > 0$ and $0 < \theta_1 \leq 1$ such that the nonlinear continuous function $f : [0, T] \times V_\alpha \times V_{\alpha-1} \rightarrow X$ satisfies

$$\|f(t, x, x_1) - f(s, y, y_1)\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x_1 - y_1\|_{\alpha-1}) \quad (5)$$

for all $x, y \in B_\alpha$, $x_1, y_1 \in B_{\alpha-1}$ and for all $s, t \in [0, T]$.

(H5) There exist constants $L_h \equiv L_h(t, u, r) > 0$ and $0 < \theta_2 \leq 1$ such that the continuous function $h : V_\alpha \times [0, T] \rightarrow [0, T]$ satisfies

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_\alpha + |t - s|^{\theta_2}), \quad (6)$$

$$h(\cdot, 0) = 0 \quad (7)$$

for all $x, y \in B_\alpha$ and for all $s, t \in [0, T]$.

(H6) There exists constant $L_g \equiv L_g(t, u_1, r_1) > 0$ such that the continuous function $g : [0, T] \times V_{\alpha-1} \rightarrow X_1$ satisfies

$$\|g(t, x_1) - g(s, y_1)\|_1 \leq L_g\{|t - s| + \|x_1 - y_1\|_{\alpha-1}\} \quad (8)$$

for all $x_1, y_1 \in B_{\alpha-1}$ and $t, s \in [0, T]$.

(H7) The function $a : [0, T] \rightarrow [0, T]$ has the following properties:

- (i) a satisfies the delay property $a(t) \leq t$ for all $t \in [0, T]$.
- (ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$\begin{aligned} |a(t) - a(s)| &\leq L_a|t - s| \quad \text{for all } t, s \in [0, T], \\ L_a\|A(0)^{\alpha-2}\| &< 1. \end{aligned}$$

We will use the following lemmas in the subsequent sections.

Lemma 2.1 [5, Lemma 1.1] Let $h \in C^\beta([t_0, T]; X)$. Define $\mathcal{F} : C^\beta([t_0, T]; X) \rightarrow C([t_0, T]; X_1)$ by

$$\mathcal{F}h(t) = \int_{t_0}^t U(t, s)h(s)ds, \quad t_0 \leq t \leq T.$$

Then \mathcal{F} is a bounded mapping and $\|\mathcal{F}h\|_{C([t_0, T]; X_1)} \leq C\|h\|_{C^\beta([t_0, T]; X)}$ for some constant $C > 0$.

Lemma 2.2 [10, Lemma 2] Let $0 < \alpha \leq 1$ and $f \in C([t_0, T]; X_\alpha)$. Define

$$w(t) = \int_{t_0}^t U(t, s)f(s)ds, \quad t_0 \leq t \leq T.$$

Then $w \in C([t_0, T]; X_1) \cap C^1((t_0, T]; X)$ and $\frac{dw(t)}{dt} + A(t)w(t) = f(t)$, $t_0 < t \leq T$.

3 Main Results

In this section, we prove the main results on the existence, uniqueness and asymptotic stability of a solution to Problem (1). Let I denote the interval $[0, T_0]$ for some positive number T_0 to be determined later. For $0 \leq \alpha \leq 1$, let \mathcal{C}_α denote the space of all X_α -valued continuous functions on I , endowed with the sup-norm $\|\cdot\|_\infty$, where

$$\|\phi\|_\infty = \sup_{t \in I} \|\phi(t)\|_\alpha, \quad \phi \in C(I; X_\alpha).$$

Let

$$Y_\alpha \equiv C_{L_\alpha}(I; X_{\alpha-1}) = \{\psi \in \mathcal{C}_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \leq L_\alpha|t - s| \text{ for all } t, s \in I\},$$

where L_α is a positive constant to be specified later. Then Y_α is a Banach space endowed with the sup-norm of \mathcal{C}_α .

Definition 3.1 A continuous function $u : I \rightarrow X_\alpha$ is said to be a mild solution to Problem (1) if

- (i) $g(\cdot, \cdot) \in X_1$;
- (ii) u satisfies the following integral equation

$$\begin{aligned} u(t) = & U(t, 0)[u(0) + g(0, u_0)] - g(t, u(a(t))) + \int_0^t U(t, s)A(s)g(s, u(a(s)))ds \\ & + \int_0^t U(t, s)f(s, u(s), u(h(u(s), s)))ds, \quad t \in I; \end{aligned}$$

- (iii) $u(0) = u_0$.

Definition 3.2 A continuous function $u : I \rightarrow X$ is said to be a solution to Problem (1) if u satisfies the following:

- (i) $u(\cdot) + g(\cdot, u(a(\cdot))) \in C_{L_\alpha}(I; X_{\alpha-1}) \cap C^1((0, T_0); X) \cap C(I; X)$;
- (ii) $u(\cdot) \in X_1$ and $g(\cdot, u(a(\cdot))) \in X_1$;
- (iii) $\frac{d}{dt}[u(t) + g(t, u(a(t)))] + A(t)u(t) = f(t, u(t), u(h(u(t), t)))$ for all $t \in (0, T_0)$;
- (iv) $u(0) = u_0$.

Let $u_0 \in X_\alpha$ and let $r > 0$ be chosen small enough such that the assumptions (H_4) – (H_6) hold for the closed balls $B_\alpha = B_\alpha(u_0, r)$ and $B_{\alpha-1} = B_{\alpha-1}(u_0, r)$. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\mathcal{S} = \left\{ v \in \mathcal{C}_\alpha \cap Y_\alpha : v(0) = u_0, \sup_{t \in I} \|v(t) - u_0\|_\alpha \leq r, \|v(t) - v(s)\|_\alpha \leq K|t - s|^\eta \text{ for all } s, t \in I \right\}.$$

It can be seen that the set \mathcal{S} is a non-empty, closed and bounded subset of \mathcal{C}_α . Based on the ideas of Friedman [4], Fu and Liu [6] and Gal [7], we have the following theorem on existence and uniqueness of a local solution to Problem (1).

Theorem 3.1 *For $0 < \alpha < \beta \leq 1$, let $u_0 \in X_\beta$. If the assumptions (H_1) – (H_7) hold, then there exist a positive number $T_0 \equiv T_0(\alpha, u_0)$ and a unique solution $u(t)$ to Problem (1) on the interval $[0, T_0]$.*

Proof. For each $v \in \mathcal{S}$ and $t \in I$, we define a map H by

$$Hv(t) = U(t, 0)[u_0 + g(0, u_0)] - g(t, v(a(t))) + \int_0^t U(t, s)A(s)g(s, v(a(s)))ds + \int_0^t U(t, s)f_v(s)ds,$$

where $f_v(t) = f(t, v(t), v(h(v(t), t)))$. If $v \in \mathcal{S}$, then the assumptions (H_4) and (H_5) imply that $f_v(t)$ is Hölder continuous on I of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. Also for $v \in \mathcal{S}$, it is clear from the assumptions (H_6) and (H_7) that $A(t)g(t, v(a(t)))$ is Hölder continuous on I of exponent η . Thus by Lemma 2.1, the map H is well defined and it can be seen that $Hv \in \mathcal{C}_\alpha$. We will claim that H maps from the set \mathcal{S} into \mathcal{S} for sufficiently small $T_0 > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned} & \|Hv(t_2) - Hv(t_1)\|_{\alpha-1} \\ & \leq \| [U(t_2, 0) - U(t_1, 0)][u_0 + g(0, u_0)] \|_{\alpha-1} \\ & \quad + \| g(t_2, v(a(t_2))) - g(t_1, v(a(t_1))) \|_{\alpha-1} \\ & \quad + \left\| \int_0^{t_2} U(t_2, s)A(s)g(s, v(a(s)))ds - \int_0^{t_1} U(t_1, s)A(s)g(s, v(a(s)))ds \right\|_{\alpha-1} \\ & \quad + \left\| \int_0^{t_2} U(t_2, s)f_v(s)ds - \int_0^{t_1} U(t_1, s)f_v(s)ds \right\|_{\alpha-1}. \end{aligned} \tag{9}$$

Since the inclusion $X \rightarrow X_{\alpha-1}$ is bounded, we get the following estimate for first term on the right hand side of (9) (cf. [4, see Lemma II. 14.1]) as

$$\| [U(t_2, 0) - U(t_1, 0)][u_0 + g(0, u_0)] \|_{\alpha-1} \leq C_1 \|u_0 + g(0, u_0)\|_\alpha (t_2 - t_1), \tag{10}$$

where C_1 is some positive constant.

Similarly, the assumptions (H_6) and (H_7) imply the following estimate

$$\| g(t_2, v(a(t_2))) - g(t_1, v(a(t_1))) \|_{\alpha-1} \leq C_2 |t_2 - t_1|, \tag{11}$$

where $C_2 = \|A(0)^{\alpha-2}\|L_g(1 + L_aL_\alpha)$.

Using [4, Lemma II. 14.4], we get the following estimates for the third and fourth term on the right hand side of (9) as

$$\begin{aligned} & \left\| \int_0^{t_2} U(t_2, s)A(s)g(s, v(a(s)))ds - \int_0^{t_1} U(t_1, s)A(s)g(s, v(a(s)))ds \right\|_{\alpha-1} \\ & \leq C_3M_g(t_2 - t_1)(|\log(t_2 - t_1)| + 1), \end{aligned} \tag{12}$$

$M_g = \sup_{t \in [0, T]} \|g(t, v(a(t)))\|_1$ and C_3 is some positive constant, and

$$\begin{aligned} & \left\| \int_0^{t_2} U(t_2, s)f_v(s)ds - \int_0^{t_1} U(t_1, s)f_v(s)ds \right\|_{\alpha-1} \\ & \leq C_4N_f(t_2 - t_1)(|\log(t_2 - t_1)| + 1), \end{aligned} \tag{13}$$

where $N_f = \sup_{t \in [0, T]} \|f_v(t)\|$ and C_4 is some positive constant.

Using estimates (10), (11), (12) and (13) in inequality (9), we get

$$\|Hv(t_2) - Hv(t_1)\|_{\alpha-1} \leq L_\alpha|t_2 - t_1|, \tag{14}$$

where $L_\alpha = \max\left\{C_1\|u_0 + g(0, u_0)\|_\alpha, \frac{\|A(0)^{\alpha-2}\|L_g}{1 - \|A(0)^{\alpha-2}\|L_\alpha}, C_3M_g(|\log(t_2 - t_1)| + 1), C_4N_f(|\log(t_2 - t_1)| + 1)\right\}$.

For sufficiently small $T_0 > 0$, we will show that

$$\sup_{t \in I} \|H(v)(t) - u_0\|_\alpha \leq r.$$

Since $u_0 + g(0, u_0) \in X_\alpha$, we can choose sufficiently small $T_1 > 0$ such that (cf. [4, Lemma II.14.1]),

$$\|[U(t, 0) - I][u_0 + g(0, u_0)]\|_\alpha \leq \frac{r}{4} \quad \text{for all } t \in [0, T_1]. \tag{15}$$

Also, it is clear from the assumptions (H_6) and (H_7) that we can choose $T_2 > 0$ small enough such that

$$\|g(t, v(a(t))) - g(0, u_0)\|_\alpha \leq \frac{r}{4} \quad \text{for all } t \in [0, T_2]. \tag{16}$$

Let $K_1 := \sup_{0 \leq t \leq T} \|f(t, u_0, u_0)\|$.

We choose $T_3 > 0$ such that

$$\left(\frac{C_5}{1 - \alpha} L_f [(1 + L_\alpha L_h)r + T_3^{\theta_2}] + \frac{C_5 K_1}{1 - \alpha} \right) T_3^{1-\alpha} \leq \frac{r}{4}$$

for some positive constant C_5 . Now from the assumptions (H_4) and (H_5) , we have for $t \in [0, T_3]$

$$\begin{aligned}
 & \left\| \int_0^t U(t, s) f(s, v(s), v(h(v(s), s))) ds \right\|_\alpha \\
 & \leq C_5 L_f \int_0^t (t-s)^{-\alpha} [\|v(s) - u_0\|_\alpha + \|v(h(v(s), s)) - u_0\|_{\alpha-1}] ds \\
 & \quad + C_5 K_1 \int_0^t (t-s)^{-\alpha} ds \\
 & \leq C_5 L_f \int_0^t (t-s)^{-\alpha} [\|v(s) - u_0\|_\alpha + L_\alpha |h(v(s), s) - h(u(0), 0)|] ds \\
 & \quad + C_5 K_1 \int_0^t (t-s)^{-\alpha} ds \\
 & \leq C_5 L_f \int_0^t (t-s)^{-\alpha} [\|v(s) - u_0\|_\alpha + L_\alpha |h(v(s), s) - h(u(0), 0)|] ds \\
 & \quad + \frac{C_5 K_1 \delta^{1-\alpha}}{1-\alpha} \\
 & \leq C_5 L_f \int_0^t (t-s)^{-\alpha} [r + L_\alpha L_h (\|v(s) - u_0\|_\alpha + s^{\theta_2})] ds + \frac{C_5 K_1 T_3^{1-\alpha}}{1-\alpha} \\
 & \leq C_5 L_f [(1 + L_\alpha L_h) r + T_3^{\theta_2}] \int_0^t (t-s)^{-\alpha} ds + \frac{C_5 K_1 T_3^{1-\alpha}}{1-\alpha} \\
 & \leq \left(\frac{C_5}{1-\alpha} L_f [(1 + L_\alpha L_h) r + T_3^{\theta_2}] + \frac{C_5 K_1}{1-\alpha} \right) T_3^{1-\alpha} \tag{17}
 \end{aligned}$$

for some positive constant C_5 . Let $K_2 = \sup_{t \in [0, T]} \|g(t, u_0)\|_1$. We choose $T_4 > 0$ small enough such that

$$C_6 (L_g L_\alpha L_a T_4 + K_2) \frac{T_4^{1-\alpha}}{1-\alpha} \leq \frac{r}{4}$$

for some positive constant C_6 . Using the assumptions (H_6) and (H_7) , we get

$$\begin{aligned}
 \left\| \int_0^t U(t, s) A(s) g(s, v(a(s))) ds \right\|_\alpha & \leq C_6 \int_0^t (t-s)^{-\alpha} (L_g (1 + L_\alpha L_a) s + \|g(s, u_0)\|_1) ds \\
 & \leq C_6 (L_g (1 + L_\alpha L_a) T_4 + K_2) \frac{T_4^{1-\alpha}}{1-\alpha}, \tag{18}
 \end{aligned}$$

where C_6 is a positive constant. Combining estimates (15), (16), (17) and (18), we obtain

$$\sup_{t \in [0, T_5]} \|Hv(t) - u_0\|_\alpha \leq r,$$

where $T_5 = \min\{T_1, T_2, T_3, T_4\}$.

It remains to show

$$\|Hv(t+h) - Hv(t)\|_\alpha \leq Kh^\eta$$

for some $K > 0$ and $0 < \eta < 1$. Let $T_6 > 0$ be a sufficiently small number. If $0 \leq \alpha < \beta \leq 1, 0 \leq t \leq t+h \leq T_6$, then we have for $t \in [0, T_6]$

$$\begin{aligned} & \|Hv(t+h) - Hv(t)\|_\alpha \\ & \leq \| [U(t+h, 0) - U(t, 0)][u_0 + g(0, u_0)] \|_\alpha \\ & \quad + \|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \\ & \quad + \left\| \int_0^{t+h} U(t+h, s)A(s)g(s, v(a(s)))ds - \int_0^t U(t, s)A(s)g(s, v(a(s)))ds \right\|_\alpha \\ & \quad + \left\| \int_0^{t+h} U(t+h, s)f(s, v(s), v(h(v(s), s)))ds - \int_0^t U(t, s)f(s, v(s), v(h(v(s), s)))ds \right\|_\alpha. \end{aligned} \tag{19}$$

The bellow estimates follow from [4, Lemma II.14.1 and Lemma II.14.4],

$$\| [U(t+h, 0) - U(t, 0)][u_0 + g(0, u_0)] \|_\alpha \leq C_7 \|u_0 + g(0, u_0)\|_\beta h^{\beta-\alpha}; \tag{20}$$

$$\begin{aligned} & \left\| \int_0^{t+h} U(t+h, s)A(s)g(s, v(a(s)))ds - \int_0^t U(t, s)A(s)g(s, v(a(s)))ds \right\|_\alpha \\ & \leq C_8 M_g h^{1-\alpha} (1 + |\log h|); \end{aligned} \tag{21}$$

$$\begin{aligned} & \left\| \int_0^{t+h} U(t+h, s)f(s, v(s), v(h(v(s), s)))ds - \int_0^t U(t, s)f(s, v(s), v(h(v(s), s)))ds \right\|_\alpha \\ & \leq C_9 N_f h^{1-\alpha} (1 + |\log h|), \end{aligned} \tag{22}$$

where C_7, C_8 and C_9 are some positive constants. Again form the assumption (H_6) and (H_7) , it is clear that

$$\|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \leq C_{10} L_g (1 + L_\alpha L_a) h \tag{23}$$

for some constant C_{10} . Combining estimates (20), (21), (22) and (23), we get for $t \in [0, T_6]$,

$$\begin{aligned} & \|Hv(t+h) - Hv(t)\|_\alpha \\ & \leq h^\eta [C_7 \|u_0 + g(0, u_0)\|_\beta T_6^{\beta-\alpha-\eta} + C_{10} L_g (1 + L_\alpha L_a) h^{1-\eta} \\ & \quad + C_8 M_g T_6^{1-\alpha-\eta} (1 + |\log h|) + C_9 N_f T_6^\nu h^{1-\alpha-\eta-\nu} (|\log h| + 1)] \end{aligned}$$

for any $\nu > 0, \nu < 1 - \alpha - \eta$. Hence, for sufficiently small $T_6 > 0$, we have

$$\|Hv(t+h) - Hv(t)\|_\alpha \leq Kh^\eta$$

for $t \in [0, T_6]$ and for some $K > 0$. Thus, we have shown that H maps from the set \mathcal{S} into \mathcal{S} .

We will now claim that the map H is a strict contraction. We choose $T_7 > 0$ such that

$$L_g \|A(0)^{-1}\| + CL_g \|A(0)^{-1}\| \frac{T_7^{1-\alpha}}{1-\alpha} + CL_f (2 + L_\alpha L_h) \frac{T_7^{1-\alpha}}{1-\alpha} \leq \frac{1}{2}$$

for some positive constant C . Using the assumptions (H_4) – (H_7) and [15, inequality (1.65), page 23], we have for $t \in [0, T_7]$ and $v_1, v_2 \in \mathcal{S}$,

$$\begin{aligned} & \|Hv_1(t) - Hv_2(t)\|_\alpha \\ & \leq L_g \|A(0)^{-1}\| \|v_1 - v_2\|_\infty \\ & + CL_g \|A(0)^{-1}\| \int_0^t (t-s)^{-\alpha} \|v_1(a(s)) - v_2(a(s))\|_\alpha ds \\ & + CL_f \int_0^t (t-s)^{-\alpha} (\|v_1(s) - v_2(s)\|_\alpha + \|v_1([h(v_1(s), s)]) - v_2([h(v_2(s), s)])\|_{\alpha-1}) ds \\ & \leq L_g \|A(0)^{-1}\| \|v_1 - v_2\|_\infty \\ & + CL_g \|A(0)^{-1}\| \|v_1 - v_2\|_\infty \frac{T_7^{1-\alpha}}{1-\alpha} + CL_f (2 + L_\alpha L_h) \frac{T_7^{1-\alpha}}{1-\alpha} \|v_1 - v_2\|_\infty \end{aligned} \tag{24}$$

for a positive constant C . Thus, the choice of T_7 implies that the map H is a strict contraction. Since \mathcal{S} is a complete metric space, by the Banach fixed-point theorem, there exists $v \in \mathcal{S}$ such that $Hv = v$. Thus Problem (1) has a unique mild solution on $[0, T_0]$ where $T_0 = \min\{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$

From Lemma 2.1 and Theorem 2.2, it follows that $v \in C^1((0, T_0); X)$. Thus v is a solution to Problem (1) on $[0, T_0]$. \square

Next we will prove the following theorem that gives the existence of a global solution to Problem (1).

Theorem 3.2 *Let the assumptions (H_1) – (H_7) hold. If there are continuous nondecreasing real valued functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\|f(t, x, y)\| \leq k_1(t)(1 + \|x\|_\alpha + \|y\|_{\alpha-1}), \tag{25}$$

$$|h(x, t)| \leq k_2(t)(1 + \|x\|_\alpha), \tag{26}$$

$$\|g(t, y)\|_1 \leq k_3(t)(1 + \|y\|_{\alpha-1}), \tag{27}$$

for all $t \geq 0$, $x \in X_\alpha$ and $y \in X_{\alpha-1}$, then Problem (1) has a unique solution and the solution exists for all $t \in [0, T]$, $T \in [0, \infty)$ for each $u_0 \in X_\beta$, where $0 < \alpha < \beta \leq 1$.

Proof. It follows from Theorem 3.1 that there exists a $T_0 \in (0, T]$ and a unique local solution $u(t)$ on $t \in [0, T_0]$ to Problem (1) is given by

$$\begin{aligned} u(t) &= U(t, 0)[u_0 + g(0, u_0)] - g(t, u(a(t))) + \int_0^t U(t, s)A(s)g(s, u(a(s)))ds \\ &+ \int_0^t U(t, s)f(s, u(s), u(h(u(s), s)))ds, \quad t \in [0, T_0]. \end{aligned}$$

If

$$\|u(t)\|_\alpha \leq \tilde{C}$$

for all $t \in [0, T_0]$ and for some constant \tilde{C} that is independent of t , then the solution $u(t)$ to Problem (1) may be continued further to the right of T_0 . Thus to show global existence of the solution $u(t)$, it is enough to show that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

Let $k_1(T) = \sup_{t \in [0, T]} k_1(t)$, $k_2(T) = \sup_{t \in [0, T]} k_2(t)$ and $k_3(T) = \sup_{t \in [0, T]} k_3(t)$. Form the assumptions $(H_4) - (H_7)$, (25), (26) and (27), we get for $t \in [0, T_0]$,

$$\begin{aligned} \|u(t)\|_\alpha &\leq \|U(t, 0)[u_0 + g(0, u_0)]\|_\alpha \\ &\quad + \|g(t, u(a(t)))\|_\alpha + \left\| \int_0^t U(t, \tau)A(\tau)g(\tau, u(a(\tau)))d\tau \right\|_\alpha \\ &\quad + \left\| \int_0^t U(t, \tau)f(\tau, u(\tau), u(h(u(\tau), \tau)))d\tau \right\|_\alpha \\ &\leq \|A^\alpha(0)A^{-\beta}(t)A^\beta(t)U(t, 0)A(0)^{-\beta}A(0)^\beta[u_0 + g(0, u_0)] \\ &\quad + k_3(T)\|A(0)^{\alpha-1}\| \left(1 + \|A(0)^{-1}\| \sup_{s \in [0, t]} \|u(s)\|_\alpha\right) \\ &\quad + k_3(T) \int_0^t (t - \tau)^{-\alpha} \left(1 + \|A(0)^{-1}\| \sup_{\zeta \in [0, \tau]} \|u(\zeta)\|_\alpha\right) d\tau \\ &\quad + k_1(T) \int_0^t (t - \tau)^{-\alpha} \left[(1 + \|u(\tau)\|_\alpha + L_\alpha|h(u(\tau), \tau) - h(u_0, 0)| + \|u_0\|_{\alpha-1})\right] d\tau \\ &\leq \left(C\|u_0 + g(0, u_0)\|_\beta + k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)\|u_0\|_{\alpha-1} \int_0^t (t - \tau)^{-\alpha} d\tau\right) \\ &\quad + \left(k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)(1 + L_\alpha k_2(T))\right) \int_0^t (t - \tau)^{-\alpha} \left(1 + \sup_{\zeta \in [0, \tau]} \|u(\zeta)\|_\alpha\right) d\tau, \end{aligned}$$

where $\tilde{L} = \max\{1, \|A(0)^{-1}\|\}$. Thus we have

$$\sup_{s \in [0, t]} \|u(s)\|_\alpha \leq \tilde{L}_1 + \tilde{M}_1 \int_0^t (t - \tau)^{-\alpha} \left(1 + \sup_{\zeta \in [0, \tau]} \|u(\zeta)\|_\alpha\right) d\tau,$$

where

$$\begin{aligned} \tilde{L}_1 &= \frac{\left(C\|u_0 + g(0, u_0)\|_\beta + k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)\|u_0\|_{\alpha-1} \int_0^t (t - \tau)^{-\alpha} d\tau\right)}{(1 - k_3(T)\|A(0)^{\alpha-2}\|)}, \\ \tilde{M}_1 &= \frac{\left(k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)(1 + L_\alpha k_2(T))\right)}{(1 - k_3(T)\|A(0)^{\alpha-2}\|)}. \end{aligned}$$

Applying Gronwall's Lemma, we get that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$. □

Next we give a theorem of existence of solutions to Problem (1) under more smoothness condition on the function f and u_0 . Denote $D(A(0))$ by X_1 . Equipped this space X_1 with the graph norm

$$\|x\|_1 := (\|x\|^2 + \|A(0)x\|^2)^{\frac{1}{2}}.$$

Then $\|\cdot\|_1$ that is equivalent to the usual norm $\|A(0) \cdot\|$.

Let V_1 and V be open sets in X_1 and X , respectively. For each $u \in V_1$ and $u_1 \in V$, there are closed balls $B_1 \equiv B_1(u, r)$ and $B \equiv B(u_1, r_1)$ such that $B_1 \subset V_1$ and $B \subset V$ for some $r, r_1 > 0$. We make the following stronger assumptions.

$(H_4)'$ There exist constants $L_f \equiv L_f(t, u, u_1, r, r_1) > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear function $f : [0, T] \times V_1 \times V \rightarrow X_\alpha$ satisfies

$$\|f(t, x, x_1) - f(s, y, y_1)\|_\alpha \leq L_f(|t - s|^{\theta_1} + \|x - y\|_1 + \|x_1 - y_1\|), \tag{28}$$

for all $x, y \in B_1$, $x_1, y_1 \in B$, for all $s, t \in [0, T]$ and $\alpha \in (0, 1)$.

(H₅)' There exist constants $L_h \equiv L_h(t, u_1, r_1) > 0$ and $0 < \theta_2 \leq 1$, such that $h : V_1 \times [0, T] \rightarrow [0, T]$ satisfies

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_1 + |t - s|^{\theta_2}), \tag{29}$$

$$h(\cdot, 0) = 0, \tag{30}$$

for all $x, y \in B_1$ and for all $s, t \in [0, T]$.

(H₆)' There exists constant $L_g \equiv L_g(t, u_1, r_1) > 0$ such that the continuous function $g : [0, T] \times V \rightarrow X_1$ satisfies

$$\|g(t, x) - g(s, y)\|_1 \leq L_g\{|t - s| + \|x - y\|\}, \tag{31}$$

for all $x, y \in B$ and $t, s \in [0, T]$.

Then we have the following theorem on the existence and uniqueness of a solution to Problem (1).

Theorem 3.3 *Let $u_0 \in X_1$ and let the assumptions (H₁)–(H₃), (H₄)' –(H₆)' and (H₇) hold. Then there exist a positive number T_0 and a unique solution $u(t)$ to Problem (1) on the interval $I \equiv [0, T_0]$ such that $u \in C_L(I; X) \cap C^1((0, T_0); X) \cap C(I; X)$, where*

$$C_L(I; X) = \{\psi \in C(I; X_1) : \|\psi(t) - \psi(s)\| \leq L|t - s| \text{ for all } t, s \in I\}$$

for some constant $L > 0$. Moreover, we assume that there are positive constants $k_4(t)$, $k_5(t)$ and $k_6(t)$ such that

$$\|f(t, x, y)\|_\alpha \leq k_4(t)(1 + \|x\|_1 + \|y\|) \text{ for } 0 < \alpha < 1, \tag{32}$$

$$|h(x, t)| \leq k_5(t)(1 + \|x\|_1), \tag{33}$$

$$\|g(t, y)\|_1 \leq k_6(t)(1 + \|y\|) \tag{34}$$

for all $t \geq 0$, $x \in X_1$ and $y \in X$. Then the unique solution of (1) exists for all $t \geq 0$.

Proof. We define a map P by

$$Pv(t) = U(t, 0)[u_0 + g(0, u_0)] - g(t, v(a(t))) + \int_0^t U(t, s)g(s, v(a(s)))ds \\ + \int_0^t U(t, s)f(s, v(s), v(h(v(s), s)))ds$$

for each $t \in I = [0, T_0]$ and for each $v \in C(I, B_1)$. By Lemma 2.2, the map P from $C(I, B_1)$ into $C(I; X_1)$ is well defined.

Let

$$\mathcal{S} = \left\{ y \in C(I; X_1) \cap C_L(I; X) : y(0) = u_0, \sup_{t \in I} \|y(t) - u_0\|_1 \leq r \right\}.$$

It is clear that \mathcal{S} is nonempty, closed, and bounded subset of $C(I; X_1) \cap C_L(I; X)$. Thus \mathcal{S} is a complete metric space. It can be proved that the map $P : \mathcal{S} \rightarrow \mathcal{S}$ is a contraction mapping. The proof can be obtained by the same argument as in the proof of Theorem 3.1 and Theorem 3.2, so we omit the details of the proof. \square

We now prove the asymptotic stability of a solution to Problem (1) that is based on ideas of Friedman [3] and Webb [19].

Theorem 3.4 *Let the assumptions (H_1) – (H_7) hold and $u_0 \in X_\beta$, where $0 < \alpha < \beta \leq 1$. Then there exists a continuous solution $u(t)$ to Problem (1) on $[0, T_0]$ for some $T_0 > 0$.*

In addition, suppose that there exist continuous functions ϵ_1 and ϵ_2 that map $[0, \infty)$ into $[0, \infty)$, and there exist constants $c_4 > 0$ and $c_5 > 0$ such that

$$\|f(t, u(t), u(h(u(t), t)))\| \leq c_4(\epsilon_1(t) + \|u(t)\|_\alpha + \|u(t)\|_{\alpha-1}) \quad \text{for } 0 < \alpha < 1, \quad (35)$$

$$\|g(t, u(a(t)))\|_1 \leq c_5(\epsilon_2(t) + \|u(t)\|_{\alpha-1}), \quad (36)$$

for $t \geq 0$. Then

- (i) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are bounded on $[0, \infty)$, then $\|u(t)\|_\alpha$ is bounded on $[0, \infty)$;
- (ii) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are of $O(e^{\sigma t})$ for some $-1 < \sigma < 0$, then $\|u(t)\|_\alpha = O(e^{\sigma t})$;
- (iii) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are of $o(1)$, then $\|u(t)\|_\alpha = o(1)$.

Proof. It can be seen that there exists $0 < \theta < \delta$ (cf. [4, see page 176]) such that

$$\|A(t)^\gamma U(t, 0)\| \leq \frac{C}{t^\gamma} e^{-\theta t}, \quad \text{if } t > 0, \quad (37)$$

for any $0 \leq \gamma \leq 1$ and some constant $C > 0$. The solution to Problem (1) is given by

$$\begin{aligned} u(t) &= U(t, 0)[u_0 + g(0, u_0)] - g(t, u(a(t))) + \int_0^t U(t, s)A(s)g(s, u(a(s)))ds \\ &\quad + \int_0^t U(t, s)f(s, u(s), u(h(u(s), s)))ds, \end{aligned}$$

for $t \in I$. Now, for $t > 0$, put $\varphi(t) = e^{\theta t}\|u(t)\|_\alpha$. Using (37) in the solution of (1), we obtain

$$\begin{aligned} \varphi(t) &\leq Ct^{-\alpha}\|u_0 + g(0, u_0)\| + c_5\|A(0)^{\alpha-1}\| \left(\|A(0)^{-1}\|\varphi(t) + e^{\theta t}\epsilon_2(t) \right) \\ &\quad + Cc_5 \int_0^t e^{\theta s}(t-s)^{-\alpha} \left(\epsilon_2(s) + \|A(0)^{-1}\|\|u(s)\|_\alpha \right) ds \\ &\quad + Cc_4 \int_0^t e^{\theta s}(t-s)^{-\alpha} \left[\epsilon_1(s) + \|u(s)\|_\alpha + \|u(s)\|_{\alpha-1} \right] ds \\ &\leq Ct^{-\alpha}\|u_0 + g(0, u_0)\| + c_5\|A(0)^{\alpha-1}\| \left(\|A(0)^{-1}\|\varphi(t) + e^{\theta t}\epsilon_2(t) \right) \\ &\quad + C \int_0^t \left[c_4\epsilon_1(s) + c_5\epsilon_2(s) \right] e^{\theta s}(t-s)^{-\alpha} ds \\ &\quad + C \left[c_4(1 + \|A(0)^{-1}\|) + c_5\|A(0)^{-1}\| \right] \int_0^t (t-s)^{-\alpha} \varphi(s) ds. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \varphi(t) &\leq \{C_0 t^{-\alpha}\|u_0 + g(0, u_0)\| + C_0 e^{\theta t}\epsilon_2(t) + C_0 \int_0^t e^{\theta s}(t-s)^{-\alpha} [c_4\epsilon_1(s) + c_5\epsilon_2(s)] ds\} \\ &\quad + C_0 \int_0^t (t-s)^{-\alpha} \varphi(s) ds, \quad (38) \end{aligned}$$

where $C_0 = \frac{\max\{C, c_5\|A(0)^{\alpha-1}\|, C[c_4(1 + \|A(0)^{-1}\|) + c_5\|A(0)^{-1}\|]\}}{(1 - c_5\|A(0)^{\alpha-1}\|\|A(0)^{-1}\|)}$. Denote

$$\chi(t) = C_0 t^{-\alpha} \|u_0 + g(0, u_0)\| + C_0 e^{\theta t} \epsilon_2(t) + C_0 \int_0^t e^{\theta s} (t - s)^{-\alpha} [c_4 \epsilon_1(s) + c_5 \epsilon_2(s)] ds.$$

Then it is clear that

$$\chi(t) \leq C_0 t^{-\alpha} \|u_0 + g(0, u_0)\| + C_0 e^{\theta t} \epsilon_2(t) + \tilde{C} e^{\theta t} \sup_{0 \leq s < \infty} \{c_4 \epsilon_1(s) + c_5 \epsilon_2(s)\},$$

for some constant $\tilde{C} > 0$. By the method of iteration, we get from (38) that

$$\varphi(t) \leq \chi(t) + \int_0^t \left[\sum_0^\infty \frac{(t - s)^{j-1-j\alpha} [\Gamma(1 - \alpha)]^j}{\Gamma(j - j\alpha)} \right] \chi(s) ds.$$

Since the series in the bracket is bounded by $D_1(t - s)^{-\alpha} \exp[D_2(t - s)^{1-\alpha}]$ for some constants $D_1, D_2 > 0$, it follows that, for $t \geq 1$ and for any $\lambda > 0$,

$$\varphi(t) \leq D_3 e^{\lambda t} \|u_0 + g(0, u_0)\| + D_4 e^{\theta t} \epsilon_2(t) + D_5 e^{\theta t} \sup_{0 \leq s < \infty} \{c_4 \epsilon_1(s) + c_5 \epsilon_2(s)\},$$

where D_3, D_4 and D_5 are some positive constants. Thus, for any $0 < \theta_0 < \theta$, we get

$$\|u(t)\|_\alpha \leq D_3 e^{-\theta_0 t} \|u_0 + g(0, u_0)\| + D_4 \epsilon_2(t) + D_5 \sup_{0 \leq s < \infty} \{c_4 \epsilon_1(s) + c_5 \epsilon_2(s)\}. \tag{39}$$

Thus the proof follows from the inequality (39). □

Remark 3.1 If $A(t)$ is a self adjoint positive definite operator in a Hilbert space X , then Theorem 3.1 and Theorem 3.2 can be strengthened. The assumptions (H_1) , (H_2) and (H_3) imply that for $0 \leq \gamma \leq 1$ and for all $s, t \in [0, T]$ [11, page 185],

$$\|A(t)^\gamma A(s)^{-\gamma}\| \leq C \|A(t)A(s)^{-1}\|^\gamma \leq \tilde{C}_1, \tag{40}$$

where $C, \tilde{C}_1 > 0$ are constants. Then Theorem 3.1 and Theorem 3.2 can be proved with less regularity assumption on u_0 .

4 Example

Consider the following problem with a deviating argument

$$\left. \begin{aligned} \frac{\partial}{\partial t} [w(t, x) + g(t, w(a(t), x))] + \frac{\partial^2}{\partial x^2} w(t, x) + b(t, x)w(t, x) &= H(x, w(t, x)) + G(t, x, w(t, x)); \\ w(t, 0) &= w(t, 1), \quad t > 0; \\ w(0, x) &= w_0(x), \quad x \in (0, 1), \end{aligned} \right\} \tag{41}$$

where $b(t, x)$ is a continuous function in x and uniformly Hölder continuous function in t . Here $H(x, w(t, x)) = \int_0^x K(x, y)w(\tilde{g}(t)|w(t, y)|, y)dy$ for all $(t, x) \in (0, \infty) \times (0, 1)$. Assume that $\tilde{g} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is locally Hölder continuous in t with $\tilde{g}(0) = 0$ and $K \in$

$C^1([0, 1] \times [0, 1]; \mathbb{R})$. The function $G : \mathbb{R}_+ \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable in x , locally Hölder continuous in t , locally Lipschitz continuous in u , uniformly in x .

Let $X = L^2((0, 1); \mathbb{R})$, $A(t)u(t)(x) = -\frac{\partial^2}{\partial x^2}u(t, x) - b(t, x)u(t, x)$. Then $X_1 = D(A(0)) = H^2(0, 1) \cap H_0^1(0, 1)$ and $X_{1/2} = D((A(0))^{1/2}) = H_0^1(0, 1)$. Then the family $\{A(t) : t > 0\}$ satisfies the assumptions (H_1) – (H_3) on each bounded interval $[0, T]$ (see [4, 6]).

Put $w(t, \cdot) \equiv u(t)$, then Problem (41) can be written as

$$\left. \begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A(t)u(t) &= f(t, u(t), u(h(u(t), t))), \quad t > 0; \\ u(0) &= u_0. \end{aligned} \right\} \quad (42)$$

We define $f : \mathbb{R}_+ \times H_0^1(0, 1) \times H^{-1}(0, 1) \rightarrow L^2(0, 1)$ by

$$f(t, \phi, \psi) = H(x, \psi) + G(t, \phi)$$

for $\phi \in H^{-1}(0, 1) \equiv H_0^1(0, 1)$ and $\psi \in H_0^1(0, 1)$. Here $H : H_0^1(0, 1) \rightarrow L^2(0, 1)$ is defined as $H(x, \psi(x, t)) = \int_0^x K(x, y)\psi(y, t)dy$ for $x \in (0, 1)$ and $\psi \in H_0^1(0, 1)$. Then it can be proved that f satisfies the assumption (H_4) for $\alpha = \frac{1}{2}$. We assume $h : H_0^1(0, 1) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by $h(\phi(x, t), t) = \tilde{g}(t)|\phi(x, t)|$ satisfies the assumption (H_5) for $\alpha = \frac{1}{2}$ (see Gal [7]). We also assume that the function $g : \mathbb{R}_+ \times L^2(0, 1) \rightarrow H_0^1(0, 1)$ satisfies the assumption (H_6) for $\alpha = \frac{1}{2}$. We can take the function $a(t)$ where $a(t) = kt$ for $t \in [0, T]$ and $0 < k \leq 1$. Thus, we can apply our the results to study the existence, uniqueness and asymptotic stability of a solution to Problem (41).

Acknowledgements

The author would like to thank the referees for their valuable remarks to improve the original manuscript. The author would like to thank Professor D. Bahuguna for his encouragement. The author would like to acknowledge the excellent facilities of Indian Institute of Technology Kanpur that were availed during the preparation of the article.

References

- [1] Dubey, R. S. Existence of a Regular Solution to Quasilinear Implicit Integrodifferential Equations in Banach Space. *Nonlinear Dynamics and System Theory* **11**(2) (2011) 137–146.
- [2] El'sgol'ts, L. E. and Norkin, S. B. *Introduction to the Theory of Differential Equations with Deviating Arguments*. Academic Press, 1973.
- [3] Friedman, A. Remarks on nonlinear parabolic equations. In: *Proceedings of Symposia in Applied Mathematics*, Vol. XVII, pp. 3–23, American Mathematical Society, Providence, R.I., 1965.
- [4] Friedman, A. *Partial Differential Equations*. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.
- [5] Friedman, A. and Shinbrot, M. Volterra integral equations in Banach space. *Transactions of the American Mathematical Society* **126** (1967) 131–179.

- [6] Fu, X. and Liu, X. Existence of periodic solutions for abstract neutral non-autonomous equations with infinite delay. *Journal of Mathematical Analysis and Applications* **325**(1) (2007) 249–267.
- [7] Gal, C. G. Nonlinear abstract differential equations with deviated argument. *Journal of Mathematical Analysis and Applications* **333**(2) (2007) 971–983.
- [8] Haloi, R., Pandey, D. N. and Bahuguna, D. Existence of solutions to a non-autonomous abstract neutral differential equation with deviated argument. *Journal Nonlinear Evolution Equations and Applications* **2011**(5) 75–90.
- [9] Haloi, R., Pandey, D. N. and Bahuguna, D. Existence, uniqueness and asymptotic stability of solutions to a non-autonomous semi-linear differential equation with deviated argument. *Nonlinear Dynamics and System Theory* **12**(2) (2012) 179–191.
- [10] Heard, M. L. An abstract parabolic Volterra integro-differential equation. *SIAM Journal on Mathematical Analysis* **13**(1) (1982) 81–105.
- [11] Krien, S. G. *Linear Differential Equations in Banach Space. Translations of Mathematical Monographs*, Vol. **29**. American Mathematical Society, 1971.
- [12] Ladas, G. E. and Lakshmikantham, V. *Differential Equations in Abstract Spaces*. Academic Press, 1972.
- [13] Pandey, D. N., Ujlayan, A. and Bahuguna, D. Semilinear Hyperbolic Integrodifferential Equations with Nonlocal Conditions. *Nonlinear Dynamics and Systems Theory* **10**(1) (2010) 77–92.
- [14] Pandey, D. N., Ujlayan, A. and Bahuguna, D. On nonlinear abstract neutral differential equations with deviated argument. *Nonlinear Dynamics and Systems Theory* **10**(3) (2010) 283–294.
- [15] Sobolevskii, P. L. Equations of parabolic type in a Banach space. *American Mathematical Society Translations* **49**(2) (1966) 1–62.
- [16] Tanabe, H. On the equations of evolution in a Banach space. *Osaka Journal of Mathematics* **12** (1960) 363–376.
- [17] Travis, C. C. and Webb, G. F. Partial differential equations with deviating arguments in the time variable. *Journal of Mathematical Analysis and Applications* **56**(2) (1976) 397–409.
- [18] Webb, G. F. Abstract Volterra integro-differential equations and class of reaction-diffusion equations. In: *Lecture Notes in Mathematics*, **737**. Springer Verlag, New York, 1979, 295–303.
- [19] Webb, G. F. Asymptotic stability in the α -norm for an abstract nonlinear Volterra integral equation, Stability of dynamical systems, theory and applications. In: *Proc. Regional Conf.*, Mississippi State Univ., Mississippi State, Miss., 1975, pp. 207–211; *Lecture Notes in Pure and Appl. Math.*, **28**, Dekker, New York, 1977.



Global Stability and Synchronization Criteria of Linearly Coupled Gyroscope

O.I. Olusola^{1,2*}, U.E. Vincent^{3,4}, A.N. Njah² and B.A. Idowu⁵

¹ CSIR - Indian Institute of Chemical Biology, 4, Raja S.C. Mullick Road, Kolkata 700032, India.

² Nonlinear Dynamics Research Group, Department of Physics, Federal University of Agriculture, P.M.B. 2240, Abeokuta, Nigeria.

³ Department of Physical Sciences, Redeemer's University, Km 46, Lagos - Ibadan Expressway P.M.B 3005, Redemption City, Nigeria.

⁴ Department of Physics, Lancaster University, Lancaster LA1 4YB, United Kingdom.

⁵ Department of Physics, Lagos state University, Ojo, Lagos State, Nigeria.

Received: August 15, 2012; Revised: July 10, 2013

Abstract: We examine the synchronization transition of a pair of unidirectionally coupled gyroscope. Based on Lyapunov stability theory and linear matrix inequalities (LMI), some necessary and sufficient criteria for stable synchronous behaviour are obtained and an exact analytic estimate of the threshold for complete chaos synchronization is derived. Finally, numerical simulation results are presented to validate the feasibility of the theoretical analysis.

Keywords: *chaos synchronization; nonlinear gyroscope; linear matrix inequality; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 70H14.

1 Introduction

In the last two decades, an intensive research activity has been devoted to the study of dynamics of coupled and driven chaotic systems. Despite the considerable body of knowledge that has already been gained and established, research on coupled nonlinear systems still remains an active field. In view of the importance of the classical results from the dynamics of driven or coupled harmonic oscillators in science and technology, the question of which phenomena emerge when chaotic oscillators are coupled or somehow

* Corresponding author: <mailto:olasunkanmi2000@gmail.com>

driven or perturbed has been and is still of great interest. The most relevant and widely studied phenomena until now are the synchronization [1–5] and the suppression of chaos [2, 4–8]. Due to the potential applications in various areas of science and technology, synchronization between two dynamical system has stimulated a wide range of research activity and many effective methods have been presented [1, 9–11].

In the past, research on chaos synchronization and its applications has intensively focused on the autonomous chaotic systems such as Lorenz, Chen, Rössler etc, but recently, the dynamics and synchronization of non-autonomous chaotic systems such as Duffing oscillator, gyroscopes, etc have witnessed tremendous research interest due to their potential applications in engineering and life sciences [12–18]. In particular, the gyroscopes, from a purely scientific viewpoint show strange and interesting properties, and from engineering viewpoint, they have great utility in the navigation of rockets, aircrafts, spacecrafts and in the control of complex mechanical system. In the past years, the gyroscope has been found with rich phenomena [12, 19, 20], for example, when subjected to harmonic vertical base excitations, it exhibits a variety of interesting dynamical behaviours that span the range from regular to chaotic motions [11, 12, 20–22].

The synchronization of the symmetric gyroscope model presented in Ref. [12] has been achieved using different methods, for example, four different kinds of one way coupling [12], active control [23], backstepping design [13, 24], fuzzy logic controller [25], sliding mode control [26, 27], sliding based fuzzy control [28] and so on. Very recently, synchronization of uncertain gyros was considered in [29]. Among the above methods, it is well known that linear feedback method provides simple control inputs for synchronization and has lately been employed to achieve stable synchronization in various unidirectionally coupled systems including, double well Duffing oscillators (DDOs) [30], parametrically excited Duffing oscillators [31] and the gyroscope. However, a crucial issue is the assessment of stability analysis for feedback controlled system and the determination of appropriate feedback gains that would guarantee stable synchronization. Since the beginning of the studies on synchronization of chaotic systems, the stability of synchronous motion was considered the most crucial question needed to be addressed, in order to furnish the proper conditions for a laboratory verification of theoretical findings. The problem of stability can be tackled in different ways and different criteria could be established, depending on specific conditions of interest.

One of the most popular and widely used criterion is the conditional Lyapunov exponents, which constitute average measurements of expansion or shrinkage of small displacements along the synchronized trajectory. However, it has been shown that negativity of the conditional Lyapunov exponents is not a sufficient condition for a stable synchronized state due to some unstable invariant sets in the stable synchronization manifold [32]. Whether this condition is necessary or not has remained an open issue (see [33] and references therein), and needs to be studied further. In [30], we proposed a linear state error feedback approach based on Lyapunov stability theory and Linear Matrix Inequality (LMI) [34], to analyze the stability of the synchronized state and also determine sufficient criteria for stable synchronous behaviour. This method is used because, it is known that many engineering optimization problem can be easily translated into linear matrix inequality (LMI) problems and a wide variety of problems arising in system and control theory can be reduced to a few standard convex or quasi-convex optimization problems involving LMI. The resulting optimization problem can be solved numerically with very high efficiency [35]. Moreover, the Lyapunov methods which are traditionally applied to the analysis of system stability, can just as well be used to determine thresh-

old coupling, k_{th} , at which complete synchronization could be reached in master-slave or mutually coupled oscillators. Critical coupling for the on-set of stable synchronization in coupled or driven oscillators is relevant for various scientific and technological applications [36].

In this paper, we consider the synchronization of unidirectionally coupled gyroscopes. We propose a novel stability criterion using Lyapunov stability theory and linear matrix inequality (LMI) to determine the threshold coupling, k_{th} , at which full and stable synchronous behaviour could be reached in the master-slave coupled gyroscope. The advantage of our method is that the coupling parameters of the system can be obtained at the same time by solving the LMI without predetermining them to check the criterion. Furthermore, the LMI can be easily solved by various optimization algorithms. The sufficient criteria can be applied to directly design the coupling strength resulting in the synchronization. The rest of the paper is structured as follows: in the next section, we present the synchronization scheme, while Section 3 is devoted to synchronization threshold and stability criteria, Section 4 is devoted to numerical results and discussions and the paper is concluded in Section 5.

2 Model and Synchronization Preliminaries

Here, we consider the motion of the symmetric gyro with linear-plus-cubic damping given as [12]

$$\ddot{\theta} + \alpha^2 \frac{(1 - \cos \theta)^2}{\sin^3 \theta} - \beta \sin \theta + c_1 \dot{\theta} + c_2 \dot{\theta}^3 = (f \sin \omega t) \sin \theta,$$

where $f \sin \omega t$ is a parametric excitation, $c_1 \dot{\theta}$ and $c_2 \dot{\theta}^3$ are linear and nonlinear damping, respectively and $\alpha^2 \frac{(1 - \cos \theta)^2}{\sin^3 \theta} - \beta \sin \theta$ is a nonlinear resilience force. After necessary transformation, the gyroscope equation in non-dimensional form can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha^2 \frac{(1 - \cos x_1)^2}{\sin^3 x_1} - c_1 x_2 - c_2 x_2^3 + (\beta + f \sin \omega t) \sin x_1, \end{aligned} \quad (1)$$

where

$$\alpha = \frac{\beta_\phi}{I_1} = \frac{I_3 \omega_z}{I_1}, \quad c_1 = \frac{D_1}{I_1}, \quad c_2 = \frac{D_1}{I_1}, \quad \beta = \frac{M_g l}{I_1}, \quad f = \frac{M_g \bar{l}}{I_1}. \quad (2)$$

The nonlinear gyro given by Eq. (1) exhibits varieties of dynamical behaviour including chaotic motion displayed in Figure 1 for the following parameters $\alpha^2 = 100$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$, and $f = 35.5$ as given in [12].

By letting $\eta(t) = \beta + f \sin \omega t$ and using the first two terms of the Taylor series expansion of $\frac{(1 - \cos x_1)^2}{\sin^3 x_1}$, system (1) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{\alpha^2 x_1}{4} - \frac{\alpha^2 x_1^3}{12} - c_1 x_2 - c_2 x_2^3 + \eta(t) \sin x_1. \end{aligned} \quad (3)$$

To facilitate the present analysis, we express system (3) in the following vector form:

$$\dot{\mathbf{x}} = A\mathbf{x} - \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}), \quad (4)$$

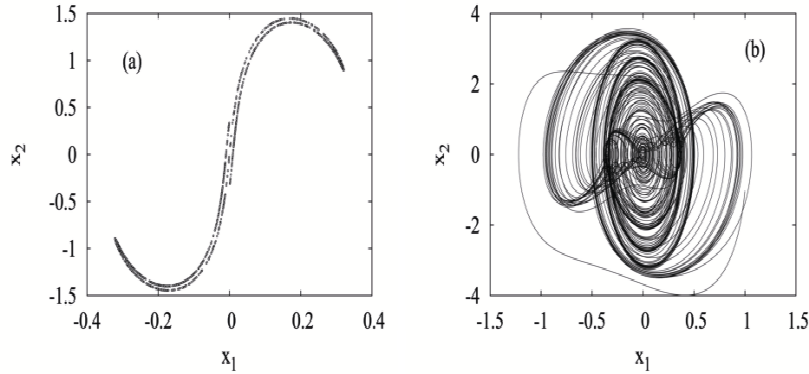


Figure 1: (a) The Poincaré map and (b) phase portrait showing a chaotic attractor of nonlinear gyroscope with the following parameters $\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2,$ and $f = 35.5.$

where $\mathbf{x} = (x_1, x_2)^T \in \mathbf{R}^2$ are state space variables and

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{\alpha^2}{4} & -c_1 \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \alpha \begin{pmatrix} 0 & 0 \\ -\frac{\alpha^2 x_1^3}{12} & -c_2 x_2^3 \end{pmatrix}, \quad \mathbf{G}(\mathbf{x}) = \eta \begin{pmatrix} 0 \\ \sin x_1 \end{pmatrix}.$$

In order to examine the synchronization between two unidirectional coupled gyroscopes, we construct a master-slave synchronization scheme for two identical chaotic gyroscopes by linear state error feedback controller in the following form:

$$\begin{aligned} M &: \dot{\mathbf{x}} = A\mathbf{x} - \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}), \\ S &: \dot{\mathbf{y}} = A\mathbf{y} - \mathbf{f}(\mathbf{y}) + \mathbf{G}(\mathbf{x}) + \mathbf{u}(t), \\ C &: \mathbf{u}(t) = K(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{5}$$

where $\mathbf{u} = K(\mathbf{x} - \mathbf{y})$ is the linear state feedback control input and $K \in \mathbf{R}^{2 \times 2}$ is a constant control matrix that determines the strength of the feedback into the response system. By defining the synchronization error variable as the difference between the relevant dynamical variables given by

$$\mathbf{e} = \mathbf{x} - \mathbf{y}, \tag{6}$$

we obtain the error dynamics for the master-slave system (5) as:

$$\dot{\mathbf{e}} = (A - K + M(\mathbf{x}, \mathbf{y}) + G(x_1, y_1))\mathbf{e}, \tag{7}$$

where

$$\begin{aligned} M(\mathbf{x}, \mathbf{y}) &= \begin{pmatrix} 0 & 0 \\ -\frac{\alpha^2 m_1(x_1, y_1)}{12} & -c_2 m_2(x_2, y_2) \end{pmatrix}, \\ m_1(x_1, y_1) &= x_1^2 + x_1 y_1 + y_1^2, \quad m_2(x_2, y_2) = x_2^2 + x_2 y_2 + y_2^2, \\ G(x_1, y_1) &= \eta \begin{pmatrix} 0 & 0 \\ g(x_1, y_1) & 0 \end{pmatrix}, \quad g(x_1, y_1) = -\frac{(\sin x_1 - \sin y_1)}{x_1 - y_1}. \end{aligned} \tag{8}$$

In the absence of the control matrix K Eq. (7) would have an equilibrium at $(0, 0)$. Our aim is to choose the appropriate coupling matrix K such that the trajectories of the master system $x(t)$ and slave one $y(t)$ satisfy

$$\lim_{t \rightarrow \infty} \|e\| = \lim_{t \rightarrow \infty} \|x(t) - y(t)\| = 0, \quad (9)$$

where $\|*\|$ represents Euclidean norm of a vector.

3 Threshold and Criteria for Synchronization

Here, we have employed the Lyapunov's direct method and linear matrix inequality (LMI) [37] to establish some criteria for global chaos synchronization in the sense of error system (7). The classical method of Lyapunov stability theory which employs Lyapunov functionals was known for the analysis and synthesis of synchronization dynamics of coupled and driven oscillators (e.g see Refs. [38,39]). In addition to the familiar approach of analyzing and synthesizing the synchronization behaviour of coupled systems; the present paper employed the Lyapunov direct method to obtain the threshold coupling at which the two systems become completely synchronized.

To begin with, we have applied the following assumption and lemma to prove the main theorem of this paper.

Assumption. The chaotic trajectory of the master gyroscope (1) is bounded i.e. for any bounded initial condition $x(0)$ within the defining domain of the drive system, there exists a positive real constant, σ , such that $\|x(t)\| \leq \sigma \forall t \geq 0$.

Remark 1 This assumption is reasonable and valid in the context of bounded feature of chaotic attractors [40].

Lemma 1 For $g(x_1, y_1)$ defined earlier, the inequality

$$|g(x_1, y_1)| \leq 1 \quad (10)$$

holds.

Proof. By the differential mean-value theorem:

$$\sin x_1 - \sin y_1 = (x_1 - y_1) \cos \phi, \quad \phi \in (x_1, y_1) \text{ or } \phi \in (y_1, x_1) \quad (11)$$

so that,

$$g(x_1, y_1) = \frac{-(\sin x_1 - \sin y_1)}{x_1 - y_1} = -(\cos \phi). \quad (12)$$

Hence, the inequality (10) holds.

Next, we proceed by utilizing the stability theory on time-varied systems [34] to derive sufficient criteria for global chaos synchronization in the sense of the error system (7). The following theorem is related to the general control matrix

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbf{R}^{2 \times 2}. \quad (13)$$

Theorem 1 The master-slave system (4) achieves global chaos synchronization if a symmetric positive matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \quad (14)$$

and a coupling matrix $K \in \mathbf{R}^{2 \times 2}$ defined in (13) are chosen such that for any $t > 0$

$$\Omega_1 = -p_{11}k_{11} - p_{12}k_{21} + |p_{12}|\omega^\beta < 0, \tag{15}$$

$$\Omega_2 = p_{12}(1 - k_{12}) - p_{22}(k_{22} + c_1 + 3c_2\sigma_2^2) < 0, \tag{16}$$

$$4\Omega_1\Omega_2 > L^2, \tag{17}$$

$$L = [p_{11}(1 - k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + 3c_2\sigma_2^2) - p_{22}k_{21}|p_{22}\omega^\beta|],$$

where $\omega^\beta = \beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}$.

Proof. Let us assume a quadratic Lyapunov function of the form:

$$V(e) = \mathbf{e}^T P \mathbf{e}, \tag{18}$$

where P is a positive definite symmetric matrix defined in (14). The derivative of the Lyapunov function with respect to time, t , along the trajectory of the error system (7) is of the form:

$$\dot{V}(e) = \dot{\mathbf{e}}^T P \mathbf{e} + \mathbf{e}^T P \dot{\mathbf{e}}. \tag{19}$$

Substituting Eq. (7) into the system (19), we have

$$\dot{V}(e) = \mathbf{e}^T [(\mathbf{A} - \mathbf{K} + \mathbf{M} + \mathbf{G})^T P + P(\mathbf{A} - \mathbf{K} + \mathbf{M} + \mathbf{G})] \mathbf{e} \tag{20}$$

$\dot{V}(e) < 0$, if

$$\lambda = (A - K + M + G)^T P + P(A - K + M + G) < 0, \tag{21}$$

that is

$$\lambda = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}, \tag{22}$$

where $\mu_{11} = -2p_{11}k_{11} + 2p_{12} \left(\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21} \right) \right)$, $\mu_{12} = p_{11}(1 - k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + c_2 m_2) + p_{22} \left(\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21} \right) \right)$ and $\mu_{22} = 2p_{12}(1 - k_{12}) - 2p_{22}(c_1 + c_2 m_2 + k_{22})$ respectively. The symmetric matrix in (22) is negative definite if and only if

$$-2p_{11}k_{11} + 2p_{12}L^\alpha < 0, \tag{23}$$

$$2p_{12}(1 - k_{12}) - 2p_{22}(c_1 + c_2 m_2 + k_{22}) < 0, \tag{24}$$

$$4L_1 L_2 - L_3 > 0, \tag{25}$$

where $L^\alpha = \eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21} \right)$,

$$L_1 = [p_{12}L^\alpha - p_{11}k_{11}],$$

$$L_2 = [p_{12}(1 - k_{12}) - p_{22}(c_1 + c_2 m_2 + k_{22})],$$

$$L_3 = [p_{11}(1 - k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + c_2 m_2) + p_{22}L^\alpha]^2.$$

It follows from the Assumption that for all $t \geq 0$,

$$|m_1(x_1, y_1)| = |x_1^2 + x_1 y_1 + y_1^2| \leq 3\sigma_1^2,$$

$$|m_2(x_2, y_2)| = |x_2^2 + x_2 y_2 + y_2^2| \leq 3\sigma_2^2$$

$$|\eta(t)| = b|\beta + f \sin \omega t| \leq \beta + |f|.$$

Since the matrix P is positive definite, we have $p_{11}p_{22} - p_{12}^2 > 0$, so that $-2p_{11}k_{11} + 2p_{12}L^\alpha \leq -2p_{11}k_{11} - 2p_{12}k_{21} + |2p_{12}|L^\alpha \leq 2\Omega_1$,

$$|p_{11}(1 - k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + c_2m_2) + p_{22} \left[\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21} \right) \right]| \leq |p_{11}(1 - k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + 3c_2\sigma_2^2) - p_{22}k_{21}| + p_{22} \left(\eta - \frac{\alpha^2}{4} + \frac{\alpha^2\sigma_1^2}{4} \right).$$

The inequalities (23)-(25) hold if the inequalities (15)-(17) are satisfied. This completes the proof.

For the purpose of applications, it is necessary that the simplest possible synchronization controllers are employed. Hence, the following corollaries can be obtained from the main theorem of this paper.

Corollary 1 *If the coupling matrix is defined by $\mathbf{K} = \text{diag}\{k_1, k_2\}$ and the symmetric positive definite matrix \mathbf{P} is as defined in (24) such that*

$$k_1 > \frac{|p_{12}| \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right)}{p_{11}}, \quad (26)$$

$$k_2 > \frac{p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}}{p_{22}}, \quad (27)$$

$$4|p_{12}| \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) - p_{11}k_1 [p_{12} - p_{22}(k_2 + c_1 + 3c_2\sigma_2^2)] > \left[|p_{11} - p_{12}(k_1 + k_2 + c_1 + 3c_2\sigma_2^2)| + p_{22} \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) \right]^2, \quad (28)$$

then the master-slave system (4) achieves global chaos synchronization.

Proof. The inequalities (26) - (28) can be obtained according to the inequalities (15)-(17) with $k_{11} = k_1$, $k_{22} = k_2$ and $k_{12} = k_{21} = 0$.

Corollary 2 *The master-slave system (4) achieves global chaos synchronization if the coupling matrix $\mathbf{K} = \text{diag}\{k, k\}$ and the positive symmetric matrix \mathbf{P} defined in (14) are chosen such that*

$$k = \max \left(\frac{|p_{12}| \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right)}{p_{11}}, \frac{p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}}{p_{22}} \right) \geq 0, \quad (29)$$

$$\begin{aligned} & 4(p_{11}p_{22} - p_{12}^2)k^2 - 4k[2p_{22}|p_{12}| \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) \\ & + p_{11}(p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}) - |p_{12}(p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12})| \\ & + 4|p_{12}| \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) (p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}) \\ & - \left[|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + p_{22} \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) \right]^2 > 0. \end{aligned} \quad (30)$$

Proof. Letting $k_1 = k_2 = k$ in the partial synchronization conditions (26) and (27), the inequality (29) can be obtained.

For $k > 0$ given by (29), we have

$$\begin{aligned} & \left[|p_{11} - p_{12}(2k + c_1 + 3c_2\sigma_2^2)| + p_{22} \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) \right]^2 \leq \\ & \left[|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + 2k|p_{12}| + p_{22} \left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4} \right) \right]^2. \end{aligned}$$

Hence, the inequality (30) can be realised by partial synchronization criterion (28) with $k_1 = k_2 = k$. Since $p_{11}p_{22} - p_{12}^2 > 0$, the solution k to the inequality (30) exists.

Remark 2 We select the elements of the positive symmetric matrix \mathbf{P} as $p_{12} = 0$, $p_{11} = p_{22} \left(\beta + |f| + \frac{\alpha^2 \sigma_1^2}{4} - \frac{\alpha^2}{4} \right)$, and obtain the following algebraic synchronization criterion via the inequalities (29) and (30).

$$K = \text{diag}\{k, k\},$$

$$k > \frac{\sqrt{(c_1 + 3c_2\sigma_2^2)^2 + 4 \left(\beta + |f| + \frac{\alpha^2 \sigma_1^2}{4} - \frac{\alpha^2}{4} \right) - (c_1 + 3c_2\sigma_2^2)}}{2} = k_{th}^1. \quad (31)$$

Corollary 3 *The synchronization scheme (5) achieves global chaos synchronization if the control matrix $K = \text{diag}\{k, 0\}$ and a symmetric positive definite matrix \mathbf{P} given in (14) are selected such that*

$$k > b \frac{|p_{12}| \gamma}{p_{11}}, \quad (32)$$

$$np_{12} - (c_1 + 3c_2\sigma_2^2)p_{22} < 0, \quad (33)$$

$$k[|p_{12}(p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12})| + |p_{12}|p_{22}\gamma - 2((c_1 + 3c_2\sigma_2^2)p_{22} - p_{12})p_{11}] + p_{12}^2 k^2 + 4|p_{12}|((c_1 + 3c_2\sigma_2^2)p_{22} - p_{12})\gamma + [|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + p_{22}\gamma]^2 < 0, \quad (34)$$

where $\gamma = \beta + |f| + \frac{\alpha^2 \sigma_1^2}{4} - \frac{\alpha^2}{4}$.

Remark 3 We select the symmetric positive definite matrix

$$\mathbf{P} = p_{22} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$$

with $p_{22} > 0$.

The following synchronization criterion is gained based on the criteria (31)-(34).

$$K = \text{diag}\{k, 0\}, \quad k > \frac{\beta + |f| + \frac{\alpha^2 \sigma_1^2}{4} - \frac{\alpha^2}{4}}{2(c_1 + 3c_2\sigma_2^2)}. \quad (35)$$

4 Results and Discussion

In this section, we present numerical simulation results to confirm the obtained criteria. We utilized the fourth order Runge-Kutta routine with the following initial conditions $(x_1(0), y_1(0)) = (1.0, -1.0)$, $(x_2(0), y_2(0)) = (1.0, -1.2)$, a time-step of 0.001 and fixing the parameter values of $\alpha^2 = 100$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$, and $f = 35.5$ as in Figure 1, to ensure chaotic motion, we solved the master-slave system (4) with the control matrices as defined in Eqs. (31) and (35). The simulation results obtained reveal that the trajectory of the master gyroscope depicted in Figure 1 is bounded and the error dynamics shown in Figure 2 oscillate chaotically with time when the two systems are decoupled. The partial variables x_1 and x_2 of the chaotic attractor satisfy $x_1(t) = x_2(t) < 1.25$ for any $t \geq 0$. Thus we find out that the constant $\sigma_1 = \sigma_2 = 1.25$.

The critical coupling at which complete synchronization could be observed is vital for many scientific and technological applications because it provides useful information regarding the operational regime for optimal performance in coupled systems. In Figure 3, we displayed a simulation result of average error, E_{ave} , against coupling, k , and noticed that as k increases and as full synchronization is approached, $E_{ave} \rightarrow 0$ asymptotically

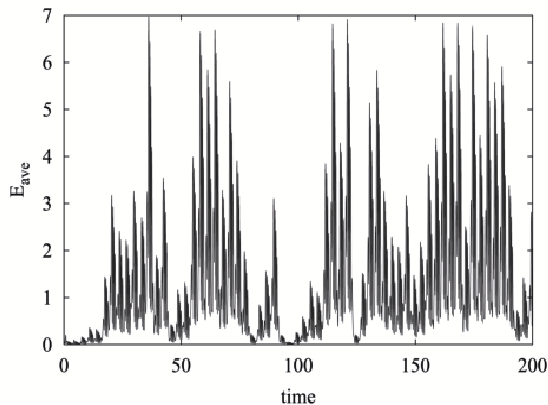


Figure 2: Average error, E_{ave} , as a function of time for the uncoupled systems with the same parameters as in Figure 1.

at the threshold coupling, $k_{th} \approx 5.98$. Then for all $k > k_{th}$, $E_{ave} = 0$ and remains stable as $t \rightarrow \infty$ implying that the oscillators are completely synchronized. Interestingly, we noticed that by direct calculations of Eq. (31) for the control matrix $K = \text{diag}\{k, k\}$, $k > k_{th} = 6.18$. Thus the obtained criterion is in good agreement with numerical simulation result.

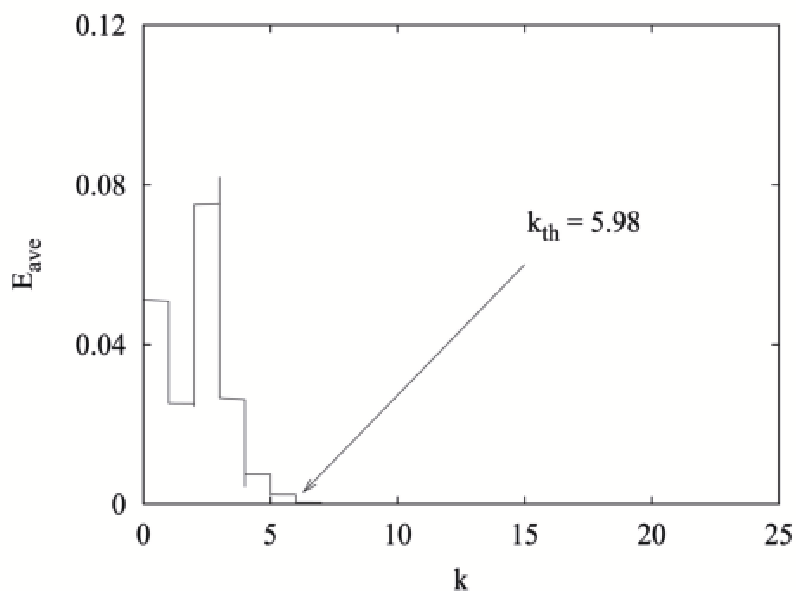


Figure 3: Average Error dynamics, E_{ave} , as a function of the coupling strength, k . Here the parameters of the system are as in Figure 1.

Using the criterion defined by Eq. (31), one readily obtains a coupling matrix $K =$

$\text{diag}\{6.18, 6.18\}$ by which the master-slave system (4) achieves chaos synchronization. Figure 4 shows the synchronization for $k = 6.2$. Finally, we depict the simulation results for the second case in which we choose constant control matrix $K = \text{diag}\{k, 0\}$, such that $k > 34.43$ which satisfies the condition in Eq. (35). The simulation results displayed in Figure 4 confirmed that complete synchronization is achieved for $k = 35.0 > k_{th}$. Notice that in both cases, the synchronization is already reached at $t = 1.0$, showing an excellent transient performance.

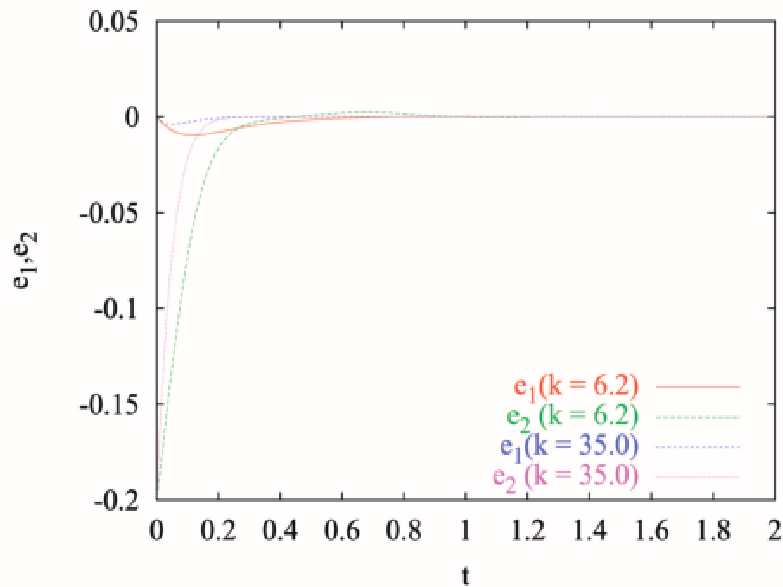


Figure 4: Chaos synchronization of two linearly coupled gyroscopes with the coupling strength $K = \text{diag}\{6.20, 6.20\}$ and $K = \text{diag}\{35.0, 0\}$.

5 Conclusions

In this paper an analytical method based on Lyapunov stability theory and linear matrix inequality (LMI) have been utilized to examine the stability of synchronized dynamics and thus determine the threshold coupling, k_{th} , at which stable synchronization regime could be observed in master-slave parametrically excited gyroscope. The criteria obtained in this paper are in algebraic form and could be easily employed for designing the feedback control gains that would guarantee complete and stable synchronization. Finally, we have presented numerical simulation results to verify the effectiveness of the obtained criteria.

Acknowledgement

O.I. Olusola gratefully acknowledges the Federation of Indian Chamber of Commerce and Industry and Department of Science and Technology, India for the award of CV Raman International Fellowship for African Researchers. He is also grateful to Federal University of Agriculture, Abeokuta, Nigeria, for granting the study leave. Finally, he acknowledges

the hospitality of Dr. S.K. Dana of Indian Institute of Chemical Biology. The work of U.E. Vincent is financially supported by the Royal Society of London, United Kingdom, under the Newton International Fellowship Alumni Scheme.

References

- [1] Pecora L. M. and Carroll T. L. Synchronization in chaotic systems. *Phys Rev. Lett.* **84** (1990) 821–824.
- [2] Chen, G. and Dong, X. *From Chaos to Order*. World Scientific, Singapore, 1998.
- [3] Pikovsky, A. S., Rosenblum, M. G. and Kurths, J. *Synchronization – A Unified Approach to Nonlinear Science*. Cambridge University Press, Cambridge, 2001.
- [4] Lakshmanan M. and Murali K. *Chaos in Nonlinear Oscillators: Controlling and Synchronization*. World Scientific, Singapore, 1996.
- [5] Fradkov, A. L. and Pogromsky, A. Yu. *Introduction to Control of Oscillations and Chaos*. World Scientific, Singapore, 1996.
- [6] Ott, E., Grebogi, C. and Yorke J. A. Controlling chaos. *Phys. Rev. Lett.* **64** (1990) 1196–1199.
- [7] Kapitaniak, T. *Controlling Chaos – Theoretical & Practical Methods in Non-linear Dynamics*. Academic Press, London, 1996.
- [8] Rosas, D., Alvarez, J. and Alvarez, E. Robust synchronization of arrays of uncertain nonlinear second-order dynamical systems. *Nonlinear Dynamics* **67** (2012) 2735–2746.
- [9] Ghosh, D. Generalized projective synchronization in time-delayed systems: Nonlinear observer approach. *Chaos* **19** (2009) 013102(1–9).
- [10] Chen G. and Dong X. *From Chaos to Order: Methodologies, Perspectives and Applications*. World Scientific, Singapore, 1998.
- [11] Filali, R.L., Hammami, S., Benrejeb, M. and Borne, P. On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized Henon Map. *Nonlinear Dynamics and Systems Theory* **12** (2012) 81–95.
- [12] Chen, H. K. Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *Journal of Sound and Vibration* **255** (2002) 719–740.
- [13] Idowu B. A., Vincent U. E. and Njah A. N. Control and Synchronization of chaos in nonlinear gyros via backstepping design. *International Journal of Nonlinear Science* **5** (2008) 11–19.
- [14] Lei, Y., Xu, W., Xu, Y. and Fang, T. Chaos control by harmonic excitation with proper random phase. *Chaos Solitons & Fractals* **21** (2004) 1175–1181.
- [15] Ge ZM and Yu TC. Chaos synchronization of a horizontal platform system. *Journal of Sound and Vibration* **268** (2003) 731–749.
- [16] Chen LJ and Li JB. Chaotic behaviour and subharmonic bifurcations for a rotating pendulum equation. *Int. J. Bifur. Chaos* **14** (2004) 3477–3488.
- [17] Yamapi, R. and Wofo, P. Dynamics and synchronization of coupled self-sustained electromechanical devices. *Journal of Sound and Vibration* **285** (2005) 1151–1170.
- [18] Vincent, U. E. and Guo, R. Adaptive Synchronization for Oscillators in ϕ^6 Potentials. *Nonlinear Dynamics and Systems Theory* **13**(1) (2012) 93–106.
- [19] Dooren, R. V. Comments on Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *Journal of Sound and Vibration* **268** (2003) 632–634.
- [20] Ge, Z. M. and Chen, H. K. Bifurcations and chaos in a rate gyro with harmonic excitation. *Journal of Sound and Vibration* **194** (1996) 107–117.

- [21] López-Mancilla, D. and Cruz-Hernandez, C. Output synchronization of chaotic systems: Model-matching approach with application to secure communication. *Nonlinear Dynamics and Systems Theory* **5** (2005) 141–156.
- [22] Boukabou, A. On Nonlinear Control and Synchronization Design for Autonomous Chaotic Systems. *Nonlinear Dynamics and Systems Theory* **8** (2008) 151–167.
- [23] Lei, Y., Xu, W. and Zheng, H. Synchronization of two chaotic nonlinear gyros using active control. *Physics Letter A* **343** (2005) 153–158.
- [24] Farivar, F., Shoorehdeli, M. A., Nekoui, M. A. and Teshnehlab M. Chaos control and modified projective synchronization of unknown heavy symmetric chaotic gyroscope systems via Gaussian radial basis adaptive backstepping control. *Nonlinear Dynamics* **67** (2012) 1913–1941.
- [25] Yau, H.-T. Nonlinear rule-based controller for chaos synchronization of two gyros with linear-plus-cubic damping. *Chaos Solitons & Fractals* **34** (2007) 1357–1365.
- [26] Salarieh, H. and Alasty, A. Chaos synchronization of nonlinear gyros in presence of stochastic excitation via sliding mode control. *Journal of Sound and Vibration* **313** (2008) 760–771.
- [27] Yau, H.-T., Kuo, C.-L., and Yan, J.-J. Fuzzy Sliding Mode Control for a Class of Chaos Synchronization with Uncertainties. *International Journal of Nonlinear Sciences and Numerical Simulation* **7** (2006) 333–338.
- [28] Farivar, F., Shoorehdeli, M.A., Nekoui, M.A., and Teshnehlab, M. Generalized Projective Synchronization of Chaotic Heavy Gyroscope Systems via Sliding Rule-Based Fuzzy Control. *ISRN Artificial Intelligence* **2012** (2012) 576873. doi:10.5402/2012/576873.
- [29] Aghababa, M. P. and Aghababa H. P. Synchronization of nonlinear chaotic electromechanical gyrostat systems with uncertainties. *Nonlinear Dynamics* **67** (2012) 2689–2701.
- [30] Olusola, O. I., Vincent, U. E. and Njah, A. N. Bistability in coupled oscillators exhibiting synchronized dynamics. *Comm. in Theor. Phys.* **53** (2010) 815–824.
- [31] Wu, X. Cai, J. and Wang, M. Global chaos synchronization of the parametrically excited Duffing oscillators by linear state error feedback control. *Chaos, Solitons & Fractals* **36** (2008) 121–128.
- [32] Heagy, J. F., Carroll, T. L. and Pecora, L. M. Desynchronization by periodic orbits. *Phys. Rev. E* **52** (1995) 1253–1256.
- [33] Shuai, J. W., Wong, K. W. and Cheng, L. M. Synchronization of spatiotemporal chaos with positive conditional Lyapunov exponents. *Phys. Rev. E* **56** (1997) 2272.
- [34] Liao, X., Wang, L. and P. Yu. Stability of Dynamical Systems: In: *Monograph series on nonlinear science and complexity* (Eds. A. C. J. Luo and G. Zaslavsky). Elsevier B.V., Netherlands, 2007.
- [35] Boyd, S., Ghaoui, L.EI., Feron, E. and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
- [36] Stefański, A. *Determination of Thresholds for Complete Synchronization and Application*. World scientific, Singapore, 2009.
- [37] Horn, R. A. and Johnson, C. R. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.
- [38] Parekh, N., Kumar, V. R., and Kulkarni, B. D. Control of Spatiotemporal Chaos: A case with an autocatalytic reaction diffusion system. *Pramana J. Phys.* **48** (1997) 303–323.
- [39] He, R. and Vaidya, P. G. Analysis and Synthesis of Synchronous Periodic and Chaotic Systems. *Phys. Rev. A* **46** (1994) 7387–7392.
- [40] Curran, P.F. and Chua, I.O. Absolute stability theory and synchronization problem. *Int. J. of bifurcation and chaos* **7** (1997) 1357–1382.



Existence of Solutions for m -Point Boundary Value Problem with p -Laplacian on Time Scales

Ozlem Batit Ozen and Ilkay Yaslan Karaca*

Department of Mathematics Ege University, 35100 Bornova, Izmir, Turkey

Received: July 6, 2012; Revised: July 15, 2013

Abstract: We consider the existence of positive solutions for a class of second-order m -point boundary value problem with p -Laplacian on time scales. By using Avery-Peterson's fixed point theorem, sufficient conditions for the positive solutions are established. Meanwhile an example is worked out to illustrate the main result.

Keywords: m -point boundary value problems; p -Laplacian operator; positive solutions; fixed point theorems; time scales.

Mathematics Subject Classification (2010): 39A10, 34B15, 34B16.

1 Introduction

Calculus on time scales was introduced by Hilger (see [6]), as a theory which is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of difference equations in the discrete case. Some preliminary definitions and theorems on time scales can be found in books [3, 4] which are excellent references for calculus of time scales. Also, there is much attention paid to the study of multipoint boundary value problem (see [1, 2, 7–13]).

In [5] the following m -point boundary value problem on time scales was studied

$$\begin{aligned} u^{\Delta\nabla}(t) + q(t)f(u(t)) &= 0, \quad t \in [0, T]_{\mathbb{T}}, \\ u^{\Delta}(0) &= \sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i), \quad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{aligned}$$

* Corresponding author: <mailto:ilkay.karaca@ege.edu.tr>

where $a_i, b_i \geq 0$ ($i = 1, 2, \dots, m - 2$), and $\xi_i \in (0, \rho(T))_{\mathbb{T}}$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \rho(T)$. And the existence of at least two positive solutions of the above problem was established by means of a fixed point theorem in a cone.

Zhao and Ge [13] studied the following m-point boundary value problem on time scales

$$\begin{aligned}
 (\phi_p(u^\Delta))^\nabla(t) + h(t)f(t, u(t), u^\Delta(t)) &= 0, \quad t \in (0, \infty)_{\mathbb{T}}, \\
 u(0) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u^\Delta(+\infty) &= \sum_{i=1}^{m-2} \beta_i u^\Delta(\eta_i),
 \end{aligned}$$

where $\alpha_i, \beta_i \geq 0$ ($i = 1, 2, \dots, m - 2$), and $\eta_i \in (0, \infty)_{\mathbb{T}}$ with $\sigma(0) < \eta_1 < \eta_2 < \dots < \eta_{m-2} < +\infty$. They established new criteria for the existence of at least three unbounded positive solutions by using Avery-Peterson’s fixed point theorem.

Ji, Bai and Ge [7] studied the following singular multipoint boundary value problem on time scales

$$\begin{aligned}
 (\phi_p(u'))'(t) + a(t)f(u(t)) &= 0, \quad t \in (0, 1), \\
 u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) &= \sum_{i=1}^{m-2} \alpha_i u(\eta_i),
 \end{aligned}$$

where $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$, $\xi_i < \eta_i$, $\alpha_i > 0$ for $i = 1, 2, \dots, m - 2$. By using fixed point index theory and the Legget-Williams fixed point theorem, sufficient conditions for the existence of countably many positive solutions are established.

Sun, Wang and Fan [10] studied the nonlocal boundary value problem with p-Laplacian of the form

$$\begin{aligned}
 (\phi_p(u^\Delta))^\nabla(t) + h(t)f(t, u(t)) &= 0, \quad t \in [t_1, t_m]_{\mathbb{T}}, \\
 u^\Delta(t_1) - \sum_{j=1}^n \theta_j u^\Delta(\eta_j) - \sum_{i=1}^{m-2} \varepsilon_i u(\xi_i) &= 0, \quad u^\Delta(t_m) = 0,
 \end{aligned}$$

where $0 \leq t_1 \leq \xi_1 \leq \xi_2 \leq \dots \leq \xi_{m-2} \leq t_m$ and $0 \leq t_1 \leq \eta_1 \leq \eta_2 \leq \dots \leq \eta_{m-2} \leq t_m$ and $\varepsilon_i > 0$, $\theta_i \geq 0$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. By using the Four functionals fixed point theorem and Five Functionals fixed point theorem, they obtained the existence criteria of at least one positive solution and three positive solutions.

Inspired by the mentioned works, in this paper we consider the following m-point boundary value problem (BVP) with p-Laplacian

$$(\phi_p(x^\Delta))^\nabla(t) + h(t)f(t, x(t), x^\Delta(t)) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \tag{1}$$

$$x^\Delta(0) - \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = 0, \quad x^\Delta(1) + \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = 0, \tag{2}$$

where \mathbb{T} is a time scale, $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, and $\frac{1}{p} + \frac{1}{q} = 1$.

We assume that the following conditions are satisfied:

(H1) $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \rho(1)$, $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < \rho(1)$, $\xi_i < \eta_i$, $\alpha_i > 0$
 for $i = 1, 2, \dots, m-2$, $\sum_{i=1}^{m-2} \alpha_i \xi_i < 1$ and $[\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)]^2 + \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) < 1$,

(H2) $f \in \mathcal{C}([0, 1]_{\mathbb{T}} \times [0, \infty) \times (-\infty, \infty), (0, \infty))$,

(H3) $h \in \mathcal{C}_{ld}([0, 1]_{\mathbb{T}}, [0, \infty))$.

By using Avery-Peterson fixed point theorem, we establish the existence of at least three positive solutions for the BVP (1)-(2). The remainder of this paper is organized as follows. Section 2 is devoted to some preliminary lemmas. We give and prove our main result in Section 3.

2 Preliminaries

To prove the main result in this paper, we will employ several lemmas. These lemmas are based on the BVP

$$(\phi_p(x^\Delta))^\nabla(t) + y(t) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \quad (3)$$

$$x^\Delta(0) - \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = 0, \quad x^\Delta(1) + \sum_{i=1}^{m-2} \alpha_i x(\eta_i) = 0. \quad (4)$$

Lemma 2.1 *Let (H1) – (H3) hold. Then for $y \in \mathcal{C}_{ld}[0, 1]_{\mathbb{T}}$, the BVP (3)-(4) has the unique solution*

$$x(t) = \frac{\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q(A_x - \int_0^s y(\tau) \nabla \tau) \Delta s}{\sum_{i=1}^{m-2} \alpha_i} - \int_t^1 \phi_q(A_x - \int_0^s y(\tau) \nabla \tau) \Delta s, \quad (5)$$

where A_x satisfies

$$\phi_q(A_x) + \phi_q\left(A_x - \int_0^1 y(s) \nabla s\right) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(A_x - \int_0^s y(\tau) \nabla \tau\right) \Delta s = 0. \quad (6)$$

Moreover, there exists a unique $A_x \in (0, \int_0^1 y(s) \nabla s)$ satisfying (6).

Proof. Integrating (3) from 0 to t , we have

$$x^\Delta(t) = \phi_q \left(\phi_p(x^\Delta(0)) - \int_0^t y(s) \nabla s \right). \tag{7}$$

Integrating (7) from t to 1, we get

$$x(t) = x(1) - \int_t^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s, \tag{8}$$

where $A_x = \phi_p(x^\Delta(0))$. Setting $t = \xi_i$ in (8) we have

$$x(\xi_i) = x(1) - \int_{\xi_i}^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s, \quad i = 1, 2, 3, \dots, m-2$$

and

$$\sum_{i=1}^{m-2} \alpha_i x(\xi_i) = \sum_{i=1}^{m-2} \alpha_i x(1) - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s$$

then

$$x(1) = \frac{\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s}{\sum_{i=1}^{m-2} \alpha_i}. \tag{9}$$

Substituting (9) into (8) we see that $x(t)$ satisfies (5) on $[0, 1]_{\mathbb{T}}$. (4) boundary conditions satisfy

$$\begin{aligned} x^\Delta(0) + x^\Delta(1) &= \sum_{i=1}^{m-2} \alpha_i x(\xi_i) - \sum_{i=1}^{m-2} \alpha_i x(\eta_i) \\ \phi_q(A_x) + \phi_q \left(A_x - \int_0^1 y(s) \nabla s \right) &= \sum_{i=1}^{m-2} \alpha_i (x(\xi_i) - x(\eta_i)) \\ &= \sum_{i=1}^{m-2} \alpha_i \left(- \int_{\xi_i}^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s + \int_{\eta_i}^1 \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s \right) \\ &= - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q \left(A_x - \int_0^s y(\tau) \nabla \tau \right) \Delta s. \end{aligned}$$

So that BVP (3)-(4) has a solution $x(t)$ where A_x satisfies (6).

For any $x \in \mathcal{C}_{ld}^\Delta[0, 1]_{\mathbb{T}}$, define

$$\begin{aligned} H_x(c) &= \phi_q(c) + \phi_q \left(c - \int_0^1 h(s) f(s, x(s), x^\Delta(s)) \nabla s \right) \\ &+ \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q \left(c - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s. \end{aligned}$$

Then $H_x : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and strictly increasing. $H_x(0) < 0$, $H_x\left(\int_0^1 h(s)f(s, x(s), x^\Delta(s))\nabla s\right) > 0$, imply the existence of a unique $c = A_x \in (0, \int_0^1 h(s)f(s, x(s), x^\Delta(s))\nabla s)$ such that $H_x(A_x) = 0$. \square

Lemma 2.2 *If (H1) – (H3) hold, then for $x \in \mathcal{C}_{1d}^\Delta[0, 1]_{\mathbb{T}}$, the unique solution $x(t)$ of BVP (3)-(4) has the following properties:*

- (i) $x(t)$ is concave on $[0, 1]_{\mathbb{T}}$,
- (ii) $x(t) > 0$.

Proof. Suppose that $x(t)$ is a solution of BVP (3)-(4), then

(i) $(\phi_p(x^\Delta))^\nabla(t) = -h(t)f(t, x(t), x^\Delta(t)) \leq 0$, $\phi_p(x^\Delta)$ is nonincreasing so that $x^\Delta(t)$ is nonincreasing. This implies that $x(t)$ is concave.

(ii) We have $x^\Delta(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) = \phi_q(A_x) > 0$ and $x^\Delta(1) = \phi_q\left(A_x - \int_0^1 h(s)f(s, x(s), x^\Delta(s))\nabla s\right) < 0$. Furthermore, we get

$$\alpha_1 x(\xi_1) - \alpha_1 x(0) = \alpha_1 \int_0^{\xi_1} x^\Delta(s)\Delta s \leq \alpha_1 \xi_1 x^\Delta(0) = \alpha_1 \xi_1 \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$$

$$\alpha_2 x(\xi_2) - \alpha_2 x(0) = \alpha_2 \int_0^{\xi_2} x^\Delta(s)\Delta s \leq \alpha_2 \xi_2 x^\Delta(0) = \alpha_2 \xi_2 \sum_{i=1}^{m-2} \alpha_i x(\xi_i).$$

If we continue like this, we have

$$\begin{aligned} \alpha_{m-2} x(\xi_{m-2}) - \alpha_{m-2} x(0) &= \alpha_{m-2} \int_0^{\xi_{m-2}} x^\Delta(s)\Delta s \leq \alpha_{m-2} \xi_{m-2} x^\Delta(0) \\ &= \alpha_{m-2} \xi_{m-2} \sum_{i=1}^{m-2} \alpha_i x(\xi_i). \end{aligned}$$

Using (H1), we obtain

$$\sum_{i=1}^{m-2} \alpha_i x(\xi_i) - \sum_{i=1}^{m-2} \alpha_i x(0) \leq \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \sum_{i=1}^{m-2} \alpha_i \xi_i < \sum_{i=1}^{m-2} \alpha_i x(\xi_i),$$

which implies that $x(0) > 0$. Similarly,

$$\alpha_1 x(1) - \alpha_1 x(\eta_1) = \alpha_1 \int_{\eta_1}^1 x^\Delta(s)\Delta s \geq \alpha_1(1 - \eta_1)x^\Delta(1) = -\alpha_1(1 - \eta_1) \sum_{i=1}^{m-2} \alpha_i x(\eta_i),$$

$$\alpha_2 x(1) - \alpha_2 x(\eta_2) = \alpha_2 \int_{\eta_2}^1 x^\Delta(s)\Delta s \geq \alpha_2(1 - \eta_2)x^\Delta(1) = -\alpha_2(1 - \eta_2) \sum_{i=1}^{m-2} \alpha_i x(\eta_i).$$

If we continue like this, we have

$$\begin{aligned} \alpha_{m-2}x(1) - \alpha_{m-2}x(\eta_{m-2}) &= \alpha_{m-2} \int_{\eta_{m-2}}^1 x^\Delta(s) \Delta s \geq \alpha_{m-2}(1 - \eta_{m-2})x^\Delta(1) \\ &= -\alpha_{m-2}(1 - \eta_{m-2}) \sum_{i=1}^{m-2} \alpha_i x(\eta_i). \end{aligned}$$

Using (H1), we have $\sum_{i=1}^{m-2} \alpha_i x(1) > 0, x(1) > 0$. Therefore, we get $x(t) > 0, t \in [0, 1]_{\mathbb{T}}$. \square

Let $E = C_{ld}^\Delta[0, 1]_{\mathbb{T}}$, then E is a Banach space with the norm

$$\|x\| = \max\left\{ \sup_{t \in [0, 1]_{\mathbb{T}}} |x(t)|, \sup_{t \in [0, 1]_{\mathbb{T}}} |x^\Delta(t)| \right\}$$

and choose the cone $P \subset E$ denoted by

$$P = \left\{ x \in E : x(t) \geq 0, x^\Delta(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), x(t) \text{ is concave on } [0, 1]_{\mathbb{T}} \right\}.$$

Define the operator $T : P \rightarrow E$ by

$$\begin{aligned} Tx(t) &= \frac{\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q(A_x - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau) \Delta s}{\sum_{i=1}^{m-2} \alpha_i} \\ &\quad - \int_t^1 \phi_q(A_x - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau) \Delta s. \end{aligned} \tag{10}$$

Lemma 2.3 *If (H1) holds, then $\sup_{t \in [0, 1]_{\mathbb{T}}} x(t) \leq M \sup_{t \in [0, 1]_{\mathbb{T}}} |x^\Delta(t)|$ for $x \in P$, where*

$$M = 1 + \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}. \tag{11}$$

Proof. For $x \in P$, one arrives at

$$x(1) - x(0) \leq \frac{x(\xi_i) - x(0)}{\xi_i}.$$

Hence,

$$\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) x(0) \leq \sum_{i=1}^{m-2} \alpha_i x(\xi_i).$$

By $x^\Delta(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$, we get

$$x(0) \leq \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} x^\Delta(0).$$

Hence

$$\begin{aligned} x(t) &= \int_0^t x^\Delta(s) \Delta s + x(0) \\ &\leq tx^\Delta(0) + x(0) \\ &\leq tx^\Delta(0) + \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} x^\Delta(0) \\ &\leq \left[1 + \frac{1}{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}\right] x^\Delta(0) \\ &= Mx^\Delta(0), \end{aligned}$$

i.e.,

$$\sup_{t \in [0,1]_{\mathbb{T}}} x(t) \leq Mx^\Delta(0) = M \sup_{t \in [0,1]_{\mathbb{T}}} x^\Delta(t) \leq M \sup_{t \in [0,1]_{\mathbb{T}}} |x^\Delta(t)|.$$

The proof is complete. \square

From Lemma 2.3, we obtain

$$\begin{aligned} \|x\| &= \max\left\{ \sup_{t \in [0,1]_{\mathbb{T}}} |x(t)|, \sup_{t \in [0,1]_{\mathbb{T}}} |x^\Delta(t)| \right\} \\ &\leq \max\left\{ M \sup_{t \in [0,1]_{\mathbb{T}}} |x^\Delta(t)|, \sup_{t \in [0,1]_{\mathbb{T}}} |x^\Delta(t)| \right\} \\ &\leq M \sup_{t \in [0,1]_{\mathbb{T}}} |x^\Delta(t)|. \end{aligned}$$

Lemma 2.4 For $x \in C_{ld}^\Delta[0,1]_{\mathbb{T}}$, let A_x satisfy (6) corresponding to x . Suppose that (H1) – (H3) hold, then $A_x : C_{ld}^\Delta[0,1]_{\mathbb{T}} \rightarrow \mathbb{R}$ is continuous about x .

Proof. Suppose $\{x_n\} \in C_{ld}^\Delta[0,1]_{\mathbb{T}}$ with $x_n \rightarrow x_0 \in C_{ld}^\Delta[0,1]_{\mathbb{T}}$, then there exists r_0 such that

$$\max\{\|x_0\|, \sup_{n \in \mathbb{N} - \{0\}} \|x_n\|\} < r_0.$$

Let A_n ($n = 0, 1, \dots$) be constants decided by (6) corresponding to x_n ($n = 0, 1, 2, \dots$). By (H2), we get that $f(t, u, v)$ is bounded on $[0,1]_{\mathbb{T}} \times [0, r_0]^2$. Set

$$B_{r_0} = \sup\{f(t, u, v) : (t, u, v) \in [0,1]_{\mathbb{T}} \times [0, r_0]^2\}.$$

Since

$$\int_0^1 h(s)f(s, x(s), x^\Delta(s))\Delta s \leq B_{r_0} \int_0^1 h(s)\Delta s = B_{r_0}\Lambda,$$

where $\Lambda = \int_0^1 h(s)\Delta s$, $A_n \in [0, \int_0^1 h(s)f(s, x(s), x^\Delta(s))\Delta s] \subseteq [0, B_{r_0}\Lambda]$, which means $\{A_n\}$ is bounded. Suppose that sequence $\{A_n\}$ does not convergence, then there exist two subsequences $\{A_{n_k}^{(1)}\}$, $\{A_{n_k}^{(2)}\}$ of $\{A_n\}$ with $A_{n_k}^{(1)} \rightarrow c_1$, $A_{n_k}^{(2)} \rightarrow c_2$, and $c_1 \neq c_2$. Combining (H2) and using the Lebesgue’s dominated convergence theorem, we get

$$\begin{aligned} \phi_q(c_1) &= - \lim_{n_k \rightarrow +\infty} \phi_q(A_{n_k}^{(1)} - \int_0^1 h(s)f(s, x_{n_k}(s), x_{n_k}^\Delta(s))\nabla s) \\ &\quad - \lim_{n_k \rightarrow +\infty} \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(A_{n_k}^{(1)} - \int_0^s h(\tau)f(\tau, x_{n_k}(\tau), x_{n_k}^\Delta(\tau))\nabla \tau)\Delta s \\ &= -\phi_q(\lim_{n_k \rightarrow +\infty} A_{n_k}^{(1)} - \lim_{n_k \rightarrow +\infty} \int_0^1 h(s)f(s, x_{n_k}(s), x_{n_k}^\Delta(s))\nabla s) \\ &\quad - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(\lim_{n_k \rightarrow +\infty} A_{n_k}^{(1)} - \lim_{n_k \rightarrow +\infty} \int_0^s h(\tau)f(\tau, x_{n_k}(\tau), x_{n_k}^\Delta(\tau))\nabla \tau)\Delta s \\ &= -\phi_q(c_1 - \int_0^1 h(s)f(s, x_0(s), x_0^\Delta(s))\nabla s) \\ &\quad - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(c_1 - \int_0^s h(\tau)f(\tau, x_0(\tau), x_0^\Delta(\tau))\nabla \tau)\Delta s. \end{aligned}$$

Since sequence $\{A_n\}$ is unique, we get $c_1 = A_0$. Similarly $c_2 = A_0$. So $c_1 = c_2$, which is a contradiction. Therefore $A_n \rightarrow A_0$ for $x_n \rightarrow x_0$, which means $A_x : \mathcal{C}_{ld}^\Delta[0, 1]_{\mathbb{T}} \rightarrow \mathbb{R}$ is continuous. The proof is complete. \square

Lemma 2.5 *Suppose that (H1) – (H3) hold, then $T : P \rightarrow P$ is completely continuous.*

Proof. We divide the proof into three steps.

Step 1. We show that $TP \subset P$. For $x \in P$, by (H1) – (H3), we have $(Tx)(t) \geq 0$ and

$$(Tx)^\Delta(0) = \sum_{i=1}^{m-2} \alpha_i (Tx)(\xi_i).$$

If $t \in [0, 1]_{\mathbb{T}}$ is left scattered, then

$$(Tx)^{\Delta\nabla}(t) = \frac{(Tx)^\Delta(t) - (Tx)^\Delta(\rho(t))}{t - \rho(t)} \leq 0$$

on $t \in [0, 1]_{\mathbb{T}}$. If $t \in [0, 1]_{\mathbb{T}}$ is left dense, then

$$(Tx)^{\Delta\nabla}(t) = \lim_{s \rightarrow t} \frac{(Tx)^\Delta(t) - (Tx)^\Delta(s)}{t - s} \leq 0$$

on $t \in [0, 1]_{\mathbb{T}}$. Hence Tx is nonnegative, concave on $[0, 1]_{\mathbb{T}}$, i.e., $TP \subset P$.

Step 2. We show that $T : P \rightarrow P$ is continuous. Let $x_n \rightarrow x$ as $n \rightarrow +\infty$ in P , then there exists r_0 such that

$$\max\{\|x\|, \sup_{n \in \mathbb{N} - \{0\}} \|x_n\|\} < r_0.$$

By (H2), we get that $f(t, u, v)$ is bounded on $[0, 1]_{\mathbb{T}} \times [0, r_0]^2$. Set

$$B_{r_0} = \sup\{f(t, u, v) : (t, u, v) \in [0, 1] \times [0, r_0]^2\}.$$

We get

$$\begin{aligned} & |\phi_p((Tx_n)^\Delta(t)) - \phi_p((Tx)^\Delta(t))| \\ &= |A_{x_n} - \int_0^t h(s)f(s, x_n(s), x_n^\Delta(s))\nabla s - A_x - \int_0^t h(s)f(s, x(s), x^\Delta(s))\nabla s| \\ &\leq |A_{x_n} - A_x| + \int_0^t h(s)|f(s, x_n(s), x_n^\Delta(s)) - f(s, x(s), x^\Delta(s))|\nabla s \\ &\leq |A_{x_n} - A_x| + 2B_{r_0}\Lambda = 2B_{r_0}\Lambda + 2B_{r_0}\Lambda = 4B_{r_0}\Lambda. \end{aligned}$$

Therefore by the Lebesgue's dominated convergence theorem, we have

$$|\phi_p((Tx_n)^\Delta(t)) - \phi_p((Tx)^\Delta(t))| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

By using Lemma 2.3 we get

$$0 \leq \|(Tx_n)(t) - (Tx)(t)\| \leq M \sup_{t \in [0, 1]_{\mathbb{T}}} |(Tx_n)^\Delta(t) - (Tx)^\Delta(t)| \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence T is continuous.

Step 3. We show that $T : P \rightarrow P$ is relatively compact. Let Ω be any bounded set of P . Then there exists $L > 0$ such that $\|x\| \leq L$ for all $x \in \Omega$. Set

$$B_L = \sup\{f(t, u, v) : (t, u, v) \in [0, 1] \times [0, r_0]^2\}.$$

For $x \in \Omega$, we have

$$\begin{aligned} \|Tx\| &= \max\left\{\sup_{t \in [0, 1]_{\mathbb{T}}} Tx(t), \sup_{t \in [0, 1]_{\mathbb{T}}} |(Tx)^\Delta(t)|\right\} \\ &\leq M(Tx)^\Delta(0) \\ &\leq M\phi_q(A_x) \leq M\phi_q(B_L\Lambda). \end{aligned}$$

Hence $T\Omega$ is uniformly bounded.

Now we show that $T\Omega$ is locally equicontinuous on $[0, 1]_{\mathbb{T}}$. For $t_1, t_2 \in [0, 1]_{\mathbb{T}}$ and $x \in \Omega$, we may assume that $t_2 > t_1$.

$$\begin{aligned} & |\phi_p((Tx)^\Delta(t_1)) - \phi_p((Tx)^\Delta(t_2))| \\ &= |A_x - \int_0^{t_1} h(s)f(s, x(s), x^\Delta(s))\nabla s - A_x + \int_0^{t_2} h(s)f(s, x(s), x^\Delta(s))\nabla s|. \end{aligned}$$

Hence,

$$|\phi_p((Tx)^\Delta(t_1)) - \phi_p((Tx)^\Delta(t_2))| \rightarrow 0 \text{ as } t_1 \rightarrow t_2.$$

Since

$$\sup_{t \in [0,1]_{\mathbb{T}}} |(Tx)^{\Delta}(t_1) - (Tx)^{\Delta}(t_2)| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2,$$

we get

$$\|(Tx)(t_1) - (Tx)(t_2)\| \longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.$$

Hence $T\Omega$ is locally equicontinuous on $[0, 1]_{\mathbb{T}}$. From step 1 – 3, we get $T : P \longrightarrow P$ is completely continuous. The proof is complete. \square

3 Existence of Three Positive Solutions

Let γ and θ be nonnegative continuous convex functionals on a cone P , α be nonnegative continuous concave functional on P and ψ be nonnegative continuous functional on P . Then for positive real numbers a, b, c and d , we define the following convex sets

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\},$$

$$P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x), \gamma(x) \leq d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$

$$R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}.$$

Theorem 3.1 (Avery-Peterson’s Fixed Point Theorem) [13] *Let \mathcal{P} be a cone in a real Banach space E . Assume that there exist two positive number M and d , two nonnegative continuous convex functionals γ and θ on P , a nonnegative continuous concave functional α on P and a nonnegative continuous functional ψ on P such that $\psi(\lambda x) \leq \lambda\psi(x)$ for all $0 \leq \lambda \leq 1$ and*

$$\alpha(x) \leq \psi(x), \quad \|x\| \leq M\gamma(x)$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)}$ is completely continuous and there exist three positive numbers a, b and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S2) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;

(S3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, \quad i = 1, 2, 3, \quad \psi(x_1) < a, \quad a < \psi(x_2) \text{ with } \alpha(x_2) < b, \quad \alpha(x_3) > b.$$

Set

$$\Omega = \int_w^v h(\tau) \nabla \tau,$$

and define the maps

$$\gamma(x) = \sup_{t \in [0,1]_{\mathbb{T}}} |x^{\Delta}(t)|, \quad \psi(x) = \theta(x) = \sup_{t \in [0,1]_{\mathbb{T}}} x(t), \quad \alpha(x) = \min_{t \in [w,v]_{\mathbb{T}}} x(t). \tag{12}$$

Theorem 3.2 Assume (H1) – (H3) hold. Let

$$\frac{2b}{w} \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\sum_{i=1}^{m-2} \alpha_i + 1 - \sum_{i=1}^{m-2} \alpha_i \xi_i \right] < c < d,$$

$$\max \left\{ \xi_i, \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[2 \sum_{i=1}^{m-2} \alpha_i - \sum_{i=1}^{m-2} \alpha_i \xi_i + \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) \sum_{i=1}^{m-2} \alpha_i (\eta_i - \xi_i) - 1 \right], \right.$$

$$\left. \frac{2b}{c} \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\sum_{i=1}^{m-2} \alpha_i + 1 - \sum_{i=1}^{m-2} \alpha_i \xi_i \right] \right\} < w < \nu < \frac{1}{2}$$

and suppose that f satisfies the following conditions

(A1) $f(t, u, v) \leq \frac{1}{2\Lambda} \phi_p(d)$ for $(t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, Md] \times [0, d]$;

(A2) $f(t, u, v) > \frac{1}{\Omega} \phi_p(\frac{b}{A})$ for $(t, u, v) \in [w, v]_{\mathbb{T}} \times [b, c] \times [0, d]$;

(A3) $f(t, u, v) < \frac{1}{2\Lambda} \phi_p(\frac{a}{M})$ for $(t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, a] \times [0, d]$;

where M, Λ are defined as in (11) and Lemma 2.4 respectively, and

$$A = \frac{1}{m-2} \left[\left(1 + \sum_{i=1}^{m-2} \alpha_i (w - \xi_i) \right) \frac{1}{2 + \sum_{i=1}^{m-2} \alpha_i (\eta_i - \xi_i)} - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) \right].$$

Then the BVP (1)-(2) has at least three positive solutions x_1, x_2 and x_3 such that

$$\gamma(x_i) \leq d, \quad i = 1, 2, 3, \quad \psi(x_1) < a, \quad a < \psi(x_2) \text{ with } \alpha(x_2) < b, \quad \alpha(x_3) > b.$$

Proof. The boundary value problem (1)-(2) has a solution $x = x(t)$ if and only if x solves the operator equation $x = Tx$. Thus we set out to verify that the operator T satisfies Avery-Peterson's fixed point theorem which will prove the existence of three fixed point of T . Now the proof is divided into four steps.

Step 1 : We will show that (A1) implies that

$$T : \overline{P(\gamma, d)} \longrightarrow \overline{P(\gamma, d)}.$$

For $x \in \overline{P(\gamma, d)}$, there is $\gamma(x) = \sup_{t \in [0, 1]_{\mathbb{T}}} |x^\Delta(t)| \leq d$. From Lemma 2.3,

$$\sup_{t \in [0, 1]_{\mathbb{T}}} x(t) \leq M \sup_{t \in [0, 1]_{\mathbb{T}}} |x^\Delta(t)| \leq Md,$$

then the condition (A1) implies

$$f(t, x(t), x^\Delta(t)) \leq \frac{\phi_p(d)}{2\Lambda}.$$

On the other hand, for $x \in P$, we get

$$\begin{aligned} \gamma(Tx) &= \sup_{t \in [0,1]_{\mathbb{T}}} |(Tx)^{\Delta}(t)| \\ &= \sup_{t \in [0,1]_{\mathbb{T}}} |\phi_q(A_x - \int_0^t h(s)f(s, x(s), x^{\Delta}(s))\nabla s)| \\ &\leq \phi_q(A_x + \int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s) \\ &\leq \phi_q(2 \int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s) \\ &\leq \phi_q(\frac{\phi_p(d)}{\Lambda} \int_0^1 h(s)\nabla s) = d. \end{aligned}$$

Step 2. We show that condition (S1) in Theorem 3.1 holds. We take

$$x(t) = \frac{c}{2} \frac{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i + \sum_{i=1}^{m-2} \alpha_i} [\frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i} t + 1]$$

for $t \in [0, 1]_{\mathbb{T}}$. By (12), we get

$$\gamma(x) = \sup_{t \in [0,1]_{\mathbb{T}}} |x^{\Delta}(t)| = \frac{c}{2} \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i + \sum_{i=1}^{m-2} \alpha_i} < d,$$

$$\psi(x) = \theta(x) = \sup_{t \in [0,1]_{\mathbb{T}}} x(t) = x(1) = \frac{c}{2} < c,$$

$$\alpha(x) = \min_{t \in [w,v]_{\mathbb{T}}} x(t) = x(w) > b.$$

Hence $\{x \in P(\gamma, \theta, \alpha, b, c, d : \alpha(x) > b)\} \neq \emptyset$.

Since

$$\begin{aligned} \phi_q(A_x) &= \phi_q\left(\int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s - A_x\right) \\ &+ \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(\int_0^s h(\tau)f(\tau, x(\tau), x^{\Delta}(\tau))\nabla \tau - A_x\right) \Delta s \\ &\geq \phi_q\left(\int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s\right) - \phi_q(A_x) \\ &+ \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q\left(\int_0^s h(\tau)f(\tau, x(\tau), x^{\Delta}(\tau))\nabla \tau\right) \Delta s \\ &- \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\eta_i} \phi_q(A_x) \Delta s \\ &\geq \phi_q\left(\int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s\right) - \phi_q(A_x) \\ &- \sum_{i=1}^{m-2} \alpha_i (\eta_i - \xi_i) \phi_q(A_x), \end{aligned}$$

we have

$$\left[2 + \sum_{i=1}^{m-2} \alpha_i (\eta_i - \xi_i)\right] \phi_q(A_x) \geq \phi_q\left(\int_0^1 h(s)f(s, x(s), x^{\Delta}(s))\nabla s\right).$$

Hence, we get

$$\phi_q(A_x) \geq \frac{1}{2 + \sum_{i=1}^{m-2} \alpha_i(\eta_i - \xi_i)} \phi_q \left(\int_0^1 h(s) f(s, x(s), x^\Delta(s)) \nabla s \right). \quad (13)$$

Case 1. If $\alpha(Tx) = \min_{t \in [w, \nu]_{\mathcal{T}}} Tx(t) = Tx(w)$ holds then from (10), (13) and (A2), we obtain

$$\begin{aligned} Tx(w) &= \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q \left(A_x - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s \right] \\ &+ \int_1^w \phi_q \left(A_x - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s \\ &= \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^w \phi_q \left(A_x - \int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s \right] \\ &\geq \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^w \phi_q(A_x) \Delta s \right. \\ &\quad \left. - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^w \phi_q \left(\int_0^s h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s \right] \\ &\geq \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\phi_q(A_x) + \sum_{i=1}^{m-2} \alpha_i (w - \xi_i) \phi_q(A_x) \right. \\ &\quad \left. - \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \Delta s \right] \\ &= \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\left(1 + \sum_{i=1}^{m-2} \alpha_i (w - \xi_i) \right) \phi_q(A_x) - \right. \\ &\quad \left. \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \right] \\ &\geq \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\left(1 + \sum_{i=1}^{m-2} \alpha_i (w - \xi_i) \right) \frac{1}{2 + \sum_{i=1}^{m-2} \alpha_i (\eta_i - \xi_i)} \right] \end{aligned}$$

$$\begin{aligned}
 & - \sum_{i=1}^{m-2} \alpha_i(1 - \xi_i) \Big] \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \\
 & = A \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \\
 & > A \phi_q \left(\int_w^\nu h(\tau) \frac{1}{\Omega} \phi_p \left(\frac{b}{A} \right) \nabla \tau \right) \\
 & = A \frac{b}{A} \phi_q \left(\frac{1}{\Omega} \int_w^\nu h(\tau) \nabla \tau \right) = b.
 \end{aligned}$$

Thus we get $Tx(w) > b$.

Case 2. If $\alpha(Tx) = \min_{t \in [w, \nu]_{\mathbb{T}}} Tx(t) = Tx(\nu)$ holds then from (10), (13) and (A2), we get

$$\begin{aligned}
 Tx(\nu) & \geq \frac{1}{\sum_{i=1}^{m-2} \alpha_i} \left[\left(1 + \sum_{i=1}^{m-2} \alpha_i(\nu - \xi_i) \right) \frac{1}{2 + \sum_{i=1}^{m-2} \alpha_i(\eta_i - \xi_i)} \right. \\
 & \quad \left. - \sum_{i=1}^{m-2} \alpha_i(1 - \xi_i) \right] \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \\
 & \geq A \phi_q \left(\int_0^1 h(\tau) f(\tau, x(\tau), x^\Delta(\tau)) \nabla \tau \right) \\
 & > A \phi_q \left(\int_w^\nu h(\tau) \frac{1}{\Omega} \phi_p \left(\frac{b}{A} \right) \nabla \tau \right) = b.
 \end{aligned}$$

Hence we get $Tx(\nu) > b$.

Therefore we get $\alpha(Tx) > b$ for all $x \in P(\gamma, \theta, \alpha, b, c, d)$. Consequently, condition (S1) in Theorem 3.1 is satisfied.

Step 3. We prove that (S2) in Theorem 3.1 holds. Since x is nonnegative and concave on $[0, 1]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
 x(w) & = x \left[\frac{\frac{1}{w}(1+t) - 1}{\frac{1}{w}(1+t)} \frac{1}{\frac{1}{w}(1+t) - 1} + \frac{1}{\frac{1}{w}(1+t)} t \right] \\
 & \geq \frac{\frac{1}{w}(1+t) - 1}{\frac{1}{w}(1+t)} x \left(\frac{1}{\frac{1}{w}(1+t) - 1} \right) + \frac{1}{\frac{1}{w}(1+t)} x(t) \\
 & \geq \frac{w}{1+t} x(t) \geq \frac{w}{2} x(t).
 \end{aligned}$$

Therefore $x(w) \geq \frac{w}{2} \sup_{t \in [0, 1]_{\mathbb{T}}} x(t) = \frac{w}{2} \theta(x)$. Similarly $x(\nu) \geq \frac{\nu}{2} \theta(x) > \frac{w}{2} \theta(x)$ holds. Hence

$$\alpha(x) \geq \frac{w}{2} \theta(x), \quad x \in [0, 1]_{\mathbb{T}}.$$

Then we get

$$\begin{aligned}
 \alpha(Tx) & \geq \frac{w}{2} \theta(Tx) > \frac{w}{2} c > \frac{w}{2} \frac{2b(L+1)}{wL} \\
 & = b \left(\frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} + 1 \right) \frac{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i}{\sum_{i=1}^{m-2} \alpha_i} > b
 \end{aligned}$$

for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$.

Step 4. Finally, we prove that (S3) in Theorem 3.1 is satisfied. Since $\psi(0) = 0 < a$, $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$, then by (A3) and Lemma 2.3 we get

$$\begin{aligned} \psi(Tx) &= \sup_{t \in [0,1]} Tx(t) \\ &\leq M \sup_{t \in [0,1]} |(Tx)^\Delta(t)| \\ &\leq M\phi_q \left[2 \int_0^1 h(s)f(s, x(s), x^\Delta(s))\nabla s \right] \\ &< M\phi_q \left[2 \int_0^1 h(s)\frac{1}{2\Lambda}\phi_p\left(\frac{a}{M}\right)\nabla s \right] \\ &< M\frac{a}{M}\phi_q\left(\frac{1}{\Lambda} \int_0^1 h(s)\nabla s\right) = a. \end{aligned}$$

Consequently condition (S3) in Theorem 3.1 holds. From steps 1 – 4 together with Theorem 3.1 we get that the boundary value problem (1)-(2) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{t \in [0,1]_{\mathbb{T}}} |x_i^\Delta(t)| \leq d, \quad i = 1, 2, 3, \quad \sup_{t \in [0,1]_{\mathbb{T}}} x_1(t) < a,$$

$$a < \sup_{t \in [0,1]_{\mathbb{T}}} x_2(t) \text{ with } \min_{t \in [w,\nu]_{\mathbb{T}}} x_2(t) < b, \quad \min_{t \in [w,\nu]_{\mathbb{T}}} x_3(t) > b.$$

The proof is complete. \square

Example 3.1 Let $\mathbb{T} = \{\frac{1}{2^n+1} : n \in \mathbb{N}\} \cup \{0, 1\}$. Consider the following problem

$$(\phi_3(x^\Delta))^\nabla(t) + 8f(t, x(t), x^\Delta(t)) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \tag{14}$$

$$x^\Delta(0) = \frac{1}{4}x\left(\frac{1}{10}\right) + \frac{1}{6}x\left(\frac{1}{5}\right), \quad x^\Delta(1) = -\frac{1}{4}x\left(\frac{1}{3}\right) - \frac{1}{6}x\left(\frac{1}{2}\right), \tag{15}$$

where

$$f(t, u, v) = \begin{cases} t[60u^7 + (\frac{v}{10^3})^4], & u \leq 1, \quad 0 \leq v, \quad v \in \mathbb{T}; \\ t[60 + (\frac{v}{10^3})^4], & u > 1, \quad 0 \leq v, \quad v \in \mathbb{T}. \end{cases}$$

It is easy to verify that (H1) – (H3) hold. Choose $a = \frac{1}{10}$, $b = 1$, $c = 40$, $d = 43$, $w = \frac{1}{4}$, $v = \frac{1}{3}$. Then by simple calculations, we can obtain that

$$M = \frac{163}{43}, \quad \Lambda = 8, \quad A = \frac{4181}{12650}, \quad \Omega = \frac{2}{3}.$$

So the nonlinear term f satisfies

$$f(t, u, v) \leq 60 + (\frac{43}{10^3})^4 = 60.00000342 < \frac{\phi_p(d)}{2\Lambda} = 115.5625, \quad (t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, 163] \times [0, 43],$$

$$f(t, u, v) \geq 20 > \phi_3(\frac{b}{2A}) = 16.43184338, \quad (t, u, v) \in [\frac{1}{4}, \frac{1}{3}]_{\mathbb{T}} \times [1, 40] \times [0, 43],$$

$f(t, u, v) < 60 \cdot \frac{1}{10^7} + (\frac{43}{10^3})^4 = 0.000009418801 < \frac{1}{2\Lambda} \phi_p(\frac{\alpha}{M}) = 0.0000434952$, $(t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, \frac{1}{10}] \times [0, 43]$.

Therefore the conditions in Theorem 3.2 are all satisfied. So BVP (14)-(15) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{t \in [0, 1]_{\mathbb{T}}} |x_i^{\Delta}(t)| \leq 43, \quad i = 1, 2, 3, \quad \sup_{t \in [0, 1]_{\mathbb{T}}} x_1(t) < \frac{1}{10},$$

$$\frac{1}{10} < \sup_{t \in [0, 1]_{\mathbb{T}}} x_2(t) \text{ with } \min_{t \in [\frac{1}{4}, \frac{1}{3}]_{\mathbb{T}}} x_2(t) < 1, \quad \min_{t \in [\frac{1}{4}, \frac{1}{3}]_{\mathbb{T}}} x_3(t) > 1.$$

References

- [1] Anderson, D.R. Existence of solutions for a first order p -Laplacian BVP on time scales. *Nonlinear Anal.* **69** (2008) 4521–4525.
- [2] Anderson, D.R., Avery R. and Henderson J. Existence of solutions for a one dimensional p -Laplacian on time scales *J. Differ. Equations Appl.* **10** (2004) 889–896.
- [3] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhauser Boston, 2001.
- [4] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales*. Birkhauser Boston, 2003.
- [5] He, Z. Existence of two solutions of m point BVP for second order dynamic equations on time scales. *J. Math. Anal. Appl.* **296** (2004) 97–109.
- [6] Hilger, S. Analysis on measure chains a unified approach to continuous and discrete calculus. *Results Math.* **18** (1990) 18–56.
- [7] Ji, D., Bai, Z. and Ge, W. The existence of countably many positive solutions for singular multipoint BVP. *Nonlinear Anal.* **72** (2010) 955–964.
- [8] Karaca, I.Y. Positive solutions to an n th order multi-point boundary value problem on time scales *Nonlinear Dynamics and Systems Theory* **11** (2011) 285–296.
- [9] Sun, B., Yang, A. and Ge, W. Weigao Successive iteration and positive solutions for some second-order three-point p -Laplacian boundary value problems. *Math. Comput. Modelling* **50** (2009) 344–350.
- [10] Sun, T.T., Wang, L.L. and Fan, Y.H. Existence of positive solutions to a nonlocal BVP with p -Laplacian on time scales. *Adv. Difference Equ.* **809497** (2010) 15 p.
- [11] Yaslan, I. Multi-point boundary value problems on time scales. *Nonlinear Dynamics and Systems Theory* **10** (2010) 305–316.
- [12] Zhang, M., Sun, S. and Han, Z. Existence of positive solutions for multipoint BVP with p -Laplacian on time scales. *Adv. Difference Equ.* **312058** (2009) 18 p.
- [13] Zhao, X. and Ge, W. Unbounded positive solutions for m point time scale BVP on infinite intervals. *J. Appl. Math. Comput.* **33** (2010) 103–123.



Huang-Hilbert Transform Based Wavelet Adaptive Tracking Control for a Class of Uncertain Nonlinear Systems Subject to Actuator Saturation

M. Sharma^{1*} and A. Verma²

¹ Sanghvi Institute of Management and Science, Indore, India

² Institute of Engineering and Technology, Indore, India

Received: October 16, 2012; Revised: July 9, 2013

Abstract: In this paper a novel Huang-Hilbert Transform (HHT) based adaptive tracking control strategy is proposed for a class of uncertain systems subjected to actuator saturation. HHT is used in this work for the online feature extraction of the uncertainties in the systems which are approximated by Wavelet Neural Networks (WNNs). Adaptation laws are developed iteratively using the Intrinsic Modal Functions (IMF) for the online tuning of wavelets parameters. The uniformly ultimate boundedness of the closed-loop tracking error is verified even in the presence of WNN approximation errors and bounded unknown disturbances, using the Lyapunov approach and with novel weight updating rules. Finally some simulations are performed to verify the effectiveness and performance of the theoretical development.

Keywords: *Hilbert-Huang transform; empirical mode decomposition; intrinsic mode function; wavelet neural networks; adaptive control; Lyapunov functional.*

1 Introduction

In many practical systems, the system model always contains some uncertain elements; these uncertainties may be due to additive unknown internal or external noise, environmental influence, nonlinearities such as hysteresis or friction, poor plant knowledge, reduced-order models, and uncertain or slowly varying parameters. The analytical study of adaptive nonlinear control systems involving online approximation structures has evolved considerably during the last decade [1–3]. The design of online approximation based controllers can be broken up into two stages: first, the unknown nonlinearity is represented by some online approximators. Hence, the designer needs to choose a specific

* Corresponding author: <mailto:er.mann24@gmail.com>

adaptive network configuration, including the general structure of the online approximator, the number of layers (in case of multi layer neural networks), the number of adjustable weights, etc. In the second stage, the designer needs to develop an appropriate feedback control law for updating the adjustable weights.

Characteristics of practical actuators are in general nonlinear, usually described by the nonlinearities such as saturation, hysteresis, backlash etc. Nonlinear behavior of the actuator causes the detuning of plant as well as controller parameters which may lead to the poor performance or even may cause the destabilization of the system. Out of these nonlinearities saturation is the frequently encountered nonlinearity and is addressed by several researchers [4, 5].

In recent years, learning-based control methodology using Neural networks (NNs) has become an alternative to adaptive control since NNs are considered as general tools for modeling nonlinear systems. Work on adaptive NN control using the universal NN approximation property is now pursued by several groups of researchers [6, 7]. By using neural network (NN) as an approximation tool, the assumptions on linear parameterized nonlinearities in adaptive controller designing aspects have greatly been relaxed. It also broadens the class of the uncertain nonlinear systems which can be effectively dealt by adaptive controllers. However there are some difficulties associated with NN based controller. The basis functions are generally not orthogonal or redundant; i.e., the network representation is not unique and is probably not the most efficient one and the convergence of neural networks may not be guaranteed. Also the training procedure for NN may be trapped in some local minima depending on the initial settings. Wavelet neural networks are the modified form of the NN having the properties of space and frequency localization properties leading to a superior learning capabilities and fast convergence. Thus WNN based control systems can achieve better control performance than NN based control systems [6–9].

Recently, a new signal analysis approach, Hilbert-Huang transform (HHT), is proposed by Huang et. al. [10, 11] which is a combination of empirical mode decomposition (EMD) and Hilbert spectral analysis (HSA). By EMD, a signal is decomposed into a series of mono-component modes defined as intrinsic mode functions (IMFs), and Hilbert transform can thus be applied to each IMF to obtain the instantaneous frequency and the instantaneous magnitude. Unlike Fourier series representation in which base functions are always sinusoidal functions, HHT adopts different IMFs to describe various signals, resulting in adaptive base functions. Also HHT is valid for nonlinear and nonstationary signals. Because of the distinct characteristics of HHT, it has attracted considerable research interest in exploring its potential as a frequency identification tool.

A straightforward method could be that, after application of HHT to a signal, comparisons are made between Fourier spectra of the obtained IMFs and that of the original signal to find out the relationships between IMFs and vibration modes. Then by computing the amplitude weighted average frequencies based on the Hilbert spectra, modal frequencies can be identified. Besides, Yang et al. [12] proposed a method in which, before they are analyzed by HHT, the signals are processed by some pre-selected bandpass filters, the thresholds of which are determined by referring to the Fourier spectra of the signals. Efficacious as they are, these two HHT-based frequency identification methods however have to rely on some a priori information about the natural frequencies to be identified, whether by comparing Fourier spectra of original signals and those of the IMFs or by selecting the thresholds of the bandpass filters. From a practical point of view, it is difficult to obtain some a priori information about the frequencies of random signals.

So the theoretical Eigen analysis techniques are not appropriate to provide a sufficiently accurate estimation of natural frequencies. That is, frequency information is usually unavailable before identification procedures are carried out. Koh et al. [13] and Chhoa et al. [14] introduced a criterion that the IMF component with the highest energy compared to other IMF components most probably represents the fundamental frequency of the system. The criterion was applied to experimental signals collected from the real time systems and successfully identified the IMFs related to the fundamental frequencies. Due to noise contamination, however, the fundamental frequency of a system may relegate from one IMF to the next IMF during the time range of the signal [15], and the identification of the relationships between IMFs and multiple physical vibration modes might be more involved as a modal frequency may be contained along specific segments of the whole time duration of one or more IMFs.

The major limitation of HHT and EMD is that the signal under analysis must be known so that its maxima and minima can be calculated. But in this work, the nonlinear function present in the dynamics of the system is uncertain in nature. To overcome this problem we have proposed a technique to estimate the uncertain function by WNN first and then through iterative EMD algorithm, the uncertain function is approximated very accurately. Multiple WNNs are cascaded to solve this problem. Every layer has different number of nodes and different tuning laws derived by gradient descent rule. The output of each WNN is used for the derivation of the adaptive tuning laws of the next cascaded WNN. This process is repeated until the residue becomes zero, which means the approximation is best possible. This novel technique of using HHT and EMD to approximate the features of an uncertainties present in the nonlinear systems has never been cited in the literature to the best of the knowledge of authors and hence reflects the contribution of this work.

This paper deals with the designing of HHT based wavelet adaptive tracking controller for a class of uncertain nonlinear systems. WNN are used for approximating the system uncertainty as well as to optimize the performance of the control strategy. HHT algorithm generates the features of these uncertainties to be fed to the consecutive WNN.

The paper is organized as follows: Section 2 deals with the system preliminaries, system description is given in Section 3. WNN based controller designing aspects are discussed in Section 4. Section 5 describes the proposed HHT based wavelet adaptive controller design. The stability analysis of the proposed control scheme is given in Section 6. Effectiveness of the proposed strategy is illustrated through an example in Section 7 while Section 8 concludes the paper.

2 System Preliminaries

2.0.1 Actuator Saturation

The output of an actuator $u(t)$ with input $v(t)$ subjected to the condition of saturation is defined as

$$u = \begin{cases} u_{\max}, & v \geq u_{\max}, \\ v, & u_{\min} < v < u_{\max}, \\ u_{\min}, & v \leq u_{\min}, \end{cases} \quad (1)$$

where u_{\max} and u_{\min} are upper and lower saturation limits as shown in Figure 1.

For symmetric actuator saturation $u_{\min} = -u_{\max}$ part of the control effort which can

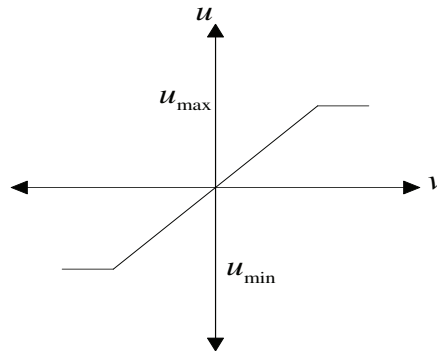


Figure 1: Saturation function.

not be implemented under this condition is defined as

$$\Delta u = \begin{cases} u_{\max} - v, & v \geq u_{\max}, \\ 0, & u_{\min} < v < u_{\max}, \\ u_{\min} - v, & v \leq u_{\min}, \end{cases} \quad (2)$$

where Δu describes the effect of actuator saturation and can be effectively approximated by using a wavelet neural network.

2.0.2 Wavelet neural network

Wavelet network is a type of building block for function approximation. The building block is obtained by translating and dilating the mother wavelet function. In contrast to conventional wavelets, a biased wavelet has a nonzero mean and can better reproduce signal components that are in the low-frequency region on the time-frequency plane since the nonzero mean enlarges low-frequency gain. Output of a biased n dimensional wavelet network with m nodes is

$$f = \alpha^T \varphi(x, w, c) + \beta^T \phi(x, w, c), \quad (3)$$

where $x = [x_1, x_2, \dots, x_n]^T \in R^n$ is the input vector, $\varphi = [\varphi_1, \varphi_2, \dots, \varphi_m]^T \in \mathfrak{R}^m$ and $\phi = [\phi_1, \phi_2, \dots, \phi_m]^T \in \mathfrak{R}^m$ are wavelet and bias functions respectively; $w = [w_1, w_2, \dots, w_m]^T \in R^{m \times n}$ and $c = [c_1, c_2, \dots, c_m]^T \in R^{m \times n}$ are dilation and translation parameters respectively ; $\alpha = [\alpha_1, \dots, \alpha_m]^T \in R^m$ and $\beta = [\beta_1, \dots, \beta_m]^T \in R^m$ are weights of wavelet and bias function respectively.

Let f^* be the optimal function approximation using an ideal wavelet approximator then

$$f = f^* + \Delta = \alpha^{*T} \varphi^* + \beta^{*T} \phi^* + \Delta, \quad (4)$$

where $\varphi^* = \varphi(x, w^*, c^*)$ and $\phi^* = \phi(x, w^*, c^*)$, $\alpha^*, \beta^*, w^*, c^*$ are the optimal parameter vectors of α, β, w, c respectively and Δ denotes the approximation error and is assumed to be bounded by $|\Delta| \leq \Delta^*$, in which Δ^* is a positive constant.

Optimal parameter vectors needed for best approximation of the function are difficult to determine so define an estimate function as

$$\hat{f} = \hat{\alpha}^T \hat{\varphi} + \hat{\beta}^T \hat{\phi}, \quad (5)$$

where $\hat{\varphi} = \varphi(x, \hat{w}, \hat{c})$, $\hat{\phi} = \phi(x, \hat{w}, \hat{c})$ and $\hat{\alpha}, \hat{\beta}, \hat{w}, \hat{c}$ are the estimates of $\alpha^*, \beta^*, w^*, c^*$ respectively. Define the estimation error as

$$\tilde{f} = f - \hat{f} = f^* - \hat{f} + \Delta = \alpha^T \tilde{\varphi} + \hat{\alpha}^T \tilde{\varphi} + \tilde{\alpha}^T \hat{\varphi} + \beta^T \tilde{\phi} + \hat{\beta}^T \tilde{\phi} + \tilde{\beta}^T \hat{\phi} + \Delta, \quad (6)$$

where $\tilde{\alpha} = \alpha^* - \hat{\alpha}$, $\tilde{\beta} = \beta^* - \hat{\beta}$, $\tilde{\varphi} = \varphi^* - \hat{\varphi}$, $\tilde{\phi} = \phi^* - \hat{\phi}$.

By properly selecting the number of nodes, the estimation error \tilde{f} can be made arbitrarily small on the compact set so that the bound $\|\tilde{f}\| = \tilde{f}_m$ holds for all $x \in \mathfrak{R}$.

Use Taylor expansion linearization technique to transform the nonlinear function into a partially linear form as a step towards the derivation of online tuning laws for the wavelet parameters to achieve the favorable estimation of system dynamics

$$\tilde{\varphi} = A_1^T \tilde{w} + B_1^T \tilde{c} + h_1 \tilde{\phi} = A_2^T \tilde{w} + B_2^T \tilde{c} + h_2, \quad (7)$$

where $\tilde{w} = w^* - \hat{w}$, $\tilde{c} = c^* - \hat{c}$ and h_1, h_2 are the vectors of higher order terms and

$$A_1 = \left[\frac{d\varphi_1}{dw}, \frac{d\varphi_2}{dw}, \dots, \frac{d\varphi_m}{dw} \right] \Big|_{w=\hat{w}}, \quad A_2 = \left[\frac{d\phi_1}{dw}, \frac{d\phi_2}{dw}, \dots, \frac{d\phi_m}{dw} \right] \Big|_{w=\hat{w}},$$

$$B_1 = \left[\frac{d\varphi_1}{dc}, \frac{d\varphi_2}{dc}, \dots, \frac{d\varphi_m}{dc} \right] \Big|_{c=\hat{c}}, \quad B_2 = \left[\frac{d\phi_1}{dc}, \frac{d\phi_2}{dc}, \dots, \frac{d\phi_m}{dc} \right] \Big|_{c=\hat{c}},$$

with

$$\frac{d\hat{\varphi}_i}{dw} = \left[0 \dots 0 \frac{d\hat{\varphi}_i}{dw_{1i}}, \frac{d\hat{\varphi}_i}{dw_{2i}}, \dots, \frac{d\hat{\varphi}_i}{dw_{ni}}, 0 \dots 0 \right]^T,$$

$$\frac{d\hat{\varphi}_i}{dc} = \left[0 \dots 0 \frac{d\hat{\varphi}_i}{dc_{1i}}, \frac{d\hat{\varphi}_i}{dc_{2i}}, \dots, \frac{d\hat{\varphi}_i}{dc_{ni}}, 0 \dots 0 \right]^T,$$

$$\frac{d\hat{\phi}_i}{dw} = \left[0 \dots 0, \frac{d\hat{\phi}_i}{dw_{1i}}, \frac{d\hat{\phi}_i}{dw_{2i}}, \dots, \frac{d\hat{\phi}_i}{dw_{ni}}, 0 \dots 0 \right]^T,$$

$$\frac{d\hat{\phi}_i}{dc} = \left[0 \dots 0, \frac{d\hat{\phi}_i}{dc_{1i}}, \frac{d\hat{\phi}_i}{dc_{2i}}, \dots, \frac{d\hat{\phi}_i}{dc_{ni}}, 0 \dots 0 \right]^T.$$

Substituting (7) into (6)

$$\tilde{f} = \tilde{\alpha}^T (\hat{\varphi} - A_1^T \tilde{w} - B_1^T \tilde{c}) + \tilde{w}^T (A_1 \hat{\alpha} + A_2 \hat{\beta}) + \tilde{c}^T (B_1 \hat{\alpha} + B_2 \hat{\beta}) + \tilde{\beta}^T (\hat{\phi} - A_2^T \tilde{w} - B_2^T \tilde{c}) + \varepsilon, \quad (8)$$

where the uncertain term is given by the following expression

$$\varepsilon = \alpha^{*T} h_1 + \tilde{\alpha}^T A_1^T w^* + \tilde{\alpha}^T B_1^T c^* + \beta^{*T} h_2 + \tilde{\beta}^T A_2^T w^* + \tilde{\beta}^T B_2^T c^*.$$

2.0.3 Hilbert-Huang Transform

This section briefly summarizes the principles and procedures of HHT. HHT is an adaptive data analysis method designed for analyzing non-stationary signals. In HHT, the signal is decomposed into a finite small number of components, called Intrinsic Mode Functions (IMF). This process of decomposition is called Empirical Mode Decomposition (EMD). Presented by Huang et al. [8], HHT essentially consists of two steps:

empirical mode decomposition and Hilbert spectral analysis. By EMD, a complicated signal is decomposed into a series of simple oscillatory modes, designated as intrinsic mode functions, and a residue. Hilbert spectral analysis is then invoked for each IMF to obtain the instantaneous frequencies and the instantaneous magnitudes, which comprise the Hilbert-Huang spectrum of the signal.

i. Empirical Mode Decomposition (EMD) The EMD decomposes the signal in terms of IMFs, each of which is a mono-component function. Given an arbitrary signal $x(t)$ following the EMD method, sifting processes are used to extract the IMFs. In a typical single sifting process, the local maxima are first identified and connected by cubic spline functions, resulting in an upper envelope $u_1^{(1)}(t)$ of the signal. A lower envelope $l_1^{(1)}(t)$ is similarly obtained based on local minima. Then a function $m_1^{(1)}(t)$ is defined as the mean of the upper and lower envelopes. Finally, subtracting the mean function $m_1^{(1)}(t)$ from signal $x(t)$, the first iterate $h_1^{(1)}(t)$, or the first proto-IMF is obtained. The above procedures are iterated until the proto-IMF $h_1^{(k+1)}(t)$ converges to the first IMF q_1 if the following conditions are satisfied:

- For $h_1^{(k+1)}(t)$, the number of extrema and the zeros differ at most by 1.
- The difference between the mean $m_1^{(k)}(t)$ and zero is within the pre-selected tolerance.

The above sifting process is shown in (9)

$$m_1^{(k+1)} = \frac{u_1^{(k+1)} + l_1^{(k+1)}}{2} h_1^{(k+1)} = h_1^k - m_1^{(k+1)}, \tag{9}$$

where $k = 0, 1, 2, \dots$ and $h_1^0 = x$. One kind of iteration stopping criterion is that the value of standard deviation SD is less than a preselected value, where SD is defined as

$$SD_k = \sum_i \frac{(h^{(k+1)}(t_i) - h^{(k)}(t_i))^2}{(h^{(k)}(t_i))^2} \tag{10}$$

or

$$SD_k = \frac{\sum_i (h^{(k+1)}(t_i) - h^{(k)}(t_i))^2}{\sum_i (h^{(k)}(t_i))^2}. \tag{11}$$

The shifting process is stopped, when SD_k becomes smaller than a pre-determined value. Once the shifting process is stopped, the first IMF q_1 can be obtained, which contains the finest scale or the shortest period component of the signal. After separating q_1 from the original signal $x(t)$, the residue of the signal is obtained

$$x(t) - q_1 = r_1. \tag{12}$$

A new sifting process is applied to r_1 , which leads to the second IMF q_2 and the residue r_2 :

$$r_1 - q_2 = r_2 \tag{13}$$

Similarly, for n^{th} IMF,

$$r_{n-1} - q_n = r_n. \tag{14}$$

The sifting processes are iterated until r_n becomes a constant, a monotonic function, or a function with only one extremum. Therefore, by EMD, the original signal $x(t)$ is denoted as

$$x(t) = \sum_{i=1}^n q_i + r_n. \quad (15)$$

Thus the decomposition of a signal in n-empirical modes is achieved. The components of the EMD are physically meaningful, as the characteristic scales are defined by the physical data. The instantaneous frequency can be computed by finding the Hilbert Transform of the IMF components.

ii. Feature Extraction using Hilbert-Huang Transform.

The features of the disturbance signals are extracted by finding the energy of the IMFs which are derived from each of the disturbance waveforms. Let q_1, q_2, q_3 be the first three IMF components and E_1, E_2 and E_3 be their corresponding energies. Energy of the IMF is calculated using the following equations

$$E_1 = \|q_1\|^2, \quad (16)$$

$$E_2 = \|q_2\|^2, \quad (17)$$

$$E_3 = \|q_3\|^2. \quad (18)$$

3 System Description

Consider a nonlinear system of the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ &\vdots \\ \dot{x}_n &= f(x) + gu, \\ y &= x_1, \end{aligned} \quad (19)$$

where $x = [x_1, x_2, \dots, x_n]^T$, u, y are state variable, control input and output respectively. $f(x)$ is a smooth unknown, nonlinear function of state variables.

Rewriting the system (19) as

$$\begin{aligned} \dot{x} &= Ax + B(f(x) + u(t)), \\ y &= Cx, \end{aligned} \quad (20)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0 \ \dots \ 0].$$

Using the actuator saturation defined in Section 2 system (20) can be transformed to

$$\begin{aligned} \dot{x} &= Ax + B(\delta(x) + (v + \Delta u)), \\ y &= Cx, \end{aligned} \quad (21)$$

where $\delta(x, \bar{y}_d) = f(x) + \Delta u$. Let $\bar{y}_d = [y_d, \dot{y}_d, \dots, \overset{n-1}{y}_d]^T$ be the vector of desired tracking trajectory. The objective is to formulate a state feedback control law to achieve the desired tracking performance simultaneously nullifying the effect of actuator saturation. The control law is formulated using the transformed system (21). The following assumptions are taken for the systems under consideration.

Assumption 3.1 Desired trajectory $y_d(t)$ is assumed to be smooth, continuous C^n and available for measurement.

4 Basic Controller Design Using Filtered Tracking Error

Define the state tracking error vector $e(t)$ as

$$e(t) = x(t) - \bar{y}_d(t). \tag{22}$$

The filter tracking error is defined as

$$r = Ke, \tag{23}$$

where $K = [k_1, k_2, \dots, k_{n-1}, 1]$ is an appropriately chosen coefficient vector such that $e \rightarrow 0$ exponentially as $\Re \rightarrow 0$.

Applying the feedback linearization method, the control laws for every iteration level are defined in the subsequent section.

5 Proposed Adaptive WNN Controller Design

A novel adaptive control strategy is proposed in this section which uses WNN to approximate the nonlinear uncertainties $\delta(x)$ present in the systems through HHT algorithm. A separate WNN network with different number of nodes and different adaptation laws is implemented for every iteration level of HHT algorithm. The tuning laws for the WNN at various iterations are derived as follows.

The cost function derived for the tuning of WNN parameters using (23) is given by

$$S = \frac{1}{2} \dot{r}^T \dot{r}. \tag{24}$$

Using the gradient descent algorithm, the online tuning laws for the WNN parameters are

$$\begin{aligned} \dot{\alpha} &= -\eta \frac{\partial S}{\partial \alpha} = -\eta \dot{r} \frac{\partial \dot{r}}{\partial \alpha}, \\ \dot{w} &= -\eta \frac{\partial S}{\partial w} = -\eta \dot{r} \frac{\partial \dot{r}}{\partial w}, \\ \dot{c} &= -\eta \frac{\partial S}{\partial c} = -\eta \dot{r} \frac{\partial \dot{r}}{\partial c}. \end{aligned} \tag{25}$$

i. First iteration.

Assuming q_1 be the WNN approximation for the first EMD, the control law can be derived as

$$u = \left(\overset{n}{y}_d - \frac{K_e e}{k_n} - r - q_1 \right), \tag{26}$$

where $K_e = [0, k_1, k_2, \dots, k_{n-1}]$.

From (23), we get

$$\dot{r} = K_e e + k_n (\delta + u + y_d^n). \quad (27)$$

And the online tuning laws for the first WNN are given by

$$\begin{aligned} \dot{\alpha}_1 &= \eta_1 k_n \dot{r} \frac{\partial q_1}{\partial \alpha_1}, \\ \dot{w}_1 &= \eta_1 k_n \dot{r} \frac{\partial q_1}{\partial w_1}, \\ \dot{c}_1 &= \eta_1 k_n \dot{r} \frac{\partial q_1}{\partial c_1}. \end{aligned} \quad (28)$$

ii. Second iteration.

Assuming q_2 be the WNN approximation for the second EMD, the control law can be derived as

$$u = (y_d^n - \frac{K_e e}{k_n} - r - q_1 - q_2). \quad (29)$$

Also the corresponding online tuning laws for WNN are derived as

$$\begin{aligned} \dot{\alpha}_2 &= \eta_2 k_n \dot{r} \frac{\partial q_2}{\partial \alpha_2}, \\ \dot{w}_2 &= \eta_2 k_n \dot{r} \frac{\partial q_2}{\partial w_2}, \\ \dot{c}_2 &= \eta_2 k_n \dot{r} \frac{\partial q_2}{\partial c_2}. \end{aligned} \quad (30)$$

iii. n^{th} iteration.

Similarly assuming q_n be the WNN approximation for the n^{th} EMD, the final control law can be derived as

$$u = (y_d^n - \frac{K_e e}{k_n} - r - (\sum_{i=1}^n q_i)). \quad (31)$$

Also the corresponding online tuning laws for WNN are derived as

$$\begin{aligned} \dot{\alpha}_n &= \eta_n k_n \dot{r} \frac{\partial q_n}{\partial \alpha_n}, \\ \dot{w}_n &= \eta_n k_n \dot{r} \frac{\partial q_n}{\partial w_n}, \\ \dot{c}_n &= \eta_n k_n \dot{r} \frac{\partial q_n}{\partial c_n}. \end{aligned} \quad (32)$$

Stability of the system (21) with the proposed control strategy will be analyzed in the next section.

5.0.4 Stability Analysis

Consider a Lyapunov functional of the form [16]

$$V = \frac{1}{2} r^2. \quad (33)$$

Differentiate it along the trajectories of the system,

$$\dot{V} = r(K_e e + K(\delta(x) + u(t) - v_r - y_d^n).$$

By the substitution of control law $u(t)$ in the above equation, we get

$$\dot{V} = r(-Kr + \tilde{\delta}(x) - v_r),$$

where $\tilde{\delta}(x)$ is the error between the actual value and the approximated value of

$$\dot{V} \leq -Kr^2 + |r| \left| \tilde{\delta}(x) \right| - rv_r.$$

Substitute the robust control term $v_r = -\frac{(\rho^2+1)r}{2\rho^2}$ in the above equation,

$$\dot{V} \leq -s_1r^2 + s_2(|r| \left| \tilde{\delta}(x) \right|)^2,$$

where $s_1 = (K + \frac{K}{2})$ and $s_2 = \frac{K\rho^2}{2}$. The system is stable as long as

$$s_1r^2 \geq s_2(|r| \left| \tilde{\delta}(x) \right|)^2. \tag{34}$$

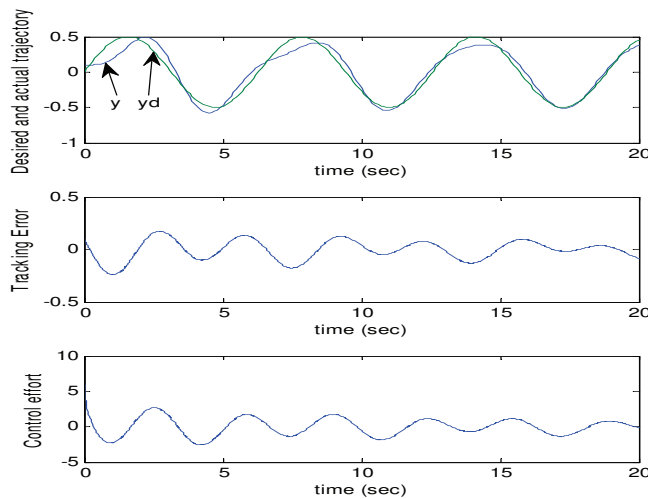


Figure 2: Desired trajectory, actual trajectory, tracking error and control effort after first iteration level.

5.0.5 Simulation results

Simulation is performed to verify the effectiveness of proposed HHT-WNN based control strategy. Consider a system of the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= 0.01x_1 \sin x_2 + u, \\ y &= x_1. \end{aligned} \tag{35}$$

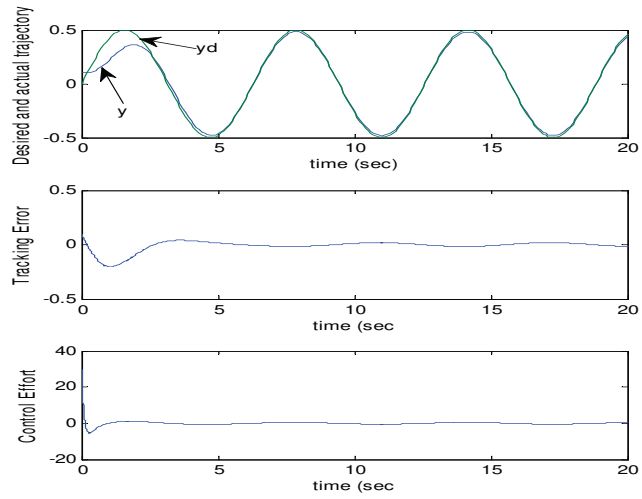


Figure 3: Desired trajectory, actual trajectory, tracking error and control effort after second iteration level.

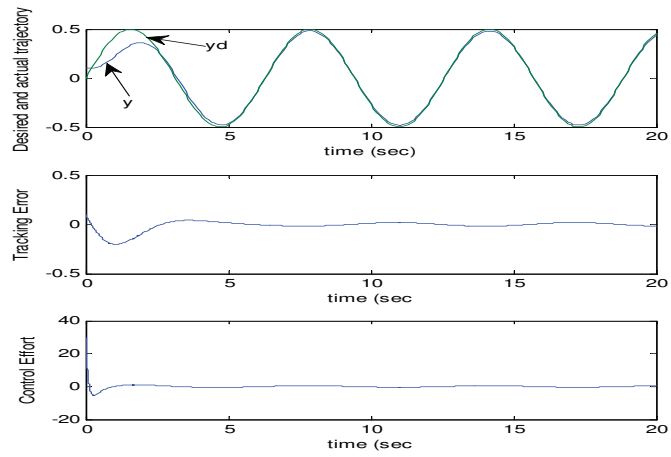


Figure 4: Desired trajectory, actual trajectory, tracking error and control effort after third iteration level.

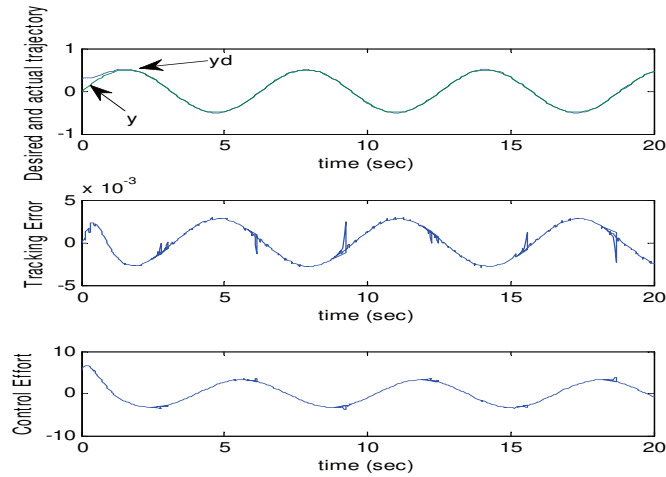


Figure 5: Desired trajectory, actual trajectory, tracking error and control effort after fourth iteration level.

The system belongs to the class of uncertain nonlinear systems defined by (21) with $n = 4$. The proposed controller strategy is applied to this system with an objective to solve the tracking problem of system. Four iteration levels are used for the simulation. The desired trajectory is taken as $y_d = 0.5 \sin t$. Initial conditions are taken as $x(0) = [0.3, 0, 0, 0]^T$. Attenuation levels for robust controller are taken as 0.01. Controller gain vector is taken as (31). Wavelet networks with Mexican Hat wavelet as the mother wavelet is used for approximating the unknown system dynamics. Wavelet parameters for these wavelet networks are tuned online using the proposed adaptation laws. Initial conditions for all the wavelet parameters are set to zero. Simulation results are shown in Figures 2–5. As observed from the figures, system response tracks the desired trajectory rapidly in consecutive iterations and after the fourth iteration the trajectory is perfectly tracked. This reflects the efficiency of the proposed control strategy.

6 Conclusion

A novel HHT based Wavelet adaptive tracking control strategy is proposed for a class of systems with unknown system dynamics and actuator saturation. Adaptive wavelet networks are used for approximating the unknown system dynamics. HHT algorithm is used for the better online feature extraction of uncertainties present in the dynamics of the system. Adaptation laws are developed for online tuning of the wavelet parameters. The stability of the overall system is guaranteed by using the Lyapunov functional. The theoretical analysis is validated by the simulation results.

References

- [1] Chang, K. M. Reference adaptive control for uncertain systems with sector-like bounded nonlinear inputs. In: *Proc. of IEEE International conference on Control and Automation (ICCA05)* (2005) 611–616.
- [2] Stefanovic, M. and Safonov, M. G. Safe Adaptive Switching Control: Stability and Convergence. *IEEE Transactions on Automatic Control* **53**(9) (2008) 2012–2021.
- [3] Chen, M., Ge, S. S. and How, B. Robust adaptive neural network control for a class of uncertain MIMO nonlinear systems with input nonlinearities. *IEEE Transaction on neural networks* **21**(5) (2010) 796–812.
- [4] He, P. and Jagannathan, S. Reinforcement learning neural-network-based controller for nonlinear discrete-time systems with input constraints. *IEEE Transactions on Systems, Man and Cybernetics – Part B: Cybernetics* **37**(2) (2007) 425–436.
- [5] Zhou, J., Joo, M. and Zhou, Y. Adaptive neural network control of uncertain nonlinear system in presence of input saturation. In: *Proceedings of the ICARCV* (2006) 895–899.
- [6] Zhang, Q. and Benveniste, A. Wavelet networks. *IEEE Transactions on Neural Networks* **3**(6) (1992) 889–898.
- [7] Zhang, J., Walter, G. G., Miao, Y. and Lee, W. Wavelet neural networks for function learning. *IEEE Transactions on Signal Processing* **43**(6) (1995) 1485–1497.
- [8] Jaddu, H. and Hiraishi, K. A Wavelet Approach for Solving linear quadratic optimal control problems. In: *SICE-ICASE International Joint Conference* (2006) 694–698.
- [9] Minu, K. K., Lineesh, M. C. and John, C. J. Wavelet Neural Networks for Nonlinear Time Series Analysis. *Applied Mathematical Sciences* **4** (1992) 2485–2495.
- [10] Huang, N. E., Shen, Z., Long, S. R., Wu, M. C., Shih, H. H., Zheng, Q., Yen, N.C., Tung, C. C. and Liu, H. H. The Empirical Mode Decomposition Method and the Hilbert Spectrum for Non-linear and Non-stationary Time Series Analysis. In: *Proceedings of Royal Society, London* (1998) 903–915.
- [11] Huang, N. E., Wu, M. C., Long, S. R., Shen, S. P., Qu, W. and Gloersen, P. A confidence limit for the empirical mode decomposition and the Hilbert spectral analysis. In: *Proceedings of Royal Society, London* (2003) 18–21.
- [12] Yang, J. N., Lei, Y., Lin, S. and Huang, N. Hilbert-Huang based approach for structural damage detection. *Journal of Engineering Mechanics, ASCE* (2004) 85-95.
- [13] Koh, S. J. A., Maalej, M. and Quek, S. T. Damage quantification of flexurally loaded RC slab using frequency response data. *Journal of Structural Health Monitoring* (2004) 293-311.
- [14] Chhoa, C. Y., Maalej, M. and Quek, S. T. Damage detection in concrete beam using Hilbert-Huang transform. In: *Proc. of 17th KKCNN symposium on civil engineering* (2004) 85-90.
- [15] Flandrin, P., Gonalvs, P. and Rilling, G. EMD equivalent filter banks, from interpretation to applications in Hibert-Huang Transform: Introduction and Application *World Scientific, Singapore* **8**(5) (2005) 67–87.
- [16] Sharma, M. and Verma, A. Wavelet Neural Network Based Adaptive Tracking Control for a Class of Uncertain Nonlinear Systems Using Reinforcement Learning. *Nonlinear Dynamics and System Theory* **12**(3) (2012) 272–279.



Numerical Research of Periodic Solutions for a Class of Noncoercive Hamiltonian Systems

M. Timoumi

Dpt of Mathematics-Faculty of Sciences-5000 Monastir-Tunisia

Received: July 30, 2012; Revised: July 11, 2013

Abstract: In this paper, we are interested in the existence of periodic solutions and approximative solutions to the Hamiltonian system $\dot{x} = JH'(t, x)$ when H is non-coercive of the type $H(t, r, p) = G(p - Ar) + h(t) \cdot (r, p)$. For the proof we use the Dual Action Principle and Critical Point Theory.

Keywords: *Hamiltonian systems; periodic solutions; non-coercive; dual action principle; discrete dual action principle; critical point theory; numerical research.*

Mathematics Subject Classification (2010): 34K28, 34K07, 34C25, 35A15.

1 Introduction

Let $G : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function such that $G' : \mathbb{R}^n \rightarrow G'(\mathbb{R}^n)$ be an homeomorphism. Let A be a matrix of order n and $h : \mathbb{R} \rightarrow \mathbb{R}^n$ be a continuous T -periodic ($T > 0$) function with zero mean value. Consider the non-coercive Hamiltonian

$$H(t, r, p) = G(p - Ar) + h(t) \cdot (r, p).$$

Here $x \cdot y$ is the usual inner product of $x, y \in \mathbb{R}^{2n}$. We are interested in the boundary value problem

$$\dot{x} = JH'(t, x) \tag{H}$$

with

$$x(0) = x(T). \tag{C}$$

The goal of this work is to prove the existence of solutions to the problem (H)(C) and to approximate these solutions.

Corresponding author: mailto:m_timoumi@yahoo.com

For T and h given, we define the dual action integral $\varphi : E \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\varphi(v) = \frac{1}{2} \int_0^T Jv \cdot \pi v dt + \int_0^T H_0^*(v - h) dt,$$

where $H_0(r, p) = G(p - Ar)$, H_0^* is the Fenchel's transformation of H_0 and E is the closed subspace of $L^2(0, T; \mathbb{R}^{2n})$ defined by:

$$E = \left\{ v \in L^2(0, T; \mathbb{R}^{2n}) / \int_0^T v(t) dt = 0 \right\}.$$

Under some suitable assumptions on G , we will prove, in Section 2, that the problem $(\mathcal{H})(\mathcal{C})$ has at least one solution and is equivalently to the following problem:

$$\text{find } v \in E \text{ such that } 0 \in \bar{\partial}\varphi(v), \quad (\mathcal{R})$$

where we introduce the notation $\bar{\partial}$ to distinguish the sub differential in E and in $L^2(0, T; \mathbb{R}^{2n})$. In Section 3, we will introduce some problems $(\mathcal{H}_N)(\mathcal{C}_N)$, (\mathcal{R}_N) , (\mathcal{P}_N) defined in a finite dimensional space and related together by a discret dual action principle. In Section 4, we will study the existence of solutions to problem (\mathcal{P}_N) , which give solutions to problem (\mathcal{R}_N) . In Section 5, we will study some convergence problems related to this discretisation. We want to know if the differences system (\mathcal{H}_N) is near to system (\mathcal{H}) for example for a very large integer N . In Section 6, we will give an example of application and in Section 7, we will conclude this work.

2 Existence of Periodic Solutions

Let $G : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable convex function, A be a symmetric matrix of order n and $h : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ be a continuous T -periodic function with zero mean value on $[0, T]$. Consider the assumptions:

Assumption 2.1

$$\lim_{|x| \rightarrow \infty} G(x) = +\infty. \quad (G_1)$$

Assumption 2.2 There exist $\alpha \in]0, \frac{\pi}{T(1+|A|^2)}[$ and $\beta \geq 0$ such that

$$\forall x \in \mathbb{R}^n, G(x) \leq \frac{\alpha}{2} |x|^2 + \beta, \quad (G_2),$$

where $|A|$ is the usual norm of A . Consider the non-coercive sub-quadratic Hamiltonian:

$$H(t, r, p) = G(p - Ar) + h \cdot (r, p).$$

We are interested in the existence of solutions for the boundary value problem

$$\dot{x} = JH'(t, x) \quad (\mathcal{H})$$

with

$$x(0) = x(T), \quad (\mathcal{C})$$

where H' is the derivative of H with respect to the second variable x and J is the standard $(2n \times 2n)$ symplectic matrix:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n .

Example 2.1 Consider a relativistic particle with a very small charge e and rest mass m_0 , subject to a uniform constant magnetic field B and a uniform electric field $E(t)$. The energy expressed as a function of (t, r, p) , i.e. a Hamiltonian, is given by

$$H(t, r, p) = c \left[m_0^2 c^2 + \left| p - \frac{e}{2} B(t) \wedge r \right|^2 \right]^{\frac{1}{2}} - E(t) \cdot r,$$

where c is the velocity of light, p the usual mechanical momentum of particle and r is its position. The particle motion is described by the associated Hamiltonian system (\mathcal{H}) .

The function

$$H_0(r, p) = G(p - Ar)$$

is convex and its Fenchel's transformation H_0^* is given for all $(s, q) \in \mathbb{R}^n \times \mathbb{R}^n$ by (see [7])

$$H_0^*(s, q) = \begin{cases} G^*(q), & \text{if } s + A^*q = 0, \\ +\infty, & \text{if } s + A^*q \neq 0. \end{cases}$$

Consider the functional

$$\psi(y) = \int_0^T \left[\frac{1}{2} J \dot{y} \cdot y + H_0^*(\dot{y} - h) \right] dt \tag{2.1}$$

defined over the space

$$\{y \in H^1(0, T; \mathbb{R}^{2n}) / y(0) = y(T)\}.$$

Note that, from the periodicity condition:

$$\forall \xi \in \mathbb{R}^{2n}, \psi(y + \xi) = \psi(y),$$

the true variable in (2.1) is \dot{y} and we can choose for y any primitive we like. The only condition on \dot{y} is:

$$\dot{y} \in L^2(0, T; \mathbb{R}^{2n}) \text{ and } \int_0^T \dot{y} dt = 0.$$

In other terms, we have

$$\psi(y) = \varphi(\dot{y}),$$

where φ is the functional

$$\varphi(v) = \int_0^T \left[\frac{1}{2} J v \cdot \pi v + H_0^*(v - h) \right] dt$$

defined on the space

$$E = \{v \in L^2(0, T; \mathbb{R}^{2n}) / \int_0^T v(t) dt = 0\},$$

where πv is the primitive of v with zero mean value:

$$\frac{d}{dt}(\pi v) = v \text{ and } \int_0^T (\pi v)(t) dt = 0$$

or also

$$(\pi v)(t) = \int_0^t v(s) ds - \frac{1}{T} \int_0^T \int_0^r v(s) ds dr.$$

This allows to introduce the following problem:

$$\text{find } v \in E \text{ such that } 0 \in \bar{\partial}\varphi(v). \quad (\mathcal{R})$$

The problems (\mathcal{R}) and $(\mathcal{H})(\mathcal{C})$ are related by a dual action principle.

Theorem 2.1 (*Dual action principle*). *Assume that the function G satisfies (G_1) , (G_2) and let $v \in E$. Then the two following assertions are equivalent:*

- (i) *v is a solution of problem (\mathcal{R}) ,*
- (ii) *there exists a constant ξ in \mathbb{R}^{2n} such that the function $x(t) = J\pi v(t) + \xi$ is a solution of problem $(\mathcal{H})(\mathcal{C})$.*

Proof. To prove this theorem, we need the following lemma.
Consider the functional

$$g(v) = \int_T^0 H_0^*(v - h)dt, \quad v \in E,$$

we have

Lemma 2.1 *The sub-differential of $g|_E$ in a point $v \in E$ where g has finite value, is given by*

$$\bar{\partial}g(v) = \{u \in L^2(0, T; \mathbb{R}^{2n}) / \exists \xi \in \mathbb{R}^{2n}, u(t) + \xi \in \partial H_0^*(v(t) - h(t)) \text{ a.e.}\}.$$

Proof. If $u \in L^2(0, T; \mathbb{R}^{2n})$, $v \in E$ and $\xi \in \mathbb{R}^{2n}$ are such that $u(t) + \xi \in \partial H_0^*(v(t) - h(t))$ a.e., we prove easily that u is in $\bar{\partial}g(v)$. Reversely, it is clear that

$$\bar{\partial}g(v) = \partial(g + \delta_E)(v),$$

where

$$\delta_E(v) = \begin{cases} 0, & \text{if } v \in E, \\ +\infty, & \text{elsewhere.} \end{cases}$$

Since it is clear that $\partial\delta_E(v)$ is the set of constant functions and it is well known that (see [3])

$$\partial g(v) = \{u \in L^2(0, T; \mathbb{R}^{2n}) / u(t) \in \partial H_0^*(v(t) - h(t)) \text{ a.e.}\},$$

the result will be proved if we have

$$\partial(g + \delta_E)(v) = \partial g(v) + \partial\delta_E(v).$$

Let us establish that $\partial(g + \delta_E)(v) = \partial g(v) + \partial\delta_E(v)$. It is enough to prove that $g^*\nabla\delta_E^*$ is exact (see [1]). By identifying the set of constant functions to \mathbb{R}^{2n} , we see that

$$\delta_E^* = \delta_{\mathbb{R}^{2n}}.$$

We deduce that for all u in L^2 :

$$(g^*\nabla\delta_E^*)(u) = \inf_{x \in \mathbb{R}^{2n}} \int_0^T H_0(u - h + x)dt$$

and by (G_2) , we obtain

$$0 \leq (g^*\nabla\delta_E^*)(u) \leq \int_0^T H_0(u - h)dt \leq \alpha(1 + |A|^2)\|u - h\|_{L^2}^2 + \beta T. \quad (2.2)$$

By convexity and (2.2), we conclude that $g^*\nabla\delta_E^*$ is continuous (see [3]). Now, let us write $u = (r, p)$ and $h = (h_1, h_2)$, we have $(g^*\nabla\delta_E^*)(u) = \inf_{\xi \in \mathbb{R}^n} F(\xi)$, where

$$F(\xi) = \int_0^T G(p - h_2 - A(r - h_1) + \xi)dt.$$

By properties of G , it is easy to see that F is continuous and $\lim_{|\xi| \rightarrow \infty} F(\xi) = +\infty$. Consequently F achieves its minimum on \mathbb{R}^n and then $g^*\nabla\delta_E^*$ is exact. On the other hand, g and δ_E are convex, l.s.c and proper, therefore for all v in E where g is finite, we have

$$\bar{\partial}g(v) = \partial g(v) + \mathbb{R}^{2n}.$$

The proof of Lemma 2.1 is complete.

Let $v \in E$ be such that

$$0 \in \bar{\partial}\varphi(v). \tag{2.3}$$

This is equivalent to

$$0 \in -J\pi v + \bar{\partial}g(v). \tag{2.4}$$

By Lemma 2.1, formula (2.4) is equivalent to the existence of $\xi \in \mathbb{R}^{2n}$ satisfying

$$J(\pi v)(t) + \xi \in \partial H_0^*(v(t) - h(t)) \text{ a.e.} \tag{2.5}$$

Let us put $x(t) = J\pi v(t) + \xi$. By Fenchel’s reciprocity, formula (2.5) can be rewritten as

$$v(t) - h(t) = H_0'(x(t))$$

or

$$\dot{x}(t) = JH'(t, x(t))$$

and it is clear that x is T -periodic. Then x is a solution of problem $(\mathcal{H})(\mathcal{C})$ and Theorem 2.1 is proved.

Now, we associate with (\mathcal{R}) the problem:

$$\text{find } \bar{v} \in E \text{ such that } \inf_{v \in E} \varphi(v) = \varphi(\bar{v}). \tag{\mathcal{P}}$$

The problem (\mathcal{P}) allows to give a solution of problem (\mathcal{R}) .

Theorem 2.2 *Assume assumptions (G_1) , (G_2) hold, then problem $(\mathcal{H})(\mathcal{C})$ has at least one solution.*

The proof of Theorem 2.2 follows immediately from Lemma 2.1 and the following lemma.

Lemma 2.2 *Problem (\mathcal{P}) possesses a solution: there exists a point $\bar{v} \in E$ such that*

$$\min_E \varphi = \varphi(\bar{v}).$$

Proof. By using assumption (G_2) and going through the conjugate, we verify that

$$\forall y \in \mathbb{R}^{2n}, H_0^*(y) \geq \frac{1}{2\alpha(1 + |A|^2)}|y|^2 - \beta. \tag{2.6}$$

On the other hand, by Wirtinger's inequality and using Fourier expansion, we have

$$\forall v \in E, \|\pi v\|_{L^2} \leq \frac{T}{2\pi} \|v\|_{L^2}. \quad (2.7)$$

We deduce from (2.6), (2.7) and Hölder's inequality that

$$\forall v \in E, \varphi(v) \geq \frac{1}{2} \left[\frac{1}{\alpha(1+|A|^2)} - \frac{T}{2\pi} \right] \|v\|_{L^2}^2 - \beta T. \quad (2.8)$$

Now, let (v_k) be a minimising sequence, then by (2.8), (v_k) is bounded. Since the space E is reflexive, then there exists a subsequence (v_{k_p}) weakly convergent to a $\bar{v} \in E$. It is well known that the functional g introduced above is l.s.c, so we have

$$\liminf_{p \rightarrow \infty} \int_0^T H_0^*(v_{k_p} - h) dt \geq \int_0^T H_0^*(\bar{v} - h) dt. \quad (2.9)$$

Elsewhere, the operator π is compact, so

$$\pi v_{k_p} \longrightarrow \pi \bar{v}, \text{ in } \mathbb{L}^2$$

and then

$$\lim_{p \rightarrow \infty} \int_0^T J v_{k_p} \cdot \pi v_{k_p} dt = \int_0^T J \bar{v} \cdot \pi \bar{v} dt. \quad (2.10)$$

Consequently, we deduce from (2.9) and (2.10) that

$$\min_E \varphi = \varphi(\bar{v}).$$

3 A Discrete Dual Action Principle

Giving a period $T > 0$ and a forcing h , we have defined in the previous section the space $E = L_0^2(0, T; \mathbb{R}^{2n})$ and the functional $\varphi : E \rightarrow \bar{\mathbb{R}}$. We will write a problem (\mathcal{R}_N) obtained by writing (\mathcal{R}) not in L^2 but in a finite dimensional space. This will allow us, having put a differences system (\mathcal{H}_N) and a constraint (\mathcal{C}_N) , to establish a "discrete dual action principle" connecting (\mathcal{R}_N) to $(\mathcal{H}_N)(\mathcal{C}_N)$.

Notations. For $x \in \mathbb{R}^{nN}$, we will adopt the following agreement writing:

$$\begin{cases} x = (x_1, x_2, \dots, x_n), \text{ where } x_i \in \mathbb{R}^N, \\ x = (x^1, x^2, \dots, x^N), \text{ where } x^j \in \mathbb{R}^n, \\ x_i^j \in \mathbb{R}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, N. \end{cases}$$

This allows us to define the space

$$E_N = \{v = (r, p) \in \mathbb{R}^{2nN} / \sum_{j=1}^N r^j = \sum_{j=1}^N p^j = 0\}.$$

Let us define on \mathbb{R} the sequence $(t^j)_{j=1,2,\dots,N}$ by

$$\begin{cases} t^1 = 0, \quad t^{N+1} = T, \\ t^{j+1} - t^j = \delta = \frac{T}{N}, \quad \forall j = 1, 2, \dots, N. \end{cases}$$

With any vector $x \in \mathbb{R}^{nN}$, we can associate a step function \tilde{x} from \mathbb{R} into \mathbb{R}^n , which will be, by construction, T -periodic, as follows:

$$\begin{cases} \tilde{x}(t) = x^j, \quad \forall t \in [t^j, t^{j+1}[, \quad \forall j = 1, 2, \dots, N; \\ \tilde{x}(t + kT) = \tilde{x}(t), \quad \forall t \in [0, T[, \quad \forall k \in \mathbb{Z}. \end{cases}$$

Then we can write φ applied to any element $v = (r, p)$ of \mathbb{R}^{2nN} . We will denote by $\varphi_N(v)$ its value (the index N in φ is to recall that we have calculated φ for elements of \mathbb{R}^{2nN}).

We obtain

$$\varphi_N(v) = \frac{\delta^2}{2} \sum_{j=1}^N \sum_{k=1}^j J v^j \cdot v^k + \delta \sum_{j=1}^N H_0^*(v^j - h^j).$$

The vector h^j is obtained by discretising h with respect to $(t^j)_{j=1,2,\dots,N}$, which is possible since h is T -periodic.

Definition 3.1 We recall the problem (\mathcal{R}_N) :

$$\text{find } v \in E_N \text{ such that } 0 \in \bar{\partial}\varphi_N(v). \tag{\mathcal{R}_N}$$

Definition 3.2 We will denote by $w_N = (r_N, p_N)$ the continuous piecewise linear functions, defined with respect to $(t^j)_{j=1,2,\dots,N}$. For these functions, we define the differences system

$$-J \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = H'_0\left(\frac{w_N(t^{j+1}) + w_N(t^j)}{2}\right) + h(t^j), \quad j = 1, \dots, N. \tag{\mathcal{H}_N}$$

Then we look for w_N satisfying (\mathcal{H}_N) and the constraint

$$w_N(0) = w_N(T). \tag{\mathcal{C}_N}$$

Theorem 3.1 (*Discrete dual action principle*). Assume G satisfies (G_1) , (G_2) . Then for $v \in E_N$ the following two assertions are equivalent:

- (i) v is a solution of (\mathcal{R}_N) ,
- (ii) there exists a constant ξ_N in \mathbb{R}^{2N} such that the function

$$w_N(t) = J \int_0^t \tilde{v}(\tau) d\tau + \xi_N$$

is a solution of $(\mathcal{H}_N)(\mathcal{C}_N)$, where \tilde{v} is defined with respect to v as above.

Proof. 1) The function w_N defined in (ii) is a continuous linear piecewise function as in Definition 3.2.

2) Given the definition of E_N , it is clear that w_N satisfies condition (\mathcal{C}_N) if and only if v belongs to this space, since

$$\int_0^T \tilde{v}(\tau) d\tau = \sum_{j=1}^N \int_{t^j}^{t^{j+1}} \tilde{v}(\tau) d\tau = \delta \sum_{j=1}^N v^j.$$

3) In the following, we will need the next result:

Lemma 3.1 *Let*

$$F(v) = \sum_{j=1}^N H_0^*(v^j),$$

then we have

$$\partial F(v) = \{u \in \mathbb{R}^{2nN} / u^j \in \partial H_0^*(v^j), \forall j = 1, \dots, N\},$$

where $u = (u^1, \dots, u^N)$.

Proof. We have

$$\begin{aligned} u \in \partial F(v) &\iff \forall x \in \mathbb{R}^{2nN} / F(x) \leq F(v) + (x - v) \cdot u \\ &\iff \forall x \in \mathbb{R}^{2nN}, \sum_{j=1}^N H_0^*(x^j) \leq \sum_{j=1}^N H_0^*(v^j) + \sum_{j=1}^N (x^j - v^j) \cdot u^j \\ &\implies \forall j = 1, \dots, N, \forall x^j \in \mathbb{R}^{2n}, H_0^*(v^1) + \dots + H_0^*(x^j) + \dots + H_0^*(v^N) \\ &\quad \leq \sum_{j=1}^N H_0^*(v^j) + (x^j - v^j) \cdot u^j \\ &\implies \forall j = 1, \dots, N, \forall x^j \in \mathbb{R}^{2n}, H_0^*(x^j) \leq H_0^*(v^j) + (x^j - v^j) \cdot u^j \\ &\implies \forall j = 1, \dots, N, u^j \in \partial H_0^*(v^j). \end{aligned}$$

Reversely, if $\forall j = 1, \dots, N, u^j \in \partial H_0^*(v^j)$, then

$$\begin{aligned} &\forall j, \forall x^j \in \mathbb{R}^{2n}, H_0^*(x^j) \leq H_0^*(v^j) + (x^j - v^j) \cdot u^j \\ &\implies \forall x \in \mathbb{R}^{2nN}, \sum_{j=1}^N H_0^*(x^j) \leq \sum_{j=1}^N H_0^*(v^j) + \sum_{j=1}^N (x^j - v^j) \cdot u^j \\ &\implies \forall x \in \mathbb{R}^{2nN}, F(x) \leq F(v) + (x - v) \cdot u \\ &\implies u \in \partial F(v). \end{aligned}$$

Now, consider the functional

$$\varphi_N(v) = Q_N(v) + \delta \sum_{j=1}^N H_0^*(v^j - h^j)$$

defined over the space E_N , with

$$Q_N(v) = \frac{\delta^2}{2} \sum_{j=1}^N \sum_{k=1}^j Jv^j \cdot v^k.$$

We have

$$Q_N(v) = \frac{\delta^2}{2} [Jv^2 \cdot v^1 + \dots + Jv^N \cdot v^1] + \text{terms without } v^1,$$

so

$$\frac{\partial Q_N}{\partial v^1} = \frac{\delta^2}{2} [Jv^2 + Jv^3 + \dots + Jv^N] = -\frac{\delta^2}{2} Jv^1.$$

Similarly for $2 \leq j \leq N$,

$$Q_N(v) = \frac{\delta^2}{2} [Jv^j \cdot (v^1 + \dots + v^{j-1}) + (Jv^{j+1} + \dots + Jv^N) \cdot v^j] + \text{terms without } v^j,$$

so

$$\begin{aligned} \frac{\partial Q_N}{\partial v^j} &= \frac{\delta^2}{2} [-J(v^1 + \dots + v^{j-1}) + J(v^{j+1} + \dots + v^N)] \\ &= \frac{\delta^2}{2} [-J \sum_{k=1}^{j-1} v^k - J \sum_{k=1}^j v^k] = -\frac{\delta^2}{2} J(2 \sum_{k=1}^{j-1} v^k + v^j). \end{aligned}$$

Therefore

$$\partial\varphi_N(v) = \{u \in \mathbb{R}^{2nN} / \forall j = 1, \dots, N, u^j \in -\frac{\delta^2}{2} (2 \sum_{k=1}^{j-1} v^k + v^j) + \delta\partial H_0^*(v^j - h^j)\}.$$

4) By writing

$$\partial\varphi_N(v) = \begin{cases} 0, & \text{if } v \in E_N, \\ +\infty, & \text{elsewhere,} \end{cases}$$

we have

$$\bar{\partial}\varphi_N(v) = \partial(\varphi_N + \delta_{E_N})(v),$$

where we introduce the notation $\bar{\partial}$ to distinguish the sub-differentials in E_N and in \mathbb{R}^{2nN} .

Lemma 3.2 *We have*

$$\bar{\partial}\varphi_N(v) = \partial\varphi_N(v) + \partial\delta_{E_N}(v).$$

Proof. By writing

$$g_N(v) = \sum_{j=1}^N H_0^*(v^j - h^j),$$

it is enough to prove that $\bar{\partial}g_N(v) = \partial g_N(v) + \partial\delta_{E_N}(v)$. It is clear that $\bar{\partial}g_N(v) = \partial(g_N + \delta_{E_N})(v)$. The result will be proved if we have

$$\partial(g_N + \delta_{E_N})(v) = \partial g_N(v) + \partial\delta_{E_N}(v).$$

For this, it is enough to prove that $g_N^* \nabla \delta_{E_N}^*$ is exact. We have $\delta_{E_N}^* = \delta_{E_N^\perp}$. Let us determine E_N^\perp . We have

$$\begin{aligned} u = (r, p) \in E_N^\perp &\iff \forall v \in E_N, u \cdot v = 0 \iff \forall (s, q) \in E_N, \sum_{j=1}^N (s^j \cdot r^j + q^j \cdot p^j) = 0 \\ &\implies [\forall i \neq j = 1, \dots, N, \forall s^i, s^j \in \mathbb{R}^n, s^i + s^j = 0 \implies s^i \cdot r^i + s^j \cdot r^j = 0] \\ &\implies [\forall i \neq j = 1, \dots, N, \forall s^i \in \mathbb{R}^n, s^i \cdot (r^i - r^j) = 0] \\ &\implies \forall i, j = 1, \dots, N, r^i = r^j. \end{aligned}$$

Similarly, $\forall i, j = 1, \dots, N$, $p^i = p^j$. Therefore we have

$$(r, p) \in E_N^\perp \implies r^1 = \dots = r^N, p^1 = \dots = p^N.$$

Reversely, if $(r, p) \in \mathbb{R}^{2nN}$ is such that $r^1 = \dots = r^N$ and $p^1 = \dots = p^N$, then

$$\forall (s, q) \in E_N, (s, q) \cdot (r, p) = \sum_{j=1}^N s^j \cdot r^j + \sum_{j=1}^N q^j \cdot p^j = \left(\sum_{j=1}^N s^j \right) \cdot r^1 + \left(\sum_{j=1}^N q^j \right) \cdot p^1 = 0.$$

Therefore, we have

$$E_N^\perp = \{(r, p) \in \mathbb{R}^{2nN} / r^1 = \dots = r^N, p^1 = \dots = p^N\}.$$

For u in \mathbb{R}^{2nN} , we have

$$\begin{aligned} (g_N^* \nabla \delta_{E_N}^*)(u) &= \inf_{u_1 + u_2 = u} (g_N^*(u_1) + \delta_{E_N}^*(u_2)) = \inf_{\xi \in E_N^\perp} g_N^*(u + \xi) = \inf_{\xi \in E_N^\perp} \sum_{j=1}^N H_0(u^j + \xi^j) \\ &= \inf_{(x, y) \in \mathbb{R}^{2n}} \sum_{j=1}^N H_0(u^j + (x, y)) = \inf_{(x, y) \in \mathbb{R}^{2n}} \sum_{j=1}^N G(u_2^j - Au_1^j + y - Ax) = \inf_{x \in \mathbb{R}^n} K(x), \end{aligned}$$

where $u^j = (u_1^j, u_2^j)$ and

$$K(x) = \sum_{j=1}^N G(u_2^j - Au_1^j + x).$$

Since K is continuous and goes to infinity as $|x| \rightarrow \infty$, then K achieves its minimum on \mathbb{R}^n . The proof of Lemma 3.2 is complete.

We have $\partial \delta_{E_N}(v) = E_N^\perp$ then $\bar{\partial} \varphi_N(v) = \partial \varphi_N(v) + E_N^\perp$. Consequently, we have

$$\begin{aligned} u \in \bar{\partial} \varphi_N(v) &\iff u \in \partial \varphi_N(v) + E_N^\perp \\ &\iff \exists \xi \in \mathbb{R}^{2n} / \begin{cases} u^1 \in \frac{-\delta^2}{2} Jv^1 + \xi + \delta \partial H_0^*(v^1 - h^1) \\ u^j \in \frac{-\delta^2}{2} J(2 \sum_{k=1}^{j-1} v^k + v^j) + \xi + \delta \partial H_0^*(v^j - h^j), \forall j = 2, \dots, N. \end{cases} \end{aligned}$$

5) v is a critical point of φ_N if and only if there exists a constant $\xi_N \in \mathbb{R}^{2n}$ such that

$$\begin{cases} 0 \in \frac{-\delta^2}{2} J\bar{v}_1 - \xi_N + \delta \partial H_0^*(\bar{v}^1 - h^1), \\ 0 \in \frac{-\delta^2}{2} J(2 \sum_{k=1}^{j-1} \bar{v}^k + \bar{v}^j) - \xi_N + \delta \partial H_0^*(\bar{v}^j - h^j), \forall j = 2, \dots, N. \end{cases}$$

$\iff \exists \xi_N \in \mathbb{R}^{2n}$ such that

$$\begin{cases} \xi_N + \frac{\delta}{2} J\bar{v}^1 \in \partial H_0^*(\bar{v}^1 - h^1), \\ \xi_N + \frac{\delta}{2} J(2 \sum_{k=1}^{j-1} \bar{v}^k + \bar{v}^j) \in \partial H_0^*(\bar{v}^j - h^j), \forall j = 2, \dots, N. \end{cases}$$

Let us associate with $v \in \mathbb{R}^{2nN}$, the step function \tilde{v} and the continuous piecewise linear function w_N defined by

$$w_N(t) = J \int_0^t \tilde{v}(\tau) d\tau + \xi_N.$$

In particular, we have

$$w_N(t^{j+1}) = J \int_0^{t^{j+1}} \tilde{v}(\tau) d\tau + \xi_N = J \sum_{k=1}^j \int_{T^k}^{t^{k+1}} \tilde{v}(\tau) d\tau + \xi_N = \delta J \sum_{k=1}^j v^k + \xi_N,$$

which implies

$$\begin{cases} w_N(t^{j+1}) - w_N(t^j) = \delta J \tilde{v}(t^j), \\ w_N(t^{j+1}) + w_N(t^j) = 2[\frac{\delta}{2}(2J \sum_{k=1}^{j-1} \tilde{v}(t^k) + \tilde{v}(t^j)) + \xi_N]. \end{cases}$$

Therefore we have

$$\begin{cases} w_N(t^{j+1}) - w_N(t^j) = \delta J \tilde{v}(t^j), \\ w_N(t^{j+1}) + w_N(t^j) \in 2\partial H_0^*(v^j - h^j). \end{cases}$$

This yields

$$\frac{w_N(t^{j+1}) + w_N(t^j)}{2} \in \partial H_0^*(-J \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} - h(t^j)).$$

By using Fenchel’s reciprocity formula, we obtain

$$-J \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = H_0'(\frac{w_N(t^{j+1}) + w_N(t^j)}{2}) + h(t^j).$$

4 Existence Results

To resolve the problem $(\mathcal{H}_N)(\mathcal{C}_N)$, it suffices, by using Section 3, to find a point \bar{v} of \mathbb{R}^{2nN} solution of (\mathcal{R}_N) , i.e.

$$\text{find } \bar{v} \in E_N \text{ such that } 0 \in \bar{\partial} \varphi_N(\bar{v}). \tag{\mathcal{R}_N}$$

For this, we can study the existence of a minimum to the associate problem

$$\text{find } \bar{v} \in E_N \text{ satisfying } \inf_{v \in E_N} \varphi_N(v) = \varphi_N(\bar{v}). \tag{\mathcal{P}_N}$$

Assume that G and h satisfy the assumptions of Section 2.

Remark 4.1 In Section 3, we have seen that we can associate with a point v in \mathbb{R}^{2nN} a step function \tilde{v} defined from \mathbb{R} into \mathbb{R}^{2n} by the relations:

$$\begin{cases} (i) \tilde{v}(t) = v^j, \forall t \in [t^j, t^{j+1}[, \forall j = 1, \dots, N, \\ (ii) \tilde{v}(t + kT) = \tilde{v}(t), \forall k \in \mathbb{Z}, \forall t \in [0, T[. \end{cases} \tag{4.1}$$

It is easy to see that the restriction $\tilde{v}|_{[0, T]}$ of \tilde{v} to $[0, T]$ is in $L^2(0, T; \mathbb{R}^{2n})$.

Definition 4.1 1) Denote by F_N the subset of $L^2(0, T; \mathbb{R}^{2n})$ defined by

$$F_N = \{\omega \in L^2(0, T; \mathbb{R}^{2n}) / \omega \text{ verifies (4.1)}\},$$

where

$$\begin{cases} (i) \omega \text{ is defined for all } t \in [0, T], \\ (ii) \omega(t) = \omega^j, \forall t \in [t^j, t^{j+1}[, \forall j = 1, \dots, N, \\ (iii) \omega(T) = \omega^1 = \omega(0). \end{cases}$$

Firstly, remark that F_N is a closed subspace of $L^2(0, T; \mathbb{R}^{2n})$.

2) Denote by η_N the function defined from \mathbb{R}^{2nN} into F_N

$$\eta_N(v) = \tilde{v}|_{[0, T]}, \quad v \in \mathbb{R}^{2nN}.$$

Remark that

$$\varphi_N(v) = \varphi(\eta_N(v)).$$

Lemma 4.1 *The function η_N establishes a diffeomorphism between F_N and \mathbb{R}^{2nN} , so we can identify \mathbb{R}^{2nN} with F_N .*

Proof. Since the partition $(t^j)_{j=1, \dots, N}$ is fixed, then η_N is a differentiable linear map and we can verify easily that it is invertible.

Lemma 4.2 *\mathbb{R}^{2nN} can be provided with the topology obtained by diffeomorphism from the topology induced from $L^2(0, T; \mathbb{R}^{2n})$ on F_N .*

Proof. It is a consequence from the fact that F_N is a closed subspace of $L^2(0, T; \mathbb{R}^{2n})$.

Remark 4.2 By denoting $\|\cdot\|_2$ the norm in $L^2(0, T; \mathbb{R}^{2n})$ and $|\cdot|_{2n}$ the norm in \mathbb{R}^{2n} , we have the equality

$$\|\eta_N(v)\|_2 = \left[\frac{1}{N} \sum_{j=1}^N |v^j|_{2n}^2 \right]^{\frac{1}{2}}.$$

The right quantity defines a norm in \mathbb{R}^{2nN} , we will denote it by $|\cdot|_{2, N}$. With these notations, η_N appears as an isometry from $(L^2(0, T; \mathbb{R}^{2n}), \|\cdot\|_2)$ into $(\mathbb{R}^{2nN}, |\cdot|_{2, N})$.

Theorem 4.1 *Under assumptions (G_1) , (G_2) , the problem (\mathcal{P}_N) has, for all integer N , a solution v_N .*

Proof. By identifying \mathbb{R}^{2nN} to F_N , the proof is the same as that of the general case (\mathcal{P}) . It is based on the following estimate:

$$\forall v \in \mathbb{R}^{2nN}, \varphi_N(v) \leq \frac{1}{2} \left[\frac{1}{\alpha(1 + |A|^2)} - \frac{T}{2\pi} \right] \|\eta_N(v)\|_2^2 - \beta T$$

or also

$$\forall v \in \mathbb{R}^{2nN}, \varphi_N(v) \leq \frac{1}{2} \left[\frac{1}{\alpha(1 + |A|^2)} - \frac{T}{2\pi} \right] |v|_{2, N}^2 - \beta T.$$

The previous theorem permits to assert that if assumptions (G_1) , (G_2) are satisfied, then for all integer N , we can find a minimum for φ_N on E_N which is also a solution of (\mathcal{R}_N) . Therefore, by the discrete dual action principle introduced in Section 3, the problem $(\mathcal{H}_N)(\mathcal{C}_N)$ has a solution.

Now we define a sequence $(v_l)_{l \in \mathbb{N}^*}$ by setting

$$\begin{cases} (i) & N = 2^l, \\ (ii) & v_l \text{ is a solution of } (\mathcal{P}_N). \end{cases}$$

The estimate of the previous theorem permits to state the following lemma:

Lemma 4.3 *Under assumption (G_2) , there exists a constant $M > 0$ such that*

$$\forall l \in \mathbb{N}^*, \|\eta_N(v_1)\|_2^2 = |v_1|_{2,N}^2 \leq M.$$

Proof. Note that, from the previous results, we have

$$\forall l \in \mathbb{N}^*, \varphi_N(v_l) = \varphi(\eta_N(v_l)) \leq \frac{k_l}{2} \|\eta_N(v_l)\|_2^2 - k_2$$

where $k_1 = \frac{1}{\alpha(1+|A|^2)} - \frac{T}{2\pi}$ and $k_2 = \beta T$. We have also

$$\forall l' \leq l, \varphi_N(v_l) \leq \varphi_{N'}(v_{l'}) \text{ with } N = 2^l.$$

Since

$$\varphi_N(v_{l'}) = \varphi_{N'}(v_{l'}) \text{ with } N' = 2^{l'},$$

we get

$$\forall l' \leq l, \varphi(\eta_N(v_l)) = \varphi_N(v_l) \leq \varphi_{N'}(v_{l'}).$$

Therefore, we have

$$\forall l \in \mathbb{N}^*, \frac{1}{2}k_1 \|\eta_N(v_l)\|_2^2 - k_2 \leq \varphi_N(v_l) \leq \varphi_1(v_1).$$

Since $\varphi_1(v_1)$ is a constant with respect to l , the proof of Lemma 4.3 is complete.

5 Convergence Results

Under assumptions (G_1) , (G_2) , we have proved in the previous section that there exists a sequence $(v_l)_{l \in \mathbb{N}^*}$ of solutions for the problems (\mathcal{P}_N) with $N = 2^l$. Consider the sequence $(w_l)_{l \in \mathbb{N}^*}$ of piecewise linear functions defined by

$$w_l(t) = \int_0^t \tilde{v}_l(\tau) d\tau + \xi_l$$

with $\xi_l \in \mathbb{R}^{2n}$ such that

$$\xi_l \in -\frac{\delta_l}{2} J\tilde{v}_l(0) + \partial H_0^*(\tilde{v}_l(0) - h_l(0)), \quad \delta_l = \frac{T}{2^l}.$$

Remark 5.1 Giving the definition of H_0 , we can assume that ξ_l is of the type $(0, \lambda_l)$ with $\lambda_l \in \mathbb{R}^n$. In fact, we have

$$\begin{aligned} (r, p) \in (a, b) + \partial H_0^*(s, q) &\iff (s, q) = H_0'((r, p) - (a, b)) \\ \iff (s, q) &= (-A^*G'(p - b - A(r - a)), G'(p - b - A(r - a))) \\ &= (-A^*G'(p - Ar - b + Aa), G'(p - Ar - b + Aa)) \\ &\iff (s, q) = H_0'(-a, p - Ar - b) \\ &\iff -(a, b) + (0, p - Ar) \in \partial H_0^*(s, q) \\ &\iff (0, p - Ar) \in (a, b) + \partial H_0^*(s, q). \end{aligned}$$

In the following, we will take ξ_l of the form $(0, \lambda_l)$, $\lambda_l \in \mathbb{R}^n$, and we will prove that the associated sequence (w_l) has a subsequence strongly convergent in $L^2(0, T; \mathbb{R}^{2n})$ to a solution \bar{w} of $(\mathcal{H})(\mathcal{C})$.

Lemma 5.1 [7] *The operator π from $L^2(0, T; \mathbb{R}^{2n})$ into itself, introduced in Section 2, is a Hilbert-Schmidt operator: it transforms quickly convergent sequences to strongly convergent sequences.*

Lemma 5.2 *Under assumptions (G_1) , (G_2) , there exists a subsequence (w_{l_k}) of (w_l) strongly convergent in $L^2(0, T; \mathbb{R}^{2n})$ to \bar{w} . Moreover \bar{w} is defined in 0 and T and satisfies $\bar{w}(0) = \bar{w}(T)$.*

Proof. It is easy to verify that the sequence (w_l) is included in $L^2(0, T; \mathbb{R}^{2n})$. By Lemma 4.3, the sequence (\tilde{v}_l) is bounded in $L^2(0, T; \mathbb{R}^{2n})$, then it possesses a subsequence (\tilde{v}_{l_k}) weakly convergent in $L^2(0, T; \mathbb{R}^{2n})$ to a point \bar{v} . In particular (\tilde{v}_{l_k}) being defined for all integer k and for all $t \in [0, T]$, the sequence $(\tilde{v}_{l_k}(t))$ is convergent in \mathbb{R}^{2n} to $\bar{v}(t)$ for all $t \in [0, T]$. Recall that we have defined ξ_l by

$$\xi_l \in -\frac{\delta_l}{2} J\tilde{v}_l(0) + \partial H_0^*(\tilde{v}_l(0) - h_l(0)), \delta_l = \frac{T}{2l}.$$

We have

$$\begin{aligned} \xi_l + \frac{\delta_l}{2} J\tilde{v}_l(0) &\in \partial H_0^*(\tilde{v}_l(0) - h_l(0)) \\ \iff \tilde{v}_l(0) - h_l(0) &= H'_0(\xi_l + \frac{\delta_l}{2} J\tilde{v}_l(0)) \\ &= H'_0\left((0, \lambda_l) + \frac{\delta_l}{2} J(\tilde{v}_l^1(0), \tilde{v}_l^2(0))\right) = H'_0\left(\frac{\delta_l}{2} \tilde{v}_l^2(0), \lambda_l - \frac{\delta_l}{2} \tilde{v}_l^1(0)\right) \iff \\ \tilde{v}_l(0) - h_l(0) &= \left(-A^* G'(\lambda_l - \frac{\delta_l}{2}(\tilde{v}_l^1(0) + A\tilde{v}_l^2(0))), G'(\lambda_l - \frac{\delta_l}{2}(\tilde{v}_l^1(0) + A\tilde{v}_l^2(0)))\right). \end{aligned}$$

Since G' is an homeomorphism from \mathbb{R}^n into $G'(\mathbb{R}^n)$ and since (δ_l) goes to zero in \mathbb{R} as l goes to infinity and $(\tilde{v}_{l_k}(0))$ is bounded and converges to $\bar{v}(0)$, the sequence (λ_{l_k}) converges to $\bar{\lambda}$ in \mathbb{R}^n with

$$\bar{\lambda} = (G')^{-1}(\bar{v}^2(0) - h^2(0)).$$

By previous Remarks and Lemma 5.1, we deduce that the sequence (w_{l_k}) converges strongly to \bar{w} in $L^2(0, T; \mathbb{R}^{2n})$. Moreover

$$\bar{w}(t) = J \int_0^t \bar{v}(\tau) d\tau + \bar{\xi} \text{ with } \bar{\xi} = (0, \bar{\lambda})$$

and then, in particular, we have $\bar{w}(0) = \bar{w}(T)$.

Lemma 5.3 *The sequence (y_{l_k}) defined by*

$$y_{l_k} = \tilde{v}_{l_k} - Jh_{l_k} \in L^2(0, T; \mathbb{R}^{2n})$$

converges strongly in $L^2(0, T; \mathbb{R}^{2n})$ to $\bar{y} = \bar{v} - Jh$.

Proof. It is an immediately consequence of previous lemma's proof.

Lemma 5.4 *With the point w_l of $L^2(0, T; \mathbb{R}^{2n})$, we associate the element ω_l of the same space defined by*

$$\begin{cases} \omega_l(t^j) = \frac{1}{2}(w_l(t^{j+1}) + w_l(t^j)), \forall j = 1, \dots, N, \\ \omega_l(0) = \omega_l(T), \\ \omega_l(t) = \omega_l(t^j), \forall t \in [t^j, t^{j+1}], \forall j = 1, \dots, N. \end{cases}$$

Under assumptions (G_1) , (G_2) , the subsequence (ω_{l_k}) of (ω_l) converges strongly in $L^2(0, T; \mathbb{R}^{2n})$ to \bar{w} .

Proof. It suffices to prove

$$\lim_{k \rightarrow \infty} \|\omega_{l_k} - w_{l_k}\|_2 = 0. \tag{5.1}$$

Then we will use the inequality

$$\|\omega_{l_k} - \bar{w}\| \leq \|\omega_{l_k} - w_{l_k}\|_2 + \|w_{l_k} - \bar{w}\|_2$$

and we conclude by using Lemma 5.2.

We have

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \int_0^T |\omega_{l_k}(t) - w_{l_k}(t)|^2 dt,$$

where $|\cdot|$ denotes $|\cdot|_{2n}$. On the other hand, we have

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \sum_{j=1}^{N_k} \int_{t^{j+1}}^{t^j} |\omega_{l_k} - w_{l_k}|^2 dt, \tag{5.2}$$

where $N_k = 2^{l_k}$. In $[t^j, t^{j+1}[$, $w_{l_k}(t)$ can be written

$$\forall t \in [t^j, t^{j+1}[, w_{l_k}(t) = w_{l_k}(t^j) + (t - t^j)\tilde{v}_{l_k}(t^j).$$

Then equality (5.2) becomes

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \sum_{j=1}^{N_k} \int_{t^{j+1}}^{t^j} |\omega_{l_k}(t^j) - w_{l_k}(t^j) - (t - t^j)\tilde{v}_{l_k}(t^j)|^2 dt.$$

This yields

$$\begin{aligned} \|\omega_{l_k} - w_{l_k}\|_2^2 &= \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |\omega_{l_k}(t^j) - w_{l_k}(t^j)|^2 dt \\ &+ 2 \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |t - t^j| |\tilde{v}_{l_k}(t^j)| |\omega_{l_k}(t^j) - w_{l_k}(t^j)| dt + \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |t - t^j|^2 |\tilde{v}_{l_k}(t^j)|^2 dt \\ &\leq \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |\omega_{l_k}(t^j) - w_{l_k}(t^j)|^2 dt + 2 \frac{T}{N_k} \sum_{j=1}^{N_k} \left[\int_{t^j}^{t^{j+1}} |\tilde{v}_{l_k}(t^j)|^2 dt \right]^{\frac{1}{2}} \left[\int_{t^j}^{t^{j+1}} |\omega_{l_k}(t^j) - w_{l_k}(t^j)|^2 dt \right]^{\frac{1}{2}} \\ &\quad + \left(\frac{T}{N_k}\right)^2 \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |\tilde{v}_{l_k}(t^j)|^2 dt. \end{aligned} \tag{5.3}$$

The expression $\omega_{l_k}(t^j) - w_{l_k}(t^j)$ can be written

$$\omega_{l_k}(t^j) - w_{l_k}(t^j) = \frac{w_{l_k}(t^j) + w_{l_k}(t^{j+1})}{2} - w_{l_k}(t^j) = \frac{w_{l_k}(t^{j+1}) - w_{l_k}(t^j)}{2}.$$

But we know that

$$\frac{w_{l_k}(t^{j+1}) - w_{l_k}(t^j)}{2} = \frac{1}{2} \delta_{l_k} \tilde{v}_{l_k}(t^j).$$

Therefore the inequality (5.3) becomes

$$\|\omega_{l_k} - w_{l_k}\|_2^2 \leq \frac{9}{4}(\delta_{l_k})^2 \int_0^T |\tilde{v}_{l_k}(t)|^2 dt. \tag{5.4}$$

Since $\delta_{l_k} = \frac{T}{N_k} = T2^{-l_k}$ goes to zero as k goes to infinity and \tilde{v}_{l_k} is bounded in $L^2(0, T; \mathbb{R}^{2n})$, the relation (5.1) is proved.

If assumption (G_2) is satisfied, Lemma 4.3 permits to write

$$\delta_{l_k} \int_0^T |v_{l_k}(t)|^2 dt = \frac{T}{2^{l_k}} [2^{l_k} \sum_{j=1}^{N_k} |v_{l_k}^j|^2] \leq \frac{T}{2^{l_k}} M.$$

Therefore we can state the following convergence result:

Theorem 5.1 *Under assumptions (G_1) , (G_2) and Lemma 5.2 notations, the subsequence (ω_{l_k}) converges strongly in $L^2(0, T; \mathbb{R}^{2n})$ to a solution \bar{w} of $(\mathcal{H})(\mathcal{C})$.*

Proof. To prove this theorem, we will need the following theorem:

Theorem 5.2 [4] *Let A be a monotone maximal operator from its domain $D(A) \subset L^2(0, T; \mathbb{R}^{2n})$ into $L^2(0, T; \mathbb{R}^{2n})$. Let (x_l) and (y_l) be two sequences satisfying*

- (i) $x_l \in \text{Dom}A, \forall l \geq l_0,$
- (ii) $y_l = A(x_l), \forall l \geq l_0,$
- (iii) (x_l) converges weakly to \bar{x} in $L^2(0, T; \mathbb{R}^{2n}),$
- (iv) (y_l) converges weakly to \bar{y} in $L^2(0, T; \mathbb{R}^{2n}),$
- (v) $\limsup_{l \rightarrow \infty} (x_l y_l) \leq \bar{x} \bar{y}.$

Then

- (j) $\bar{x} \in \text{Dom}A,$
- (jj) $\bar{y} = A(\bar{x}).$

By Section 3, we know that for all integer l , the following system is verified:

$$\begin{cases} (i) \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = J[H'_0 \frac{w_N(t^{j+1}) + w_N(t^j)}{2}] + h^j, \forall j = 1, \dots, 2^l \\ \text{and} \\ (ii) \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = v_l^j, \forall j = 1, \dots, 2^l. \end{cases}$$

By using the notations of Lemma 5.3, equation (i) can be rewritten

$$\forall t \in [0, T], -Jy_l(t) = H'_0(\omega(t)).$$

Since the operator " $-J$ " from \mathbb{R}^{2n} into \mathbb{R}^{2n} is an isometry, we deduce from the previous Lemmas that the sequences $(-Jy_{l_k})$ and (ω_{l_k}) as the operator H'_0 verify assumptions of the previous Theorem, therefore we can assert that

$$\forall t \in [0, T], -J\bar{y}(t) = H'_0(\bar{w}(t))$$

or also

$$\forall t \in [0, T], \bar{v}(t) = J(H'_0(\bar{w}(t)) + h(t)),$$

where

$$\bar{w}(t) = \int_0^t \bar{v}(\tau) d\tau + (0, \bar{\lambda}).$$

Therefore \bar{w} is a solution of $(\mathcal{H})(\mathcal{C})$.

6 Conclusion

In this paper, we first prove the existence of solutions of a problem of non-coercive convex Hamiltonian systems $(\mathcal{H})(\mathcal{C})$ through the theory of critical point theory and the dual action principle. Then we associate with $(\mathcal{H})(\mathcal{C})$ a sequence of problems $(\mathcal{H}_N)(\mathcal{C}_N)$, (R_N) , (P_N) defined in a finite dimensional space and related together by a discrete dual action principle. We prove that problems $(\mathcal{H}_N)(\mathcal{C}_N)$ possess a sequence of solutions which converges to a solution of problem $(\mathcal{H})(\mathcal{C})$.

References

- [1] Aubin, J.P. and Ekeland, I. *Applied Nonlinear Analysis*. Wiley, 1986.
- [2] Benhassine, A. and Timoumi, M. Homoclinics orbits for a class of second order Hamiltonian systems. *Nonlinear Dynamics and Systems Theory* **12** (2012) 145–157.
- [3] Ekeland, I. and Temam, R. *Analyse convexe et problèmes variationnels*. Dunod-Gauthier-Villars, 1974. [French].
- [4] Gaussens, E. *Numerical research of periodic solutions for an Hamiltonian system*. In: *Proceeding du Colloque de Rome*. BIRHÖSER, 1981.
- [5] Timoumi, M. Oscillations d’une particule relativiste. *J. Méca. Théorique et Appliquée*. **7** (1988) 531–544. [French]
- [6] Timoumi, T. Periodic and subharmonic solutions for a class of noncoercive superquadratic Hamiltonian systems. *Nonlinear Dynamics and Systems Theory* **11** (2011) 319–336.
- [7] Timoumi, M. Solutions périodiques de systèmes Hamiltoniens convexes non coercitifs. *Bulletin de la classe des sciences* **12** (1989) 463–481. [French]



\mathcal{F} Mixing and \mathcal{F} Scattering

Xinhua Yan*

*Department of Mathematics and Physics, Beijing Institute of Petro-Chemical Technology,
Beijing 102617, P.R. China*

Received: March 27, 2012; Revised: July 21, 2013

Abstract: In this paper, we study the complexity of group actions from the viewpoint of Furstenberg families, we characterize the \mathcal{F} uniform rigidity and \mathcal{F} equicontinuity using topological sequence complexity function, and we establish the connection between \mathcal{F} mixing and \mathcal{F} scattering.

Keywords: \mathcal{F} uniform rigidity; \mathcal{F} mixing; \mathcal{F} scattering.

Mathematics Subject Classification (2010): 37B05, 54H20.

1 Introduction

Blanchard, Host and Maass used open covers to define a complexity function for a continuous map on a compact metric space, and discussed the equicontinuity and scattering properties. Subsequently, Yang discussed the relations of \mathcal{F} mixing and \mathcal{F} scattering of a continuous map (see [1–3]). We study the complexity of group actions from the viewpoint of Furstenberg families. The results are as follows: we characterize the \mathcal{F} uniform rigidity and \mathcal{F} equicontinuity using topological sequence complexity function, and we establish the connection between \mathcal{F} mixing and \mathcal{F} scattering.

Suppose (X, T) is a semi-dynamical system, where X is a compact metric space, T is a topological semigroup and contains the unit element.

- Suppose X is a topological space, T is a topological semigroup, if a map

$$\pi : X \times T \rightarrow X$$

satisfies

$$\pi(\pi(x, t), s) = \pi(x, ts), \forall x \in X, \forall t, s \in T,$$

*Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (PHR201008363).

Corresponding author: <mailto:yxh823@163.com>

then we call π a right action of T on X . If the right action π is continuous, then (X, T, π) is called a semi-dynamical system (abbreviation: (X, T)). Often we write $\pi(x, t) = xt$.

• We denote by \mathcal{P} the collection of all subsets of T . Subset \mathcal{F} of \mathcal{P} is called a family, if \mathcal{P} has hereditary upward, i.e., if $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$, then $F_2 \in \mathcal{F}$. The family \mathcal{F} is a proper family when it is a proper subset of \mathcal{P} , neither empty nor all of \mathcal{P} .

For a family \mathcal{F} , we define the dual family:

$$\begin{aligned} k\mathcal{F} &= \{F|F \cap F_1 \neq \emptyset, \text{ for all } F_1 \in \mathcal{F}\} \\ &= \{F|T \setminus F \notin \mathcal{F}\}. \end{aligned}$$

• For $t \in T$ define $g^t : T \rightarrow T$ by $g^t(s) = ts, \forall s \in T$, g^t is called a translation map. If for any $t \in T$ and any $F \in \mathcal{F}$, we have $(g^t)^{-1}(F) \in \mathcal{F}$, then a family \mathcal{F} is called translation invariant. Write $\tau\mathcal{F} = \{F|(g^{t_1})^{-1}(F) \cap \dots \cap (g^{t_k})^{-1}(F) \in \mathcal{F}, \text{ for any finite subset } \{t_1, t_2, \dots, t_k\} \text{ of } T\}$. Let \mathcal{B} be a family of infinite subset of T , if $k\mathcal{B} \cdot \mathcal{F} = \{A \cap F|A \in k\mathcal{B}, F \in \mathcal{F}\} \subset \mathcal{F}$, then a proper family \mathcal{F} is called full.

• Assume that \mathcal{F} is a family, $x \in X$. Write $\omega_{\mathcal{F}}(x) = \bigcap_{F \in k\mathcal{F}} \overline{xF}$, then $\omega_{\mathcal{F}}(x)$ is called a \mathcal{F} limit set of x ; $y \in \omega_{\mathcal{F}}(x)$, i.e., for any neighborhood U of $y, D(x, U) = \{t|xt \in U\} \in \mathcal{F}$, then y is called a \mathcal{F} limit point of x . Recall that the continuous action π on X induces a continuous action π_* of T on $C^u(X, X)$ by $(\pi_t)_*(h) = \pi_t \circ h$. We call (X, T) \mathcal{F} uniformly rigid, if $id \in \omega_{\mathcal{F}}(id)$, i.e., for any $\varepsilon > 0, \{t|d(\pi_t, id) < \varepsilon\} \in \mathcal{F}$ (where $d(\pi_t, id) = \sup\{d(\pi_t(x), x)|x \in X\}$).

• Let $C = \{U_1, \dots, U_k\}$ be an open cover of X . If S is a infinite subset of T , denote the set of all finite subsets of S by $F(S)$. For $A \in F(S)$, denote $C_0^A = \bigvee_{t \in A} (\pi_t)^{-1}C$. Let $r_S(T, C, A)$ denote the number of sets in a finite subcover of C_0^A with smallest cardinality. We get a map $r_S(T, C, \cdot) : F(S) \rightarrow Z^+, A \mapsto r_S(T, C, A)$. $r_S(T, C, \cdot)$ is said to be the topological complexity function of the cover C along S . Put $E = \{1, \dots, k\}$. One defines a map $\omega : T \rightarrow E, t \mapsto \omega(t)$. If $x \in \bigcap_{t \in S} \pi_t^{-1}U_{\omega(t)}$, then ω is called a C_S -name of x . Denote $J^*(\omega) = \bigcap_{t \in T} \pi_t^{-1}U_{\omega(t)}, J_S^*(\omega) = \bigcap_{t \in S} \pi_t^{-1}U_{\omega(t)}$. If $\bigcup_{i \in I} J_S^*(\omega_i) = X$, then we say that the set of C_S -names ω_i covers X . Let $M(T, E)$ be the set of maps from T to E and $M(S, E)$ be the set of maps from S to E .

• For any open set U, V of X , if $D(U, V) = \{t \in T|U \cap \pi_t^{-1}V \neq \emptyset\} \in \mathcal{F}$, then (X, T) is called \mathcal{F} transitive. If $(X \times X, T)$ is \mathcal{F} transitive, then (X, T) is called \mathcal{F} mixing; If for any $S \in \mathcal{F}$, and any finite cover C of X by non-dense open sets, we have $r_S(T, C, \cdot)$ is unbounded, then (X, T) is called \mathcal{F} scattering.

2 \mathcal{F} Uniformly Rigid, \mathcal{F} Mixing and \mathcal{F} Scattering

Lemma 2.1 *Suppose T is countable, a finite cover $C = (U_1, \dots, U_k)$ has complexity bounded by m if and only if there exist $\omega_1, \dots, \omega_m \in M(T, E)$ such that $\bigcup_{i=1}^m J^*(\omega_i) = X$.*

Proof. Since T is countable, suppose $T = \{t_1, t_2, \dots, t_n, \dots\}$. Take $A_n = \{t_1, \dots, t_n\}$, then $r_T(T, C, A_n) \leq m$.

Denote by $H(n)$ the set of m -tuples (v_1, \dots, v_m) of elements of $M(T, E)$ such that $(J_{A_n}^*(v_1), \dots, J_{A_n}^*(v_m))$ covers X , the set $H(n)$ is non-empty and a closed subset of $M(T, E)^m$. If $(J_{A_n}^*(v_1), \dots, J_{A_n}^*(v_m))$ covers X , then $(J_{A_{n-1}}^*(v_1), \dots, J_{A_{n-1}}^*(v_m))$ covers X too, hence $H(n) \subseteq H(n-1)$, the intersection $H = \bigcap_{n=0}^{\infty} H(n)$ is non-empty, so there is $\omega = (\omega_1, \dots, \omega_m) \in H$. Obviously $\bigcup_{i=1}^m J^*(\omega_i) = \lim_{n \rightarrow \infty} \bigcup_{i=1}^m J_{A_n}^*(\omega_i) = X$.

Theorem 2.1 *Suppose T is a topological group satisfying the second axiom of countability. Then (X, T) is \mathcal{F} uniformly rigid if and only if there is a set $S \in \mathcal{F}$ containing a unit element, for any finite cover C of X , $r_S(T, C, \cdot)$ is bounded and C_S -names ω_i covering X are k instant.*

Proof. \Rightarrow . Since (X, T) is \mathcal{F} uniformly rigid, $id \in \omega_{\mathcal{F}}(id)$. Let ε be a Lebesgue number of C , then $S = \{t \in T | \sup_{x \in X} d(\pi_t(x), x) < \frac{\varepsilon}{2}\} \in \mathcal{F}$. Let $x_1, \dots, x_m \in X$ be such that the open balls $\{B(x_i, \frac{\varepsilon}{2}) | i = 1, 2, \dots, m\}$ cover X . For any $t \in S$, we have $B(x_i, \frac{\varepsilon}{2})t \subset B(x_i, \varepsilon)$, and for any $1 \leq i \leq m$, there is $U_{l(i)} \in C$ such that $B(x_i, \varepsilon) \subset U_{l(i)}$. Then for any finite set A of S , we have $B(x_i, \frac{\varepsilon}{2}) \subset \bigcap_{t \in A} \pi_t^{-1} U_{l(i)}$, suppose the number of $U_{l(i)}$ is k . Since $\{\bigcap_{t \in A} \pi_t^{-1}(U_{l(i)}) | i = 1, \dots, m\}$ is a finite cover of $\bigvee_{t \in A} \pi_t^{-1}(C)$, then $r_S(T, C, A) \leq k$. By Lemma 2.1, for a countable dense set D of S , we have $\bigcup_{i=1}^k \bigcap_{t \in D} \pi_t^{-1}(U_{l(i)}) = X$. By the denseness of D , $\bigcup_{i=1}^k \bigcap_{t \in S} \pi_t^{-1}(U_{l(i)}) = X$.

\Leftarrow . If (X, T) is not \mathcal{F} uniformly rigid, then there is $\varepsilon > 0$, such that $\{t | d(\pi_t, id) < \varepsilon\} \notin \mathcal{F}$, then $S' = \{t | d(\pi_t, id) \geq \varepsilon\} \in k\mathcal{F}$. Let $C = \{U_1, \dots, U_m\}$ be a finite cover by open balls with radius $\frac{\varepsilon}{4}$. If there is $S \in \mathcal{F}$, for any finite set A of S , we have $r_S(T, C, A) \leq k$ and C_S -names ω_i covering X are instant. Then by Lemma 2.1, there exists a closed cover $\{X_1, \dots, X_k\}$ of X , where $X_i = \bigcap_{t \in S} \pi_t^{-1}(\overline{U_{i'}})$. Because of $S \cap S' \neq \emptyset$, take $t \in S \cap S'$, then $d(\pi_t, id) \geq \varepsilon$, that is there is $x_t \in X$ such that $d(x_t t, x_t) \geq \varepsilon$. Let $x_t \in X_i$, then $x_t \in \overline{U_{i'}}$ and for any $s \in S$ we have $x_t s \in \overline{U_{i'}}$ that is $d(x_t s, x_t) \leq \frac{\varepsilon}{2}$, which contradicts the assumption $d(x_t t, x_t) \geq \varepsilon$.

Theorem 2.2 *(X, T) is \mathcal{F} equicontinuous if and only if there is $F \in \mathcal{F}$, and for any finite open cover C , $r_F(T, C, \cdot)$ is bounded.*

Proof. The proof is similar to the proof of Proposition 2.2 in [4].

Remark 2.1 In the case $T = Z_+$, \mathcal{F} is the family of infinite subsets. If X is represented as the unit circle in C , then $\hat{\theta}^1$ is given by $\hat{\theta}^1(Z) := \alpha z (z \in C, |z| = 1)$ with $\alpha := \exp(2\pi i \theta)$, let θ be irrational, then $(X, Z_+, \hat{\theta}^1)$ is \mathcal{F} equicontinuous.

In the following we discuss the existence of \mathcal{F} equicontinuous point, and the connection between \mathcal{F} mixing and \mathcal{F} scattering.

Lemma 2.2 *Assume \mathcal{F} is a translation invariant proper family, (X, T) is not $k\mathcal{F}$ mixing if and only if there is a non-empty open set U, V of X and $S \in \mathcal{F}$, such that for any $t \in S$ either $\pi_t^{-1}U \cap U = \emptyset$ or $\pi_t^{-1}V \cap U = \emptyset$.*

Proof. The proof is similar to the proof of Lemma 3.1 of [2].

Theorem 2.3 *Assume that \mathcal{F} is a translation invariant proper family, if there is $F \in \mathcal{F}$, such that there is a F equicontinuous point x , then (X, T) is not $k\mathcal{F}$ mixing.*

Proof. Take $y \in X$ and $y \neq x$, let $\varepsilon < d(y, x)$. Since x is a F equicontinuous point, there is δ , $0 < \delta < \frac{\varepsilon}{4}$, if $d(x, z) < \delta$, we have $d(xt, zt) < \frac{\varepsilon}{4} (\forall t \in F)$. Let $U = B(y, \delta)$, $V = B(x, \delta)$, if there is $t \in F$ such that $\pi_t^{-1}U \cap V \neq \emptyset$, then $\pi_t V \cap U \neq \emptyset$, thus $\pi_t V \cap V = \emptyset$, that is $\pi_t^{-1}V \cap V = \emptyset$. By Lemma 2.2, (X, T) is not $k\mathcal{F}$ mixing.

Lemma 2.3 *If the family \mathcal{F} is full, then (X, T) is \mathcal{F} mixing if and only if (X, T) is $\tau\mathcal{F}$ transitive.*

Proof. The proof can be found in [4].

Theorem 2.4 Assume that T is commutative, \mathcal{F} is full, and (X, T) is \mathcal{F} mixing, then (X, T) is $k\tau\mathcal{F}$ scattering.

Proof. For any non-trivial closed cover $\alpha = (W_1, \dots, W_n)$ of X . Let U_1, U_2, V_1, V_2 be non-empty open sets of X , since (X, T) is \mathcal{F} mixing,

$$F = D(U_1, U_2) \cap D(V_1, V_2) \in \mathcal{F}.$$

Take $t \in F$, let $U = U_1 \cap \pi_t^{-1}U_2, V = V_1 \cap \pi_t^{-1}V_2$. By Lemma 2.3, (X, T) is $\tau\mathcal{F}$ transitive, then $D(U, V) \in \tau\mathcal{F}$. Because of $D(U, V) \subset D(U_1, U_2) \cap D(V_1, V_2)$, and $\tau\mathcal{F}$ is a family, then $D(U_1, U_2) \cap D(V_1, V_2) \in \tau\mathcal{F}$.

Now we take U, V such that U, V do not simultaneously belong to any element of α .

Let $S_1 = D(U, U) \cap D(U, V) \in \tau\mathcal{F}$, for any $S \in k\tau\mathcal{F}$ there are $t_1 \in S_1 \cap S$ and $x_1, x'_1 \in U$ such that

$$x_1 t_1 \in U, x'_1 t_1 \in V.$$

So one takes $A_1 = \{t_1\}$, then $r_S(T, \alpha, A_1) \geq 2$. By the continuity of π , there exists a neighbourhood $U_1 \subset U$ of x'_1 such that $U_1 t_1 \subset V$. Let $S_2 = D(U_1, U) \cap D(U_1, V) \in \tau\mathcal{F}$, then there are $t_2 \in S_2 \cap S$ and $x_2, x'_2 \in U_1$ such that

$$x_2 t_1 \in V, x'_2 t_1 \in V, x_2 t_2 \in V, x'_2 t_2 \in U.$$

Obviously $t_1 \neq t_2$. so we take $A_2 = \{t_1, t_2\}$ then $r_S(T, \alpha, A_2) \geq 3$. By the continuity of π , there exists a neighbourhood $U_2 \subset U_1$ of x'_2 such that $U_2 t_1 \subset U_1$. Let $S_3 = D(U_2, U) \cap D(U_2, V) \in \tau\mathcal{F}$, then there are $t_3 \in S_3 \cap S$ and $x_3, x'_3 \in U_2$ such that

$$x_3 t_1 \in V, x'_3 t_1 \in V, x_3 t_2 \in V, x'_3 t_2 \in U, x_3 t_3 \in U, x'_3 t_3 \in V.$$

so one takes $A_3 = \{t_1, t_2\}$ then $r_S(T, \alpha, A_3) \geq 4$.

Using similar arguments repeatedly, we can get an infinite sequence

$$\{x_1, x_2, \dots, x_n, \dots\} \text{ and } \{t_1, \dots, t_n, \dots\} \text{ satisfy}$$

$$x_n \in U, i = 1, 2, \dots,$$

$$x_1 t_1 \in U, x_i t_1 \in V, i = 2, 3, \dots,$$

$$x_2 t_2 \in V, x_i t_2 \in U, i = 3, 4, \dots,$$

$$x_3 t_3 \in U, x_i t_3 \in V, i = 4, 5, \dots,$$

...

For any $N \geq 1$, take $A_N = \{t_1, t_2, \dots, t_N\}$ then $r_S(T, \alpha, A_N) \geq N + 1$.

Example 2.1 In the case $T = Z_+$, \mathcal{F} is the family of infinite subsets. Let S be a finite set with at least two elements, say $S = \{0, \dots, s - 1\}$ with $s \in N, s \geq 2$. Consider S as a finite discrete topological space and put $\Omega := S^{Z_+}$. Endowed with the product topology. Define a mapping $\sigma : \Omega \rightarrow \Omega, (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, \dots)$. Clearly (Ω, Z_+, σ) is \mathcal{F} mixing, then (Ω, Z_+, σ) is $k\tau\mathcal{F}$ scattering.

3 Concluding Remarks

In this paper, we study the complexity of group actions. We characterize the \mathcal{F} uniform rigidity and \mathcal{F} equicontinuity using topological sequence complexity function, and we show that \mathcal{F} mixing implies $k\tau\mathcal{F}$ scattering.

References

- [1] Blanchard, F., Host, B. and Maass, A. Topological complexity. *Ergod. Th. and Dynam. Sys.* **20** (2000) 641–662.
- [2] Yang, R. Topological sequence complexity and mixing. *Chinese Annals of Mathematics* **25A** (6) (2004) 809–816.
- [3] Canovas, J. S. Topological sequence entropy and chaos of star maps. *Nonlinear Dynamics and Systems Theory* **5**(1) (2005) 1–8.
- [4] Akin, E. *Recurrence in topological dynamics: Furstenberg and Ellis actions*. Plenum Press, New York, 1997.
- [5] J. de Vries. *Elements of Topological Dynamics*. Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.



BOOK REVIEW

“Comparison Method and Stability of Motions
of Nonlinear Systems”

by A.Yu. Aleksandrov and A.V. Platonov

A.G. Mazko^{1*} and V.N. Shchennikov²

¹ *Institute of Mathematics National Academy of Science of Ukraine,
Tereschenkivska Str. 3, Kyiv, 01601, Ukraine*

² *Ogarev Mordovia State University, Bolshevistskaya Str. 68, Saransk, 430005, Russia*

Received: June 11, 2013; Revised: July 21, 2013

The monograph “**The Comparison Method and Stability of Motions of Non-linear Systems**” by A.Yu. Aleksandrov and A.V. Platonov.

Publishing house: Saint Petersburg University Press, 2012.

Pages: 263.

Language: Russian.

Readership: pure and applied mathematicians, applied physicists, industrial and control engineers, communication network specialists, and upper-level undergraduate and graduate students studying stability theory and control theory.

Contents: Introduction, Differential inequalities and comparison systems, Absolute stability and ultimate boundedness of nonlinear systems, Stability of motions of complex systems by nonlinear approximation, References.

This self-contained monograph is devoted to the problem of stability analysis for broad classes of systems of nonlinear differential equations. Along with the describing of classical results, the book presents recently developed novel approaches for stability investigation of motions of nonlinear systems in critical, in the Lyapunov sense, cases. The developments in this area are remarkable, from both the theoretical and the practical point of view.

In Chapter 1, the basic notions and principal results of the differential inequalities theory and the comparison method are presented. Several classical approaches for the decomposition and aggregation of complex systems are considered. A special attention is devoted to stability criteria of linear and nonlinear Wazewskii systems. Furthermore,

* Corresponding author: <mailto:mazko@imath.kiev.ua>

new conditions of ultimate boundedness of solutions for autonomous Wazewskii systems are proposed.

The absolute stability and absolute ultimate boundedness problems for nonlinear systems are studied in Chapter 2. The Persidskii-type systems and some of their generalizations are considered. Several approaches for the constructing of Lyapunov functions for such systems are proposed. By means of these functions, the conditions of absolute stability and absolute ultimate boundedness are found. Moreover, an emphasis is placed on the analysis of nonlinear switched systems. To provide the absolute stability or the absolute ultimate boundedness uniform with respect to a switching law for a system of such type, it is sufficient to construct a common Lyapunov function for the corresponding family of subsystems. In the monograph, new conditions in terms of linear inequalities of a special form are presented to guarantee the existence of common Lyapunov functions. The problem of the solvability for the obtained inequalities systems is investigated, and constructive algorithms for finding their solutions are proposed.

Chapter 3 is devoted to the problem of stability analysis of complex (large-scale, multiconnected) systems by nonlinear approximation. First, classical results by N.N. Krasovskii and V.I. Zubov on the stability by homogeneous and generalized homogeneous approximation are presented. Next, the case when a system of the first approximation is nonlinear and inhomogeneous is studied, and original stability conditions are obtained. After that, new forms of decomposition and aggregation of complex systems are proposed, and the effectiveness of their usage for the stability investigation of essentially nonlinear complex systems is demonstrated. Moreover, the approaches for finding the estimates of transient times for multiconnected systems in critical cases are developed. Finally, the problem of stability analysis of equilibrium positions of mechanical systems on the base of decomposition is studied.

The monograph contains a lot of examples of the applications of obtained results and proposed approaches in control problems, mechanics and population dynamics.

In summary, this book is valuable for all those who are interested in stability theory and its applications. It covers a broad spectrum of important topics. The book is well written and the presentation of the material is well organized. The book is issued only in Russian, however it will be also definitely interesting for the English-speaking specialists.

Professor V. G. Miladzhyanov (1953 – 2013). Obituary



On May 13, 2013 the known scholar, doctor of physical and mathematical sciences, professor Miladzhyanov Vakhobzhon Ganizhonovich died suddenly.

He was born on June 21, 1953 in Russkoe village of Markhamatsky district of Andizhan region into a peasant family. From 1970 to 1974, after finishing school, he studied at the Physical and Mathematical Department of Andizhan State Pedagogical Institute (ASPI). On graduating the Institute, he started to work as a teacher of mathematics at secondary school N3 of Markhamatsky district. From November, 1974 to November 31, 1975 he did his military service in the Soviet Army, and then he proceeded with his work at the same school N3. In April, 1977 he was invited to join the Chair of Algebra and Theory of Numbers of ASPI in the capacity of a technician and in February, 1981 he moved to the position of assistant.

In November, 1985 he entered a post-graduate course at the Stability of Processes Department of the Institute of Mechanics of AN UkrSSR. Under the tutelage of doctor of physical and mathematical sciences, professor A.A. Martynyuk, he prepared the Candidate thesis entitled “The application of matrix Lyapunov functions in stability investigation of systems with slow and quick motions” which was a result of his intensive persevering work. After defending the Candidate thesis he was conferred a Candidate of Science degree (PhD) in Physics and Mathematics, speciality 01.02.01 — Theoretical mechanics.

In December, 1988 he started as a senior lecturer at the Chair of Algebra and Computer Science of Andizhan State University and in September, 1990 he was appointed a senior research fellow of the Chair.

V.G. Miladzhyanov managed to combine fruitfully teaching and guiding, academic and research activity. He spared neither time nor effort, throwing himself into work. Many long days, sleepless nights and months of hard work enabled him to carry out intensive research and to prepare in a short term his Doctor thesis “Stability analysis of nonlinear systems under structural perturbations”. On December 21, 1993 he successfully defended the thesis at the Special Council of the Institute of Mathematics of NAS of Ukraine and

took the degree of Doctor of Physical and Mathematical Sciences in speciality 01.02.01 — Theoretical Mechanics.

In August, 1994 V.G. Miladzhanov was elected the Head of the Chair of Algebra and Theory of Numbers and starting from December, 1996 he fulfilled the duties of the Dean of the Mathematical Department. In February, 1998 he was elected the Head of the Chair of Applied Mathematics and Mechanics of Andizhan State University.

From September 1, 2000 he became the Dean of the Physical and Mathematical Department and until the very last day of his life he headed the Chair of Mathematics of Andizhan State University.

V.G. Miladzhanov has made a major contribution to the preparation of young teachers for the Republic of Uzbekistan in the whole, and for Andizhan region in particular. During his professional life (for almost forty years) he brought up and trained more than eight thousand highly qualified young specialists working in the secondary and high education. Scientific results of Miladzhanov are associated with the development of the method of matrix Lyapunov functions for stability investigation of systems with quick and slow motions; stability analysis of large-scale systems under structural perturbations; stability analysis of large-scale discrete systems under structural perturbations; stability of impulsive system under structural perturbations. He proposed a method of constructing hierarchical matrix Lyapunov function for nonautonomous systems which is used in stability investigation of nonlinear mechanical systems.

Scientific achievements attained by Miladzhanov in the fundamental and applied problems of mechanics are widely known in many countries including, of course, Uzbekistan and Ukraine. Together with academician A.A. Martynyuk he worked over and has prepared a generalizing monograph which is intended to be published in English. Besides, he wrote and published more than 150 scientific and methodological papers.

Cherished memory of the well-known scientist and a remarkable man Miladzhanov Vakhobzhon Ganizhonovich will linger on in the memory of those who knew him and worked together with him.

Stability Analysis via Matrix Functions Method. Part I and Part II

Bookboon.com Ltd, 2013, Part I, ISBN: 978-87-403-0446-6

1 edition Pages: 262 Price: Free, Part II, ISBN: 978-87-403-0447-3

1 edition Pages: 130, Price: Free

A.A. Martynyuk

*Institute of Mechanics, National Academy of Sciences of
Ukraine, Kyiv, Ukraine*

The monograph presents a generalization of the well-known Lyapunov function method and related concepts to the matrix auxiliary functions case within the framework of systematic stability analysis of dynamical systems (differential equations). The book is organized in five chapters as follows.

In Chapter 1, the author starts with some explanations on Lyapunov's original definition of stability of systems. This culminates in comments, the well-accepted definitions and concepts of stability, and the concept of reference solution and equilibrium states. After brief outline of trends in Lyapunov's stability theory the author reports on the development of the fairly new view on the method of vector Lyapunov functions.

Chapter 2 is designed to give a survey on the matrix Lyapunov function method in general, using the calculus of Dini derivatives and a technique originated at the works of Yoshizawa. The concept of matrix Lyapunov functions and the theory of differential inequalities provide a very general comparison principle which is described in detail here.

Chapter 3 is devoted to stability analysis of singularly perturbed systems. In particular, the author treats oscillating systems of solid bodies and Lur'e-Postnikov systems (absolute stability in hydraulic servo systems).

In Chapter 4, the author continues with probabilistic stability analysis of differential equations with Markovian random perturbations (Ito-interpreted SDEs) in the spirit of Katz-Krasovskij.

Chapter 5 exhibits illustrations of the versatility, applicability and efficiency of matrix-valued Lyapunov functions in stability investigations with respect to equilibrium states. Here the author discusses asymptotic stability of population models (predator-prey), an orbital astronomical observatory, n-generator power system and the motion in space of winged aircrafts. Besides, each chapter is accompanied by numerous examples and notes on the locally related bibliography.

Thus it can be recommended to any specialist in nonlinear dynamical systems and differential equations, both in deterministic and stochastic analysis.

Download 1000+ FREE eBooks
Easy and without registration
bookboon.com