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PERSONAGE IN SCIENCE

Academician A.M. Samoilenko

On His 75th Birthday

A.A. Boichuk [1], A.G. Mazko [1], A.A. Martynyuk [2*] and M.O. Perestyuk [3]

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The paper contains biographical data and a survey of scientific achievements of Anatoly Mykhailovych Samoilenko, a prominent expert in the field of differential equations.

1 Brief Biography of A.M. Samoilenko

Anatoly Mykhailovych Samoilenko was born on January 2, 1938 in the village of Potiivka (Zhytomyr Region, Ukraine) to the family of Mykhailo Grygorovych and Mariya Vasylivna Samoilenko. Somewhat later, his family moved to the city of Malyn (Zhytomyr Region).

In 1955, he finished school and entered the Geological Faculty of the Shevchenko Kyiv State University. Quite soon he understood that mathematics is his vocation and continued his education at the Faculty of Mechanics and Mathematics of the same university and graduated from this faculty with honors in 1960.

By the invitation of Academician Yu.O. Mitropolsky, Anatoly Samoilenko entered a the post-graduate course at the Institute of Mathematics of the Ukrainian Academy of Sciences, where he became a member of the Krylov–Bogolyubov Kyiv Scientific School. In 1961, he published his first scientific works. In 1963, he defended his Candidate–Degree Thesis "Application of Asymptotic Methods to the Investigation of Nonlinear Differential Equations with Irregular Right-Hand Sides."

In 1965, A.M. Samoilenko started his pedagogic career at the Chair of Differential Equations of the Shevchenko Kyiv State University.

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For a fairly short period of four years, he prepared and defended (in 1967) his Doctoral-Degree Thesis "Some Problems of Periodic and Quasiperiodic Systems."

In 1963–1974, A.M. Samoilenko worked at the Kyiv Institute of Mathematics. In 1974, he received the academic title of professor and headed the Chair of Integral and Differential Equations at the Kyiv University, where he was a teacher of a great number of future scientists. Many of his former students are now well known throughout the world. Together with his colleagues from the Chair of Integral and Differential Equations, he prepared a series of textbooks on the theory of differential equations. Several editions of these textbooks were published, and they still remain popular in Ukraine and in the countries of the former USSR.

In 1978, Prof. Samoilenko was elected to become a Corresponding Member of the Academy of Sciences of the Ukrainian SSR.

In 1987, he returned to the Institute of Mathematics, where he headed the Department of Ordinary Differential Equations.

In 1988, he was elected by the staff of the Institute to become the Director of the Institute, and he occupies this position up to now.

In 1995, Prof. Samoilenko was elected to become a Full Member of the Ukrainian National Academy of Sciences.

In 1997, he became the Editor-in-Chief of the "Nonlinear Oscillations" journal founded on his initiative.

In 1998–2011, he also headed the Chair of Differential Equations at the "Kyiv Polytechnic Institute" National Technical University of Ukraine. Under his guidance, researchers of the Institute of Mathematics started to teach students at this chair, and its scientific life was significantly intensified.

Since 2006, he works as the Academician-Secretary of the Department of Mathematics of the Ukrainian National Academy of Sciences.

Academician Samoilenko is the author of more than 600 scientific works, including 30 monographs and 15 textbooks. Most of his works are translated into English and other languages. He is a member of the editorial boards of several Ukrainian and foreign journals. As an excellent teacher, he gives much attention to training highly qualified scientific personnel. Among his disciples, there are 33 Doctors and Candidates of Science in Physics and Mathematics. They successfully work in numerous mathematical centers throughout the world. He is also deeply involved in the social life. His activities are aimed at the support of young Ukrainian mathematicians and talented children.

Tremendous scientific achievements of A.M. Samoilenko are explained by his great mathematical talent and persistence and efficiency in his work.

A significant role in his life is played by his strong family. His wife, Lypa Hryhorivna, also a scientific researcher, worked for many years at the Institute of Cybernetics of the Ukrainian National Academy of Sciences. His son is a talented geneticist, and the father of two children.

We wish Academician Samoilenko good health, family happiness, creative inspiration, and subsequent successes in his scientific work.

2 Main Scientific Interests

2.1 Theory of invariant manifolds of differential systems

The notion of the Green function of the problem of invariant torus for the linear extension of a dynamical system on the torus introduced by A.M. Samoilenko at the Fifth

International Conference on Nonlinear Oscillations in Kyiv and investigated in detail in his work [6] appeared to be extremely fruitful and gave a new impetus to the development of various aspects of the theory of perturbations and stability of toroidal manifolds. In the mathematical literature, this notion is known as the "Green–Samoilenko function." This name was introduced by a Moldavian mathematician I.U. Bronshtein. A survey of the main subsequent results in the field of toroidal manifolds of autonomous differential systems can be found in the monograph [XI].

The works of A.M. Samoilenko in the theory of multifrequency oscillations made an important contribution to this theory and opened new directions in their investigation and development. In the works written with V.L. Kulyk, the authors developed the theory of alternating Lyapunov functions for the investigation of the solutions of linear autonomous differential systems bounded on the entire axis and linear extensions of dynamical systems on the torus. The results obtained in this field were generalized, together with Yu.V. Teplinskii, to the case of countable systems and, together with O.M. Stanzhitskii, to the case of stochastic differential equations.

2.2 Asymptotic methods of nonlinear mechanics

Continuing the investigations of M.M. Krylov, M.M. Bogolyubov, A.M. Kolmogorov, V.I. Arnold, J. Moser, and Yu.O. Mitropolsky, A.M. Samoilenko proposed a modification of the asymptotic method of successive changes of variables, which was called the "method of accelerated convergence" in 1969 in the monograph [I]. In [9], together with Yu.O. Mitropolsky, he generalized the asymptotic averaging method and established sufficient conditions for the "averaging" operator under which the asymptotic solutions are separated into naturally varying and slowly varying components. The theory has been further developed, in particular, in his joint works with R.I. Petryshyn.

2.3 Nonlinear boundary-value problems

In 1965–1966, the papers [2, 3] were published, in which an original method was proposed for the determination of periodic solutions of ordinary differential systems. In subsequent publications, the Soviet mathematicians called it "the Samoilenko numerical-analytic method." Later, in joint works with M.I. Ronto, V.I. Trofimchuk, and their disciples, this method was generalized to a broad class of boundary-value problems.

On the basis of the theory of generalized inverse operators, A.M. Samoilenko, together with O.A. Boichuk, developed the theory of Fredholm boundary-value problems for differential equations, delay equations, impulsive equations, and singularly perturbed systems. The obtained results are presented in the monograph [XXIX]. This theory was later developed for the determination of solutions, bounded on the entire real axis, for systems of differential and difference equations under the condition of dichotomy on semiaxes for the corresponding homogeneous system.

2.4 Theory of impulsive differential systems

Apparently, the best-known series of works of A.M. Samoilenko is devoted to the theory of impulsive differential equations. This field of investigations is traditionally associated with the Kiev Mathematical School. As early as 1937, M.M. Krylov and M.M. Bogolyubov showed that asymptotic methods of nonlinear mechanics can be efficiently applied to impulsive equations. However, the systematic study of these problems is associated with the name of A.M. Samoilenko. The first scientific paper of A.M. Samoilenko, published in 1961, was devoted to these problems. In 1967, in the joint work [4], A.D. Myshkis and A.M. Samoilenko formulated general theorems on the existence of solutions and their extendability and also on the uniqueness of a solution of the Cauchy problem for impulsive systems. In 1987, the monograph of A.M. Samoilenko and M.O. Perestyuk (complemented and translated into English in 1995 [XVI]) became the first monograph in the world literature in which fundamental results of the theory of impulsive systems were presented.

2.5 Integrability of dynamical systems on symplectic manifolds

A.M. Samoilenko and Ya.A. Prykarpatsky proposed and described new analytic and topological-geometric approaches to the problem of imbedding of integral manifolds for completely integrable dynamical systems and their perturbations. The main results are presented in the monograph [XXV].

2.6 Linear theory of ordinary differential equations

In 2011, the paper [21] was published, in which A.M. Samoilenko considered problems of the linear theory of systems of ordinary differential equations related to the investigation of invariant hyperplanes of these systems, the notion of equivalence for these systems, and the Floquet–Lyapunov theory for periodic systems of linear equations. In particular, a new Floquet-type formula was proposed for periodic systems.

2.7 Theory of functions

In 1968, the paper [5] was published, in which A.M. Samoilenko solved a problem posed by V.I. Arnold, namely, he gave a purely analytic proof of the equivalence of a smooth function and its Taylor polynomial in a neighborhood of a critical point of finite type. This investigation was continued in the paper [20] published in 2007. The local behavior of smooth functions in the neighborhoods of their regular and critical points was investigated, and theorems on the average values of the considered functions of the type of the Lagrange theorem on finite increments were proved. The symmetry of the derivative of an analytic function in a neighborhood of its multiple zero was also studied, and new statements of the Weierstrass preparation theorem related to the critical point of a function of finite smoothness were proved. The nongradient vector field in the neighborhood of the critical point was determined, and one critical case of stability of the equilibrium position of a nonlinear system was considered.

3 Participation in Scientific Institutions and Editorial Boards

A.M. Samoilenko is a full member of the National Academy of Sciences of Ukraine and the European Academy of Sciences, a foreign member of the Academy of Sciences of the Republic of Tajikistan, and a member of the Ukrainian Mathematical Society and the American Mathematical Society. He is the editor-in-chief of the journals "Ukrains'kyi Matematychnyi Zhurnal," "Neliniini Kolyvannya," and "Ukrains'kyi Matematychnyi Visnyk" and a member of the editorial boards of the journals "Dopovidi Natsional'noi Akademii Nauk Ukrainy," "U Sviti Matematyky," "Nonlinear Mathematical Physics," "Memoirs on Differential Equations and Mathematical Physics," etc.

4 Awards

A.M. Samoilenko was awarded the Order of Friendship of Peoples (1984), the third class Order of Merit (2003), the fifth class Order of Prince Yaroslav the Wise (2008), and a Certificate of Honor of the Presidium of the Supreme Soviet of Ukraine (1987). He was also awarded the State Prize of Ukraine in the Field of Science and Engineering (1985, 1996), the State Prize of Ukraine in the Field of Education (2012), M. Ostrovsky Prize (1968), M. Krylov Prize (1981), M. Bogolyubov Prize (1998), M. Lavrent'ev Prize (2000), M. Ostrogradsky Prize (2004), Yu. Mitropolsky Prize (2010), and the titles of a "Soros Professor" (1998) and an Honored Scientist of Ukraine (1998).

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Nonlinear Dynamics and Systems Theory, **13** (2) (2013) 115–134

Orthogonal Functions Approach for Model Order Reduction of LTI and LTV Systems

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Abstract: In this paper, we elaborate new methods for model-order reduction of linear time invariant (LTI) and time variant (LTV) systems by using orthogonal functions. These techniques which can be efficiently applied in SISO (single-input single-output) and MIMO (multi-input multi-output) cases are based on the projection of the system parameters and variables on an orthogonal functions basis. The useful properties of the orthogonal functions basis such as operational matrices combined with the Kronecker product permit the conversion of the system differential equations into algebraic ones allowing the determination of the reduced model parameters.

Keywords: *model-order reduction; LTI and LTV systems; orthogonal functions; operational matrices; shifted Legendre polynomials.*

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1 Introduction

In all engineering fields, an accurate modeling is necessary to have good results in control and analysis of complex systems. If the system is internally complex, the use of modern control techniques such as optimal control, μ-synthesis or robust control may lead to a controller having a comparable order as the considered system. In order to study, simulate and control those systems and to avoid time consuming in computing procedures, it is convenient and sometimes necessary to reduce their complexity, preserving the input-output behavior.

The primary problem of interest in model reduction is the efficient computation of an accurate low-order model approximating a given dynamical system. The low-order model must match the original one in some sense. However, the conditions of accuracy, speed,

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stability and low order cannot always be reached at the same time. The model-order reduction (MOR) reaches far from electrical engineering and touches various disciples of science and engineering fields such as aerospace science [1, 2], chemical processes [3], protection of civil structures, modeling of biological systems [4], power systems [47] and mechanical engineering [5].

So far, the main MOR techniques were introduced and developed for linear systems and precisely LTI systems and were lately extended to LTV systems and nonlinear systems [6, 7]. The main MOR methods fall into three classes [8]:

– Singular Value Decomposition (SVD) or Gramian-based methods including optimal Hankel MOR [9, 10], and balanced truncation realization first introduced by Moore [11] and improved during the last decades [12–14].

– Krylov subspace-based methods [15] including techniques based on Lanczos procedure [16, 17] or Arnoldi algorithm [18, 19].

– Proper orthogonal decomposition (POD) or Karhunen-Loève expansion [2, 20].

Many recent techniques give an alternative to these classical methods such as the MOR by least squares [21] and using LMIs [22]. The MOR techniques for LTI systems were later extended to modeling linear time varying (LTV) systems [7, 23, 24].

In this paper, we introduce new analytic methods for model-order reduction (MOR) of linear time invariant (LTI) and time variant (LTV) systems starting from a state space realization or a transfer function system description. Those approaches using the orthogonal functions as a tool of approximation can be applied not only for SISO systems but also for the MIMO ones. This paper is organized as follows: in Section 2, the orthogonal functions are presented with their interesting properties. The dynamical systems description by orthogonal functions is introduced in Section 3. The proposed methods for model order reduction of LTI and LTV systems using orthogonal functions are derived in Section 4. The last section is devoted to simulation examples to emphasize the effectiveness of the presented approaches.

2 Orthogonal Functions for Dynamical Systems Description

In recent decades, the approximation of time functions by orthogonal functions has been considered by many researchers to solve modeling and control problems [48]. The main characteristic of this technique is that it reduces the differential equations to algebraic ones, thus greatly simplifying the problem.

This approach originated from the use of Walsh [25] and block-pulse [26] functions was later extended to orthogonal polynomial series such as the Laguerre [27], the Chebychev [28], the Hermite [29] and the Legendre polynomials [30]. They were also used with nonlinear systems [31] and for PID control of LTI [32] and LTV systems [33]. The development in Fourier or Taylor polynomial series can give convenient results but their quick convergence is not always guaranteed or their use can be sometimes inadequate.

2.1 Orthogonal functions and properties

2.1.1 Approximation using orthogonal functions

Orthogonal functions were introduced in the field of system control because of their interesting properties as a sharp tool of approximation. Given $\Phi = \{\phi_i(t), i \in \mathbb{N}\}$ a set of functions defined over a certain interval $[a, b]$. Any function $f(t)$ absolutely integrable over $[a,b]$ can be developed as follows

$$f(t)=\sum_{i=0}^{\infty} f_i\phi_i(t), \tag{1}$$

where $f_i=\displaystyle\int_a^b w(t)f(t)\phi_i(t)\,dt$, for $i\in\mathbb{N}$, $w(t)$ is a positive and integrable function as the weighting function of the scalar product. For practical use, the development (1) is truncated up to an order N, thus giving the following time approximation of the function

$$f(t)\cong\sum_{i=0}^{N-1} f_i\phi_i(t)=F_N^T\Phi_N(t) \tag{2}$$

with

$$F_N=\begin{bmatrix} f_0 & f_1 & \dots & f_{N-1}\end{bmatrix}^T,\ \Phi_N(t)=\begin{bmatrix}\phi_0(t) & \phi_1(t) & \dots & \phi_{N-1}(t)\end{bmatrix}^T,$$

where $\Phi_N(t)$ is the vector of the orthogonal functions basis. The coefficients f_i and the orthogonal functions $\{\phi_i(t), i\in\mathbb{N}\}$ have the particularity to minimize the error:

$$\varepsilon=\int_a^b\left(f(t)-\sum_{i=0}^{N-1} f_i\ \phi_i(t)\right)^2 dt. \tag{3}$$

The orthogonal functions obey the orthogonality relation

$$<\phi_i(t),\phi_j(t)>=\int_a^b w(t)\phi_i(t)\phi_j(t)\,dt=\delta_{ij}\ c_i, \tag{4}$$

where δ_{ij} is the Kronecker delta. If $c_i=1$, then the functions are not only orthogonal, but orthonormal.

2.1.2 Shifted definition interval

If the function $f(t)$ is defined over an interval $[t_0,t_f]$ and the orthogonal functions $\phi_i(t)$ over the interval $[a,b]$, we can shift the defining domain by considering the functions :

$$\forall i\in\mathbb{N}, \psi_i(t)=\phi_i\left(\frac{t-\mu}{\lambda}\right)$$

with $t\in[t_0,t_f]$, $\lambda=\dfrac{t_0-t_f}{a-b}$ and $\mu=\dfrac{at_f-bt_0}{a-b}$.

The functions $\psi_i(t),\forall i\in\mathbb{N}$ are also orthogonal over $[t_0,t_f]$ with the weighting function $w'(t)=w(\dfrac{t-\mu}{\lambda})$.

2.1.3 Matrix functions approximation

A time dependent matrix function $A(t)\in\mathbb{R}^{n\times m}$ given by $A(t)=[a_{ij}(t)]$ where $a_{ij}(t)$ are integrable over an interval $[a,b]$. The matrix $A(t)$ can be developed into orthogonal functions series with a truncation to an order N under the following relation

$$A(t)\cong\sum_{i=0}^{N-1} A_{Ni}\phi_i(t), \tag{5}$$

where $A_{Ni}\in\mathbb{R}^{n\times m}$ for $i\in\{0,1,\dots,N-1\}$ are matrices with constant coefficients.

2.1.4 Operational matrix of integration

The operational matrix of integration is a constant coefficient function $P_N \in \mathbb{R}^{N\times N}$ verifying the integral property of the orthogonal functions basis vector $\Phi_N(t)$:

$$\underbrace{\int \cdots \int_\alpha^t}_{k \text{ times}} \Phi_N(t)\, dt^k \cong P_N^k \Phi_N(t). \tag{6}$$

Clearly, the form of P_N depends on the particular choice of the basis vector $\Phi_N(t)$.

2.1.5 Operational matrix of product

The operational vectors of product K_{ij} [35] have constant coefficients and verify the property:

$$\forall\, i,j \in \{0,1,\cdots,N-1\},\ \ \phi_i(t)\phi_j(t) \cong K_{ij}^T \Phi_N(t). \tag{7}$$

From the relationship (7), we can readily get the operational matrix of product:

$$M_{iN} = \begin{bmatrix} K_{0i}^T \\ \vdots \\ K_{N-1,i}^T \end{bmatrix} \tag{8}$$

that allows the approximation

$$\phi_i(t)\Phi_N(t) \cong M_{iN}\Phi_N(t). \tag{9}$$

2.1.6 Legendre polynomials

The Legendre polynomials may have advantages over other orthogonal functions. This was shown by way of examples [30] where Legendre polynomials converge to the exact solution of a differential equation faster than the other types of orthogonal functions, such as, for example, Walsh functions, Hermite and Laguerre polynomials. The Legendre polynomials are defined for the time interval $x \in [-1,1]$ and they have the following analytical form given by the Olinde-Rodrigues formula [36]:

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}. \tag{10}$$

Using the above expression for $L_n(x)$, one may readily determine the first few Legendre polynomials : $L_0(x) = 1$, $L_1(x) = x$, ...

The Legendre polynomials are also given by the recursive formula [34]:

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x). \tag{11}$$

The polynomials $L_i(x)$ form a complete set and are orthogonal [30] with

$$\int_{-1}^{1} L_i(x)L_j(x)dx = \frac{2}{2i+1}\delta_{ij}. \tag{12}$$

2.1.7 Shifted Legendre polynomials

For practical use of Legendre polynomials in the time interval $t \in [0, t_f]$, it is necessary to shift the defining domain of Legendre polynomials from the interval $[-1, 1]$ to $[0, t_f]$ through the variable transformation:

$$x = \frac{2t}{t_f} - 1, \quad 0 \le t \le t_f. \tag{13}$$

Thus, the shifted Legendre polynomials $s_i(t)$ ($\forall i \in \mathbb{N}$) for $0 \le t \le t_f$ are thus given by

$$s_{n+1}(t) = \frac{2n+1}{n+1} \frac{2t - t_f}{t_f} s_n(t) - \frac{n}{n+1} s_{n-1}(t) \tag{14}$$

with $s_0(t) = 1$ and $s_1(t) = \dfrac{2t}{t_f} - 1$.

It is apparent that polynomials $s_n(t)$ also constitute a complete set and are orthogonal [37] with

$$\int_0^{t_f} s_i(t) s_j(t)\, dt = \frac{t_f}{2i+1} \delta_{ij}. \tag{15}$$

Any time function $f(t)$ that is absolutely integrable on the time interval $[0, t_f]$ may be expanded into shifted Legendre series as follows

$$f(t) = \sum_{i=0}^{\infty} f_i s_i(t), \tag{16}$$

where [38]

$$f_i = \frac{2i+1}{t_f} \int_0^{t_f} f(t) s_i(t)\, dt. \tag{17}$$

If equation (16) is truncated up to its first N terms, then it may be written as

$$f(t) \cong \sum_{i=0}^{N-1} f_i s_i(t) = F_N^T S_N(t) \tag{18}$$

with $F_N = \begin{bmatrix} f_0 & f_1 & \dots & f_{N-1} \end{bmatrix}^T$ and $S_N(t) = \begin{bmatrix} s_0(t) & s_1(t) & \dots & s_{N-1}(t) \end{bmatrix}^T$. The shifted Legendre polynomials and coefficients f_i, ($i = 0, 1, \dots, N-1$) have the particularity to minimize the integral squared-error:

$$\varepsilon = \int_0^{t_f} \left(f(t) - \sum_{i=0}^{N-1} f_i s_i(t) \right)^2 dt. \tag{19}$$

2.1.8 Operational matrix of integration

Since the shifted Legendre polynomials $s_i(t)$, ($i = 0, 1, \dots$) satisfy [34] the differential equation:

$$s_i(t) = \frac{t_f}{2(2i+1)} \left[\frac{ds_{i+1}}{dt} - \frac{ds_{i-1}}{dt} \right] \tag{20}$$

and $s_i(0) = (-1)^i$, it can be easily shown that the integrals of $s_i(t)$, $(i = 0, 1, \dots)$ are given by

$$\int_0^t s_i(\tau)d\tau = \begin{cases} \frac{t_f}{2}\left[s_1(t) - s_0(t)\right] & \text{, for } i = 0, \\ \frac{t_f}{2(2i+1)}\left[s_{i+1}(t) - s_{i-1}(t)\right] & \text{, for } i > 0. \end{cases} \tag{21}$$

From equation (21) we can obtain the integral of truncated shifted Legendre vector as follows

$$\int_0^t S_N(\tau)d\tau \cong P_N S_N(t), \tag{22}$$

where P_N is the constant operational matrix of integration given in [39] and [40].

3 Dynamical systems description using orthogonal functions

3.1 LTI systems described by a transfer function

Given a linear time invariant system described by a transfer function:

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + ... + b_m s^m}{a_0 + a_1 s + ... + a_n s^n} \tag{23}$$

with $m \leq n$, or a linear differential equation in time domain with constant coefficients, the input $u(t)$ and the output $y(t)$:

$$a_0 y(t) + a_1 y'(t) \cdots + a_n y^{(n)}(t) = b_0 u(t) + b_1 u'(t) \cdots + b_m u^{(m)}(t). \tag{24}$$

Upon integration of both sides of equation (24) n times, we have:

$$\begin{aligned} &a_0 \underbrace{\int \cdots \int_0^t}_{n \text{ times}} y(\tau)d\tau^n + a_1 \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} y(\tau)d\tau^{n-1} + \cdots + a_n y(t) = \\ &b_0 \underbrace{\int \cdots \int_0^t}_{n \text{ times}} u(\tau)d\tau^n + b_1 \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} u(\tau)d\tau^{n-1} + \cdots + b_m \underbrace{\int \cdots \int_0^t}_{n-m \text{ times}} u(\tau)d\tau^{n-m}. \end{aligned} \tag{25}$$

The projection of the input $u(t)$ and the output $y(t)$ on an orthogonal functions basis with truncated developments to an order N over a time interval $[0, t_f]$ yields:

$$y(t) \cong Y_N \Phi_N(t), \tag{26}$$

$$u(t) \cong U_N \Phi_N(t), \tag{27}$$

where Y_N and U_N are constant coefficient vectors.

By introducing the projections (26) and (27) in equation (25) and considering the case where the initial conditions are equal to zero, we obtain the relation

$$\begin{aligned} &a_0 Y_N \underbrace{\int \cdots \int_0^t}_{n \text{ times}} \Phi_N(\tau)d\tau^n + a_1 Y_N \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} \Phi_N(\tau)d\tau^{n-1} + \cdots + a_n y(t) = \\ &b_0 U_N \underbrace{\int \cdots \int_0^t}_{n \text{ times}} \Phi_N(\tau)d\tau^n + \cdots + b_m U_N \underbrace{\int \cdots \int_0^t}_{n-m \text{ times}} \Phi_N(\tau)d\tau^{n-m}. \end{aligned} \tag{28}$$

In the case where the initial conditions are different from zero, they can be projected on the orthogonal basis and then integrated in the equation (28). By using the operational matrix of integration and the property (6), the equation (28) yields:

$$\begin{aligned} &Y_N \left(a_0 P_N^n + a_1 P_N^{n-1} \cdots + a_n I_N\right) \Phi_N(t) = \\ &\qquad U_N \left(b_0 P_N^n + b_1 P_N^{n-1} + \cdots + b_m P_N^{n-m}\right) \Phi_N(t). \end{aligned} \tag{29}$$

This equality is available for all time $t \in [0, t_f]$ then the simplification by $\Phi_N(t)$ in the equality (29) leads to the following description of the considered system:

$$Y_N \, \mathbb{M} = U_N \, \mathbb{T} \quad \text{or} \quad Y_N = U_N \, \mathbb{T} \, \mathbb{M}^{-1} \tag{30}$$

with

$$\begin{aligned} \mathbb{M} &= a_0 P_N^n + a_1 P_N^{n-1} + ... + a_n I_N, \\ \mathbb{T} &= b_0 P_N^n + b_1 P_N^{n-1} + ... + b_m P_N^{n-m}. \end{aligned} \tag{31}$$

3.2 LTI systems described by a state representation

Consider a linear time invariant (LTI) MIMO system given by the following state realization:

$$\begin{cases} \dot{X}(t) = A\, X(t) + B\, U(t), & X(0) = 0, \\ Y(t) = C\, X(t), & t \in [0, t_f], \end{cases} \tag{32}$$

with the state vector $X(t) \in \mathbb{R}^n$, the inputs vector $U(t) \in \mathbb{R}^m$ and the output one $Y(t) \in \mathbb{R}^p$. The matrices A, B, and C have respectively the dimensions $n \times n$, $n \times m$ and $p \times n$. The integration of the state equation (32) with zero initial conditions gives:

$$X(t) = A \int_0^t X(\tau) d\tau + B \int_0^t U(\tau) d\tau. \tag{33}$$

The projection of the state vector $X(t)$, the input U and the output Y, on an orthogonal basis functions $\{\varphi_i(t), i \in \{0, 1, \ldots, N-1\}\}$ with a truncated development to an order N over the interval $[0, t_f]$ leads to:

$$X(t) \cong X_N \Phi_N(t), \tag{34}$$

$$U(t) \cong U_N \Phi_N(t), \tag{35}$$

$$Y(t) \cong Y_N \Phi_N(t), \tag{36}$$

where the matrices X_N and U_N are constant coefficients matrices. With the developments (34) and 35), the integrated state equation (33) can be written under the following form

$$X_N \Phi_N(t) = A \int_0^t X_N \Phi_N(\tau) d\tau + B \int_0^t U_N \Phi_N(\tau) d\tau. \tag{37}$$

The use of the operational matrix of integration that approximates the integration of the orthogonal basis vector $\Phi_N(t)$:

$$\int_0^t \Phi_N(\tau)\, d\tau \cong P_N \; \Phi_N(t) \tag{38}$$

leads to the relation:

$$X_N\ \Phi_N(t) = A\ X_N\ P_N \Phi_N(t) + B\ U_N\ P_N\ \Phi_N(t). \tag{39}$$

Simplifying by the orthogonal functions basis vector $\Phi_N(t)$ yields:

$$X_N - A\ X_N\ P_N = B\ U_N\ P_N. \tag{40}$$

For rearranging the equation (40), we use the Vec operator [41] that reshapes a matrix by stacking its columns into a long vector. This vector denoted by $Vec(A)$ is associated with a matrix A and has the following property:

$$Vec(E\ F\ G) = \left(G^T \otimes E\right) Vec(F), \tag{41}$$

where E, F and G are matrices having appropriate dimensions and $\otimes$ is the Kronecker product.

Mathematically, let $R = [r_{ij}] \in \mathbb{R}^{m\times n}$ and $W = [w_{ij}] \in \mathbb{R}^{p\times q}$, the Kronecker product of R and W, denoted by $R \otimes W \in \mathbb{R}^{mp\times nq}$ is defined by [41]:

$$R \otimes W = \begin{bmatrix} r_{11}W & r_{12}W & \dots & r_{1n}W \\ r_{21}W & r_{21}W & \dots & r_{2n}W \\ \vdots & \vdots & \vdots & \vdots \\ r_{m1}W & r_{m1}W & \dots & r_{mn}W \end{bmatrix}. \tag{42}$$

By applying the property (41) to the equation (40), we get the following algebraic relation

$$Vec(X_N) = \left[I_{n\times N} - \left(P_N^T \otimes A\right)\right]^{-1} \left(P_N^T \otimes B\right) Vec(U_N) \tag{43}$$

and in the same way, the output relation in (32) can be written as:

$$Vec(Y_N) = (I_N \otimes C)\, Vec(X_N). \tag{44}$$

3.3 LTV systems described by a state representation

In this section, we consider the linear time varying (LTV) systems described by the following state space realization

$$\begin{cases} \dot{X}(t) = A(t)\ X(t) + B(t)\ U(t), \\ Y(t) = C(t)\ X(t), \end{cases} \tag{45}$$

with $A(t)$, $B(t)$ and $C(t)$ varying in time t with respective dimensions $n \times n$, $n \times m$ and $p \times n$. The expressions of matrices $A(t)$, $B(t)$ and $C(t)$ are supposed to be known with:

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \dots & \dots & \dots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad C(t) = \begin{bmatrix} c_1(t) & \dots & c_n(t) \end{bmatrix}.$$

Notice that this state description can be derived from an input-output LTV differential equation. A technique of LTV systems identification was proposed in [42]. The integration of the state equation gives:

$$X(t) = \int_0^t A(\tau)X(\tau)d\tau + \int_0^t B(\tau)U(\tau)d\tau. \tag{46}$$

By exploiting the matrix functions approximation (5), the variable in time parameters of the system can be projected into th orthogonal basis and then written under the form:

$$A(t) \cong \sum_{i=0}^{N-1} A_{Ni}\; \varphi_i(t), \tag{47}$$

$$B(t) \cong \sum_{i=0}^{N-1} B_{Ni}\; \varphi_i(t), \tag{48}$$

$$C(t) \cong \sum_{i=0}^{N-1} C_{Ni}\; \varphi_i(t), \tag{49}$$

where A_{Ni}, B_{Ni} and C_{Ni} are constant coefficients matrices having respectively the same dimensions as $A(t)$, $B(t)$ and $C(t)$.

With the same projections (34), (35) and (36) of the state vector, the input vector and the output vector, the equation (46) becomes:

$$X_N\,\Phi_N(t) = \int_0^t \sum_{i=0}^{N-1} A_{iN}\phi_i(\tau) X_N \Phi_N(\tau) d\tau + \int_0^t \sum_{i=0}^{N-1} B_{iN}\phi_i(\tau) U_N \Phi_N(\tau) d\tau. \tag{50}$$

The orthogonal functions $\phi_i(t)$ are scalar functions, so:

$$X_N\,\Phi_N(t) = \int_0^t \sum_{i=0}^{N-1} A_{iN} X_N \phi_i(\tau) \Phi_N(\tau) d\tau + \int_0^t \sum_{i=0}^{N-1} B_{iN} U_N \phi_i(\tau) \Phi_N.(\tau) d\tau \tag{51}$$

By using the operational matrix of product [35] and the property (9), one has:

$$X_N\,\Phi_N(t) = \int_0^{t_f} \sum_{i=0}^{N-1} A_{iN} X_N M_{iN} \Phi_N(t) dt + \int_0^{t_f} \sum_{i=0}^{N-1} B_{iN} U_N M_{iN} \Phi_N(t) dt \tag{52}$$

and with the operational matrix of integration [39, 40], it comes out:

$$X_N\,\Phi_N(t) = \sum_{i=0}^{N-1} A_{iN} X_N M_{iN} P_N \Phi_N(t) + \sum_{i=0}^{N-1} B_{iN} U_N M_{iN} P_N \Phi_N(t). \tag{53}$$

We can simplify by the orthogonal function basis vector and eliminate the time depending parameters in the equation (53). The application of the property of the Vec operator (41) yields:

$$\begin{gathered} \left[I_{n\times N} - \left(\sum_{i=0}^{N-1} \left(M_{iN} P_N\right)^T \otimes A_{iN} \right) \right] Vec\,(X_N) = \\ \left(\sum_{i=0}^{N-1} \left(M_{iN} P_N\right)^T \otimes B_{iN} \right) Vec\,(U_N) \end{gathered} \tag{54}$$

or

$$Vec\,(X_N) = \mathbb{G}^{-1}\, \mathbb{H}\, Vec\,(U_N) \tag{55}$$

with the constant matrices

$$\mathbb{G} = I_{n\times N} - \left(\sum_{i=0}^{N-1} \left(M_{iN} P_N\right)^T \otimes A_{iN} \right),\;\; \mathbb{H} = \sum_{i=0}^{N-1} \left(M_{iN} P_N\right)^T \otimes B_{iN}.$$

On the other hand, we have

$$Y(t) \cong Y_N\, \Phi_N(t) = \sum_{i=0}^{N-1} C_{iN}\, \phi_i(t)\, X_N \Phi_N(t) \cong \sum_{i=0}^{N-1} C_{iN}\, X_N\, M_{iN} \Phi_N(t). \tag{56}$$

The application of the Vec operator yields:

$$Vec(Y_N) = \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN} \right) Vec\,(X_N)\,. \tag{57}$$

4 Model-order reduction (MOR) using orthogonal functions

4.1 MOR with a transfer function representation

Consider a linear time invariant system described by the transfer function (23). The order of reduction can be chosen by the Hankel singular values. The reduced-order model have an order k and the following transfer function:

$$\frac{Y_r(p)}{U(p)} = \frac{d_0 + d_1 p + ... + d_l p^l}{c_0 + c_1 p + ... + c_{q-1} p^{q-1} + p^q} \tag{58}$$

with $l \leq q < n$. The input-output differential equation of the reduced order system will be written as:

$$c_0 y_r(t) + c_1 y_r'(t) \cdots + c_{q-1} y_r^{(q-1)}(t) + y_r^{(q)}(t) = d_0 u(t) + d_1 u'(t) \cdots + d_l u^{(l)}(t), \tag{59}$$

where $u(t)$ is the input and $y_r(t)$ is the output of the reduced order system.

The description of the reduced order system by orthogonal functions will have an analogue form to (30), given by the following relation:

$$Y_{rN} = U_N \mathbb{T}_r\, \mathbb{M}_r^{-1}, \tag{60}$$

where $\mathbb{M}_r(c_0, \ldots, c_{q-1}) = c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N + I_N$ and $\mathbb{T}_r(d_0, \ldots, d_l) = d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l}$ are matrices depending on the parameters of the reduced order system and P_N the operational matrix of integration depending of the chosen orthogonal functions basis.

The reduced-order system is computed such that it has a similar input-output dynamical behavior to the original system for all inputs $u(t)$. When projected into the orthogonal functions basis, this condition yields:

$$Y_N\, \Phi_N(t) = Y_{rN}\, \Phi_N(t) \quad \Leftrightarrow \quad Y_N = Y_{rN}. \tag{61}$$

The developments (60) and (30) lead to:

$$U_N \mathbb{T}\mathbb{M}^{-1} = U_N \mathbb{T}_r \mathbb{M}_r^{-1}, \tag{62}$$

where $\mathbb{T}$ and $\mathbb{M}$ are constant matrices depending on the known parameters of the original system and the operational matrix of integration P_N given by (31).

The relation (62) must be verified for any input $u(t)$ (i.e. to any U_N). Therefore, we can formulate the equality (62) as:

$$\mathbb{T}\mathbb{M}^{-1} \left(c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N + I_N \right) = d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l} \tag{63}$$

or

$$\left(d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l}\right) - \mathbb{TM}^{-1}\left(c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N\right) = \mathbb{TM}^{-1}. \tag{64}$$

Let Θ be the vector of reduced order system parameters:

$$\Theta^T = \left[\begin{array}{cccccc} d_0 & d_1 & \cdots & d_l & c_0 & \cdots & c_{q-1} \end{array}\right], \tag{65}$$

$\mathbb{A} = \left[\begin{array}{cc} A_l & A_r \end{array}\right]$ with

$$A_l = \left[\begin{array}{ccc} Vec(P_N^q) & Vec(P_N^{q-1}) \cdots & Vec(P_N^{q-l}) \end{array}\right],$$

$$A_r = \left[\begin{array}{ccc} Vec(-\mathbb{TM}^{-1} P_N^q) & \cdots & Vec(-\mathbb{TM}^{-1} P_N) \end{array}\right],$$

and

$$\mathbb{B} = Vec(\mathbb{TM}^{-1}).$$

Then the equation (64) can be written as

$$\mathbb{A}\,\Theta = \mathbb{B}. \tag{66}$$

The vector of the unknown parameters Θ are derived by means of least square resolution

$$\Theta = (\mathbb{A}^T\,\mathbb{A})^{-1}\mathbb{A}^T\,\mathbb{B}. \tag{67}$$

Remark 4.1 Extension to the MIMO LTI system case.
For MIMO LTI system described by a transfer matrix

$$H(s) = \left[\begin{array}{ccc} H_{11}(s) & \cdots & H_{1p}(s) \\ \cdots & \cdots & \cdots \\ H_{k1}(s) & \cdots & H_{kp}(s) \end{array}\right] \tag{68}$$

the order reduction of $H(s)$ can be led by considering the order reduction of each partial transfer function $H_{ij}(s)$ between the i-input and j-output. Note that the reduced order choice of each transmittance $H_{ij}(s)$ can be made using the Hankel singular values technique [44].

4.2 Model order reduction with a state space LTI realization

Consider a linear time invariant (LTI) system described by the state realization (32). We are searching for a reduced order system having an order $r < n$ and the following realization

$$\left\{\begin{array}{l} \dot{X}_r(t) = A_r\, X_r(t) + B_r\, U(t), \\ Y_r(t) = C_r\, X_r(t). \end{array}\right. \tag{69}$$

Using the orthogonal functions (43) for the reduced-order model description, one obtains:

$$Vec(X_{Nr}) = \left[I_{r\times N} - \left(P_N^T \otimes A_r\right)\right]^{-1}\left(P_N^T \otimes B_r\right) Vec(U_N). \tag{70}$$

The reduced system is computed such that it has the same dynamical output as the original system for any input $U(t)$. This condition is equivalent to

$$Y = Y_r \quad \text{or} \quad C\,X = C_r\,X_r. \tag{71}$$

By projecting the relation (71) in the orthogonal functions basis, one has:

$$CX_N\Phi_N(t) = C_rX_{Nr}\Phi_N(t). \tag{72}$$

The simplification by the vector of orthogonal functions $\Phi_N(t)$ and the application of the Vec operator yields:

$$(I_N \otimes C)Vec(X_N) = (I_N \otimes C_r)Vec(X_{Nr}). \tag{73}$$

With combination by substitution of the equations (43), (70) and (73), we obtain the relation

$$\begin{aligned}(I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B)Vec(U_N) =\\ (I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r)Vec(U_N).\end{aligned} \tag{74}$$

The relation (74) must be verified to get a convenient reduced system for any input U (i.e. for any matrix $U_N\ \Phi_N(t)$). Therefore, it gives the following equation which must be verified by the parameters of the reduced system:

$$(I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B) = (I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r). \tag{75}$$

The parameters of the reduced system with the realization (A_r, B_r, C_r) derived [45] by minimizing the norm ξ of the difference between both parts of the equation (75). This unconstrained minimization can be led by using the functions of the optimization tools or genetic algorithms. Then, the reduced model determination is brought back to the following optimization problem: derive A_r, B_r and C_r such that they minimize:

$$\xi = \left\| \begin{array}{c}(I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B)\\ -(I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r)\end{array} \right\|. \tag{76}$$

4.3 Model order reduction of LTV systems

In this section, we consider the order model reduction of the LTV systems defined by the realization (45). The reduced order system is taken equal to r and the state space description of the reduced system is the following:

$$\begin{cases}\dot{\tilde{X}}(t) = \tilde{A}(t)\,\tilde{X}(t) + \tilde{B}(t)\,U(t),\\ \tilde{Y}(t) = \tilde{C}(t)\,\tilde{X}(t),\end{cases} \tag{77}$$

with $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$ varying in time with respective dimensions $r \times r$, $r \times m$ and $p \times r$. The description of the original system (45) using an orthogonal functions basis $\Phi_N(t)$ is given by the relations (55) and (57).

In the same manner, the variable in time parameters of the reduced LTV system will be defined by their projections on the orthogonal functions basis truncated to an order N:

$$\tilde{A}(t) \cong \sum_{i=0}^{N-1} \tilde{A}_{Ni}\,\varphi_i(t), \quad \tilde{B}(t) \cong \sum_{i=0}^{N-1} \tilde{B}_{Ni}\,\varphi_i(t), \quad \tilde{C}(t) \cong \sum_{i=0}^{N-1} \tilde{C}_{Ni}\,\varphi_i(t), \tag{78}$$

where $\tilde{A}_{Ni}$, $\tilde{B}_{Ni}$ and $\tilde{C}_{Ni}$ are constant with the same dimensions as $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$. Then, the description of the reduced-order model using the orthogonal functions basis can be written as:

$$\begin{aligned}\left[I_{r\times N}-\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{A}_{iN}\right)\right]Vec\left(\tilde{X}_N\right)=\\ \left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{B}_{iN}\right)Vec\left(U_N\right)\end{aligned}\tag{79}$$

and

$$Vec(\tilde{Y}_N)=\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes\tilde{C}_{iN}\right)Vec\left(\tilde{X}_N\right).\tag{80}$$

The equalization between the original system and the reduced system outputs can be expressed by the following relation : $\tilde{Y}_N=Y_N$ which can be written as

$$\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes C_{iN}\right)Vec\left(X_N\right)=\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes\tilde{C}_{iN}\right)Vec\left(\tilde{X}_N\right).\tag{81}$$

The substitution in (81) of $Vec(X_N)$ and $Vec(\tilde{X}_N)$ by their expressions (55) and (79) yields the following equality:

$$\begin{aligned}&\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes C_{iN}\right)\mathbb{G}^{-1}\mathbb{H}\,Vec\left(U_N\right)=\\ &\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes\tilde{C}_{iN}\right)Q\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{B}_{iN}\right)Vec\left(U_N\right),\end{aligned}\tag{82}$$

where $Q=\left[I_{r\times N}-\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{A}_{iN}\right)\right]^{-1}$. This relation must be verified for any input $U(t)$. Then, one obtains:

$$\begin{aligned}&\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes C_{iN}\right)\mathbb{G}^{-1}\mathbb{H}=\\ &\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes\tilde{C}_{iN}\right)\left[I_{r\times N}-\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{A}_{iN}\right]^{-1}\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{B}_{iN}\right)\end{aligned}\tag{83}$$

with the constant matrices

$$\mathbb{G}=I_{n\times N}-\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes A_{iN}\right),\ \ \mathbb{H}=\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes B_{iN}.$$

The parameters of the reduced order system $\tilde{A}_{iN}$, $\tilde{B}_{iN}$ and $\tilde{C}_{iN}$ can be derived by the minimization of the norm of the difference between both sides of the equation (83). Thus, the problem of the reduced model determination can be formulated as follows: determine $\tilde{A}_{iN}$, $\tilde{B}_{iN}$ and $\tilde{C}_{iN}$ in order to minimize

$$\xi=\left\|\begin{array}{c}\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes C_{iN}\right)\mathbb{G}^{-1}\mathbb{H}\\ -\left(\sum_{i=0}^{N-1}M_{iN}^T\otimes\tilde{C}_{iN}\right)Q\left(\sum_{i=0}^{N-1}\left(M_{iN}P_N\right)^T\otimes\tilde{B}_{iN}\right)\end{array}\right\|.\tag{84}$$

Remark 4.2 Stability of the reduced model.
The reduced-order model is determined such that the error between its output vector and that of the original full-order system is minimized regardless the input signals. When the time horizon of approximation is sufficiently large to take in consideration the transient response and the steady state, then one may conclude that if the original system is stable, the reduced order one will be also stable.

Remark 4.3 Number of the orthogonal basis functions.
The accuracy and validity of the reduced model depend on the number of the orthogonal basis functions. The higher is the number of the basis functions, the more accurate is the obtained reduced order model. However, the size of the matrices interfering in the computation of the reduced order parameters and the calculus time cost increase with respect to the orthogonal basis dimension. Thus, the number of the basis functions is generally chosen such that it satisfies a compromise between the accuracy of the searched model and the computational constraints.

5 Simulation study

In order to illustrate the availability of the developed approaches for system order reduction, we consider in this section different examples of high order systems that we will reduce using a set of shifted Legendre polynomials with order $N = 16$ as an orthogonal functions basis.

5.1 LTI SISO system example

We consider the LTI system studied in [46] and given by the following transfer function

$$G(s) = \frac{s^4+35s^3+291s^2+1093s+1700}{s^9+9s^8+66s^7+294s^6+1029s^5+2541s^4+4689s^3+5856s^2+4620s+1700}. \tag{85}$$

The order reduction of this system has been led by both approaches developed in paragraph 4.1 using the transfer function representation and paragraph 4.2 using the state space description. The reduced order is taken $r = 3$.

The first approach yields the following reduced transfer function:

$$H_r(p) = \frac{0.3298\,s^2 - 1.713\,s + 3.232}{s^3 + 3.05\,s^2 + 4.992\,s + 3.232}$$

and by the second approach technique, we obtain the reduced state space description (32) with

$$A_r = \begin{bmatrix} 0.01032 & 1.349 & 8.391 \\ 0.08643 & -0.1717 & 3.45 \\ -0.5394 & 0.07006 & -2.741 \end{bmatrix}, \quad B_r = \begin{bmatrix} 3.508 \\ 3.375 \\ -1.235 \end{bmatrix},$$

$$C_r = \begin{bmatrix} 2.338 & 1.138 & 9.559 \end{bmatrix}.$$

Figure 1 shows the step responses of the original system (85) and the reduced systems (by transfer function and by state space methods). It appears from these simulations that the behavior of the reduced models obtained by the developed methods in this paper is very close to that of the original system which shows the availability of the proposed techniques. This property can also be verified with different inputs applied to the reduced order model.

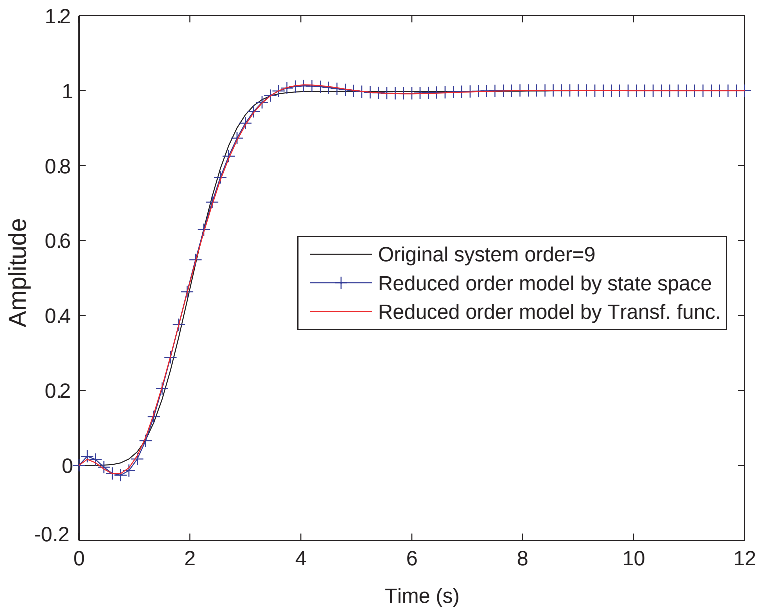

Figure 1: Step responses of the original system order $n = 9$ and reduced order models $r = 3$ obtained by the proposed techniques : starting from a state space realization and from a transfer function.

5.2 LTI MIMO system example: CD player

The proposed technique using orthogonal functions has been applied to the model of a CD player. This example is widely treated in many papers concerning MOR [49]. The considered model of CD player describes the dynamics between the lens actuator and the radial arm position as shown in Figure 2 and it is obtained using finite element approximation. Detailed description can be found in [50].

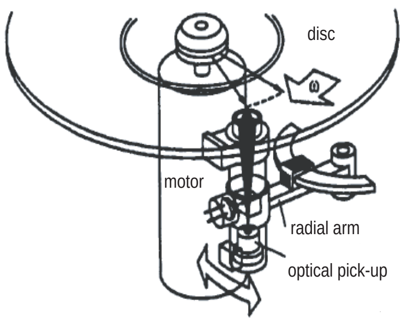

Figure 2: **CD player model. Source [50].**

The full-order model of the CD player is LTI MIMO having 120 states, 2 inputs and

2 outputs [21]. Gugercin and Antoulas proposed a reduced model having an order $r = 12$ by considering the system as LTI SISO. Chu and al. [51] reduced the LTI MIMO model to an order $r = 12$. The proposed technique has been applied to the full-order MIMO model and the reduction order is chosen to $r = 10$. The simulation of the error between the step responses of the original model and the reduced order ones obtained by the following techniques:
– The proposed technique using a shifted Legendre polynomials basis truncated to an order 10 on the time domain $[0, 50s]$,
– Balanced Hankel based (HMR) model reduction via square root method,
– Stochastic model truncation via Schur method (BST),
shows that the proposed reduction technique using orthogonal functions gives a minimal error converging to zero and the behavior of the obtained reduced model is close to the original 120-states full-model for any control input.

5.3 LTV system example

We consider now the LTV SISO system described by the state space realization (45) with

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 + e^{-t} & -2 + \frac{1}{8t} & -3 - 0.7\cos(-0.01t) & -2 + 0.5\cos(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 + 0.15\, cos(1.2t - 0.5) \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The variable in time system parameters are projected on the orthogonal functions basis. The obtained matrices $A_{iN} \in \mathbb{R}^{4\times 4}, i \in \{0, 1, \dots, 15\}$ and vectors $B_{iN} \in \mathbb{R}^{4\times 1}, i \in \{0, 1, \dots, 15\}$ resulting from this truncated development to the order 16, are used for computing the reduced order LTV model as shown in Section 4.3. The reduced order is chosen equal 2 ($r = 2$).

Figure 3 represents the time plot of the variable in time parameters of the reduced order model. Figure 4 shows that the step response of the reduced-order model (order $r = 2$) is close to the original system with order $n = 4$.

6 Conclusion

In this paper, new approaches have been introduced for the model order reduction of LTI and LTV systems using orthogonal functions as a tool of approximation. The proposed techniques can be applied to the order simplification of models defined either by an input-output relation or a state representation. Indeed, the projection of the input, the output and system variables on an orthogonal functions basis and the use of the operational matrices of integration and product have permitted the conversion of the system model from differential equations to algebraic ones. The minimization of the difference between the algebraic original system description and the algebraic reduced model have allowed the determination of the reduced order parameters.

Notice that the proposed order reduction techniques constitute an important contribution in the field of dynamical model simplification. These techniques come to reinforce

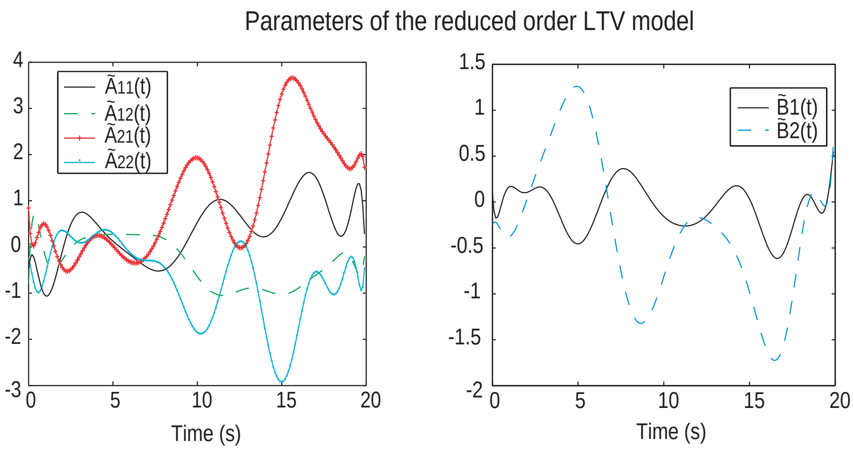

Figure 3: Time plots of the reduced order LTV system parameters obtained by the proposed technique.

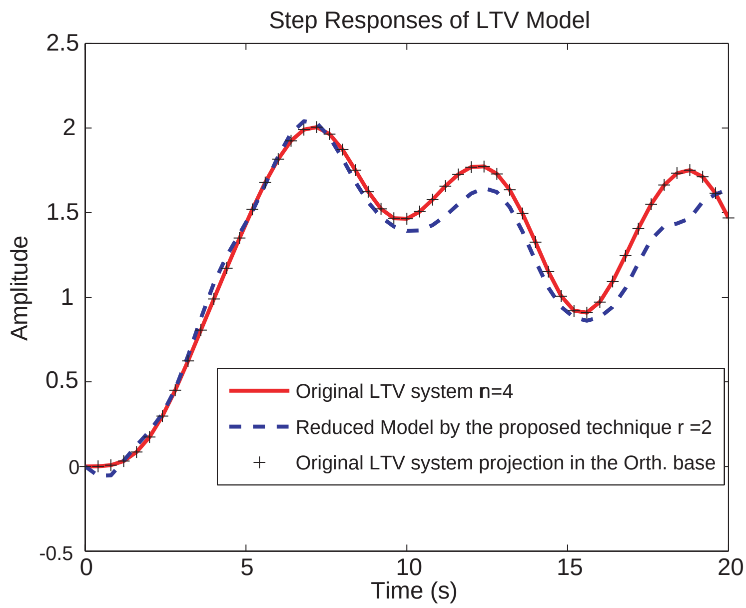

Figure 4: Step responses of the original LTV system, the projected system into Legendre shifted polynomials and the reduced order LTV model.

the existing approaches, especially in the case of LTV systems where only few methods with limited efficiency are published on the order reduction subject.

Finally, let us point out that the presented results in this paper are concerned with linear systems in both cases : time invariant and time variant parameters. However, it seems that they can be extended to some classes of nonlinear systems as bilinear systems.

This subject will be considered in our further works.

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A Separation Principle of a Class of Time-Varying Dynamical Systems

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Abstract: This paper studies the separation principle for a class of nonlinear time-varying dynamical systems whose dynamics are in general bounded in time. The resultant observer-based state feedback control guarantees practical stability of the state oscillation given that the system is both uniformly controllable and observable. Our separation principle relies on stability results for cascades systems.

Keywords: *nonlinear differential equations; stabilization; Lyapunov functions; practical stability; Riccatti equation.*

Mathematics Subject Classification (2010): 34H15, 37B25.

1 Introduction

The stability problem of nonlinear time-varying systems has attracted the attention of several authors and has produced many important results [8], [11], [12], [13] and [14] and the references therein. The problem of state trajectory control for nonlinear systems by output feedback has received much attention. For systems with non-periodically time-varying parameters, an output feedback control design is proposed in [4] for linear time-varying systems based on the gradient algorithm. In [5], a new design is proposed for the state feedback control of multivariable linear time-varying systems. The new design is based on inversion state transformation and a forward differential Riccati equation.

The condition that we impose on the globally stabilizing state feedback control law is that it does not vanish asymptotically for large values. Then, we will give a separation principle based on analysis results for cascaded systems, as done for instance in [1] , [2], [3], [6], [7], [9] and [10]. However, in contrast to [11] we stress that our results will be formulated for time-varying systems and hence are applicable to tracking problems. Moreover as mentioned above, in [15] the author imposes the more restrictive assumption ISS. Our cascades criteria lead to milder conditions.

The main contribution of this paper is the separation principle of nonlinear systems by a linear output feedback under a generalized conditions. A practical stability approach is obtained. Furthermore, we give an example to show the applicability of our result

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2 General Definitions

We consider the system

$$\dot{x}(t) = F(t, x), \quad x(t_0) = x_0, \tag{1}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ is the state. The function $F : [0, +\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x.

We now introduce the notions of uniform boundedness and uniform ultimate boundedness of a trajectory of (1) (see [8]).

Definition 2.1 The system (1) is uniformly bounded if for all $R_1 > 0$, there exists a $R_2 = R_2(R_1) > 0$, such that for all $t_0 \geq 0$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R_2, \quad \forall t \geq t_0.$$

Definition 2.2 The system (1) is uniformly ultimately bounded if there exists $R > 0$, such that for all $R_1 > 0$, there exists a $T = T(R_1)$, such that for all $t_0 \geq 0$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R, \quad \forall t \geq t_0 + T.$$

Let $r \geq 0$ and $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$. First, we give the definition of uniform stability and uniform attractivity of B_r.

Definition 2.3 (Uniform stability of B_r) (i) B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all $t_0 \geq 0$

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (1) are globally uniformly bounded.

Definition 2.4 (Uniform attractivity of B_r) B_r is globally uniformly attractive, if for all $\varepsilon > r$ and $c > 0$, there exists $T(\varepsilon, c) > 0$, such that for all $t_0 \geq 0$

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, c), \quad \|x_0\| < c.$$

Definition 2.5 The system (1) is globally uniformly practically asymptotically stable if there exists $r \geq 0$, such that B_r is globally uniformly stable and globally uniformly attractive.

Definition 2.6 B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k \geq 0$, such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)) + r.$$

The system (1) is globally practically uniformly exponentially stable if there exists $r > 0$, such that B_r is globally uniformly exponentially stable.

3 Basic Results

We consider now the following dynamical system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \\ y(t) = C(t)x(t), \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ is the system output, $u(t) \in \mathbb{R}^m$ is the control input and $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$ are matrices whose elements are bounded continuous or piecewise continuous functions of time. The function $f(t, x)$ is continuous, locally Lipschitz in x and there exists a non negative constant f_0, such that

$$\|f(t, 0)\| \leq f_0, \ \forall t \geq 0.$$

The corresponding nominal system is described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases} \tag{3}$$

3.1 Stabilization

We prove in this subsection the stabilization of system (2) by a state feedback control candidate. It is assumed that the system (3) is uniformly controllable (see [5]).

Definition 3.1 The pair $(A(t), B(t))$ is uniformly controllable if there exist Δ and another constant α depending on Δ, such that the controllability grammian $I(t - \Delta, t)$ satisfies

$$I(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau)B(\tau)B^T(\tau)\psi^T(t-\Delta, \tau)d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$ and is defined by

$$\frac{\partial \psi(t, t_0)}{\partial t} = A(t)\psi(t, t_0), \quad \psi(t, t) = I,$$

$$\psi(t, t_0)\psi(t_0, s) = \psi(t, s)$$

and

$$\psi(t_0, t) = \psi^{-1}(t, t_0).$$

We find from [5] the state feedback gain $K(t)$, such that the control input

$$u(t) = K(t)x(t) \tag{4}$$

with

$$K(t) = R_1^{-1}(t)\overline{B}^T(t)P(t),$$

where $P(t)$ is the solution of the forward differential Riccati equation

$$\dot{P}(t) = -\overline{A}^T(t)P(t) - P(t)\overline{A}(t) + R_1(t) - P(t)\overline{B}(t)R_2^{-1}(t)\overline{B}^T(t)P(t), \quad P(0) = P_0 > 0, \tag{5}$$

in which
$$\overline{A}(t) = -T(x)A(t)T^{-1}(x), \quad \overline{B}(t) = T(x)B(t),$$
with
$$T(x) = I - 2\frac{x(t)x^T(t)}{x^T(t)x(t)},$$
$R_1(t) > 0$, $R_2(t) > 0$ and $R_1(t), R_2(t), R_1^{-1}(t), R_2^{-1}(t)$ are all uniformly bounded.

Proposition 3.1 *(see [6]) Consider the system (3) and the state feedback control (4) and (5), if the system (3) is uniformly controllable, the closed-loop system is globally exponentially stable.*

Notice that, the system (3) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally exponentially stable, then from [6] we have for all positive definite symmetric matrix $Q_1(t)$,
$$Q_1(t) \geq c_1 I > 0, \quad \forall t \geq 0,$$
there exists a positive definite symmetric matrix $P_1(t)$,
$$0 < c_2 I < P_1(t) < c_3 I, \quad \forall t \geq 0,$$
which satisfies
$$A_K^T(t)P_1(t) + P_1(t)A_K(t) + \dot{P}_1(t) = -Q_1(t), \quad \text{where } A_K(t) = A(t) + B(t)K(t). \tag{6}$$

Now, we prove the global practical uniform stabilizability of (2). We shall suppose the following.
$(\mathcal{A}_1)$ Assume that
$$\|f(t,x) - f(t,y)\| \leq \gamma(t)\|x - y\| + \delta(t) + \varepsilon, \quad \forall t \geq 0, \ \forall \ x, y \in \mathbb{R}^n, \tag{7}$$
where $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are continuous non-negative functions with
$$\int_0^{+\infty} \gamma(s)\, ds \leq M_\gamma < +\infty$$
and
$$\int_0^{+\infty} \delta^2(s)\, ds \leq M_\delta < +\infty.$$

Theorem 3.1 *Under assumption $(\mathcal{A}_1)$, the system (3) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable.*

Proof. Let us consider the Lyapunov function $V(t, x(t)) = x^T(t)P_1(t)x(t)$. The derivative of V along the trajectories of system (2) is given by
$$\dot{V}(t, x(t)) \leq -\left(\frac{c_1}{c_3} - \frac{2c_3\gamma(t)}{c_2}\right)V(t, x(t)) + 2\frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)\sqrt{V(t, x(t))}.$$
Use the following change $v(t) = \sqrt{V(t, x(t))}$. Then, $v(t)$ satisfies the following estimation
$$v(t) \leq v(t_0)e^{-\displaystyle\int_{t_0}^t \alpha(s)\, ds} + \frac{c_3}{\sqrt{c_2}}\left(\int_{t_0}^t (\delta(s) + f_0 + \varepsilon)e^{\displaystyle\int_{t_0}^s \alpha(\tau)\, d\tau} ds\right) e^{-\displaystyle\int_{t_0}^t \alpha(s)\, ds}$$

with

$$\alpha(t) = \frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}\cdot$$

A simple computation shows that,

$$\left(\int_{t_0}^{t} (\delta(s) + f_0 + \varepsilon) e^{\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds\right) e^{-\int_{t_0}^{t} \alpha(s)\, ds} \leq \left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0 + \varepsilon)\frac{c_3}{c_1}\right) e^{\frac{c_3 M_\gamma}{c_2}}.$$

Thus, we obtain

$$v(t) \leq v(t_0) e^{\frac{c_3 M_\gamma}{c_2}} e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0 + \varepsilon)\frac{c_3}{c_1}\right) e^{\frac{c_3 M_\gamma}{c_2}}.$$

It follows that

$$\|x(t)\| \leq \sqrt{\frac{c_3}{c_2}} e^{\frac{c_3 M_\gamma}{c_2}} \|x_0\| e^{-\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0 + \varepsilon)\frac{c_3}{c_1}\right) e^{\frac{c_3 M_\gamma}{c_2}}.$$

This implies the global uniform exponential stability of B_κ with

$$\kappa = \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0 + \varepsilon)\frac{c_3}{c_1}\right) e^{\frac{c_3 M_\gamma}{c_2}}.$$

Hence, the system (2) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable. □

3.2 Conception of the observer

For the concept of observer, we aim at simplifying the design of this system by exploiting the linear form of the nominal system. The system (3) is assumed to be uniformly observable (see [5]).

Definition 3.2 The pair $(A(t), C(t))$ is uniformly observable if there exist Δ and another constant α depending on Δ, such that the observability grammian $J(t-\Delta, t)$ satisfies

$$J(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau) C(\tau) C^T(\tau) \psi^T(t-\Delta, \tau) d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$.

Definition 3.3 (Practical exponential observer) A practical exponential observer for (2) is a dynamical system which has the following form

$$\dot{\hat{x}}(t) = F(t, \hat{x}(t), u(t)) - L(t)(C(t)\hat{x}(t) - y(t)), \tag{8}$$

where $L(t)$ is the gain matrix and the error equation with $e(t) = \hat{x}(t) - x(t)$, is given by

$$\dot{e}(t) = F(t, \hat{x}(t), u(t)) - F(t, x(t), u(t)) - L(t)C(t)e(t) \tag{9}$$

a Luenberger observer which is expected to produce an estimation of the state in the sense of global practical exponential stability. It means that, the system (9) is globally practically uniformly exponentially stable and the following estimation holds:

$$\|e(t)\| \leq \lambda_1 \|e(t_0)\| \; e^{-\lambda_2 (t-t_0)} + r, \quad \forall\, t \geq t_0,$$

with $\lambda_1, \ \lambda_2, \ r > 0$.

To design an observer, we shall consider the system

$$\dot{\hat{x}} = A(t)\hat{x}(t) + B(t)u(t) + f(t, \hat{x}(t)) - L(t)(C(t)\hat{x}(t) - y(t)), \tag{10}$$

where $\hat{x}(t)$ is the state estimate of $x(t)$ and $L(t) \in \mathbb{R}^{n\times p}$ is the observer feedback gain to be determined so that $\hat{x}(t)$ tends to $x(t)$ exponentially. One such design is the well known Kalman filter design ([3]), in which the observer feedback gain $L(t)$ is chosen as

$$L(t) = Q(t)C^T(t)V_2^{-1}(t), \tag{11}$$

where $Q(t)$ satisfies a forward differential Riccati equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t), \quad Q(0) = Q_0 > 0, \tag{12}$$

in which $V_1(t) > 0$, $V_2(t) > 0$ and $V_1(t)$, $V_2(t)$, $V_1^{-1}(t)$, $V_2^{-1}(t)$ are all uniformly bounded. The error equation is given by

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A(t) - L(t)C(t))e(t) + f(t, \hat{x}(t)) - f(t, x(t)). \tag{13}$$

Proposition 3.2 *(see [9]) Consider the system (3) and the observer (11) and (12). If $(A(t), C(t))$ is uniformly observable, the closed-loop system is globally exponentially stable.*

Notice that, if the system (3) in closed-loop with the observer (11) and (12) is globally uniformly exponentially stable, then for all positive definite symmetric matrix $Q_2(t)$,

$$Q_2(t) \geq b_1 I > 0, \quad \forall t \geq 0,$$

there exists a positive definite symmetric matrix $P_2(t)$,

$$0 < b_2 I < P_2(t) < b_3 I, \quad \forall t \geq 0,$$

which satisfies

$$A_L^T(t)P_2(t) + P_2(t)A_L(t) + \dot{P}_2(t) = -Q_2(t), \quad \text{where } A_L(t) = A(t) - L(t)C(t). \tag{14}$$

Theorem 3.2 *Under assumption ($\mathcal{A}_1$), the system* (10) *is a practical exponential observer for the system (2).*

Proof. Let us consider the Lyapunov function $Y(t, e(t)) = e^T(t)P_2(t)e(t)$. The derivative of Y along the trajectories of system (13) is given by

$$\dot{Y}(t, e(t)) \leq -\left(\frac{b_1}{b_3} - \frac{2b_3}{b_2}\gamma(t)\right) Y(t, e(t)) + 2\frac{b_3}{\sqrt{b_2}}(\delta(t) + \varepsilon)\sqrt{Y(t, e(t))}.$$

Use the following change $y(t) = \sqrt{Y(t, e(t))}$. Then, $y(t)$ satisfies the following estimation

$$y(t) \leq y(t_0) e^{-\int_{t_0}^{t} \beta(s)\, ds} + \frac{b_3}{\sqrt{b_2}} \left(\int_{t_0}^{t} (\delta(s) + \varepsilon) e^{\int_{t_0}^{s} \beta(\tau)\, d\tau} ds \right) e^{-\int_{t_0}^{t} \beta(s)\, ds}$$

with

$$\beta(t) = \frac{b_1}{2b_3} - \frac{b_3 \gamma(t)}{b_2}\cdot$$

A simple computation shows that,

$$\left(\int_{t_0}^{t} (\delta(s) + \varepsilon) e^{\int_{t_0}^{s} \beta(\tau)\, d\tau} ds \right) e^{-\int_{t_0}^{t} \beta(s)\, ds} \leq \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

Thus, we obtain

$$y(t) \leq y(t_0) e^{\frac{b_3 M_\gamma}{b_2}} e^{-\frac{b_1}{2b_3}(t - t_0)} + \frac{b_3}{\sqrt{b_2}} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

Hence,

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}} e^{\frac{b_3 M_\gamma}{b_2}} \|e(t_0)\| e^{-\frac{b_1}{2b_3}(t - t_0)} + \frac{b_3}{b_2} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

This implies the global uniform exponential stability of B_η with

$$\eta = \frac{b_3}{b_2} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

We deduce that, the system (13) is globally practically exponentially stable. Hence, the system (10) is a practical exponential observer for the system (2). □

3.3 Separation principle

Now, we obtain a separation principle for (2). We consider the system (2) controlled by the linear feedback control $u(t) = K(t)\hat{x}(t)$ and estimated with the observer (10).

Theorem 3.3 *Under assumption ($\mathcal{A}_1$), the system*

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + f(t, \hat{x}(t)) - L(t)C(t)e(t), \\ \dot{e}(t) = (A(t) - L(t)C(t))e(t) + f(t, \hat{x}(t)) - f(t, x(t)), \end{cases} \tag{15}$$

is globally practically uniformly exponentially stable.

Proof. In order to study the stabilization problem via an observer, we consider the system

$$\dot{\hat{x}}(t) = \psi(t, \hat{x}(t)) - L(t)C(t)e(t), \tag{16}$$

where

$$\psi(t, \hat{x}(t)) = (A(t) + B(t)K(t))\hat{x}(t) + f(t, \hat{x}(t)).$$

Let us consider the Lyapunov function $v(t, \hat{x}(t)) = \sqrt{\hat{x}^T(t)P_1(t)\hat{x}(t)}$, which satisfies

$$\sqrt{c_2}\|\hat{x}(t)\| \leq v(t, \hat{x}(t)) \leq \sqrt{c_3}\|\hat{x}(t)\|,$$

$$\frac{\partial v}{\partial t}(t, \hat{x}(t)) + \frac{\partial v}{\partial \hat{x}(t)}\psi(t, \hat{x}(t)) \leq \ -\alpha(t)v(t, \hat{x}(t)) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)$$

and

$$\left\| \frac{\partial v}{\partial \hat{x}}(t, \hat{x}(t)) \right\| \leq \ \frac{c_3}{\sqrt{c_2}},$$

where

$$\alpha(t) = \frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}\cdot$$

The derivative of v along the trajectories of system (16) is given by

$$\begin{aligned}\dot{v}(t, \hat{x}(t)) &\leq -\alpha(t)v(t, \hat{x}(t)) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)\\ &+ \frac{c_3}{\sqrt{c_2}}\|L(t)C(t)\| \left(\sqrt{\frac{b_3}{b_2}}\ e^{\dfrac{b_3M_\gamma}{b_2}} \|e(t_0)\| e^{-\dfrac{b_1}{2b_3}(t-t_0)} \right.\\ &\left. + \frac{b_3}{b_2}\left(\sqrt{\frac{b_3M_\delta}{b_1}} + 2\varepsilon\frac{b_3}{b_1}\right)\ e^{\dfrac{b_3M_\gamma}{b_2}} \right).\end{aligned}$$

Since $L(t)C(t)$ is bounded for all $t \geq t_0$, then there exists $R_1 > 0$, such that

$$\|L(t)C(t)\| \leq R_1, \quad \forall t \geq t_0 \geq 0.$$

Then

$$\dot{v}(t, \hat{x}(t)) \leq -\alpha(t)v(t, \hat{x}(t)) + \lambda\|e(t_0)\|e^{-\dfrac{b_1}{2b_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}}\delta(t) + R$$

with

$$\lambda = \frac{c_3}{\sqrt{c_2}}R_1\sqrt{\frac{b_3}{b_2}}\ e^{\dfrac{b_3M_\gamma}{b_2}}$$

and

$$R = \frac{c_3}{\sqrt{c_2}}(f_0 + \varepsilon) + \frac{b_3c_3}{b_2\sqrt{c_2}}R_1\left(\sqrt{\frac{b_3M_\delta}{b_1}} + 2\varepsilon\frac{b_3}{b_1}\right)\ e^{\dfrac{b_3M_\gamma}{b_2}}.$$

Using the following change

$$y(t) = v(t)e^{\displaystyle\int_{t_0}^{t}\alpha(s)\,ds},$$

we obtain

$$y(t) \leq y(t_0) + \frac{c_3}{\sqrt{c_2}} \int_{t_0}^{t} \delta(s) e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds + \lambda \|e(t_0)\| \int_{t_0}^{t} e^{-\displaystyle\frac{b_1}{2b_3}(s-t_0)} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds$$
$$+ R \int_{t_0}^{t} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds.$$

Then

$$v(t) \leq v(t_0) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds} + \frac{c_3}{\sqrt{c_2}} \left(\int_{t_0}^{t} \delta(s) e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}$$
$$+ \lambda \|e(t_0)\| \left(\int_{t_0}^{t} e^{-\displaystyle\frac{b_1}{2b_3}(s-t_0)} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}$$
$$+ R \left(\int_{t_0}^{t} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}.$$

A simple computation shows that

$$v(t) \leq v(t_0) e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\displaystyle\frac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}}$$
$$+ \lambda \|e(t_0)\| \frac{2b_3 c_3}{c_1 b_3 - b_1 c_3}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\displaystyle\frac{b_1}{2b_3}(t-t_0)} + 2\frac{R c_3}{c_1}.$$

Let

$$\theta = \min\left(\frac{c_1}{2c_3},\ \frac{b_1}{2b_3} \right).$$

Then

$$v(t) \leq \sqrt{c_3} \|\hat{x}_0\| e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\theta(t-t_0)} + \frac{2\lambda b_3 c_3}{c_1 b_3 - b_1 c_3}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} \|e(t_0)\|\, e^{-\theta(t-t_0)}$$
$$+ \frac{c_3}{\sqrt{c_2}} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} + 2\frac{R c_3}{c_1}.$$

Let

$$k = \max\left(\sqrt{c_3},\ \frac{2\lambda b_3 c_3}{c_1 b_3 - b_1 c_3} \right).$$

Hence,

$$\|\hat{x}(t)\| \leq \frac{k}{\sqrt{c_2}} e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} (\|\hat{x}_0\| + \|e(t_0)\|)\, e^{-\theta(t-t_0)} + \frac{c_3}{c_2} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} + 2\frac{R c_3}{c_1 \sqrt{c_2}}.$$

Then, the cascade system (15) is globally practically uniformly exponentially stable. □

Example 3.1 Consider the system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \\ y(t) = C(t)x(t) \end{cases} \tag{17}$$

with $x(t) = (x_1(t), x_2(t))^T$,

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 1 & e^{-2t} \end{pmatrix}$$

and

$$f(t, x(t)) = e^{-kt}x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad k > 0.$$

The proposed control (4) is then applied to the system with the following design parameters $P(0) = I$, $R_1(t) = I$, $R_2(t) = I$ in (5). The matrix $P(t)$ is calculated by solving the Ricatti equation (5). The function $f(t, x(t))$ is continuous and satisfies assumption $(\mathcal{A}_1)$ because

$$\int_0^{+\infty} e^{-kt} = \frac{1}{k}, \quad k > 0.$$

We conclude that the system (2) can be globally practically uniformly exponentially stable. The observer feedback gain $L(t)$ is chosen as (11) by solving the Riccati equation (12). We conclude that the system (10) is a practical exponential observer for the system (17). Thus, Theorem 3.3 is satisfied. We conclude that, the system (15) is globally uniformly practically exponentially stable.

4 Conclusion

This paper presents a separation principle for a class of nonlinear controls systems. It is shown that the system can be globally exponentially stabilizable by means of an estimated state feedback control given by an observer design.

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AIDTC Techniques for Induction Motors

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Abstract: Artificial intelligent systems are widely used in control applications. The proposed techniques controller of Induction Motor are used to reduce torque and flux ripples producing by the hysteresis comparators in the conventional DTC at very low speed. In addition the proposed speed controllers are presented in this paper to guarantee that the motor speed converges very well to the desired speed. The simulation results confirm the validity of the proposed techniques.

Keywords: *ANN; DTC; fuzzy logic; PI; IM; speed controller.*

Mathematics Subject Classification (2010): 03B52, 93C42, 94D05.

1 Introduction

Induction motors have been widely applied in industry because of the advantages of simple construction, ruggedness, reliability, low cost, and minimum maintenance. The recent challenge is to apply induction motors to precision servo machines such as robots and NC machines. The problem arises from the load variation during the motion of the motor [3]. The apparition of the field oriented control (FOC) made induction machine drives a major candidate in high performance motion control applications. However, the complexity of field oriented algorithms led to the development in recent years of many studies to find out different solutions for the induction motor control having the features of precise and quick torque response [4]. Direct torque control (DTC) of induction machines (IM) is a powerful control method for motor drives. Featuring a direct control of the stator flux and torque instead of the conventional current control technique, it

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provides a systematic solution to improve operating characteristics of the motor and the voltage inverter source [1, 5]. It has emerged over the last decade to become one possible alternative to the well-known Vector Control of Induction Machines. Its main characteristic is the good performance, obtaining results as good as the classical vector control but with several advantages based on its simpler structure and control diagram [6]. In addition, direct torque control minimizes the use of machine parameters, so it is very little sensible to the parameters variation [7]. Several solutions with modified DTC are presented in the literature. Due to its simple structure, DTC can be easily integrated with an artificial intelligence control strategy [8]. The fuzzy logic solution of flux and torque control is given in [9, 13] During years, many solutions have focused to reduce high level of torque and flux ripples producing by the hysteresis comparators in the traditional DTC. Both open and closed loop speed and position estimators are widely used in literature [8]. This paper investigates the control by neural network and fuzzy logic in order to reduce torque ripples and to compare the traditional DTC with DTC based on Artificial Intelligent systems (DTAIC) with and without speed regulation.

2 Direct Torque Control Strategy

Direct torque control (DTC) of induction motors has aroused significant interest among researchers looking for an efficient and high performance ac motor drive [10].

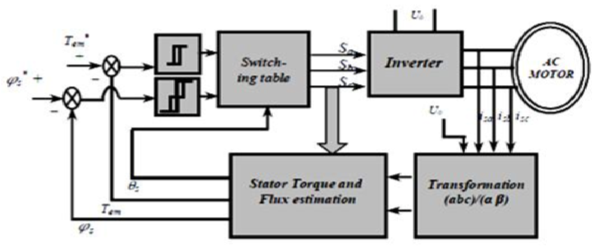

Figure 1: Schematic diagram of CDTC strategy.

The scheme of the classical DTC (Figure 1) consists of two hysteresis controllers. Stator flux controller defines the time duration of the active voltage vectors, which moves the stator flux along the reference trajectory, and torque controller determinates the time duration of the inverter zero states, which keep the motor torque in the defined range by hysteresis value tolerance band. Finally, in every sampling time the voltage vector selection block defines the inverter switching state ($S_A,\ S_B,\ S_C$), which reduces the instantaneous flux and torque errors.

3 Estimated Torque and Flux

In the DTC, the stator flux vector is estimated by taking the integral of difference between the input voltage and the voltage drop across the stator resistance given by [11].

The component α and β of vector φ_s can be obtained:

$$\varphi_{s\alpha} = \int_0^t (V_{s\alpha} - Ri_{s\alpha})\, dt, \quad \varphi_{s\beta} = \int_0^t (V_{s\beta} - Ri_{s\beta})\, dt. \tag{1}$$

Stator flux amplitude and phase angle are calculated in expression (2):

$$\varphi_s = \sqrt{\varphi_{s\alpha}^2 + \varphi_{s\beta}^2}\,\arg\varphi_s = \arctan\left(\frac{\varphi_{s\beta}}{\varphi_{s\alpha}}\right). \tag{2}$$

Once the two components of flux are obtained, the electromagnetic torque can be estimated from the relationship cited below:

$$T_e = \frac{3}{2}p(\varphi_{s\alpha}i_{s\beta} - \varphi_{s\beta}i_{s\alpha}). \tag{3}$$

4 Proposed DTAIC

To resolve torque ripple problem, this paper propose a direct torque control based on artificial intelligent of induction motor to replace the hysteresis comparators and switching table in open loop and artificial intelligent speed controller to replace the PI controller in closed loop as shown Figure 2.

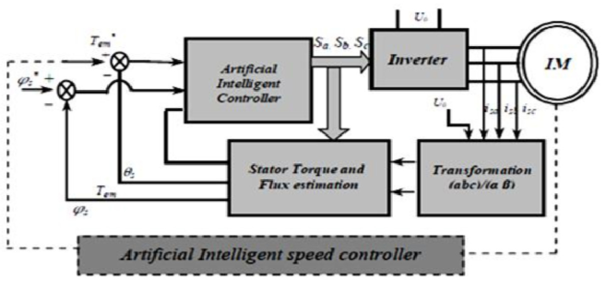

Figure 2: Schematic diagram of DTC-PMSM control.

The proposed IA has three inputs:

1. ***Flux linkage errors:*** The error of flux linkage E_φ is related value of stator's flux φ_s^* and real value of stator's φ_s,

$$E_{\varphi_s} = \varphi_s^* - \varphi_s. \tag{4}$$

2. ***Electromagnetic torque errors:*** Error of torque E_{Te} is related to desired torque value T_e^* and real torque value T_e,

$$E_{Te} = T_e^* - T_e. \tag{5}$$

3. ***Angle of flux linkage θs:*** The angle of flux linkage θ_s is an angle between stator's flux Φs and a reference axis is defined by the equation

$$\theta_s = \arctan\left(\frac{\varphi_{s\beta}}{\varphi_{s\alpha}}\right). \tag{6}$$

The output is: 1. The Boolean switching controls (***Sa, Sb, Sc***).

So we have three controllers based on artificial intelligent as fuzzy, neural fuzzy and neural networks.

4.1 Fuzzy controller

Fuzzy control is a way for controlling a system without the need of knowing the plant mathematic model. It uses the experience of people's knowledge to form its control rule base [12]. The fuzzy logic controller is comprised of fuzzification part, fuzzy inference part and defuzzification part [13].

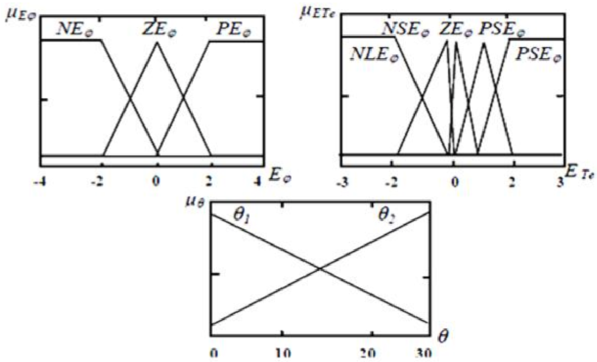

Figure 3: Membership distribution of fuzzy variable.

Fuzzification: The fuzzification is the process of a mapping from measured or estimated input to corresponding fuzzy set in the universe of discourse as shown in Figure 3.

Fuzzy inference: The fuzzy reasoning used is Mamdani's method. The fuzzy control rule-base is shown in Table I.

Deffuzification: The Mamdani's minimum operation rule is used as the interface method and, the output obtained by the center of gravity method used for defuzzification.

	P	*Z*	*N*			*P*	*Z*	*N*
PL	*V1*	*V2*	*V2*		*PL*	*V1*	*V1*	*V2*
PS	*V1*	*V2*	*V3*		*PS*	*V1*	*V2*	*V2*
ZE	-	-	-		*ZE*	-	-	-
NS	*V6*	-	*V4*		*NS* *NL*	*V5*	-	*V4*
NL	*V6*	*V5*	*V5*			*V5*	*V5*	*V4*

Table 1: Set of fuzzy rules.

4.2 Neural fuzzy controller

Artificial Neural Networks (ANNs) tend to imitate the human learning process in a very limited way by a computer program or electronic circuit. The ANNs do not require the mathematical model of the system [14], they just use experimental or simulated input/output data to be trained.

The Neural Fuzzy proposed in this paper based on desired fuzzy output, consists of three input nodes (torque error, flux error and stator flux angle); one hidden layer and an output layer with three neurons with (3-11-3).

4.3 Neural network controller

The third proposed controller is composed of three principal layers: the input layer, the hidden layer and the output layer, as shown Figure 4 with 10 neurons in hidden layer.

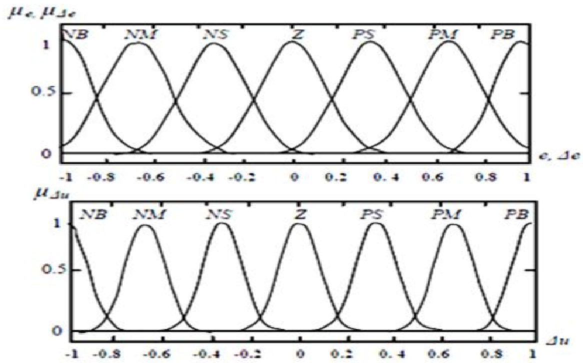

Figure 4: Architecture of neural network controller.

5 Proposed Speed Controller Based on Artificial Intelligent (AI)

The motor speed can be controlled indirectly by controlling the torque with a traditional controller such as PI or controller based on Artificial Intelligent including Fuzzy Logic and Neural Networks.

So this paper presents direct torque control of induction motor based on AI in closed loop, under transient and steady state uncertainties caused by the variation in load torque, the proposed speed controllers are PI, Fuzzy Logic, Neural Fuzzy logic and Neural networks.

5.1 The Proportional Integral (PI) controllers

The PI controller for the above system can be expressed as

$$u = K_p e(t) + k_i \int e(t)dt, \tag{7}$$

where Kp and Ki are the proportional and integral gain constants [15]. The speed error e(k) and $\triangle e(k)$ are defined as:

$$e(k) = \omega_r^*(k) - \omega_r(k), \tag{8}$$

$$\Delta e(k) = e(k+1) - e(k). \tag{9}$$

5.2 Fuzzy controller

The input linguistic variables speed error e(k), change in speed error $\triangle e(k)$ and output linguistic variable u(k) membership functions will be divided into seven fuzzy sets with the linguistic values NL (negative large), NM (negative medium), NS (negative small), ZE (zero), PS (positive small), PM (positive medium), PL (positive large) respectively and are given in Table 2.

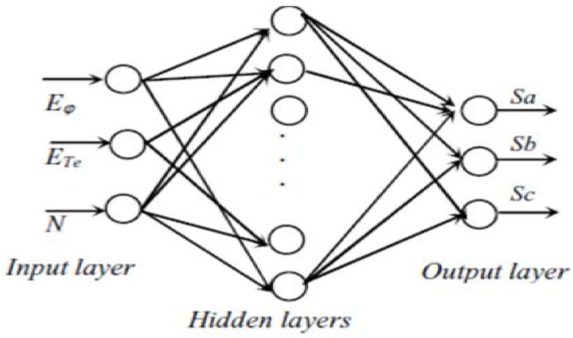

Figure 5: Membership functions of the Fuzzy controller.

	e	PL	PM	PS	ZE	NS	NM	NL
Δe								
PL		PL	PL	PL	PL	PM	PS	ZE
PM		PL	PL	PL	PM	PS	ZE	NS
PS		PL	PL	PM	PS	ZE	NS	NM
ZE		PL	PM	PS	ZE	NS	NM	NL
NS		PM	PS	ZE	NS	NM	NL	NL
NM		PS	ZE	NS	NM	NL	NL	NL
NL		ZE	NS	NM	NL	NL	NL	NL

Table 2: Fuzzy control table.

5.3 Neural fuzzy controller

This proposed controller is obtained by learning ANN controller based on data inputs/outputs of fuzzy controller cited below, the NF controller has two inputs/one output (error and variation error of speed rotor/ torque command) as shown in Figure 6 and the hidden layer contains 25 neuron.

5.4 Neural network

The controller presented in this section has also two inputs/one output and one hidden layer (Figure 7) contains 20 neurons. The sum squared error falls under 5.34.e-5 after 2201 iterations.

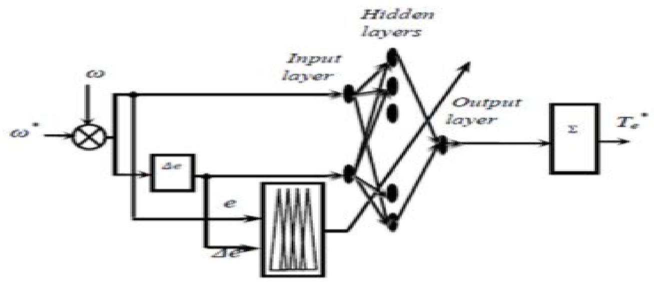

Figure 6: Basic structure of neural fuzzy speed controller.

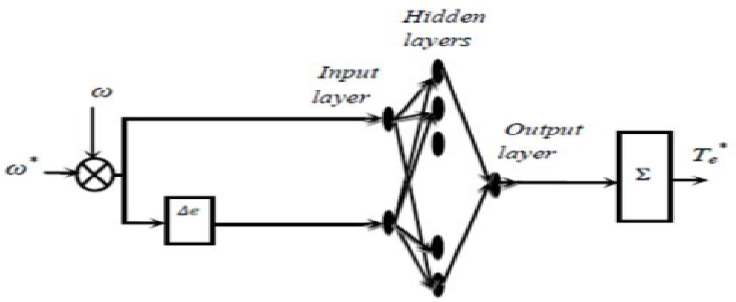

Figure 7: Basic structure of neural speed controller.

6 Simulation Result

A simulation configured SIMULINK environment has been carried out for the evaluation of the proposed system. The parameters of the induction motors used in this study are shown in Table 3.

Parameters of IM setting		
P	Power	1.5 Kw
Rs	Stator resistance	4.85 Ohm
Rr	rotor resistance	4.85 Ohm
j	Inertia	0.031Kg.m2
f	frequency	50Hz
P	Poles	2

Table 3: Parameters of IM setting.

The constant load torque of 10Nm and a constant flux of 1.1Wb were used, Figure 8 shows comparison between fuzzy (DTFLC), neural fuzzy (DTNFC), neural network (DTNNC) and traditional DTC (CDTC), in this figure the stator current is nearly sinusoidal and stator flux trajectory is evidently circular. It can be seen that the torque and flux ripple is significantly reduced by using fuzzy logic controller compared to neural network, neural fuzzy logic and traditional controller in open loop, we have nearly the same remarks in closed loop, the estimated values of electromagnetic torque and stator flux track the references with load torque applied (10Nm) as shown (Figure 9, Figure 10 and Figure 11) and the ripple in torque, current and flux is less by using fuzzy controller compared to the others controller in Figure 11 the rotor speed flow the reference quickly and without overshoot by using neural and neural based on fuzzy logic. We find finally that the neural speed controller based on fuzzy logic gives better performances.

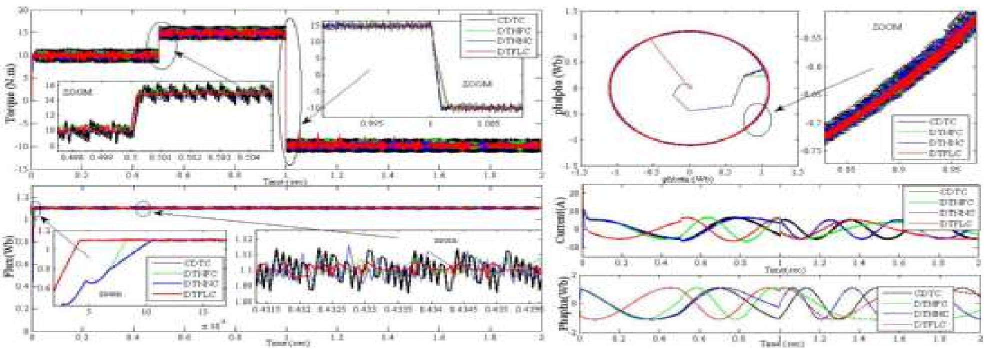

Figure 8: Responses of flux, torque and stator current with a load torque applied.

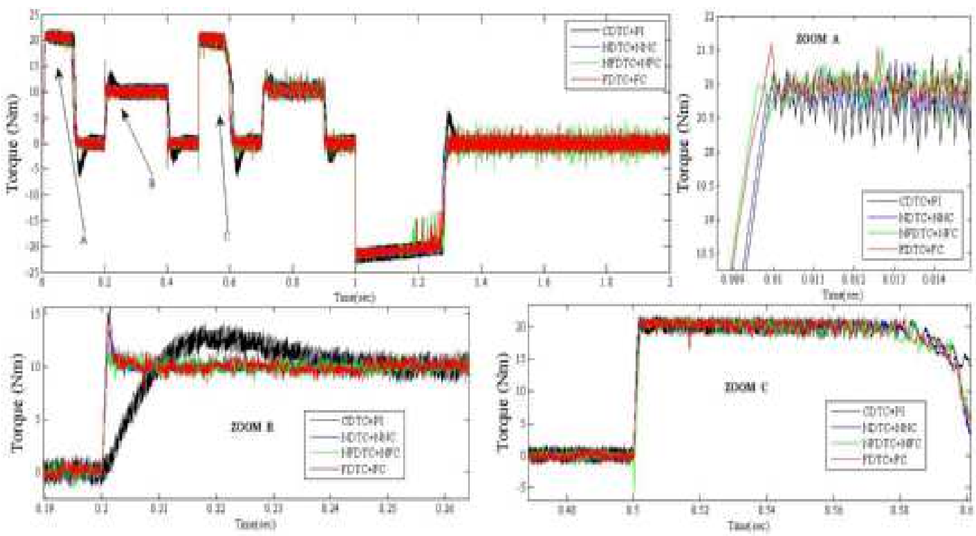

Figure 9: Torque response.

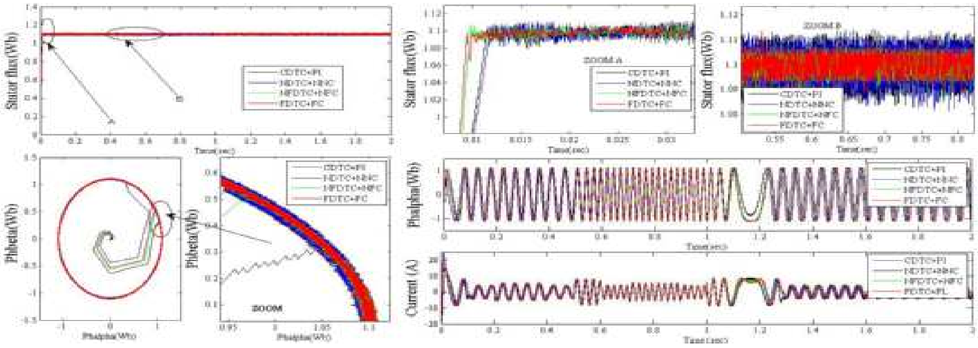

Figure 10: Rotor speed response.

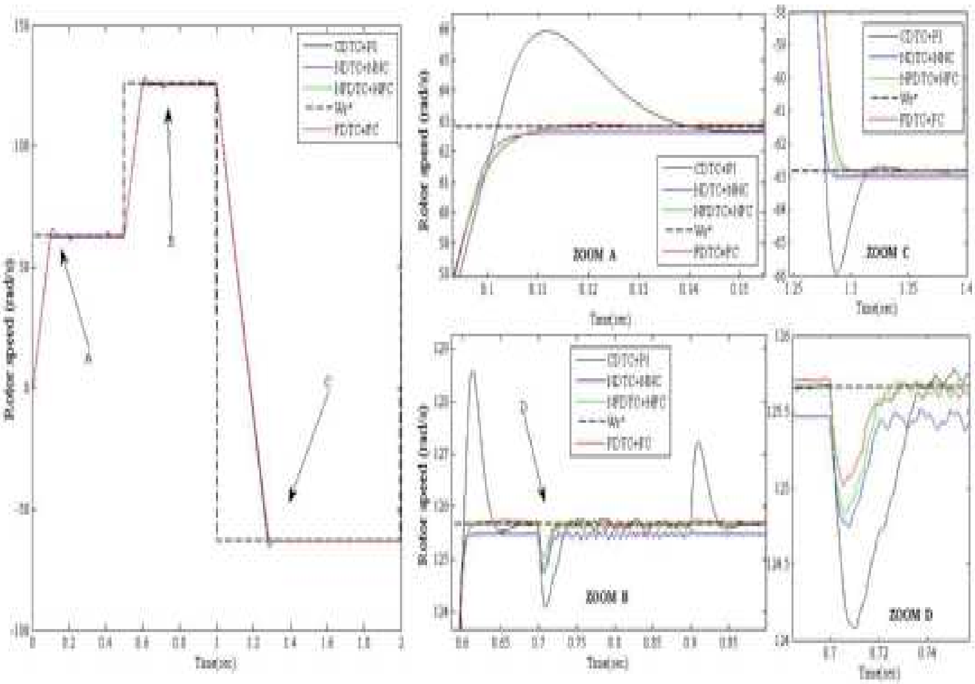

Figure 11: Stator flux response.

7 Conclusion

This paper presents comparative study of DTC-IM with and without speed controller based on artificial intelligent system such as fuzzy logic, neural network and neural fuzzy logic.

The obtained simulation results show good performance of proposed methods which are better than conventional method. The motor reaches the reference speed rapidly and without overshoot, load disturbances are rapidly rejected; torque and flux ripples are significantly attenuated.

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On Parameterized Lyapunov and Control Lyapunov Functions for Discrete-Time Systems

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Abstract: This paper deals with the existence and synthesis of parameterized-(control) Lyapunov functions (p-(C)LFs) for discrete-time nonlinear systems that are possibly subject to constraints. A p-LF is obtained by associating a finite set of parameters to a standard LF. A set-valued map, which generates admissible sets of parameters, is defined such that the corresponding p-LF enjoys standard Lyapunov properties. It is demonstrated that the so-obtained p-LFs offer non-conservative stability analysis conditions, even when Lyapunov functions with a particular structure, such as quadratic forms, are considered. Furthermore, possible methods for synthesizing p-CLFs are discussed. These methods require solving on-line a low-complexity convex optimization problem.

Keywords: *difference equations; asymptotic stability; Lyapunov methods; convex optimization.*

Mathematics Subject Classification (2010): 39A30, 37B25, 37L25.

1 Introduction

The problems considered in this paper are stability analysis and stabilizing controller synthesis via Lyapunov methods for discrete-time nonlinear systems that are possibly subject to constraints. It is well known that such methods rely on the existence and construction of a proper Lyapunov function (LF) [8, 11, 12, 19] and control Lyapunov function (CLF) [1, 9, 24], respectively. Unfortunately, the construction of LFs for general nonlinear systems is a very challenging problem. In particular, even linear systems with hard state/input constraints pose a non-trivial challenge to finding a non-conservative LF. As such, it would be desirable to identify a non-conservative class of Lyapunov

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functions that leads to a tractable implementation for nonlinear systems. As our interest lies mainly within the discrete-time domain, the following brief account of advances in Lyapunov methods is restricted to discrete-time systems.

For unconstrained linear systems, existence of a LF with a fixed structure (e.g., quadratic or polyhedral) and parameter set (e.g., common weight matrix for all states) is necessary and sufficient for stability. However, when constraints are present, the unconstrained solution usually provides a conservative domain of attraction. Moreover, when other classes of nonlinear systems are considered, such as systems with polytopic uncertainty, piecewise affine (PWA) or switched systems, existence of a fixed LF with a common set of parameters is known to be conservative. For such relevant classes of nonlinear systems it was already shown that enriching the set of admissible parameters for the Lyapunov weight matrix leads to a less conservative LF, even when the structure of the LF is fixed. For example, parameter dependent quadratic Lyapunov functions were constructed in [6] for uncertain linear systems by parameterizing the weight matrix of a quadratic LF as a function of the uncertain parameter. This idea was further used to construct switched quadratic LFs for switched systems in [7]. For recent results on parameter dependent Lyapunov functions for uncertain linear systems we refer the interested reader to [23], [4] and the references therein. A different type of relaxation was developed for PWA systems in [13]. To deal with a switching law defined by a state-space partition, the weight matrix of a quadratic LF was allowed to have different values (within a finite set of admissible matrices), which yielded a piecewise quadratic (PWQ) LF. More recently, a method to synthesize trajectory-dependent time-variant CLFs defined using the infinity norm was proposed in [18].

This paper continues on this line of research and proposes a definition of a parameterized LF (p-LF), without a fixed structure, that is applicable to general discrete-time nonlinear systems. The term parameterized LF denotes the fact that the LF candidate is endowed with a set of parameters, not necessarily structured in a particular form (e.g., a matrix of certain dimensions), which can take multiple values within an admissible set that depends on each state. As such, the Lyapunov conditions for stability can be formulated in terms of the set valued map that generates an admissible set of parameters for each state. In contrast to the set-up of [18], the conditions that define a p-LF are time-invariant. The non-conservatism of the proposed p-LFs, even with a fixed structure, is indicated by a converse theorem, which establishes that exponentially stable nonlinear systems always admit a p-quadratic LF. A corresponding definition of a parameterized control Lyapunov function (p-CLF) is also provided, which leads to several possibilities for synthesizing trajectory-dependent stabilizing control laws. Furthermore, it is shown that for p-quadratic-CLFs and input affine nonlinear systems, a synthesis solution based on solving on-line a single low-complexity semi-definite program (SDP) can be obtained, under the assumption of recursive feasibility.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \mathbb{R} \cap \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\text{int}(\mathcal{S})$ denote the interior of $\mathcal{S}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A polytope is a closed and bounded polyhedron. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|\cdot\|$ denotes an arbitrary p-norm, $p \in \mathbb{Z}_{\geq 1} \cup \infty$. Let $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$ and $\|x\|_2 := \sqrt{\sum_{i=1}^n |[x]_i|^2}$, where $|\cdot|$ denotes the absolute value. $I_n \in \mathbb{R}^{n\times n}$ denotes the n-th dimensional identity matrix. For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). Moreover, $*$ is used to denote the symmetric part of a matrix. For the definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions we refer the reader to [11].

Next, consider the discrete-time autonomous system

$$x(k+1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at the discrete-time instant k and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary map with $\Phi(0) = 0$.

Definition 2.1 Let $\lambda \in \mathbb{R}_{[0,1]}$. We call a set $\mathbb{X} \subseteq \mathbb{R}^n$ λ-*contractive* (or shortly, contractive) for system (1) if for all $x \in \mathbb{X}$ it holds that $\Phi(x) \in \lambda\mathbb{X}$. When this property holds with $\lambda = 1$ we call $\mathbb{X}$ a *positively invariant (PI)* set.

Definition 2.2 Let $\mathbb{X}$ with $0 \in \mathrm{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) asymptotically stable in $\mathbb{X}$, or shortly, AS($\mathbb{X}$), if there exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that, for each $x(0) \in \mathbb{X}$ it holds that the corresponding state trajectory of (1) satisfies $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{Z}_+$. We call system (1) exponentially stable in $\mathbb{X}$, or shortly, ES($\mathbb{X}$), if $\beta(s,k) := \theta\mu^k s$ for some $\theta \in \mathbb{R}_{\geq 1}$, $\mu \in \mathbb{R}_{[0,1)}$.

Theorem 2.1 [11,14] *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with* $0 \in \mathrm{int}(\mathbb{X})$. *Furthermore, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \ \forall x \in \mathbb{X}, \tag{2a}$$

$$V(\Phi(x)) \leq \rho V(x), \quad \forall x \in \mathbb{X}. \tag{2b}$$

Then system (1) *is AS($\mathbb{X}$).*

A function V that satisfies (2) is called a *Lyapunov function* and ρ is called the *rate of decrease* of V. Notice that in discrete-time, continuity of the dynamics or Lyapunov function is not necessary (except at the origin) for stability, as pointed out in [14]. As such, in what follows we do not explicitly require this property. However, as pointed out recently in [17], one must take additional precautions with respect to inherent robustness, when discontinuous Lyapunov functions are involved.

3 Parameterized Lyapunov Functions

Let $\mathbb{P}$ denote a set of parameter sets, where each parameter set (or element of $\mathbb{P}$) contains a finite number of parameters with an arbitrary structure, e.g., a parameter set or element in $\mathbb{P}$ can be a matrix of certain fixed dimensions. Let us now define a function $V : \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}_+$, which is zero at zero for all elements in $\mathbb{P}$. Next, let $(P_1, P_2) \in \mathbb{P} \times \mathbb{P} =: \mathbb{P}^2$ and consider the following inequalities for some $x \in \mathbb{X}$:

$$\alpha_1(\|x\|) \leq V(x, P_1) \leq \alpha_2(\|x\|), \tag{3a}$$

$$V(\Phi(x), P_2) \leq \rho V(x, P_1). \tag{3b}$$

Consider the set-valued map $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{P} \times \mathbb{P}$,

$$\mathcal{P}(x) := \{(P_1, P_2) \in \mathbb{P}^2 \mid \text{(3a) and (3b) hold}\}. \tag{4}$$

For any $x \in \mathbb{X}$, $\mathcal{P}(x) \neq \emptyset$ denotes the fact that there exists at least one pair $(P_1, P_2) \in \mathbb{P}^2$ that satisfies (3). To distinguish between the two outputs of $\mathcal{P}$ we will use $[\mathcal{P}(x)]_1$ and $[\mathcal{P}(x)]_2$ to denote the sets where the first and the second component of a pair $(P_1, P_2) \in \mathbb{P}^2$ that satisfies (3) take values, respectively. $[\mathcal{P}(x)]_\bullet$ denotes an element of $\mathcal{P}(x)$. With a slight abuse of notation we will use $P(x)$ to denote any $P_1 \in [\mathcal{P}(x)]_1$.

Definition 3.1 A function $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ is called a parameterized Lyapunov function (p-LF) in $\mathbb{X} \subseteq \mathbb{R}^n$ for system (1) if

$$\mathcal{P}(x) \neq \emptyset, \quad \forall x \in \mathbb{X}, \tag{5a}$$

$$[\mathcal{P}(x)]_2 \cap [\mathcal{P}(\Phi(x))]_1 \neq \emptyset, \quad \forall x \in \mathbb{X}. \tag{5b}$$

Theorem 3.1 *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with* $0 \in \text{int}(\mathbb{X})$. *Suppose that system* (1) *admits a parameterized Lyapunov function in* $\mathbb{X}$. *Then system* (1) *is AS($\mathbb{X}$).*

Proof. The claim is proven using standard arguments [8,11,14]. From (5a) we obtain that for all $x \in \mathbb{X}$

$$\alpha_1(\|x\|) \leq V(x, P(x)) \leq \alpha_2(\|x\|), \ \forall P(x) \in [\mathcal{P}(x)]_1.$$

From (5b) we obtain that for all $x \in \mathbb{X}$ there exists at least one pair $(P(x), P_2) \in \mathcal{P}(x)$ such that $P_2 \in [\mathcal{P}(\Phi(x))]_1$, which yields that

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) \leq 0, \quad \forall x \in \mathbb{X},$$

with $P(\Phi(x)) = P_2 \in [\mathcal{P}(\Phi(x))]_1$ and $P(x) \in [\mathcal{P}(x)]_1$. As $\mathbb{X}$ is a PI set, the above inequality can be applied recursively for any trajectory $\{x(k)\}_{k \in \mathbb{Z}_+}$ with $x(0) \in \mathbb{X}$, which yields:

$$\begin{aligned}\alpha_1(\|x(k+1)\|) &\leq V(\Phi(x(k)), P(\Phi(x(k)))) \\ &\leq \rho^{k+1} V(x(0), P(x(0))) \leq \rho^{k+1}\alpha_2(\|x(0)\|),\end{aligned}$$

for all $x(0) \in \mathbb{X}$. Hence, $\|x(k)\| \leq \beta(\|x(0)\|, k)$ for all $x(0) \in \mathbb{X}$, where $\beta(s,k) := \alpha_1^{-1}(\rho^k \alpha_2(s)) \in \mathcal{KL}$, which completes the proof. $\square$

To illustrate the relaxation with respect to existing approaches, consider the case when one adds a particular structure to the parameter set $\mathbb{P}$ and the candidate p-LF. As such, let us consider p-quadratic-LFs, defined as $V(x, P(x)) := x^\top P(x) x$, $P(x) \in [\mathcal{P}(x)]_1$, $\mathcal{P}(x) \subseteq \mathbb{P}^2$ for all x, where $\mathbb{P} \subseteq \mathbb{R}^{n \times n}$. Consider now the case of a PWA system with a fixed switching law defined by a partition of the state-space, i.e., $\{\Omega_j\}_{j \in \mathcal{S}}$, with $\mathcal{S}$ a finite set of indices, and let $\mathbb{P} := \{P_i\}_{i \in \mathcal{S}}$, $P_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathcal{S}$. If one sets $\mathcal{P}(x) := \{(P_i, P_j)\}$ for all $(x, \Phi(x)) \in \Omega_i \times \Omega_j$ and imposes (3a) for all $x \in \Omega_j$ and (3b) for all $(x, \Phi(x)) \in \Omega_i \times \Omega_j$, $(i,j) \in \mathcal{S} \times \mathcal{S}$, one obtains a PWQ Lyapunov function with an $\mathcal{S}$-procedure relaxation [13], as a particular case of a p-quadratic-LF. Similarly, it can be shown that quadratic periodic Lyapunov functions [3] are a particular case of p-quadratic-LFs. Moreover, it can be shown that parameter dependent Lyapunov functions form a particular type of p-LFs, by allowing the map $\mathcal{P}$ to depend on both the state and the uncertain parameter. It would be interesting to further relate p-LFs with polynomial Lyapunov functions, which can be obtained if $P(x)$ is allowed to be a particular polynomial of x. Then, the map $\mathcal{P}(x)$ would assign the coefficients of the polynomial. As the relation to polynomial LFs is beyond the scope of this paper, we will not pursue it any further.

The following converse result reveals the non-conservatism of p-LFs, even when a particular structure is imposed. We will consider two of the most popular type of structures for candidate LFs, i.e., p-quadratic-LFs and p-polyhedral-LFs, defined as $V(x, P(x)) := \|P(x)x\|_\infty$, $P(x) \in [\mathcal{P}(x)]_1$, $\mathcal{P}(x) \subseteq \mathbb{P}^2$ for all x, where $\mathbb{P} \subseteq \mathbb{R}^{r\times n}$ ($r \in \mathbb{Z}_{\geq n}$). Let $\mathcal{N}$ denote an arbitrary neighborhood of the origin (i.e., a bounded set with a non-empty interior that contains the origin in its interior).

Assumption 3.1 *There exists a positively invariant neighborhood of the origin $\mathcal{N}$ such that system* (1) *admits a p-quadratic-LF (p-polyhedral-LF) in $\mathcal{N}$.*

The above assumption is reasonable, as most nonlinear systems can be approximated around the origin by a linear system, PWA system or a polytopic difference inclusion and then one can use the above indicated results to obtain a local p-quadratic-LF. Next, suppose that system (1) is either ES($\mathbb{X}$) or AS($\mathbb{X}$) and, as such, by a standard converse theorem, see, e.g., Theorem 1 in [12] or Lemma 4 in [20] (AS) and Theorem 2 (ES) in [12], it admits a Lyapunov function in $\mathbb{X}$. Notice that the above-mentioned converse theorems require AS($\mathbb{R}^n$). In what follows we implicitly assume that these results can be applied to an invariant subset $\mathbb{X}$ of $\mathbb{R}^n$.

Let V_1 denote a LF established by a converse theorem and let $V_L(x, P_L(x))$ with $P_L(x) \in [\mathcal{P}_L(x)]_1$ for some $\mathcal{P}_L(x) \subseteq \mathbb{P}^2$ denote a p-quadratic-LF (or p-polyhedral-LF) in $\mathcal{N}$.

Theorem 3.2 *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with $0 \in \mathrm{int}(\mathbb{X})$.*

(i) *Suppose that system* (1) *is ES($\mathbb{X}$). Then, there exists a p-quadratic-LF in $\mathbb{X}$ for system* (1)*.*

(ii) *Suppose that system* (1) *is AS($\mathbb{X}$), Assumption 3.1 holds and there exists a $c \in \mathbb{R}_{(0,1]}$ such that $V_1(x) \geq cV_L(x, P_L(x))$ for all $x \in \mathcal{N}$ and all $P_L(x) \in [\mathcal{P}_L(x)]_1$. Then, there exists a p-quadratic-LF (p-polyhedral-LF) in $\mathbb{X}$ for system* (1)*.*

Proof. Let us begin with the proof of *(i)*. As system (1) is ES($\mathbb{X}$), by Theorem 2 in [12] it admits a standard Lyapunov function V_1 that satisfies (2) for all $x \in \mathbb{X}$. Moreover, V_1 satisfies (2a) with $\alpha_1(s) := s^2$ and $\alpha_2(s) := ls^2$ for some $l \in \mathbb{R}_{\geq 1}$. Using this function define

$$\mathcal{P}(x) := \left\{ \left(\frac{V_1(x)}{\|x\|_2^2} I_n, \frac{V_1(\Phi(x))}{\|\Phi(x)\|_2^2} I_n \right) \right\}, \quad \forall x \in \mathbb{X}. \tag{6}$$

Note that $\Phi(x) \in \mathbb{X}$ for all $x \in \mathbb{X}$ and

$$l = \frac{\alpha_2(\|x\|)}{\|x\|_2^2} \geq \frac{V_1(x)}{\|x\|_2^2} \geq \frac{\alpha_1(\|x\|)}{\|x\|_2^2} = 1, \quad \forall x \in \mathbb{X}.$$

Thus, $\mathcal{P}(x)$ is well-defined for all $x \in \mathbb{X}$. Observing that the candidate p-quadratic-LF $V(x, P(x)) := x^\top P(x)x$ with $P(x) \in [\mathcal{P}(x)]_1$ satisfies $V(x, P(x)) = V_1(x)$ for all $x \in \mathbb{X}$ and V_1 is a LF in $\mathbb{X}$ for system (1) completes the proof.

Consider now hypothesis *(ii)*. As system (1) is AS($\mathbb{X}$), by Theorem 1 in [12] it admits a standard Lyapunov function V_1 that satisfies (2) for all $x \in \mathbb{X}$. Using this function define $\bar{P}(x) := \frac{V_1(x)}{\|x\|_2^2} I_n$ for a p-quadratic-LF, or $\bar{P}(x) := \frac{V_1(x)}{\|x\|_\infty} I_n$ for a p-polyhedral-LF. Note that, as $\frac{V_1(x)}{\|x\|_2^2} \leq \frac{\alpha_2(\|x\|)}{\|x\|_2^2}$ for all $x \in \mathbb{X}$ and $\mathcal{N}$ is bounded and contains the origin in its interior, $\frac{V_1(x)}{\|x\|_2^2}$ is well defined for all $x \in \mathbb{X} \setminus \mathcal{N} =: \mathbb{X}_\mathcal{N}$. Similarly, $\frac{V_1(x)}{\|x\|_\infty}$ is well defined for all $x \in \mathbb{X}_{\mathcal{N}}$. Next, consider the candidate p-quadratic-LF (p-polyhedral-LF) $V(x, P(x)) = x^\top P(x)x$ ($V(x, P(x)) = \|P(x)x\|_\infty$), where $P(x) \in [\mathcal{P}(x)]_1$ and

$$\mathcal{P}(x) := \begin{cases} c\mathcal{P}_L(x), & \forall (x, \Phi(x)) \in \mathcal{N}^2, \\ \{(\bar{P}(x), c[\mathcal{P}_L(\Phi(x))]_1)\}, & \forall (x, \Phi(x)) \in \mathbb{X}_{\mathcal{N}} \times \mathcal{N}, \\ \{(\bar{P}(x), \bar{P}(\Phi(x)))\}, & \forall (x, \Phi(x)) \in (\mathbb{X}_{\mathcal{N}})^2. \end{cases} \tag{7}$$

Then, for all $(x, \Phi(x)) \in (\mathbb{X}_{\mathcal{N}})^2$, as $V(x, P(x)) = V_1(x)$ for all $x \in \mathbb{X}_{\mathcal{N}}$, (2b) yields

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) = V_1(\Phi(x)) - \rho V_1(x) \le 0.$$

Moreover, for all $(x, \Phi(x)) \in \mathbb{X}_{\mathcal{N}} \times \mathcal{N}$, (2b) also yields

$$\begin{aligned} V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) & \\ &= cV_L(\Phi(x), P_L(\Phi(x))) - \rho V_1(x) \\ &\le V_1(\Phi(x)) - \rho V_1(x) \le 0, \end{aligned}$$

for all $P_L(\Phi(x)) \in [\mathcal{P}_L(\Phi(x))]_1$. As $\mathcal{N}$ is a PI set for system (1), the last case to be analyzed is when $(x, \Phi(x)) \in \mathcal{N}^2$. Then

$$\begin{aligned} V(\Phi(x), &P(\Phi(x))) - \rho V(x, P(x)) \\ &= c(V_L(\Phi(x), P_L(\Phi(x))) - \rho V_L(x, P_L(x))) \le 0, \end{aligned}$$

where $P_L(x) \in [\mathcal{P}_L(x)]_1$ for all $x \in \mathcal{N}$. Thus, we conclude that $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ for all $x \in \mathbb{X}$ and $\mathcal{P}(x)$ as defined in (7) satisfies

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)), \quad \forall x \in \mathbb{X}.$$

Observing that

$$\alpha_{1,L}(\|x\|) \le V_L(x, P_L(x)) \le \alpha_{2,L}(\|x\|), \, \forall x \in \mathcal{N},$$

for some $\alpha_{1,L}, \alpha_{2,L} \in \mathcal{K}_\infty$, yields that

$$\alpha_{1,p}(\|x\|) \le V(x, P(x)) \le \alpha_{2,p}(\|x\|), \, \forall x \in \mathbb{X},$$

where $\alpha_{1,p}(s) := \min(\alpha_1(s), c\alpha_{1,L}(s)) \in \mathcal{K}_\infty$ and $\alpha_{2,p}(s) := \alpha_2(s) \in \mathcal{K}_\infty$. This further implies that $V(x, P(x))$ (i.e., the constructed p-quadratic-LF or p-polyhedral-LF candidate) satisfies the conditions of Definition 3.1, which completes the proof. □

For clarity of exposition, in this section we have considered discrete-time systems of the form (1) that are described by a difference equation. However, all the developed results apply *mutatis mutandis* to the case when $\Phi(x)$ is a compact and non-empty set-valued map and yield *strong* asymptotic stability in $\mathbb{X}$ (i.e., AS($\mathbb{X}$) for all possible trajectories generated by the set-valued map). Then, the converse theorem in [10] should be used instead of the ones in [12]. In the next section we will deal with a difference equation that involves a set-valued control input and refer to the results established in this section, as in fact, these results hold for difference inclusions as well.

4 Parameterized Control Lyapunov Functions

Consider the discrete-time system

$$x(k+1) = \phi(x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{8}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input, $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary map with $\phi(0,0) = 0$ and $\mathbb{X}$, $\mathbb{U}$ contain the origin in their interior.

Definition 4.1 We call a set $\mathbb{X} \subseteq \mathbb{R}^n$ constrained control invariant with respect to $\mathbb{U}$ (CCI($\mathbb{X}$,$\mathbb{U}$)) for system (8) if for all $x \in \mathbb{X}$, $\exists u \in \mathbb{U}$ such that $\phi(x,u) \in \mathbb{X}$.

Assumption 4.1 *$\mathbb{X} \subseteq \mathbb{R}^n$ is a CCI($\mathbb{X}$,$\mathbb{U}$) set for the discrete-time system* (8).

Notice that the above assumption is made only for ease of exposition. If $\mathbb{X}$ is not a CCI($\mathbb{X}$,$\mathbb{U}$), the results simply apply for the largest subset of $\mathbb{X}$ with this property.

Next, consider the following inequalities corresponding to (3) for some $x \in \mathbb{X}$:

$$\alpha_1(\|x\|) \leq V(x, P_1) \leq \alpha_2(\|x\|), \tag{9a}$$

$$V(\phi(x,u), P_2) \leq \rho V(x, P_1). \tag{9b}$$

Consider the set-valued map $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{P} \times \mathbb{P}$,

$$\mathcal{P}(x) := \{(P_1, P_2) \in \mathbb{P}^2 \mid \exists u \in \mathbb{U} \text{ s.t. (9) holds}\}. \tag{10}$$

Furthermore, let $\pi : \mathbb{R}^n \times \mathbb{P} \times \mathbb{P} \rightrightarrows \mathbb{U}$ denote

$$\pi(x, [\mathcal{P}(x)]_\bullet) := \{u \in \mathbb{U} \mid \text{(9) holds for } [\mathcal{P}(x)]_\bullet\}.$$

Definition 4.2 A function $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ is called a parameterized control Lyapunov function (p-CLF) in $\mathbb{X}$ for system (8) if

$$\mathcal{P}(x) \neq \emptyset, \quad \forall x \in \mathbb{X}, \tag{11a}$$

$$\exists [\mathcal{P}(x)]_\bullet \in \mathcal{P}(x), \exists u \in \pi(x, [\mathcal{P}(x)]_\bullet) \text{ s.t.}$$
$$[\mathcal{P}(x)]_2 \cap [\mathcal{P}(\phi(x,u))]_1 \neq \emptyset, \quad \forall x \in \mathbb{X}. \tag{11b}$$

In what follows we will focus on the synthesis of p-CLFs. Although these methods will also provide useful insights for stability analysis via synthesis of p-LFs, exploring this path further is beyond the scope of this paper.

Next, we will formulate an optimization problem to be solved on-line that yields a trajectory-dependent p-CLF (td-p-CLF). By trajectory-dependent we mean that the computed sequence of parameter sets $\{P(x(k))\}_{k \in \mathbb{Z}_+}$, with $P(x(k)) \in [\mathcal{P}(x(k))]_1$ for all $k \in \mathbb{Z}_+$, will only be valid along the trajectory $\{x(k)\}_{k \in \mathbb{Z}_+}$. The advantage of this approach is that the non-conservatism of a p-CLF is preserved. The challenge, which is common to all optimization based controllers, is to guarantee recursive feasibility. Unfortunately, the problem of constructing a set of *a priori* verifiable conditions for recursive feasibility is non-trivial and it is not solved in this paper. Instead, we propose a heuristic solution for attaining recursive feasibility, which requires minimization of the decrease rate of the td-p-CLF. Simulations conducted on several challenging case studies indicate that *not* enforcing a steep decrease of the p-CLF is beneficial in terms of recursive feasibility. This is in contrast with the classical CLF approach of [1], where the optimal decrease is required.

Let the structure, e.g., p-quadratic-CLF, the set of parameter sets $\mathbb{P}$, the functions α_1, α_2 and the rate of decrease $\rho \in \mathbb{R}_{[0,1)}$ of an arbitrary candidate p-CLF V be specified.

Problem 4.1 Let $x(k) \in \mathbb{X}$ be known at each $k \in \mathbb{Z}_+$. Let $x^+(k) := \phi(x(k), u(k))$ for all $k \in \mathbb{Z}_+$ and consider the following inequalities:

$$x^+(k) \in \mathbb{X},\ \ u(k) \in \mathbb{U}, \tag{12a}$$

$$\alpha_1(\|x(k)\|) \leq V(x(k), P(x(k))) \leq \alpha_2(\|x(k)\|), \tag{12b}$$

$$\alpha_1(\|x^+(k)\|) \leq V(x^+(k), P(x^+(k))) \leq \alpha_2(\|x^+(k)\|), \tag{12c}$$

$$V(x^+(k), P(x^+(k))) \leq \rho V(x(k), P(x(k))). \tag{12d}$$

If $k = 0$ find a $u(0) \in \mathbb{U}$ and a $(P(x^+(0)), P(x(0))) \in \mathbb{P}^2$ that satisfy (12). If $k \in \mathbb{Z}_{\geq 1}$ *set* $P(x(k)) = P(x^+(k-1))$ and find a $u(k) \in \mathbb{U}$ and a $P(x^+(k)) \in \mathbb{P}$ that satisfy (12a)-(12c)-(12d). □

In the above problem, $x^+(k)$ can be interpreted as the one-step ahead prediction calculated at time $k \in \mathbb{Z}_+$ using the measured state $x(k)$, the input $u(k)$ and the plant model $\phi(\cdot,\cdot)$. Obviously, in the ideal case $x^+(k-1) = x(k)$ and then the assignment $P(x(k)) = P(x^+(k-1))$ becomes redundant. However, this assignment plays a very important role if a perturbation $w \in \mathbb{R}^n$ acts on state $x^+(k-1)$, which yields $x(k) = x^+(k-1)+w$. In this case, by setting $P(x(k)) = P(x^+(k-1))$, one can exploit continuity of $V(\cdot, P(x^+(k-1)))$ in its first argument to establish inherent input to state stability [11, 17].

Next, let us propose a cost function that penalizes the decrease of the p-CLF. Let $x^+ := \phi(x, u)$ and let $\bar{\phi}(x) := \{\phi(x, u) \mid u \in \pi(x, [\mathcal{P}(x)]_\bullet),\ [\mathcal{P}(x)]_\bullet \in \mathcal{P}(x)\}$. Furthermore, suppose that we augment Problem 4.1 with the following cost function that guides the choice of $(u(k), P(x^+(k)))$. For any known $x(k) \in \mathbb{X}$ and $P(x(k)) = P(x^+(k-1))$, $k \in \mathbb{Z}_{\geq 1}$, consider the cost function

$$J(x(k), u(k), P(x^+(k))) := \rho V(x(k), P(x(k))) - V(x^+(k), P(x^+(k))) \tag{13}$$

and let

$$(u^*(k), P^*(x^+(k))) := \arg \inf_{u \in \pi(x(k), [\mathcal{P}(x(k))]_\bullet), P \in [\mathcal{P}(x(k))]_2} J(x(k), u, P)$$

denote the corresponding infimizer. Notice that due to (12d) J is bounded by zero from below and thus, the infimum is a minimum. For brevity we assume the minimum is attainable, which is true if V and ϕ are continuous in both arguments, respectively, and $\pi(x(k), [\mathcal{P}(x(k))]_\bullet) \subseteq \mathbb{U}$, $[\mathcal{P}(x(k))]_2 \subseteq \mathbb{P}$ are compact sets for all $k \in \mathbb{Z}_+$, and unique. Alternatively, one can always infimize J over $u \in \mathbb{U}$ and $P \in \mathbb{P}$, for some known compact sets $\mathbb{U}$ and $\mathbb{P}$. Several examples are presented in the next section, which illustrate the benefits of augmenting Problem 4.1 with the cost J, as defined in (13).

Remark 4.1 The p-CLFs defined in this section, along with the corresponding synthesis problem, i.e., Problem 4.1, bring a significant relaxation with respect to the trajectory dependent time-variant CLF construction proposed in [18]. The conditions imposed on the td-CLF therein translate into $P_1 = P_2$ for all $(P_1, P_2) \in \mathcal{P}(x)$, for all $x \in \mathbb{X}$. With respect to Problem 4.1 these conditions would require that for each $k \in \mathbb{Z}_+$, $P(x^+(k)) = P(x(k))$, which is obviously more conservative. It should be mentioned that the benefit of the conditions in [18] is that the corresponding Problem 4.1 can be rendered tractable for a polyhedral V as well. □

It is also worth to point out that the concept of a state-dependent Riccati equation [5] can be related to a particular setting of the proposed parameterized Lyapunov inequality, i.e., to a corresponding parameterized Lyapunov equation.

4.1 Synthesis of p-quadratic-CLFs

In what follows we will restrict our attention to input affine nonlinear systems, i.e.,

$$\phi(x, u) := f(x) + g(x)u \tag{14}$$

for some $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with $f(0) = 0$. Also, we will consider p-quadratic-CLF candidates of the form $V : \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}_+$,

$$V(x, P(x)) = x^\top P(x)x,\ P(x) \in [\mathcal{P}(x)]_1,\ \mathcal{P}(x) \subseteq \mathbb{P}^2,$$

where $\mathbb{P} = \mathbb{R}^{n\times n}$. Notice that such a function satisfies $V(0, P) = 0$ for all $P \in \mathbb{P}$, but it does not already satisfy (9a). Next, we will present an LMI based formulation of Problem 4.1. Let $\gamma \in \mathbb{R}_{>0}$ and $\Gamma \in \mathbb{R}_{\geq\gamma}$ denote positive constants and suppose that $\mathbb{X}$ and $\mathbb{U}$ are polytopes. As such, constraint (12a) becomes a set of linear inequalities in $u(k)$ for each $x(k)$, $k \in \mathbb{Z}_+$. So, we will only focus on fulfillment of the inequalities (12b), (12c) and (12d). Consider now the following inequalities

$$\begin{aligned} x(k)^\top(P(x(k)) - \gamma I_n)x(k) &\geq 0, \\ x(k)^\top(\Gamma I_n - P(x(k)))x(k) &\geq 0, \end{aligned} \tag{15a}$$

$$Z(k) - \Gamma^{-1}I_n \succeq 0,\ \gamma^{-1}I_n - Z(k) \succeq 0, \tag{15b}$$

$$\begin{pmatrix} \rho x(k)^\top P(x(k))x(k) & * \\ f(x(k)) + g(x(k))u(k) & Z(k) \end{pmatrix} \succeq 0. \tag{15c}$$

Lemma 4.1 *Let $k \in \mathbb{Z}_+$ and let $x(k) \in \mathbb{X}$, γ, Γ and ρ be known. Suppose that $\{u(k), P(x(k)), Z(k)\}$ are a feasible solution of the LMI* (15)*. Then, $V(x(k), P(x(k))) = x(k)^\top P(x(k))x(k)$, $P(x^+(k)) = Z^{-1}(k)$ and $u(k)$ are a feasible solution of* (12b)*,* (12c) *and* (12d) *with $\alpha_1(s) := \gamma s^2$ and $\alpha_2(s) := \Gamma s^2$.*

Proof. Notice that (15a) is equivalent to (12b) for the specified α_1, α_2 and, by applying the Schur complement to (15c) one obtains (12d). (15b) yields that $\Gamma I_n \succeq P(x^+(k)) = Z^{-1}(k) \succeq \gamma I_n$. Thus, (12c) holds for the specified α_1, α_2, which completes the proof. □

The advantage of the solution of Lemma 4.1 is that (15c) offers a translation of the decreasing condition (12d) that does not introduce any conservatism. However, (15c) yields $P(x^+(k)) \succeq 0$, which is not necessary for (12c) to hold.

Notice that the resulting receding horizon control law is stabilizing only if the corresponding optimization problem is recursively feasible. In that respect, minimization of the cost (13) is advised. For example, using some non-trivial facts about positive semi-definite matrices it can be proven that by adding the LMI

$$\varepsilon(k)I_{n+1} - \begin{pmatrix} \rho x(k)^\top P(x(k))x(k) & * \\ f(x(k)) + g(x(k))u(k) & Z(k) \end{pmatrix} \succeq 0$$

to (15) and minimizing $\varepsilon(k)$ at each $k \in \mathbb{Z}_+$, minimization of the cost J in (13) is attained.

5 Illustrative Examples

In this section we present several examples of nonlinear systems that pose a non-trivial challenge to the problem of synthesizing a stabilizing control law. For each example we will provide a plot of the state trajectory and of the sub-level sets $\{z \in \mathbb{R}^n \mid V(z, P(x(k))) \leq 1\}_{k \in \mathbb{Z}_+}$ in Figure 1 and Figure 2, respectively.

Example 5.1 The first example consists of an uncertain linear system defined by

$$x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} \delta(k) \\ 1 \end{pmatrix} u(k), \quad k \in \mathbb{Z}_+,$$

where $\delta(k) \in \mathbb{R}_{[-c,c]}$ for all $k \in \mathbb{Z}_+$ ($c \in \mathbb{R}_{>0}$) is an unknown time-varying parameter. If $c \leq 1$, the system admits a quadratic CLF. However, for any $c > 1$, this no longer holds, i.e., *this system exhibits an infinite gap in the existence of a (robust) quadratic CLF*. However, the uncertain system does admit a parameter dependent quadratic CLF, which can be computed as shown in [6], but the implementation of the corresponding control law requires knowledge of $\delta(k)$, for all $k \in \mathbb{Z}_+$. To design a stabilizing controller for the above system with $c = 1.15$ we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.99$. In (15c) we made use of the extreme realizations $\delta(k) = 1.15$ and $\delta(k) = -1.15$ for all $k \in \mathbb{Z}_+$. Notice that this is sufficient for (15c) to hold for all $\delta(k) \in \mathbb{R}_{[-c,c]}$. To optimize convergence, we added the one-step cost $J_1 + J$ to (15), with $J_1(x(k), u(k)) := x^+(k)^\top Q x^+(k)$ ($Q = I_2$) and J defined as in (13), which still allows a conversion into a SDP. Only the extreme realizations of $\delta(k)$ were used to implement minimization of the above cost. A state trajectory plot obtained for $x(0) = [4 \ -4]^\top$ is given in Figure 1.

Example 5.2 The second example is taken from [14,18] and it consists of *a piecewise linear system that does not admit a common quadratic or PWQ CLF*. For brevity, we refer to the above references for the numerical details regarding the system. As shown in [14] the problem of computing such a CLF requires solving a bilinear matrix inequality. To design a stabilizing controller for this system we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.9$. The cost J as defined in (13) was added to (15). A state trajectory plot obtained for $x(0) = [5 \ -5]^\top$ is given in Figure 1.

Example 5.3 The third example is taken from [15] and it consists of a nonlinear system subject to state and input constraints. This system corresponds to (8)-(14) with $\mathbb{X} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 5\}$, $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 1\}$ and

$$f(x) = \begin{pmatrix} [x]_1 + 0.7[x]_2 + ([x]_2)^2 \\ [x]_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0.245 + \sin([x]_2) \\ 0.7 \end{pmatrix}.$$

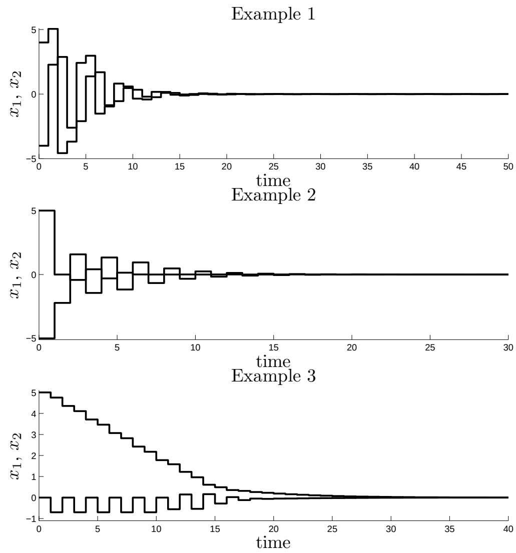

Figure 1: Simulation results – State trajectories.

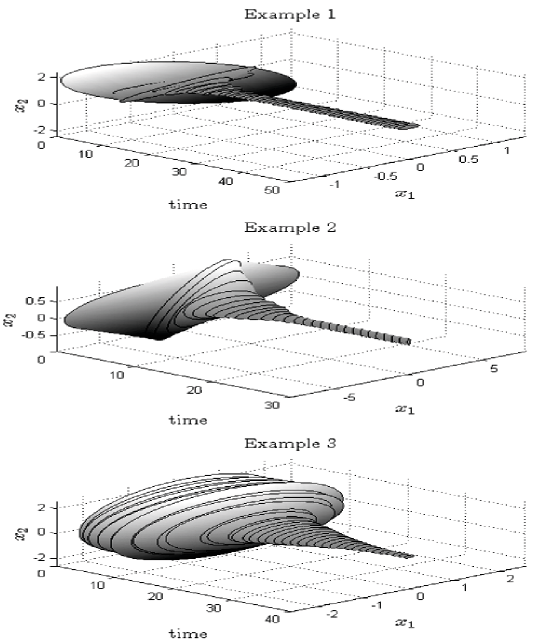

Figure 2: Simulation results – Evolution in time of the sub-level sets of V.

To design a stabilizing controller for this system we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.8$. To optimize convergence and improve feasibility, we added the cost $J_1 + J$ to (15). A state trajectory plot obtained for $x(0) = [5\ 0]^\top$, which lies on the boundary of $\mathbb{X}$, is given in Figure 1. Notice that input and state constraints are fulfilled at all times. In [15], *a non-monotone* CLF with a fixed parameter set was employed to stabilize the system for a similar initial condition.

6 Conclusions

This paper has provided results on existence and preliminary results on synthesis of parameterized-(control) Lyapunov functions for discrete-time nonlinear systems that are possibly subject to constraints. A p-LF was defined by assigning a finite set of parameters to a standard LF, which can take different values for each state. It was demonstrated that the so-obtained p-LFs offer non-conservative stability analysis conditions, even when Lyapunov functions with a particular structure, such as quadratic forms, are considered. Furthermore, a method for synthesizing p-CLFs for discrete-time nonlinear systems was proposed. It was shown that this method can be implemented by solving on-line a single low-complexity semi-definite program. Deriving *a priori* verifiable conditions under which the developed synthesis method yields a recursively feasible optimization problem makes the object of future research.

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Act-and-Wait Control Theory for Continuous-Time Systems with Random Feedback Delays

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Abstract: Continuous-time systems with random state feedback delay are difficult to control in general because of its infinite poles. In this paper, the act-and-wait controller is well developed to solve this problem. If the infinite dimensional pole placement problem can be reduced to a finite dimensional one, it would be facility to make the system stable by the aid of pole placement method. The mechanism of the act-and-wait concept is that the state feedback is periodically switched on (act) and off (wait) during the control procedure. By using the act and wait controller, the stability of system can be represented by a finial dimensional monodromy matrix when the interval between two successive act moments is larger than the maximum state feedback delay. The aim of this paper is to design the periodic controller so that a finite number of eigenvalues can describe stability of the delay system, so the stability of the system can be achieved by use of pole placement method. The efficiency of the method is shown by a simulation.

Keywords: *act-and-wait controller; random delay; pole placement; stability.*

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1 Introduction

Pole placement method is a very important tool in control theory [3], which is used in stabilizing plants and improving the performance of the controlled systems [16]. In the real situations, instability and poor performance of system are often led by time delay in the feedback loop of control systems. Many researchers have studied various kinds of delay systems [9, 19, 20, 22, 23]. Classic pole placement technique is well developed and common when it is applied to the systems without delay, while it is complex and crucial in delayed systems [5, 14]. In delayed systems the number of poles to be controlled is much larger than the degrees of freedom in the controller [13], so classical pole placement techniques of ordinary -differential equations can not be applied for delayed systems.

Periodic control method has shown advantages in stabilizing linear time-invariant (LTI) systems [18]. Several papers have been published which have used periodic feedback controller to control systems. Recently much attention [2, 10] has been attracted to the stabilization of continuous LTI systems with feedback delays by applying a periodic controller. In [15] it has been shown that the output feedback controller was used to make the system stable, which contains a periodic gain related to a cosine function.

On the other hand, lots of literature in which the stabilization problems of systems were studied by using act-and-wait concept [4, 6–8, 11, 21] focuses on this problem. The scholar in [4] made a comparison between the act-and-wait control and Intermittent control. Other researchers in [6–8,11,21] used the act-and-control mechanism to deal with the stabilization problems in different system, such as LTI systems, robotics systems, chaotic oscillator systems, and so on. In this approach the controller is periodically switched on (act) and off (wait). If the duration of waiting (switched off) time is longer than time delays in a system, then the problem about stabilization of the system is simplified to pole placement.

This paper discussed the act-and-wait controller applied to linear n-dimensional order system with random feedback delay. In general case, the n-dimensional system with feedback delay has infinite number of poles, which is hard to handle by the finite control parameters. As introduced in this paper, we can make the infinite poles of the n-dimensional system with random feedback delay reduce to n-dimensional one. The act-and-wait control mechanism means that the controller can periodically switch off and switch on, so the stability properties of the system can be described by n eigenvalues decided by an $N \times N$ monodromy matrix. It's assumed that the duration of waiting is larger than the maximum feedback delay.

2 Problem Statement

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given constant matrices. Firstly, we consider the autonomous delayed state feedback controller

$$u(t) = Dx(t - \tau_r), \tag{2}$$

where $D \in \mathbb{R}^{m\times n}$ is a constant matrix and τ_r is the random delay of state feedback. Because of the noise, information transmission, online data processing, computation, the problem of application of actuator and so on, delays always occur in feedback control, which are hardly eliminated or tuned during the control design. In general case, the delay is not a fixed parameter of the system with the changing of the environment. The range of random delay can often be estimated before designing the controller. In this paper, we assumed that τ_r is random non-negative integer, i.e. $0 \leq \tau_r \leq \tau_{max}$, where τ_{max} is the maximum delay.

By using controller (2), system (1) yields the closed-loop equation:

$$\dot{x}(t) = Ax(t) + BDx(t - \tau_r). \tag{3}$$

There is an infinite number of characteristic roots in the transcendental characteristic equation with the random time delay:

$$det(\lambda I - A - BDe^{-\tau\lambda}) = 0. \tag{4}$$

When all the poles of this system are located in the left half of the complex plane, the system will be asymptotically stable. Poles optimization method was used to deal with this type of problem [1, 17].

For the given system matrices A, B and random feedback delay τ_r, we want to find an appropriate parameters matrix D in order to get satisfied control effect. The specialty of this feedback delay system is that an infinite poles should be placed by use of finite control parameters from D. As introduced in the first section of this paper, a special case of periodic feedback controller called act-and-wait controller will be studied here.

3 Act-and-Wait Mechanism

The form of the act-and-wait controller is

$$u(t) = g(t)Dx(t - \tau_r), \tag{5}$$

where g(t) is the T-periodic switching function, which is defined as

$$g(t) = \begin{cases} 0, & [0,\ t_w), \\ 1, & [t_w,\ T]. \end{cases} \tag{6}$$

In the above function, t_w represents the switched off period of the controller, and t_a represents the switched on period of the controller. The whole period is

$$T = t_w + t_a, \tag{7}$$

By using the act-and-wait controller (5), the system (1) can be written as

$$\dot{x}(t) = Ax(t) + BDg(t)x(t - \tau_r). \tag{8}$$

In the classic control stability theory, this system with act-and-wait controller will be stable if all the eigenvalues of the transcendental characteristic equation are located in the left half of the complex plane. Now the stable problem is how to find the appropriate control parameters, such as t_w, t_a, and control matrix D. In this paper, in order to stabilize the system we focus on the optimization of feedback gain matrix D.

When t_w is smaller than τ_r, the characteristic equation will still have infinite poles. If $t_w \geq \tau_r$, the monodromy operator of system equation (8) can be presented as an $N \times N$ matrix.

Assumed that $t_w \geq \tau_r$, there are still two cases here which will be discussed separately below.

(**a**) $0 < t_a \leq \tau_r$; Without loss of generality, the first period of the controller will be studied here. When $t \in [0, t_w)$ and $g(t) = 0$, the system equation with the initial state $x(0)$ can be given as

$$x(t) = e^{At} x_0 \ t \in [0, t_w). \tag{9}$$

When $t \in [t_w, T)$, then the controller is switched on (g(t) = 1). In other words, the delayed term is active in system

$$\dot{x}(t) = Ax(t) + BDe^{A(s-\tau)} x(0). \tag{10}$$

The initial state of ordinary differential equation (10) is $x(t_w) = e^{At_w} x(0)$. Solve the equation (10), it is derived that

$$x(T) = \Big(e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds \Big) x(0). \tag{11}$$

Let

$$\Phi = e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds, \tag{12}$$

where Φ is the transition matrix of the system with N eigenvalues during the acting time of the controller. This means that all the other eigenvalues of the monodromy matrix of (8) are zeros but n eigenvalues in Φ. Actually, Φ is the monodromy matrix of the system.

(**b**) $t_a > \tau_r$; Assumed that t_a is between $k\tau_r$ and $(k+1)\tau_r$, so the transition matrix Φ can be obtained by step-by-step integration over every succeeding small interval. For example, the situation about $k = 1$ will be discussed below. Firstly, the solution over $[0, t_w)$ can be determined similarity to equation (8), then the $N \times N$ monodromy matrix can be obtained by the piecewise integration over the consecutive interval $[0, t_w)$, $[t_w, t_w + \tau)$, $[t_w + \tau_r, T)$:

$$\begin{aligned} \Phi &= e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds \\ &\quad + \int_{t_w+\tau}^{T} e^{A(T-s_1)} BD \int_{t_w}^{s_1-\tau} e^{A(s_1-s_2-\tau)} BDe^{A(s_2-\tau)} ds_1 ds_2. \end{aligned} \tag{13}$$

In this way, n eigenvalues of Φ can be placed using the control parameters D, so that the stability of the system can be achieved. But with the company of increasing of the k, the monodromy matrix becomes more and more complex. Because Φ depends nonlinearly on the control parameters D, it's impossible that arbitrary pole placement of the Φ can be obtained. Therefore, the simply case($t_a \leq \tau_r$) was studied here, and a simulation of the pole optimization of this case was shown in the next section.

4 Simulation

There is a second-order system with delayed feedback described by (1), (5), and (6). In the system:

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix}, \quad \tau \in \ \{1, \ 0.9, \ 0.8\} \ .$$

If an autonomous controller is used in the system, then the characteristic equation is

$$\lambda^2 - a + d_1 e^{-\lambda} + d_2 \lambda e^{-\lambda} = 0. \tag{14}$$

The number of poles is infinite, so the poles can't be arbitrarily placed using the only two parameters d_1 and d_2.

So we discussed this system with act-and-wait controller when $a = 0$ and $a = -4$ are applied in system matrix A. Optimal control parameters will be investigated for the act-and-wait case with $t_w = 1.2$ and $t_a = 0.3$.

1. When $a = 0$. Actually this system is a feedback stabilized double integrator with input delay. When the act-and-wait controller is used with $t_w = 1.2s$, $t_a = 0.3s$. The delay called τ_r which belongs to $\{1\ 0.9\ 0.8\}$is a random variable. In order to study the performance of the system with different control parameters, the monodromy matrices are calculated separately with different delays 1s, 0.9s, 0.8s. Firstly, we assume that the delay τ_r is 1s, then the system can be presented by the 2×2 monodromy matrix

$$A = \begin{bmatrix} 1 - 0.045d_1 & 1.5 - 0.0135d_1 - 0.045d_2 \\ -0.3d_1 & 1 - 0.105d_1 - 0.3d_2 \end{bmatrix}$$

given by (10). It can be seen that the pole placement problem is now reduced to the placement of the two eigenvalues of Φ using the D called feedback matrix. By using the appropriate D, the both eigenvalues of the Φ can be moved to zero. By calculating the Φ, it can be obtained that these optimal parameters are $d_1 = 2.2157$ and $d_2 = 5.5588$. The simulation is shown in Figure 1. It can be seen that the system with act-and-wait controller actually converges to zero within period $2T$.

In the same way, the optimal parameters for $\tau_r = 0.9s$ and $\tau_r = 0.8s$ can be achieved:

$$D_{(\tau_r=0.9)} = \begin{bmatrix} -2.2146 \\ -5.3377 \end{bmatrix}^T,$$
$$D_{(\tau_r=0.8)} = \begin{bmatrix} -2.2157 \\ -5.1157 \end{bmatrix}^T.$$

And the simulations are shown in Figures 2 and 3. In Figures 1–3, it implies that the system can converge to zero with the random delay. In period $2T$ the system stops at zero completing the deadbeat convergence.

2. When $a = 4$. In this situation, system matrix A is unstable. Because of the complexity of the system, it can't be stabilized using an autonomous controller since $a > 2$. If the act-and-wait control mechanism is applied with $t_w = 1.2s$ and $t_a = 0.3s$, then we can achieve the monodromy matrix (15). For simplicity, only the situation Φ with $\tau_r = 1$ is calculated and shown.

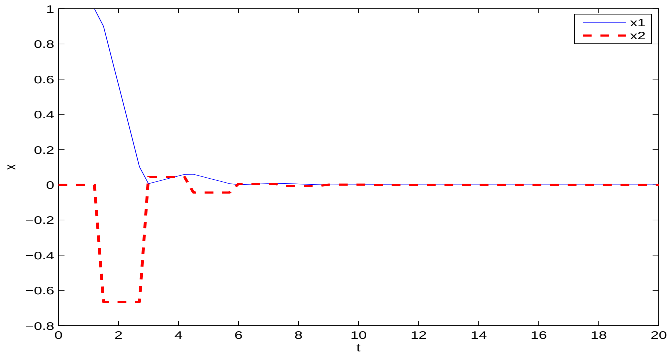

Figure 1. $d_1 = 2.2157$; $d_2 = 5.5588$ (τ= $1s$).

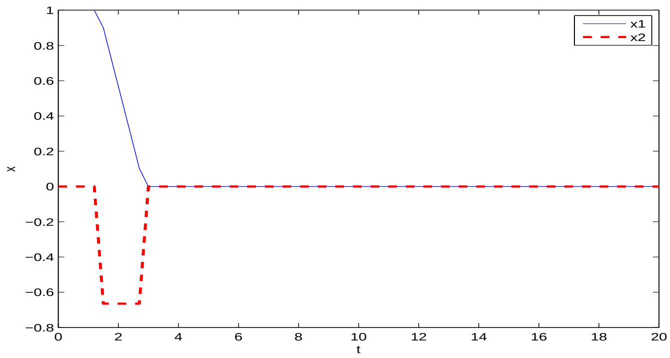

Figure 2. $d_1 = 2.2146$; $d_2 = 5.3377$ (τ= $0.9s$).

By applying the act-and-wait controller, the system can be stabilized, and both eigenvalues can be moved to the origin. With this condition $d_1 = 7.4635$ and $d_2 = 9.8639$ can be obtained. In Figure 4 it can be seen that the state $x_1(t)$ and $x_2(t)$ converges to zero at about 10.5s. It's explicit that the state $x_1(t)$ and $x_2(t)$ grows very quickly in wait period in which the controller is switched off. But the growing tendency of $x_1(t)$ and $x_2(t)$ is restrained in act period. The system is stabilized after several periods of the controller.

In order to show the stability region of the system, the decay ratio $\rho = e^{Re(\lambda_1)}$ is introduced, where λ_1 is rightmost eigenvalue in the pole in the roots figure. In other words, this means $Re(\lambda_1) \geq Re(\lambda_i)$, $i = 2, 3, ..., \infty$. This decay ratio is a measure of the average error decay over a unit period, since $|x(t + 1)| \leq \rho|x(t)|$. Figure 6 shows the stable region of the LTI system with random delay ($a = 0$).

5 Conclusions

In this paper, we consider the stability problem in a continuous LTI system with random feedback delays. In the simulation part, a second-order linear time-invariant system with

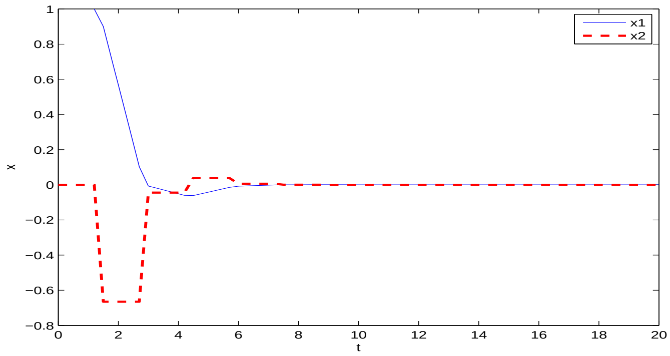

Figure 3. $d_1 = 2.2157$; $d_2 = 5.1157$. (τ= 0.8s).

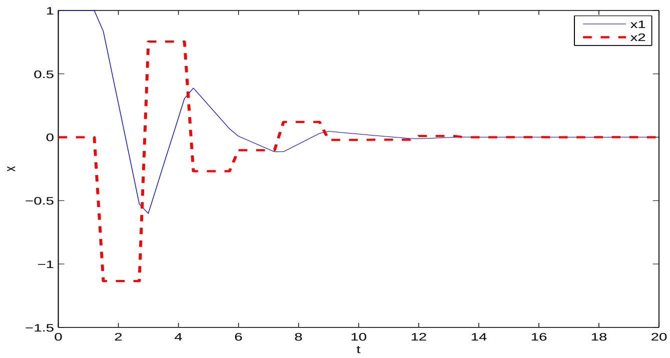

Figure 4. State of the system ($a = 4$).

random feedback delay is introduced to verify availability of the act-and-wait controller. By applying the periodic controller the monodromy matrix only have 2 eigenvalues which are easily placed to original by the control parameters. And the periodic controller can still stabilize the system in some cases while the autonomous one can't work.

Generally speaking, large gain in the controller can result in quick convergence, but continued large gain input may make the system become unstable. So in a control period, large gain can only be used in the acting period. In this way, the controller can keep higher performance and the stability. In the future research work, the algorithm about how to design control parameters and how to select the period of the controller should be investigated in order to obtain optimal performance of the system.

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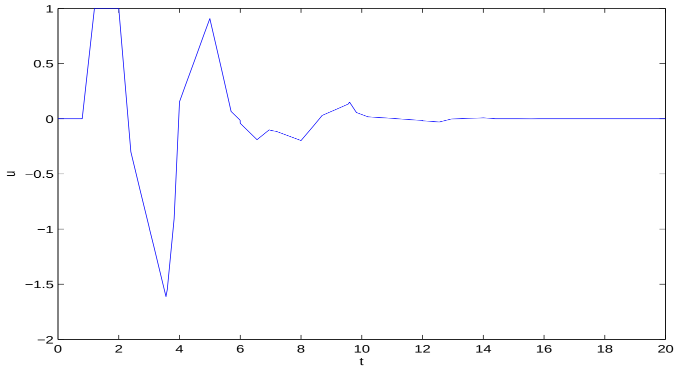

Figure 5. Output of the system ($a = 4$).

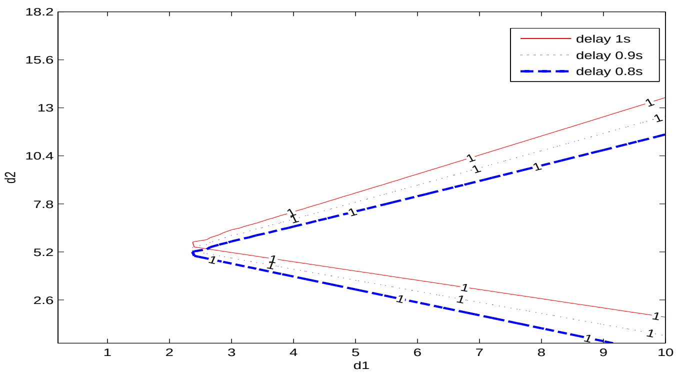

Figure 6. Deterministic stability for three different delays $\tau = 1$, $\tau = 0.9$, $\tau = 0.8$. The intersecting region of these triangular boundaries is stability region in deterministic sense.

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Acceleration Control in Nonlinear Vibrating Systems Based on Damped Least Squares

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Abstract: A discrete time control algorithm using the damped least squares is introduced for acceleration and energy exchange controls in nonlinear vibrating systems. It is shown that the damping constant of least squares and sampling time step of the controller must be inversely related to insure that vanishing the time step has little effect on the results. The algorithm is illustrated on two linearly coupled Duffing oscillators near the 1:1 internal resonance. In particular, it is shown that varying the dissipation ratio of one of the two oscillators can significantly suppress the nonlinear beat phenomenon.

Keywords: *damped least squares; acceleration control.*

Mathematics Subject Classification (2010): 34H05.

1 Introduction

The damped least squares is a simple but effective analytical manipulation that helps to avoid singularity in practical minimization and control algorithms. It is also known as Levenberg-Marquardt method [11]. In order to illustrate the idea in simple terms, let us consider the minimization problem

$$\|E - A\delta u\|^2 \rightarrow \min, \tag{1}$$

where $E \in R^n$ is a given vector, the notation $\|...\|$ indicates the Euclidean norm in R^n, A is typically a Jacobian matrix of n rows and m columns, and $\delta u \in R^m$ is an unknown minimization vector. Although a formal solution of this problem is given by $\delta u = (A^T A)^{-1} A^T E$, the matrix product $A^T A$ may appear to be singular so that no unique solution is possible. This fact usually points to multiple possibilities of achieving

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the same result unless specific conditions are imposed on the vector δu. The idea of damped least squares is to avoid such conditioning by adding one more quadratic form to the left hand side of expression (1) as follows

$$\|E - A\delta u\|^2 + \lambda \|\delta u\|^2 \to \min, \tag{2}$$

where λ is a positive scalar number, which is often called *damping constant*; note that the term 'damping' has no relation to the physical damping or energy dissipation effects in vibrating systems usually characterized by *damping ratios*.

Now the inverse matrix includes the damping constant λ which can provide the uniqueness of solution given by

$$\delta u = (A^T A + \lambda I)^{-1} A^T E, \tag{3}$$

where I is $n \times n$ identity matrix.

Different arguments are discussed in the literature regarding the use of damped least squares and best choice for the damping parameter λ [1], [2], [3], [4], [6], [7], [9], [10], [15], [16], [17], [23], [24]. In particular, it was noticed that the parameter λ may affect convergence properties of the corresponding algorithms. The parameter λ can be used also for other reason such as shifting the solution δu into desired area in R^m. In this case, the meaning of λ is rather close to that of Lagrangian multiplier imposing constraints on control inputs.

In case of dynamical systems, when all the quantities in (2) may depend on time, a continuous time analogue of (2) can be written in the integral form

$$\min_{\delta u} \int_0^T (\|E - A\delta u\|^2 + \lambda \|\delta u\|^2) dt, \tag{4}$$

where the interval of integration is manipulated as needed, for instance, T can be equal to sampling time of the controller [12].

However, in the present work, a discrete time algorithm based on the damped least squares solution (3), which is used locally at every sample time t_n, is introduced. Such algorithm appears to be essentially discrete namely using different time step h may lead to different results. Nevertheless, if the parameters λ and h are coupled by some condition then the control input and system response show no significant dependence on the time step.

A motivation for the present work is as follows. In order to comply with the standard tool of dynamical systems dealing with differential equations, the methods of control are often formulated in continuous time by silently assuming that a discrete time analogous is easy to obtain one way or another whenever it is needed for practical reasons. For instance, data acquisition cards and on-board computers of ground vehicles usually acquire and process data once per 0.01 sec. Typically, based on the information, which is known about the system dynamic states and control inputs by the time instance t_n, the computer must calculate control adjustments for the next active time instance, t_{n+1}. The corresponding computational time should not therefore exceed $t_{n+1} - t_n = 0.01$ sec. Generally speaking, it is possible to memorize snapshots of the dynamic states and control inputs at some of the previous times $\{..., t_{n-2}, t_{n-1}\}$. However, increasing the volume of input data may complicate the code and, as a result, slow down the calculation process. Therefore, let us assume that updates for the control inputs are obtained by processing the system states, controls, and target states given only at the current time instance, t_n. The corresponding algorithm can be built on the system model described by its differential equations of motion and some rule for minimizing the deviation (error) of the current dynamic states from the target. Recall that, in the present work, such a rule will be defined according to the damped least squares (2). Illustrating physical example of two linearly coupled Duffing oscillators is considered. It is shown that the corresponding algorithm, which is naturally designed and effectively working in discrete time, may face a problem of transition to the continuous time limit.

2 Problem Formulation

Consider the dynamical system

$$\ddot{x} = f(x, \dot{x}, t, u), \tag{5}$$

where $x = x(t) \in R^n$ is the system position (configuration) vector, the overdot indicates derivative with respect to time t, the right-hand side $f \in R^n$ represents a vector-function that may be interpreted as a force per unit mass of the system, and $u = u(t) \in R^m$ is a control vector, whose dimension may differ from that of the positional vector x so that generally $n \neq m$.

In common words, the purpose of control $u(t)$ is to keep the acceleration $\ddot{x}(t)$ of system (5) as close as possible to the target $\ddot{x}^*(t)$. The term 'close' will be interpreted below through a specifically designed target function of the following error vector

$$E(t) = \ddot{x}^*(t) - \ddot{x}(t). \tag{6}$$

As discussed in Introduction, for practical implementations, the problem must be formulated in terms of the discrete time $\{t_k\}$ as follows. Let $x_k = x(t_k)$, $\dot{x}_k = \dot{x}(t_k)$, and $u_k = u(t_k)$ are observed at some time instance t_k. The corresponding target acceleration, $\ddot{x}^*_k = \ddot{x}^*(t_k)$, is assumed to be known. Then, taking into account (5) and (6), gives the following error at the same time instance

$$E_k = \ddot{x}^*_k - f(x_k, \dot{x}_k, t_k, u_k). \tag{7}$$

Now the purpose of control is to minimize the following target function

$$\begin{aligned} P_k &= \frac{1}{2} E_k^T W_k E_k \qquad (8) \\ &= \frac{1}{2}[\ddot{x}^*_k - f(x_k, \dot{x}_k, t_k, u_k)]^T W_k [\ddot{x}^*_k - f(x_k, \dot{x}_k, t_k, u_k)], \end{aligned}$$

where W_k is $n \times n$ diagonal weight matrix whose elements are positive or at least non-negative functions of the system states, $W_k = W(x_k, \dot{x}_k, t_k)$.

Note that all the quantities in expression (8) represent a snapshot of the system at $t = t_k$ while including no data from the previous time step t_{k-1}. Since the control vector u_k cannot be already changed at time t_k then quantity P_k is out of control at time t_k. In other words expression (8) summarizes all what is observed *now*, at the time instance t_k. The question is how to adjust the control vector u for the next step t_{k+1} based on the information included in (8) while the system state at $t = t_{k+1}$ is yet unknown, and no information from the previous times $\{..., t_{n-2}, t_{n-1}\}$ is available.

Let us represent such an update for the control vector in the form

$$u_{k+1} = u_k + \delta u_k, \tag{9}$$

were δu_k is an unknown adjustment of the control input.

Replacing u_k in (8) by (9) and taking into account that

$$\begin{aligned} f(x_k, \dot{x}_k, t_k, u_{k+1}) &= f(x_k, \dot{x}_k, t_k, u_k) + A_k \delta u_k + O(\|\delta u_k\|^2), \\ A_k &= \partial f(x_k, \dot{x}_k, t_k, u_k)/\partial u_k, \end{aligned} \tag{10}$$

gives

$$P_k = \frac{1}{2}(E_k - A_k \delta u_k)^T W_k (E_k - A_k \delta u_k), \tag{11}$$

where A_k is the Jacobian matrix of n rows and m columns.

Although the replacement u_k by u_{k+1} in (10) may look artificial, this is how the update rule for the control vector u is actually defined here. Namely, if u_k did not provide a minimum for $P_k(\ddot{x}_k^*, x_k, \dot{x}_k, t_k, u_k)$, then let us minimize $P_k(\ddot{x}_k^*, x_k, \dot{x}_k, t_k, u_k + \delta u_k)$ with respect to δu_k and then apply the adjusted vector (9) at least the next time, t_{n+1}. Assuming that the variation δu_k is small, in other words, u_k is still close enough to the minimum, expansion (10) is applied. Now the problem is formulated as a minimization of the quadratic form (11) with respect to the adjustment δu_k. However, what often happens practically is that function (11) has no unique minimum so that equation

$$\frac{dP_k}{d\delta u_k} = 0 \tag{12}$$

has no unique solution. In addition, even if the unique solution does exist, it may not satisfy some conditions imposed on the control input due to the physical specifics of actuators. As a result, some constraint conditions may appear to be necessary to impose on the variation of control adjustment, δu_k. However, the presence of constraints would drastically complicate the problem. Instead, the target function (11) can be modified in order to move solution δu_k into the allowed domain. For that reason, let us generalize function (11) as

$$\begin{aligned} P_k &= \frac{1}{2}(E_k - A_k \delta u_k)^T W_k (E_k - A_k \delta u_k) \\ &\quad + \frac{1}{2}(B_k + C_k \delta u_k)^T \Lambda_k (B_k + C_k \delta u_k), \end{aligned} \tag{13}$$

where $\Lambda_k = \Lambda(x_k, \dot{x}_k, t_k)$ is a diagonal regularization matrix, $B_k = B(x_k, \dot{x}_k, t_k)$ is a vector-function of n elements, and $C_k = C(x_k, \dot{x}_k, t_k)$ is a matrix of n rows and m columns.

Note that the structure of new function (13) is a generalization of (2). Substituting (13) in (12), gives a linear set of equations in the matrix form whose solution δu_k brings relationship (9) to the form

$$u_{k+1} = u_k + (A_k^T W_k A_k + C_k^T \Lambda_k C_k)^{-1} (A_k^T W_k E_k - C_k^T \Lambda_k B_k). \tag{14}$$

The entire discrete time system is obtained by adding a discrete version of the dynamical system (5) to (14) . Assuming that the time step is fixed, $t_{k+1} - t_k = h$, a simple discrete version can be obtained by means of Euler explicit scheme as follows

$$\begin{aligned} x_{k+1} &= x_k + h v_k, \\ v_{k+1} &= v_k + h f(x_k, v_k, t_k, u_k). \end{aligned} \tag{15}$$

Finally, equations (14) and (15) represent a discrete time dynamical system, whose motion should follow the target acceleration $\ddot{x}_k^* = \ddot{x}^*(t_k)$.

It will be shown in the next section that the structure of equation (14) does not allow for the transition to continuous limit of the entire dynamic system (14) through (15), unless some specific assumption are imposed on the parameters in order to guarantee that $\delta u_k = O(h)$ as $h \to 0$.

3 Illustrating Example

The algorithm, which is designed in the previous section, is applied now to a two-degrees-of-freedom nonlinear vibrating system for an *active* control of the energy exchange (nonlinear beats) between the two oscillators. The problem of *passive* control of energy flows in vibrating systems is of great interest [22], and it is actively discussed from the standpoint of nonlinear beat phenomena [14]. The beating phenomenon takes place when frequencies of the corresponding linear oscillators are either equal or at least close enough to each other.

For illustrating purposes, let us consider two unit-mass Duffing oscillators of the same linear stiffness K coupled by the linear spring of stiffness γ. The system position is described by the vector-function of coordinates, $x(t) = (x_1(t), x_2(t))^T$. Introducing the parameters $\Omega = (\gamma + K)^{1/2}$ and $\varepsilon = \gamma/(\gamma + K)$, brings the differential equations of motion to the form

$$\begin{aligned} \dot{x}_1 &= v_1, \\ \dot{x}_2 &= v_2, \\ \dot{v}_1 &= -2\zeta\Omega v_1 - \Omega^2 x_1 + \varepsilon(\Omega^2 x_2 - \alpha x_1^3) \equiv f_1(x_1, x_2, v_1), \\ \dot{v}_2 &= -2u\Omega v_2 - \Omega^2 x_2 + \varepsilon(\Omega^2 x_1 - \alpha x_2^3) \equiv f_2(x_1, x_2, v_2, u), \end{aligned} \tag{16}$$

where α is a positive parameter, ζ and u are damping ratios of the first and the second oscillators, respectively; the damping ratio u, which is explicitly shown as an argument of the function $f_2(x_1, x_2, v_2, u)$, will be considered as a control input.

The problem now is to find such variable damping ratio $u = u(t)$ under which the second oscillator accelerates as close as possible to the given (target) acceleration, $\ddot{x}_2^*(t)$.

Following the discussion of the previous section, let us consider the problem in the discrete time $\{t_k\}$. In order to avoid confusion, the iterator k will be separated from the vector component indexes by coma, for instance, $x_k = (x_{1,k}, x_{2,k})^T$. Since only the second mass acceleration is of interest and the system under consideration includes only one control input u, then, assuming the weights to be constant, gives

$$W_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_k = \frac{\partial}{\partial u_k} \begin{bmatrix} f_{1,k} \\ f_{2,k} \end{bmatrix},$$

where $f_{1,k} \equiv f_1(x_{1,k}, x_{2,k}, v_{1,k})$ and $f_{2,k} \equiv f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k)$, and other matrix terms become scalar quantities, say, $\Lambda_k = \lambda$, $B_k = b$, and $C_k = 1$. The unities in W_k and C_k can always be achieved by re-scaling the target function and parameters λ and b. Note that re-scaling the target function by a constant factor has no effect on the solution of equation (12).

As mentioned in Introduction, the damping (dissipation) ratio should not be confused with the damping coefficient λ.

As a result, the target function (13) takes the form

$$P_k = \frac{1}{2}\left(\ddot{x}^*_{2,k} - f_{2,k} - \frac{\partial f_{2,k}}{\partial u_k}\delta u_k\right)^2 + \frac{\lambda}{2}(b+\delta u_k)^2. \tag{17}$$

In this case, equation (12) represents a single linear equation with respect to the scalar control adjustment, δu_k. Substituting the corresponding solution in (14) and taking into account (15), gives the discrete time dynamical system

$$u_{k+1} = u_k - \frac{(f_{2,k} - \ddot{x}^*_{2,k})(\partial f_{2,k}/\partial u_k) + \lambda b}{(\partial f_{2,k}/\partial u_k)^2 + \lambda} \tag{18}$$

and

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + hv_{1,k}, \\ x_{2,k+1} &= x_{2,k} + hv_{2,k}, \\ v_{1,k+1} &= v_{1,k} + hf_{1,k}, \\ v_{2,k+1} &= v_{2,k} + hf_{2,k}. \end{aligned} \tag{19}$$

Let us assume now that the target acceleration $\ddot{x}^*_2$ is zero, in other words, the purpose of control is to minimize acceleration of the second oscillator at any sample time t_k as much as possible. Let us set still arbitrary parameter b also to zero. Then the target function (17) and dynamical system (18) and (19) take the form

$$P_k = \frac{1}{2}\left[f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k) + \frac{\partial f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k)}{\partial u_k}\delta u_k\right]^2 + \frac{\lambda}{2}(\delta u_k)^2, \tag{20}$$

$$\begin{aligned} u_{k+1} &= u_k + \frac{2\Omega v_{2,k}}{4\Omega^2 v^2_{2,k} + \lambda} f_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k), \\ x_{1,k+1} &= x_{1,k} + hv_{1,k}, \\ x_{2,k+1} &= x_{2,k} + hv_{2,k}, \\ v_{1,k+1} &= v_{1,k} + hf_1(x_{1,k}, x_{2,k}, v_{1,k}), \\ v_{2,k+1} &= v_{2,k} + hf_2(x_{1,k}, x_{2,k}, v_{2,k}, u_k), \end{aligned} \tag{21}$$

where the functions f_1 and f_2 are defined in (16).

As follows from the first equation in (21), transition to the continuous time limit for the entire system (21) would be possible under the condition that

$$\frac{2\Omega v_{2,k}}{4\Omega^2 v^2_{2,k} + \lambda} = O(h), \quad \text{as} \quad h \to 0. \tag{22}$$

Condition (22) can be satisfied by assuming that $\Omega = O(h)$. Such an assumption, however, makes little if any physical sense. As an alternative choice, the condition $\lambda = O(h^{-1})$ can be imposed by setting, for instance,

$$\lambda h = \lambda_0, \tag{23}$$

where λ_0 remains finite as $h \to 0$.

However, condition (23) essentially shifts the weight on control to the second term of the target function (17) so that the function asymptotically takes the form

$$P_k \simeq \frac{\lambda_0}{2h}(\delta u_k)^2, \quad \text{as} \quad h \to 0. \tag{24}$$

Such a target function leads to the solution $\delta u_k = 0$, which effectively eliminates the control equation. In other words, the iterative algorithm seems to be essentially discrete. As a result, the control input u_k, generated by the first equation in (21), depends upon sampling time interval h. Let us illustrate this observation by implementing the iterations (21) under the fixed set of parameters, $\varepsilon = 0.1$, $\Omega = 1.0$, $\alpha = 1.5$, $\zeta = 0.025$, and initial conditions, $u_0 = 0.025$, $x_{1,0} = 1.0$, $x_{2,0} = 0.1$, $v_{1,0} = v_{2,0} = 0$. The values to vary are two different sampling time intervals, $h = 0.01$ and $h = 0.001$, and three different values of the damping constant, $\lambda = 0.1$, $\lambda = 1.0$, and $\lambda = 10.0$. For comparison reason, Figure 1 shows time histories of the system coordinates under the fixed control variable $u = \zeta$. This (no control) case corresponds to free vibrations of the model (16) whose dynamics represent a typical beat-wise decaying energy exchange between the two oscillators. As mentioned at the beginning of this section, the beats are due to the 1:1 resonance in the generating system ($\varepsilon = 0$, $u = \zeta = 0$); more details on non-linear features of this phenomenon, the related analytical tools, and literature overview can be found in [20] and [14]. In particular, the standard averaging method was applied to the no damping case of system (16) in [20].

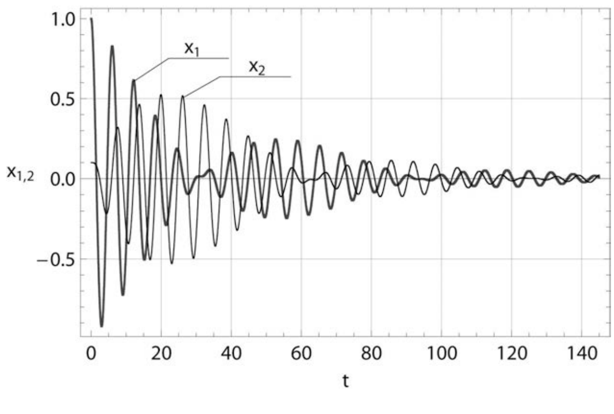

Figure 1: No control beat dynamics with the decaying energy exchange between two Duffing's oscillators; $u = \zeta = 0.025$.

Now the problem is to suppress the beat phenomenon by preventing the energy flow from the first oscillator into the second oscillator. As follows from Figures 2 through 5, such a goal can be achieved by varying the damping ratio of the second oscillator, $\{u_k\}$, during the vibration process according to the algorithm (21) ⋆. First, the diagrams in Figures 2 and 3 confirm that the sampling time interval h represents an essential parameter of the entire control loop. In particular, decreasing the sampling interval from $h = 0.01$ to $h = 0.001$ effectively increases the strength of the control; compare fragments

⋆ Note that, although the algorithm is designed to suppress accelerations of the second oscillator, acceleration and energy levels of vibrating systems are related.

(b) in Figures 2 and 3. However, if such decrease of the sampling time is accompanied by the increase of λ according to condition (23), then the strength of control remains practically unchanged; compare now fragments (b) in Figures 2 and 4. As follows from fragments (a) in Figures 2 and 4, the above modification of both parameters, h and λ, also brings some difference in the system response during the interval $80 < t < 150$, but this is rather due to numerical effect of the time step.

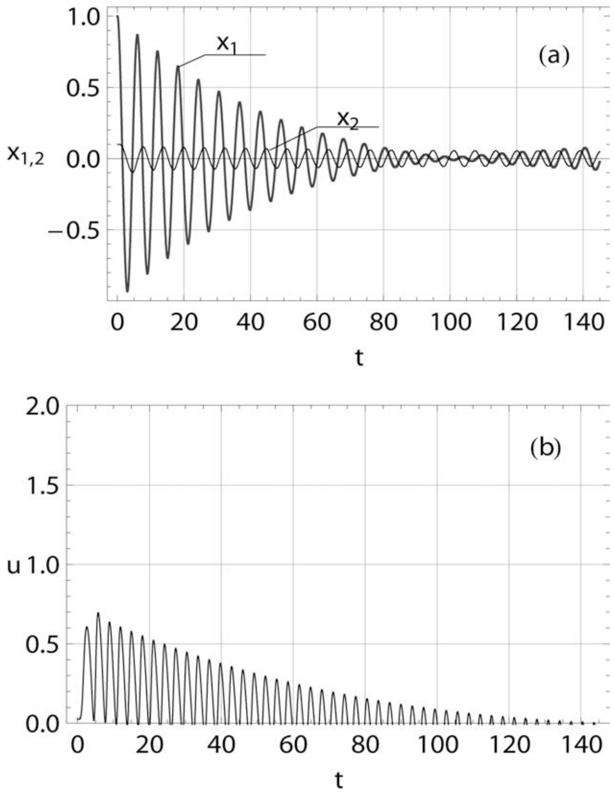

Figure 2: Beat suppression under the time increment $h = 0.01$ and weight parameter $\lambda = 1.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

Finally, analyzing the diagrams in Figures 3 and 5, shows that reducing the parameter λ as many as ten times under the fixed time step h leads to a significant increase of the control input $\{u_k\}$ with a minor effect on the system response though. Therefore the parameter λ can be used for the purpose of satisfying some constraint conditions on the control inputs $\{u_k\}$ in case such conditions are due to physical limits of the corresponding actuators. In addition, let us show that parameter λ may affect the convergence of algorithm (21) based on the following convergence criterion [18]:

For a fixed point z_* to be a point of attraction of the algorithm $z_{k+1} = G(z_k)$ a sufficient condition is that the Jacobian matrix of G at the point z_* has all its eigenvalues numerically less than 1, and a necessary condition is that they are numerically at most 1. The geometric rate of convergence is the numerically largest eigenvalue of this Jacobian.

Applying this criterion to the algorithm (21) at zero point, gives that one of the

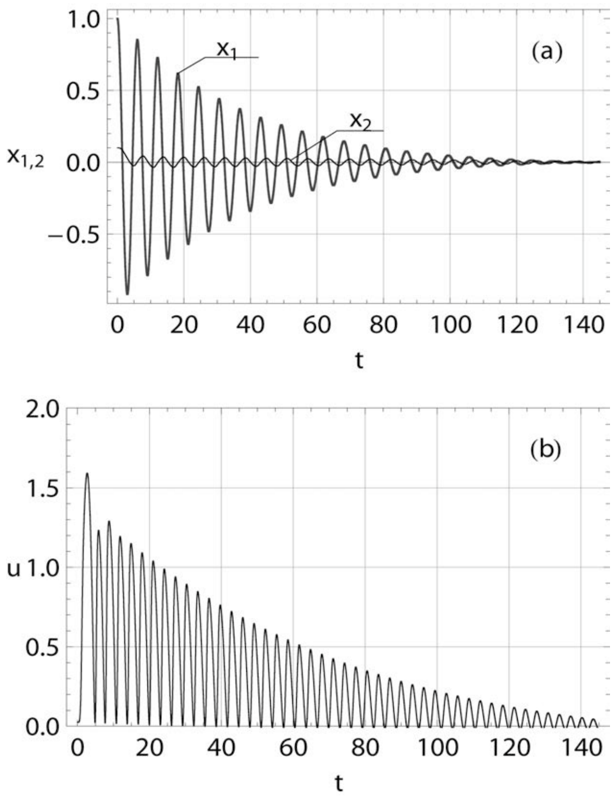

Figure 3: Beat suppression under the reduced time increment $h = 0.001$ and the same weight parameter $\lambda = 1.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

eigenvalues is always zero, $q_0 = 0$, whereas another four eigenvalues, q_i $(i = 1, ..., 4)$ are proportional to the time step, $q_i = hp_i$, where the coefficients p_i are given by the roots of algebraic equation

$$p^4 + 2\zeta\Omega p^3 + 2\Omega^2 p^2 + 2\zeta\Omega^3 p + (1 - \varepsilon^2)\Omega^4 = 0. \tag{25}$$

As follows from (25), the damping coefficient λ has no influence on the convergence condition near the equilibrium point, and the convergence can always be achieved under a small enough time step h. Nevertheless, the damping coefficient may appear to affect the convergence away from the equilibrium point. In this case, analytical estimates for eigen values of the Jacobian become technically complicated unless $\varepsilon = 0$, when four of the five eigenvalues vanish as $h \to 0$, except one eigenvalue, which is estimated by

$$q = -\left(1 + \frac{\lambda}{4\Omega^2 v_2^2}\right)^{-1}. \tag{26}$$

This root gives $q \to q_0 = 0$ as $v_2 \to 0$. However, when $v_2 \neq 0$, equation (26) gives the estimate $0 < q \leq 1$ as $\infty > \lambda \geq 0$. Therefore, only the necessary convergence condition is satisfied for $\lambda = 0$.

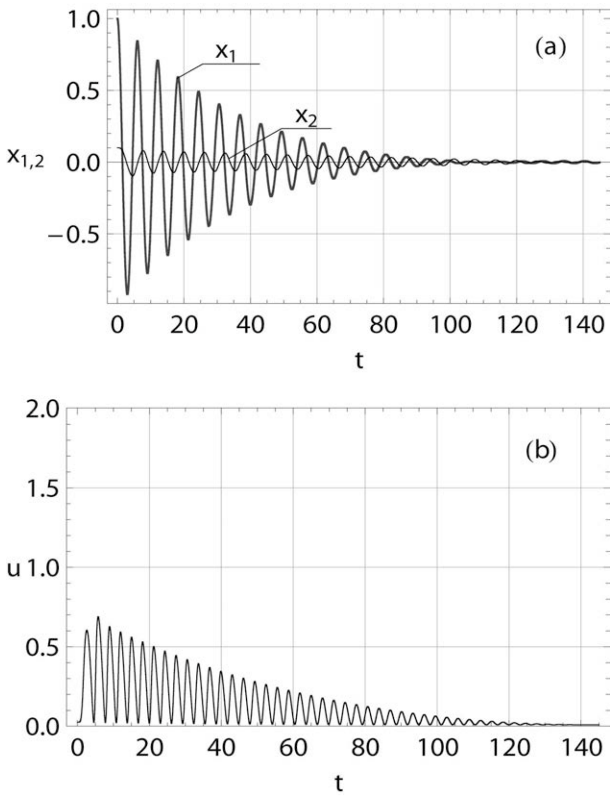

Figure 4: Beat suppression under the reduced time increment $h = 0.001$ but increased weight parameter $\lambda = 10.0$: (a) the system response, (b) control input - the damping ratio of second oscillator.

4 Conclusions

In this work, a discrete time control algorithm for nonlinear vibrating systems using the damped least squares is introduced. It is shown that the corresponding damping constant λ and sampling time step h must be coupled by the condition $\lambda h = constant$ in order to preserve the result of calculation when varying the time step. In particular, the above condition prohibits a direct transition to the continuos time limit. This conclusion and other specifics of the algorithm are illustrated on the nonlinear two-degrees-of-freedom vibrating system in the neighborhood of 1:1 resonance. It is shown that the dissipation ratio of one of the two oscillators can be controlled in such way that prevents the energy exchange (beats) between the oscillators. From practical standpoint, controlling the dissipation ratio can be implemented by using devices based on the physical properties of magnetorheological fluids (MRF) [8], [19]. In particular, different MRF dampers are suggested to use for semi-active ride controls of ground vehicles and seismic response reduction.

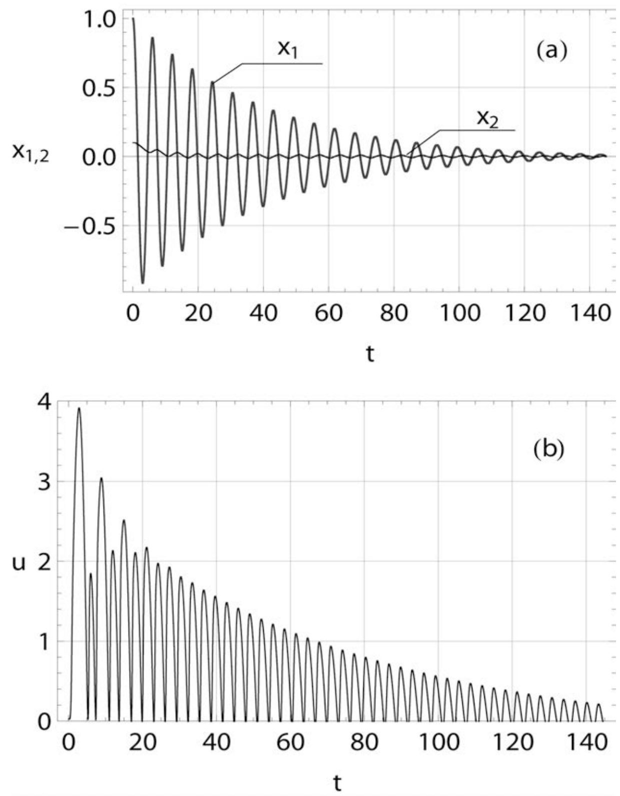

Figure 5: Beat suppression under the reduced time increment $h = 0.001$ and vanishing weight parameter $\lambda = 0.1$: (a) the system response, (b) control input - the damping ratio of second oscillator.

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Existence and Uniqueness of a Solution of Fisher-KKP Type Reaction Diffusion Equation

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Abstract: In this paper we prove the existence and uniqueness of a strong solution of a Fisher-KKP type reaction diffusion equation with Dirichlet boundary conditions using the method of semidiscretization.

Keywords: *method of semidiscretization; reaction diffusion equation; strong solution; A priori estimate.*

Mathematics Subject Classification (2010): 35K57, 65N40, 35B45, 35D35.

1 Introduction

In this paper we concerned with the following reaction diffusion equation of KPP-Fisher type with Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + ku(t,x)[1-u(t,x)] + f(t,x), \quad t \in (0,T], \quad x \in (0,\pi), \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in (0,\pi), \tag{2}$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in (0,T], \tag{3}$$

where k is a positive constant and $u_0 \in L_2(0,\pi)$.

Since 1930, various classical types of initial boundary value problem have been investigated by many authors using the method of semidiscretization; see for instance [11,15,16] and references therein.

The method of semidescretization in time is a very efficient tool in the study of an approximate solution and its convergence to the solution of the problem. In this

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method we replace the time derivative by the corresponding difference quotients giving rise to a system of time independent operator equations. With the help of the theory of semigroups, these systems are guaranteed to have unique solutions. An approximate solution to the given problem is defined in terms of the solutions of these time independent systems. After proving a priori estimates for the approximate solution, the convergence of the approximate solution to a unique strong solution is established.

In this paper my aim is to apply the method of semidiscretization to a reaction diffusion equation of KPP-Fisher type with Dirichlet boundary conditions. Fisher-KKP equations are most simple case of nonlinear reaction diffusion equation that was first shown to have traveling wave front by Fisher [18].

This work is motivated by the work of Fisher [18], in which he has considered the Fisher-KKP type reaction diffusion equation:

$$\frac{\partial u}{\partial t} = ru(t,x)\left[1 - \frac{u(t,x)}{K}\right] + D\frac{\partial^2 u}{\partial x^2},$$

where r and D are positive parameters.

Dubey [3], has established the existence and uniqueness of a strong solution for the following nonlinear nonlocal functional differential equation in a Banach X, using the method of semidiscretization:

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t), u_t), \quad t \in (0,T], \\ h(u_0) &= \phi \quad \text{on} \quad [-\tau, 0], \end{aligned}$$

where $0 < T < \infty$, $\phi \in C_0 := C([-\tau,0];X)$, $\tau > 0$, the nonlinear operator A is singlevalued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0,T] \times X \times C_0 := C([-\tau,0];X)$ into X, the map h is defined from C_0 into C_0. For $u \in C_T := C([-\tau,T];X)$, function $u_t \in C_0$ is given by $u_t(s) = u(t+s)$ for $s \in [-\tau, 0]$. Here $C_t := C([-\tau,t];X)$ for $t \in [0,T]$ is the Banach space of all continuous functions from $[-\tau,t]$ into X endowed with the supremum norm

$$\|\phi\|_t = \sup_{-\tau \le \eta \le t} \|\phi(\eta)\|, \quad \phi \in C_t.$$

Bouziani, Merchri [17] and Lakoud, Chaoui [14] have applied the method of semidiscretization to integrodifferential equations, and prove the existence and uniqueness of a weak solution. For the application of method of semidiscretization to delayed cooperation diffusion system with Dirichlet boundary conditions, we refer readers to [19]. For the more applications of Rothe method to integrodifferential equations, parabolic problems, hyperbolic problems, we refer readers to [9, 10, 12, 13] and references therein.

By literature, it is clear that method of semidiscretization is applicable in many physical, mathematical, biological problems modeled by partial differential equations.

The plan of the rest paper is as follows. In Section 2, we state some basic results and definitions that will be used in the next sections. In Section 3, we state the main result. In the last section, we state and prove all the lemmas that are required to prove the main result and at the end of this section, we prove the main result.

2 Preliminaries

We define

$$B_R(0) = \{u \in L^2(0,\pi) : \|u\| \le R\}.$$

Now we define a function $F : (0, T] \times B_R(0) \to B_R(0)$ by

$$F(t, \chi)(x) = k\chi[1 - \chi](x) + f(t, x).$$

Consider that $H := L^2[0, \pi]$ is the real Hilbert space of all real-valued square-integrable functions on the interval $[0, \pi]$, let the linear operator A be defined by

$$D(A) := \{u \in H : u'' \in H, u(0) = u(\pi) = 0\}, \quad Au = -u''.$$

Then we know that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t),\ t \geq 0$ of contractions in H.

If we identify $u : (0, T] \to H$, by $u(t)(x) = u(t, x)$, and $f : (0, T] \to H$ by $f(t)(x) = f(t, x)$, then (1)-(3) reduce to:

$$\frac{\partial u(t)}{\partial t} + Au(t) = F(t, u(t)), \tag{4}$$

$$u(0) = u_0. \tag{5}$$

Lemma 2.1 *There exists a constant $L_F(R) > 0$ such that*

$$\|F(t, \chi_1) - F(t, \chi_2)\| \leq L_F(R)\|\chi_1 - \chi_2\|,$$

for all $\chi_1, \chi_2 \in B_R(0), \quad t \in (0, T]$.

Proof. Now for any $\chi_1, \chi_2 \in B_R(0)$ and $t \in (0, T]$, we have

$$\begin{aligned}
&\|F(t, \chi_1) - F(t, \chi_2)\|_2^2 \\
&= \int_0^\pi |F(t, \chi_1)(x) - F(t, \chi_2)(x)|^2 dx \\
&= \int_0^\pi |k\chi_1(1 - \chi_1)(x) - k\chi_2(1 - \chi_2)(x)|^2 dx \\
&\leq k^2 \int_0^\pi (|\chi_1(x) - \chi_2(x)|^2 + |\chi_2^2(x) - \chi_1^2(x)|^2) dx \\
&\leq k^2 \int_0^\pi |\chi_1(x) - \chi_2(x)|^2 (1 + |\chi_1(x) + \chi_2(x)|^2) dx \\
&\leq k^2 \int_0^\pi |\chi_1(x) - \chi_2(x)|^2 dx \int_0^\pi (1 + |\chi_1(x) + \chi_2(x)|^2) dx \\
&\leq k^2 \|\chi_1 - \chi_2\|_2^2 (\pi + \|\chi_1 + \chi_2\|^2) \\
&\leq k^2 (\pi + 2R^2) \|\chi_1 - \chi_2\|_2^2.
\end{aligned}$$

This implies that

$$\|F(t, \chi_1) - F(t, \chi_2)\|_2 \leq L'_F(R)\|\chi_1 - \chi_2\|_2,$$

where $L'_F(R) = k\sqrt{\pi + 2R^2}$. $\square$

Lemma 2.2 *If f satisfies a Lipschitz-like condition, i.e., there exists a constant $k_1 > 0$ such that*

$$\|f(t) - f(s)\| \leq k_1 \mid t - s \mid, \quad \forall t, s \in (0, T],$$

then F also satisfies a Lipschitz condition in $(0, T]$, i.e.,

$$\|F(t, \chi) - F(s, \chi)\| \leq k_1 \mid t - s \mid, \quad \forall t, s \in (0, T].$$

Remark 2.1 From Lemma 2.1 and Lemma 2.2, we conclude that F satisfies a local Lipschitz condition, i.e., there exists a constant $L_F(R) > 0$ such that

$$\|F(t,\chi_1) - F(s,\chi_2)\| \leq L_F(R)[|t-s| + \|\chi_1 - \chi_2\|_2], \quad \forall t,s \in (0,T], \quad \forall \chi_1,\chi_2 \in B_R(0).$$

Definition 2.1 Let X be a Banach space and let X^* be its dual. For every $x \in X$ we define the duality map J as:

$$J(x) = \{x^* : \ x^* \in X^* \quad \text{and} \quad (x^*, x) = \|x\|^2 = \|x^*\|^2\},$$

where (x^*, x) denotes the value of x^* at x.

Lemma 2.3 ([1], Theorem 1.4.3) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions then A is m-accretive, i.e.,*

$$(Au, J(u)) \geq 0 \quad \textit{for} \ \ u \in D(A),$$

where J is the duality mapping and $R(I + \lambda A) = X$ for $\lambda > 0$, I is the identity operator on X and $R(.)$ is the range of an operator.

Lemma 2.4 ([2], Lemma 2.5(a)) *If $-A$ is the infinitesimal generator of a C_0-semigroup of contractions, $X^n \in D(A)$, $n = 1, 2, 3, ...$, $X^n \to u \in H$ and $\|AX^n\|$ are bounded, then $u \in D(A)$ and $AX^n \rightharpoonup Au$.*

A function $u \in C([0,T], H)$ such that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(s, u(s))ds, \quad \text{if} \quad t \in [0,T].$$

is called a mild solution of (4)-(5).

By a strong solution of (4)-(5) we mean a function $u \in C([0,T], X)$ such that $u(t) \in D(A)$ for a.e. $t \in [0,T]$, u is differentiable a.e. on $[0,T]$ and

$$u'(t) + Au(t) = F(t, u(t)), \qquad \text{a.e.} \ \ t \in [0,T].$$

3 Main Result

Theorem 3.1 *Under the conditions of Lemma 2.1 and Lemma 2.2, problem (4)-(5) has a unique strong solution on the interval $[0, t_0]$, $0 < t_0 < T$ which can be uniquely continued either on [0,T], or on the maximal interval of existence $[0, t_{max}[$, $0 < t_{max} \leq T$. If $0 < t_{max} < T$, then*

$$\lim_{t \uparrow t_{max}} \|u(t)\| = \infty.$$

We will prove this result by using the method of semidiscretization.

4 Discretization and A priori Estimates

To apply the method of semidiscretization we divide the interval $[0, t_0]$ into the subintervals of length $h_n = \frac{t_0}{n}$ and replace (4) and (5) by the following approximate equations

$$\begin{aligned} \frac{u_j^n - u_{j-1}^n}{h_n} + Au_j^n &= F(t_j^n, u_{j-1}^n), &(6)\\ u_0^n &= u_0. &(7)\end{aligned}$$

Existence of a unique $u_j^n \in H$, satisfying (6) and (7) is a consequence of Lemma 2.3.

Now we construct Rothe's sequence

$$u_n(t) = u_{j-1}^n + \frac{u_j^n - u_{j-1}^n}{h_n}(t - t_j^n), \quad t \in [t_{j-1}^n, t_j^n]. \tag{8}$$

Also, we construct a sequence of step functions:

$$X^n(t) = \begin{cases} u_0, & \text{if} \quad t = 0, \\ u_j^n, & \text{if} \quad t \in (t_{j-1}^n, t_j^n]. \end{cases} \tag{9}$$

Now we state and prove the following two lemmas which are required to prove the main result.

Lemma 4.1 *There exists a constant C_1 (independent of n, j and h_n) such that $\|u_j^n - u_0\| \leq C_1$ (note that here C_1 is a generic constant that may have different value in the same discussion).*

Proof. Substituting $j = 1$ in (6), we get

$$\frac{u_1^n - u_0^n}{h_n} + Au_1^n = F(t_1^n, u_0^n).$$

Subtracting Au_0 from both sides and applying $J(u_1^n - u_0)$ on both sides, we get

$$\begin{aligned}\left(\frac{u_1^n - u_0}{h_n}, J(u_1^n - u_0)\right) + (A(u_1^n - u_0), J(u_1^n - u_0)) = (F(t_1^n, u_0), J(u_1^n - u_0))\\ -(Au_0, J(u_1^n - u_0)).\end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\begin{aligned} \frac{1}{h_n}\|u_1^n - u_0\|^2 &\leq \|F(t_1^n, u_0)\|\|u_1^n - u_0\| + \|Au_0\|\|u_1^n - u_0\| \\ \Longrightarrow \|u_1^n - u_0\| &\leq h_n[\|F(t_1^n, u_0)\| + \|Au_0\|]. \end{aligned}$$

Using Remark 2.1, we can obtain

$$\begin{aligned} \|F(t_1^n, u_0)\| &\leq \|F(t_1^n, u_0) - F(0, u_0)\| + \|F(0, u_0)\| \\ &\leq L_F(R)|t_1^n| + \|F(0, u_0)\| \\ &\leq L_F(R)t_0 + \|F(0, u_0)\|. \end{aligned}$$

Using the above inequality, we get

$$\begin{aligned}\|u_1^n - u_0\| &\leq h_n[L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|] \\ &\leq t_0[L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|] = C_1.\end{aligned}$$

To prove this lemma, we will use induction method, for this we assume that

$$\|u_i^n - u_0\| \leq C_1, \quad i = 1, \cdots, j-1.$$

We have to show that

$$\|u_j^n - u_0\| \leq C_1.$$

Subtracting Au_0 from both sides of (6), and applying $J(u_j^n - u_0)$, we get

$$\begin{aligned}&\left(\frac{u_j^n - u_0}{h_n}, J(u_j^n - u_0)\right) + (A(u_j^n - u_0), J(u_j^n - u_0)) \\ &= \left(\frac{u_{j-1}^n - u_0}{h_n}, J(u_j^n - u_0)\right) + (F(t_j^n, u_{j-1}^n), J(u_j^n - u_0)) - (Au_0, J(u_j^n - u_0)).\end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\begin{aligned}\frac{1}{h_n}\|u_j^n - u_0\|^2 &\leq \frac{1}{h_n}\|u_{j-1}^n - u_0\|\|u_j^n - u_0\| + \|F(t_j^n, u_{j-1}^n)\|\|u_j^n - u_0\| \\ &\quad +\|Au_0\|\|u_j^n - u_0\| \\ \implies \|u_j^n - u_0\| &\leq \|u_{j-1}^n - u_0\| + h_n[\|F(t_j^n, u_{j-1}^n)\| + \|Au_0\|].\end{aligned}$$

By using induction hypothesis, we obtain

$$\|u_j^n - u_0\| \leq C_1 + t_0[\|F(t_j^n, u_{j-1}^n)\| + \|Au_0\|].$$

Using Remark 2.1, we get

$$\begin{aligned}\|F(t_j^n, u_{j-1}^n)\| &\leq \|F(t_j^n, u_{j-1}^n) - F(0,u_0)\| + \|F(0,u_0)\| \\ &\leq L_F(R)[|t_j^n| + \|u_{j-1}^n - u_0\|] + \|F(0,u_0)\| \\ &\leq L_F(R)[t_0 + C_1] + \|F(0,u_0)\|.\end{aligned}$$

Using the above inequality, we get

$$\|u_j^n - u_0\| \leq C_1 + t_0[L_F(R)(t_0 + C_1) + \|F(0,u_0)\| + Au_0].$$

This completes the proof of the lemma. □

Lemma 4.2 *There exists a constant C_2 (independent of n, j and h_n) such that $\left\|\frac{u_j^n - u_{j-1}^n}{h_n}\right\| \leq C_2$ (note that here C_2 is a generic constant that may have different value in the same discussion).*

Proof. As in the previous lemma, we can show that

$$\left\|\frac{u_1^n - u_0^n}{h_n}\right\| \leq [L_F(R)t_0 + \|F(0,u_0)\| + \|Au_0\|].$$

We will prove this result by induction. For this we assume that

$$\left\| \frac{u_i^n - u_{i-1}^n}{h_n} \right\| \le C_2, \quad i = 1, \cdots, j-1.$$

We have to show that

$$\left\| \frac{u_j^n - u_{j-1}^n}{h_n} \right\| \le C_2.$$

Subtracting from (6) the same equation written for $j-1$, and applying $J(u_j^n - u_{j-1}^n)$ on both sides, we get

$$\begin{aligned} \left(\frac{u_j^n - u_{j-1}^n}{h_n}, J(u_j^n - u_{j-1}^n) \right) \quad &\le \quad \left(\frac{u_{j-1}^n - u_{j-2}^n}{h_n}, J(u_j^n - u_{j-1}^n) \right) \\ &\quad + (F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n), J(u_j^n - u_{j-1}^n)). \end{aligned}$$

Using Lemma 2.3 and the definition of duality map, we get

$$\left\| \frac{u_j^n - u_{j-1}^n}{h_n} \right\| \le \left\| \frac{u_{j-1}^n - u_{j-2}^n}{h_n} \right\| + \|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\|.$$

By using induction hypothesis, we get

$$\left\| \frac{u_j^n - u_{j-1}^n}{h_n} \right\| \le C_2 + \|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\|.$$

By using Remark 2.1, we get

$$\begin{aligned} \|F(t_j^n, u_{j-1}^n) - F(t_{j-1}^n, u_{j-2}^n)\| \quad &\le \quad L_F(R)[t_0 + C_2 h_n] \\ &\le \quad L_F(R)[t_0 + C_2 t_0]. \end{aligned}$$

Using the above inequality, we get

$$\left\| \frac{u_j^n - u_{j-1}^n}{h_n} \right\| \le C_2 + L_F(R)[t_0 + C_2 t_0].$$

This completes the proof of the lemma. □

Remark 4.1 By using Lemma 4.1 and Lemma 4.2, we conclude that sequence $\{u^n(t)\}$ is uniformly Lipschitz continuous and $u^n(t) - X^n(t) \to 0$, as $n \to \infty$, $t \in (0, t_0]$.

If we denote that

$$f^n(t) = F(t_j^n, u_{j-1}^n),$$

and using (8) and (9), then (4) reduces to:

$$\frac{d^-}{dt} U^n(t) + AX^n(t) = f^n(t), \quad t \in (0, t_0], \tag{10}$$

where $\frac{d^-}{dt}$ denotes the left derivative in $(0, t_0]$.
Also, for $t \in (0, t_0]$, we have

$$\int_0^t AX^n(s) ds = u_0 - U^n(t) + \int_0^t f^n(s) ds. \tag{11}$$

Next we prove the convergence of U^n to u in $C([0, t_0], H)$.

Lemma 4.3 ([3], Lemma 3.4) *There exists $u \in C([0,t_0],H)$, such that $U^n \to u$ in $C([0,t_0],H)$ as $n \to \infty$. Moreover, u is Lipschitz continuous on $[0,t_0]$.*

Remark 4.2 Clearly $X^n(t) \in D(A)$, for each n. As $u^n(t) - X^n(t) \to 0$ as $n \to \infty$, $X^n(t) \to u(t) \in H$. Also $\|AX^n\|$ are bounded therefore by Lemma 2.4, it is clear that $AX^n \rightharpoonup Au$.

So for every $x^* \in X^*$ and $t \in (0,t_0]$, we have

$$\int_0^t (AX^n(s),x^*)ds = (u_0,x^*) - (U^n(t),x^*) + \int_0^t (f^n(s),x^*)ds.$$

Using Lemma 4.3, Remark 4.2 and the bounded convergence theorem, we obtain as $n \to \infty$,

$$\int_0^t (Au(s),x^*)ds = (u_0,x^*) - (u(t),x^*) + \int_0^t (F(s,u(s)),x^*)ds. \tag{12}$$

As $Au(t)$ is Bochner integrable on $[0,t_0]$, from (12) we have

$$\frac{d}{dt}u(t) + Au(t) = F(t,u(t)), \quad \text{a.e.} \quad t \in (0,t_0]. \tag{13}$$

Clearly $u \in C([0,t_0];H)$ and differentiable a.e. on $(0,t_0]$ with $u(t) \in D(A)$ a.e. on $(0,t_0]$ satisfying (13). Hence u is a strong solution of (6)-(7) on $[0,t_0]$.

Now we show the uniqueness of this strong solution. For this we assume that u_1 and u_2 are two strong solutions of (6)-(7) on the interval $[0,t_0]$. Let $u = u_1 - u_2$

$$\begin{aligned}&\left(\frac{du(t)}{dt}, J(u(t))\right) + (A(u_1(t)-u_2(t)), J(u_1(t)-u_2(t)))\\ &= (F(t,u_1(t)) - F(t,u_2(t)), J(u(t))).\end{aligned}$$

By Lemma 2.3 and by the definition of duality mapping, we get

$$\frac{d}{dt}\|u(t)\|^2 \le \|F(t,u_1(t)) - F(t,u_2(t))\|\|u(t)\|.$$

Using Lemma 2.1, we get

$$\frac{d}{dt}\|u(t)\|^2 \le K\|u(t)\|^2.$$

This implies that

$$\|u(t)\|^2 \le K\int_0^t \|u(t)\|^2 ds.$$

Applying Grownwall's inequality, we get $u \equiv 0$ on $[0,t_0]$. Hence we get a unique strong solution on the interval $[0,t_0]$.

Strong solution u of (6)–(7) on interval $[0,t_0]$ can be extended on the larger interval $[0,t_0+\delta]$, $\delta > 0$ [[1], Theorem 6.2.2]. Continuing this process, we obtain a unique strong solution either on the whole interval or on the maximal interval of existence $[0,t_{max}]$. If $t_{max} < \infty$, then $\lim\limits_{t \uparrow t_{max}} \|u(t)\| = \infty$, otherwise we get contradiction [[1], Theorem 6.1.4].

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On Stability Conditions of Singularly Perturbed Nonlinear Lur’e Discrete-Time Systems

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Abstract: This paper deals with stability of discrete-time nonlinear Lur’e-type systems. Through the singular perturbations technique, the original system is reduced to a block-diagonal form with slow and fast decoupled modes. Stability conditions of the two-time-scale decoupled model based on Borne-Gentina practical stability criterion and the use of matrices in the Benrejeb arrow form are developed and compared with those concerning the original discrete-time system. It is shown that these results are practical and less conservative then the existing ones. A third order system is introduced to illustrate the efficiency of the proposed approach.

Keywords: *discrete Lur’e systems; singular perturbations technique; two-time-scale systems; stability; arrow form matrix.*

Mathematics Subject Classification (2010): 34H15, 34K35.

1 Introduction

During the past several decades, the stability problem of dynamical systems has attracted an immense attention in the control society. A great majority of the encountered problems is concerned with the closed-loop behavior of feedback nonlinear systems. An important and typical class of such systems is Lur’e-type systems introduced by Lur’e and Postnikov [39], and described by combinations of a dynamic linear bloc and a feedback interconnected to a static nonlinearity, assumed to lie in a given sector. Since that,

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Lur'e systems have become an attractive research subject and have received a series of results in many relevant nonlinear engineering applications, such as mechanical, electrical, economic and biological [55].

The original analysis was motivated by the need to understand the effect of nonlinearities on control systems due to elements such as imperfect actuators or sensors that have gain or amplification that can vary over time. Within this framework, the nonlinearities are most commonly modeled as gain bounded or sector bounded uncertainties and the absolute stability is analyzed via the formulation of finite system of quadratic equations.

Defined as a global asymptotic stability tolerating any nonlinear perturbations with special constraints [57], the absolute stability problem has been the subject of extensive research for continuous Lur'e systems [11, 15, 22, 24, 25, 31, 36, 44, 49, 50, 53]. One of the most main results related to absolute stability has been the Popov criterion [43], which is a graphical construction that provides a simple approach to maximize the nonlinear sector. Popov proved that the analysis can be done in the frequency domain and the stability is derived by Lyapunov's direct method. The circle criterion [21,29], dealing with time varying nonlinearity, analyzes the absolute stability via a suitable strict positive-realness condition on the linear part and a given sector condition on the nonlinear part. Recently, more results about the stability analysis for Lur'e systems with slope-restricted are introduced in [3, 33, 41, 42, 48, 55], and with time-delays and model uncertainties in [7, 17, 23, 26, 32, 51].

Because of their wide applications in many practical processes, a great number of results in control and system theory have been extended successfully to singular systems [13]. The two-time-scale nature of such systems is exploited to decompose the design problem into two lower-order design problems for the slow and fast modes. Some results on singular perturbed nonlinear Lur'e systems in continuous-time are developed in the field [13,52,54] where the stability criterion is deduced by mean of Lyapunov functional method. However, the stability investigation on Lur'e type discrete time systems is limited [31].

The paper is organized as follows. The class of discrete Lur'e-type systems will be introduced in Section 2. In Section 3 a two-time-scale decoupling procedure for the original Lur'e-type system based on singular perturbation technique is presented. In Section 4 stability conditions of original Lur'e-type system and decoupled model, are derived and compared. The synthesized results are formulated by the use of the Benrejeb arrow form matrix and the Borne-Gentina practical stability criterion. In Section 5 the proposed model decoupling strategy is applied to a nonlinear system of order three. Stability conditions of original system and reduced order subsystems are developed and discussed.

2 System Description and Problem Statement

Consider the Lur'e type discrete-time system described by state space representation (1). The model consists of a static nonlinearity in cascade with a dynamic linear time invariant system according to [11] and [29]:

$$S : \begin{cases} x_{k+1} = A_L x_k + B_L u_k, \\ u_k = h\left(\varepsilon_k\right)\varepsilon_k, \\ \varepsilon_k = r_k - C_L x_k, \end{cases} \tag{1}$$

where A_L, B_L and C_L are known matrices of appropriate dimensions, $x_k \in \Re^n$ denotes the state vector, $u_k \in \Re$ is the vector input, $r_k \in \Re$ is the reference input, $r_k = 0$ and $\varepsilon_k \in \Re$ is the control error of the closed-loop system, $h\,(.) : \Re \to \Re$ represents memoryless nonlinear matrix valued function.

The investigated Lur'e-type discrete-time system can be represented by the nonlinear regression equation:

$$\varepsilon_{k+n} + \sum_{i=1}^{n} g_i\,(\varepsilon_{k+n-i})\varepsilon_{k+n-i} = 0, \tag{2}$$

where the corresponding expression in terms of state space representation (1) becomes:

$$S : x_{k+1} = A\,(\varepsilon_k)\,x_k \tag{3}$$

with

$$A\,(\varepsilon_k) = A_L - B_L h\,(\varepsilon_k)\,C_L, \tag{4}$$

$A\,(\varepsilon_k)$ denotes the instantaneous characteristic matrix expressed in Frobenius form as:

$$A\,(\varepsilon_n) = \begin{bmatrix} 0 & \cdots & 0 & -g_n\,(\varepsilon_n) \\ 1 & 0 & \vdots & -g_{n-1}\,(\varepsilon_n) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -g_1\,(\varepsilon_n) \end{bmatrix}. \tag{5}$$

In the design of complex and/or large scale systems, models are usually of high order. Model reduction techniques can be used to obtain a low-order approximation of these models, allowing for efficient analysis or control design. Many order reduction techniques can be found in the literature: reduced order models synthesized via aggregation and dominant modes approaches neglect fast stable dynamics and some of the poorly controllable and observable slow dynamics. With the singular perturbation method [1, 14, 35, 38, 47], both slow and fast dynamics are retained; analysis and design problems are solved in two steps, first for the fast and then for the slow dynamics. These methods for model reduction of nonlinear systems have in common that the stability of the reduced-order model is not guaranteed. In the present work, model reduction procedure, based on singular perturbation technique, for discrete Lur'e-type systems is presented, and conditions to ensure asymptotic stability of the fast and reduced-order decoupled subsystem as well as the original system (1) are given.

3 Two-Time-Scale Decoupling

By reordering and/or rescaling of states, let the nonlinear discrete system be structured in the two-time-scale model:

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}, \tag{6}$$

where x_k^1 and x_k^2 are n_1 and n_2 dimensional state vectors, respectively, and the overall system is of dimension $n = n_1 + n_2$, $x_k^1 \in \Re^{n_1}$ and $x_k^2 \in \Re^{n_2}$. This system is assumed to possess a two-time-scale property, which means that the n eigenvalues of the system can be separated into n_1 slow modes and n_2 stable fast modes related to x_k^1, and x_k^2, respectively. The fast subsystem x_k^2, assumed to be stable, is active only during a short initial period, after, it is negligible and the characterization of the system can be described by its slow subsystem x_k^1.

An explicit two-time-scale property of this model can be introduced by assuming that:

$$A_{11}^* = \mu^{-1} \left(A_{11} - I_{n_1} \right), \tag{7}$$

$$A_{12}^* = \mu^{-1} A_{12}, \tag{8}$$

$$A_{21}^* = A_{21}, \tag{9}$$

$$A_{22}^* = A_{22}. \tag{10}$$

The transformed system is expressed in the standard singular perturbation system structure [38] and [27,28,34,37]:

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} I_{n1} + \mu A_{11}^* & \mu A_{12}^* \\ A_{21}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}, \tag{11}$$

where μ is a small positive singular perturbation parameter and $\det\left(I_{n_2} - A_{22}^*\right) \neq 0$ [47].

As $\mu \to 0$, the eigenvalues of (11) cluster into two groups and, the original system (6) can be decoupled in slow subsystem S_s and fast subsystem S_f candidates:

$$S_s: \quad x_{k+1}^s = (I_{n_1} + \mu A_s) x_k^s, \tag{12}$$

$$S_f: \quad x_{k+1}^f = A_{22}^* x_k^f, \tag{13}$$

with

$$A_s = A_{11}^* + A_{12}^* \left(I_{n_2} - A_{22}^* \right)^{-1} A_{21}^*, \tag{14}$$

where $x_s \in \Re^{n_1}$ and $x_f \in \Re^{n_2}$ are, respectively, the slow and the fast subsystems state vectors defined using a decoupling transformation [12,40,47], if it exists.

The slow subsystem is defined by neglecting the fast stable dynamics, which is equivalent to replace the second equation of (11) by its steady-state algebraic equation. The fast subsystem, supposed to be stable, is derived by assuming that slow variables are constant during fast transients and $\mu = 0$.

4 Main Results

By considering the instantaneous characteristic polynomial $P_S(.,\lambda)$ of (1), (2) or (3):

$$P(\,.\,,\lambda) = \lambda^n + \sum_{i=1}^{n} g_i(.) \lambda^{n-i} \tag{15}$$

and distinct arbitrary constant parameters α_j, $j = 1, 2, \cdots, n-1$, $\alpha_i \neq \alpha_j$, $\forall i \neq j$, it comes the following notations:

$$\beta_j = \prod_{\substack{k=1 \\ k \neq j}}^{n-1} (\alpha_j - \alpha_k)^{-1}, \forall j = 1, 2, \cdots, n-1, \tag{16}$$

$$\gamma_j(.) = -P(\,.\,,\alpha_j), \forall j = 1, 2, \cdots, n-1, \tag{17}$$

$$\delta_n(.) \;=\; -g_1(.) - \sum_{i=1}^{n-1} \alpha_i. \tag{18}$$

Let S be a Lur'e- type system of the form (1)-(3) and let S_s be the decoupled Lur'e-type subsystem (12). By applying the Borne-Gentina practical stability criterion [8,9,20] to the discrete Lur'e type systems characterized by the Benrejeb arrow form matrix [2–6, 10, 18], we obtain the following theorems and corollary.

Theorem 4.1 *The discrete nonlinear system S of the form (1) is asymptotically stable, if there exist constant parameters $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j, \forall i \neq j$, such that*

$$|\alpha_i| < 1 \qquad \forall i = 1, \cdots, n-1 \tag{19}$$

and

$$1 - \left|\delta_{_n}(\cdot)\right| - \sum_{j=1}^{n-1} \left|\gamma_j(\cdot)\right| |\beta_j| \left(1 - |\alpha_j|\right)^{-1} > 0. \tag{20}$$

Theorem 4.2 *For chosen stable fast subsystem, i.e., $|\alpha_i| < 1$ $\forall i = n_1, \cdots, n-1$, the discrete nonlinear decoupled system (12) is asymptotically stable if there exist arbitrary constant parameters $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j, \forall i \neq j$, such that the following conditions are satisfied and*

$$|\alpha_i| < 1 \;\forall i = 1, ..., n_1 - 1, \tag{21}$$

$$1 - \left|\delta_{_n}(\cdot) + \sum_{j=n_1}^{n-1} \gamma_j(\cdot)\,\beta_j \left(1 - \alpha_j\right)^{-1}\right| - \sum_{j=1}^{n_1-1} \left|\gamma_j(\cdot)\right| |\beta_j| \left(1 - |\alpha_j|\right)^{-1} > 0. \tag{22}$$

Corollary 4.1 *For chosen stable fast subsystem, i.e., $|\alpha_i| < 1$ $\forall i = n_1, \cdots, n-1$, the discrete nonlinear decoupled subsystem (12) (respectively the original system (1)) is asymptotically stable if the original system (1) (respectively decoupled subsystem (12)) is asymptotically stable and, if there exists constant parameter $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j$, $\forall i \neq j$, such that the following conditions are satisfied*

$$\begin{cases} \alpha_j > 0, & \forall j = 1, ..., n_1 - 1, \\ \sum\limits_{i=1}^{n-1} \alpha_i > -g_1(.), & \\ \gamma_j(.)\,\beta_j > 0, & \forall j = 1, ..., n_1 - 1. \end{cases} \tag{23}$$

Proof. (Theorem 4.1) Let us consider the Lur'e-type system S of the form (1)-3). A change of coordinate defined by:

$$y_k = T x_k \tag{24}$$

with $y_k \in \Re^n$ and

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_{n-2} & \alpha_{n-2}^2 & \cdots & \alpha_{n-2}^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \end{bmatrix}. \tag{25}$$

leads to the following state space description

$$y_{k+1} = G\left(.\right) y_k. \tag{26}$$

Allowing the synthesis of sufficient stability conditions easy to test, the new instantaneous characteristic matrix $G\left(.\right)$ is chosen to be in the arrow form [2–6,10,18], Appendix 2, as follows

$$G\left(.\right) = T\,A(.)\,T^{-1} = \begin{bmatrix} \delta_n\left(\cdot\right) & \beta_1 & \cdots & \beta_{n-1} \\ \gamma_1\left(\cdot\right) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n-1}\left(\cdot\right) & & & \alpha_{n-1} \end{bmatrix}, \tag{27}$$

where β_i, γ_i, δ_n and α_i, $\forall i = 1, 2, \cdots, n-1$ are defined (16)–(18).

A pseudo-overvaluing matrix $M\left(G\left(\cdot\right)\right)$ of the system (26), corresponding to the use of the vector norm (Appendix 1):

$$p\left(y\right) = \left[\left|y_1\right|, \left|y_2\right|, \dots, \left|y_n\right|\right]^T, \tag{28}$$

$y = \left[y_1,\, y_2,\, \dots,\, y_n\right]^T$, for the stability study, can be obtained from the inequality:

$$p(y_{k+1}) \le M\left(G\left(\cdot\right)\right) p(y_k) \tag{29}$$

satisfied for each corresponding component; that leads to the following comparison system

$$z_{k+1} = M\left(G\left(\cdot\right)\right) z_k \tag{30}$$

with

$$M\left(G\left(.\right)\right) = \begin{bmatrix} \left|\delta_n\left(\cdot\right)\right| & \left|\beta_1\right| & \cdots & \left|\beta_n\right| \\ \left|\gamma_1\left(\cdot\right)\right| & \left|\alpha_1\right| & & \\ \vdots & & \ddots & \\ \left|\gamma_n\left(\cdot\right)\right| & & & \left|\alpha_n\right| \end{bmatrix} \tag{31}$$

such as: $z_0 = p\left(y_0\right)$.

If the nonlinearities of the comparison nonlinear system (30) are isolated in one row of $M\left(G\left(\cdot\right)\right)$, the verification of the Kotelyanski condition (Appendix 1) enables to conclude about the stability of the original system characterized by $G\left(\cdot\right)$ [3,9,10].

It comes the following sufficient asymptotic stability condition of original system:

$$\left(I_n - M\left(G\left(\cdot\right)\right)\right)\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \dots & j \end{pmatrix} > 0 \quad \forall j = 1, \dots, n. \tag{32}$$

This ends the proof of Theorem 4.1.

Proof. (Theorem 4.2) Note that the satisfaction of the condition (19), i.e. $\left|\alpha_i\right| < 1, i = 1, \cdots, n-1$, means that the fast system characterized by a diagonal matrix $\{\alpha_i\}$, $i = n_1, \cdots, n-1$ is stable. Conditions $\left|\alpha_i\right| < 1, i = 1, \cdots, n_1 - 1$ are necessary to satisfy for the reduced slow subsystem stability.

In order to synthesize the stability conditions of the two-time-scale decoupled system S, we first, reorder the transformed nonlinear system states (3). Resulting A_{11}, A_{12},

A_{21} and A_{22} matrices are then in the form (33), where the matrix A_{11} is candidate to characterize the slow subsystem of (6) and A_{22} the fast one:

$$\begin{array}{l} A_{_{11}} = \begin{bmatrix} \delta_n(\cdot) & \beta_1 & \cdots & \beta_{n_1-1} \\ \gamma_1(\cdot) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n_1-1}(\cdot) & & & \alpha_{n_1-1} \end{bmatrix}, \; A_{_{12}} = \begin{bmatrix} \beta_{n_1} & \cdots & \beta_{n-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \\ A_{_{21}} = \begin{bmatrix} \gamma_{n_1}(\cdot) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1}(\cdot) & 0 & \cdots & 0 \end{bmatrix}, \; A_{_{22}} = \begin{bmatrix} \alpha_{n_1} & & \\ & \ddots & \\ & & \alpha_{n-1} \end{bmatrix}. \end{array} \tag{33}$$

Arbitrary constant parameters α_i, $i = n_1, \cdots, n-1$, are chosen in concordance with the estimation of the dynamics that we consider physically fast for the studied system.

Substituting the relations (33), (7)-(10) and (14) into (12) and (13), yields the following discrete slow and fast subsystems, respectively:

$$\begin{array}{l} x_{k+1}^s = A_s(.)\, x_k^s, \\ x_{k+1}^f = A_f x_k^f, \end{array} \tag{34}$$

and then comparison systems, respectively:

$$y_{k+1}^s = M(A_s(\cdot))\, y_k^s, \tag{35}$$

$$y_{k+1}^f = M(A_f)\, y_k^f, \tag{36}$$

where $A_s \in \Re^{n_1 x n_1}$ and $A_f \in \Re^{n_2 x n_2}$ are given by

$$A_s = \begin{bmatrix} \delta_n(.) + \sum\limits_{j=n_1}^{n-1} \frac{\gamma_j(.)\beta_j}{(1-\alpha_j)} & \beta_1 & \cdots & \beta_{n_1-1} \\ \gamma_1(.) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n_1-1}(.) & & & \alpha_{n_1-1} \end{bmatrix}, \tag{37}$$

$$A_f = \begin{bmatrix} \alpha_{n_1} & & \\ & \ddots & \\ & & \alpha_{n-1} \end{bmatrix}, \tag{38}$$

and $M(A_s(\cdot))$ and $M(A_f(\cdot))$ are respectively the pseudo-overvaluing matrices of the slow and fast subsystems (12) and (13), corresponding to the use of the vector norm (28). By applying the practical Borne-Gentina criterion [3, 9, 10, 16] to the comparison systems (35) and (36) of (34), we deduce the stability conditions of the decoupled discrete systems. Theorem 4.2 is then proved.

Proof. (Corollary 4.1) The proof can be easily obtained by substituting the relations (23) in (22).

5 Illustrative Example

To show the effectiveness of the derived theorems, a numerical example is studied below. Consider the discrete nonlinear Lur'e system described by means of the following block-oriented nonlinear model (Figure 1), where $f(.) : \Re \to \Re$ is a nonlinear function, $B_0(s) =$

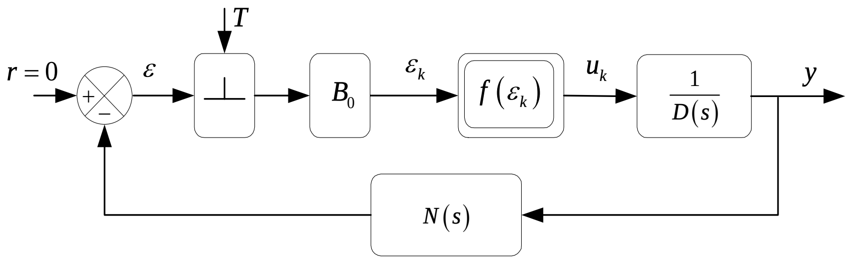

Figure 1: Lur'e systems.

$\frac{1-e^{-Ts}}{s}$ is a zero order holder, $T = 0.2s$ the sampling time, and $D(s)$ and $N(s)$ are polynomials defined by:

$$D(s) = s(1+\tau_1 s)(1+\tau_2 s), \tag{39}$$

$$N(s) = \lambda_2 s^2 + \lambda_1 s + \lambda_0. \tag{40}$$

A state space representation (3) synthesized in the canonical Frobenius form gives:

$$A(\varepsilon_k) = \begin{bmatrix} 0 & 0 & -1,19.10^{-6} f(\varepsilon_k) \\ 1 & 0 & -0,13 + 0,23.10^{-1} f(\varepsilon_k) \\ 0 & 1 & 1,13 - 1,92 f(\varepsilon_k) \end{bmatrix}. \tag{41}$$

By choosing $\alpha_1 = 0.9$ and $\alpha_2 = 0.1$ satisfying (19), the synthesized transformed state space representation in the arrow form is defined by:

$$N(\varepsilon_k) = \begin{bmatrix} 0,14 - 0,19 f(\varepsilon_k) & 1,20 & -1,20 \\ 0,69.10^{-1} - 0,14 f(\varepsilon_k) & 0,90 & 0 \\ -0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k) & 0 & 0,10 \end{bmatrix}. \tag{42}$$

Furthermore, by taking $\mu = 0.1$, the decoupled slow and the fast subsystems are given respectively by

$$N_s = \begin{bmatrix} 0,14 - 0,19 f(\varepsilon_k) & 1,20 \\ 0,69.10^{-1} - 0,14 f(\varepsilon_k) & 0,90 \end{bmatrix}, \quad N_f = 0,10. \tag{43}$$

The stability conditions of the original system deduced from Theorem 4.1, are, for chosen α_1 and α_2:

$$1 - |0,14 - 0,19 f(\varepsilon_k)| - 12 \times \left|0,69.10^{-1} - 0,14 f(\varepsilon_k)\right| \\ -1.33 \times \left|-0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k)\right| > 0$$

or

$$-0.01 < f(\varepsilon_k) < 1.05. \tag{44}$$

Now, by applying Theorem 4.2, the stability conditions of the decoupled nonlinear system (43) are:

$$1 - |0,14 - 0,19 f(\varepsilon_k)| - 12 \times \left|0,69.10^{-1} - 0,14 f(\varepsilon_k)\right| > 0$$

or

$$-0.01 < f(\varepsilon_k) < 1.05. \tag{45}$$

Furthermore, according to the corollary, if we impose the synthesized conditions (23)

$$\begin{cases} 0,14 - 0,19 f(\varepsilon_k) > 0, \\ -0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k) < 0, \end{cases} \tag{46}$$

we obtain

$$-8.64 < f(\varepsilon_k) < 0.73. \tag{47}$$

Consequently, the original Lur'e discrete-time system (42) and the decoupled system (43) are asymptomatically stable for the common stability domain:

$$-0.01 < f(\varepsilon_k) < 0.73. \tag{48}$$

Stability domain (D1) of the original system (42) and the common stability domain (D2) are introduced in Figure 2.

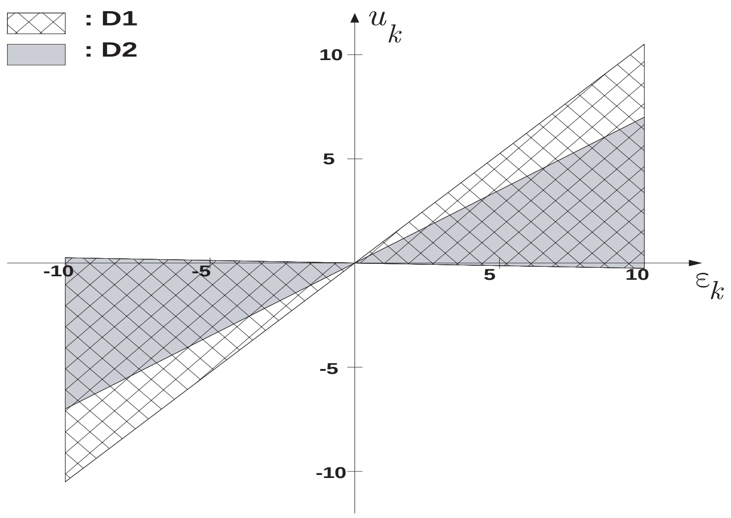

Figure 2: Stability domains.

6 Conclusion

The problem of singular perturbed nonlinear Lur'e discrete-time systems is addressed and a model reduction procedure based on the singular perturbation technique is introduced. Sufficient conditions for stability of the decoupled system as well as the original nonlinear Lur'e type discrete system(1) are then derived. Supplementary stability conditions are synthesized to ensure a common stability domain for the original and the decoupled system. An example is studied to illustrate the efficiency of the proposed results.

Appendix 1

Definition 6.1 (**Vector Norm** [45, 46]) Let $E = \Re^n$ be a vector space and $E_1, E_2, \cdots, E_k$ be subspaces of E which verify: $E = E_1 \cup E_2 \cup \cdots \cup E_k$. Let $x \in E$ be an n vector defined on E with a projection in the subspace E_i denoted by x_i, $x_i = P_i x$, where P_i is a projection operator from E into E_i, p_i is a scalar norm $(i = 1, \cdots, k)$ defined on the subspace E_i and p denotes the vector norm of dimension k and with i^{th} component, $p_i(x) = p_i(x_i)$, $p_i(x) : R^n \to R_+^k$, where $p_i(x_i)$ is a scalar norm of x_i.

Lemma 6.1 *(**Kotelyanski** [19, 30]) The real parts of the eigenvalues of matrix A, with non negative off diagonal elements, are less than a real number μ if and only if all those of matrix $M = \mu I_n - A$ are positive, with I_n being the n identity matrix.*

When successive principal minors of matrix $(-A)$ are positive, Kotelyanski lemma permits to conclude on stability property of the system characterized by A.

Appendix 2

Let us consider the observable nonlinear system:

$$z_{k+1} = A\,(.)\, z_k,$$

$$A\,(.) = \begin{bmatrix} 0 & \cdots & 0 & -a_n\,(.) \\ 1 & 0 & \vdots & -a_{n-1}\,(.) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -a_1\,(.) \end{bmatrix},$$

where $a_i(.)$ are the instantaneous characteristic polynomial $P_A(., \lambda)$ coefficients of $A\,(.)$, such that:

$$P_A(\,.\,, \lambda) = \lambda^n + \sum_{i=1}^{n} a_i(.)\, \lambda^{n-i}.$$

A change of base defined by:

$$\hat{z}_k = T z_k,$$

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_{n-2} & \alpha_{n-2}^2 & \cdots & \alpha_{n-2}^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \end{bmatrix},$$

where α_j, $j = 1, 2, \cdots, n-1$ are distinct arbitrary constant parameters, allows the new state matrix, denoted by $F\,(.)$, to be in arrow form [2–6, 10, 18]:

$$F(.) = T\,A(.)\,T^{-1} = \begin{bmatrix} \delta_n(.) & \beta_1 & \cdots & \beta_{n-1} \\ \gamma_1(.) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n-1}(.) & & & \alpha_{n-1} \end{bmatrix}$$

with

$$\beta_j = \prod_{\substack{k=1 \\ k \neq j}}^{n-1} (\alpha_j - \alpha_k)^{-1}, \forall j = 1, 2, \ldots, n-1,$$

$$\delta_j(.) = -P_A(\,.\,, \alpha_j), \forall j = 1, 2, \ldots, n-1,$$

$$\delta_n(.) = -a_1(.) - \sum_{i=1}^{n-1} \alpha_i.$$

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