

## NONLINEAR DYNAMICS AND SYSTEMS THEORY

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## CONTENTS

State Feedback Controller of Robinson Nuclear Plant with States and Control Constraints .....	1
<i>A.A. Abouelsoud, H. Abdelfattah, M. El Metwally and M. Nasr</i>	
$\mathcal{H}_\infty$ Filtering for Discrete-time Nonlinear Singularly-Perturbed Systems ....	19
<i>M.D.S. Aliyu, M. Perrier and L. Baron</i>	
Optimal State Observer Design for Nonlinear Dynamical Systems .....	37
<i>R. Aloui and N. Benhadj Braiek</i>	
Existence and Uniqueness Conditions for a Class of $(k + 4j)$ -Point $n$ -th Order Boundary Value Problems .....	49
<i>P. Eloe, J. Henderson and R.A. Khan</i>	
Asymptotic Behavior of $n$ -th Order Dynamic Equations .....	63
<i>L. Erbe, G. Hovhannisyann and A. Peterson</i>	
On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized Hénon Map .....	81
<i>R.L. Filali, S. Hammami, M. Benrejeb and P. Borne</i>	
Existence of Positive Solutions for the $p$ -Laplacian with Nonlinear Boundary Conditions .....	97
<i>Sihua Liang and Jihui Zhang</i>	
Global Stability Given Local Stability Via Curvature of Some Nonautonomous Differential Equations .....	105
<i>N. Nilakantan and V. Raghavendra</i>	

NONLINEAR DYNAMICS &amp; SYSTEMS THEORY

Volume 12, No. 1, 2012

# Nonlinear Dynamics and Systems Theory

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# Nonlinear Dynamics and Systems Theory

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<i>P. Eloe, J. Henderson and R.A. Khan</i>	
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## State Feedback Controller of Robinson Nuclear Plant with States and Control Constraints

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**Abstract:** This paper deals with the problem of finding a stabilizing feedback controller for nuclear reactor power plant. A mathematical model of the H. B. Robinson pressurized water reactor plant is formulated. The model includes representations for point kinetics, core heat transfer, piping, pressurizer, and the steam generator. The designed linear state feedback controller accounts for constraints on neutron flux level, steam pressure in steam generator, hot leg temperature and constraints on control inputs of reactivity and electric heater to pressurizer. Simulation results show the effectiveness of the proposed design.

**Keywords:** *H.B. Robinson nuclear plant; stabilization; state feedback controller; state constraints.*

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### 1 Introduction

Currently, there are more than 80 pressurized water reactors (PWRs) operating as important contributors to electricity supply worldwide. But, in this type of reactor, safety margins obstruct the optimal exploitation of the plant because instability may occur under particular operating conditions. The stability of PWR reactor systems has been of a great concern from the safety and the design point of view [1].

Stability problems may only arise during start up or during transients which significantly shift the operating point. Instructions for PWRs contain clear rules on how to

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avoid operating points (regions) that may produce power-void oscillations. The current trend of increasing reactor powers and of applying natural circulation core cooling, however, has major consequences for the stability of new PWR designs. These modifications have allowed PWRs to work at high nominal power, but they have also favored an increase in the reactivity feedback and a decrease in the response time, resulting in a lower stability margin when the reactor is operated at low mass flow and high nominal power [2]. The objective of improved control is to obtain higher plant productivity. Increasing 1) the plant availability, 2) the economic utilization of the nuclear fuel, and 3) the operational flexibility.

A new intelligent nonlinear control for power system stabilizers that improves the transient stability. This permits the most possible simple design implementation of an adaptive-fuzzy logic passivity-based controller which is developed on power system obtained by a suitable use of the backstepping technique [19]. It is difficult to overstate the importance of considering control constraints in control system design: such constraints have well-known implications for the behaviour of the resulting closed-loop system, and ignoring these constraints can lead to a dramatic loss of performance and, potentially, stability. Hassan and Boukas [20] show that the problem of stabilizing a linear quadratic regulator is subject to constraints on the state and the input vectors, Our technique relies on an iterative approach that uses the solution of the standard linear quadratic regulator as an initial guess for the optimal solution and then iteratively, the solution is improved by designing a controller that compensates for the violation of the constraints at each iteration .

Recently, several controller design techniques for constrained linear systems have been proposed. We provide a critical review of constraint compensation techniques for control systems with an emphasis on methods which have been successfully applied to process control problems. Most of these methods can be classified as: (i) anti-windup techniques; (ii) model predictive control techniques; and (iii) hybrid feedback linearization/model predictive control techniques. Anti-windup methods usually are based on applying linear anti-windup compensation to the linear system obtained from feedback linearization [12–16]. Model predictive control provides a very convenient framework for the control of constrained systems as input and output constraints can be incorporated directly into the associated controller [13–17]. Hybrid feedback linearization/model predictive control techniques utilize feedback linearization to generate a constrained linear system which is regulated with a linear model predictive controller [14–18].

Many approaches demonstrate the design of a robust controller using the linear quadratic gaussian with loop transfer recovery (LQG/ LTR) for nuclear reactors with the objective of keeping a desirable performance for reactor fuel temperature and temperature of the coolant leaving the reactor for a wide range of reactor power [15].

This paper deals with the problem of designing a stabilizing feedback controller for continuous H. B. Robinson pressurized water reactor plant which is in the form of linear state-variable model, where the control inputs (reactivity and electric heater to pressurizer) act additively. The model is based on mass, and energy balance; design data from the safety analysis report are used to evaluate the necessary coefficients. The model includes representations for point kinetics equations (six delayed neutron groups), core heat transfer, piping, pressurizer, and the steam generator [3].

The H. B. Robinson Nuclear Plant produces 2200 MW at full power. It includes a pressurized water reactor (PWR), pressurizer, and three vertical U-tube recirculation-type steam generators [3]. The practicality of the control schemes is demonstrated on

the problem of finding a stabilizing controller for continuous H. B. Robinson pressurized water reactor plant subject to both state and control constraints.

Meanwhile the problem of stabilization with state and input constraints has been solved recently [4–6, 7–9]. Saberi [5] generalized Kaliora’s result to a general linear system. Diao [6] constructed a semi-global stabilizing controller subject to both amplitude and rate constraints. Lin [10] constructed a semi global stabilizing controller subject to both amplitude and rate constraints. Castelan et al. [7] showed that the problem of designing a state feedback controller to constrain linear system  $\dot{x} = Ax + Bu$  to a symmetric state constraint set  $S = \{-w \prec Gx \prec w\}$  is solvable if rank  $G$  is less than or equal to the number of controls and the null space of  $G$ ;  $\ker G$  is  $A, B$  invariant [11]. Thus there exists an  $F$  such that  $\ker G$  is  $A + BF$  invariant, the eigenvalues of  $(A + BF)_{\ker G}$  are in the open left-half plane. Abouelsoud [8] generalized this result to both state and input constraints.

This paper is organized as follows. In Section 2 a state and control constrained controller is designed. Section 3 presents description of H. B. Robinson Nuclear power Plant model. In Section 4 simulation results and discussions are provided. Conclusion is given in Section 5.

## 2 Stabilization with State and Control Constraints

Given a continuous-time linear system

$$\dot{x} = Ax(t) + Bu(t), \tag{1}$$

where  $x \in R^n, u \in R^m, (A, B)$  is a controllable pair, and symmetric constraint state and control sets

$$S_x = \{x \in R^n : -w_x \leq G_x x \leq w_x\}, \tag{2}$$

$$S_u = \{u \in R^m : -w_u \leq E_u u \leq w_u\}. \tag{3}$$

By scaling we can make  $w_x = \bar{1}$  and  $w_u = \bar{1}$ , where  $\bar{1}$  is a column with elements unity,

$$S_x = \{x \in R^n : -\bar{1} \leq G_x x \leq \bar{1}\}, \tag{4}$$

$$S_u = \{u \in R^m : -\bar{1} \leq E_u u \leq \bar{1}\}, \tag{5}$$

$G_x \in R^{(s_1 \times n)}, E_u \in R^{(r_1 \times m)}$  are both full rank, we consider the problem of designing a linear state feedback controller

$$u(t) = Fx(t) \tag{6}$$

such that the closed loop system

$$\dot{x} = A_C x(t), \tag{7}$$

where  $A_C = A + BF$ , is asymptotically stable and both the state and control constraints (4) and (5) are satisfied. We use the results of [8] to design  $F$ . First choose the closed loop poles according to the following criterion.

**Lemma 2.1** [7] *A necessary and sufficient condition for*

$$S_x = \{x \in R^n : -\bar{1} \leq G_x x \leq \bar{1}\}$$

*to be positively invariant for system (7) is that the eigenvalues  $\lambda_i = \mu_i \pm j\sigma_i$  of matrix  $A_C$  satisfy*

$$\mu_i \leq -|\sigma_i|. \tag{8}$$

**Proof** See [7].

Let

$$G = \begin{bmatrix} G_x \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ E_u \end{bmatrix}.$$

Then the state and control constraints become

$$-\bar{1} \leq \begin{pmatrix} G \\ EF \end{pmatrix} x \leq \bar{1} \quad \text{or} \quad -\bar{1} \leq Gx + Eu \leq \bar{1}.$$

Assume that the invariant zeros of the system  $\sum_1 : (A, b, G, E)$  are in the open left-half plane. (i.e.  $\sum_1$  is minimum phase), then we can choose the closed loop poles as those invariant zeros; the remaining closed-loop poles are chosen to satisfy condition (8). Let  $\lambda_i$  be an invariant zero of  $\sum_1$ , then there exist a state direction  $v_i$  and a control direction  $w_i$  such that

$$P(\lambda_i) \begin{pmatrix} v_i \\ w_i \end{pmatrix} = \begin{pmatrix} \lambda_i I - A & -B \\ G & E \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix} = 0. \quad (9)$$

for  $i = 1, \dots, n - s$ , where  $s = \text{rank} \begin{pmatrix} G_x B \\ E_u \end{pmatrix}$ ,  $P(\lambda_i)$  is the system matrix. Hence the feedback matrix satisfies

$$Fv_i = w_i \quad \text{or} \quad FV_1 = W_1, \quad (10)$$

where  $V_1 = (v_1, \dots, v_{n-s})$ ,  $W_1 = (w_1, \dots, w_{n-s})$ . The remaining closed-loop poles are chosen to satisfy conditions (8). Thus there exist closed-loop eigenvectors  $V_2$  satisfying

$$GV_2 + EW_2 = I_{S \times S}, \quad (11)$$

$$V_2 \Lambda_2 = AV_2 + BW_2, \quad (12)$$

where

$$\Lambda_2 = \text{blockdiag} \begin{pmatrix} \mu_i & -\sigma_i \\ \sigma_i & \mu_i \end{pmatrix}.$$

For simple complex poles or real poles of the feedback matrix  $A_C$ , let

$$FV_2 = W_2. \quad (13)$$

Hence

$$F = (W_1 \quad W_2) (V_1 \quad V_2)^{-1}. \quad (14)$$

The feedback matrix  $F$  (14) ensures that the closed loop system is asymptotically stable and the state and control constraints are satisfied [8]. The control is now

$$u = Fx. \quad (15)$$

### 3 Robinson Nuclear Power Model

A linear differential equations of pressurized water reactor model that includes the reactor core, pressurizer, primary system piping, and a  $U$ -tube recirculation-type steam generator.



### 3.1 Core point kinetics equations

The point kinetics equations with six groups of delayed neutrons and reactivity feedbacks due to changes in fuel temperature, coolant temperature, and primary coolant system pressure. For model with one fuel node and two coolant nodes [3]

$$\begin{aligned} \frac{d\delta P}{dt} = & -400\delta P + 0.0125\delta C_1 + 0.035\delta C_2 + 0.111\delta C_3 + 0.301\delta C_4 + 1.140\delta C_5 \\ & + 3.01\delta C_6 - 1781\delta T_f - 13700\delta T_{C1} - 13700\delta T_{C2} + 411\delta P_P + 10^6\delta\rho_{Rod}, \end{aligned} \quad (16)$$

$$\frac{d\delta C_1}{dt} = 13.125\delta P - 0.0125\delta C_1, \quad (17)$$

$$\frac{d\delta C_2}{dt} = 87.5\delta P - 0.0305\delta C_2, \quad (18)$$

$$\frac{d\delta C_3}{dt} = 78.125\delta P - 0.111\delta C_3, \quad (19)$$

$$\frac{d\delta C_4}{dt} = 158.125\delta P - 0.301\delta C_4, \quad (20)$$

$$\frac{d\delta C_5}{dt} = 46.25\delta P - 1.140\delta C_5, \quad (21)$$

$$\frac{d\delta C_6}{dt} = 16.875\delta P - 3.01\delta C_6, \quad (22)$$

$$\frac{d\delta T_f}{dt} = 0.0756\delta P - 0.16466\delta T_f + 0.16466\delta T_{C1}, \quad (23)$$

$$\frac{d\delta T_{C1}}{dt} = 0.05707\delta T_f + 2.3832\delta T_{LP} - 2.4403\delta T_{C1}, \quad (24)$$

$$\frac{d\delta T_{C2}}{dt} = 0.05707\delta T_f - 2.3832\delta T_{C2} + 2.3262\delta T_{C1}. \quad (25)$$

### 3.2 Pressurizer equations

The pressurizer model is based on mass, energy, and volume balances with the assumption that saturation conditions always apply for steam-water mixture in the pressurizer,

$$\begin{aligned} \frac{d\delta P_P}{dt} = & 0.0207\delta T_f - 0.0207\delta T_{C1} + 0.0103\delta T_{C2} + 0.240\delta T_{UP} - 0.130\delta T_{IP} \\ & - 0.509\delta T_P + 0.634\delta T_m - 0.116\delta T_{OP} + 0.121\delta T_{LP} - 0.279\delta T_{HL} \\ & + 0.0235\delta T_{CL} - 0.0062\delta Q. \end{aligned} \quad (26)$$

### 3.3 Steam generator equations

The steam generator model with three regions: primary fluid, tube metal, and secondary fluid,

$$\frac{d\delta T_P}{dt} = 0.2238\delta T_{IP} - 0.76642\delta T_P - 0.53819\delta T_m, \quad (27)$$

$$\frac{d\delta T_m}{dt} = 3.07017\delta T_P - 5.3657\delta T_m - 0.33272\delta P_s, \quad (28)$$

$$\frac{d\delta P_s}{dt} = 1.349\delta T_m - 0.2034\delta P_s - 0.0384\delta W_{FW}. \quad (29)$$

### 3.4 Piping equations

All piping sections are modeled as well-mixed volumes,

$$\frac{d\delta T_{UP}}{dt} = 0.33645\delta T_{C2} - 0.33645\delta T_{UP}, \quad (30)$$

$$\frac{d\delta T_{HL}}{dt} = 2.5\delta T_{UP} - 2.5\delta T_{HL} - 0.0016\delta W_P, \quad (31)$$

$$\frac{d\delta T_{IP}}{dt} = 1.45\delta T_{HL} - 1.45\delta T_{IP}, \quad (32)$$

$$\frac{d\delta T_{OP}}{dt} = 1.45\delta T_P - 1.45\delta T_{OP}, \quad (33)$$

$$\frac{d\delta T_{CL}}{dt} = 1.48\delta T_{OP} - 1.48\delta T_{CL}, \quad (34)$$

$$\frac{d\delta T_{LP}}{dt} = 0.516\delta T_{CL} - 0.516\delta T_{LP}, \quad (35)$$

where

$\delta P$ : deviation in reactor power from its initial steady -state value,

$\delta C_i$ : deviation of normalized precursor concentrations,

$\delta T_f$ : deviation of fuel temperature in the fuel node,

$\delta T_{C1}$ : deviation of coolant temperature in the first coolant node,

$\delta T_{C2}$ : deviation of coolant temperature in the second coolant node,

$\delta P_P$ : deviation of primary system pressure,

$\delta T_P$ : deviation of temperature of primary coolant node in the steam generator,

$\delta T_m$ : deviation of the steam generator tube metal temperature,

$\delta P_s$ : deviation of steam pressure from its initial steady-state value,

$\delta T_{UP}$ : deviation of the reactor upper plenum temperature,

$\delta T_{LP}$ : deviation of the reactor lower plenum temperature,

$\delta T_{HL}$ : deviation of hot leg temperature,

$\delta T_{IP}$ : deviation of temperature of primary coolant in the steam generator or inlet plenum,

$\delta T_{OP}$  : deviation of temperature of primary coolant in the steam generator or outlet plenum,

$\delta T_{CL}$ : deviation of cold leg temperature,

$\delta \rho_{Rod}$ : reactivity due to control rod movement,

$\delta Q$  : rate of heat addition to the pressurizer fluid with electric heater,

$\delta W_{FW}$ : deviation of feedwater flow rate in steam generator,

$\delta W_P$  : deviation of primary water flow rate to the steam generator.

Eqs (16)–(35) describing the H. B. Robinson nuclear power system formed in state space model as follow:

$$\dot{x} = Ax + Bu, \quad (36)$$

where

$$x = [ x_1 \quad x_2 ]^T,$$

$$x_1 = [ \delta P \quad \delta C_1 \quad \delta C_2 \quad \delta C_3 \quad \delta C_4 \quad \delta C_5 \quad \delta C_6 \quad \delta T_f \quad \delta T_{C1} \quad \delta T_{C2} ],$$

$$x_2 = [ \delta P_P \quad \delta T_P \quad \delta T_m \quad \delta P_s \quad \delta T_{UP} \quad \delta T_{HL} \quad \delta T_{IP} \quad \delta T_{OP} \quad \delta T_{CL} \quad \delta T_{LP} ],$$

$$u = [ \delta \rho_{Rod} \quad \delta W_{FW} \quad W_P \quad \delta Q ]^T,$$

$$B = \begin{bmatrix} 10^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0016 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

and

$$A = [ A_1 \quad A_2 ],$$

$$A_1 = \begin{bmatrix} -400 & 0.0125 & 0.0305 & 0.111 & 0.301 & 1.14 & 3.01 & -1781 & -13700 & -13700 \\ 13.125 & -0.0125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 87.5 & 0 & -0.0305 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 78.125 & 0 & 0 & -0.111 & 0 & 0 & 0 & 0 & 0 & 0 \\ 158.125 & 0 & 0 & 0 & -0.301 & 0 & 0 & 0 & 0 & 0 \\ 46.25 & 0 & 0 & 0 & 0 & -1.14 & 0 & 0 & 0 & 0 \\ 16.875 & 0 & 0 & 0 & 0 & 0 & -3.01 & 0 & 0 & 0 \\ 0.07 & 0 & 0 & 0 & 0 & 0 & 0 & -0.16466 & 0.16466 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05707 & -2.4403 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05707 & 2.3262 & -2.3832 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0207 & -0.0207 & 0.0103 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.33645 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 411 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.3832 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.634 & -0.509 & 0 & 0.240 & -0.279 & -0.130 & -0.116 & 0.0235 & 0.121 & 0 \\ 0 & -5.3657 & 3.07017 & 0.3372 & 0 & 0 & 0.2238 & 0 & 0 & 0 & 0 \\ 0 & 0.53819 & -0.76442 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.349 & 0 & -0.2034 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.33645 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & -2.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.45 & -1.45 & 0 & 0 & 0 & 0 \\ 0 & 1.45 & 0 & 0 & 0 & 0 & 0 & -1.45 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.48 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.516 & -0.516 & 0 \end{bmatrix}.$$

We can apply the technique in (14) for designing a linear state feedback controller with state and control constraints to the system (36). The state constraints are on neutron flux level ( $\delta P$ ), steam pressure in steam generator ( $\delta P_s$ ) and hot leg temperature ( $\delta T_{HL}$ ). Thus

$$-400 \leq \delta P \leq 400, \quad -10.07 \leq \delta P_s \leq 10.07, \quad -347.24 \leq \delta T_{HL} \leq 347.24.$$

By scaling we can make

$$-1 \leq 0.0025\delta P \leq 1, \quad -1 \leq 0.099\delta P_s \leq 1, \quad -1 \leq 0.0021\delta T_{HL} \leq 1.$$

The control constraint depends on reactivity ( $\delta\rho_{Rod}$ ) and electric heater to pressurizer ( $\delta Q$ ),

$$-1 \leq 200\delta\rho_{Rod} + 0.006\delta Q \leq 1.$$

Thus,

$$G_x = \begin{bmatrix} 0.0025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.099 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0021 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.0025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.099 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0021 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$E_u = [ 200 \quad 0 \quad 0 \quad 0.006 ], \quad \text{rank} \left( \begin{array}{c} G_x B \\ E_u \end{array} \right) = 4.$$

Thus,

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 200 & 0 & 0 & 0.006 \end{bmatrix}.$$

The transfer function of system (36) has 16 zeros at -0.3365, -0.0849, -2.4362, -2.3832, -1.48, -1.45, -0.516, -0.1688, -5.701, -0.4315, -0.0305, -0.111, -0.301, -1.14, -3.01, -1.45, which are in left hand poles thus the transfer function is minimum phase.

The system matrix  $P(\lambda) = \begin{pmatrix} \lambda I - A & -B \\ G & E \end{pmatrix}$  has 16 state direction  $v_i$  and 16 control direction  $w_i, i = 1, \dots, 16$ .

Let  $V_1 = [ v_1 \quad v_2 \quad \dots \quad v_{16} ] = [ v_{11} \quad v_{12} ]$ , where

$$v_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3e-016 & -5e-012 & -2e-013 & -6e-011 & 3.6e-12 & -0.005 & \\ 0 & 0 & -4e-010 & -4e-014 & -0.0006 & -0.0006 & 0.0003 & -3.6e-015 & \\ 0 & 0 & 0.0022 & -0.0021 & -0.0015 & 0.0015 & 0.0003 & -0.0003 & \\ 0.0006 & -0.0730 & -0.0025 & 0.0026 & 0.0041 & -0.0042 & -0.0039 & 0.0381 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -0.00006 & 0.00001 & -0.0003 & 0.0003 & 0.0004 & -0.0004 & -0.0006 & -0.0005 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & -3e-015 & -6e-011 & 3e-019 & 4e-015 & 3e-012 & -6e-018 & \\ 0 & 0 & -6e-011 & -8e-014 & 0.0004 & -0.0004 & 2e-011 & -2e-017 & \\ 0 & 0 & 2e-013 & -3e-012 & -0.0002 & 0.0002 & 0.0002 & 6e-019 & \end{bmatrix},$$

$$v_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -8e-011 & 0.0001 & 0 & 0 & 0 & 0 & 0 & -0.0001 \\ -9e-015 & 0.0001 & 0 & 0 & 0 & 0 & 0 & -0.0006 \\ -3e-012 & 0.0002 & 0 & 0 & 0 & 0 & 0 & -0.0015 \\ -0.0035 & -0.0024 & 0.0001 & -0.0009 & -0.0003 & -0.0002 & -0.0002 & 0.0042 \\ -0.0285 & 6e-018 & 0 & 0 & 0 & 0 & 0 & 7e-018 \\ -0.0031 & 0.00002 & 0 & 0 & 0 & 0 & 0 & 6e-018 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9e-018 & -0.0006 & 7e-019 & -6e-020 & 7e-018 & -6e-018 & -4e-012 & 0.0004 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5e-011 \\ 0.0011 & 6e-017 & 0 & 0 & 0 & 0 & 0 & 6e-016 \\ -0.0004 & 8e-019 & 0 & 0 & 0 & 0 & 0 & 0.0004 \\ 2e-013 & 0.0001 & 0 & 0 & 0 & 0 & 0 & -0.0002 \end{bmatrix},$$

$$W_1 = [ w_1 \ w_2 \ \dots \ w_{16} ] = [ w_{11} \ w_{12} ],$$

$$w_{11} = \begin{bmatrix} -6e-018 & 2e-016 & 1e-013 & -6e-015 & -2e-012 & 5e-014 & 6e-019 & 7e-018 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8e-017 & 0.5418 & -0.5426 & -0.6841 & 0.6923 & 1 & 0.8253 \\ 0.0084 & -1 & -1 & 1 & 1 & -1 & -0.3402 & 1 \end{bmatrix},$$

$$w_{12} = \begin{bmatrix} 6e-018 & 6e-018 & -2e-016 & 1e-019 & 5e-012 & 4e-015 & 2e-011 & -2e-020 \\ 1 & 0.0002 & 0 & 0 & 0 & 0 & 0 & 6e-018 \\ 0.0019 & 1 & -4e-013 & -6e-017 & -5e-013 & 0 & -3e-019 & -0.6923 \\ -0.0438 & -0.1894 & 0.0006 & -0.0158 & -0.0140 & -0.0411 & -0.1032 & 1 \end{bmatrix}.$$

The corresponding state and control direction are 4 state direction  $v_i$  and 4 control direction  $w_i, i = 17, \dots, 20$ .

$$V_2 = [ v_{17} \ v_{18} \ v_{19} \ v_{20} ] = \begin{bmatrix} 400 & -1.9e-016 & -6.9e-016 & 2.7e-015 \\ -769.6 & 3.6e-015 & -2.9e-015 & -1e-014 \\ -5144.4 & 6e-014 & -2.5e-014 & -7e-014 \\ -4648.2 & 2.2e-014 & -1e-013 & -6.1e-014 \\ -9681.6 & 1.2e-013 & -2.1e-013 & -1.3e-013 \\ -3249 & 4.2e-014 & -5.1e-014 & -5.1e-014 \\ -1765.2 & 2.1e-014 & -1.4e-014 & -8.2e-014 \\ -4.5327 & -7.9e-004 & 11e-004 & -6.1e-017 \\ 0.0589 & -0.0262 & 0.0054 & 3e-018 \\ 0.0273 & 0.0191 & -0.0023 & -3.3e-018 \\ -4.9772 & -2.9407 & 15.6583 & 0.2972 \\ -8.7e-021 & 23.5195 & -4.2434 & -2.9e-020 \\ 4.6e-021 & -2.6368 & 3.6039 & -1.7e-020 \\ -6.5e-022 & 10.10 & 2.6e-017 & 9.1e-021 \\ -0.0014 & -0.0012 & 9.9e-005 & 3.3e-019 \\ 7.1e-020 & -3.1e-020 & 476.1905 & -4.6e-019 \\ -8.8e-020 & -2.3e-021 & -105.8493 & 3.2e-019 \\ -1.5e-021 & 0.9287 & -0.8011 & 1.3e-020 \\ 2.5e-022 & -0.3363 & 0.1826 & -9.1e-021 \\ 6.6e-024 & 0.0344 & -0.0126 & 1.5e-021 \end{bmatrix}$$

and

$$W_2 = [ w_{17} \ w_{18} \ w_{19} \ w_{20} ] = \begin{bmatrix} 0.1650 & 0.0011 & -0.0064 & -1.22e-004 \\ -4.3e-019 & 2.2e+003 & -148.956 & -2.8e-019 \\ 2.2128 & 1.9233 & -1.6289e+006 & -2.2e-016 \\ -5501.1 & -37.0275 & 213.0615 & 170.7381 \end{bmatrix}.$$

The remaining closed loop poles are chosen as follows:  $-6.834, -5.567, -7.9732, -0.15$ . The state and control feedback controller

$$F = ( W_1 \ W_2 ) ( V_1 \ V_2 )^{-1}, \tag{37}$$

$$F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 3.8e-4 & -7.6e-6 & -3e-8 & -1e-7 & -3e-7 & -1.1e-6 & -3e-6 \\ 2e-4 & 1.01e-4 & 0 & 0 & 0 & 0 & 0 \\ 11.70 & 6.08 & 6.8e-18 & 5e-16 & 1.8e-16 & 7e-16 & 2.6e-15 \\ -6.5e+16 & -3.4e+16 & 0.001 & 0.0037 & 0.01 & 0.038 & 0.1003 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.0018 & 0.0137 & 0.0137 & -4.1e-4 & -6e-16 & -2.6e-15 & 8.1e-17 \\ 0 & 0 & 0 & 0 & 35.10 & 0 & 139.56 \\ -2.9e-11 & 8e-10 & 1.5e-12 & 9.5e-14 & 8.1e-10 & 6e-9 & 7.1e-11 \\ -59.36 & -456.66 & -456.66 & 13.7 & 1.1e-11 & 5.1e-11 & 11.006 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 0 & 1.1e-7 & 5.9e-7 & 3.1e-15 & 4e-17 & 1.2e-16 \\ 0 & 4.4e-16 & 9e-16 & 0 & 0 & 0 \\ -1562.5 & -3420.8 & -0.38 & -4.5e-9 & 6e-10 & -1e-9 \\ 0 & -0.30 & -0.0039 & -6.9e-11 & 2.6e-13 & -7.1e-12 \end{bmatrix}.$$

#### 4 Result and Discussions

This section presents the simulation and numerical results based on linear state feedback controller (37) applied to the system (36). The system is simulated for initial state variables values as follows

$$x = [ 400 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 10.07 \ 0 \ 347.24 \ 0 \ 0 \ 0 \ 0 ],$$

where the initial values satisfy the defined constraints of deviations of neutron flux, steam pressure and hot leg temperature.

The following figures represent the responses (deviations of the system state variables with time, where it is clear that all deviations decay with time and tend to zero, satisfying both performance criterion stability and zero steady state error (1 sec=1000 iterations).

Figure 1 shows deviation of neutron flux with time, we observe the curve has speed response (maximum overshoot is within the acceptable constraint  $400 \leq x_1 \leq 400$ ).

Figure 2 shows deviation of generations from the first to sixth of the delayed neutron fractions within range  $[-150, 300]$ . It is clear that the increase in the maximum overshoot of responses as the increase of generations of the delayed neutron fractions, while the damping frequency and the settling time of all delayed neutron fractions are the same.

Figure 3 shows the deviation of fuel temperature, it is clear that from graph the maximum over shoot are very small and settling time ( $\leq 400$  iterations).

Figure 4 shows the deviations of coolant temperature in first node and second node, also show the deviations of metal and primary temperatures in steam generator, we observe the curves have speed response, maximum overshoot is within the range  $[0,5]$ .

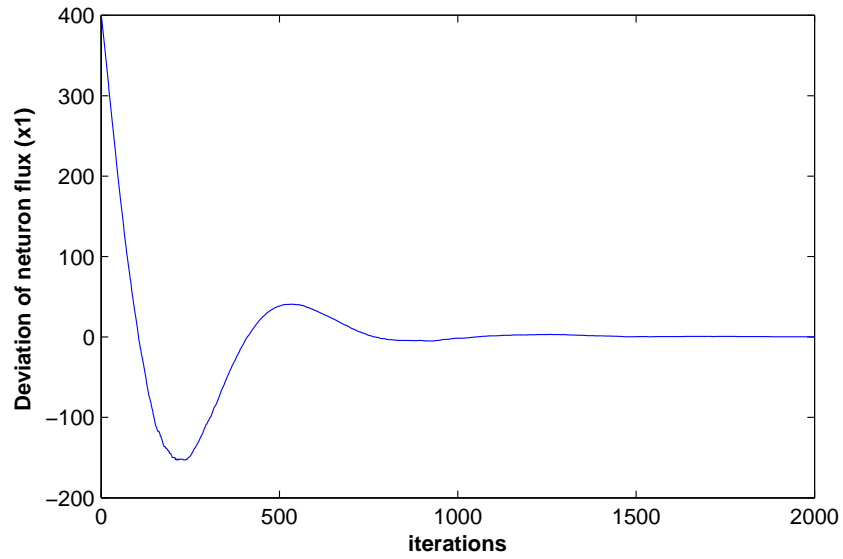
Figure 5 shows the deviation of primary pressure, it is clear that from graph the maximum over shoot is within the interval  $[0,150]$ , decays with time and tends to zero.

Figure 6 shows the deviation of steam pressure in steam generator, we observe the curve has speed response (maximum overshoot is within the acceptable constraint  $10.07 \leq x_{14} \leq 10.07$  and settling time ( $\leq 800$  iterations).

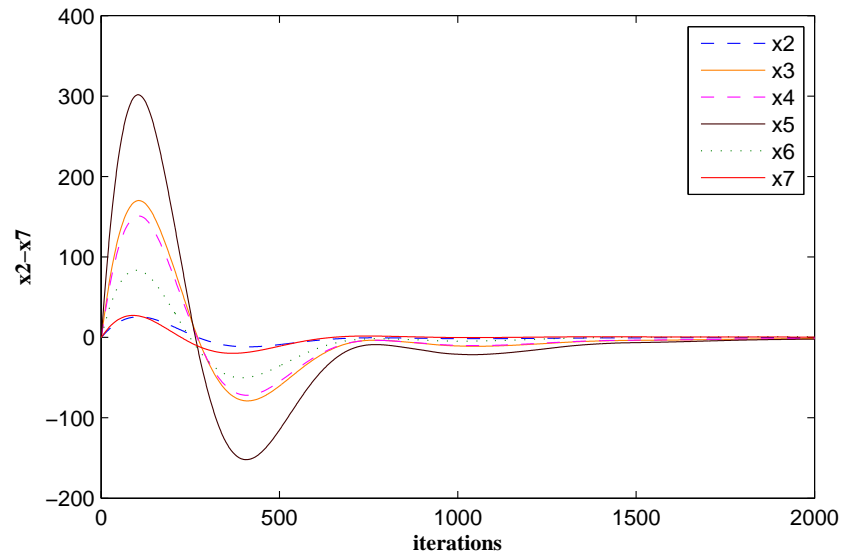
Figure 7 shows the deviations of the reactor upper, outlet plenum, cold leg and lower plenum temperatures in steam generator, we observe the figures have the same range, the maximum over shoot are small and settling time ( $\leq 14000$  iterations).

Figure 8 shows the deviation of hot leg temperature, it is clear that the graph has speed response with (maximum overshoot is within the acceptable constraint  $-347.24 \leq x_{16} \leq 347.24$ , and very small settling time ( $\leq 600$  iterations).

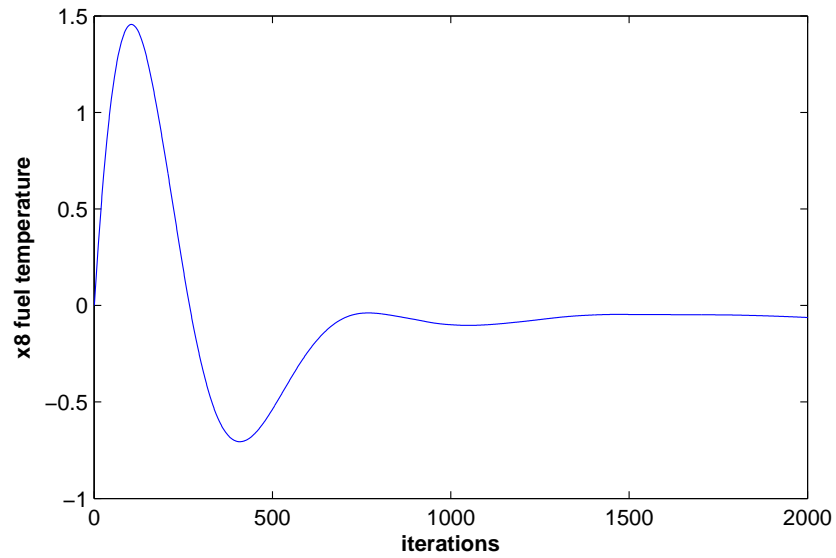
Figure 9 shows deviation of inlet plenum temperature, we observe the figure has maximum over shoot are small, also its decays with time and tends to zero.



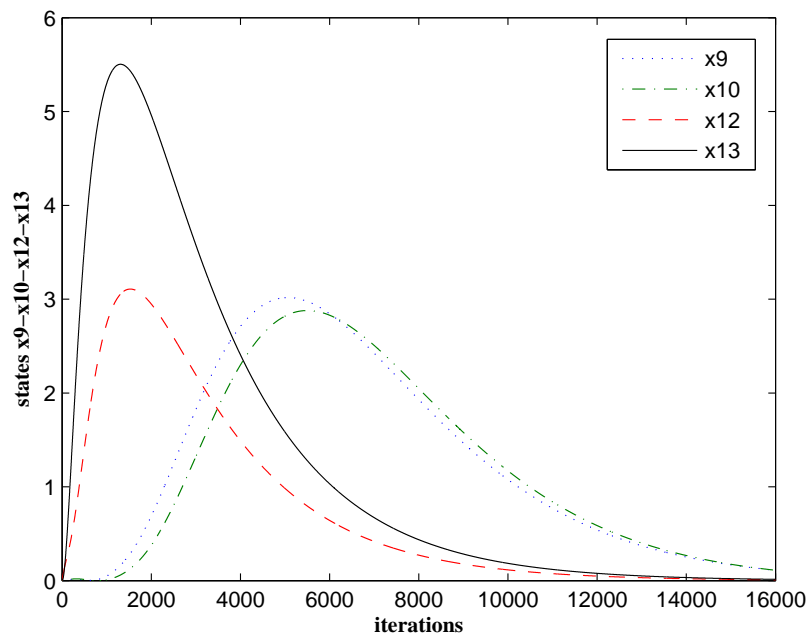
**Figure 1:** Deviation of neutron flux for simulations with the state feedback controller (37).



**Figure 2:** Deviation of normalized precursor concentrations for simulations with the state feedback controller (37).

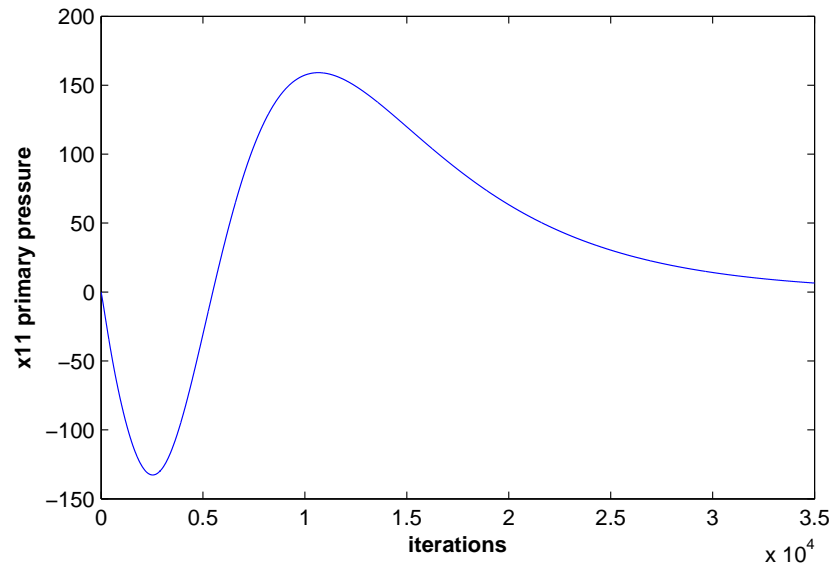


**Figure 3:** Deviation of fuel temperature for simulations with the state feedback controller (37).

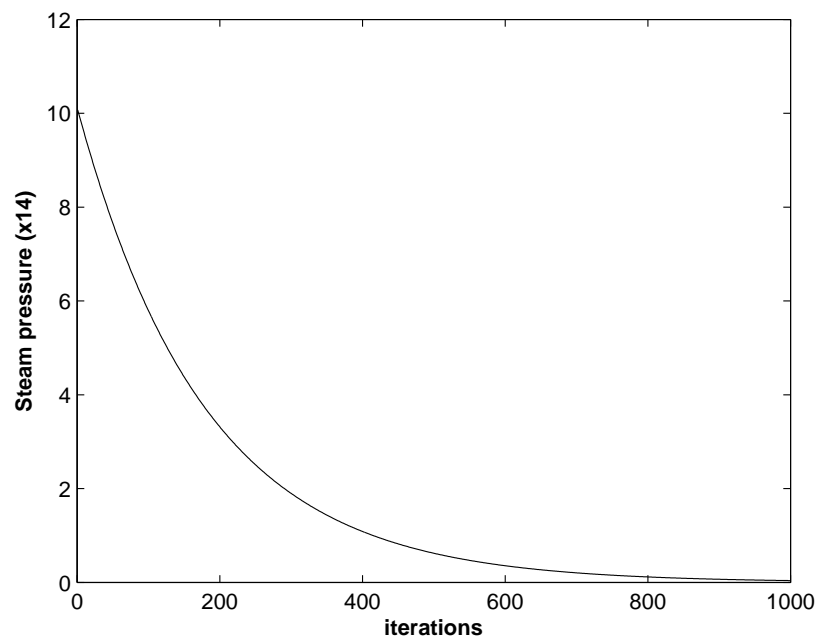


**Figure 4:** Deviations of coolant in first node ,second node , metal and primary steam generator temperatures for feedback controller (37).

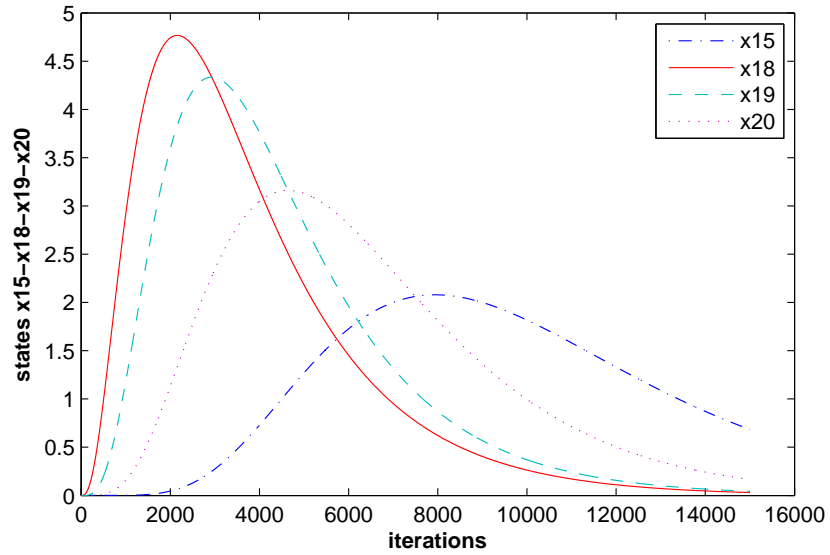




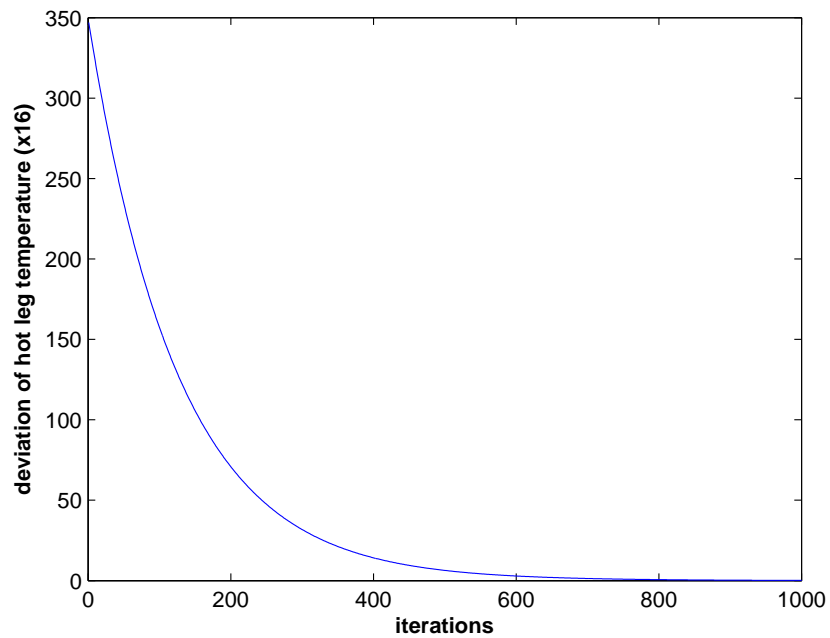
**Figure 5:** Deviation of primary pressure for simulations with the state feedback controller (37).



**Figure 6:** Deviation of steam pressure for simulations with the state feedback controller (37).



**Figure 7:** Deviation of the reactor upper, outlet plenum, cold leg and lower plenum temperatures for controller (37).



**Figure 8:** Deviation of hot leg temperature for simulations with the state feedback controller (37).

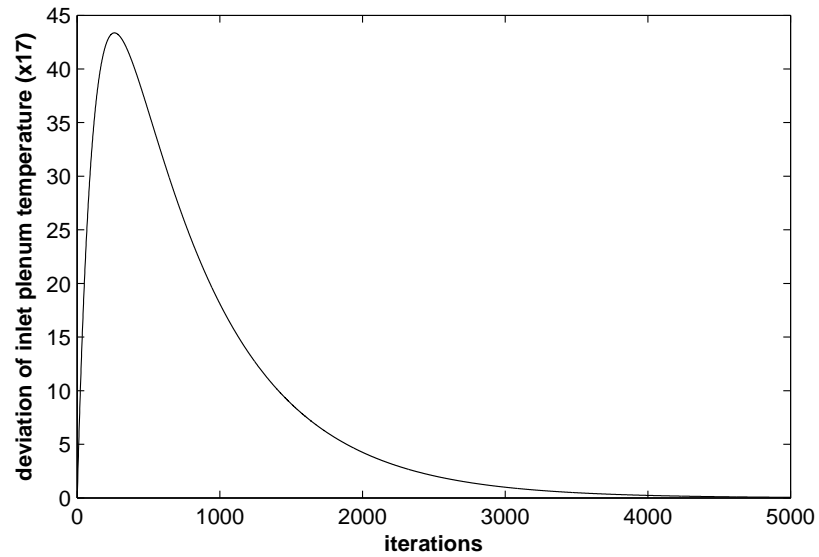


Figure 9: Deviation of inlet plenum temperature for simulations with controller (37).

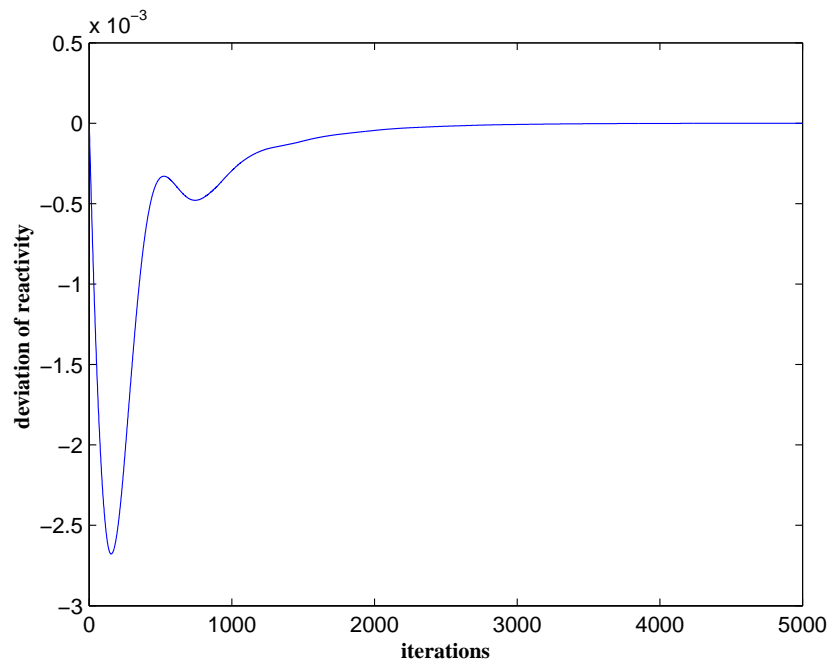
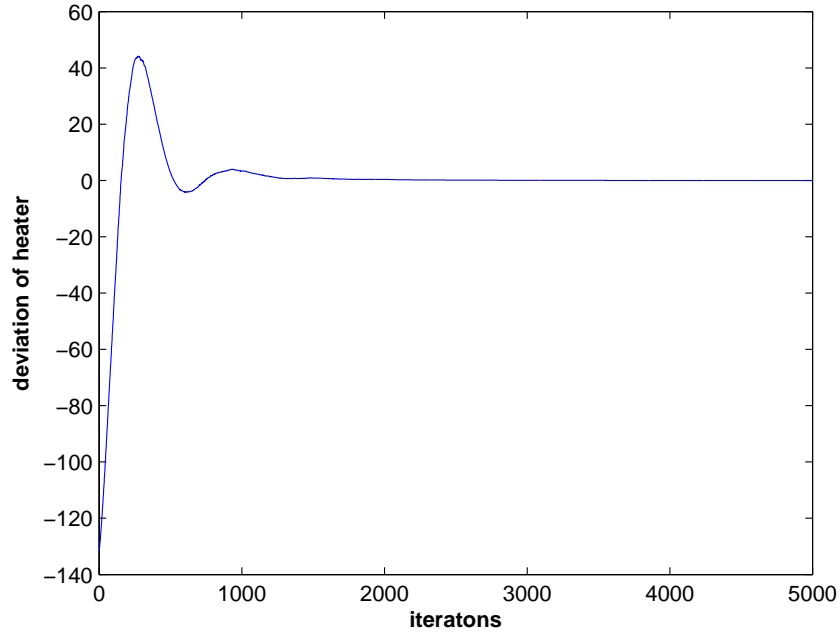


Figure 10: Deviation of reactivity for a simulation with the state feedback controller (37).



**Figure 11:** Deviation of electric heater input to the pressurizer with the state feedback controller (37).

Figures 10 and 11 show the deviations of reactivity, electric heater control inputs where the two curves satisfy the acceptable constraints,  $-0.005 \leq \delta\rho_{Rod} \leq 0.005$ ,  $-170 \leq \delta Q \leq 170$  which interprets the superior effect of the control technique used.

These figures demonstrate stability of the state feedback system, while the neutron flux level, steam pressure in steam generator, hot leg temperature, control input reactivity and control input electric heater to pressurizer constraints are satisfied.

The simulation results provide us with an important practical implication, that is the nuclear power plant has reached its desired steady state value in a very small time as the neutron flux of our theoretical system under study as shown in Figure 1 has reached the desired steady state value in about, which indicates a relative importance of our control algorithm for practical implementation of different systems.

## 5 Conclusions

A linear state feedback controller [9, 10] has been designed to globally asymptotically stabilize H.B. Robinson pressurized water reactor plant [2] subject to symmetrical neutron level flux, steam pressure in steam generator, hot leg temperature, control input reactivity and electric heater input constraints. Simulation results show the effectiveness of the proposed technique.

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## $\mathcal{H}_\infty$ Filtering for Discrete-time Nonlinear Singularly-Perturbed Systems

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**Abstract:** In this paper, we consider the  $\mathcal{H}_\infty$  filtering problem for discrete-time singularly-perturbed (two time-scale) nonlinear systems. Two types of filters, namely, (i) decomposition; and (ii) aggregate, are discussed, and sufficient conditions for the approximate solvability of the problem in terms of discrete-time Hamilton–Jacobi–Isaacs equations (DHJIEs) are presented. In addition, for each type of filter above, reduced-order filters are also derived in each case. The results are also specialized to linear systems, in which case the HJIEs reduce to a system of linear-matrix-inequalities (LMIs) which are computationally efficient. An example is also given to demonstrate the approach.

**Keywords:** *discrete-time nonlinear filtering;  $\mathcal{H}_\infty$ -norm; discrete-time singularly-perturbed nonlinear system; decomposition filters; aggregate filters; discrete-time Hamilton–Jacobi–Issacs equations (DHJIEs).*

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### 1 Introduction

The optimal control problem for linear and nonlinear discrete-time singularly-perturbed systems has been considered by several authors [8–10], [16, 18]. On the other hand, the filtering problem for linear singularly-perturbed systems has received little attention [5, 18, 22]. Kalman filtering techniques have generally been considered, and various types of filters have been proposed, including composite and reduced-order filters. However, to the best of our knowledge, the nonlinear filtering problem and in particular the problem for affine nonlinear singularly-perturbed systems has not been considered by any authors.

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Moreover, recently the authors have discussed the Kalman filtering problem for this class of systems and it is therefore our aim in this paper to discuss the nonlinear  $\mathcal{H}_\infty$  filtering problem for discrete-time singularly-perturbed systems.

Singularly-perturbed systems are those class of systems that are characterized by a discontinuous dependence of the system properties on a small perturbation parameter  $\epsilon$ . They arise in many physical systems such as electrical power systems and electrical machines (eg. an asynchronous generator, a dc motor, electrical converters), electronic systems (e.g. oscillators) mechanical systems (eg. fighter aircrafts), biological systems (eg. bacterial-yeast-virus cultures, heart) and also economic systems with various competing sectors. This class of systems has two time-scales, namely, a “fast” and a “slow” dynamics. This makes their analysis and control more complicated than regular systems. Nevertheless, they have been studied extensively [15, 17].

Furthermore, statistical discrete-time nonlinear filtering techniques developed using minimum-variance, Bayesian and maximum-likelihood criteria [6, 19, 21] are too complicated, and approximations [14, 20] are still computationally intensive to implement. On the other hand, the nonlinear  $\mathcal{H}_\infty$  filter is easy to derive using a deterministic approach and relies on finding a smooth solution to a discrete-time Hamilton–Jacobi–Isaac’s (DHJI) partial-differential-equation (PDE) or DHJIE in short, which can be found using polynomial approximations or other methods. Therefore,  $\mathcal{H}_\infty$  filtering techniques for nonlinear discrete-time systems have been considered by several authors [24–26] including the authors [2, 3]. As is well-known, the  $\mathcal{H}_\infty$  filter has several advantages over the extended-Kalman filter [4], among which are robustness towards  $\mathcal{L}_2$ -bounded disturbances and uncertainties, as well as the fact that it is derived from a completely deterministic setting.

A solution to the discrete-time (sub-optimal) nonlinear  $\mathcal{H}_\infty$  filtering problem is given in [24] under the simplifying assumption that the solution to the DHJIE is quadratic in the estimation error. This approach is very useful for practical applications, but a complete solution to the problem is also desirable in its own right. Hence recently, the authors have presented exact and approximate solutions to the problem [2, 3]. Moreover, the authors have proposed two-degree-of-freedom (2-DOF) proportional-derivative (PD) and proportional-integral (PI)-filters, and the advantages of these approaches over the 1-DOF filters have also been demonstrated. Thus, in this paper, we extend some of these results to discrete-time singularly-perturbed nonlinear systems which hitherto have not been considered by any authors.

In this paper, we propose to discuss the  $\mathcal{H}_\infty$  filtering problem for discrete-time affine nonlinear singularly-perturbed systems. Two types of filters, namely, (i) decomposition, and (ii) aggregate filters will be considered, and sufficient conditions for the solvability of the problem in terms of Hamilton–Jacobi–Isaacs equations (HJIEs) will be presented. The paper is organized as follows. In the remainder of this section, we introduce notations. Then in Section 2, we define the problem and give also some other preliminary definitions. In Section 3, we present a solution to the filtering problem using decomposition filters. This is followed in Section 4 by an alternative solution using aggregate filters. An example is then presented in Section 5, and finally in Section 6, we give conclusions.

The notation is standard, except where otherwise stated. Moreover,  $\|(\cdot)\|$  will denote the standard Euclidean vector norm on  $\mathfrak{R}^n$ , the spaces  $\ell_2([k_0, \infty), \mathfrak{R}^n)$   $\ell_\infty([k_0, \infty), \mathfrak{R}^n)$  are the time-domain standard Lebesgue spaces of square-summable and essentially bounded vector-valued sequences. While  $\mathcal{H}_\infty(j\mathfrak{R})$  is the corresponding frequency-domain subspace of the counterpart frequency-domain space of  $\ell_\infty([k_0, \infty), \mathfrak{R}^n)$  of vector functions that



are analytic on the open right-hand complex plane  $\mathbf{C}_+$ . We shall only use this notation to refer to stable input-output maps and when there is no confusion. The norm on the above  $\ell_2$ , and  $\ell_\infty$ -spaces are defined accordingly as  $\|(\cdot)\|_2^2 \triangleq \sum_{k_0}^\infty \|(\cdot)\|^2$ ,  $\|(\cdot)\|_\infty \triangleq \sup_k \|(\cdot)\|$ . Other notations will be defined accordingly.

## 2 Problem Definition and Preliminaries

The general set-up for studying  $\mathcal{H}_\infty$  filtering problems is shown in Figure 1, where  $\mathbf{P}_k$  is the plant, while  $\mathbf{F}_k$  is the filter. The noise signal  $w \in \mathcal{P}'$  is in general a bounded power signal (e.g. a Gaussian white-noise signal) which belongs to the set  $\mathcal{P}'$  of bounded spectral signals, and similarly  $\tilde{z} \in \mathcal{P}'$ , is also a bounded power signal or  $\ell_2$  signal. Thus, the induced norm from  $w$  to  $\tilde{z}$  (the penalty variable to be defined later) is the  $\ell_\infty$ -norm of the interconnected system  $\mathbf{F}_k \circ \mathbf{P}_k$ , i.e.

$$\|\mathbf{F}_k \circ \mathbf{P}_k\|_{\ell_\infty} \triangleq \sup_{0 \neq w \in \mathcal{S}'} \frac{\|\tilde{z}\|_{\mathcal{P}'}}{\|w\|_{\mathcal{P}'}} \tag{1}$$

where

$$\mathcal{P}' \triangleq \{w : w \in \ell_\infty, R_{ww}(k), S_{ww}(j\omega) \text{ exist for all } k \text{ and all } \omega \text{ resp.}, \|w\|_{\mathcal{P}'} < \infty\},$$

$$\|z\|_{\mathcal{P}'}^2 \triangleq \lim_{K \rightarrow \infty} \frac{1}{2K} \sum_{k=-K}^K \|z_k\|^2$$

and  $R_{ww}, S_{ww}(j\omega)$  are the autocorrelation and power spectral density matrices of  $w$ . Notice also that,  $\|(\cdot)\|_{\mathcal{P}'}$  is a seminorm. In addition, if the plant is stable, we replace the induced  $\ell_\infty$ -norm above by the equivalent  $\mathcal{H}_\infty$  subspace norms.

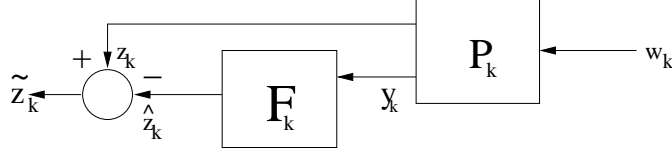
At the outset, we consider the following singularly-perturbed affine nonlinear causal discrete-time state-space model of the plant which is defined on  $\mathcal{X} \subseteq \mathbb{R}^{n_1+n_2}$  with zero control input:

$$\mathbf{P}_{sp}^{da} : \begin{cases} x_{1,k+1} &= f_1(x_{1,k}, x_{2,k}) + g_{11}(x_{1,k}, x_{2,k})w_k; & x_1(k_0, \varepsilon) = x^{10}, \\ \varepsilon x_{2,k+1} &= f_2(x_{1,k}, x_{2,k}, \varepsilon) + g_{21}(x_{1,k}, x_{2,k})w_k; & x_2(k_0, \varepsilon) = x^{20}, \\ y_k &= h_{21}(x_{1,k}) + h_{22}(x_{2,k}) + k_{21}(x_{1,k}, x_{2,k})w_k, \end{cases} \tag{2}$$

where  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X}$  is the state vector with  $x_1$  the slow state which is  $n_1$ -dimensional and  $x_2$  the fast, which is  $n_2$ -dimensional;  $w \in \mathcal{W} \subseteq \mathbb{R}^r$  is an unknown disturbance (or noise) signal, which belongs to the set  $\mathcal{W} \subset \ell_2[k_0, \infty) \subset \mathcal{P}'$  of admissible exogenous inputs;  $y \in \mathcal{Y} \subset \mathbb{R}^m$  is the measured output (or observation) of the system, and belongs to  $\mathcal{Y}$ , the set of admissible measured-outputs; while  $\varepsilon$  is a small perturbation parameter.

The functions  $f_1 : \mathcal{X} \rightarrow \mathbb{R}^{n_1}$ ,  $\mathcal{X} \subset \mathbb{R}^{n_1+n_2}$ ,  $f_2 : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}^{n_2}$ ,  $g_{11} : \mathcal{X} \rightarrow \mathcal{M}^{n_1 \times r}(\mathcal{X})$ ,  $g_{21} : \mathcal{X} \rightarrow \mathcal{M}^{n_2 \times r}(\mathcal{X})$ , where  $\mathcal{M}^{i \times j}$  is the ring of  $i \times j$  smooth matrices over  $\mathcal{X}$ ,  $h_{21}, h_{22} : \mathcal{X} \rightarrow \mathbb{R}^m$ , and  $k_{21} : \mathcal{X} \rightarrow \mathcal{M}^{m \times r}(\mathcal{X})$  are real  $C^\infty$  functions of  $x$ . More specifically,  $f_2$  is of the form  $f_2(x_{1,k}, x_{2,k}, \varepsilon) = (\varepsilon x_{2,k} + \bar{f}_2(x_{1,k}, x_{2,k}))$  for some smooth function  $\bar{f}_2 : \mathcal{X} \rightarrow \mathbb{R}^{n_2}$ . Furthermore, we assume without any loss of generality that the system (2) has an isolated equilibrium-point at  $(x_1^T, x_2^T) = (0, 0)$  such that  $f_1(0, 0) = 0$ ,  $f_2(0, 0) = 0$ ,  $h_{21}(0, 0) = h_{22}(0, 0) = 0$ . We also assume that there exists a unique solution  $x(k, k_0, x_0, w, \varepsilon) \forall k \in \mathbf{Z}$  for the system, for all initial conditions  $x(k_0) \triangleq x^0 = (x^{10^T}, x^{20^T})^T$ , for all  $w \in \mathcal{W}$ , and all  $\varepsilon \in \mathbb{R}$ .

The suboptimal  $\mathcal{H}_\infty$  local filtering/state estimation problem is defined as follows.



**Figure 1:** Set-up for discrete-time  $\mathcal{H}_\infty$  filtering.

**Definition 2.1** (Sub-optimal  $\mathcal{H}_\infty$  Local State Estimation (Filtering) Problem). Find a filter,  $\mathbf{F}_k$ , for estimating the state  $x_k$  or a function of it,  $z_k = h_1(x_k)$ , from observations  $\mathbf{Y}_k \triangleq \{y_i : i \leq k\}$  of  $y_i$  up to time  $k$ , to obtain the estimate

$$\hat{x}_k = \mathbf{F}_k(\mathbf{Y}_k),$$

such that, the  $\mathcal{H}_\infty$ -norm from the input  $w \in \mathcal{W}$  to some suitable penalty function  $z$  is locally rendered less than or equal to a given number  $\gamma$  for all initial conditions  $x^0 \in \mathcal{O} \subset \mathcal{X}$ , for all  $w \in \mathcal{W}$ . Moreover, if the filter solves the problem for all  $x^0 \in \mathcal{X}$ , we say the problem is solved globally.

In the above definition, the condition that the  $\mathcal{H}_\infty$ -norm is less than or equal to  $\gamma$ , is more correctly referred to as the  $\ell_2$ -gain condition

$$\sum_{k_0}^{\infty} \|z_k\|^2 \leq \gamma^2 \sum_{k_0}^{\infty} \|w_k\|^2, \quad x^0 \in \mathcal{O} \subset \mathcal{X}, \quad \forall w \in \mathcal{W}. \quad (3)$$

We shall adopt the following definition of observability.

**Definition 2.2** For the nonlinear system  $\mathbf{P}_{sp}^a$ , we say that, it is locally zero-input observable, if for all states  $x_1, x_2 \in U \subset \mathcal{X}$  and input  $w(\cdot) = 0$ ,

$$y(k; x_1, w) \equiv y(k; x_2, w) \implies x_1 = x_2,$$

where  $y(\cdot, x_i, w)$ ,  $i = 1, 2$  is the output of the system with the initial condition  $x_{k_0} = x_i$ . Moreover, the system is said to be zero-input observable, if it is locally observable at each  $x^0 \in \mathcal{X}$  or  $U = \mathcal{X}$ .

### 3 Solution to the $\mathcal{H}_\infty$ Filtering Problem Using Decomposition Filters

In this section, we present a decomposition approach to the  $\mathcal{H}_\infty$  estimation problem defined in the previous section, while in the next section, we present an aggregate approach.

We construct two time-scale filters corresponding to the decomposition of the system into a “fast” and “slow” subsystems. As in the linear case [5, 12, 16, 18, 22], we first assume that there exists locally a smooth invertible coordinate transformation (a diffeomorphism)  $\varphi : x \mapsto \xi$ , i.e.

$$\xi_1 = \varphi_1(x), \quad \varphi_1(0) = 0, \quad \xi_2 = \varphi_2(x), \quad \varphi_2(0) = 0, \quad \xi_1 \in \mathfrak{R}^{n_1}, \xi_2 \in \mathfrak{R}^{n_2}, \quad (4)$$

such that the system (2) is locally decomposed into the form

$$\tilde{\mathbf{P}}_{sp}^{da} : \begin{cases} \xi_{1,k+1} &= \tilde{f}_1(\xi_{1,k}, \varepsilon) + \tilde{g}_{11}(\xi_k, \varepsilon)w_k, & \xi_1(k_0) = \varphi_1(x^0, \varepsilon), \\ \varepsilon \xi_{2,k+1} &= \tilde{f}_2(\xi_{2,k}, \varepsilon) + \tilde{g}_{21}(\xi_k, \varepsilon)w_k; & \xi_2(k_0) = \varphi_2(x^0, \varepsilon), \\ y_k &= \tilde{h}_{21}(\xi_{1,k}, \xi_{2,k}, \varepsilon) + \tilde{h}_{22}(\xi_{1,k}, \xi_{2,k}, \varepsilon) + \tilde{k}_{21}(\xi_k, \varepsilon)w. \end{cases} \quad (5)$$

**Remark 3.1** It is virtually impossible to find a coordinate transformation such that  $\tilde{h}_{2j} = \tilde{h}_{2j}(\xi_j), j = 1, 2$ . Thus, we have made the more practical assumption that  $\tilde{h}_{2j} = \tilde{h}_{2j}(\xi_1, \xi_2), j = 1, 2$ .

Necessary conditions that such a transformation must satisfy are given in [1]. The filter is then designed based on this transformed model as follows

$$\mathbf{F}_{1c}^{da} : \begin{cases} \hat{\xi}_{1,k+1} &= \tilde{f}_1(\hat{\xi}_{1,k}, \varepsilon) + \tilde{g}_{11}(\hat{\xi}_k, \varepsilon)w_k^* + L_1(\hat{\xi}_k, y_k, \varepsilon)[y_k - \tilde{h}_{21}(\hat{\xi}_k, \varepsilon) - \tilde{h}_{22}(\hat{\xi}_k, \varepsilon)]; \\ \hat{\xi}_1(k_0, \varepsilon) &= 0, \\ \varepsilon \hat{\xi}_{2,k+1} &= \tilde{f}_2(\hat{\xi}_{2,k}, \varepsilon) + \tilde{g}_{21}(\hat{\xi}_k, \varepsilon)w_k^* + L_2(\hat{\xi}_k, y_k, \varepsilon)[y_k - \tilde{h}_{21}(\hat{\xi}_k, \varepsilon) - \tilde{h}_{22}(\hat{\xi}_k, \varepsilon)]; \\ \hat{\xi}_2(k_0, \varepsilon) &= 0, \end{cases} \quad (6)$$

where  $\hat{\xi} \in \mathcal{X}$  is the filter state,  $L_1 \in \mathbb{R}^{n_1 \times m}$ ,  $L_2 \in \mathbb{R}^{n_2 \times m}$  are the filter gains, and  $w^*$  is the worst-case noise, while all the other variables have their corresponding previous meanings and dimensions. We can then define the penalty variable or estimation error at each instant  $k$  as

$$z_k = y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k). \quad (7)$$

The problem can then be formulated as a dynamic optimization problem with the following cost functional

$$\begin{cases} \min_{L_1, L_2 \in \mathbb{R}^{n \times m}} \sup_{w \in \mathcal{W}} J_1(L_1, L_2, w) = \frac{1}{2} \sum_{k=k_0}^{\infty} \{ \|z_k\|^2 - \gamma^2 \|w_k\|^2 \}, \\ s.t. \quad (6) \text{ and with } w = 0 \quad \lim_{k \rightarrow \infty} \{ \hat{\xi}_k - \xi_k \} = 0. \end{cases} \quad (8)$$

To solve the problem, we form the Hamiltonian function  $H : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \mathbb{R}^{n_1 \times m} \times \mathbb{R}^{n_2 \times m} \times \mathbb{R} \rightarrow \mathbb{R}$ :

$$\begin{aligned} H(\hat{\xi}, w, y, L_1, L_2, V, \varepsilon) &= V\left(\tilde{f}_1(\hat{\xi}_1, \varepsilon) + \tilde{g}_{11}(\hat{\xi}, \varepsilon)w + L_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}_1, \varepsilon) - \right. \\ &\quad \left. h_{22}(\hat{\xi}_2 \varepsilon)), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon) + \tilde{g}_{21}(\hat{\xi}, \varepsilon)w + \frac{1}{\varepsilon} L_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - \tilde{h}_{22}(\hat{\xi}, \varepsilon)), y\right) - \\ &\quad V(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|z\|^2 - \gamma^2 \|w\|^2) \end{aligned} \quad (9)$$

for some  $C^1$  positive-definite function  $V : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}_+$  and where  $\hat{\xi}_1 = \hat{\xi}_{1,k}$ ,  $\hat{\xi}_2 = \hat{\xi}_{2,k}$ ,  $y = y_k$ ,  $z = \{z_k\}$ ,  $w = \{w_k\}$ . We then determine the worst-case noise  $w^*$  and the optimal gains  $L_1^*$  and  $L_2^*$  by maximizing and minimizing  $H$  with respect to  $w$  and  $L_1, L_2$  respectively in the above expression (9), as

$$w^* = \arg \sup_w H(\hat{\xi}, w, y, L_1, L_2, V, \varepsilon), \quad (10)$$

$$[L_1^*, L_2^*] = \arg \min_{L_1, L_2} H(\hat{\xi}, w^*, y, L_1, L_2, V, \varepsilon). \quad (11)$$

However, because the Hamiltonian function (9) is not a linear or quadratic function of  $w$  and  $L_1, L_2$ , only implicit solutions may be obtained [1]. Thus, the only way to obtain an explicit solution is to use an approximate scheme. In [1] we have used a second-order Taylor series approximation of the Hamiltonian about  $(\tilde{f}_1(\hat{\xi}_1), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2), y)$  in the direction of the state vectors  $(\hat{\xi}_1, \hat{\xi}_2)$ . It is believed that, this would capture most,

if not all, of the system dynamics. However, for the  $\mathcal{H}_\infty$  problem at hand, such an approximation becomes too messy and the solution becomes more involved. Therefore, instead we would rather use a first-order Taylor approximation which is given by

$$\begin{aligned} \widehat{H}(\hat{\xi}, \hat{w}, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &= \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\ &\quad \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)[\tilde{g}_{11}(\hat{\xi}, \varepsilon)\hat{w} + \\ &\quad \hat{L}_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))] + \\ &\quad \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2, \varepsilon}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)[\tilde{g}_{21}(\hat{\xi}, \varepsilon)\hat{w} + \\ &\quad \hat{L}_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))] + \\ &\quad \frac{1}{2}(\|\tilde{z}\|^2 - \gamma^2\|\hat{w}\|^2) + O(\|\hat{\xi}\|^2), \end{aligned} \quad (12)$$

where  $\hat{V}$ ,  $\hat{w}$ ,  $\hat{L}_1$ ,  $\hat{L}_2$  are the corresponding approximate functions, and  $\hat{V}_{\hat{\xi}_1}$ ,  $\hat{V}_{\hat{\xi}_2}$  are the row vectors of first-partial derivatives of  $\hat{V}$  with respect to  $\hat{\xi}_1$ ,  $\hat{\xi}_2$  respectively. We can now obtain  $w^*$  as

$$\hat{w}^* = \frac{1}{\gamma^2}[\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \frac{1}{\varepsilon}\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)]. \quad (13)$$

Then substituting  $\hat{w} = \hat{w}^*$  in (12), we have

$$\begin{aligned} \widehat{H}(\hat{\xi}, \hat{w}^*, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &\approx \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\ &\quad \frac{1}{2\gamma^2} \left[ \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \right. \\ &\quad \left. \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2, \varepsilon}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right] + \\ &\quad \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\hat{L}_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) + \\ &\quad \frac{1}{2\gamma^2} \left[ \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{21}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \right. \\ &\quad \left. \frac{1}{\varepsilon^2}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{21}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right] + \\ &\quad \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\hat{L}_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) + \frac{1}{2}\|\tilde{z}\|^2. \end{aligned} \quad (14)$$

Completing the squares now for  $\hat{L}_1(\hat{\xi}, y)$  and  $\hat{L}_2(\hat{\xi}, y)$  in (14), we get

$$\begin{aligned} \widehat{H}(\hat{\xi}, \hat{w}^*, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &\approx \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\ &\quad \frac{1}{2\gamma^2} \left[ \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right. \\ &\quad \left. + \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right] \\ &\quad + \frac{1}{2} \left\| \hat{L}_1^T(\hat{\xi}, y)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + (y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) \right\|^2 + \end{aligned}$$

$$\begin{aligned} & \frac{1}{2\gamma^2} \left[ \frac{1}{\varepsilon} \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \tilde{g}_{21}(\hat{\xi}, \varepsilon) \tilde{g}_{11}^T(\hat{\xi}, \varepsilon) \hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right. \\ & \left. + \frac{1}{\varepsilon^2} \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \tilde{g}_{21}(\hat{\xi}) \tilde{g}_{21}^T(\hat{\xi}) \hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right] - \\ & \frac{1}{2} \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \hat{L}_1(\hat{\xi}, y, \varepsilon) \hat{L}_1^T(\hat{\xi}, y, \varepsilon) \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \\ & \frac{1}{2\varepsilon^2} \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \hat{L}_2(\hat{\xi}, y, \varepsilon) \hat{L}_2^T(\hat{\xi}, y, \varepsilon) \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \\ & + \frac{1}{2} \left\| \frac{1}{\varepsilon} \hat{L}_2^T(\hat{\xi}, y, \varepsilon) \hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + (y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) \right\|^2 - \\ & \frac{1}{2} \|z\|^2. \end{aligned}$$

Hence, setting the optimal gains as

$$\hat{V}_{\hat{\xi}_1, \varepsilon}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \hat{L}_1^*(\hat{\xi}, y, \varepsilon) = -(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))^T, \quad (15)$$

$$\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \hat{L}_2^*(\hat{\xi}, y, \varepsilon) = -\varepsilon(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))^T, \quad (16)$$

minimizes the Hamiltonian  $\hat{H}(\cdot, \cdot, \hat{L}_1, \hat{L}_2, \cdot, \cdot)$  and guarantees that the saddle-point condition [7]

$$\hat{H}(\cdot, \hat{w}^*, \hat{L}_1^*, \hat{L}_2^*, \cdot, \cdot) \leq \hat{H}(\cdot, \hat{w}^*, \hat{L}_1, \hat{L}_2, \cdot, \cdot) \quad \forall \hat{L}_1 \in \mathfrak{R}^{n_1 \times m}, \hat{L}_2 \in \mathfrak{R}^{n_2 \times m} \quad (17)$$

is satisfied. Finally, substituting the above optimal gains in (12) and setting

$$\hat{H}(\hat{\xi}, w^*, y, \hat{L}_1^*, \hat{L}_2^*, \hat{V}, \varepsilon) = 0,$$

results in the following discrete Hamilton-Jacobi-Isaacs equation (DHJIE)

$$\begin{aligned} & \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\ & \frac{1}{2\gamma^2} \left[ \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \quad \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \right] \times \\ & \left[ \begin{array}{cc} \tilde{g}_{11}(\hat{\xi}) \tilde{g}_{11}^T(\hat{\xi}, \varepsilon) & \frac{1}{\varepsilon} \tilde{g}_{11}(\hat{\xi}, \varepsilon) \tilde{g}_{21}^T(\hat{\xi}, \varepsilon) \\ \frac{1}{\varepsilon} \tilde{g}_{21}(\hat{\xi}, \varepsilon) \tilde{g}_{11}^T(\hat{\xi}, \varepsilon) & \frac{1}{\varepsilon^2} \tilde{g}_{21}(\hat{\xi}, \varepsilon) \tilde{g}_{21}^T(\hat{\xi}, \varepsilon) \end{array} \right] \left[ \begin{array}{c} \hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \\ \hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon} \tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \end{array} \right] - \\ & \frac{3}{2} (y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))^T (y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) = 0 \quad \hat{V}(0, 0, 0) = 0. \quad (18) \end{aligned}$$

We then have the following result.

**Proposition 3.1** *Consider the nonlinear discrete system (2) and the  $\mathcal{H}_\infty$ -filtering problem for this system. Suppose the plant  $\mathbf{P}_{sp}^{da}$  is locally asymptotically stable about the equilibrium-point  $x = 0$  and zero-input observable. Further, suppose there exist a local diffeomorphism  $\varphi$  that transforms the system to the partially decoupled form (5), a  $C^1$  positive-semidefinite function  $\hat{V} : \hat{N} \times \hat{Y} \rightarrow \mathfrak{R}_+$  locally defined in a neighborhood  $\hat{N} \times \hat{Y} \subset \mathcal{X} \times \mathcal{Y}$  of the origin  $(\hat{\xi}, y) = (0, 0)$ , and matrix functions  $\hat{L}_i : \hat{N} \times \hat{Y} \rightarrow \mathfrak{R}^{n_i \times m}$ ,  $i = 1, 2$ , satisfying the DHJIE (18) together with the side-conditions (15), (16) for some  $\gamma > 0$ . Then, the filter  $\mathbf{F}_{1c}^{da}$  solves the  $\mathcal{H}_\infty$  filtering problem for the system locally in  $\hat{N}$ .*

**Proof** The optimality of the filter gains  $\hat{L}_1^*$ ,  $\hat{L}_2^*$  has already been shown above. It remains to show that the saddle-point conditions [7]

$$\begin{aligned} \widehat{H}(\cdot, \hat{w}, \hat{L}_1^*, \hat{L}_2^*, \cdot, \cdot) \leq \widehat{H}(\cdot, \hat{w}^*, \hat{L}_1^*, \hat{L}_2^*, \cdot, \cdot) \leq \widehat{H}(\cdot, \hat{w}^*, \hat{L}_1, \hat{L}_2, \cdot, \cdot), \\ \forall \hat{L}_1 \in \mathfrak{R}^{n_1 \times m}, \hat{L}_2 \in \mathfrak{R}^{n_2 \times m}, \forall w \in \ell_2[k_0, \infty). \end{aligned} \quad (19)$$

and the  $\ell_2$ -gain condition (3) hold for all  $w \in \mathcal{W}$ . Moreover, that there is asymptotic convergence of the estimation error vector.

Now, the right-hand-side of the above inequality (19) has already been shown. It remains to show that the left hand side also holds. Accordingly, it can be shown from (12), (18) that

$$\widehat{H}(\hat{\xi}, \hat{w}, \hat{L}_1^*, \hat{L}_2^*, \hat{V}, \varepsilon) = \widehat{H}(\hat{\xi}, \hat{w}^*, \hat{L}_1^*, \hat{L}_2^*, \hat{V}, \varepsilon) - \frac{1}{2}\gamma^2\|\hat{w} - \hat{w}^*\|^2.$$

Therefore, we also have the left-hand side of (19) satisfied, and the pair  $(\hat{w}^*, [\hat{L}_1^*, \hat{L}_2^*])$  constitute a saddle-point solution to the dynamic game (8), (6).

Next, let  $\hat{V} \geq 0$  be a  $C^1$  solution of the DHJIE (18). Then, consider the time-variation of  $\hat{V}$  along a trajectory of (6), with  $\hat{L}_1 = \hat{L}_1^*$ ,  $L_2 = \hat{L}_2^*$ , and  $w \in \mathcal{W}$ , to get

$$\begin{aligned} \hat{V}(\hat{\xi}_{1,k+1}, \hat{\xi}_{2,k+1}, y) &\approx \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \\ &\quad \cdot [\tilde{g}_{11}(\hat{\xi}, \varepsilon)\hat{w} + \hat{L}_1^*(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))] \\ &+ \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) [\tilde{g}_{21}(\hat{\xi}, \varepsilon)w + \hat{L}_2^*(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))] \\ &= \hat{V}(\hat{\xi}, y_{k-1}) - \frac{\gamma^2}{2}\|\hat{w} - \hat{w}^*\|^2 + \frac{1}{2}(\gamma^2\|\hat{w}\|^2 - \|\tilde{z}\|^2) \\ &\leq \hat{V}(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\gamma^2\|\hat{w}\|^2 - \|\tilde{z}\|^2) \quad \forall \hat{w} \in \mathcal{W}, \end{aligned} \quad (20)$$

where we have used the first-order Taylor approximation in the above, and the last inequality follows from using the DHJIE (18). Moreover, the last inequality is the discrete-time dissipation-inequality [?] which also implies that the  $\ell_2$ -gain inequality (3) is satisfied.

In addition, setting  $w = 0$  in (20) implies that

$$\hat{V}(\hat{\xi}_{1,k+1}, \hat{\xi}_{2,k+1}, y) - \hat{V}(\hat{\xi}_{1,k}, \hat{\xi}_{2,k}, y_{k-1}) = -\frac{1}{2}\|z_k\|^2.$$

Therefore, the filter dynamics is stable, and  $V(\hat{\xi}, y)$  is non-increasing along a trajectory of (6). Further, the condition that  $\hat{V}(\hat{\xi}_{1,k+1}, \hat{\xi}_{2,k+1}, y) \equiv \hat{V}(\hat{\xi}_{1,k}, \hat{\xi}_{2,k}, y_{k-1}) \forall k \geq k_s$  (say!) implies that  $\tilde{z}_k \equiv 0$ , which further implies that  $y_k = \tilde{h}_{21}(\hat{\xi}_k) + \tilde{h}_{22}(\hat{\xi}_k) \forall k \geq k_s$ . By the zero-input observability of the system, this implies that  $\hat{\xi} = \xi$ . Finally, since  $\varphi$  is invertible and  $\varphi(0) = 0$ ,  $\hat{\xi} = \xi$  implies  $\hat{x} = \varphi^{-1}(\hat{\xi}) = \varphi^{-1}(\xi) = x$ .  $\square$

Next, we consider the limiting behavior of the filter (6) and the corresponding DHJIE (18). Letting  $\varepsilon \downarrow 0$ , we obtain from (6)

$$0 = \tilde{f}_2(\hat{\xi}_{2,k}) + L_2(\hat{\xi}_k, y_k)(y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k)) \quad \forall k,$$

and since  $\tilde{f}_2(\cdot)$  is asymptotically stable, we have  $\hat{\xi}_2 \rightarrow 0$ . Therefore  $H(\cdot, \dots, \cdot)$  in (9) becomes

$$H_0(\hat{\xi}, w, y, L_1, L_2, V, 0) = V\left(\tilde{f}_1(\hat{\xi}_1) + \tilde{g}_{11}(\hat{\xi})w + L_1(\hat{\xi}, y)(y - \tilde{h}_{21}(\hat{\xi}_1) - h_{22}(\hat{\xi}_2)), 0, y\right) - V(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|z\|^2 - \gamma^2\|w\|^2). \tag{21}$$

A first-order Taylor approximation of this Hamiltonian about  $(\tilde{f}_1(\hat{\xi}_1), 0, y)$  similarly yields

$$\begin{aligned} \hat{H}_0(\hat{\xi}, \hat{w}, y, \hat{L}_{10}, \bar{V}, 0) &= \bar{V}(\tilde{f}_1(\hat{\xi}_1), 0, y) + \bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\hat{L}_{10}^T(\hat{\xi}, y)(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi})) \\ &\quad + \bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\tilde{g}_{11}(\hat{\xi})w - \bar{V}(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|z\|^2 - \gamma^2\|\hat{w}\|^2) + \\ &\quad O(\|\hat{\xi}\|^2) \end{aligned} \tag{22}$$

for some corresponding positive-semidefinite function  $\bar{V} : \mathcal{X} \times \mathcal{Y} \rightarrow \Re$ , and gain  $\hat{L}_{10}$ . Minimizing again this Hamiltonian, we obtain the worst-case noise  $w_{10}^*$  and optimal gain  $\hat{L}_{10}^*$  given by

$$\hat{w}_{10}^* = -\tilde{g}_{11}^T(\hat{\xi})\bar{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), 0, y), \tag{23}$$

$$\bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\hat{L}_{10}^*(\hat{\xi}, y) = -(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi}))^T \tag{24}$$

where  $\bar{V}$  satisfies the reduced-order DHJIE

$$\begin{aligned} \bar{V}(\tilde{f}_1(\hat{\xi}_1), 0, y) + \frac{1}{2\gamma^2}\bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\tilde{g}_{11}(\hat{\xi})\tilde{g}_{11}^T(\hat{\xi})\bar{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), 0, y) - \bar{V}(\hat{\xi}_1, 0, y_{k-1}) - \\ \frac{3}{2}(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi}))^T(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi})) = 0, \quad \bar{V}(0, 0, 0) = 0. \end{aligned} \tag{25}$$

The corresponding reduced-order filter is given by

$$\bar{\mathbf{F}}_{1r}^{da} : \begin{cases} \dot{\hat{\xi}}_1 &= \tilde{f}_1(\hat{\xi}_1) + \hat{L}_{10}^*(\hat{\xi}_1, y)(y - \tilde{h}_{21}(\hat{\xi}) - \tilde{h}_{22}(\hat{\xi})) + O(\varepsilon). \end{cases} \tag{26}$$

Moreover, since the gain  $\hat{L}_{10}^*$  is such that the estimation error  $e_k = y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k) \rightarrow 0$ , and the vector-field  $\tilde{f}_2(\hat{\xi}_2)$  is locally asymptotically stable, we have  $\hat{L}_2^*(\hat{\xi}_k, y_k) \rightarrow 0$  as  $\varepsilon \downarrow 0$ . Correspondingly, the solution  $\bar{V}$  of the DHJIE (25) can be represented as the asymptotic limit of the solution of the DHJIE (18) as  $\varepsilon \downarrow 0$ , i.e.,

$$\hat{V}(\hat{\xi}, y) = \bar{V}(\hat{\xi}_1, y) + O(\varepsilon).$$

We can specialize the result of Proposition 3.1 to the following discrete-time linear singularly-perturbed system (DLSPS) [5, 16, 18, 22] in the slow coordinate:

$$\mathbf{P}_{dsp}^l : \begin{cases} x_{1,k+1} &= A_1x_{1,k} + A_{12}x_{2,k} + B_{11}w_k; \quad x_1(k_0) = x^1, \\ \varepsilon x_{2,k+1} &= A_{21}x_{1,k} + (\varepsilon I_{n_2} + A_2)x_{2,k} + B_{21}w_k; \quad x_2(k_0) = x^2, \\ y_k &= C_{21}x_{1,k} + C_{22}x_{2,k} + w_k, \end{cases} \tag{27}$$

where  $A_1 \in \Re^{n_1 \times n_1}$ ,  $A_{12} \in \Re^{n_1 \times n_2}$ ,  $A_{21} \in \Re^{n_2 \times n_1}$ ,  $A_2 \in \Re^{n_2 \times n_2}$ ,  $B_{11} \in \Re^{n_1 \times s}$ , and  $B_{21} \in \Re^{n_2 \times s}$ , while the other matrices have compatible dimensions. Then, an explicit form of the required transformation  $\varphi$  above is given by the Chang transformation [12]:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & -\varepsilon \mathbf{HL} & -\varepsilon \mathbf{H} \\ & \mathbf{L} & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{28}$$

where the matrices  $\mathbf{L}$  and  $\mathbf{H}$  satisfy the equations

$$\begin{aligned} 0 &= (\varepsilon I_{n_2} + A_2)\mathbf{L} - A_{21} - \varepsilon\mathbf{L}(A_1 - A_{12}\mathbf{L}), \\ 0 &= -\mathbf{H}[(\varepsilon I_{n_2} + A_2) + \varepsilon\mathbf{L}A_{12}] + A_{12} + \varepsilon(A_1 - A_{12}\mathbf{L})\mathbf{H}. \end{aligned}$$

The system is then represented in the new coordinates by

$$\tilde{\mathbf{P}}_{dsp}^l : \begin{cases} \xi_{1,k+1} &= \tilde{A}_1\xi_{1,k} + \tilde{B}_{11}w_k; & \xi_1(k_0) = \xi^{10}, \\ \varepsilon\xi_{2,k+1} &= \tilde{A}_2\xi_{2,k} + \tilde{B}_{21}w_k; & \xi_2(k_0) = \xi^{20}, \\ y_k &= \tilde{C}_{21}\xi_{1,k} + \tilde{C}_{22}\xi_{2,k} + w_k, \end{cases} \quad (29)$$

where

$$\begin{aligned} \tilde{A}_1 &= A_1 - A_{12}\mathbf{L} = A_1 - A_{12}(\varepsilon I_{n_2} + A_2)^{-1}A_{21} + O(\varepsilon), \\ \tilde{B}_{11} &= B_{11} - \varepsilon\mathbf{H}LB_{11} - \mathbf{H}B_{21} = B_{11} - A_{12}A_2^{-1}B_{21} + O(\varepsilon), \\ \tilde{A}_2 &= (\varepsilon I_{n_2} + A_2) + \varepsilon\mathbf{L}A_{12} = A_2 + O(\varepsilon), \\ \tilde{B}_{21} &= B_{21} + \varepsilon\mathbf{L}B_{11} = B_{21} + O(\varepsilon), \\ \tilde{C}_{21} &= C_{21} - C_{22}\mathbf{L} = C_{21} - C_{22}(\varepsilon I_{n_2} + A_2)^{-1}A_{21} + O(\varepsilon), \\ \tilde{C}_{22} &= C_{22} + \varepsilon(C_{21} - C_{22})\mathbf{H} = C_{22} + O(\varepsilon). \end{aligned}$$

Adapting the filter (6) to the system (29) yields the following filter

$$\mathbf{F}_{1c}^{dl} : \begin{cases} \hat{\xi}_{1,k+1} &= \tilde{A}_1\hat{\xi}_{1,k} + \tilde{B}_{11}w_k^* + \hat{L}_1(y_k - \tilde{C}_{21}\hat{\xi}_{1,k} - \tilde{C}_{22}\hat{\xi}_{2,k}), \\ \varepsilon\hat{\xi}_{2,k+1} &= \tilde{A}_2\hat{\xi}_{2,k} + \tilde{B}_{21}w_k^* + \hat{L}_2(y_k - \tilde{C}_{21}\hat{\xi}_{1,k} - \tilde{C}_{22}\hat{\xi}_{2,k}). \end{cases} \quad (30)$$

Taking

$$\hat{V}(\hat{\xi}, y) = \frac{1}{2}(\hat{\xi}_1^T \hat{P}_1 \hat{\xi}_1 + \hat{\xi}_2^T \hat{P}_2 \hat{\xi}_2 + y^T \hat{Q} y),$$

for some symmetric positive-definite matrices  $\hat{P}_1$ ,  $\hat{P}_2$ ,  $\hat{Q}$ , the DHJIE (18) reduces to the following algebraic equation

$$\begin{aligned} &(\hat{\xi}_1^T \tilde{A}_1^T \hat{P}_1 \tilde{A}_1 \hat{\xi}_1 + \frac{1}{\varepsilon^2} \hat{\xi}_2^T \tilde{A}_2^T \hat{P}_2 \tilde{A}_2^T \hat{\xi}_2 + y^T \hat{Q} y) - (\hat{\xi}_1^T \hat{P}_1 \hat{\xi}_1 + \hat{\xi}_2^T \hat{P}_2 \hat{\xi}_2 + y_{k-1}^T \hat{Q} y_{k-1}) + \\ &\frac{1}{\gamma^2} \left[ \hat{\xi}_1^T \tilde{A}_1^T \hat{P}_1 \tilde{B}_{11} \tilde{B}_{11}^T \hat{P}_1 \tilde{A}_1 \hat{\xi}_1 + \frac{1}{\varepsilon^2} \hat{\xi}_2^T \tilde{A}_2^T \hat{P}_2 \tilde{B}_{21} \tilde{B}_{21}^T \hat{P}_1 \tilde{A}_1 \hat{\xi}_1 + \frac{1}{\varepsilon^2} \hat{\xi}_1^T \tilde{A}_1^T \hat{P}_1 \tilde{B}_{11} \tilde{B}_{21}^T \hat{P}_2 \tilde{A}_2 \hat{\xi}_2 \right. \\ &\left. + \frac{1}{\varepsilon^4} \hat{\xi}_2^T \tilde{A}_2^T \hat{P}_2 \tilde{B}_{21} \tilde{B}_{21}^T \hat{P}_2 \tilde{A}_2 \hat{\xi}_2 \right] - 3(y^T y - \hat{\xi}_1^T \tilde{C}_{21}^T y - y^T \tilde{C}_{21}^T \hat{\xi}_1 - y^T \tilde{C}_{22}^T \hat{\xi}_1 - y^T \tilde{C}_{22}^T \hat{\xi}_2 - \\ &\hat{\xi}_2^T \tilde{C}_{22}^T y + \hat{\xi}_1^T \tilde{C}_{21}^T \tilde{C}_{21} \hat{\xi}_1 + \hat{\xi}_1^T \tilde{C}_{21}^T \tilde{C}_{22} \hat{\xi}_2 + \hat{\xi}_2^T \tilde{C}_{22}^T \tilde{C}_{21} \hat{\xi}_1 + \hat{\xi}_2^T \tilde{C}_{22}^T \tilde{C}_{22} \hat{\xi}_2) = 0. \end{aligned} \quad (31)$$

Subtracting now  $\frac{1}{2}y^T \hat{R} y$  for some symmetric matrix  $\hat{R} > 0$  from the left-hand side of the above equation (and absorbing  $\hat{R}$  in  $\hat{Q}$ ), we have the following matrix-inequality

$$\begin{bmatrix} \tilde{A}_1^T \hat{P}_1 A_1 - \hat{P}_1 + \frac{1}{\gamma^2} \tilde{A}_1^T \hat{P}_1 \tilde{B}_{11} \tilde{B}_{11}^T \hat{P}_1 \tilde{A}_1 - 3\tilde{C}_{21}^T \tilde{C}_{21} \\ \frac{1}{\gamma^2 \varepsilon^2} \tilde{A}_2^T \hat{P}_2 \tilde{B}_{21} \tilde{B}_{21}^T \hat{P}_1 \tilde{A}_1 + 3\tilde{C}_{22}^T \tilde{C}_{21} \\ 3\tilde{C}_{21} \\ 0 \\ \frac{1}{\gamma^2 \varepsilon^2} \tilde{A}_1^T \hat{P}_1 \tilde{B}_{11} \tilde{B}_{21}^T \hat{P}_2 \tilde{A}_2 + 3\tilde{C}_{21}^T \tilde{C}_{22} & 3\tilde{C}_{21}^T & 0 \\ \frac{1}{\varepsilon^2} \tilde{A}_2^T \hat{P}_2 \tilde{A}_2 - \hat{P}_2 + \frac{1}{\gamma^2 \varepsilon^4} \tilde{A}_2^T \hat{P}_2 \tilde{B}_{21} \tilde{B}_{21}^T \hat{P}_2 \tilde{A}_2 - 3\tilde{C}_{22}^T \tilde{C}_{22} & 3\tilde{C}_{22}^T & 0 \\ 3\tilde{C}_{22} & \hat{Q} - 3I & 0 \\ 0 & 0 & -\hat{Q} \end{bmatrix} \leq 0. \quad (32)$$



While the side conditions (15), (16) reduce to the following LMIs

$$\begin{bmatrix} 0 & 0 & \frac{1}{2}(\tilde{A}_1^T \hat{P}_1 \hat{L}_1 - \tilde{C}_{21}^T) \\ 0 & 0 & -\frac{1}{2}\tilde{C}_{22}^T \\ \frac{1}{2}(\tilde{A}_1^T \hat{P}_1 \hat{L}_1 - \tilde{C}_{21}^T)^T & -\frac{1}{2}\tilde{C}_{22}^T & (1 - \delta_1)I \end{bmatrix} \leq 0, \quad (33)$$

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2}\tilde{C}_{21}^T \\ 0 & 0 & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \hat{P}_2 \hat{L}_2 - \tilde{C}_{22}^T) \\ -\frac{1}{2}\tilde{C}_{21} & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \hat{P}_2 \hat{L}_2 - \tilde{C}_{22}^T)^T & (1 - \delta_2)I \end{bmatrix} \leq 0 \quad (34)$$

for some numbers  $\delta_1, \delta_2 \geq 1$ . The above matrix inequality (32) can be further simplified using Schur's complements, but cannot be made linear because of the off-diagonal and coupling terms. This is primarily because the assumed transformation  $\varphi$  can only achieve a partial decoupling of the original system, and a complete decoupling of the states will require more stringent assumptions and conditions.

Consequently, we have the following corollary to Proposition 3.1.

**Corollary 3.1** *Consider the DLSPS (27) and the  $\mathcal{H}_\infty$  filtering problem for this system. Suppose the plant  $\mathbf{P}_{sp}^l$  is locally asymptotically stable about the equilibrium-point  $x = 0$  and observable. Suppose further, it is transformable to the form (29), and there exist symmetric positive-definite matrices  $\hat{P}_1 \in \mathfrak{R}^{n_1 \times n_1}$ ,  $\hat{P}_2 \in \mathfrak{R}^{n_2 \times n_2}$ ,  $\hat{Q} \in \mathfrak{R}^{m \times m}$ , and matrices  $\hat{L}_1 \in \mathfrak{R}^{n_1 \times m}$ ,  $\hat{L}_2 \in \mathfrak{R}^{n_2 \times m}$ , satisfying the matrix inequalities (32), (33), (34) for some numbers  $\delta_1, \delta_2 \geq 1$  and  $\gamma > 0$ . Then, the filter  $\mathbf{F}_{1c}^{dl}$  solves the  $\mathcal{H}_\infty$  filtering problem for the system.*

Similarly, for the reduced-order filter (26) and the DHJIE (25), we have respectively

$$\mathbf{F}_{1r}^{dl} : \hat{\xi}_{1,k+1} = \tilde{A}_1 \hat{\xi}_{1,k} + \hat{L}_{10}^* (y_k - \tilde{C}_{21} \hat{\xi}_{1,k} - \tilde{C}_{22} \hat{\xi}_{2,k}), \quad (35)$$

$$\begin{bmatrix} \tilde{A}_1^T \hat{P}_{10} \tilde{A}_1 - \hat{P}_{10} - 3\tilde{C}_{21}^T \tilde{C}_{21} & \tilde{A}_1^T \hat{P}_{10} \tilde{B}_{11} & 3\tilde{C}_{21} & 0 \\ \tilde{B}_{11}^T \hat{P}_{10} \tilde{A}_1 & -\gamma^{-2}I & 0 & 0 \\ 3\tilde{C}_{21}^T & 0 & \hat{Q} - 3I & 0 \\ 0 & 0 & 0 & \hat{Q} \end{bmatrix} \leq 0, \quad (36)$$

$$\begin{bmatrix} 0 & 0 & \frac{1}{2}(\tilde{A}_1^T \hat{P}_{10} \hat{L}_{10} - \tilde{C}_{21}^T) \\ 0 & 0 & -\frac{1}{2}\tilde{C}_{22}^T \\ \frac{1}{2}(\tilde{A}_1^T \hat{P}_{10} \hat{L}_{10} - \tilde{C}_{21}^T)^T & -\frac{1}{2}\tilde{C}_{22}^T & (1 - \delta_{10})I \end{bmatrix} \leq 0 \quad (37)$$

for some symmetric positive-definite matrices  $\hat{P}_{10}, \hat{Q}_{10}$ , gain matrix  $\hat{L}_{10}$  and some number  $\delta_{10} > 0$ .

Proposition 3.1 has not yet exploited the benefit of the coordinate transformation in designing the filter (6) for the system (5). We shall now design separate reduced-order filters for the decomposed subsystems which should be more efficient than the previous one. If we let  $\varepsilon \downarrow 0$  in (5), we obtain the following reduced system model:

$$\tilde{\mathbf{P}}_r^a : \begin{cases} \hat{\xi}_{1,k+1} &= \tilde{f}_1(\hat{\xi}_1) + \tilde{g}_{11}(\hat{\xi})w, \\ 0 &= \tilde{f}_2(\hat{\xi}_2) + \tilde{g}_{21}(\hat{\xi})w, \\ y_k &= \tilde{h}_{21}(\hat{\xi}) + \tilde{h}_{22}(\hat{\xi}) + \tilde{k}_{21}(\hat{\xi})w. \end{cases} \quad (38)$$

Then, we assume the following [15, 17].

**Assumption 3.1** The system (2), (38) is in the “*standard form*”, i.e., the equation

$$0 = \tilde{f}_2(\xi_2) + \tilde{g}_{21}(\xi)w \quad (39)$$

has  $l \geq 1$  isolated roots, we can denote any one of these solutions by

$$\bar{\xi}_2 = q(\xi_1, w) \quad (40)$$

for some  $C^1$  function  $q : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X}$ .

Under Assumption 3.1, we obtain the reduced-order slow subsystem

$$\mathbf{P}_r^a : \begin{cases} \xi_{1,k+1} &= \tilde{f}_1(\xi_{1,k}) + \tilde{g}_{11}(\xi_{1,k}, q(\xi_{1,k}, w_k))w_k + O(\varepsilon), \\ y_k &= \tilde{h}_{21}(\xi_{1,k}, q(\xi_{1,k}, w_k)) + \tilde{h}_{22}(\xi_{1,k}, q(\xi_{1,k}, w_k)) + \\ &\quad \tilde{k}_{21}(\xi_{1,k}, q(\xi_{1,k}, w_k))w_k + O(\varepsilon) \end{cases}$$

and a boundary-layer (or quasi-steady-state) subsystem as

$$\bar{\xi}_{2,m+1} = \tilde{f}_2(\bar{\xi}_{2,m}, \varepsilon) + \tilde{g}_{21}(\xi_{1,m}, \bar{\xi}_{2,m})w_m, \quad (41)$$

where  $m = \lfloor k/\varepsilon \rfloor$  is a stretched-time parameter. This subsystem is guaranteed to be asymptotically stable for  $0 < \varepsilon < \varepsilon^*$  (see Theorem 8.2 in Ref. [15]) if the original system (2) is asymptotically stable.

We can then proceed to redesign the filter (6) for the composite system (41), (41) separately as

$$\tilde{\mathbf{F}}_{2c}^{da} : \begin{cases} \check{\xi}_{1,k+1} = \tilde{f}_1(\check{\xi}_{1,k}) + \tilde{g}_{11}(\check{\xi}_{1,k})\check{w}_{1,k}^* + \check{L}_1(\check{\xi}_{1,k}, y_k)(y_k - \tilde{h}_{21}(\check{\xi}_{1,k}) - \tilde{h}_{22}(\check{\xi}_{1,k})), \\ \varepsilon\check{\xi}_{2,k+1} = \tilde{f}_2(\check{\xi}_{2,k}, \varepsilon) + \tilde{g}_{21}(\check{\xi}_k)\check{w}_{2,k}^* + \check{L}_2(\check{\xi}_{2,k}, y_k)(y_k - \tilde{h}_{21}(\check{\xi}_k) - \tilde{h}_{22}(\check{\xi}_k)), \end{cases} \quad (42)$$

where

$$\tilde{h}_{21}(\check{\xi}_{1,k}) = \tilde{h}_{21}(\check{\xi}_{1,k}, q(\check{\xi}_{1,k}, \hat{w}_{1,k}^*)), \quad \tilde{h}_{22}(\check{\xi}_{1,k}) = \tilde{h}_{21}(\check{\xi}_{1,k}, q(\check{\xi}_{1,k}, \hat{w}_{2,k}^*)).$$

Notice also that,  $\xi_2$  cannot be estimated from (40) since this is a “quasi-steady-state” approximation. Then, using a similar approximation procedure as in Proposition 3.1, we arrive at the following result.

**Theorem 3.1** Consider the nonlinear system (2) and the  $\mathcal{H}_\infty$  estimation problem for this system. Suppose the plant  $\mathbf{P}_{sp}^{da}$  is locally asymptotically stable about the equilibrium-point  $x = 0$  and zero-input observable. Further, suppose there exists a local diffeomorphism  $\varphi$  that transforms the system to the partially decoupled form (5), and Assumption 3.1 holds. In addition, suppose for some  $\gamma > 0$ , there exist  $C^1$  positive-semidefinite functions  $\check{V}_i : \check{N}_i \times \check{Y}_i \rightarrow \mathbb{R}_+$ ,  $i = 1, 2$ , locally defined in neighborhoods  $\check{N}_i \times \check{Y}_i \subset \mathcal{X} \times \mathcal{Y}$  of the origin  $(\check{\xi}_i, y) = (0, 0)$   $i = 1, 2$  respectively, and matrix functions  $\check{L}_i : \check{N}_i \times \check{Y}_i \rightarrow \mathbb{R}^{n_i \times m}$ ,

$\check{Y}_i \subset \mathcal{Y}$ ,  $i = 1, 2$  satisfying the pair of DHJIEs:

$$\begin{aligned} & \check{V}_1(\tilde{f}_1(\hat{\xi}_1), y) + \frac{1}{2\gamma^2} \check{V}_{1,\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), y) \tilde{g}_{11}(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)) \tilde{g}_{11}^T(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)) \check{V}_{1,\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), y) - \\ & \bar{V}_1(\hat{\xi}_1, y_{k-1}) - \frac{3}{2}(y - \tilde{h}_{21}(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)) - h_{22}(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)))^T (y - \tilde{h}_{21}(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)) - \\ & h_{22}(\hat{\xi}_1, q(\xi_1, \check{w}_1^*))) = 0, \quad \check{V}_1(0, 0) = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} & \check{V}_2\left(\frac{1}{\varepsilon} \tilde{f}_2(\check{\xi}_2, \varepsilon), y\right) + \frac{1}{2\gamma^2} \check{V}_{2,\check{\xi}_2}\left(\frac{1}{\varepsilon} \tilde{f}_2(\check{\xi}_2, \varepsilon), y\right) \tilde{g}_{21}(\check{\xi}_2, \varepsilon) \tilde{g}_{21}^T(\check{\xi}_2, \varepsilon) \check{V}_{2,\check{\xi}_2}^T\left(\frac{1}{\varepsilon} \tilde{f}_2(\check{\xi}_2, \varepsilon), y\right) - \\ & \check{V}_2(\check{\xi}_2, y_{k-1}) - \frac{3}{2}(y - \tilde{h}_{21}(\check{\xi}_2, \varepsilon) - \tilde{h}_{22}(\check{\xi}_2, \varepsilon))^T (y - \tilde{h}_{21}(\check{\xi}_2, \varepsilon) - h_{22}(\check{\xi}_2, \varepsilon)) = 0, \\ & \check{V}_2(0, 0) = 0 \end{aligned} \tag{44}$$

together with the side-conditions

$$\check{w}_1^* = \frac{1}{\gamma^2} \tilde{g}_{11}^T(\hat{\xi}_1, q(\xi_1, \check{w}_1^*)) \check{V}_{1,\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), y), \tag{45}$$

$$\check{w}_2^* = \frac{1}{\gamma^2} \tilde{g}_{21}^T(\check{\xi}_2, \varepsilon) \check{V}_{2,\check{\xi}_2}^T\left(\frac{1}{\varepsilon} \tilde{f}_2(\check{\xi}_2, \varepsilon), y\right), \tag{46}$$

$$\hat{V}_{1,\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1)) \check{L}_1^*(\hat{\xi}_1, y) = -(y - \tilde{h}_{21}(\hat{\xi}_1) - \tilde{h}_{22}(\hat{\xi}_1))^T, \tag{47}$$

$$\check{V}_{2,\check{\xi}_2}^T\left(\frac{1}{\varepsilon} \tilde{f}_2(\check{\xi}_2, \varepsilon), y\right) \check{L}_2^*(\check{\xi}_2, \varepsilon) = -\varepsilon(y - \tilde{h}_{21}(\check{\xi}_2, \varepsilon) - \tilde{h}_{22}(\check{\xi}_2, \varepsilon))^T. \tag{48}$$

Then the filter  $\tilde{\mathbf{F}}_{2c}^{da}$  solves the  $\mathcal{H}_\infty$  filtering problem for the system locally in  $\cup \check{N}_i$ .

**Proof** We define separately two Hamiltonian functions  $H_i : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \mathbb{R}^{n_i \times m} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $i = 1, 2$  for each of the two separate components of the filter (42). Then the rest of the proof follows along the same lines as Proposition 3.1.  $\square$

**Remark 3.2** Comparing (43), (47) with (24), (25), we see that the two reduced-order filter approximations are similar. Moreover, notice that  $\check{\xi}_1$  appearing in (48), (44) is not considered as an additional variable, because it is assumed to be known from (42a), (47) respectively, and is therefore regarded as a parameter. In addition, we observe that, the DHJIE (43) is implicit in  $\check{w}_1^*$ , and therefore, some sort of approximation is required in order to obtain an explicit solution.

**Remark 3.3** Notice also that, in the determination of  $\check{w}_1^*$ , we assume  $\bar{\xi}_2 = q(\xi_1, w)$  is frozen in the Hamiltonian  $H_2$ , and therefore the contribution to  $\check{w}_1^*$  from  $\tilde{g}_{11}(\cdot, \cdot)$ ,  $\tilde{h}_{21}(\cdot, \cdot)$  is neglected.

We can similarly specialize the result of Theorem 3.1 to the discrete-time linear system (27) in the following corollary.

**Corollary 3.2** Consider the DLSPS (27) and the  $\mathcal{H}_\infty$  filtering problem for this system. Suppose the plant  $\mathbf{P}_{sp}^l$  is locally asymptotically stable about the equilibrium-point  $x = 0$  and observable. Suppose further, it is transformable to the form (29) and Assumption 3.1 is satisfied, i.e.,  $\tilde{A}_2$  is nonsingular. In addition, suppose for some  $\gamma > 0$

there exist symmetric positive-definite matrices  $\check{P}_i \in \mathfrak{R}^{n_i \times n_i}$ ,  $\check{Q}_i \in \mathfrak{R}^{m \times m}$ , and matrix  $\check{L}_i \in \mathfrak{R}^{n_i \times m}$ ,  $i = 1, 2$  satisfying the following LMIs

$$\begin{bmatrix} \check{A}_1^T \check{P}_1 \check{A}_1 - \check{P}_1 - 3\check{C}_{21}^T \check{C}_{21} & \check{A}_1^T \check{P}_1 \check{B}_{11} & 3\check{C}_{21}^T & 0 \\ \check{B}_{11}^T \check{P}_1 \check{A}_1 & -\gamma^2 I & 0 & 0 \\ 3\check{C}_{21} & 0 & \check{Q}_1 - 3I & 0 \\ 0 & 0 & 0 & -\check{Q} \end{bmatrix} \leq 0, \quad (49)$$

$$\begin{bmatrix} -3\check{C}_{21}^T \check{C}_{21} & -3\check{C}_{21}^T \check{C}_{22} & 0 & 3\check{C}_{21}^T & 0 \\ -3\check{C}_{22}^T \check{C}_{21} & \check{A}_2^T \check{P}_2 \check{A}_2 - \check{P}_2 - 3\check{C}_{22}^T \check{C}_{22} & \check{A}_2^T \check{P}_2 \check{B}_{21} & 3\check{C}_{22}^T & 0 \\ 0 & \check{B}_{21}^T \check{P}_2 \check{A}_2 & \gamma^2 \varepsilon^2 I & 0 & 0 \\ 3\check{C}_{21} & 3\check{C}_{22} & 0 & \check{Q}_2 - 3I - \check{R}_2 & 0 \\ 0 & 0 & 0 & 0 & -\check{Q}_2 \end{bmatrix} \leq 0, \quad (50)$$

$$\begin{bmatrix} 0 & \frac{1}{2}(\check{A}_1^T \check{P}_1 \check{L}_1 - \check{C}_{21}^T) \\ \frac{1}{2}(\check{A}_1^T \check{P}_1 \check{L}_1 - \check{C}_{21}^T)^T & (1 - \delta_3)I \end{bmatrix} \leq 0, \quad (51)$$

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2}\check{C}_{21}^T \\ 0 & 0 & \frac{1}{2\varepsilon^2}(\check{A}_2^T \check{P}_2 \check{L}_2 - \check{C}_{22}^T) \\ -\frac{1}{2}\check{C}_{21} & \frac{1}{2\varepsilon^2}(\check{A}_2^T \check{P}_2 \check{L}_2 - \check{C}_{22}^T)^T & (1 - \delta_4)I \end{bmatrix} \leq 0, \quad (52)$$

for some numbers  $\delta_3, \delta_4 > 0$ , where

$$\check{B}_{11} = \tilde{B}_{11} + \check{C}_{22} \check{A}_2^{-1} \tilde{B}_{21}, \quad \check{C}_{21} = \tilde{C}_{21} - \frac{1}{\gamma^2} \check{C}_{22} \check{A}_2^{-1} \tilde{B}_{21} \tilde{B}_{11}^T \check{P}_1 \check{A}_1.$$

Then, the filter  $\mathbf{F}_{2c}^{dl}$  solves the  $\mathcal{H}_\infty$  filtering problem for the system.

**Proof** We take similarly

$$\check{V}_1(\hat{\xi}_1, y) = \frac{1}{2}(\check{\xi}_1^T \check{P}_1 \check{\xi}_1 + y^T \check{Q}_1 y),$$

$$\check{V}_2(\hat{\xi}_2, y) = \frac{1}{2}(\check{\xi}_2^T \check{P}_2 \check{\xi}_2 + y^T \check{Q}_2 y)$$

and apply the result of the Theorem.  $\square$

#### 4 Aggregate Filters

In the absence of the coordinate transformation,  $\varphi$  discussed in the previous section, a filter has to be designed to solve the problem for the aggregate system (2). We discuss this class of filters in this section. Accordingly, consider the following class of filters  $\mathbf{F}_{3ag}^{da}$  :

$$\begin{cases} \dot{x}_{1,k+1} = f_1(\dot{x}_k) + g_{11}(\dot{x}_k) \dot{w}_k^* + \check{L}_1(\dot{x}_k, y_k, \varepsilon)(y_k - h_{21}(\dot{x}_{1,k}) - h_{22}(\dot{x}_{2,k})); \\ \dot{x}_1(k_0) = \bar{x}^{10}, \\ \varepsilon \dot{x}_{2,k+1} = f_2(\dot{x}_k, \varepsilon) + g_{21}(\dot{x}_k) \dot{w}_k^* + \check{L}_2(\dot{x}_k, y_k, \varepsilon)(y_k - h_{21}(\dot{x}_{1,k}) - h_{22}(\dot{x}_{2,k})); \\ \dot{x}_2(k_0) = \bar{x}^{20}, \\ \dot{z}_k = y_k - h_{21}(\dot{x}_{1,k}) - h_{22}(\dot{x}_{2,k}), \end{cases} \quad (53)$$

where  $\check{L}_1, \check{L}_2 \in \mathfrak{R}^{n \times m}$  are the filter gains, and  $\dot{z}$  is the new penalty variable. We can repeat the same kind of derivation above to arrive at the following.

**Theorem 4.1** Consider the nonlinear system (2) and the  $\mathcal{H}_2$  estimation problem for this system. Suppose the plant  $\mathbf{P}_{\text{sp}}^{\text{da}}$  is locally asymptotically stable about the equilibrium-point  $x = 0$ , and zero-input observable. Further, suppose there exist a  $C^1$  positive-definite function  $\dot{V} : \dot{N} \times \dot{\Upsilon} \rightarrow \mathfrak{R}_+$ , locally defined in a neighborhood  $\dot{N} \times \dot{\Upsilon} \subset \mathcal{X} \times \mathcal{Y}$  of the origin  $(\dot{x}_1, \dot{x}_2, y) = (0, 0, 0)$ , and matrix functions  $\dot{L}_i : \dot{N} \times \dot{\Upsilon} \rightarrow \mathfrak{R}^{n_i \times m}$ ,  $i = 1, 2$ , satisfying the DHJIE:

$$\begin{aligned} & \dot{V}(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) - \dot{V}(\dot{x}, y_{k-1}) + \\ & \frac{1}{2\gamma^2} [ \dot{V}_{\dot{x}_1}(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) \quad \dot{V}_{\dot{x}_2}(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) ] \times \\ & \begin{bmatrix} g_{11}(\dot{x})g_{11}^T(\dot{x}) & \frac{1}{\varepsilon} g_{11}(\dot{x})g_{21}^T(\dot{x}) \\ \frac{1}{\varepsilon} g_{21}(\dot{x})g_{11}^T(\dot{x}) & \frac{1}{\varepsilon^2} g_{21}(\dot{x})g_{21}^T(\dot{x}) \end{bmatrix} \begin{bmatrix} \dot{V}_{\dot{x}_1}^T(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) \\ \dot{V}_{\dot{x}_2}^T(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) \end{bmatrix} \\ & - \frac{3}{2}(y - h_{21}(\dot{x}_1) - h_{22}(\dot{x}_2))^T(y - h_{21}(\dot{x}_1) - h_{22}(\dot{x}_2)) = 0, \quad \dot{V}(0, 0) = 0, \end{aligned} \quad (54)$$

together with the side-conditions

$$\dot{V}_{\dot{x}_1}(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) \dot{L}_1^*(\dot{x}, y) = -(y - h_{21}(\dot{x}_1) - h_{22}(\dot{x}_2))^T, \quad (55)$$

$$\dot{V}_{\dot{x}_2}(f_1(\dot{x}), \frac{1}{\varepsilon} f_2(\dot{x}, \varepsilon), y) \dot{L}_2^*(\dot{x}, y) = -\varepsilon(y - h_{21}(\dot{x}_1) - h_{22}(\dot{x}_2)). \quad (56)$$

Then, the filter  $\mathbf{F}_{3ag}^a$  solves the  $\mathcal{H}_\infty$  filtering problem for the system locally in  $\dot{N}$ .

**Proof** Proof follows along the same lines as Proposition 3.1.  $\square$

For the DLSPS (27), the Chang transformation  $\varphi$  is always available as given by (28). Moreover, the result of Theorem 4.1 specialized to the DLSPS is horrendous, in the sense that, the resulting inequalities are not linear and too involved. Thus, it is more useful to consider the reduced-order filter which will be introduced shortly as a special case of the nonlinear.

Using similar procedure as outlined in the previous section, we can obtain the limiting behavior of the filter  $\mathbf{F}_{3ag}^a$  as  $\varepsilon \downarrow 0$

$$\bar{\mathbf{F}}_{5ag}^{\text{da}} : \begin{cases} \dot{x}_{1,k+1} &= f_1(\dot{x}_k) + g_{11}(\dot{x}_k) \dot{w}_{10,k}^* + \dot{L}_{10}(\dot{x}_k, y_k)(y_k - h_{21}(\dot{x}_{1,k})); \\ \dot{x}_1(k_0) &= \bar{x}^{10}, \\ \dot{x}_{2,k} &\rightarrow 0 \end{cases} \quad (57)$$

with

$$\dot{w}_{10}^* = \frac{1}{\gamma^2} g_{11}^T(\dot{x}) \dot{V}_{\dot{x}_1}^T(f_1(\dot{x}))$$

and the DHJIE (54) reduces to the DHJIE

$$\begin{aligned} & \dot{V}(f_1(\dot{x}_1), y) + \frac{1}{2\gamma^2} \bar{V}_{\dot{x}_1}(f_1(\dot{x}_1), y) g_{11}(\dot{x}) g_{11}^T(\dot{x}) \dot{V}_{\dot{x}_1}^T(f_1(\dot{x})) - \dot{V}(\dot{x}_1, y) - \\ & \frac{3}{2}(y - h_{21}(\dot{x}_1))^T(y - h_{21}(\dot{x}_1)) = 0, \quad \dot{V}(0) = 0. \end{aligned} \quad (58)$$

together with the side-conditions

$$\dot{V}_{\dot{x}_1}(f_1(\dot{x}_1)) \dot{L}_{10}^*(\dot{x}, y) = -(y - h_{21}(\dot{x}_1))^T, \quad (59)$$

$$\dot{L}_2(\dot{x}, y) \rightarrow 0. \quad (60)$$

Similarly, specializing the above result to the DLSPS (27), we obtain the following reduced-order filter

$$\mathbf{F}_{6agr}^{dl} : \left\{ \begin{array}{l} \dot{\tilde{x}}_{1,k+1} = A_1 \dot{\tilde{x}}_{1,k} + B_{11} \dot{w}_{10,k}^* + \dot{L}_{10}^*(y_k - \tilde{C}_{21} \dot{\tilde{x}}_{1,k}) \end{array} \right. \quad (61)$$

with

$$\dot{w}_{10}^* = \frac{1}{\gamma^2} B_{11}^T \dot{P}_1 A_1 \dot{\tilde{x}}_1$$

and the DHJIE (58) reduces to the LMI

$$\begin{bmatrix} A_1^T \dot{P}_{10} \tilde{A}_1 - \dot{P}_{10} - 3C_{21}^T C_{21} & A_1^T \dot{P}_{10} B_{11} & 3C_{21}^T & 0 \\ B_{11}^T \dot{P}_{10} A_1 & -\gamma^2 I & 0 & 0 \\ 3C_{21} & 0 & \dot{Q}_{10} - 3I & 0 \\ 0 & 0 & 0 & -\dot{Q}_{10} \end{bmatrix} \leq 0, \quad (62)$$

$$\begin{bmatrix} 0 & \frac{1}{2}(A_1^T \dot{P}_{10} \dot{L}_{10} - C_{21}^T) \\ \frac{1}{2}(A_1^T \dot{P}_{10} \dot{L}_{10} - C_{21}^T)^T & (1 - \delta_5)I \end{bmatrix} \leq 0 \quad (63)$$

for some symmetric positive-definite matrices  $\dot{P}_{10}$ ,  $\dot{Q}_{10}$ , gain matrix  $\dot{L}_{10}$  and some number  $\delta_5 \geq 1$ .

**Remark 4.1** If the nonlinear system (2) is in the standard form, i.e., the equivalent of Assumption 3.1 is satisfied, and there exists at least one root  $\bar{x}_2 = \sigma(x_1, w)$  to the equation

$$0 = f_2(x_1, x_2) + g_{21}(x_1, x_2)w,$$

then reduced-order filters can also be constructed for the system similar to the result of Section 3 and Theorem 3.1. Such filters would take the following form

$$\mathbf{F}_{7agr}^a : \left\{ \begin{array}{l} \check{x}_{1,k+1} = f_1(\check{x}_{1,k}, \sigma(\check{x}_1, \check{w}_{1,k}^*)) + g_{11}(\check{x}_1, \sigma(\check{x}_1, \check{w}_{1,k}^*))\check{w}_{1,k}^* + \\ \quad \check{L}_1(\check{x}_{1,k}, y_k, \varepsilon)(y_k - h_{21}(\check{x}_{1,k}) - h_{22}(\sigma(\check{x}_1, \check{w}_{1,k}^*))); \quad \check{x}_1(k_0) = \bar{x}_{10}, \\ \varepsilon \check{x}_{2,k+1} = f_2(\check{x}_k, \varepsilon) + g_{21}(\check{x}_1, \check{x}_2)\check{w}_{2,k}^* + \check{L}_2(\check{x}_k, y_k, \varepsilon)(y_k - h_{21}(\check{x}_{1,k}) - \\ \quad h_{22}(\check{x}_{2,k})); \quad \check{x}_2(k_0) = \bar{x}_{20}, \\ \check{z}_k = y_k - h_{21}(\check{x}_{1,k}) - h_{22}(\check{x}_{2,k}). \end{array} \right.$$

However, this filter would fall into the class of decomposition filters, rather than aggregate, and because of this, we shall not discuss it further in this section.

In the next section, we consider an example.

## 5 Examples

Consider the following singularly-perturbed nonlinear system

$$\begin{aligned} x_{1,k+1} &= x_{1,k}^{\frac{1}{3}} + x_{2,k}^{\frac{1}{2}} + w, \\ \varepsilon x_{2,k+1} &= -x_{2,k}^{\frac{1}{2}} - x_{2,k}^{\frac{1}{3}}, \\ y_k &= x_{1,k} + x_{2,k} + w, \end{aligned}$$

where  $w \in \ell_2[0, \infty)$  is a noise process,  $\varepsilon \geq 0$ . We construct the aggregate filter  $\mathbf{F}_{3ag}^a$  presented in the previous section for the above system. It can be checked that the

system is locally observable, and with  $\gamma = 1$ , the function  $\check{V}(\check{x}) = \frac{1}{2}(\check{x}_1^2 + \varepsilon\check{x}_2^2)$ , solves the inequality form of the DHJIE (54) corresponding to system. Subsequently, we calculate the gains of the filter as

$$\dot{L}_1(\check{x}, y) = -\frac{(y - \check{x}_1 - \check{x}_2)}{\check{x}_1^{\frac{1}{3}} + \check{x}_2^{\frac{1}{2}}}, \quad \dot{L}_2(\check{x}, y) = \frac{(y - \check{x}_1 - \check{x}_2)}{\check{x}_2^{\frac{1}{2}} + \check{x}_2^{\frac{1}{3}}}, \quad (64)$$

where the gains  $\dot{L}_1, \dot{L}_2$  are set equal to zero if  $\|\check{x}\| < \epsilon$  (small) to avoid the singularity at the origin  $\check{x} = 0$ .

## 6 Conclusion

In this paper, we have presented a solution to the  $\mathcal{H}_\infty$  filtering problem for discrete-time affine nonlinear singularly-perturbed systems. Two classes of filters, namely, decomposition and aggregate filters, have been discussed, and in each case, first-order approximate filters have been presented. Reduced-order filters have also been derived as limiting cases of the above filters as the singular parameter  $\varepsilon \downarrow 0$ . Sufficient conditions for the solvability of the problem using each filter have been given in terms of DHJIEs. The results have also been specialized to linear systems, in which case, the sufficient conditions reduce to a system of matrix-inequalities or LMIs which are computationally efficient to solve. In addition, an example has been presented to illustrate the approach.

Future efforts would concentrate in finding an explicit form for the coordinate transformation discussed in Section 3, and developing computationally efficient algorithms for solving the DHJIEs.

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## Optimal State Observer Design for Nonlinear Dynamical Systems

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**Abstract:** This paper investigates the synthesis and the performance study of the optimal state observer designed for nonlinear dynamical systems to reconstruct the unmeasurable state variables and to stabilize rapidly the observation error system. The proposed nonlinear optimal state observer is based on the determination of the optimal observation gain matrix which is derived by minimizing a quadratic criterion formulated as an output feedback control problem of the observation error system. The gradient matrix operations is applied to the Lagrangian function in order to obtain necessary and sufficient conditions, for minimizing the proposed criterion, to perform the optimal gain matrix. The necessary and sufficient conditions are presented by coupled equations which resolution, by a numerical efficient algorithm, allows the calculus of the optimal observation gain. The effectiveness and the availability of the observer design approach are illustrated through numerical simulation to reconstruct the state variables of a robot with flexible link.

**Keywords:** *nonlinear observer design; optimal control; output feedback control; flexible robot.*

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## 1 Introduction

When the exact and complete knowledge on current states of a dynamic plant is impossible by different reasons, the use of a state observer (estimator) is compulsory to realize a successful closed-loop control [1–4].

Hence, the problem of state observation for nonlinear systems is of main importance in automatic control. In recent years many contributions have been presented in literature that investigate this problem for different classes of nonlinear systems. Generally, there are two approaches dealing with the nonlinear observer design. The first one is based on a nonlinear transformation by which the error dynamic is linear so that the design of state observer can be performed using linear techniques [5]. Necessary and sufficient conditions for the existence of the state transformation have been established in [5]. The second approach does not need any transformation and the observer design is directly based on the original system [2, 6].

For linear and nonlinear dynamical systems, a number of methods for observing the state variables and especially for the determination of the observation gain matrix, such that the asymptotic stability is ensured, have been proposed in the literature as the linear matrix inequality (LMI) approach [7–10], the Lyapunov equation method [11–13], the algebraic Riccati equation [5, 14, 15] and the min-max approach [16–19].

In synthesizing a control law and/or observation one two goals are focused: maximizing performances and minimizing costs of implementation. Hence, a simple control law, which is less complicated and less costly to implement than a full state feedback controller for example, may be preferred. Indeed, there is a number of structural alternatives such as full output feedback or low order dynamic compensation [20, 21].

In this paper we have considered the optimal state observer design for nonlinear dynamical systems which the non-linearity satisfy a globally Lipschitz condition. This approach is based on the minimizing of a quadratic criterion formulated as a quadratic output feedback control problem of the observation error in order to obtain an optimal gain. This proposed quadratic criterion has a direct signification and interpretation regarding to the desired observer. Thus, this optimal gain is calculated from the gradient resolution of the designed Lagrangian function in order to obtain necessary and sufficient conditions.

These necessary and sufficient conditions for the proposed nonlinear optimal state observer are derived, using the gradient techniques, in the form of Lyapunov and Riccati equations which resolution, by a proposed efficient iterative numerical algorithm, allows the calculus of the optimal gain matrix.

This paper is organized as follows: the proposed nonlinear optimal state observer is presented in Section 2. In Section 3, an illustrate example of a robot with flexible link is presented to highlight the performance of the proposed nonlinear optimal state observation approach.

## 2 Nonlinear Optimal State Observer

### 2.1 Problem formulation

We consider the class of nonlinear systems described by the following state equations

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x, u), \\ y(t) = Cx(t), \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^p$  is the control vector,  $y(t) \in \mathbb{R}^m$  is the output vector,  $A$  and  $C$  are constant matrices of appropriate dimensions. The nonlinear function  $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is Lipschitz with respect to the state  $x(t)$ , uniformly in the control  $u(t)$ , that is, there exists a constant  $\gamma > 0$  such that

$$\|f(t, x_1, u) - f(t, x_2, u)\| \leq \gamma \|x_1 - x_2\| \tag{2}$$

for all  $x_1(t), x_2(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^p$ .

Systems with Lipschitz nonlinearity are common in many practical applications. Many nonlinear systems satisfy the Lipschitz property at least locally by representing them by a linear part plus a Lipschitz nonlinearity around their equilibrium points.

We assume that the pair  $(A, C)$  is observable. Then the state observer for the nonlinear system (1) may be written as follows

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + f(t, \hat{x}, u) + L(y(t) - \hat{y}(t)), \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{3}$$

with  $\hat{x}(t) \in \mathbb{R}^n$  the state observer of  $x(t)$  and  $L \in \mathbb{R}^{n \times m}$  the observer gain matrix to be determined.

The observation error between the real state and the observed one is defined by

$$e(t) = x(t) - \hat{x}(t). \tag{4}$$

Subtracting (1) from (3) gives the dynamical reconstruction error

$$\dot{e}(t) = (A - LC)e(t) + f(t, x, u) - f(t, \hat{x}, u). \tag{5}$$

In literature, several methods can be used for the determination of the observer gain matrix, such that the asymptotic stability of the observation error is ensured, as the Lyapunov equation method, the algebraic Riccati equation approach and the linear matrix inequality (LMI) technique. The drawback of these methods is that the observation gain to determine can be practically not acceptable and where the minimization of a quadratic criterion is used has no direct physical interpretation regarding to the observation error dynamic.

In what follows we propose a new formulation of the dynamical observation error (5). Thus, the dynamical observation error can be considered as the following system

$$\begin{cases} \dot{e}(t) = Ae(t) + \eta(t) + f(t, x, u) - f(t, \hat{x}, u), \\ \eta(t) = -L\nu(t), \\ \nu(t) = Ce(t). \end{cases} \tag{6}$$

The system (6) expresses an output feedback control problem of the nonlinear system of order  $n$  with  $n$  dimensional input vector  $\eta(t)$  and  $m$  dimensional output vector  $\nu(t)$ .

The proposed output feedback control problem scheme can be optimized by minimizing the following quadratic criterion defined by

$$\begin{aligned} J &= \int_0^\infty \left( e^T(t)Q_0e(t) + \eta^T(t)R_0\eta(t) \right) dt \\ &= \int_0^\infty e^T(t) \left( Q_0 + C^T L^T R_0 LC \right) e(t) dt \end{aligned} \tag{7}$$

with  $Q_0 = Q_0^T \geq 0$  and  $R_0 = R_0^T > 0$ .

Then, we have the following result.

**Theorem 2.1** *Consider the dynamical observation error (6). If there exists a matrix  $P = P^T$  solution of the following algebraic Riccati equation*

$$(A - LC)^T P + P(A - LC) + \delta^{-1} P^2 + \delta \gamma^2 I + Q_0 + C^T L^T R_0 LC + Q = 0 \quad (8)$$

with  $Q_0 = Q_0^T \geq 0$ ,  $Q = Q^T \geq 0$ ,  $R_0 = R_0^T$  and  $\delta$  a positive scalar.

Then the state observation error is globally asymptotically stable and the quadratic criterion (7) satisfies

$$J \leq e_0^T P e_0, \quad (9)$$

where  $e_0 = e(0)$  is the initial state observation error vector.

**Proof** In order to prove the asymptotic stability of the observation error (4), we consider the following quadratic Lyapunov function candidate

$$V(e(t)) = e(t)^T P e(t). \quad (10)$$

The observation error converges asymptotically towards zero if  $V(e(t)) > 0$  and  $\dot{V}(e(t)) < 0$  for all  $e(t) \neq 0$ .

The time derivative of  $V(e(t))$  along any trajectory of (5) is given by

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t) P e(t) + e^T(t) P \dot{e}(t) \\ &= e^T(t) \left[ (A - LC)^T P + P(A - LC) \right] e(t) + 2e^T(t) P \left[ f(t, x, u) - f(t, \hat{x}, u) \right] \\ &\leq e^T(t) \left[ (A - LC)^T P + P(A - LC) + \delta^{-1} P^2 + \delta \gamma^2 I \right] e(t). \end{aligned} \quad (11)$$

The inequality (11) is obtained by using the following relation

$$\begin{aligned} 2e^T(t) P \left[ f(t, x, u) - f(t, \hat{x}, u) \right] &\leq \delta^{-1} e^T(t) P P e(t) \\ &\quad + \delta \left[ f(t, x, u) - f(t, \hat{x}, u) \right]^T \left[ f(t, x, u) - f(t, \hat{x}, u) \right] \\ &\leq e^T(t) \left[ \delta^{-1} P P + \delta \gamma^2 I \right] e(t). \end{aligned}$$

The inequality (11) can be written as

$$\begin{aligned} \dot{V}(e(t)) &\leq -e^T(t) Q e(t) - e^T(t) \left( Q_0 + C^T L^T R_0 LC \right) e(t) \\ &\leq -e^T(t) \left( Q_0 + C^T L^T R_0 LC \right) e(t) \\ &< 0, \end{aligned} \quad (12)$$

where  $(A - LC)^T P + P(A - LC) + \delta^{-1} P P + \delta \gamma^2 I + Q_0 + C^T L^T R_0 LC = -Q$ .

Hence,  $V(e(t))$  is a Lyapunov function for the system (5). Therefore, the observation error (4) is asymptotically stable. Furthermore, by integrating both sides of the inequality (12) from 0 to  $T$  and using the initial conditions, we have

$$V(e(T)) - V(e(0)) < -\int_0^T e^T(t) \left( Q_0 + C^T L^T R_0 L C \right) e(t) dt.$$

Since the system (4) is asymptotically stable, that is,  $e(T) \rightarrow 0$ , when  $T \rightarrow \infty$ , we obtain  $V(e(T)) \rightarrow 0$ . Thus we get

$$\begin{aligned} J &= \int_0^T e^T(t) \left( Q_0 + C^T L^T R_0 L C \right) e(t) dt \\ &< V(e(0)) \\ &< e_0^T P e_0. \end{aligned}$$

The proof of Theorem 2.1 is completed.  $\square$

At this stage, (6) and (9) form an optimization problem which, given an  $e_0$ , can be solved in order to obtain an optimal observation gain  $L$  for the nonlinear system. Unfortunately, this optimal gain  $L$  will in general depend on  $e_0$ . Thus, it would not really be a feedback control. In order to find an optimal observation gain that is independent of the initial observation error, it is necessary to overcome this problem. Then, we attempt to determine the optimal gain  $L$  in an average sense, if we view the initial observation error  $e_0$  as a random variable uniformly distributed over the surface of an  $n$  dimensional unit sphere, it follows that

$$E\{e_0 e_0^T\} = I. \tag{13}$$

Then, the expected value of the quadratic criterion  $\bar{J}$  of the cost function (9) is simply evaluated as follows

$$\bar{J} = E\{J\} \leq E\{e_0^T P e_0\} = \text{trace}\{P\}. \tag{14}$$

Thus, that may have appeared to be a dynamical problem (5) is formulated as a static quadratic criterion (14) which is minimized with respect to the observation gain matrix  $L$  and the symmetric positive definite matrix  $P$  subject to the constraint (8).

## 2.2 Gain matrix optimization

The optimal observation gain matrix of the state observation (3), which ensures the asymptotic convergence of the state observation error (4), is given by the following theorem

**Theorem 2.2** *We consider Theorem 2.1 and if there exists a matrix  $\Gamma = \Gamma^T \geq 0$  solution of the Lyapunov equation*

$$(A - LC)\Gamma + \Gamma(A - LC)^T + 2\delta^{-1}P + I = 0. \tag{15}$$

*Then the optimal observation gain matrix of the system (3) is given by*

$$L = R_0^{-1} P \Gamma C^T (C \Gamma C^T)^{-1}. \tag{16}$$



- Update  $L_{n+1}$ , for the obtained values  $P_n$  and  $\Gamma_n$ , with the relation  $F_1(L_{n+1}, P_n, \Gamma_n)$ .

3.  $n = n + 1$  :

Repeat the step 2 for  $n = n + 1$  to obtain the optimal values.

4. Terminate :

Stop the algorithm if  $\|P_n - P_{n-1}\| \leq \varepsilon$  ( $\varepsilon$  is a prescribed small number used to check the convergence of the algorithm).

So, for  $n = 1, 2, \dots$ , we have

$P_n$  is found from the Riccati equation  $F_3(L_n, P_n) = 0$ ,

$\Gamma_n$  is found from the Lyapunov equation  $F_2(L_n, P_n, \Gamma_n) = 0$ ,

$L_{n+1}$  is found from  $F_1(L_{n+1}, P_n, \Gamma_n)$ .

### 3 Numerical Example

To illustrate the availability and the efficiency of the proposed nonlinear optimal state observer design, we consider the system of a single link robot with a revolute elastic joint rotating in a vertical plane which is modelled by [8, 24]:

$$\begin{cases} \dot{\theta}_m = \omega_m, \\ \dot{\omega}_m = -\frac{F_m}{J_m}\omega_m + \frac{K}{J_m}(\theta_l - \theta_m) + \frac{K_\tau}{J_m}u, \\ \dot{\theta}_l = \omega_l, \\ \dot{\omega}_l = -\frac{F_l}{J_l}\omega_l - \frac{K}{J_l}(\theta_l - \theta_m) - \frac{Mgh}{J_l}\sin(\theta_l), \end{cases} \quad (22)$$

where  $\theta_m, \omega_m, \theta_l$  and  $\omega_l$  are the motor angular displacement, the angular velocity of the motor, the link angular displacement and the angular velocity of the link respectively.  $J_m$  and  $J_l$  are the inertia of the motor and link respectively,  $2h$  and  $M$  represent the length and mass of the link,  $F_m$  and  $F_l$  are the viscous friction coefficients,  $K$  is the elastic constant,  $g$  is the gravity constant and  $K_\tau$  is the amplifier gain. The control  $u$  is the torque delivered by the motor.

The system (22) can be rewritten under the form (1) in the following state representation

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{J_m} & -\frac{F_m}{J_m} & \frac{K}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{J_l} & 0 & -\frac{K}{J_l} & -\frac{F_l}{J_l} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{ghM}{J_l} \end{bmatrix} \sin(x_3), \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \end{cases} \quad (23)$$

with  $[x_1 \ x_2 \ x_3 \ x_4]^T = [\theta_m \ \omega_m \ \theta_l \ \omega_l]^T$ .

The performances of the proposed nonlinear optimal state observer with the optimal gain obtained by the proposed iterative algorithm was investigated by simulation for the flexible link robot (23) characterized by the following numerical parameters (table 1) [8]:

Parameter	Numerical value
$K$	$1.8Nm/rad$
$K_\tau$	$0.8Nm/V$
$J_m$	$37.9 \times 10^{-3}Kgm^2$
$J_l$	$94.6 \times 10^{-3}Kgm^2$
$h$	$0.15m$
$M$	$0.21Kg$
$F_m$	$47.3 \times 10^{-3}Nm/rad/s$
$F_l$	$0Nm/rad/s$

**Table 1:** Numerical parameters of the flexible link robot.

In the following, the procedure for the nonlinear optimal state observer design is presented. For the computation of the observation gain matrix  $L$ , we select the parameters  $Q_0 = 0.75 \cdot I_4$ ,  $Q = 0.75 \cdot I_4$ ,  $R_0 = I_4$  and  $\delta = 0.25$ .

Using the proposed iterative algorithm described above for the given  $Q_0, Q, R_0$  and  $\delta$  the observation gain matrix can be found using MATLAB. If the results are not satisfactory,  $Q_0$  and  $R_0$  are modified and the procedure is repeated. After some design repetition and with the selected parameters, the outcomes of the iterative algorithm resolution after  $N = 27$  iterations are the following:

- the optimal observation gain matrix:

$$L_{opt} = \begin{bmatrix} -11.6291 & 10.1573 \\ 9.8738 & -0.4123 \\ 39.1030 & -11.1101 \\ -5.3396 & 5.9703 \end{bmatrix},$$

- the symmetric positive definite matrix:

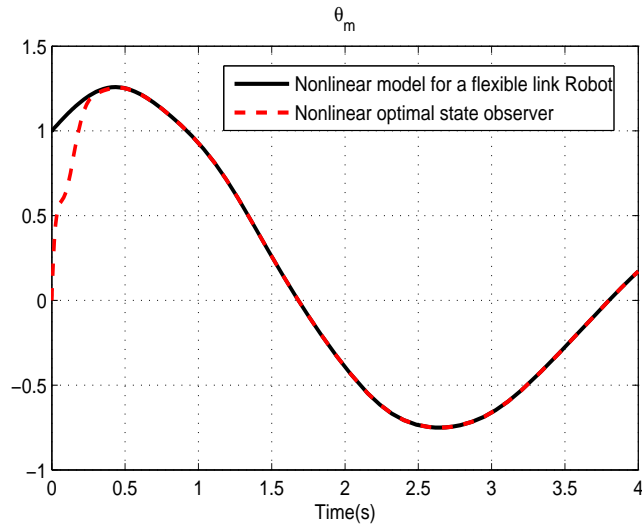
$$P_{opt} = \begin{bmatrix} 4.8652 & -0.2200 & -1.4939 & 0.9748 \\ -0.2200 & 0.9352 & -0.0126 & 0.1866 \\ -1.4939 & -0.0126 & 14.8564 & -4.2233 \\ 0.9748 & 0.1866 & -4.2233 & 2.7364 \end{bmatrix},$$

- the matrix of Lagrangian multiplier:

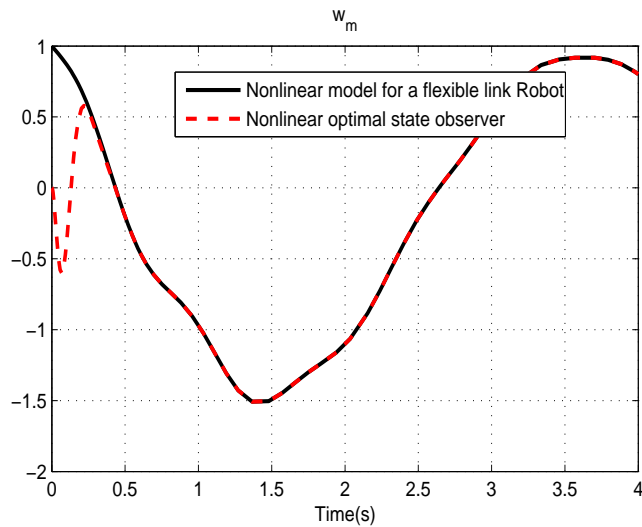
$$\Gamma_{opt} = \begin{bmatrix} 7.9319 & -4.2409 & 2.2174 & -2.1123 \\ -4.2409 & 6.2424 & -2.3009 & 2.3777 \\ 2.2174 & -2.3009 & 3.6516 & -1.8314 \\ -2.1123 & 2.3777 & -1.8314 & 1.6209 \end{bmatrix}.$$

The performances of the proposed nonlinear optimal state observer, tested by numerical simulation, are shown in Figures 1 to 4 which depict the evolution of the actual and the observed state variables of the studied flexible link robot: the motor angular position





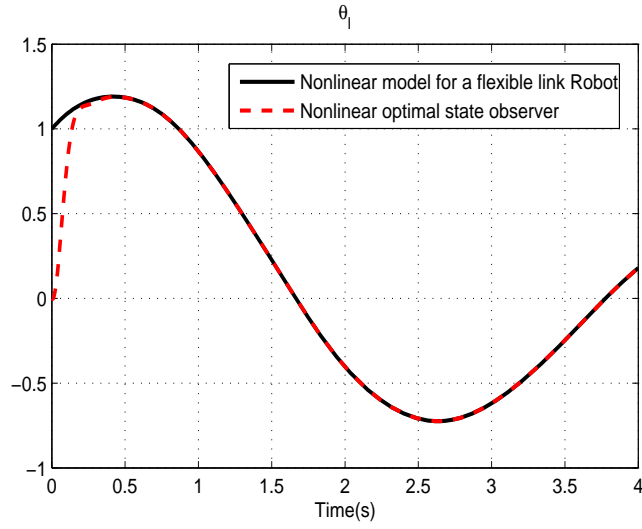
**Figure 1:** Actual and observed angular position  $\theta_m$  of the motor.



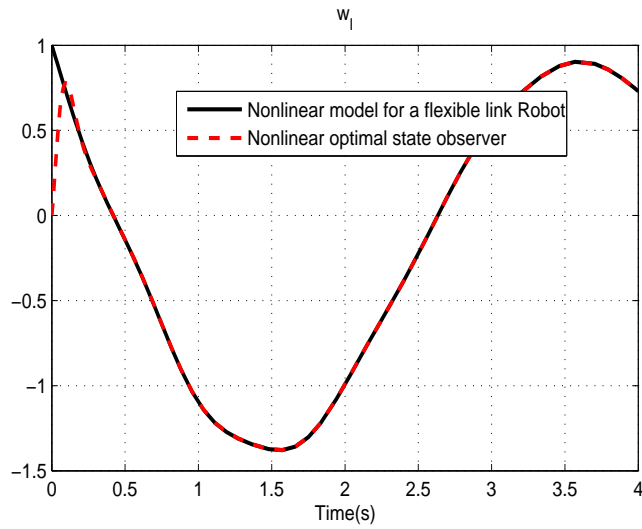
**Figure 2:** Actual and observed angular velocity  $\omega_m$  of the motor.

$\theta_m$ , the motor angular velocity  $\omega_m$ , the link angular position  $\theta_l$  and the link angular velocity  $\omega_l$ .

It appears, from these simulations, that the nonlinear optimal state observation approach allows a well reconstruction of the actual states. It can converge rapidly towards the state variable of the flexible link robot. Indeed, the high performances of the proposed nonlinear optimal state observer show the improvement led by the use of the proposed iterative algorithm permitting the calculus of the optimal gain matrix.



**Figure 3:** Actual and observed angular position  $\theta_l$  of the link.



**Figure 4:** Actual and observed angular velocity  $\omega_l$  of the link.

#### 4 Conclusion

Nonlinear optimal state observer design for a class of continuous-time nonlinear systems, where the nonlinearity satisfy the Lipschitz condition, has been studied in this paper. The nonlinear optimal state observer is based on the determination of an optimal observing gain matrix derived by minimizing a quadratic criterion characterized by a quadratic output feedback control problem of the observation error system.

It has been shown from the simulation results that the proposed nonlinear optimal state observer allows the reconstruction of the unmeasurable state variables of the flexible link robot. Indeed, the performance improvement of the nonlinear optimal state observer is due to the design of a numerical efficient algorithm leading to the calculus of the optimal gain matrix.

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## Existence and Uniqueness Conditions for a Class of $(k + 4j)$ -Point $n$ -th Order Boundary Value Problems

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**Abstract:** For the  $n$ th order nonlinear differential equation

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

we consider uniqueness implies uniqueness and existence results for solutions satisfying certain  $(k + 4j)$ -point boundary conditions,  $1 \leq j \leq n - 1$  and  $1 \leq k \leq n - 2j$ . We define  $(j; k; j)$ -point unique solvability in analogy to  $k$ -point disconjugacy and we show that  $(j; n - 2j; j)$ -point unique solvability implies  $(j; k; j)$ -point unique solvability for  $1 \leq k \leq n - 2j$ . This result is in analogy to  $n$ -point disconjugacy implies  $k$ -point disconjugacy,  $2 \leq k \leq n - 1$ .

**Keywords:** boundary value problem; uniqueness; existence; unique solvability; non-linear interpolation.

**Mathematics Subject Classification (2010):** 34B15, 34B10, 65D05.

### 1 Introduction

In this paper, we are concerned with uniqueness and existence of solutions for a class of boundary value problems for  $n$ th order ordinary differential equation,  $n \geq 3$ ,

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad a < x < b, \quad (1)$$

subject to  $n - 2j$  conjugate boundary conditions and  $2j$  nonlocal boundary conditions, where  $j \geq 1$ . In particular, given  $1 \leq k \leq n - 2j$ , positive integers  $m_1, \dots, m_k$  such that  $m_1 + \dots + m_k = n - 2j$ , points  $a < t_1 < \dots < t_{2j} < x_1 < x_2 < \dots < x_k < s_1 < \dots < s_{2j} < b$

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real values  $y_i, 1 \leq i \leq j, y_{il}, 1 \leq i \leq m_l, 1 \leq l \leq k$ , and real values  $y_{n-(i-1)}, 1 \leq i \leq j$ , we are concerned with uniqueness implies uniqueness and existence questions for solutions of (1) satisfying the conjugate and nonlocal boundary conditions of the type

$$\begin{aligned} a_i y(t_{2i-1}) - b_i y(t_{2i}) &= y_i, \quad 1 \leq i \leq j, \quad j \text{ nonlocal conditions,} \\ y^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \quad k\text{-point, } n-2 \text{ conjugate conditions,} \\ c_i y(s_{2i-1}) - d_i y(s_{2i}) &= y_{n-(i-1)}, \quad 1 \leq i \leq j, \quad j \text{ nonlocal conditions,} \end{aligned} \quad (2)$$

where  $a_i, b_i, c_i, d_i, 1 \leq i \leq j$  are positive real numbers. We shall refer to the boundary conditions, (2), as  $(j; k; j)$ -point boundary conditions. The  $(0; k; 0)$ -point boundary conditions are referred to as *conjugate* type boundary conditions [18].

Questions of the types with which we deal in this paper have been considered for solutions of (1) satisfying  $\alpha$ -point conjugate boundary conditions; in particular, for boundary value problems for (1) satisfying, for  $2 \leq \alpha \leq n$ , conjugate boundary conditions of the form,

$$y^{(i-1)}(t_l) = r_{il}, \quad 1 \leq i \leq p_l, \quad 1 \leq l \leq \alpha, \quad (3)$$

where  $p_1, \dots, p_\alpha$  are positive integers such that  $p_1 + \dots + p_\alpha = n, a < t_1 < \dots < t_\alpha < b$ , and  $r_{ij} \in \mathbf{R}, 1 \leq i \leq p_j, 1 \leq j \leq \alpha$ . These questions have involved: **(i)** whether uniqueness of solutions of (1), (3), for  $\alpha = n$ , implies uniqueness of solutions of (1), (3), for  $2 \leq \alpha \leq n-1$ , and **(ii)** whether uniqueness of solutions of (1), (3), for  $\alpha = n$ , implies existence of solutions of (1), (3), for  $2 \leq \alpha \leq n$ . Of course, a main reason for considering question (i) would be in resolving question (ii).

**Hypothesis 1.1** *With respect to equation (1), we assume throughout that*

(A)  $f(t, s_1, \dots, s_n) : (a, b) \times \mathbf{R}^n \rightarrow \mathbf{R}$  is continuous;

(B) Solutions of initial problems for (1) are unique and extend to  $(a, b)$ .

Given Hypothesis 1.1, Jackson [18] established that indeed (i) is true. In independent works, Hartman [7, 8] and Klaasen [21] provided a positive answer to question (ii).

Several other papers have been devoted to uniqueness questions of these types as well as uniqueness implies existence questions for boundary value problems. These works have dealt not only with ordinary differential equations [2, 4, 9, 10, 19, 22, 23], but also with boundary value problems for finite difference equations [11]–[13], and recently with dynamic equations on time scales [6, 17]. Some questions of these types have also received recent attention for nonlocal boundary value problems for (1), for the cases of  $n = 2, 3, 4$ ; see [1, 5, 15, 16]. Recently, [3, 20] the case of nonlocal conditions for equations of arbitrary order  $n$  have been addressed.

Referring to the methods employed in the papers cited above as shooting methods, the authors shoot from one boundary point with one boundary condition. The contribution in this article is that we shoot from two boundary points, to the left from  $x_1$  and to the right from  $x_k$ . New arguments for uniqueness of solutions implies existence of solutions are given to allow for multiple shooting.

## 2 Uniqueness of Solutions

In the first result of this section, we shall obtain continuous dependence of solutions of (1) on boundary conditions.

**Theorem 2.1** *Assume that for some  $1 \leq k \leq n-2j$ , and positive integers  $m_1, \dots, m_k$  such that  $m_1 + \dots + m_k = n - 2j$ , solutions of the corresponding boundary value problem (1), (2) are unique, when they exist. Given a solution  $y(x)$  of (1), an interval  $[c, d]$ , points  $c < x_1 < \dots < x_k < \dots < x_{k+4j} < d$  and an  $\epsilon > 0$ , there exists  $\delta(\epsilon, [c, d]) > 0$  such that, if  $|x_i - \xi_i| < \delta$ ,  $1 \leq i \leq k + 4j$ , and  $c < \xi_1 < \dots < \xi_k < \dots < \xi_{k+4j} < d$ , and if*

$$\begin{aligned} &|a_i y(x_{2i-1}) - b_i y(x_{2i}) - z_i| < \delta, \quad i = 1, 2, \dots, j, \\ &|y^{(i-1)}(x_{2j+l}) - z_{il}| < \delta, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \quad \text{and} \\ &|c_i y(x_{k+2j+2i-l}) - d_i y(x_{k+2j+2i}) - z_{n-(i-1)}| < \delta, \quad i = 1, 2, \dots, j, \end{aligned}$$

then there exists a solution  $z(x)$  of (1) satisfying

$$\begin{aligned} &a_i z(\xi_{2i-l}) - b_i z(\xi_{2i}) = z_i, \quad 1 \leq i \leq j, \\ &z^{(i-1)}(\xi_l) = z_{il}, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \\ &c_i z(\xi_{k+2j+2i-l}) - d_i z(\xi_{k+2j+2i}) = z_{n-(i-1)}, \quad 1 \leq i \leq j, \end{aligned}$$

and  $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$  on  $[c, d]$ ,  $1 \leq i \leq n$ .

**Proof** Fix a point  $p_0 \in (c, d)$  and define the set

$$G = \{(s_1, \dots, s_{k+4j}, c_1, \dots, c_n) \mid c < s_1 < \dots < s_{k+4j} < d, \quad c_1, \dots, c_n \in \mathbf{R}\}.$$

$G$  is an open subset of  $\mathbf{R}^{k+4j+n}$ . Let  $u(x)$  be a solution of the initial value problem for (1) satisfying the initial conditions  $u^{(i-1)}(p_0) = c_i$ ,  $1 \leq i \leq n$ . Define a mapping  $\phi : G \rightarrow \mathbf{R}^{k+4j+n}$  by

$$\begin{aligned} \phi(s_1, \dots, s_{k+4j}, c_1, \dots, c_n) &= (s_1, \dots, s_{k+4j}, a_1 u(s_1) - b_1 u(s_2), \dots, a_j u(s_{2j-l}) - b_j u(s_{2j}), \\ &u(s_{2j+1}), \dots, u^{(m_1-1)}(s_{2j+1}), \dots, u(s_{2j+k}), \dots, u^{(m_k-1)}(s_{2j+k}), \\ &c_1 u(s_{k+2j+l}) - d_1 u(s_{k+2j+2}), \dots, c_j u(s_{k+4j-l}) - d_j u(s_{k+4j})). \end{aligned}$$

The continuity of  $\phi$  follows from Condition (B) in Hypothesis 1.1. Moreover, the uniqueness assumption on solutions of (1), (2), for the given  $k$  and  $m_1, \dots, m_k$ , implies that  $\phi$  is one-one. Hence, from the Brouwer theorem on invariance of domain [25], it follows that  $\phi(G)$  is an open subset of  $\mathbf{R}^{k+4j+n}$ , and that  $\phi$  is a homeomorphism from  $G$  to  $\phi(G)$ . The conclusion of the theorem follows directly from the continuity of  $\phi^{-1}$  and the fact that  $\phi(G)$  is open.

We now establish that for  $k = n - 2j$ , uniqueness of solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2), implies uniqueness of solutions of the  $(j - i; n - 2j + i, j)$ -point BVP (1), (2), for  $i = 1, 2, \dots, j$ .

**Theorem 2.2** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, for each  $i = 1, 2, \dots, j$ , solutions of the  $(j - i; n - 2j + i, j)$ -point BVP (1), (2) are unique, when they exist.*

**Proof** Assume uniqueness of solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2). Firstly, we show that solutions of the  $(j - 1; n - 2j + 1, j)$ -point BVP (1), (2) are unique. Assume the conclusion is not true and there exist points  $a < t_1 < \dots < t_{2j-2} < x_1 < \dots <$

$x_{n-2j+1} < s_1 < \dots < s_{2j} < b$  for which there exist distinct solutions  $y(x)$  and  $z(x)$  of the  $(j-1; n-2j+1, j)$ -point BVP such that

$$\begin{aligned} a_i y(t_{2i-1}) - b_i y(t_{2i}) &= a_i z(t_{2i-1}) - b_i z(t_{2i}), \quad i = 1, 2, \dots, j-1, \\ y(x_1) &= z(x_1), \\ y(x_l) &= z(x_l), \quad 2 \leq l \leq n-2j+1, \\ c_i y(s_{2i-1}) - d_i y(s_{2i}) &= c_i z(s_{2i-1}) - d_i z(s_{2i}), \quad i = 1, 2, \dots, j. \end{aligned}$$

Defining  $w = y - z$ , we obtain

$$\begin{aligned} a_i w(t_{2i-1}) - b_i w(t_{2i}) &= 0, \quad i = 1, 2, \dots, j-1, \\ w(x_1) &= 0, \\ w(x_l) &= 0, \quad 2 \leq l \leq n-2j+1, \\ c_i w(s_{2i-1}) - d_i w(s_{2i}) &= 0, \quad i = 1, 2, \dots, j. \end{aligned}$$

If there exists some  $p_1 \in (t_{2j-2}, x_1)$  such that  $w(p_1) = 0$ , then we have

$$a_j w(p_1) - b_j w(x_1) = 0, \quad a_j, b_j \in \mathbf{R}.$$

This implies that  $y(x)$  and  $z(x)$  are distinct solutions of the  $(j; n-2j; j)$ -point BVP at the points  $t_1, \dots, t_{2j-2}, p_1, x_1, \dots, x_{n-2j}, s_1, \dots, s_{2j}$ , which is a contradiction. Hence,  $w(t) \neq 0$  on  $(t_{2j-2}, x_1)$ . Let  $w(t) > 0$  on  $(t_{2j-2}, x_1)$ . The case  $w(t) < 0$  on  $(t_{2j-2}, x_1)$  is dealt with similarly. Then,

$$\max\{w(t) : t \in [t_{2j-2}, x_1]\} = w(\tau_1) > 0.$$

Define

$$v(t) = \begin{cases} a_j w(t) - b_j w(\tau_1), & \text{if } a_j \geq b_j, \\ b_j w(t) - a_j w(\tau_1), & \text{if } a_j \leq b_j. \end{cases}$$

Then,  $v(\tau_1) > 0$  and  $v(x_1) < 0$ . By the mean value theorem, there exists  $p' \in (\tau_1, x_1)$  such that  $v(p') = 0$  which implies that  $a_j w(p') - b_j w(\tau_1) = 0$ . Hence, there are distinct solutions of the  $(j; n-2j; j)$ -point BVP at the points

$$t_1, \dots, t_{2j-2}, \tau_1, p'_1, x_2, \dots, x_{n-2j}, s_1, \dots, s_{2j},$$

which is again a contradiction. Hence, solutions of the  $(j-1; n-2j+1, j)$ -point BVP (1), (2) are unique.

Now, using the uniqueness of solutions of the  $(j-1; n-2j+1, j)$ -point BVP, by the same process, we can show uniqueness of solutions of the  $(j-2; n-2j+2, j)$ -point BVP (1), (2). Continuing in the same fashion, we obtain uniqueness of solution of the  $(j-i; n-2j+i, j)$ -point BVP for each  $i = 1, 2, \dots, j$ .

**Corollary 2.1** *Let  $j \geq 1$ . Assume that for  $k = n-2j$ , solutions of the  $(j; n-2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, solutions of the  $(0; n-j; j)$ -point BVP (1), (2) are unique, when they exist.*

**Theorem 2.3** *Let  $j \geq 1$ . Assume that for  $k = n-2j$ , solutions of the  $(j; n-2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, for each  $i = 1, 2, \dots, j$ , solutions of the  $(j; n-2j+i, j-i)$ -point BVP (1), (2) are unique, when they exist.*



**Proof** Assume uniqueness of solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2). Firstly, we show that solutions of the  $(j; n - 2j + 1, j - 1)$ -point BVP (1), (2) are unique. Assume the conclusion is not true and there exist points

$$a < t_1 < \dots < t_{2j} < x_1 < \dots < x_{n-2j+1} < s_1 < \dots < s_{2j-2} < b$$

for which there exist distinct solutions  $y(x)$  and  $z(x)$  the  $(j; n - 2j + 1, j - 1)$ -point BVP such that

$$\begin{aligned} a_i y(t_{2i-1}) - b_i y(t_{2i}) &= a_i z(t_{2i-1}) - b_i z(t_{2i}), \quad i = 1, 2, \dots, j, \\ y(x_l) &= z(x_l), \quad 1 \leq l \leq n - 2j, \\ y(x_{n-2j+1}) &= z(x_{n-2j+1}), \\ c_i y(s_{2i-1}) - d_i y(s_{2i}) &= c_i z(s_{2i-1}) - d_i z(s_{2i}), \quad i = 1, 2, \dots, j - 1. \end{aligned}$$

Defining  $w = y - z$ , then we obtain

$$\begin{aligned} a_i w(t_{2i-1}) - b_i w(t_{2i}) &= 0, \quad i = 1, 2, \dots, j, \\ w(x_l) &= 0, \quad 1 \leq l \leq n - 2j, \\ w(x_{n-2j+1}) &= 0, \\ c_i w(s_{2i-1}) - d_i w(s_{2i}) &= 0, \quad i = 1, 2, \dots, j - 1. \end{aligned}$$

If there exists some  $q_1 \in (x_{n-2j+1}, s_1)$  such that  $w(q_1) = 0$ , then we have

$$c_0 w(x_{n-2j+1}) - d_0 w(q_1) = 0, \quad c_0, d_0 \in \mathbf{R}.$$

This implies that  $y(x)$  and  $z(x)$  are distinct solutions of the  $(j; n - 2j; j)$ -point BVP at the points

$$t_1, \dots, t_{2j}, x_1, \dots, x_{n-2j}, x_{n-2j+1}, q_1, s_1, \dots, s_{2j-2},$$

which is a contradiction. Hence,  $w(t) \neq 0$  on  $(x_{n-2j+1}, s_1)$ . Let  $w(t) > 0$  on  $(x_{n-2j+1}, s_1)$ . The case  $w(t) < 0$  on  $(x_{n-2j+1}, s_1)$  can be dealt with similarly. Then,

$$\max\{w(t) : t \in [x_{n-2j+1}, s_1]\} = w(\tau) > 0.$$

Define

$$v(t) = \begin{cases} c_0 w(t) - d_0 w(\tau), & \text{if } c_0 \geq d_0, \\ d_0 w(t) - c_0 w(\tau), & \text{if } c_0 \leq d_0. \end{cases}$$

Then,  $v(\tau) > 0$  and  $v(x_{n-2j+1}) < 0$ . By the mean value theorem, there exists  $q' \in (x_{n-2j+1}, \tau)$  such that  $v(q') = 0$  which implies that  $c_0 w(q') - d_0 w(\tau) = 0$ . Hence, there are distinct solutions of the  $(j; n - 2j; j)$ -point BVP at the points

$$t_1, \dots, t_{2j}, x_1, \dots, x_{n-2j}, q', \tau, s_1, \dots, s_{2j-2},$$

which is again a contradiction. Hence, solutions of the  $(j; n - 2j + 1, j - 1)$ -point BVP (1), (2) are unique.

Now, using the uniqueness of solutions of the  $(j; n - 2j + 1, j - 1)$ -point BVP, by the same process, we can show uniqueness of solutions of the  $(j; n - 2j + 2, j - 2)$ -point BVP (1), (2). Continuing in the same fashion, we obtain uniqueness of solution of the  $(j; n - 2j + i, j - i)$ -point BVP for each  $i = 1, 2, \dots, j$ .

**Corollary 2.2** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, solutions of the  $(j; n - j; 0)$ -point BVP (1), (2) are unique, when they exist.*

**Corollary 2.3** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, solutions of the  $n$ -point conjugate BVP (1), (3) (that is, the  $(0; n; 0)$ -point BVP), are unique, when they exist.*

In view of the uniqueness implies existence results due to Hartman [7, 8] and Klassen [21] as discussed in regard to question (ii), we have an immediate corollary concerning existence of solutions for  $k$ -point conjugate boundary value problems for (1).

**Corollary 2.4** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, solutions of the  $l$ -point conjugate BVP (1), (3) (that is, the  $(0; l; 0)$ -point BVP), for  $2 \leq l \leq n$ , are unique, when they exist.*

We now establish that uniqueness of solutions of (1), (2), when  $k = n - 2j$ , implies uniqueness of solutions of (1), (2), when  $1 \leq k \leq n - 2j - 1$ .

**Theorem 2.4** *Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, for each  $1 \leq k \leq n - 2j - 1$ , solutions of the  $(j; k; j)$ -point BVP (1), (2) are unique, when they exist.*

**Proof** Assume that solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique. Assume that, for some  $1 \leq k \leq n - 2j - 1$ , some  $(j; k; j)$ -point BVP (1), (2) has distinct solutions. Let

$$h = \max\{k = 1, \dots, n - 2j - 1 \mid (j; k; j) \text{-point BVP has distinct solutions}\}.$$

Then, there are positive integers,  $m_1, \dots, m_h$ , such that  $m_1 + \dots + m_h = n - 2j$ , and points  $a < t_1 < \dots < t_{2j} < x_1 < \dots < x_h < s_1 < \dots < s_{2j} < b$ , for which there exist distinct solutions  $y(x)$  and  $z(x)$  of the  $(j; h; j)$ -point boundary value problem (1), (2), for these  $m_1, \dots, m_h$ ; that is,

$$\begin{aligned} a_i y(t_{2i-1}) - b_i y(t_{2i}) &= a_i z(t_{2i-1}) - b_i z(t_{2i}), \quad i = 1, 2, \dots, j, \\ y^{(i-1)}(x_l) &= z^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq h, \\ c_i y(s_{2i-1}) - d_i y(s_{2i}) &= c_i z(s_{2i-1}) - d_i z(s_{2i}), \quad i = 1, 2, \dots, j. \end{aligned}$$

Since  $h \leq n - 2j - 1$ , so some  $m_l \geq 2$ . Let

$$m_{l_0} = \max\{m_l \mid 1 \leq l \leq h\};$$

then  $m_{l_0} \geq 2$ . Since,  $x_{l_0}$  is a zero of  $y - z$  of exact multiplicity  $m_{l_0}$ ,  $1 \leq l \leq h$  and  $y$  and  $z$  are distinct solutions of (1), we may assume, with no loss of generality, that

$$y^{(m_{l_0})}(x_{l_0}) > z^{(m_{l_0})}(x_{l_0}).$$

Now fix  $a < \tau < x_1$ . By the maximality of  $h$ , solutions of the  $(j; h + 1; j)$ -problems (1), (2) at the points  $t_1, \dots, t_{2j}, \tau, x_1, \dots, x_h, s_1, \dots, s_{2j}$  are unique. Hence, it follows from Theorem 2.1 that, for each  $\epsilon > 0$ , there is a  $\delta > 0$  and there is a solution  $z_\delta(x)$  of the  $(j; h + 1; j)$ -point problem (1), (2), (corresponding to  $k = h + 1$ ), satisfying at the points  $t_1, \dots, t_{2j}, \tau, x_1, \dots, x_h, s_1, \dots, s_{2j}$ ,

$$\begin{aligned} a_i z_\delta(t_{2i-1}) - b_i z_\delta(t_{2i}) &= a_i z(t_{2i-1}) - b_i z(t_{2i}) = a_i y(t_{2i-1}) - b_i y(t_{2i}), \quad i = 1, 2, \dots, j, \\ z_\delta(\tau) &= z(\tau), \\ z_\delta^{(i-1)}(x_l) &= z^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq h, \quad l \neq l_0, \\ z_\delta^{(i-1)}(x_{l_0}) &= z^{(i-1)}(x_{l_0}) = y^{(i-1)}(x_{l_0}), \quad 1 \leq i \leq m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2), \\ z_\delta^{(m_{l_0}-2)}(x_{l_0}) &= z^{(m_{l_0}-2)}(x_{l_0}) + \delta = y^{(m_{l_0}-2)}(x_{l_0}) + \delta, \\ c_i z_\delta(s_{2i-1}) - d_i z_\delta(s_{2i}) &= c_i z(s_{2i-1}) - d_i z(s_{2i}) = c_i y(s_{2i-1}) - d_i y(s_{2i}), \quad i = 1, 2, \dots, j, \end{aligned}$$

and  $|z_\delta(x) - z(x)| < \epsilon$  on  $[t_1, s_{2j}]$ . For  $\epsilon > 0$ , sufficiently small, there exist points  $x_{l_0-1} < \rho_1 < x_{l_0} < \rho_2 < x_{l_0+1}$  such that

$$\begin{aligned} a_i z_\delta(t_{2i-1}) - b_i z_\delta(t_{2i}) &= a_j y(t_{2i-1}) - b_i y(t_{2i}), \quad i = 1, 2, \dots, j, \\ z_\delta^{(i-1)}(x_l) &= y^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq l_0 - 1, \\ z_\delta(\rho_1) &= y(\rho_1), \\ z_\delta^{(i-1)}(x_{l_0}) &= y^{(i-1)}(x_{l_0}), \quad 1 \leq i \leq m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2), \\ z_\delta(\rho_2) &= y(\rho_2), \\ z_\delta^{(i-1)}(x_l) &= y^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad l_0 + 1 \leq l \leq h, \\ c_i z_\delta(s_{2i-1}) - d_i z_\delta(s_{2i}) &= c_i y(s_{2i-1}) - d_i y(s_{2i}), \quad i = 1, 2, \dots, j. \end{aligned}$$

Thus,  $z_\delta(x)$  and  $y(x)$  are distinct solutions of the  $(j; h + 1; j)$ -point boundary value problem at the points  $t_1, \dots, t_{2j}, x_1, \dots, x_{l_0-1}, \rho_1, \rho_2, x_{l_0+1}, \dots, x_h, s_1, \dots, s_{2j}$ , which is a contradiction because of the maximality of  $h$ . The proof is complete.

In view of Theorem 2.2 and Theorem 2.4, we have the following corollaries.

**Corollary 2.5** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, for  $1 \leq k \leq n - 2j$  and  $1 \leq i \leq j$ , solutions of the  $(j; k + i; j - i)$ -point BVP are unique, when they exist.*

**Corollary 2.6** *Let  $j \geq 1$ . Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2) are unique, when they exist. Then, for  $1 \leq k \leq n - 2j$  and  $1 \leq i \leq j$ , solutions of the  $(j - i; k + i; j)$ -point BVP are unique, when they exist.*

### 3 Existence of Solutions

Now we deal with uniqueness implies existence for these problems. For such existence results, continuous dependence as in Theorem 2.1 plays a role. In addition, we shall make use of a Schrader [24] precompactness result on bounded sequences of solutions of (1) which is stated as follows:

**Theorem 3.1** *Assume the uniqueness of solutions for (1), (3), when  $\ell = n$ . If  $\{y_\nu(x)\}$  is a sequence of solutions of (1) which is uniformly bounded on a nondegenerate compact subinterval  $[c, d] \subset (a, b)$ , then there is a subsequence  $\{y_{\nu_i}(x)\}$  such that  $\{y_{\nu_i}^{(i)}(x)\}$  converges uniformly on each compact subinterval of  $(a, b)$ , for each  $i = 0, \dots, n - 1$ .*

We have as a corollary a precompactness condition in terms of (1), (2), when  $k = n - 2j$ .

**Corollary 3.1** *Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP (1), (2), are unique. If  $\{y_\nu(x)\}$  is a sequence of solutions of (1) which is uniformly bounded on a nondegenerate compact subinterval  $[c, d] \subset (a, b)$ , then there is a subsequence  $\{y_{\nu_i}(x)\}$  such that  $\{y_{\nu_i}^{(i)}(x)\}$  converges uniformly on each compact subinterval of  $(a, b)$ , for each  $i = 0, \dots, n - 1$ .*

We now present our uniqueness implies existence result for the  $(j; k; j)$ -point boundary value problems.

**Theorem 3.2** *Let  $j \geq 1$ . Assume that solutions of (1), (2), when  $k = n - 2j$ , are unique. Then, for each  $1 \leq k \leq n - 2j$ , positive integers  $m_1, \dots, m_k$  such that  $m_1 + \dots + m_k = n - 2j$ , points  $a < t_1 < \dots < t_{2j} < x_1 < \dots < x_k < s_1 < \dots < s_{2j} < b$ , real values  $y_i, 1 \leq i \leq j, y_{il}, 1 \leq i \leq m_l, 1 \leq l \leq k$  and  $y_{n-i}, 0 \leq i \leq j - 1$ , there exists a unique solution of (1), (2).*

**Proof** Let  $1 \leq k \leq n - 2j$ , positive integers  $m_1, \dots, m_k$  such that  $m_1 + \dots + m_k = n - 2j$ , points  $a < t_1 < \dots < t_{2j} < x_1 < \dots < x_k < s_1 < \dots < s_{2j} < b$ , real values  $y_i, 1 \leq i \leq j, y_{il}, 1 \leq i \leq m_l, 1 \leq l \leq k$  and  $y_{n-i}, 0 \leq i \leq j - 1$ , be given.

Assume that for  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP, (1), (2), are unique. For  $1 \leq k \leq n - 2j$ , in view of Corollary 2.4, solutions of the  $(0; l; 0)$ -point BVP ( $l$ -point conjugate BVP) for  $2 \leq l \leq n$ , are also unique. Let  $z(x)$  be the unique solution of (1) satisfying the  $(k + 2j + 2)$ -point conjugate boundary conditions (3) at the points  $t_1, p_1, t_2, \dots, t_j, x_1, \dots, x_k, s_1, \dots, s_{j+1}$  if  $m_1 > 1, m_k > 1$  (or alternatively, if  $m_1 = 1, m_k = 1$ ,  $z(x)$  satisfies the  $(k + 2j)$ -point conjugate boundary conditions and if one of  $m_1 = 1, m_k = 1$  hold, then  $z(x)$  satisfies the  $(k + 2j + 1)$ -point conjugate boundary conditions), that is,

$$\begin{aligned} z(t_1) &= \frac{y_1}{a_1}, z(p_1) = 0, \\ z(t_i) &= \frac{y_i}{a_i}, 2 \leq i \leq j, \\ z^{(i-1)}(x_1) &= y_{i1}, 1 \leq i \leq m_1 - 1, \\ z^{(i-1)}(x_l) &= y_{il}, 1 \leq i \leq m_l, 2 \leq l \leq k - 1, \\ z^{(i-1)}(x_k) &= y_{ik}, 1 \leq i \leq m_k - 1, \\ z(s_i) &= \frac{y_{n-(i-1)}}{c_i}, 1 \leq i \leq j - 1, \\ z(s_j) &= \frac{y_{n-(j-1)}}{c_j}, z(s_{j+1}) = 0. \end{aligned}$$

From the first and the last lines, we obtain

$$a_1 z(t_1) - b_1 z(p_1) = y_1, c_j z(s_j) - d_j z(s_{j+1}) = y_{n-(j-1)}.$$

Now, define the set

$$\begin{aligned} S = \{ & (u^{(m_1-1)}(x_1), u^{(m_k-1)}(x_k)) \mid u \text{ is a solution of (1) satisfying} \\ & a_1 u(t_1) - b_1 u(p_1) = y_1, u(t_i) = \frac{y_i}{a_i}, 2 \leq i \leq j, \\ & u^{(i-1)}(x_1) = y_{i1}, 1 \leq i \leq m_1 - 1, \\ & u^{(i-1)}(x_l) = y_{il}, 1 \leq i \leq m_l, 2 \leq l \leq k - 1, \\ & u^{(i-1)}(x_k) = y_{ik}, 1 \leq i \leq m_k - 1, \\ & u(s_i) = \frac{y_{n-(i-1)}}{c_i}, 1 \leq i \leq j - 1, c_j u(s_j) - d_j u(s_{j+1}) = y_{n-(j-1)} \}. \end{aligned}$$

Clearly,  $(z^{(m_1-1)}(x_1), z^{(m_k-1)}(x_k)) \in S$ , and so  $S$  is a nonempty subset of  $\mathbf{R}^2$ .

Next, choose  $(\rho_0, \sigma_0) \in S$ . Then, there is a solution  $u_0(x)$  of (1) satisfying

$$\begin{aligned} a_1 u_0(t_1) - b_1 u_0(p_1) &= y_1, u_0(t_i) = \frac{y_i}{a_i}, 2 \leq i \leq j, \\ u_0^{(i-1)}(x_1) &= y_{i1}, 1 \leq i \leq m_1 - 1, \\ u_0^{(m_1-1)}(x_1) &= \rho_0, \\ u_0^{(i-1)}(x_l) &= y_{il}, 1 \leq i \leq m_l, 2 \leq l \leq k - 1, \\ u_0^{(i-1)}(x_k) &= y_{ik}, 1 \leq i \leq m_k - 1, \\ u_0^{(m_k-1)}(x_k) &= \sigma_0, \\ u_0(s_i) &= \frac{y_{n-(i-1)}}{c_i}, 1 \leq i \leq j - 1, c_j u_0(s_j) - d_j u_0(s_{j+1}) = y_{n-(j-1)}. \end{aligned}$$

By the uniqueness of solutions of the  $(1; k + 2j - 2; 1)$ -point BVP by Corollary 2.6, and in view of Theorem 2.1, there exists a  $\delta > 0$  such that, for each  $|\rho - \rho_0| < \delta, |\sigma - \sigma_0| < \delta$ , there is a solution  $u_{\rho\sigma}(x)$  of (1) satisfying

$$\begin{aligned} a_1 u_{\rho\sigma}(t_1) - b_1 u_{\rho\sigma}(p_1) &= y_1, u_{\rho\sigma}(t_i) = \frac{y_i}{a_i}, 2 \leq i \leq j, \\ u_{\rho\sigma}^{(i-1)}(x_1) &= y_{i1}, 1 \leq i \leq m_1 - 1, \\ u_{\rho\sigma}^{(m_1-1)}(x_1) &= \rho, \\ u_{\rho\sigma}^{(i-1)}(x_l) &= y_{il}, 1 \leq i \leq m_l, 2 \leq l \leq k - 1, \\ u_{\rho\sigma}^{(i-1)}(x_k) &= y_{ik}, 1 \leq i \leq m_k - 1, \\ u_{\rho\sigma}^{(m_k-1)}(x_k) &= \sigma, \\ u_{\rho\sigma}(s_i) &= \frac{y_{n-(i-1)}}{c_i}, 1 \leq i \leq j - 1, c_j u_{\rho\sigma}(s_j) - d_j u_{\rho\sigma}(s_{j+1}) = y_{n-(j-1)} \end{aligned}$$

and  $|u_{\rho\sigma} - u_0| < \delta$  on  $[t_1, s_{j+1}]$ , which implies that  $(u_{\rho\sigma}^{(m_1-1)}(x_1), u_{\rho\sigma}^{(m_k-1)}(x_k)) \in S$ , that is,  $(\rho, \sigma) \in S$ . Hence,  $\{(\rho, \sigma) : |\rho - \rho_0| < \delta, |\sigma - \sigma_0| < \delta\} \subset S$ . Thus,  $S$  is an open, nonempty subset of  $\mathbb{R}^2$ .

Now, we show that  $S$  is also a closed subset of  $\mathbb{R}^2$ . To do this, assume that  $S$  is not closed and assume there exists  $r_0 = (p_0, q_0) \in \overline{S} \setminus S$ . Let  $\{r_n\} = \{(p_n, q_n)\} \subset S$  such that

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} (p_n, q_n) = (p_0, q_0) = r_0.$$

We can assume that each sequence  $\{p_n\}, \{q_n\}$  is monotone. For the sake of this argument, we shall assume that each of  $\{p_n\}$  and  $\{q_n\}$  is monotone nondecreasing; the arguments for the other three cases,  $\{p_n\}$  nondecreasing and  $\{q_n\}$  nonincreasing,  $\{p_n\}$  nonincreasing and  $\{q_n\}$  nondecreasing, and each of  $\{p_n\}, \{q_n\}$  nonincreasing are analogous.

So assume  $p_n < p_{n+1} \leq p_0, q_n < q_{n+1} \leq q_0$  and assume one of the inequalities,  $p_{n+1} \leq p_0, q_{n+1} \leq q_0$ , is strict. By the definition of  $S$ , for each term  $r_n, n \in \mathbb{N}$ , there exists a unique solution  $u_n(x)$  of (1) satisfying

$$\begin{aligned} a_1 u_n(t_1) - b_1 u_n(p_1) &= y_1, u_n(t_i) = \frac{y_i}{a_i}, 2 \leq i \leq j, \\ u_n^{(i-1)}(x_1) &= y_{i1}, 1 \leq i \leq m_1 - 1, \\ u_n^{(m_1-1)}(x_1) &= p_n, \\ u_n^{(i-1)}(x_l) &= y_{il}, 1 \leq i \leq m_l, 2 \leq l \leq k - 1, \\ u_n^{(i-1)}(x_k) &= y_{ik}, 1 \leq i \leq m_k - 1, \\ u_n^{(m_k-1)}(x_k) &= q_n, \\ u_n(s_i) &= \frac{y_{n-(i-1)}}{c_i}, 1 \leq i \leq j - 1, c_j u_n(s_j) - d_j u_n(s_{j+1}) = y_{n-(j-1)}. \end{aligned}$$

Set  $w_n = u_n - u_{n+1}$ . Then

$$\begin{aligned} a_1 w_n(t_1) - b_1 w_n(p_1) &= 0, \quad w_n(t_i) = 0, \quad 2 \leq i \leq j, \\ w_n^{(i-1)}(x_1) &= 0, \quad 1 \leq i \leq m_1 - 1, \\ w_n^{(m_1-1)}(x_1) &= p_n - p_{n+1} \leq 0, \\ w_n^{(i-1)}(x_l) &= 0, \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k - 1, \\ w_n^{(i-1)}(x_k) &= 0, \quad 1 \leq i \leq m_k - 1, \\ w_n^{(m_k-1)}(x_k) &= q_n - q_{n+1} \leq 0, \\ w_n(s_i) &= 0, \quad 1 \leq i \leq j - 1, \quad c_j w_n(s_j) - d_j w_n(s_{j+1}) = 0. \end{aligned}$$

First assume  $p_{n+1} < p_0$  and  $q_{n+1} < q_0$ . By the uniqueness of solutions of the  $(1; k + 2j - 2; 1)$ -point BVP, there exists  $\epsilon_n > 0$  such that

- (a)  $u_n(x) < u_{n+1}(x)$  on  $(x_1 - \epsilon_n, x_1) \cup (x_1, x_2)$ , if  $m_1$  is odd,
- (b)  $u_n(x) > u_{n+1}(x)$  on  $(x_1 - \epsilon_n, x_1)$  and  $u_n(x) < u_{n+1}(x)$  on  $(x_1, x_2)$ , if  $m_1$  is even,
- (c)  $u_n(x) < u_{n+1}(x)$  on  $(x_{k-1}, x_k) \cup (x_k, x_k + \epsilon_n)$ , if  $m_k$  is odd,
- (d)  $u_n(x) > u_{n+1}(x)$  on  $(x_{k-1}, x_k)$  and  $u_n(x) < u_{n+1}(x)$  on  $(x_k, x_k + \epsilon_n)$ , if  $m_k$  is even.

For the sake of this argument, we shall assume that  $m_1$  and  $m_k$  are odd; the other cases are argued analogously. We also note that either  $u_n(x) < u_{n+1}(x)$  on  $(t_j, x_1)$  or  $u_n(x) < u_{n+1}(x)$  on  $(x_k, s_1)$ . If neither of these inequalities hold, then there exist  $t_j < \hat{t} < x_1$  and  $x_k < \hat{s} < s_1$  such that  $u_n(\hat{t}) - u_{n+1}(\hat{t}) = 0 = u_n(\hat{s}) - u_{n+1}(\hat{s})$  violating the uniqueness of solutions of  $(1; k + 2j; 1)$ -point BVPs. For the sake of this argument, let us assume that  $u_n(x) < u_{n+1}(x)$  on  $(t_j, x_1)$ . The sequence  $\{r_n\}$  converges to  $r_0$  and  $r_0 \notin S$ . In view of Corollary 3.1, the sequence  $\{u_n(x)\}$  is not uniformly bounded on any compact subset of each of  $(t_j, x_1)$ ,  $(x_1, x_2)$ , and  $(x_{k-1}, x_k)$ .

Now, let  $w(x)$  be the unique solution of the  $(0; k + 2j; 0)$ -point conjugate BVP (1),(3) satisfying at the points  $t_1, p_1, t_2, \dots, t_j, x_1, \dots, x_k, s_1, \dots, s_j$ ,

$$\begin{aligned} w(t_1) &= \frac{y_1}{a_1}, \quad w(p_1) = 0, \\ w(t_i) &= \frac{y_i}{a_i}, \quad 2 \leq i \leq j, \\ w^{(i-1)}(x_1) &= y_{i1}, \quad 1 \leq i \leq m_1 - 1, \quad (\text{if } m_1 > 1), \\ w^{(m_1-1)}(x_1) &= p_0, \\ w^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k - 1, \\ w^{(i-1)}(x_k) &= y_{ik}, \quad 1 \leq i \leq m_k - 1, \quad (\text{if } m_k > 1) \\ w_n^{(m_k-1)}(x_k) &= q_0, \\ w(s_i) &= \frac{y_{n-(i-1)}}{c_i}, \quad 1 \leq i \leq j - 1. \end{aligned}$$

From the monotonicity and unboundedness property of the sequence  $\{u_n(x)\}$ , it follows that, for some large  $n_0$ , there exist a solution  $u_{n_0}$  of (1) and points  $t_j < \tau_1 < x_1 < \tau_2 < x_2$ ,  $x_{k-1} < \rho_1 < x_k$  such that

$$u_{n_0}(\tau_1) = w(\tau_1), \quad u_{n_0}(\tau_2) = w(\tau_2), \quad u_{n_0}(\rho_1) = w(\rho_1).$$

In particular,

$$\begin{aligned} a_1 u_{n_0}(t_1) - b_1 u_{n_0}(p_1) &= y_1 = aw(t_1) - b_1 w(p_1), \\ u_{n_0}(t_i) &= \frac{y_i}{a_i} = w(t_i), \quad 2 \leq i \leq j, \\ u_{n_0}(\tau_1) &= w(\tau_1), \\ u_{n_0}^{(i-1)}(x_1) &= y_{i1} = w^{(i-1)}(x_1), \quad 1 \leq i \leq m_1 - 1, \\ u_{n_0}(\tau_2) &= w(\tau_2), \\ u_{n_0}^{(i-1)}(x_l) &= y_{il} = w^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k - 1, \\ u_{n_0}(\rho_1) &= w(\rho_1), \\ u_{n_0}^{(i-1)}(x_k) &= y_{ik} = w^{(i-1)}(x_k), \quad 1 \leq i \leq m_k - 1, \\ u_{n_0}(s_i) &= \frac{y_{n-(i-1)}}{c_i} = w(s_i), \quad 1 \leq i \leq j - 1. \end{aligned}$$

Thus,  $u_{n_0}(x)$  and  $w(x)$  are distinct solutions of the same  $(1; k + 2j + 1; 0)$ -point (or if  $m_1 = 1$  and  $m_k = 1$ , the same  $(1; k + 2j + 2; 0)$ -point) BVP which contradicts Corollary 2.5.

If  $q_{n+1} = q_0$ , (and keeping with the assumptions that  $m_1, m_k$  odd) then

$$u_n(x) < u_{n+1}(x), \quad t_j < x < x_2.$$

Now  $w$  is already constructed and as before, find  $u_{n_0}, t_j < \tau_1 < x_1 < \tau_2 < x_2$ , such that

$$u_{n_0}(\tau_1) = w(\tau_1), \quad u_{n_0}(\tau_2) = w(\tau_2).$$

Then,

$$\begin{aligned} a_1 u_{n_0}(t_1) - b_1 u_{n_0}(p_1) &= y_1 = aw(t_1) - b_1 w(p_1), \\ u_{n_0}(t_i) &= \frac{y_i}{a_i} = w(t_i), \quad 2 \leq i \leq j, \\ u_{n_0}(\tau_1) &= w(\tau_1), \\ u_{n_0}^{(i-1)}(x_1) &= y_{i1} = w^{(i-1)}(x_1), \quad 1 \leq i \leq m_1 - 1, \\ u_{n_0}(\tau_2) &= w(\tau_2), \\ u_{n_0}^{(i-1)}(x_l) &= y_{il} = w^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k, \\ u_{n_0}(s_i) &= \frac{y_{n-(i-1)}}{c_i} = w(s_i), \quad 1 \leq i \leq j - 1, \end{aligned}$$

and Corollary 2.5 is contradicted.

The conclusion then is that  $S$  contains all its limit points and is a closed subset of  $\mathbb{R}^2$ ; since  $S$  is open and nonempty,  $S \equiv \mathbb{R}^2$ .

By choosing  $(y_{m_1 1}, y_{m_k k}) \in S$ , there is a corresponding solution  $y(x)$  of (1) such that

$$\begin{aligned} a_1 y(t_1) - b_1 y(p_1) &= y_1, \\ y(t_i) &= \frac{y_i}{a_i}, \quad 2 \leq i \leq j, \\ y^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \\ y(s_i) &= \frac{y_{n-(i-1)}}{c_i}, \quad 1 \leq i \leq j - 1, \\ c_j y(s_j) - d_j y(s_{j+1}) &= y_{n-(j-1)}, \end{aligned}$$

which is the desired solution of the  $(1; k + 2j - 2; 1)$ -point BVP.

Now, let  $z_1(x)$  be the unique solution of the  $(1; k + 2j - 2; 1)$ -point BVP satisfying the  $(k + 2j - 2)$ -point conjugate boundary conditions (or the  $(k + 2j)$ -point conjugate boundary conditions if  $m_1 > 1$  and  $m_k > 1$ ) at the points

$$t_1, p_1, t_2, p_2, t_3, \dots, t_j, x_1, \dots, x_k, s_1, \dots, s_{j-1}, q_1, s_j, s_{j+1},$$

that is,

$$\begin{aligned}
a_1 z(t_1) - b_1 z(p_1) &= y_1, \\
z_1(t_2) &= \frac{y_2}{a_2}, \quad z_1(p_2) = 0, \\
z_1^{(i-1)}(x_1) &= y_{i1}, \quad 1 \leq i \leq m_1 - 1, \\
z_1^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k - 1, \\
z_1^{(i-1)}(x_k) &= y_{ik}, \quad 1 \leq i \leq m_k - 1, \\
z_1(s_i) &= \frac{y_{n-(i-1)}}{c_i}, \quad 1 \leq i \leq j - 2, \\
z_1(s_{j-1}) &= \frac{y_{n-(j-2)}}{c_{j-1}}, \quad z_1(q_1) = 0, \\
c_j z_1(s_j) - d_j z_1(s_{j+1}) &= y_{n-(j-1)}.
\end{aligned}$$

We have

$$a_2 z_1(t_2) - b_2 z_1(p_2) = y_2, \quad c_{j-1} z_1(s_{j-1}) - d_{j-1} z_1(q_1) = y_{n-(j-2)}.$$

Define the set

$$\begin{aligned}
S_1 &= \{(u^{(m_1-1)}(x_1), u^{(m_k-1)}(x_k)) \mid u \text{ is a solution of (1) satisfying} \\
&\quad a_1 u(t_1) - b_1 u(p_1) = y_1, \quad a_2 u(t_2) - b_2 u(p_2) = y_2, \\
&\quad u(t_i) = \frac{y_i}{a_i}, \quad 3 \leq i \leq j, \\
&\quad u^{(i-1)}(x_1) = y_{i1}, \quad 1 \leq i \leq m_1 - 1, \\
&\quad u^{(i-1)}(x_l) = y_{il}, \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k - 1, \\
&\quad u^{(i-1)}(x_k) = y_{ik}, \quad 1 \leq i \leq m_k - 1, \\
&\quad u(s_i) = \frac{y_{n-(i-1)}}{c_i}, \quad 1 \leq i \leq j - 2, \\
&\quad c_{j-1} u(s_{j-1}) - d_{j-1} u(q_1) = y_{n-(j-2)}, \quad c_j u(s_j) - d_j u(s_{j+1}) = y_{n-(j-1)}\}.
\end{aligned}$$

Clearly,  $(z_1^{(m_1-1)}(x_1), z_1^{(m_k-1)}(x_k)) \in S_1$ , and so  $S_1$  is a nonempty subset of  $\mathbf{R}^2$ . By the same process as we did previously, we can show that  $S_1 = \mathbf{R}^2$ . Hence,  $(y_{m_1,1}, y_{m_k,k}) \in S_1$ , which implies that there is a solution  $y_1(x)$  of (1) such that

$$\begin{aligned}
a_1 y(t_1) - b_1 y(p_1) &= y_1, \quad a_2 y(t_2) - b_2 y(p_2) = y_2, \\
y(t_i) &= \frac{y_i}{a_i}, \quad 3 \leq i \leq j, \\
y_1^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \\
y(s_i) &= \frac{y_{n-(i-1)}}{c_i}, \quad 1 \leq i \leq j - 2, \\
c_{j-1} y(s_{j-1}) - d_{j-1} y(q_1) &= y_{n-(j-2)}, \quad c_j y(s_j) - d_j y(s_{j+1}) = y_{n-(j-1)},
\end{aligned}$$

which is the desired solution of the  $(2; k+2j-4; 2)$ -point BVP. Continuing in the same way, we obtain a unique solution of the  $(j; k; j)$ -point BVP, that is, a solution  $y(x)$  of (1) such that at the points  $t_1, \dots, t_{2j}, x_1, \dots, x_k, s_1, \dots, s_{2j}$ , satisfies

$$\begin{aligned}
a_i y(t_{2i-1}) - b_i y(t_{2i}) &= y_i, \quad i = 1, 2, \dots, j, \\
y^{(i-1)}(x_l) &= y_{il}, \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq k, \\
c_i y(s_{2i-1}) - d_i y(s_{2i}) &= y_{n-(i-1)}, \quad i = 1, 2, \dots, j.
\end{aligned}$$

We restate Theorem 3.2 in the terminology introduced in Introduction.

**Corollary 3.2** *Assume that  $k = n - 2j$ , solutions of the  $(j; n - 2j; j)$ -point BVP, are unique. Then, for each  $1 \leq k \leq n - 2j$ , (1) is  $(j; k; j)$ -point uniquely solvable.*



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# Asymptotic Behavior of $n$ -th Order Dynamic Equations

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**Abstract:** We are concerned with the asymptotic behavior of solutions of an  $n$ -th order linear dynamic equation on a time scale in terms of Taylor monomials. In particular, we describe the asymptotic behavior of the so-called (first) principal solution in terms of the Taylor monomial of degree  $n - 1$ . Several interesting properties of the Taylor monomials are established so that we can prove our main results.

**Keywords:** *asymptotic behavior; dynamic equations; time scale; Taylor monomials, oscillation.*

**Mathematics Subject Classification (2010):** 34E10, 39A10.

## 1 Introduction

We shall first consider the two term  $n$ -th order linear dynamic equation

$$u^{\Delta^n} + p(t)u(t) = 0, \quad p(t) > 0, \quad t \geq t_0 \quad (1)$$

on a time scale  $\mathbb{T}$ . Later (see Theorem 2.4) we consider a more general  $n$ -th order linear dynamic equation with  $n + 1$  terms. For the sake of completeness, we recall some basic definitions from the theory of time scales [7, 14].

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers. Since we are interested in oscillation results, we will consider time scales which are unbounded above, i.e.,  $\sup(\mathbb{T}) = \infty$ . We use the notation  $\mathbb{T} := [t_0, \infty)$ .

For  $t \in \mathbb{T}$  we define the forward and backward jump operators

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}. \quad (2)$$

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The (forward) graininess function  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by

$$\mu(t) = \sigma(t) - t. \quad (3)$$

If  $\mathbb{T}$  has a left-scattered minimum  $m$ , then  $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ , otherwise  $\mathbb{T}^\kappa = \mathbb{T}$ . For  $f : \mathbb{T} \rightarrow \mathbb{R}$  and  $t \in \mathbb{T}^\kappa$  define the delta derivative  $f^\Delta(t)$  to be the number (provided it exists) with the property that for any  $\epsilon > 0$ , there exists a  $\delta > 0$  and a neighborhood  $U = (t - \delta, t + \delta) \cap \mathbb{T}$  of  $t$  such that

$$|f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon |\sigma(t) - s|, \quad f^\sigma(t) \equiv f(\sigma(t)), \quad (4)$$

for all  $s \in U$  (see [7]). A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be *rd*-continuous provided it is continuous at right-dense points in  $\mathbb{T}$  and at each left-dense point  $t$  in  $\mathbb{T}$  the left hand limit at  $t$  exists (finite). The set of *rd*-continuous functions on  $\mathbb{T}$  will be denoted by  $C_{rd}$ . The set of functions such that their  $n$ -th delta derivative exists and is *rd*-continuous on  $\mathbb{T}$  is denoted by  $C_{rd}^n$ . In (1) we assume that  $p \in C_{rd}$  and we say  $x$  is a solution provided  $x \in C_{rd}^n$  and  $u^{\Delta^n}(t) + p(t)u(t) = 0$  for  $t \in \mathbb{T}^\kappa$ . We say that a function  $f$  is regressive on  $\mathbb{T}$  if  $1 + \mu(t)f(t) \neq 0$  for all  $t \in \mathbb{T}$ . The set of regressive functions on  $\mathbb{T}$  which belong to  $C_{rd}$  is denoted by  $\mathcal{R}$ . The set of regressive functions in  $C_{rd}^n$  will be denoted by  $\mathcal{R}^n$ .

A solution  $u$  of (1) is said to have a zero at  $a \in \mathbb{T}$  if  $u(a) = 0$ , and it has a generalized zero at  $a$  if either  $u(t)$  has a zero at  $a$  or if  $u(\rho(a))u(a) < 0$ . A solution of (1) is said to be oscillatory if it has an infinite sequence of generalized zeros in  $\mathbb{T}$ , and nonoscillatory otherwise. Equation (1) is said to be **oscillatory** if all solutions are oscillatory and is said to be **nonoscillatory** if all solutions are nonoscillatory. An interesting question is what conditions guarantee the existence of both (i.e, coexistence). Oscillation theorems for  $n$ -th order differential equations have been established by many authors. One often finds criteria under which all solutions are oscillatory. The approach here is somewhat different in that we are interested in establishing sufficient conditions for the existence of at least one oscillatory solution or conditions which guarantee that all solutions are nonoscillatory with a certain asymptotic form. We refer to the results of W. Leighton and Z. Nehari [21], I. M. Glazman [12], G. V. Anan'eva and V. I. Balaganskii [2], V. A. Kondrat'ev [17], I. T. Kiguradze [16], the book of Swanson [24], and the many references therein.

Oscillation theorems for second order dynamic equations on a time scale have been studied by many authors since the introduction of the time scale calculus by Hilger [14]. As examples, we refer to the results in [4, 11, 18]. In this paper we establish some sufficient conditions for the existence of an oscillatory solution and for nonoscillation of the  $n$ -th order equation (1) on a time scale in terms of the Taylor monomials. We also mention that some oscillation results for (1) were obtained in [20]. For additional related results on the asymptotic behavior of solutions of dynamic equations see [6, 15, 23, 25].

## 2 Main Results

We recall the definition of the Taylor monomials (these Taylor monomials were first introduced by Agarwal and Bohner in [1]) as follows:

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau, \quad k = 0, 1, \dots, \quad h_0(t, s) = 1, \quad t \geq s. \quad (5)$$

The solution  $u = u(\cdot, t_1)$  of the IVP (1),

$$u(t_1) = u^\Delta(t_1) = \dots = u^{\Delta^{n-2}}(t_1) = 0, \quad u^{\Delta^{n-1}}(t_1) = 1, \quad t_1 > t_0. \quad (6)$$

is called the **principal solution** of (1) at  $t_1$ .

Our first result gives a ‘smallness’ condition (7) on an integral involving the Taylor monomials which guarantees that the principal solution is nonoscillatory.

**Theorem 2.1** *If  $p \in C_{rd}$ , and*

$$\int_{t_0}^{\infty} h_{n-1}(s, t_0)p(s)\Delta s < \infty, \tag{7}$$

*then the principal solution  $u$  of (1) is eventually positive. Moreover, (7) holds if and only if*

$$\lim_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} = C > 0. \tag{8}$$

**Theorem 2.2** *If  $p \in C_{rd}$ , and  $u$  is a solution of (1) which is eventually positive, then*

$$\lim_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} = \lim_{t \rightarrow \infty} u^{\Delta^{n-1}}(t) := L, \tag{9}$$

*where  $0 < L < +\infty$ . That is, both limits are finite and positive.*

**Theorem 2.3** *If  $p \in C_{rd}$ , and*

$$\int_{t_1}^{\infty} h_{n-2}(t, t_1)p(t)\Delta t = \infty, \tag{10}$$

*then equation (1) has at least one oscillatory solution.*

**Remark 2.1** *If*

$$\lim_{t \rightarrow \infty} \frac{\mu(t)}{t} = 0, \tag{11}$$

*then Theorem 2.3 is true if, instead of (10), the simpler condition*

$$\int_{t_1}^{\infty} t^{n-2}p(t)\Delta t = \infty \tag{12}$$

*is satisfied. More generally, if for some number  $K \in (0, 1)$ ,*

$$\frac{\mu(t)}{t} \leq (n-1)^{\frac{1}{n-2}} \left( K^{\frac{1}{2-n}} - 1 \right), \quad n \geq 4, \tag{13}$$

*and (12) are satisfied then the conclusion of Theorem 2.3 is true. In general, however (12) does not imply (10) as is shown in the following example.*

**Example 2.1** Consider the time scale  $\mathbb{T}_1 = \{t_k = 2^{2^k}, k = 0, 1, 2, 3, \dots\}$  (see [7]). For this time scale there are functions  $p$  such that

$$\int_1^{\infty} h_2(t, t_1)p(t)\Delta t < \infty \quad \text{but} \quad \int_1^{\infty} t^2p(t)\Delta t = \infty.$$

The proof of this example is given at the end of Section 3.

Using the asymptotic representation method [9, 10, 22] one can prove the following theorem.

**Theorem 2.4** *Assume that for all  $j = 1, \dots, n$ , we have  $p_j \in C_{rd}$ , and*

$$\int_{t_0}^{\infty} |p_j(t)| h_{[j/2]}(t, t_0) h_{j-1-[j/2]}(t, t_0) \frac{h_{j-1}^{\sigma}(t, t_0)}{h_{j-1}(t, t_0)} \left( \frac{h_1^{n-1}(t, t_0)}{h_{n-1}(t, t_0)} \right)^{\sigma} \Delta t < \infty, \quad (14)$$

where  $[j/2]$  is the integral part of  $\frac{j}{2}$ . Then the equation

$$u^{\Delta^n} + p_1(t)u^{\Delta^{n-1}} + \dots + p_{n-1}(t)u^{\Delta}(t) + p_n(t)u(t) = 0, \quad t \in \mathbb{T} \quad (15)$$

is nonoscillatory on  $\mathbb{T} \cap [t_1, \infty)$ .

**Remark 2.2** If (13) is true, then equation (15) is nonoscillatory if the simpler condition

$$\int_t^{\infty} \sigma^{j-1}(s) |p_j(s)| \Delta s < \infty, \quad j = 1, \dots, n. \quad (16)$$

is satisfied.

Note that under assumption (16), the asymptotic behavior of solutions of (15) on a continuous time scale ( $\sigma(s) = s$ ) was described by Ghizzetti [12].

**Remark 2.3** When  $n = 3$ , equation (1) is nonoscillatory if

$$\int_{t_1}^{\infty} \frac{t^2 \sigma^2(t) p(t) \Delta t}{h_2(t, t_1)} < \infty, \quad (17)$$

and it has at least one oscillatory solution if

$$\int_{t_1}^{\infty} t p(t) \Delta t = \infty. \quad (18)$$

Before beginning the proofs, we would like to mention some consequences for the  $n$ -th order linear difference equation

$$\Delta^n x(k) + p(k)x(k) = 0, \quad (19)$$

where  $p(k) \geq 0$ . It was shown in [20] that all solutions are oscillatory in case

$$\sum_1^{\infty} k^{n-1-\epsilon} p(k) = \infty, \quad (20)$$

for some  $0 < \epsilon < n - 1$  when  $n$  is even, and every solution is either oscillatory or  $\lim_{n \rightarrow \infty} x(n) = 0$  when  $n$  is odd. However, when  $\epsilon = 0$ , the result is no longer valid. The results in the present paper show that if  $\sum_1^{\infty} k^{n-2} p(k) = \infty$ , then there exists at least one oscillatory solution. If  $\sum_1^{\infty} k^{n-1} p(k) < \infty$ , then the equation is nonoscillatory.

### 3 Proofs

In the proof of the main results we use the methods developed in [21]. We shall need various estimates on the Taylor monomials which we collect in the following lemma.

**Lemma 3.1** *The Taylor monomials satisfy the following properties:*

$$h_n(t_1, s) \geq h_n(t_2, s), \quad t_1 \geq t_2 \geq s, \quad h_n(t, s_1) \leq h_n(t, s_2), \quad t \geq s_1 \geq s_2, \quad (21)$$

$$h_n(t, t_1) \geq h_{n-1}(t, t_1), \quad \lim_{t \rightarrow \infty} h_n(t, t_1) = \infty, \quad t \geq t_1 + 1, \quad n = 1, 2, \dots, \quad (22)$$

$$\lim_{t \rightarrow \infty} \frac{h_k(t, t_1)}{h_{n-1}(t, t_1)} = 0, \quad k = 0, 1, \dots, n-2, \quad \lim_{t \rightarrow \infty} \frac{h_k(t, t_2)}{h_k(t, t_1)} = 1, \quad (23)$$

$$(t-s)^n \leq \frac{((t-s)^{n+1})^\Delta}{n+1}, \quad \int_s^t (\tau-s)^n \Delta\tau \leq \frac{(t-s)^{n+1}}{n+1}, \quad n = 0, 1, \dots, \quad t \geq s, \quad (24)$$

$$h_n(t, s) \leq \frac{(t-s)^n}{n!} = \frac{h_1^n(t, s)}{n!}, \quad n = 0, 1, \dots, \quad t \geq s > 0, \quad (25)$$

$$\frac{h_{k-1}(t, s)}{h_k^\sigma(t, s)} \leq \frac{h_k(t, s)}{h_{k+1}^\sigma(t, s)}, \quad h_{q-1}(t, s)h_{j-q}(t, s) \leq h_q(t, s)h_{j-q-1}(t, s), \quad t \geq s, \quad (26)$$

where  $1 \leq k \leq n, \quad 1 \leq q \leq j/2$ .

Suppose that for some positive integer  $m$  there exists a number  $A \in (0, 1)$  such that

$$\frac{\mu(t)}{t} \leq S_m, \quad S_m = (m+1)^{\frac{1}{m}}(A^{-\frac{1}{m}} - 1), \quad t > 0. \quad (27)$$

Then

$$S_{k+1} < S_k, \quad k = 1, 2, \dots, \quad (28)$$

and

$$t^k \geq A \frac{(t^{k+1})^\Delta}{k+1}, \quad k = 1, 2, \dots, m. \quad (29)$$

If (27) is true for  $m = n$ , then

$$h_n(t, s) \geq B_{n-1}t^n - (1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1}) \frac{t^{n-1}s}{(n-1)!}, \quad (30)$$

where

$$B_n = \frac{A^n}{(n+1)!}, \quad B_0 = 1, \quad n = 0, 1, 2, \dots. \quad (31)$$

**Proof** The statement concerning the monotone increasing nature of  $h_n(t, s)$  in the first argument is trivial. We prove the monotone decreasing property of  $h_n(t, s)$  in the second argument by induction. That is, we will show

$$h_{n-1}(t, s_1) \leq h_{n-1}(t, s_2), \quad s_1 \geq s_2, \quad n = 1, 2, \dots.$$

If  $n = 1$  the statement is trivial. Assuming that the result is true for  $n - 1$ , we see that (21) holds for  $n$  since

$$h_n(t, s_1) = \int_{s_1}^t h_{n-1}(\tau, s_1) \Delta\tau \leq \int_{s_1}^t h_{n-1}(\tau, s_2) \Delta\tau \leq \int_{s_2}^t h_{n-1}(\tau, s_2) \Delta\tau = h_n(t, s_2).$$

We also establish property (22) by induction. For  $n = 1$ , (22) follows from the formula  $h_1(t, t_1) = t - t_1 \geq 1$ . Assuming that (22) is true for  $n = 1, 2, \dots, k$ , we obtain

$$h_{k+1}(t, t_1) = \int_{t_1}^t h_k(\tau, t_1) \Delta\tau \geq \int_{t_1}^t h_{k-1}(\tau, t_1) \Delta\tau = h_k(t, t_1),$$

which completes the induction.

From these inequalities we get

$$h_n(t, t_1) \geq h_1(t, t_1) = t - t_1, \quad n \geq 1,$$

and the property  $\lim_{t \rightarrow \infty} h_n(t, t_1) = \infty$ .

To prove (23) we will use L'Hospital's rule:

**Lemma 3.2** [7] *Assume  $f$  and  $g$  are differentiable on  $\mathbb{T}$  with*

$$\lim_{t \rightarrow \infty} g(t) = \infty,$$

$$g(t) > 0, \quad g^\Delta(t) > 0, \quad t \in \mathbb{T}.$$

*Then*

$$\lim_{t \rightarrow \infty} \frac{f^\Delta(t)}{g^\Delta(t)} = r$$

*implies*

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = r.$$

Indeed, since  $h_n^\Delta(t, t_1) = h_{n-1}(t, t_1)$ , then using (22) we have

$$\lim_{t \rightarrow \infty} \frac{h_1(t, t_1)}{h_2(t, t_1)} = \lim_{t \rightarrow \infty} \frac{h_0(t, t_1)}{h_1(t, t_1)} = \lim_{t \rightarrow \infty} \frac{1}{t - t_1} = 0.$$

The general case of (23) is proved similarly.

To prove (24) we note that

$$((t - s)^{n+1})^\Delta = \sum_{k=0}^n (\sigma(t) - s)^k (t - s)^{n-k} \geq \sum_{k=0}^n (t - s)^k (t - s)^{n-k} = (n + 1)(t - s)^n.$$

The second inequality in (24) is proved by integration of the previous inequality.

Inequality (25) may again be established by induction (see also [6, Theorem 4.1] for a proof of this result). For  $n = 0$  it is clear. Assuming

$$h_{n-1}(t, s) \leq \frac{(t - s)^{n-1}}{(n - 1)!}$$

we have

$$h_n(t, s) = \int_s^t h_{n-1}(\tau, s) \Delta\tau \leq \int_s^t \frac{(\tau - s)^{n-1}}{(n - 1)!} \Delta\tau \leq \int_s^t \frac{((\tau - s)^n)^\Delta}{n!} \Delta\tau = \frac{(t - s)^n}{n!}.$$



To prove the first inequality (26) it is enough to prove that

$$\frac{h_{k-1}(t, s)}{h_k(t, s)} \leq \frac{h_k(t, s)}{h_{k+1}(t, s)}, \quad k = 1, 2, \dots \tag{32}$$

in view of

$$\frac{h_k(t, s)}{h_{k+1}^\sigma(t, s)} - \frac{h_{k-1}(t, s)}{h_k^\sigma(t, s)} = \frac{h_k^2(t, s) - h_{k+1}(t, s)h_{k-1}(t, s)}{h_{k+1}^\sigma(t, s)h_k^\sigma(t, s)}.$$

We will prove (32) by induction, and that the sequence  $\frac{h_{k-1}(t, s)}{h_k(t, s)}$ ,  $k = 1, 2, \dots$  is decreasing with respect to  $t$ .

For  $k = 1$  we have the sequence  $\frac{1}{h_1(t, s)} = \frac{1}{t-s}$  is decreasing with respect to  $t$ , and

$$\frac{h_0(t, s)}{h_1(t, s)} \leq \frac{h_1(t, s)}{h_2(t, s)}$$

which follows from (25):  $h_2(t, s) \leq \frac{h_1^2(t, s)}{2}$ .

Assuming that  $\frac{h_{k-1}(t, s)}{h_k(t, s)}$  is decreasing with respect to  $t$  and (32) is true for  $k$  we have

$$\left( \frac{h_k(t, s)}{h_{k+1}(t, s)} \right)^\Delta = \frac{h_{k-1}(t, s)h_{k+1}(t, s) - h_k^2(t, s)}{h_{k+1}^\sigma(t, s)h_{k+1}(t, s)} \leq 0.$$

That is,  $\frac{h_k(t, s)}{h_{k+1}(t, s)}$  is decreasing with respect to  $t$ , and

$$h_{k+2}(t, s) = \int_s^t \frac{h_{k+1}(\tau, s)}{h_k(\tau, s)} h_k(\tau, s) \Delta\tau \leq \frac{h_{k+1}(t, s)}{h_k(t, s)} \int_s^t h_k(\tau, s) \Delta\tau = \frac{h_{k+1}(t, s)h_{k+1}(t, s)}{h_k(t, s)}$$

which gives (32) with  $k \rightarrow k + 1$  :

$$\frac{h_k(t, s)}{h_{k+1}(t, s)} \leq \frac{h_{k+1}(t, s)}{h_{k+2}(t, s)}.$$

The second inequality (26) may be proved by using the property of Taylor monomials that the ratio  $\frac{h_{j-q-1}(t, s)}{h_{q-1}(t, s)}$  is increasing in  $t$  if  $j - q - 1 \geq q - 1$ , or  $q \leq j/2$ . Indeed,

$$h_{j-q}(t, s) = \int_{t_0}^t \frac{h_{j-q-1}(z, s)}{h_{q-1}(z, s)} h_{q-1}(z, s) \Delta z \leq \frac{h_{j-q-1}(t, s)}{h_{q-1}(t, s)} \int_{t_0}^t h_{q-1}(z, s) \Delta z = \frac{h_{j-q-1}(t, s)}{h_{q-1}(t, s)} h_q(t, s).$$

To prove (28) note that both sequences  $(k + 1)^{1/k}$  and  $A^{-1/k}$  are decreasing with respect to  $k$  for  $k \geq 1$ .

First we prove (29) for the case  $k = m$ . Since

$$(t^{m+1})^\Delta = \sum_{k=0}^m \sigma^k(t) t^{m-k} = t^m + \sigma(t) t^{m-1} + \dots + \sigma^{m-1}(t) t + \sigma^m(t),$$

to prove (28) with  $k = m$  from (27) it is enough to prove

$$t^m \geq \frac{A}{m+1} \sum_{k=0}^m \sigma^k(t) t^{m-k}.$$

If  $\mu \equiv 0$ , it is trivial with  $A = 1$ . Assuming  $\mu \neq 0$  and dividing the inequality by  $t^m$ , we have

$$1 \geq \frac{A}{m+1} \sum_{k=0}^m (x+1)^k, \quad \text{where } x = \frac{\mu(t)}{t},$$

so that summing the right hand side gives

$$1 \geq \frac{A}{m+1} \frac{(x+1)^{m+1} - 1}{x} = \frac{A}{m+1} (x^m + C_{m+1}^1 x^{m-1} + \cdots + C_{m+1}^{m-2} x^2 + C_{m+1}^{m-1} x + C_{m+1}^m),$$

where  $C_{m+1}^k$  is the binomial coefficient. Hence the inequality holds if

$$1 \geq A \left( \frac{x^m}{m+1} + \frac{C_{m+1}^1 x^{m-1}}{m+1} + \cdots + \frac{C_{m+1}^{m-1} x}{m+1} + 1 \right).$$

Now this inequality is true if

$$\begin{aligned} 1 &\geq A \left( \frac{x}{(m+1)^{\frac{1}{m}}} + 1 \right)^m & (33) \\ &= A \left( \frac{x^m}{m+1} + \frac{C_m^1 x^{m-1}}{(m+1)^{\frac{m-1}{m}}} + \frac{C_m^2 x^{m-2}}{(m+1)^{\frac{m-2}{m}}} + \frac{C_m^3 x^{m-3}}{(m+1)^{\frac{m-3}{m}}} + \cdots + 1 \right) \end{aligned}$$

is satisfied, since

$$\frac{C_m^k x^{m-k}}{(m+1)^{\frac{m-k}{m}}} \geq \frac{C_{m+1}^k x^{m-k}}{m+1},$$

or

$$(m+1-k)^m \geq (m+1)^{m-k}, \quad k = 0, 1, \dots, m, \quad m = 0, 1, 2, \dots.$$

To see this note that if  $m = 0$ ,  $k = 0$ , it is true, and if it is true for  $m$ ,  $k = 0, 1, \dots, m$ , then it is true for  $m \rightarrow m+1$ ,  $k = 0, 1, \dots, m+1$ . Now we need to show that

$$(m+2-k)^{m+1} \geq (m+2)^{m+1-k}, \quad k = 0, 1, \dots, m+1.$$

To prove this, we do another induction on  $k$ : If  $k = 0$ , it is true. Assuming

$$(m+2)^{m+1-k} \leq (m+2-k)^{m+1}$$

is true, we obtain the result for  $k \rightarrow k+1$  as follows:

$$(m+2)^{m-k} = \frac{(m+2)^{m+1-k}}{m+2} \leq \frac{(m+2-k)^{m+1}}{m+2} \leq (m+3-k)^{m+1},$$

or dividing by  $(m+2-k)^{m+1}$ , we get

$$\frac{1}{m+2} \leq \left( \frac{m+3-k}{m+2-k} \right)^{m+1}.$$

which is true, since the left side is less than one, and the right side is greater than 1. Furthermore, from (33) we have

$$1 \geq A^{\frac{1}{m}} \left( \frac{x}{(m+1)^{\frac{1}{m}}} + 1 \right),$$

so by assumption (27) we have

$$\frac{t}{\mu(t)} = \frac{1}{x} \geq \frac{A^{\frac{1}{m}}}{(m+1)^{\frac{1}{m}}(1 - A^{\frac{1}{m}})}.$$

To prove (29) for all  $k = 1, \dots, m$ , note that if (27) is true for  $m > 1$ , then it is true for all  $k = 1, 2, \dots, m$  since the sequence  $S_k$  is decreasing.

The last property (30)

$$h_n(t, s) \geq B_{n-1}t^n - (1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1})\frac{t^{n-1}s}{(n-1)!},$$

we prove again by induction. When  $n = 1$ , it is obvious. Assuming (29) is true we prove it for  $n \rightarrow n + 1$ . From (27)

$$\begin{aligned} h_{n+1}(t, s) &= \int_s^t h_n(\tau, s)\Delta\tau \\ &\geq \int_s^t \left( B_{n-1}\tau^n - (1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1})\frac{t^{n-1}s}{(n-1)!} \right) \Delta\tau \\ &\geq \frac{A_n}{n+1}B_{n-1}(t^{n+1} - s^{n+1}) - \frac{1}{n!}(1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1})(t^n - s^n)s \\ &\geq B_n t^{n+1} - B_n t^n s - \frac{1}{n!}(1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1})t^n s \end{aligned}$$

in view of  $t^{n-1} \leq \frac{(t^n)^\Delta}{n}$ . That is,

$$h_{n+1}(t, s) \geq B_n t^{n+1} - (1 + B_1 + 2!B_2 + \dots + (n-1)!B_{n-1} + n!B_n)\frac{t^n s}{n!}.$$

which completes the proof.

**Proof of Theorem 2.1**

Since (7) holds, we may take  $t_1$  large enough so that

$$\int_t^\infty h_{n-1}(s, t_1)p(s)\Delta s \leq \frac{1}{2}, \quad t \geq t_1. \tag{34}$$

Assume the principal solution of (1) is oscillatory. Then there are two possibilities:

1. There exists a point  $t_2 \in \mathbb{T}, t_2 > t_1$ , where the principal solution has a zero:  $u(t_2, t_1) = 0$  and  $u(t, t_1) > 0$  on  $(t_1, t_2)$ .
2. There exists a point  $t_2 \in \mathbb{T}, t_2 > t_1$ , where  $u(t_2, t_1) > 0$  and  $u(\sigma(t_2), t_1) < 0$ .

In the first case, from Taylor's formula

$$u(t) = u(t_1) + u^\Delta(t_1)(t - t_1) + u^{\Delta\Delta}(t_1)h_2(t, t_1) + \cdots + u^{\Delta^{n-1}}(t_1)h_{n-1}(t, t_1) + \int_{t_1}^t u^{\Delta^n}(s)h_{n-1}(t, \sigma(s))\Delta s, \quad h_k^\Delta(t, s) = h_{k-1}(t, s), \quad k = 0, 1, 2, \dots, n-1,$$

so that the principal solution of equation (1) can be written in the form

$$u(t, t_1) = h_{n-1}(t, t_1) - \int_{t_1}^t h_{n-1}(t, \sigma(s))p(s)u(s, t_1)\Delta s. \quad (35)$$

From (35) we have

$$\begin{aligned} u(t, t_1) &\leq h_{n-1}(t, t_1), \\ h_{n-1}(t_2, t_1) &= \int_{t_1}^{t_2} h_{n-1}(t_2, \sigma(s))p(s)u(s, t_1)\Delta s \leq h_{n-1}(t_2, \sigma(t_1)) \int_{t_1}^{t_2} p(s)u(s, t_1)\Delta s \\ &\leq h_{n-1}(t_2, \sigma(t_1)) \int_{t_1}^{t_2} p(s)h_{n-1}(s, t_1)\Delta s. \end{aligned}$$

Dividing this inequality by  $h_{n-1}(t_2, \sigma(t_1))$ , we get

$$\int_{t_1}^{t_2} h_{n-1}(s, t_1)p(s)\Delta s \geq \frac{h_{n-1}(t_2, t_1)}{h_{n-1}(t_2, \sigma(t_1))}.$$

Using the monotonicity of the Taylor monomial with respect to its second argument, we get

$$\int_{t_1}^{t_2} h_{n-1}(s, t_1)p(s)\Delta s \geq 1,$$

which contradicts (34).

In the second case, from (35)

$$u(\sigma(t_2), t_1) = h_{n-1}(\sigma(t_2), t_1) - \int_{t_1}^{\sigma(t_2)} h_{n-1}(\sigma(t_2), \sigma(s))p(s)u(s, t_1)\Delta s < 0$$

$$h_{n-1}(\sigma(t_2), t_1) < \int_{t_1}^{\sigma(t_2)} h_{n-1}(\sigma(t_2), \sigma(s))p(s)u(s, t_1)\Delta s \leq$$

$$h_{n-1}(\sigma(t_2), \sigma(t_1)) \int_{t_1}^{\sigma(t_2)} p(s)u(s, t_1)\Delta s \leq h_{n-1}(\sigma(t_2), \sigma(t_1)) \int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s$$

$$\int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s \geq \frac{h_{n-1}(\sigma(t_2), t_1)}{h_{n-1}(\sigma(t_2), \sigma(t_1))} \geq 1,$$

or

$$\int_{t_1}^{\sigma(\infty)} h_{n-1}(s, t_1)p(s)\Delta s \geq \int_{t_1}^{\sigma(t_2)} h_{n-1}(s, t_1)p(s)\Delta s \geq 1,$$

so we get a contradiction again. Therefore, we conclude that the principal solution is nonoscillatory.

From (35)

$$\begin{aligned} h_{n-1}(t, t_1) &= u(t, t_1) + \int_{t_1}^t h_{n-1}(t, \sigma(s))p(s)u(s, t_1)\Delta s \\ &\leq u(t, t_1) + h_{n-1}(t, \sigma(t_1)) \int_{t_1}^t p(s)u(s, t_1)\Delta s \\ &\leq u(t, t_1) + h_{n-1}(t, t_1) \int_{t_1}^t p(s)h_{n-1}(s, t_1)\Delta s \\ &\leq u(t, t_1) + \frac{1}{2}h_{n-1}(t, t_1), \end{aligned}$$

so we get the inequality

$$\frac{1}{2}h_{n-1}(t, t_1) \leq u(t, t_1) \leq h_{n-1}(t, t_1). \tag{36}$$

Before proving the second part of Theorem 2.1, we prove Theorem 2.2.

**Proof of Theorem 2.2.**

Assuming that  $u(t)$  is a positive solution of (1) on  $[t_1, \infty)$ , we have from Taylor’s formula and (1) that

$$R(t) = u(t) + \int_{t_1}^t h_{n-1}(t, \sigma(s))p(s)u(s)\Delta s, \tag{37}$$

where  $R(t)$  is the polynomial

$$R(t) = \sum_{k=0}^{n-1} h_k(t, t_1)u^{\Delta^k}(t_1), \tag{38}$$

or

$$R(t) = u(t_1) + (t - t_1)u^{\Delta}(t_1) + h_2(t, t_1)u^{\Delta\Delta}(t_1) + \dots + h_{n-1}(t, t_1)u^{\Delta^{n-1}}(t_1).$$

Since  $u > 0$ ,  $t \geq t_1$ , and  $h_{n-1}(t, s)$  is decreasing in the second argument, we have from (37)

$$\begin{aligned} R(t) &\leq u(t) + h_{n-1}(t, t_1) \int_{t_1}^t p(s)u(s)\Delta s \\ &= u(t) + h_{n-1}(t, t_1) \left( u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(t) \right), \end{aligned}$$

where (1) has been used in the last step. Dividing by  $h_{n-1}(t, t_1)$  we get

$$\frac{R(t)}{h_{n-1}(t, t_1)} \leq \frac{u(t)}{h_{n-1}(t, t_1)} + u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(t).$$

In view of  $\lim_{t \rightarrow \infty} \frac{R(t)}{h_{n-1}(t, t_1)} = u^{\Delta^{n-1}}(t_1)$ , if  $t$  tends to infinity through a sequence of points for which  $\frac{u(t)}{h_{n-1}(t, t_1)}$  approaches its lower limit we have

$$\lim_{t \rightarrow \infty} u^{\Delta^{n-1}}(t) \leq \liminf_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)}. \tag{39}$$

Note that the limit  $u^{\Delta^{n-1}}(t)$  as  $t \rightarrow \infty$  exists since, by (1),  $u^{\Delta^{n-1}}(t)$  decreases. Choosing  $t_1 < \xi < t$  from (37), we have

$$\begin{aligned} R(t) &\geq u(t) + \int_{t_1}^{\xi} h_{n-1}(t, \sigma(s)) p(s) u(s) \Delta s \\ &\geq u(t) + h_{n-1}(t, \sigma(\xi)) \int_{t_1}^{\xi} p(s) u(s) \Delta s \\ &= u(t) + h_{n-1}(t, \sigma(\xi)) \left( u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(\xi) \right). \end{aligned}$$

From

$$\frac{R(t)}{h_{n-1}(t, t_1)} \geq \frac{u(t)}{h_{n-1}(t, t_1)} + \frac{h_{n-1}(t, \sigma(\xi))}{h_{n-1}(t, t_1)} \left( u^{\Delta^{n-1}}(t_1) - u^{\Delta^{n-1}}(\xi) \right)$$

we get

$$\frac{h_{n-1}(t, \sigma(\xi)) u^{\Delta^{n-1}}(\xi)}{h_{n-1}(t, t_1)} + \frac{R(t)}{h_{n-1}(t, t_1)} \geq \frac{u(t)}{h_{n-1}(t, t_1)} + \frac{h_{n-1}(t, \sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t, t_1)}$$

or

$$u^{\Delta^{n-1}}(\xi) + \frac{R(t)}{h_{n-1}(t, t_1)} \geq \frac{u(t)}{h_{n-1}(t, t_1)} + \frac{h_{n-1}(t, \sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t, t_1)}.$$

since

$$\frac{h_{n-1}(t, \sigma(\xi))}{h_{n-1}(t, t_1)} \leq 1.$$

Now as  $t \rightarrow \infty$  using (23) we get

$$u^{\Delta^{n-1}}(\xi) + u^{\Delta^{n-1}}(t_1) \geq \limsup_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} + \limsup_{t \rightarrow \infty} \frac{h_{n-1}(t, \sigma(\xi)) u^{\Delta^{n-1}}(t_1)}{h_{n-1}(t, t_1)}$$

and since from (23)

$$\lim_{t \rightarrow \infty} \frac{h_{n-1}(t, \sigma(\xi))}{h_{n-1}(t, t_1)} = 1$$

we have

$$u^{\Delta^{n-1}}(\xi) + u^{\Delta^{n-1}}(t_1) \geq \limsup_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} + u^{\Delta^{n-1}}(t_1)$$

or

$$\limsup_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} \leq \lim_{\xi \rightarrow \infty} u^{\Delta^{n-1}}(\xi), \quad (40)$$

which with (39) proves Theorem 2.2.

Returning to the proof of Theorem 2.1, recall that under assumption (7) it was shown that the principal solution  $u(t, t_1)$  satisfies inequalities (36).

By Theorem 2.2

$$\lim_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)}$$

exists and it is positive. So condition (7) is sufficient for the existence of a solution with the prescribed asymptotic behavior.

To prove the necessity, we assume that (1) has a solution such that

$$\lim_{t \rightarrow \infty} \frac{u(t)}{h_{n-1}(t, t_1)} = c > 0. \tag{41}$$

Evidently this assumption ensures that  $u(t)$  is ultimately nonoscillatory, otherwise the limit in question would be zero. Now we may assume that  $u(t)$  is positive, and from Theorem 2.2  $\lim_{t \rightarrow \infty} u^{\Delta^{n-1}}(t) = c$ . Integrating (1), we get that

$$\int_{t_1}^{\infty} p(t)u(t)\Delta t = u^{\Delta^{n-1}}(t_1) - c.$$

From our assumption,  $u(t) \geq (c - \varepsilon)h_{n-1}(t, t_1)$  for some  $\varepsilon > 0$ , and so

$$u^{\Delta^{n-1}}(t_1) - c \geq (c - \varepsilon) \int_{t_1}^{\infty} t^{n-1}p(t)h_{n-1}(t, t_1)\Delta t,$$

and it follows that  $\int_{t_1}^{\infty} h_{n-1}(t, t_1)p(t)\Delta t < \infty$ . This completes the proof of Theorem 2.1.

**Proof of Theorem 2.3.**

Assume that (10) holds but equation (1) is nonoscillatory on  $(t_1, \infty)$ . Then the principal solution  $u(t, t_1)$  will be positive for  $t > t_1$ . From (1)  $u^{\Delta^n}(t, t_1) < 0$ ,  $u^{\Delta^{n-1}}(t, t_1)$  is decreasing. By Theorem 2.2  $\lim_{t \rightarrow \infty} u^{\Delta^{n-1}}(t, t_1)$  is positive, so  $u^{\Delta^{n-1}}(t, t_1) > 0$ ,  $t > t_1$  and

$$u^{\Delta^{n-2}}(t, t_1) = \int_{t_1}^t u^{\Delta^{n-1}}(s)\Delta s > A > 0, \quad t > t_1,$$

and since  $u(b, t_1) > 0$  if  $b$  is slightly larger than  $t_1$ , we have

$$u(t) = u(b) + \dots + h_{n-3}(t, b)u^{\Delta^{n-3}}(b) + \int_b^t h_{n-3}(t, \sigma(s))u^{\Delta^{n-2}}(s)\Delta s$$

$$u(t, t_1) \geq u(b) + h_1(t, b)u^{\Delta}(b) + \dots + h_{n-3}(t, b)u^{\Delta^{n-2}}(b) + Ah_{n-2}(t, b) > Ah_{n-2}(t, b).$$

On the other hand

$$u^{\Delta^{n-1}}(b, t_1) = u^{\Delta^{n-1}}(t, t_1) + \int_b^t p(t)u(t, t_1)\Delta t > \int_b^t p(t)u(t, t_1)\Delta t > A \int_b^t p(t)h_{n-2}(t, b)\Delta t,$$

and when  $t \rightarrow \infty$  we get

$$\int_b^t p(t)h_{n-2}(t, b)\Delta t < \infty,$$

which contradicts (10).

**Proof of Remark 2.1.**

If condition (13) is satisfied, then (27) is true for  $m = n - 2$ , and from (30) we have for some small positive  $\varepsilon > 0$

$$h_{n-2}(t, t_1) \geq t^{n-2}(B_{n-3} - \varepsilon)$$

which implies Remark 2.1.

**Proof of Example 2.1.**

For the time scale  $\mathbb{T}_1 = \{t_k = 2^{2^k}, k = 0, 1, 2, 3, \dots\}$ , we have

$$\sigma(t) = t^2, \quad \mu(t) = t^2 - t, \quad h_1(t) = h_1(t, 2) = t - 2$$

and for  $m \geq 1$

$$h_2(t_m, 2) = \sum_{k=0}^{m-1} h_1(t_k) \mu(t_k) = \sum_{k=0}^{m-1} (t_k - 2)(t_k^2 - t_k) \leq \sum_{k=0}^{m-1} t_k(t_k^2 - t_k) \leq t_{m-1}^3 = t_m^{3/2},$$

where we used the inequality

$$\sum_{k=0}^{m-1} (t_k^3 - t_k^2) \leq t_{m-1}^3, \quad m \geq 1,$$

which may be proved by induction. To see this note that it is true for  $m = 1$ , and if it is true for  $m$ , then it is true for  $m \rightarrow m + 1$  as well:

$$\sum_{k=0}^m (t_k^3 - t_k^2) = \sum_{k=0}^{m-1} (t_k^3 - t_k^2) + t_m^3 - t_m^2 \leq t_{m-1}^3 + t_m^3 - t_m^2 \leq t_m^3.$$

Further choosing  $p(t) = t^{-4-\varepsilon_k}$ ,  $\varepsilon_k = 2^{-k} = \frac{1}{\log_2(t_k)} > 0$ , we have

$$\begin{aligned} \int_1^\infty h_2(t) p(t) \Delta t &= \sum_{k=1}^\infty h_2(t_k) p(t_k) \mu(t_k) \leq \sum_{k=1}^\infty t_k^{3/2} t_k^{-4-\varepsilon} (t_k^2 - t_k) \leq \\ &\sum_{k=1}^\infty t_k^{-1/2-\varepsilon} = \sum_{k=1}^\infty \frac{1}{2 \cdot 2^{2^{k-1}}} < \infty. \end{aligned}$$

However

$$\begin{aligned} \int_1^\infty t^2 p(t) \Delta t &= \sum_{k=1}^\infty t_k^2 p(t_k) \mu(t_k) = \\ &\sum_{k=1}^\infty t_k^2 t_k^{-4-\varepsilon} (t_k^2 - t_k) \geq \frac{1}{2} \sum_{k=1}^\infty t_k^{-\varepsilon} = \frac{1}{2} \sum_{k=1}^\infty 2^{-1} = \infty. \end{aligned}$$

This establishes the validity of Example 2.1.

**4 Proof of Theorem 2.4**

To prove Theorem 2.4 we will construct explicit nonoscillating asymptotic solutions of (15). Since different asymptotic methods [4, 10, 22] are used in the proof of Theorem 2.4, we include the proof of it in this special case.

The equation

$$u^{\Delta^n} + p_1(t) u^{(\Delta)^{n-1}} + \dots + p_{n-1}(t) u^\Delta(t) + p_n(t) u(t) = 0, \quad t \in \mathbb{T} \quad (42)$$

may be written as a system

$$x^\Delta(t) = (J + P(t))x(t), \quad (43)$$



where

$$x(t) = \begin{pmatrix} u^{(\Delta)^{n-1}} \\ \dots \\ u^\Delta \\ u \end{pmatrix}, \quad P(t) = \begin{pmatrix} -p_1 & -p_2 & \dots & -p_n \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & \dots & \dots & 1 & 0 \end{pmatrix}.$$

Using the transformation

$$x(t) = \Lambda(t)y(t), \tag{44}$$

where

$$\Lambda(t) = e^J D(t), \quad D(t) = \text{diag}\{h_0, h_1, \dots, h_{n-1}\},$$

that is

$$\Lambda(t) = \begin{pmatrix} h_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ h_1 & h_1 & 0 & 0 & \dots & 0 & 0 \\ h_2 & h_1^2 & h_2 & 0 & \dots & 0 & 0 \\ h_3 & h_2 h_1 & h_1 h_2 & h_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ h_{n-1} & h_{n-1} h_1 & \dots & \dots & \dots & \dots & h_{n-1} \end{pmatrix},$$

(here we suppress the dependence on  $t$ :  $h_j = h_j(t, t_0)$ ,  $p_j = p_j(t)$ ), we get

$$y^\Delta(t) = (E(t) + B(t))y(t), \tag{45}$$

where by direct calculations

$$E(t) = (\Lambda^{-1})^\sigma (J\Lambda(t) - \Lambda^\Delta(t)) = -(D^\sigma)^{-1} D^\Delta(t), \quad B(t) = (\Lambda^{-1})^\sigma P\Lambda. \tag{46}$$

Here

$$E(t) = \text{diag}\{\theta_1(t), \dots, \theta_n(t)\}, \tag{47}$$

where

$$\theta_1(t) = 0, \quad \theta_k(t) = -\frac{h_{k-2}(t, t_0)}{h_{k-1}(\sigma(t), t_0)}, \quad k = 2, 3, \dots, n.$$

From property (26) the sequence  $\theta_k$ ,  $k = 1, 2, \dots$ , is decreasing with respect to  $k$ , that is,

$$\theta_{k+1}(t) < \theta_k(t), \quad k = 1, 2, \dots, n-1, \quad t \geq t_1 > t_0.$$

Note that  $\theta_k \in \mathcal{R}$  since  $1 + \mu\theta_k = \frac{h_{k-1}(t, t_0)}{h_{k-1}(\sigma(t), t_0)} > 0$ . Consider the solutions of the  $n^2$  initial value problems

$$w_{ij}^\Delta(t) = q(t)w_{ij}(t), \quad w_{ij}(t_1) = 1, \quad q(t) = \frac{\theta_j(t) - \theta_i(t)}{1 + \mu(t)\theta_i(t)}. \tag{48}$$

Note that solutions of (48) exist and are unique, if  $q \in \mathcal{R}$ .

To find asymptotic representations of solutions of (45) we will apply a time scale version of Levinson’s theorem (for further results on the time scale version of Levinson’s Theorem see [3]):

**Theorem 4.1** [4] *Assume that  $\theta_k \in \mathbb{R}$ ,  $1 \leq k \leq n$ ,*

$$\int_{t_1}^{\infty} \left| \frac{B(t)\Delta t}{1 + \mu(t)\theta_j(t)} \right| < \infty, \quad 1 \leq j \leq n, \quad (49)$$

*and suppose that there exists a number  $m > 0$  such that for each pair  $(i, j)$  with  $i \neq j$ , solutions  $w_{ij}(t)$  of (48) satisfy*

$$\lim_{t \rightarrow \infty} w_{ij}(t) = 0, \quad \left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| \geq m, \quad t_1 \leq s \leq t. \quad (50)$$

*Then the linear system (45) has a fundamental matrix  $Y(t)$  such that*

$$Y(t) = [I + \varepsilon(t)]V(t), \quad \lim_{t \rightarrow \infty} \varepsilon(t) = 0, \quad (51)$$

*where  $\varepsilon(t)$  is the error matrix-function, and  $V(t)$  satisfies*

$$V^\Delta(t) = E(t)V(t), \quad V(t_1) = I. \quad (52)$$

Since the matrix  $E$  is diagonal (see (47)), and  $\theta_j \in \mathcal{R}$ , one can find solutions of (52) in terms of the Euler exponential functions:

$$v_j(t) = e_{\theta_j}(t, t_1), \quad j = 1, 2, 3, \dots, n,$$

or in terms of the Taylor monomials:

$$e_{\theta_j}(t, t_1) = \frac{h_{j-1}(t_1, t_0)}{h_{j-1}(t, t_0)}, \quad j = 1, \dots, n, \quad t \geq t_1 > t_0. \quad (53)$$

Note that  $v_j(t) = e_{\theta_j}(t, t_1)$ ,  $j = 1, 2, \dots, n$  are nonoscillatory. Solutions of (48) are

$$w_{ij}(t) = e_q(t, t_1) = \frac{e_{\theta_j}(t, t_1)}{e_{\theta_i}(t, t_1)}, \quad q = \frac{\theta_j - \theta_i}{1 + \mu\theta_i}, \quad j > i.$$

Since  $\theta_j < \theta_i$ ,  $j > i$ , we have  $q(t) < 0$ , but  $q \in \mathcal{R}$ , in view of  $1 + q\mu = \frac{1 + \mu\theta_j}{1 + \mu\theta_i} < 0$ . From (48), (53) we get

$$w_{ij}(t) = \frac{h_{j-1}(t_1, t_0)h_{i-1}(t, t_0)}{h_{j-1}(t, t_0)h_{i-1}(t_1, t_0)}, \quad t \geq t_1 > t_0. \quad (54)$$

Before applying Theorem 4.1 let us check the conditions. From (23), condition (50) is satisfied:

$$\lim_{t \rightarrow \infty} w_{ij}(t) = \lim_{t \rightarrow \infty} \frac{h_{j-1}(t_1, t_0)h_{i-1}(t, t_0)}{h_{j-1}(t, t_0)h_{i-1}(t_1, t_0)} = 0, \quad j > i \geq 1,$$

$$\left| \frac{w_{ij}(s)}{w_{ij}(t)} \right| = \frac{h_{i-1}(s, t_0)h_{j-1}(t, t_0)}{h_{j-1}(s, t_0)h_{i-1}(t, t_0)} \geq \frac{h_{i-1}(s, t_0)h_{i-1}(t, t_0)}{h_{i-1}(s, t_0)h_{i-1}(t, t_0)} = 1, \quad j > i \geq 1.$$

To check condition (49) note that by direct calculations from (46)

$$B_{n,k} = h_{k-1} \sum_{j=k}^n p_j h_{j-k} \left( \frac{h_1^{n-1} - (n-2)h_1^{n-3}h_2 - \dots}{h_{n-1}} \right)^\sigma.$$

In view of (25), (26) we have

$$h_{k-1}h_{j-k} \leq h_{[j/2]}h_{j-1-[j/2]} \quad 1 \leq k \leq j,$$

$$\left| \frac{h_1^{n-1} - (n-2)h_1^{n-3}h_2 - \dots}{h_{n-1}} \right| \leq C \frac{h_1^{n-1}}{h_{n-1}},$$

so

$$h_{k-1} \sum_{j=k}^n |p_j| h_{j-k} \leq \sum_{j=1}^n |p_j| h_{[j/2]} h_{j-1-[j/2]}, \quad \text{for all } 1 \leq k \leq j,$$

and

$$|B_{nk}| \leq C \sum_{j=1}^n |p_j| h_{[j/2]} h_{j-1-[j/2]} \left( \frac{h_1^{n-1}}{h_{n-1}} \right)^\sigma.$$

Therefore

$$\|B(t)\| = C \sum_{j=1}^n |p_j| h_{[j/2]} h_{j-1-[j/2]} \left( \frac{h_1^{n-1}}{h_{n-1}} \right)^\sigma. \tag{55}$$

So condition (49) becomes

$$\int_{t_0}^\infty |p_j(t)| h_{[j/2]}(t, t_0) h_{j-1-[j/2]}(t, t_0) \frac{h_{j-1}^\sigma(t, t_0)}{h_{j-1}(t, t_0)} \left( \frac{h_1^{n-1}(t, t_0)}{h_{n-1}(t, t_0)} \right)^\sigma \Delta t < \infty, \quad j = 1, \dots, n,$$

which is condition (14).

From Theorem 4.1 we get the asymptotic representation (51)

$$Y(t) = \Lambda(t)(I + \varepsilon(t))V(t).$$

The fundamental matrix solution  $X$  of (43) in view of (44) may be written in the form

$$X(t) = \Lambda(t)Y(t) = \Lambda(t)(I + \varepsilon(t))V(t),$$

and solutions  $u$  of equation (42) are not oscillatory.

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# On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized Hénon Map

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**Abstract:** Suitable stabilization conditions obtained for continuous chaotic systems are generalized, in this paper, to discrete-time chaotic systems. The proposed approach, leading to these conditions for complete synchronization, anti-synchronization and hybrid synchronization phenomena studies, is based on the use of state feedback and aggregation techniques for stability and stabilizability studies associated with the Benrejeb arrow form matrix for system description. The results, easy to use, are successfully applied for two identical 3D generalized Hénon maps.

**Keywords:** *hyperchaotic discrete-time systems; stability; Benrejeb arrow form matrix; complete synchronization; anti-synchronization; hybrid synchronization.*

**Mathematics Subject Classification (2010):** 34C28, 93C55.

## 1 Introduction

Chaos synchronization has received a significant attention due to its potential applications [12, 27] in various fields, for instance, application to control theory, secure communication, chemical reaction and encoding message [13, 24]. There exist many types of synchronization, such as Complete Synchronization (CS) [27], Anti-Synchronization (AS) [19], Hybrid Synchronization (HS) [21, 22], Phase Synchronization [29], Lag Synchronization [30], Generalized Synchronization [31], Projective Synchronization [25] and Q-S Synchronization [32].

Given the two following chaotic systems:  
the master one:

$$x_m(k+1) = F(x_m(k)), \quad (1)$$

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the slave one:

$$y_s(k+1) = G(y_s(k)), \quad (2)$$

$x_m(k) = [x_{m1}(k) \dots x_{mn}(k)]^T$  is the  $n$ -dimensional state vector of the master system and  $y_s(k) = [y_{s1}(k) \dots y_{sn}(k)]^T$  is the  $n$ -dimensional state vector of the slave system.  $F : R^n \rightarrow R^n$  and  $G : R^n \rightarrow R^n$  are vector functions in  $n$ -dimensional space. If the following conditions are satisfied:

$$\lim_{k \rightarrow \infty} \|y_i(k) - \alpha_i x_i(k)\| = 0, \quad \forall i = 1, \dots, n, \quad (3)$$

then the complete synchronization is achieved when all the values of  $\alpha_i$  are equal to 1, the anti-synchronization when all  $\alpha_i$  are equal to -1 and the hybrid synchronization [21,22] when some pairs of the state variables achieve (AS) and the other pairs of state variables, simultaneously, achieve (CS).

In this paper, the proposed approach is based on establishing a new state feedback stabilizing conditions for nonlinear discrete-time hierarchical systems, which constitute an extension of previous results on synchronization studies of continuous chaotic processes [17, 19]. This approach is based on the Borne and Gentina practical criterion for stability study [7-9] (Appendix) associated with the Benrejeb arrow form matrix for system description [3-6,10,11,16,18].

In fact, the main purpose in this work is to design an adaptive state feedback controller guaranteeing the asymptotic stability followed by the complete synchronization, the anti-synchronization and the hybrid synchronization of the nonlinear discrete-time error of two identical hyperchaotic systems.

The paper is organized as follows. After a brief description of the third order generalized Hénon map in Section 2, sufficient conditions leading to conclude on the asymptotic stability of dynamic nonlinear discrete-time processes characterized, in the state space, by a thin arrow form matrix [11], are given in Section 3. In Section 4, the design of a complete synchronous state feedback stabilizing controller of two identical Hénon maps is proposed. The case of anti-synchronization of two identical Hénon maps is also considered in Section 5, and hybrid synchronization between two identical Hénon maps in Section 6.

## 2 3D Generalized Hénon Map

In this section, two identical hyperchaotic discrete-time Hénon map master and slave systems are described as follows [2, 15, 23, 26]:

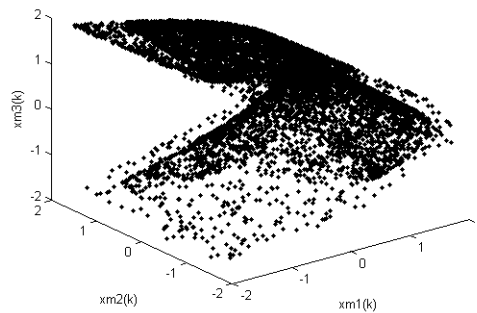
the master one:

$$\begin{cases} x_{m1}(k+1) = \mu - x_{m2}^2(k) - bx_{m3}(k), \\ x_{m2}(k+1) = x_{m1}(k), \\ x_{m3}(k+1) = x_{m2}(k). \end{cases} \quad (4)$$

the slave one:

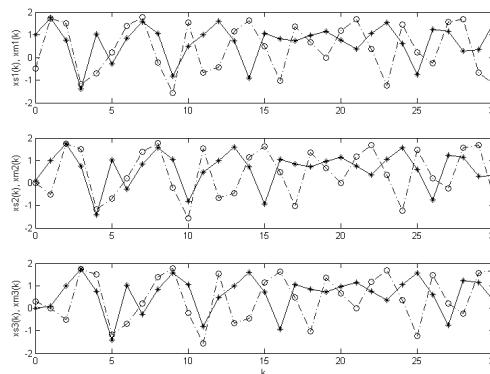
$$\begin{cases} x_{s1}(k+1) = \mu - x_{s2}^2(k) - bx_{s3}(k) + u_1(k), \\ x_{s2}(k+1) = x_{s1}(k) + u_2(k), \\ x_{s3}(k+1) = x_{s2}(k) + u_3(k), \end{cases} \quad (5)$$

$x_m(k) = [x_{m1}(k) \ x_{m2}(k) \ x_{m3}(k)]^T$  is the state vector of master system,  $x_s(k) = [x_{s1}(k) \ x_{s2}(k) \ x_{s3}(k)]^T$  is the state vector of slave system and  $u(k) = [u_1(k) \ u_2(k) \ u_3(k)]^T$  the control vector to be designed later for achieving synchronization, anti-synchronization or hybrid synchronization. The hyperchaotic attractor of system (4) characterized by:  $b = 0.1$  and  $\mu = 1.76$ , with the initial values  $x_m(0) = (1, 0.1, 0)$  [15], is shown in Figure 2.1.



**Figure 2.1:** Hyperchaotic attractor of system (4) in  $x_m(k)$  hyperplane.

The simulation results of the two identical Hénon maps hyperchaotic systems, shown in Figure 2.2, illustrate the systems (4) and (5) responses when the control is turned off and for initial states  $x_m(0) = (1, 0.1, 0)$ ,  $x_s(0) = (-0.5, 0, 0.3)$ [15]. Then, the states are not synchronized.



**Figure 2.2:** Responses of the master (—) and slave (---) systems.

### 3 Sufficient Conditions of Asymptotic Stability of Error Dynamics for Chaotic Discrete-time System

Let us consider the following error vector:

$$e_i(k) = x_{si}(k) - \alpha_i x_{mi}(k), \quad i = 1, 2, 3, \quad \alpha_i \in \{1, -1\}, \quad (6)$$

described in state space by:

$$e(k + 1) = A(k, x(k))e(k) + Bu(k). \quad (7)$$

When system (7) is stabilized by the feedback law  $u(k)$ , the error converges to zero such as:

$$\lim_{k \rightarrow \infty} \|e_i(k)\| = 0, \quad i = 1, 2, 3. \quad (8)$$

Then, systems (4) and (5) achieve (CS), (AS) or (HS) according to the values of  $\{\alpha_i\}$ . To reach this goal, the control law  $u(k)$  is chosen such as [6,20]:

$$u(k) = -K(k, x(k))e(k), \quad (9)$$

thus:

$$e(k+1) = A_a(k, x(k))e(k) \quad (10)$$

with

$$A_a(k, x(k)) = A(k, x(k)) - BK(k, x(k)) \quad (11)$$

and the Borne and Gentina criterion [7-9], associated with the particular canonical Ben-rejeb arrow form matrix  $A_a(k, x(k))$  [3-6,10,11,16,18], is used for the formulation of the following theorem [6,16,18].

**Theorem 3.1** *The error process, described by (7) is stabilized by the control law defined by (9), if the matrix  $A_a(k, x(k))$ , defined by (11), is in the arrow form such that:*

- i. the nonlinear elements are isolated in one row of the matrix  $A_a(k, x(k))$ ;*
- ii. the diagonal elements,  $a_{a_{ii}}(k, x(k))$ , of the matrix  $A_a(k, x(k))$  are such that:*

$$1 - |a_{a_{ii}}(k, x(k))| > 0, \quad \forall i = 1, \dots, n-1, \quad (12)$$

- iii. there exist  $\varepsilon > 0$  such that:*

$$1 - |a_{a_{nn}}(k, x(k))| - \sum_{i=1}^{n-1} (|a_{a_{in}}(k, x(k))a_{a_{ni}}(k, x(k))| \times (1 - |a_{a_{ii}}(k, x(k))|)^{-1}) > \varepsilon. \quad (13)$$

**Proof** The overvaluing system  $M(A_a(k, x(k)))$ , associated with the vectorial norm  $p(z(k))$  is defined (Appendix), in this case, by the following equation

$$z(k+1) = M(A_a(k, x(k)))z(k). \quad (14)$$

The process, described by (7), is stabilized by the control law (9), if the matrix  $(I - M(A_a(k, x(k))))$  is an M matrix [28] or if, by application of the stability criterion of Borne and Gentina [7-9], we have

$$\begin{cases} 1 - |a_{a_{ii}}(k, x(k))| > 0, \quad \forall i = 1, \dots, n-1, \\ \det(I - M(A_a(k, x(k)))) > \varepsilon \end{cases} \quad (15)$$

The computation of the first member of the last inequality leads to the following expression

$$\begin{aligned} \det(I - M(A_a(k, x(k)))) = \\ (1 - |a_{a_{nn}}(k, x(k))| - \sum_{i=1}^{n-1} (|a_{a_{in}}(k, x(k))a_{a_{ni}}(k, x(k))| \times (1 - |a_{a_{ii}}(k, x(k))|)^{-1})) \\ \times \prod_{j=1}^{n-1} (1 - |a_{a_{jj}}(k, x(k))|) \end{aligned} \quad (16)$$

and achieves easily the proof of the theorem.



#### 4 Synchronization of Two Identical 3D Generalized Hénon Maps

In this section, we propose a systematic procedure to synchronize two identical third-order generalized Hénon maps. This approach determines a control vector  $u(k)$  which makes the slave system achieve synchronization with the master system [1,14].

##### 4.1 Problem statement of synchronization of two identical Hénon maps

Let us consider the synchronization error between systems (4) and (5) described by

$$e_i(k) = x_{si}(k) - x_{mi}(k), \forall i = 1, 2, 3, \tag{17}$$

$$\begin{cases} e_1(k+1) = -(x_{s2}(k) + x_{m2}(k))e_2(k) - 0.1e_3(k) + u_1(k), \\ e_2(k+1) = e_1(k) + u_2(k), \\ e_3(k+1) = e_2(k) + u_3(k), \end{cases} \tag{18}$$

or, in state space, by

$$e(k+1) = A_s(x(k))e(k) + Bu(k) \tag{19}$$

with

$$A_s(x(k)) = \begin{bmatrix} 0 & -(x_{s2}(k) + x_{m2}(k)) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

and

$$B = I_{3 \times 3}. \tag{21}$$

Figure 4.1 shows the error states between systems (4) and (5) when the control is turned off. It is obvious that the error grows chaotically with time.

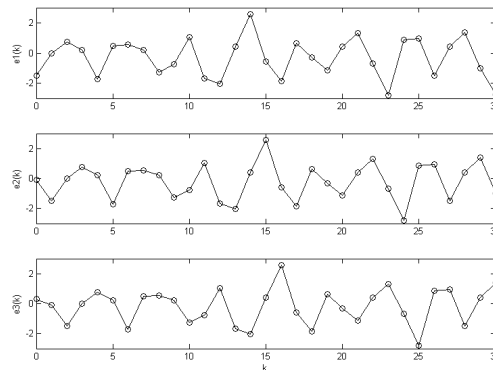


Figure 4.1: Error dynamics for 3D generalized Hénon maps for deactivated controller.

##### 4.2 Chaos synchronization via feedback control law

From the control theory viewpoint, the synchronization of system (18) is equivalent to the stabilization of system (19) by the feedback control law  $u(k)$ . To achieve this goal,

let consider  $u(k)$  introduced in (9) such that:

$$K(x(k)) = \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix}. \quad (22)$$

Then, the error system becomes:

$$e(k+1) = A_{sc}(x(k))e(k) \quad (23)$$

with

$$A_{sc}(x(k)) = A_s(x(k)) - BK(x(k)). \quad (24)$$

$A_{sc}(x(k))$  can be rewritten as

$$A_{sc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & -(x_{s2}(k) + x_{m2}(k)) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \quad (25)$$

A circular permutation on the components of state vector and the choice of correction parameters  $k_{23}$  and  $k_{32}$  constant as follows

$$\begin{cases} 1 - k_{32} = 0, \\ k_{23} = 0, \end{cases} \quad (26)$$

make the matrix  $A_{sa}(x(k))$  in Benrejeb arrow form:

$$A_{sa}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & -(x_{s2}(k) + x_{m2}(k)) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & 0 \\ -k_{31}(x(k)) & 0 & -k_{33}(x(k)) \end{bmatrix}. \quad (27)$$

The system characterized by (27) is asymptotically stable, if the control gains  $k_{ij}(x(k))$ ,  $i, j = 1, 2, 3$ , are chosen so that the following constraints are satisfied:

- i. the nonlinear elements are isolated in one row of the matrix  $A_{sa}(x(k))$ ;
- ii. the diagonal elements of the matrix  $A_{sa}(x(k))$  are such that:

$$\begin{cases} 1 - |k_{33}(x(k))| > 0, \\ 1 - |k_{22}(x(k))| > 0, \end{cases} \quad (28)$$

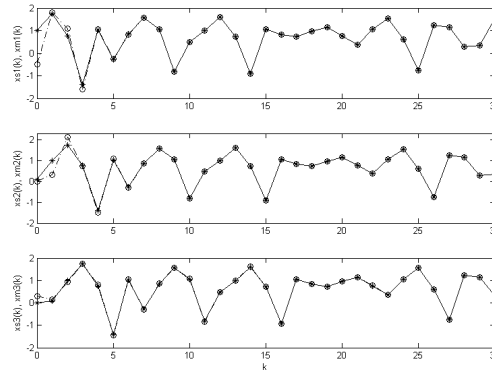
- iii. there exist  $\varepsilon > 0$  such that:

$$\begin{aligned} & 1 - |k_{11}(x(k))| - \frac{|k_{31}(x(k))(0.1 + k_{13}(x(k)))|}{1 - |k_{33}(x(k))|} \\ & - \frac{|(k_{12}(x(k)) + x_{s2}(k) + x_{m2}(k))(1 - k_{21}(x(k)))|}{1 - |k_{22}(x(k))|} > \varepsilon. \end{aligned} \quad (29)$$

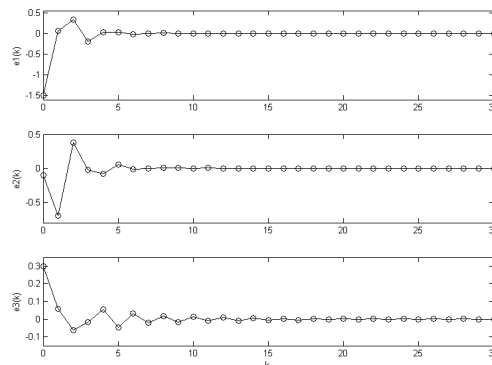
Then, instantaneous gains  $k_{ij}(x(k))$ ,  $\forall i, j = 1, 2, 3$ , satisfying inequalities (28) and (29) such as:

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 - x_{s2}(k) - x_{m2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix} \quad (30)$$

guaranty the synchronization, between systems (4) and (5), as shown in Figures 4.2 and 4.3.



**Figure 4.2:** Time responses of spatiotemporal chaos synchronization master (—) and slave (---) outputs.



**Figure 4.3:** Error dynamics of the the 3D generalized Hénon maps for activated controller.

Figure 4.3 shows that  $e_1(k)$  converges to zero after 4 iterations and  $e_2(k)$  and  $e_3(k)$  after 5 iterations.

## 5 Anti-synchronization of Two Identical 3D Generalized Hénon Maps

In this section, the objective is to design a controller such that the controlled third order generalized Hénon map (5) is anti-synchronous with the third order generalized Hénon map (4), i.e., to make the sum of the oscillating signals converge to zero, when  $k \rightarrow \infty$ .

### 5.1 Problem statement of anti-synchronization of two identical Hénon maps

Let us consider, in the present case, the error vector as

$$e(k + 1) = x_{si}(k) + x_{mi}(k), \forall i = 1, 2, 3 \tag{31}$$

and the error system as

$$\begin{cases} e_1(k+1) = -x_{m2}^2(k) - x_{s2}^2(k) - 0.1e_3(k) + 3.52 + u_1(k), \\ e_2(k+1) = e_1(k) + u_2(k), \\ e_3(k+1) = e_2(k) + u_3(k). \end{cases} \quad (32)$$

The previous equations (32) can be rewritten under the following matrix description

$$e(k+1) = A_{As}(x(k))e(k) + Bu(k) + C_{As}(x(k)) \quad (33)$$

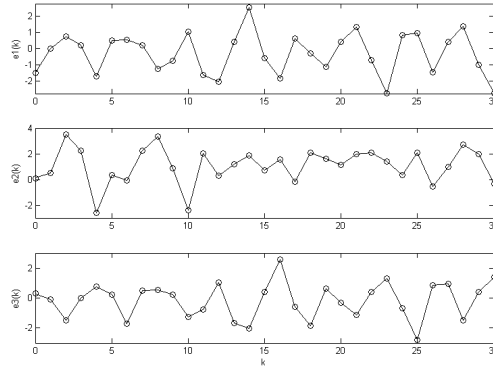
with

$$A_{As}(x(k)) = \begin{bmatrix} 0 & x_{s2}(k) - x_{m2}(k) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (34)$$

and

$$C_{As}(x(k)) = \begin{bmatrix} 3.52 - 2x_{s2}^2(k) \\ 0 \\ 0 \end{bmatrix}, \quad (35)$$

$B = I_{3 \times 3}$ . Figure 5.1 shows the states error between systems (4) and (5) when the control is turned off. It is obvious that the error grow chaotically with time.



**Figure 5.1:** Error dynamics of the 3D generalized Hénon maps for deactivated controller.

## 5.2 Anti-synchronization using state feedback control law

To achieve the property of anti-synchronization between the identical Hénon maps (4) and (5) and by referring to the hypothesis mentioned in the theorem announced in Section 3, let us define the active control functions as follows:

$$u_i(k) = -f_i(x(k)) - \sum_{j=1}^3 k_{ij}(x(k))e_j(k), \quad \forall i = 1, 2, 3, \quad (36)$$

$$u(k) = - \begin{bmatrix} 3.52 - 2x_{s2}^2(k) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix} e(k). \quad (37)$$

Hence, the error system (32) becomes:

$$e(k + 1) = A_{Asc}(x(k))e(k) \tag{38}$$

with

$$A_{Asc}(x(k)) = A_{As}(x(k)) - BK(x(k)). \tag{39}$$

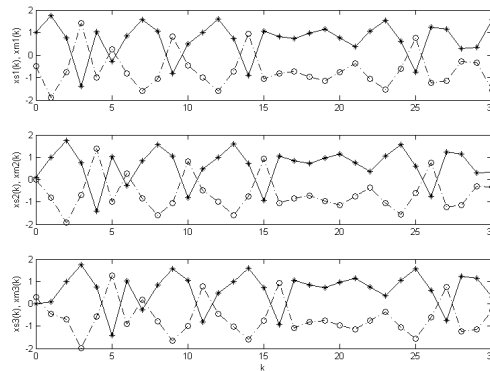
$A_{Asc}(x(k))$  can be rewritten as

$$A_{Asc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & x_{s2}(k) - x_{m2}(k) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \tag{40}$$

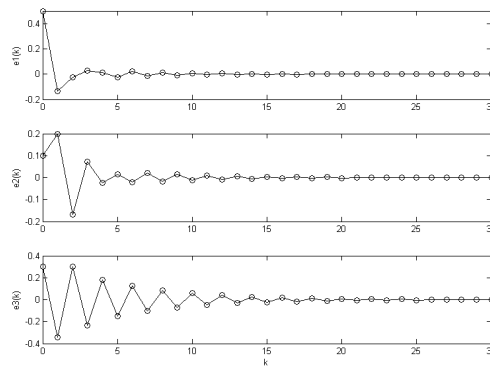
Proceeding as before, we make the appropriate choice of the instantaneous control gains  $k_{23}$  and  $k_{32}$  as shown in equalities (26). Then, with the gains  $k_{ij}(x(k))$ ,  $\forall i, j = 1, 2, 3$ , satisfying inequalities (12) and (13) of the theorem announced in Section 3 such as

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 - x_{m2}(k) + x_{s2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix} \tag{41}$$

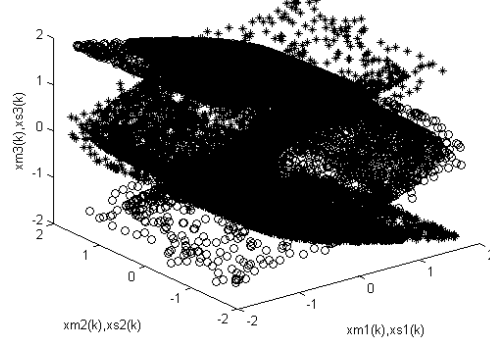
the system (31) converges and hence, the anti-synchronization of (4) and (5) is realized as shown in Figures 5.2, 5.3 and 5.4.



**Figure 5.2:** Time responses of spatiotemporal chaos anti-synchronization of master (—) and slave (---) outputs.



**Figure 5.3:** Error dynamics of the 3D generalized Hénon for activated controller.



**Figure 5.4:** Hyperchaotic attractor of system (4)( $o$ ) and (5)( $*$ ).

Figure 5.3 shows the time response of the anti-synchronization errors, one can observe that  $e_1(k)$ ,  $e_2(k)$  and  $e_3(k)$  converges to zero respectively in 2, 4 and 11 iterations. Figure 5.4 depicts the projection of the anti-synchronized attractors onto the  $x_{mi}(k)$  and  $x_{si}(k)$ ,  $\forall i = 1, 2, 3$  hyperplane, where the state vectors of the master and slave systems evolve in the opposite directions.

## 6 Hybrid Synchronization of Two Identical 3D Generalized Hénon Maps

In this section, we focus on the problem of hybrid synchronization process of two identical chaotic Hénon maps.

### 6.1 Problem statement of hybrid synchronization of two identical Hénon maps

The error vector defined as

$$\begin{cases} e_1(k+1) = x_{s1}(k) - x_{m1}(k), \\ e_2(k+1) = x_{s2}(k) + x_{m2}(k), \\ e_3(k+1) = x_{s3}(k) - x_{m3}(k), \end{cases} \quad (42)$$

leads to the following error system

$$\begin{cases} e_1(k+1) = (x_{m2}(k) - x_{s2}(k))e_2(k) - 0.1e_3(k) + u_1(k), \\ e_2(k+1) = e_1(k) + 2x_{m1}(k) + u_2(k), \\ e_3(k+1) = e_2(k) - 2x_{m2}(k) + u_3(k), \end{cases} \quad (43)$$

The previous equations (43) can be rewritten under the following matrix form

$$e(k+1) = A_{Hs}(x(k))e(k) + Bu(k) + C_{Hs}(x(k)) \quad (44)$$

with

$$A_{Hs}(x(k)) = \begin{bmatrix} 0 & x_{m2}(k) - x_{s2}(k) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_{Hs}(x(k)) = \begin{bmatrix} 0 \\ 2x_{m1}(k) \\ -2x_{m2}(k) \end{bmatrix}, \quad (45)$$

$B = I_{3 \times 3}$ . Figure 6.1 shows the error dynamics when the control is turned off. One can observe that errors grow chaotically with time.

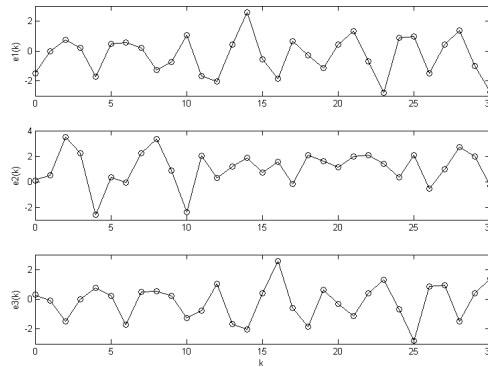


Figure 6.1: Error dynamics of the 3D generalized Hénon maps for deactivated controller.

### 6.2 Hybrid synchronization via state feedback control law

We seek, as before, for an error system stabilizing control law:

$$u_i(k) = -f t_i(x(k)) - \sum_{j=1}^3 k_{ij}(x(k))e_j(k), \quad \forall i = 1, 2, 3, \quad (46)$$

$$u(k) = - \begin{bmatrix} 0 \\ 2x_{m1}(k) \\ -2x_{m2}(k) \end{bmatrix} - \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix} e(k). \quad (47)$$

It comes to the following error dynamical system

$$e(k+1) = A_{Hsc}(x(k))e(k) \quad (48)$$

with:

$$A_{Hsc}(x(k)) = A_{Hs}(x(k)) - BK(x(k)). \quad (49)$$

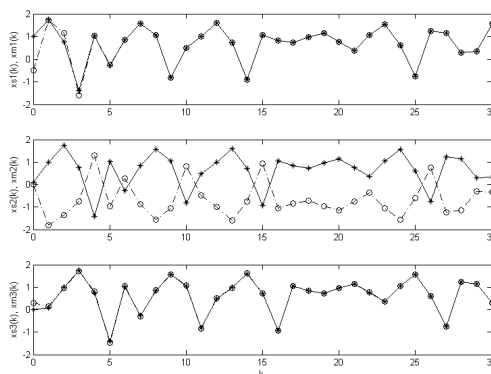
$A_{Hsc}(x(k))$  can be rewritten as

$$A_{Hsc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & x_{m2}(k) - x_{s2}(k) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \quad (50)$$

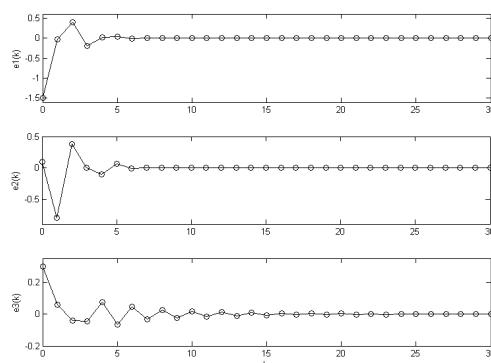
These feedback laws stabilize system (43).  $e_1(k)$ ,  $e_2(k)$  and  $e_3(k)$  converging to zero as time tends to infinity, imply that the hybrid synchronization of the two identical Hénon map systems (4) and (5) is obtained. To achieve this goal, the instantaneous gain matrix  $k_{ij}(x(k))$ ,  $\forall i, j = 1, 2, 3$ , have to satisfy equalities (26) for system description and inequalities (12) and (13) of the theorem announced in Section 3 for stability study such as

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 + x_{m2}(k) - x_{s2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix}. \quad (51)$$

The guaranteed hybrid synchronization is shown in Figures 6.2 and 6.3.



**Figure 6.2:** Time responses of spatiotemporal chaos hybrid synchronization master (—) and slave (---) outputs.



**Figure 6.3:** Error dynamics of the 3D generalized Hénon map for activated controller.

One can observe, in Figure 6.3 that  $e_1(k)$ ,  $e_2(k)$  and  $e_3(k)$  converge to zero respectively after 4, 6 and 5 iterations.

## 7 Conclusion

Stability and stabilisability analysis of discrete-time chaotic systems approaches leading to suitable stabilization conditions is proposed, in this paper, for synchronization studies using the practical stability criterion of Borne and Gentina associated with the particular matrix description, namely the Benrejeb arrow form matrix. Numerical simulations illustrate the efficiency of above stabilization conditions for synchronization studies of two identical Hénon maps. Obtained results can be applied to secure communication and message encoding.



### 8 Appendix

*Borne-Gentina practical stability criterion [7-9]:*

Let us consider the nonlinear discrete-time system described in the state space by

$$x(k + 1) = A(k, x(k))x(k), \tag{52}$$

where  $A(k, x(k))$  is a  $n \times n$  matrix,  $A(k, x(k)) = \{a_{ij}(k, x(k))\}$  and  $x(k) = [x_1(k) \dots x_n(k)]^T \in R^n$  is the state vector. Consider the overvaluing matrix  $M(A(k, x(k)))$ , associated with the vectorial norm  $p(z(k)) = [|z_1(k)| \dots |z_n(k)|]^T$ ,  $z(k) = [z_1(k) \dots z_n(k)]^T$ , such that

$$M(A(k, x(k))) : \{a_{ij}^*(k, x(k)) = |a_{ij}(k, x(k))|, \forall i, j = 1, \dots, n\}. \tag{53}$$

If non-constant elements are isolated in only one row of the overvaluing matrix  $M(A(k, x(k)))$ , asymptotic stability is ensured if all the successive principal minors of the matrix  $(I - M(A(k, x(k))))$  are positive.

Thus, the stability conditions of the initial system (53) are the following

$$1 - a_{11}^* \geq \varepsilon > 0, \quad \left| \begin{array}{cc} 1 - a_{11}^* & -a_{12}^* \\ -a_{21}^* & 1 - a_{22}^* \end{array} \right| \geq \varepsilon > 0, \dots, \tag{54}$$

$$\left| \begin{array}{cccc} 1 - a_{11}^* & -a_{12}^* & \dots & -a_{1n}^* \\ -a_{21}^* & 1 - a_{22}^* & \dots & -a_{2n}^* \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1}^* & -a_{n2}^* & \dots & 1 - a_{nn}^* \end{array} \right| \geq \varepsilon > 0 \quad \forall (k, x(k)).$$

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# Existence of Positive Solutions for the $p$ -Laplacian with Nonlinear Boundary Conditions

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**Abstract:** In this paper, we consider a class of nonlinear elliptic problem with nonlinear boundary condition. The existence of positive solutions are established by sub-supersolution method and the Mountain Pass Lemma.

**Keywords:**  $p$ -Laplacian equations; sub-supersolution; Mountain Pass Lemma; nonlinear boundary condition; positive solutions.

**Mathematics Subject Classification (2010):** 35J55.

## 1 Introduction

In this paper, we are concerned with the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f(x, u), & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = g(x, u), & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N (N \geq 3)$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplacian with  $p > 1$  and  $\frac{\partial}{\partial \nu}$  is the out normal derivative.

Recently, Afrouzi and Alizadeh [1] considered  $p$ -Laplacian equations with a nonlinear boundary condition, they developed a quasilinearization method in order to construct an iterative scheme that converges to a solution. They extended the results of [2] with  $p \neq 2$ . When  $p = 2$ , Song, Wang and Zhao [3] considered problem (1). By the sub-supersolution method, the existence of a positive solution was established. In [4], they

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presented necessary and sufficient conditions of existence for positive solutions of the system with  $p$ -Laplacian. For other nonlinear boundary conditions problems, we cite [5–7]. In [8–11], they offered some applications in physics and engineering.

In this paper, we consider a class of nonlinear elliptic problems with nonlinear boundary condition (1). The existence of positive solutions are established by sub-supersolution method and the Mountain Pass Lemma.

The precise assumptions on the source terms  $f$  and  $g$  are as follows:

(C<sub>1</sub>) For all  $s \geq 0$ , there exist some nonnegative constants  $A_1, A_2, B_1$  and  $B_2$  such that

$$\begin{aligned} 0 \leq f(x, s) &\leq A_1 s^{q_1-1} + A_2, & \text{a.e. in } \Omega, \\ 0 \leq g(x, s) &\leq B_1 s^{q_2-1} + B_2, & \text{a.e. on } \partial\Omega, \end{aligned}$$

where  $2 < p < q_1 < 2^* := \frac{2N}{N-2}$  and  $2 < p < q_2 < \bar{2}^* = \frac{2(N-1)}{N-2}$ ;

(C<sub>2</sub>) The function  $x \mapsto f(x, 0) + g(x, 0)$  is not identically zero;

(C<sub>3</sub>) For all  $s \in \mathbb{R}$ , the functions  $f(\cdot, s), g(\cdot, s) : \bar{\Omega} \rightarrow \mathbb{R}$  are continuous and for every  $x \in \bar{\Omega}$ , the functions  $f(x, \cdot), g(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$  are local Lipschitz continuous.

## 2 Preliminary Lemmas

Let  $W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$  with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left( \int_{\Omega} |\nabla u|^p + |u|^p dx \right)^{\frac{1}{p}},$$

then  $W^{1,p}(\Omega)$  is a Banach space.

Now, we define the concepts of sub-solution and super-solution. We say that  $u \in W^{1,p}(\Omega)$  is a weak sub-solution (weak super-solution) of problem (1) if it satisfies

$$\begin{cases} \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} u v dx \leq (\geq) \int_{\Omega} f(x, u) v dx, \\ \int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} v \leq (\geq) \int_{\partial\Omega} g(x, u) v d\sigma, \end{cases}$$

for all  $v \in W^{1,p}(\Omega)$  with  $v \geq 0$ .

We give the following lemmas which are similar to [1], so we omit the proof here.

**Lemma 2.1** *Assume that  $\lambda > 0, \mu > 0$  and  $u \in W^{2,p}(\Omega)$  satisfies*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u \geq 0, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \mu |u|^{p-2} u \geq 0, & \text{on } \partial\Omega. \end{cases}$$

*Then  $u \geq 0$ .*

**Lemma 2.2** *Assume that  $\xi \in L^p(\Omega)$  and  $\zeta \in L^p(\partial\Omega)$ . Then, for any  $\lambda, \mu > 0$  the Robin problem:*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = \xi, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \mu |u|^{p-2} u = \zeta, & \text{on } \partial\Omega \end{cases}$$

*admits a unique solution  $u \in W^{2,p}(\Omega)$ .*

**Lemma 2.3** *Let  $\lambda, \mu > 0$ ,  $\xi \in L^p(\Omega)$  and  $\zeta \in L^p(\partial\Omega)$ . Then, there exists a constant  $C$  such that if  $u$  is a weak solution of*

$$\begin{cases} -\Delta_p u + \lambda|u|^{p-2}u = \xi, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \mu|u|^{p-2}u = \zeta, & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|u\|_{W^{1,p}(\Omega)} \leq C [\|\xi\|_{L^p(\Omega)} + \|\zeta\|_{L^p(\partial\Omega)}].$$

**Remark 2.1** By the compactness of the imbedding  $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$  and the result of Lemma 2.3, we know that the operator  $T : L^p(\Omega) \times L^p(\partial\Omega) \rightarrow W^{1,p}(\Omega)$  given by  $F(\xi, \zeta) = u$  is compact.

In order to obtain the super-solution of problem (1), we use the following Mountain Pass Lemma.

**Lemma 2.4** [12] *Let  $X$  be a Banach space and let  $I \in C^1(X, \mathbb{R})$  satisfy the Palais-Smale condition. If the following conditions hold:*

- (I)  $I(0) = 0$ ;
- (II) *there exist constants  $r, a > 0$  such that  $I(u) \geq a$ , if  $\|u\| = r$ ;*
- (III) *there exists an element  $\theta \in X$  with  $\|\theta\| > r$ ,  $I(\theta) \leq 0$ .*

Define  $\Gamma := \{g \in C([0, 1], X); g(0) = 0, g(1) = \theta\}$ . Then

$$c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I[g(t)]$$

is a critical value of  $I$ .

### 3 Main Results

Our main results are as follows:

**Theorem 3.1** *Let conditions  $(C_1)$ - $(C_3)$  be satisfied. Then problem (1) has one positive solution  $u$  for  $A_2$  and  $B_2$  small enough.*

**Proof** Firstly, from condition  $(C_1)$ , we know that 0 is a subsolution of problem (1), and 0 is not a solution of problem (1) by condition  $(C_2)$ . In order to use sub-supersolution method, we need a positive supersolution which comes from the Mountain Pass Lemma. Now, we consider the following problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = A_1 u^{q_1-1} + A_2, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = B_1 u^{q_2-1} + B_2, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

the functional associated with the problem (2) is

$$J(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p + |u|^p) dx - \frac{A_1}{q_1} \int_{\Omega} u^{q_1} dx - A_2 \int_{\Omega} u dx - \frac{B_1}{q_2} \int_{\partial\Omega} u^{q_2} d\sigma - B_2 \int_{\partial\Omega} u d\sigma.$$

We claim that  $J$  satisfies the  $(PS)_c$  condition. In fact, let  $\{u_n\}$  be a Palais-Smale sequence in  $W^{1,p}(\Omega)$ , that is  $J(u_n) \rightarrow c$  and  $J'(u_n) \rightarrow 0$ , then we have

$$\begin{aligned} J(u_n) &= \frac{1}{q} \langle J'(u_n), u_n \rangle \\ &= \left( \frac{1}{p} - \frac{1}{q} \right) \|u_n\|_{W^{1,p}(\Omega)}^p - \left( \frac{1}{q_1} - \frac{1}{q} \right) A_1 \int_{\Omega} u_n^{q_1} dx - \left( 1 - \frac{1}{q} \right) A_2 \int_{\Omega} u_n dx \\ &\quad - \left( \frac{1}{q_2} - \frac{1}{q} \right) B_1 \int_{\partial\Omega} u_n^{q_2} d\sigma - \left( 1 - \frac{1}{q} \right) B_2 \int_{\partial\Omega} u_n d\sigma \\ &= c + o(1), \end{aligned}$$

where  $q := \min\{q_1, q_2\}$ ,  $A_1, A_2, B_1, B_2 > 0$ . By the Sobolev embedding theorem and Sobolev trace embedding theorem, we can choose a constant  $\tau > 0$  such that

$$c + 1 + \tau \|u_n\|_{W^{1,p}(\Omega)} \geq \left( \frac{1}{p} - \frac{1}{q} \right) \|u_n\|_{W^{1,p}(\Omega)}^p.$$

Hence  $\{u_n\}$  is bounded in  $W^{1,p}(\Omega)$ . So  $\{u_n\}$  admits a weakly convergent subsequence. Since all the growths in problem (2) are subcritical, by the standard argument we deduce that  $\{u_n\}$  admits a strongly convergence subsequence.

Next, we verify the conditions of Mountain Pass Lemma. By the Hölder's inequality, the Sobolev embedding theorem and Sobolev trace embedding theorem, we have

$$\begin{aligned} \int_{\Omega} |u|^{q_1} dx &= \|u\|_{L^{q_1}(\Omega)}^{q_1} \leq C_1 \|u\|_{W^{1,p}(\Omega)}^{q_1}, \quad \int_{\partial\Omega} |u|^{q_2} d\sigma = \|u\|_{L^{q_2}(\partial\Omega)}^{q_2} \leq C_2 \|u\|_{W^{1,p}(\Omega)}^{q_2}, \\ \int_{\Omega} |u| dx &\leq C_3 \|u\|_{W^{1,p}(\Omega)}, \quad \int_{\partial\Omega} |u| d\sigma \leq C_4 \|u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} J(u) &\geq \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - C_1 \|u\|_{W^{1,p}(\Omega)}^{q_1} - C_2 \|u\|_{W^{1,p}(\Omega)}^{q_2} \\ &\quad - C_3 A_2 \|u\|_{W^{1,p}(\Omega)} - C_4 B_2 \|u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Assume that  $\|u\|_{W^{1,p}(\Omega)} < 1$ , then we have

$$J(u) \geq \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - C_5 \|u\|_{W^{1,p}(\Omega)}^q - C_3 A_2 \|u\|_{W^{1,p}(\Omega)} - C_4 B_2 \|u\|_{W^{1,p}(\Omega)}.$$

Consider the function  $g(s) := \frac{1}{p}s^p - C_5s^q - C_6\rho s$ , if we take  $s = s_0 = (2pC_6\rho)^{\frac{1}{p-1}}$  such that  $g(s_0) = a = C_7\rho^{\frac{p}{p-1}} - C_8\rho^{\frac{q}{p-1}} > 0$ , since  $\frac{q}{p-1} > \frac{p}{p-1} > 1$ ,  $\rho$  is small enough. This fact implies that  $J(u) \geq a > 0$  for all  $\|u\|_{W^{1,p}(\Omega)} = s_0$  and  $A_2, B_2$  small enough.

Let  $\psi \in C_0^\infty(\Omega)$  with  $\psi > 0$  on  $\Omega$ . Then for any  $t \geq 0$ , we have

$$\begin{aligned} J(t\psi) &= \frac{t^p}{p} \int_{\Omega} (|\nabla\psi|^p + |\psi|^p) dx - \frac{A_1 t^{q_1}}{q_1} \int_{\Omega} \psi^{q_1} dx - A_2 t \int_{\Omega} \psi dx \\ &\quad - \frac{B_1 t^{q_2}}{q_2} \int_{\partial\Omega} \psi^{q_2} d\sigma - B_2 t \int_{\partial\Omega} \psi d\sigma \rightarrow -\infty \quad \text{as } t \rightarrow +\infty, \end{aligned}$$

since  $p < p_1, p_2$ . Then we take  $\psi_0 = k\psi$ , with  $k$  large enough, we have  $\|\psi_0\|_{W^{1,p}(\Omega)} > s_0$  and  $J(\psi_0) < a$ . Thus we have a solution  $\beta(x)$  of the problem (1) by the Mountain Pass



Lemma. It is easy to see by using standard elliptic regularity that  $\beta(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ , and  $\beta(x)$  is a positive supersolution of problem (1) by condition  $(C_1)$ .

Denote  $N := \max_{x \in \bar{\Omega}} \beta(x)$ , by condition  $(C_3)$ , there exists a constant  $\lambda > 0$  such that  $|f(x, s_1) - f(x, s_2)| \leq \lambda|s_1 - s_2|$ , for all  $(x, s_1), (x, s_2) \in \bar{\Omega} \times [0, N]$ . So  $f(x, s) + \lambda s$  is increasing on  $s \in [0, N]$ . We choose  $\mu$  in the same way, and define the function  $Q : \bar{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$Q(x, u) = \begin{cases} 0, & \text{if } u < 0, \\ u, & \text{if } 0 \leq u \leq \beta(x), \\ \beta(x), & \text{if } u > \beta(x). \end{cases}$$

Consider the compact operator  $T : C(\bar{\Omega}) \rightarrow C(\bar{\Omega})$  given by  $Tv = u$ , where  $u$  is the unique solution of the Robin problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda|u|^{p-2}u = f(x, Q(x, v)) + \lambda Q(x, v), & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \mu|u|^{p-2}u = g(x, Q(x, v)) + \mu Q(x, v), & \text{on } \partial\Omega. \end{cases}$$

Let  $v \leq u$ , since  $f(x, s) + \lambda s$  is increasing on  $s \in [0, N]$ , so we have

$$\begin{aligned} & -\Delta_p(Tu) + |Tu|^{p-2}(Tu) + \lambda|Tu|^{p-2}(Tu) \\ &= f(x, Q(x, u)) + \lambda Q(x, u) \geq f(x, Q(x, v)) + \lambda Q(x, v) \\ &= -\Delta_p(Tv) + |Tv|^{p-2}(Tv) + \lambda|Tv|^{p-2}(Tv), \quad \text{in } \Omega. \end{aligned}$$

On the other hand, by nonlinear boundary condition, we have

$$\begin{aligned} & |\nabla(Tu)|^{p-2} \frac{\partial(Tu)}{\partial \nu} + \mu|(Tu)|^{p-2}(Tu) \\ &= g(x, Q(x, u)) + \mu Q(x, u) \geq g(x, Q(x, v)) + \mu Q(x, v) \\ &= |\nabla(Tv)|^{p-2} \frac{\partial(Tv)}{\partial \nu} + \mu|(Tv)|^{p-2}(Tv), \quad \text{on } \partial\Omega. \end{aligned}$$

From the maximum principle, it follows that  $Tu \geq Tv$ . This fact implies that  $T$  is increasing.

We claim that  $T : \langle 0, \beta(x) \rangle \rightarrow \langle 0, \beta(x) \rangle$ , where  $\langle 0, \beta(x) \rangle = \{u \in C(\bar{\Omega}) : 0 \leq u(x) \leq \beta(x)\}$ ,  $\beta(x)$  is the supersolution of problem (1). In fact, from the definition of supersolution, we have

$$\begin{aligned} & -\Delta_p \beta + |\beta|^{p-2} \beta + \lambda|\beta|^{p-2} \beta \\ & \geq f(x, \beta) + \lambda Q(x, \beta) \geq f(x, Q(x, \beta)) + \lambda Q(x, \beta) \\ & = -\Delta_p(T\beta) + |T\beta|^{p-2}(T\beta) + \lambda|T\beta|^{p-2}(T\beta), \quad \text{in } \Omega. \end{aligned}$$

In a similar way, we have

$$|\nabla \beta|^{p-2} \frac{\partial \beta}{\partial \nu} + \mu|\beta|^{p-2} \beta \geq |\nabla(T\beta)|^{p-2} \frac{\partial(T\beta)}{\partial \nu} + \mu|(T\beta)|^{p-2}(T\beta), \quad \text{on } \partial\Omega.$$

From the maximum principle, we have  $T\beta \leq \beta$ . So  $T : \langle 0, \beta(x) \rangle \rightarrow \langle 0, \beta(x) \rangle$ . Notice that the positive cone  $K$  of  $C(\bar{\Omega})$  is regular and the interior of  $K$  is not empty, therefore  $T$  has a fixed point  $u$  satisfying  $0 \leq u \leq \beta(x)$  and hence  $u$  is a positive solution of problem (1).

**Theorem 3.2** *Assume that  $f(x, s), g(x, s)$  are nonnegative continuous functions in  $\overline{\Omega} \times \mathbb{R}$ . Let condition  $(C_2)$  hold and problem (1) have a continuous weak supersolution. Then problem (1) has a positive solution.*

**Proof** Firstly, we know that 0 is a subsolution of problem (1), let  $\beta(x)$  be a supersolution of problem (1). For a variational approach, the functional associated with problem (1) is

$$J(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p + |u|^p) dx - \int_{\Omega} F(x, u) dx - \int_{\partial\Omega} G(x, u) d\sigma,$$

where  $F(x, u) = \int_0^u f(x, z) dz$ ,  $G(x, u) = \int_0^u g(x, z) d\sigma$  and  $d\sigma$  is the surface measure.

Let  $w \in W^{1,p}(\Omega)$  and define the function  $Q : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$Q(x, w) = \begin{cases} 0, & \text{if } w < 0, \\ w, & \text{if } 0 \leq w \leq \beta(x), \\ \beta(x), & \text{if } w > \beta(x). \end{cases} \quad (3)$$

Now we consider

$$\begin{aligned} I(w) &= \frac{1}{p} \int_{\Omega} (|\nabla w|^p + |w|^p) dx - \int_{\Omega} F(x, Q(x, w(x))) dx - \int_{\partial\Omega} G(x, Q(x, w(x))) d\sigma \\ &= \frac{1}{p} \|w\|_{W^{1,p}(\Omega)}^p - \left( \int_{\Omega} F(x, Q(x, w(x))) dx + \int_{\partial\Omega} G(x, Q(x, w(x))) d\sigma \right) \\ &= I_1(w) - I_2(w). \end{aligned}$$

We note that  $I_1(w)$  is weakly lower semi-continuous. In the following we prove that  $I_2(w)$  is weakly continuous. Let  $H(w) := \int_{\Omega} F(x, Q(x, w(x))) dx$  and  $w_n \rightharpoonup w$  in  $W^{1,p}(\Omega)$ , then we have  $w_n \rightarrow w$  a.e. in  $\Omega$  and  $Q(x, w_n(x)) \rightarrow Q(x, w(x))$ . Since

$$|F(x, Q(x, w_n(x)))| \leq \sup_{0 \leq w(x) \leq \beta(x)} |F(x, w(x))| = N.$$

So, by the Dominated Convergence Theorem, we get

$$\lim_{n \rightarrow \infty} H(w_n) = \lim_{n \rightarrow \infty} \int_{\Omega} F(x, Q(x, w_n(x))) dx = \int_{\Omega} \lim_{n \rightarrow \infty} F(x, Q(x, w_n(x))) dx = H(w),$$

so  $I_2(w)$  is weakly continuous. Thus  $I(w)$  is weakly lower semi-continuous. Since  $f(x, s), g(x, s)$  are continuous and  $\beta(x)$  is bounded in  $\overline{\Omega}$ , we know that  $H(w)$  is bounded and we have that  $I(w) \rightarrow +\infty$  as  $\|w\|_{W^{1,p}(\Omega)} \rightarrow \infty$ , this implies that  $I(w)$  is a coercive functional, therefore there exists  $w_0 \in W^{1,p}(\Omega)$  such that  $I'(w_0) = 0$ . By (3), we have  $0 \leq w_0 \leq \beta(x)$ . Thus  $I'(w_0) = 0$ . Notice that 0 is not a solution of problem (1), so  $w_0$  is a positive solutions of problem (1).

For the special case of problem (1):

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = A_1 u^{q_1-1} + A_2, & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \quad (4)$$

we can also obtain the nonexistence results.

**Theorem 3.3** *There exists a positive constant  $D = D(A_1, A_2, q_1)$  such that the problem (4) has no positive solution for all  $A_2 > D$ .*

**Proof** Let  $A := \{A_2 > 0 : \text{the problem (4) has a positive solution}\}$ . Theorem 3.1 implies that  $A \neq \emptyset$ . So we can define  $D := \sup A$ . We claim that  $0 < D < +\infty$ . Obviously  $D > 0$ . Let

$$A^* = \max_{s>0} \{s^{p-1} - A_1 s^{q_1-1}\} < +\infty. \quad (5)$$

If  $A_2 \in A$ , then we have

$$\int_{\Omega} u^{p-1} dx = A_1 \int_{\Omega} u^{q_1-1} dx + A_2 |\Omega|.$$

From (5), we have  $A_2 \leq A^*$ . So  $0 < D \leq A^* < +\infty$ .

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# Global Stability Given Local Stability Via Curvature of Some Nonautonomous Differential Equations

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**Abstract:** In this article, the global stability, (given local stability) of a class of nonautonomous differential equations is obtained. The boundedness of the curvature of the trajectories on sets with certain properties is used to determine the stability.

**Keywords:** *stability; nonautonomous differential equation.*

**Mathematics Subject Classification (2010):** 34D23, 37C75.

## 1 Introduction

In this study, we revisit the asymptotic stability of ordinary differential equations via the curvature properties of the trajectories. We have studied the global stability (given the local stability) of the zero solution of a class of non-autonomous linear differential equations of the type

$$x'(t) = A(t)x$$

under certain conditions on the matrix  $A$ . The usual conditions on  $A + A^T$  appear to be restrictive, whereas the arguments via the curvature yield the global stability given the local stability. Apparently, the theory depends on the boundedness of the curvature and a property of the trajectory called *the negative property on compact sets*. The idea of the proof is borrowed from [1], although it deals with only the autonomous systems. In this paper, the stress is on the nonautonomous differential equations.

Section 2 deals with the necessary preliminaries. The main result is stated in Section 3. Examples have been given here for illustration.

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## 2 Preliminaries

Let the curve  $\Gamma$  be represented by the  $C^1$  map  $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ . The curvature at each point  $x = \phi(t)$ , for some  $t \in \mathbb{R}$  on a  $C^1$  curve  $\Gamma$ , where  $\phi'(t) \neq 0$ , is given by

$$\kappa(x) = \frac{\| \|\phi'(t)\|^2 \phi''(t) - \langle \phi'(t), \phi''(t) \rangle \phi'(t) \|}{\|\phi'(t)\|^4}.$$

Let  $K(x)$  be the curvature at a point  $x \in \Gamma$ , where  $\Gamma$  is a  $C^1$  curve given by  $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ . A consequence of the curvature being bounded everywhere along  $\Gamma$  is:

**Proposition 2.1** *Let  $K$  be an upper bound for the curvature on a  $C^1$  curve  $\Gamma$ . If there exist  $t_-, t_0$  and  $t_+ \in \mathbb{R}$  with  $t_- < t_0 < t_+$  such that the tangent at  $\phi(t_0)$  is orthogonal to the tangents at  $\phi(t_-)$  and  $\phi(t_+)$  and  $\langle \phi'(t), \phi'(t_0) \rangle > 0$  for all  $t \in (t_-, t_+)$ , then  $d(\phi(t_-), \phi(t_0)) \geq \frac{\sqrt{2}}{K}$  and  $d(\phi(t_0), \phi(t_+)) \geq \frac{\sqrt{2}}{K}$ . Here,  $d(\phi(t_-), \phi(t_0))$  denotes the euclidean distance between  $\phi(t_-)$  and  $\phi(t_0)$ .*

**Proof** Let  $T_t$  denote the unit tangent vector to the curve  $\Gamma$  at  $\phi(t)$ , and let  $\theta(t)$  be the angle between  $T_{t_0}$  and  $T_t$ .

$$\langle (\phi(t_+) - \phi(t_0)), T_{t_0} \rangle = \int_{t_0}^{t_+} \langle \phi'(t), T_{t_0} \rangle dt = \int_{t_0}^{t_+} \|\phi'(t)\| \cos \theta(t) dt.$$

Let  $s(t)$  denote the arc length along the curve  $\Gamma$ , at time  $t$  from the fixed point  $\phi(t_0)$ . From [2], we know that  $\left| \frac{d\theta}{ds} \right| \leq K$ . Hence,

$$\langle (\phi(t_+) - \phi(t_0)), T_{t_0} \rangle = \int_{s(t_0)}^{s(t_+)} \cos \theta(s) ds \geq \frac{1}{K} \int_{\theta(t_0)}^{\theta(t_+)} \cos \theta d\theta.$$

Since  $T_{t_0}$  and  $T_{t_+}$  are orthogonal to each other,  $\theta(t_0) = 0$  and  $\theta(t_+) = \frac{\pi}{2}$ .

So,  $\langle \phi(t_+) - \phi(t_0), T_{t_0} \rangle \geq \frac{1}{K}$ . By a similar argument,  $\langle \phi(t_+) - \phi(t_0), T_{t_+} \rangle \geq \frac{1}{K}$  or  $d(\phi(t_0), \phi(t_+)) \geq \frac{\sqrt{2}}{K}$ . By a similar argument  $d(\phi(t_0), \phi(t_-)) \geq \frac{\sqrt{2}}{K}$ , thereby proving the proposition.

A consequence of Proposition 2.1 is

**Definition 2.1** Let  $\Omega \subset \mathbb{R}^n$  be a set such that the curvature at each point along any regular curve in  $\Omega$  is bounded above by  $K$ . If for each point  $x_0$  in  $\Omega$  there do not exist points  $x_1$  and  $x_2$  on any regular curve  $\phi$  through  $x_0$ , (where  $\phi(t_0) = x_0$ ,  $\phi(t_1) = x_1$  and  $\phi(t_2) = x_2$ ) with  $\langle \phi'(t_0), \phi'(t_1) \rangle = 0$ ,  $\langle \phi'(t_0), \phi'(t_2) \rangle = 0$  and  $d(x_0, x_1), d(x_0, x_2) \geq \frac{2}{K}$ , then  $\Omega$  is said to have the *negative property*.

We now turn our attention to nonautonomous ordinary differential equations of the type

$$x' = A(t)x, \tag{1}$$

where  $A : [0, \infty) \rightarrow M_n(\mathbb{R})$  is a  $C^1$  matrix, such that  $\lim_{t \rightarrow \infty} A(t)$  exists and  $\lim_{t \rightarrow \infty} A(t) \neq 0$ .

A solution of equation (1) passing through  $x_0$  at time  $t_0$  is denoted by  $x(t, t_0, x_0)$ .

Henceforth, if  $\Omega$  has the negative property, we assume that it is with respect to the solution curves of the concerned differential equation.

The following is a result on the  $\omega$  limit points of trajectories in a set with the negative property.

**Lemma 2.1** *Let  $\Omega \subset \mathbb{R}^n$  have no equilibrium points of equation (1) and have the negative property. Let  $\Gamma = \{\phi(t) : t \in [0, \infty)\}$  be any forward trajectory of equation (1) which is contained entirely in  $\Omega$ . Then, the omega limit set of  $\Gamma$  consists entirely of equilibrium points which lie in the closure of  $\Omega$  but not in  $\Omega$ .*

**Proof** Let  $\bar{x}$  be an omega limit point of  $\Gamma$ . Assume that  $\bar{x}$  is not an equilibrium point. There exists a sequence  $(t_n) \uparrow \infty$  such that  $\phi(t_n) \rightarrow \bar{x}$ . By using the fundamental theorem of calculus, we get

$$\lim_{n \rightarrow \infty} \int_{t_n}^{t_{n+1}} A(t)\phi(t) dt = 0. \tag{2}$$

There exists a  $T \in \mathbb{R}$  such that for all  $t > T$ ,  $\langle A(t)\bar{x}, A(t)\bar{x} \rangle > 0$ . Hence, there exists a  $\delta > 0$ ,  $t_0 \geq T$ , such that  $\langle A(s)y, A(r)z \rangle > 0$ , for all  $s, r > t_0$  and  $y, z \in B(\bar{x}, \delta)$ , i.e., there exists a  $\tau > 0$ , such that  $\forall |t - t_0| < \tau$ ,  $s, r > t_0$  and  $y, z \in B(\bar{x}, \delta)$

$$\langle A(s)x(t, t_0, y), A(r)z \rangle > 0. \tag{3}$$

Fix a  $s > t_0$  and  $y \in B(\bar{x}, \delta)$  and call the vector  $A(s)y$  as  $v$ . From equation (2), we see that

$$\lim_{n \rightarrow \infty} \int_{t_n}^{t_{n+1}} \langle A(t)\phi(t), v \rangle dt = 0. \tag{4}$$

When  $n$  is sufficiently large,  $\phi(t_n) \rightarrow \bar{x}$ . We get from equation (3):

$$\int_{t_n}^{t_n + \tau} \langle A(t)\phi(t), v \rangle dt > 0 \tag{5}$$

For  $n$  sufficiently large, there exists an interval  $(t_n, t_{n+1})$ , where the integrand must change sign, i.e., there exist  $t_{m-}, t_{m+}$  with  $t_{m-} < t_m < t_{m+}$  such that  $\langle A(t_m)\phi(t_m), A(t_{m+})\phi(t_{m+}) \rangle = 0$  and  $\langle A(t_{m-})\phi(t_{m-}), A(t_m)\phi(t_m) \rangle = 0$ .

Let  $x_0 = A(t_m)\phi(t_m)$ ,  $x_1 = A(t_{m-})\phi(t_{m-})$  and  $x_2 = A(t_{m+})\phi(t_{m+})$ . Now,  $\langle \phi'(t_m), \phi'(t_{m+}) \rangle = 0$  and  $\langle \phi'(t_m), \phi'(t_{m-}) \rangle = 0$ . From Proposition 2.1, we conclude that  $d(x_-, x_0)$  and  $d(x_0, x_+) \geq \frac{\sqrt{2}}{K}$ , which implies that  $\Omega$  does not have the negative property, a contradiction. Hence  $\bar{x}$  has to be an equilibrium point. Since  $\bar{x} \notin \Omega$ ,  $\bar{x}$  has to belong to  $\bar{\Omega}$ .

We turn our attention to a few applications. Let us now consider nonautonomous linear differential equations

$$x'(t) = A(t)x + g(x), \tag{6}$$

where  $A : [0, \infty) \rightarrow M_n(\mathbb{R})$  is a  $C^1$  matrix, such that  $\lim_{t \rightarrow \infty} A(t)$  exists with  $\lim_{t \rightarrow \infty} A(t) \neq 0$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $C^1$  function and satisfies the smallness conditions.

On the lines of the proof of Lemma 2.1, we have the following result.

**Corollary 2.1** *Let  $\Omega \subset \mathbb{R}^n$  have the negative property with respect to equation (6). Let  $\Gamma = \{\phi(t) : t \in \mathbb{R}\}$  be any forward trajectory which is contained entirely in  $\Omega$ . Then, the omega limit set of  $\Gamma$  consists entirely of equilibrium points which lie in the closure of  $\Omega$  and not in  $\Omega$ .*

**Remark 2.1** It is interesting to note that, if  $\Omega$  is compact and has the negative property, then every forward trajectory has to leave  $\Omega$ .

From [3], we have the following result for stability.

**Proposition 2.2** *Let us consider equation (1) where  $A(t)$  is a continuous real valued  $n \times n$  matrix on  $[0, \infty)$ . Let  $M(t)$  be the maximum eigenvalue of  $A(t) + A(t)^T$ .*

*If  $\lim_{t \rightarrow \infty} \int_{t_0}^t M(s) ds = -\infty$ , where  $t_0$  is fixed, then every solution of equation (1) tends to zero as  $t \rightarrow \infty$ .*

### 3 Stability

In this section, we study the global stability given the local stability of equations of the type (1) or (6).

**Theorem 3.1** *Let  $D \subset \mathbb{R}^n$  be eventually positively invariant under either of the equations (1) or (6) and  $\Omega \subset D$  be compact, with no equilibrium points of equation (1) and have the negative property. If the solutions of  $D \setminus \Omega$  approach the equilibrium solution asymptotically, then the solutions in  $\Omega$  must also approach the equilibrium solution asymptotically.*

**Proof** Since,  $\Omega$  is compact and has the negative property, from Lemma 2.1 and Corollary 2.1, we know that every forward trajectory in  $\Omega$  must leave  $\Omega$ . As  $\Omega \subset D$ , which is a positively invariant set in  $\mathbb{R}^n$ , the result follows.

We now have a result on global stability given the local stability.

**Corollary 3.1** *Let  $D$  be positively invariant and contain a unique equilibrium  $\bar{x}$  (with respect to either equation (1) or (6) and  $\Omega \subset D$  be compact and have the negative property. If the solutions of  $D \setminus \Omega$  approach  $\bar{x}$  asymptotically, then  $\bar{x}$  is globally stable in  $D$ .*

**Remark 3.1** Consider the ordinary differential equation  $x' = Ax$ , where  $A$  is a stable matrix. Let  $B$  be a closed ball of radius  $\frac{1}{2}$  in  $\mathbb{R}^n$  such for each  $y \in B$ ,  $\|y\| \geq 2$ . We see that in  $B$  the curvature along any solution curve of  $x' = Ax$  is bounded above by 1.  $B$  does not contain the critical point and has the negative property. Therefore, by Theorem 3.1, the zero solution is globally stable.

**Remark 3.2** The autonomous equation  $y' = y(y-1)$  has critical points at  $y = 0$  and  $y = 1$ .  $(-\infty, 1)$  is a positively invariant set under  $y' = y(y-1)$  and  $(-2, -1)$  satisfies the negative property. By Theorem 3.1, we see that the zero solution is stable in  $(-\infty, 1)$ .



**Example 3.1**  $x' = A(t)x$  where  $t > 0$  and  $A(t)$  is

$$\begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{1+t} \end{bmatrix}.$$

The only equilibrium point is  $(0, 0)$ . The maximum eigenvalue,  $\lambda(t)$  of  $A + A^T$  is  $\frac{-2}{1+t}$ .

Since,  $\lim_{s \rightarrow \infty} \int_0^s \lambda(t) dt = 0$ , we see from Proposition 2.2 that we cannot conclude whether the zero solution is stable or not.

However, the set  $\overline{B((1, 0), \frac{1}{2})}$  has the negative property and is compact. Using Corollary 3.1, the zero solution is globally asymptotically stable on  $\mathbb{R}^2$ .

**Example 3.2** Consider  $x' = A(t)x$  where  $t > 0$  and  $A(t)$  is

$$\begin{bmatrix} 0 & -\frac{3}{(1+t)^2} \\ 0 & -\frac{5}{(1+t)} \end{bmatrix}.$$

The only equilibrium point is  $(0, 0)$ . The maximum eigenvalue  $\lambda(t)$  of  $A + A^T$  is  $\frac{-5 + \sqrt{13}}{2(1+t)}$ .  $\lim_{s \rightarrow \infty} \int_0^s \lambda(t) dt = 0$ .

Since, the set  $\overline{B((1, 1), \frac{1}{2})}$  has the negative property and is compact, by Corollary 3.1, the zero solution is globally asymptotically stable on  $\mathbb{R}^2$ .

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