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PERSONAGE IN SCIENCE

Professor A.N. Golubentsev

(On the occasion of his 95th Birthday)

Ya.M. Grigorenko, V.B. Larin and A.A. Martynyuk *

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March 29, 2011 marked the 95th Birthday of the well-known scientist in the area of general mechanics, Professor A.N. Golubentsev who was born to the family of a railway worker in Raskatikha railway station of Tomsk railway (Russia).

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He started working at a very early age. He became a mechanic's assistant at a locomotive depot in the Topki railway station of Tomsk railway when he was just 13. After finishing a factory-and-workshop training course he entered the Kemerov Mining Secondary School. Later on, he graduated with distinction from Tomsk Industrial Institute.

In 1941 Golubentsev married Valentina Grigorievna Pozhidayeva and lived with her until his death in 1971. The Golubentsev had two children — daughter Eleonora (1942 – 2008) and son Aleksander (1945 – 2008), granddaughter Helen and three grand grandchildren.

From 1933 to 1953, Golubentsev was employed at several enterprises of coil mining industry in the former USSR. He was a master mechanic and a chief mechanic of coal mining enterprises in Kuzbass (Russia) and Donbass (Ukraine).

In 1953 he defended his candidate thesis (Ph.D.) on the problems of electric drive of a winder and in 1956 he received his doctoral degree (Habilitation Degree) defending his thesis "Dynamics of machines with elastic constraints". From 1955 to 1958 Golubentsev was the Head of the Department at Gostekhnika of USSR (Moscow) and then he became a deputy chairman of the State Scientific and Technical Committee of the Council of Ministers of Ukrainian SSR.

Starting from 1959, A.N. Golubentsev sank into scientific and scientific-organizational activities at the Institute of Building Mechanics of the Academy of Sciences of UkrSSR (now the S.P.Timoshenko Institute of Mechanics of the National Academy of Sciences of Ukraine). When he was a deputy director of the Institute, he proposed a program reorganizing areas of research conducted at the Institute. As a result, the Institute of Structural Mechanics of AN of UkrSSR was renamed the "Institute of Mechanics of AN of UkrSSR" and the research fellows of the Institute became involved in modern analysis of continuum mechanics, mechanics of composite materials and general mechanics with the applications to rocket science and other applied areas. In view of the significance of new investigations and due to Golubentsev's efforts the research team of the Institute was granted a BESM-2M computer which was the first one in the Ukrainian SSR. This fact was of great importance for the fulfillment of current tasks of national economy and the defense industry. In 1959 – 1965, Golubentsev chaired the Department of Motion Dynamics and Stability of the Institute. He upheld the development of new perspective areas, but he was not always supported by a number of scholars who adhered to the traditional directions of investigations. In 1965, the Department of Motion Dynamics and Stability was integrated into the Institute of Hydrodynamics of AN of UkrSSR at which the problem of motion stability of ekranoplans, i.e. ram wing surface effect vehicles serving for the military, was tackled at that time. Later on, Golubentsev, together with the corresponding member of the Academy of Sciences of UkrSSR S.N. Kozhevnikov founded the Sector of Mechanics of Machines at the Institute of Geotechnical Mechanics of AN UkrSSR.

Golubentsev's intense scientific research gave rise to the development of the theory of transitional processes in machines with elastic links and yielded new significant results on optimization of processes in the parameter space of the machines. These results have been presented in a series of his monographs [1 – 4].

Golubentsev's keen sense of responsibility for all the matters he dealt with, including his analysis of the model of the socialist system at that time, motivated him to make an attempt to improve the economy in the former USSR on a strictly mathematical basis. It is clear that such an unveiled intention was fruitless, since many advocates of a more conservative model of socialism did not share his enthusiasm.

Nevertheless, these attempts resulted in development of a new direction in mathematical economics — the econo-thermodynamics. The prime postulate of this theory is the well-known statement of Karl Marx that the economic epochs do not differ by what is produced but by how (i.e. with what means) it is produced. The theory of econo-thermodynamics developed by Golubentsev is laid out in his monograph [5].

Alongside his research activity, Golubentsev spent much time nurturing his postgraduate students. He produced a total of 18 people with Ph.D. and 2 people with Habilitation degrees in the areas of mechanics of machines and theoretical mechanics.

Golubentsev's merits were recognized with seven prestigious Government awards.

Golubentsev suddenly died on October 11, 1971 due to a heart failure.

He always upheld new ideas of important problems in mechanics and encouraged young researchers striving to develop appropriate methods of their solutions. His talks presented at seminars and conferences were always well-spoken, while sometimes irrespective of ranks and positions of the people he referred to. In these cases he was guided by only his scientific conscious and wisdom.

Golubentsev was a good-hearted and benevolent person. He was a true patriot of his country being totally dedicated to its service. He made a significant contribution to the development of his country. His scientific discoveries will always belong to the treasures of world science and remain in demand by young researches in mechanics and mathematical economics.

The list of principle publications by A.N. Golubentsev:

1. A.N. Golubentsev. *Start of Asynchronous Engine of a Winder.* Kiev, GITTL UkrSSR, 1959.
2. A.N. Golubentsev. *Dynamics of Transient Processes in Machines with Many Masses.* Moscow, GNTI, 1959.
3. A.N. Golubentsev. *Integral Methods in Dynamics.* Kiev, Tekhnika, 1967.
4. A.N. Golubentsev. *Generalized Input in Dynamics.* Kiev, Tekhnika, 1971.
5. A.N. Golubentsev. *Thermodynamics of Production Process.* Kiev, Tekhnika, 1969.

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Weak Solutions for Boundary-Value Problems with Nonlinear Fractional Differential Inclusions

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Abstract: This paper deals with the existence of solutions, under the Pettis integrability assumption, for a class of boundary value problems for fractional differential inclusions involving nonlinear integral conditions. Our results are based on the technique of measures of weak noncompactness and a fixed point theorem of Mönch type.

Keywords: *boundary value problem; differential inclusion; Caputo fractional derivative; measure of weak noncompactness; Pettis integrals; weak solution.*

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1 Introduction

This note is concerned with the existence of solutions of the boundary value problem with fractional order differential inclusions and nonlinear integral conditions of the form

$$^{c}D^{\alpha}x(t) \in F(t, x(t)), \quad \text{for a.e.} \quad t \in J = [0, T], \ 1 < \alpha \leq 2, \tag{1}$$

$$x(0) - x'(0) = \int_0^T g(s, x(s))ds, \tag{2}$$

$$x(T) + x'(T) = \int_0^T h(s, x(s))ds, \tag{3}$$

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where ${}^cD^\alpha$, $1 < \alpha \leq 2$, is the Caputo fractional derivative, $F : J \times E \to P(E)$ is a multivalued map, E is a Banach space with the norm $\| \cdot \|$, $P(E)$ is the family of all nonempty subsets of E, and g, $h : J \times E \to E$ are given functions satisfying some assumptions that will be specified later.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc. (see [18, 24, 30]). There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monographs of Kilbas *et al.* [21], Lakshmikantham *et al.* [23], Podlubny [30], and the papers [2, 3, 11, 17, 28].

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics [13] and cellular systems [1]. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors such as Arara and Benchohra [4], Benchohra *et al.* [10], Infante [20], and the references therein.

In our investigation we apply the method associated with the technique of measures of weak noncompactness and a fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Banaŝ and Goebel [6] and subsequently developed and used in many papers; see, for example, Banaŝ *et al.* [7], Guo *et al.* [19], Krzyska and Kubiaczyk [22], Lakshmikantham and Leela [23], Mönch [25], O'Regan [26, 27], Szufla [32], Szufla and Szukala [33], and the references therein. In [8, 12] Benchohra *et al.* considered some classes of boundary value problems for fractional order differential equations in Banach space by means of the strong measure of noncompactness. As far as we know, they are very few results devoted to weak solutions of boundary value problems for nonlinear fractional differential equations [9]. The present results complement and extend those considered with the strong measure of noncompactness [8, 12].

2 Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel. Let E be the real Banach space with norm $\| \cdot \|$ and dual space E^*, and let $(E, w) = (E, \sigma(E, E^*))$ denote the space E with its weak topology. Here, $C(J, E)$ is the Banach space of all continuous functions $x : J \to E$ with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

We let $L^1(J, E)$ denote the Banach space of functions $x : J \to E$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_0^T \|x(t)\| dt,$$

and $L^\infty(J, E)$ denote the Banach space of bounded measurable functions $x : J \to E$ equipped with the norm

$$\|x\|_{L^\infty} = \inf\{c > 0 : \|x(t)\| \leq c \ \ a.e. \ t \in J\}.$$

Also, $AC^1(J, E)$ will denote the space of functions $x : J \to E$ that are absolutely continuous and whose first derivative, x', is absolutely continuous.

Let $(E, \|\cdot\|)$ be a Banach space and let $P_{cl}(E) = \{Y \in P(E) : Y \text{ is closed}\}$, $P_b(E) = \{Y \in P(E) : Y \text{ is bounded}\}$, $P_{cp}(E) = \{Y \in P(E) : Y \text{ is compact}\}$, and $P_{cp,cv}(E) = \{Y \in P(E) : Y \text{ is compact and convex}\}$. A multivalued map $F : E \to P(E)$ is *convex* (*closed*) valued if $F(x)$ is convex (closed) for all $x \in E$. We say that F is *bounded on bounded sets* if $F(B) = \cup_{x\in B} F(x)$ is bounded in E for all $B \in P_b(E)$ (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$). The mapping F is called *upper semi-continuous* (u.s.c.) on E if for each $x_0 \in E$, the set $F(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. The mapping F has a *fixed point* if there is $x \in E$ such that $x \in F(x)$.

For more details on multivalued maps see the books of Aubin and Frankowska [5] and Deimling [15]. We will need the following definitions in the sequel.

Definition 2.1 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n(t) \to x(t)$ in (E, w) for each $t \in J$, we have $h(x_n(t)) \to h(x(t))$ in (E, w) for each $t \in J$).

Definition 2.2 A function $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph if for any sequence $(x_n, y_n)_1^\infty \in Q \times Q$, $y_n \in F(x_n)$ for $n \in \{1, 2, ...\}$ with $x_n(t) \to x(t)$ in (E, ω) for each $t \in J$ and $y_n(t) \to y(t)$ in (E, ω) for each $t \in J$, then $y \in F(x)$.

Definition 2.3 [29] The function $x : J \to E$ is said to be Pettis integrable on J if and only if there is an element $x_I \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J, E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Proposition 2.1 [16, 29] *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis integrable.*

Definition 2.4 [14] Let E be a Banach space, Ω_E be the bounded subsets of E, and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty]$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

Properties: The De Blasi measure of noncompactness satisfies the following properties:

(a) $A \subset B \implies \beta(A) \le \beta(B)$;

(b) $\beta(A) = 0 \iff A$ is relatively compact;

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(d) $\beta(\overline{A}^\omega) = \beta(A)$, where $\overline{A}^\omega$ denotes the weak closure of A;

(e) $\beta(A + B) \le \beta(A) + \beta(B)$;

(f) $\beta(\lambda A) = |\lambda|\beta(A)$;

(g) $\beta(conv(A)) = \beta(A)$;

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The following result follows directly from the Hahn–Banach theorem.

Proposition 2.2 *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For completeness, we recall the definitions of the Pettis-integral and the Caputo derivative of fractional order.

Definition 2.5 ([31]) Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where the sign "$\int$" denotes the Pettis integral and Γ is the Gamma function.

Definition 2.6 ([21]) For a function $h : I \to E$, the Caputo fractional-order derivative of h is defined by

$$^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)ds}{(t-s)^{1-n+\alpha}},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The following theorem will be used to prove our main result.

Theorem 2.1 *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,T],E)$. Suppose $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup F(V)) \Longrightarrow V \quad \textit{is relatively weakly compact} \tag{4}$$

holds for every subset $V \subset Q$, then the operator inclusion $x \in F(x)$ has a solution in Q.

3 Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)–(3).

Definition 3.1 A function $x \in AC^1(J,E)$ is said to be a solution of (1)–(3), if there exists a function $v \in L^1(J,E)$ with $v(t) \in F(t,x(t))$ for a.e. $t \in J$, such that

$$^cD^\alpha x(t) = v(t) \text{ a.e. } t \in J, \quad 1 < \alpha \leq 2,$$

and the function x satisfies the boundary conditions (2) and (3).

For any $x \in C(J,E)$, we define the set

$$S_{F,x} = \{v \in L^1(J,E) : v(t) \in F(t,x(t)) \ \text{ for a.e. } t \in J\}.$$

This is known as the set of *selection functions.*

For the existence of solutions to the problem (1)–(3), we need the following auxiliary lemmas.

Lemma 3.1 [34] *Let $\alpha > 0$; then the differential equation ${}^cD^\alpha h(t) = 0$ has the solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, and $n = [\alpha] + 1$.*

Lemma 3.2 [34] *Let $\alpha > 0$; then $I^{\alpha c} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.*

As a consequence of Lemmas 3.1 and 3.2, we have the following result which will be useful in the remainder of the paper.

Lemma 3.3 *Let $1 < \alpha \le 2$ and let σ, σ_1, $\sigma_2 : J \to E$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = P(t) + \int_0^T G(t,s)\sigma(s)ds \tag{5}$$

with

$$P(t) = \frac{(T+1-t)}{T+2}\int_0^T \sigma_1(s)ds + \frac{(t+1)}{T+2}\int_0^T \sigma_2(s)ds \tag{6}$$

and

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & 0 \le s \le t, \\ -\frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & t \le s < T, \end{cases} \tag{7}$$

if and only if x is a solution of the fractional boundary value problem

$$ {}^cD^\alpha x(t) = \sigma(t), \quad t \in J,$$

$$x(0) - x'(0) = \int_0^T \sigma_1(s)ds, \quad x(T) + x'(T) = \int_0^T \sigma_2(s)ds.$$

Let

$$\tilde{G} = \sup\left\{\int_0^T |G(t,s)|ds,\ t \in J\right\}.$$

We are now in a position to state and prove our existence result for the problem (1)–(3). We first list the following hypotheses:

(H1) $F : J \times E \to P_{cp,cl,cv}(E)$ has weakly sequentially closed graph.

(H2) For each $t \in J$, $g(t,\cdot)$ and $h(t,\cdot)$ are weakly sequentially continuous.

(H3) For each continuous $x : J \to E$, there exists a scalarly measurable function $v : J \to E$ with $v(t) \in F(t, x(t))$ a.e. on J and v is Pettis integrable on J.

(H4) For each $x \in C(J, E)$, $g(\cdot, x(\cdot))$ and $h(\cdot, x(\cdot))$ are Pettis integrable on J.

(H5) There exist $p \in L^\infty(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le p(t)\psi(\|x\|).$$

(H6) There exist $\phi_g \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi^* : [0, \infty) \to [0, \infty)$ such that
$$\|g(t, x)\| \le \phi_g(t)\psi^*(\|x\|).$$

(H7) There exist $\phi_h \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\bar{\psi} : [0, \infty) \to [0, \infty)$ such that
$$\|h(t, x)\| \le \phi_h(t)\bar{\psi}(\|x\|).$$

(H8) there exists a number $R > 0$ such that
$$\frac{R}{a\psi^*(R) + b\bar{\psi}(R) + c\tilde{G}\psi(R)} > 1, \tag{8}$$
where
$$a = \frac{T+1}{T+2}\int_0^T \phi_g(s)ds, \quad b = \frac{T+1}{T+2}\int_0^T \phi_h(s)ds, \quad \text{and} \quad c = \|p\|_{L^\infty}.$$

(H9) For each bounded set $Q \subset E$ and each $t \in J$,
$$\beta(F(t, Q)) \le p(t)\beta(Q), \tag{9}$$
$$\beta(g(t, Q)) \le \phi_g(t)\beta(Q), \tag{10}$$
$$\beta(h(t, Q)) \le \phi_h(t)\beta(Q). \tag{11}$$

Theorem 3.1 *Let E be a Banach space. Assume that hypotheses (H1)–(H9) hold. If*
$$\frac{T+1}{T+2}\int_0^T [\phi_g(s) + \phi_h(s)]ds + \tilde{G}\|p\|_{L^\infty} < 1, \tag{12}$$
then the problem (1)–(3) has at least one solution.

Proof We transform the problem (1)–(3) into fixed point problem by considering the multivalued operator $N : C(J, E) \to P_{cl,cv}(C(J, E))$ defined by
$$N(y) = \left\{ h \in C(J, E) : h(t) = P_x(t) + \int_0^T G(t, s)v(s)ds, v \in S_{F,x} \right\}, \tag{13}$$
where
$$P_x(t) = \frac{T+1-t}{T+2}\int_0^T g(s, x(s))ds + \frac{t+1}{T+2}\int_0^T h(s, x(s))ds$$
and the function $G(\cdot, \cdot)$ is given by (7). Clearly, from Lemma 3.3, the fixed points of N are solutions of the problem (1)–(3). We first show that (13) makes sense. To see this, let $x \in C(J, E)$; by (H3) there exists a Pettis integrable function $v : J \to E$ such that $v(t) \in F(t, x(t))$ for a.e. $t \in J$. Since $G(t, \cdot) \in L^\infty(J)$, then $G(t, \cdot)v(\cdot)$ is Pettis integrable and thus N is well defined.

Let $R \in \mathbb{R}_+^*$, and consider the set
$$Q = \left\{ x \in C(J, E) : \|x\|_\infty \le R \ \text{and}\ \|x(t_1) - x(t_2)\| \le \frac{|t_1 - t_2|}{T+2}\psi^*(R)\int_0^T \phi_g(s)ds \right.$$
$$\left. + \frac{|t_1 - t_2|}{T+2}\bar{\psi}(R)\int_0^T \phi_h(s)ds + \|p\|_{L^\infty}\psi(R)\int_0^T \|G(t_2, s) - G(t_1, s)\|ds \ \text{ for }\ t_1, t_2 \in J \right\}.$$

Notice that Q is a closed, convex, bounded and equicontinuous subset of $C(J,E)$. We shall show that N satisfies the assumptions of Theorem 2.1.

Step 1: $N(x)$ is convex for each $x \in Q$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exists Pettis integrable functions $\upsilon_1(t)$, $\upsilon_2(t) \in F(t,x(t))$ such that, for all $t \in J$, we have:

$$h_i(t) = P_x(t) + \int_0^T G(t,s)\upsilon_i(s)ds, \ i = 1,2.$$

Let $0 \le \lambda \le 1$; then, for each $t \in J$, we have:

$$(\lambda h_1 + (1-\lambda)h_2)(t) = P_y(t) + \int_0^T G(t,s)[\lambda\upsilon_1(s) - (1-\lambda)\upsilon_2(s)]ds.$$

Since F has convex values, $(\lambda\upsilon_1 + (1-\lambda)\upsilon_2)(t) \in F(t,x(t))$, and we have $\lambda h_1 + (1-\lambda)h_2 \in N(x)$.

Step 2: N maps Q into Q.

To see this, take $u \in NQ$. Then there exists $x \in Q$ with $u \in N(x)$ and there exists a Pettis integrable function $\upsilon : J \to E$ with $\upsilon(t) \in F(t,x(t))$ for a.e. $t \in J$. Without loss of generality, we assume $u(s) \neq 0$ for all $s \in J$. Then, there exists $\varphi_s \in E^*$ with $\|\varphi_s\| = 1$ and $\varphi_s(u(s)) = \|u(s)\|$. Hence, for each fixed $t \in J$, we have:

$$\begin{aligned}
\|u(t)\| &= \varphi_t(u(t)) \\
&= \varphi_t\left(P_x(t) + \int_0^T G(t,s)\upsilon(s)ds\right) \\
&\le \varphi_t(P_x(t)) + \varphi_t\left(\int_0^T G(t,s)\upsilon(s)ds\right) \\
&\le \|P_x(t)\| + \int_0^T \|G(t,s)\|\varphi_t(\upsilon(s))ds \\
&\le \frac{T+1}{T+2}\psi^*(\|x\|_\infty)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\bar{\psi}(\|x\|_\infty)\int_0^T \phi_h(s)ds \\
&\quad + \tilde{G}\psi(\|x\|_\infty)\|p\|_{L^\infty}.
\end{aligned}$$

Therefore, by (H8),

$$\|u\|_\infty \le \frac{T+1}{T+2}\psi^*(R)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\bar{\psi}(R)\int_0^T \phi_h(s)ds + \tilde{G}\psi(R)\|p\|_{L^\infty} \le R.$$

Next suppose $u \in NQ$ and t_1, $t_2 \in J$ with $t_1 < t_2$ so that $u(t_2) - u(t_1) \neq 0$. Then, there exist $\varphi \in E^*$ such that $\|u(t_2) - u(t_1)\| = \varphi(u(t_2) - u(t_1))$. Thus,

$$\begin{aligned}
\|u(t_2) - u(t_1)\| &= \varphi\left(P_x(t_2) - P_x(t_1) + \int_0^T (G(t_2, s) - G(t_1, s))\upsilon(s)ds\right) \\
&\leq \varphi(P_x(t_2) - P_x(t_1)) + \varphi\left(\int_0^T (G(t_2, s) - G(t_1, s))\upsilon(s)ds\right) \\
&\leq \|P_x(t_2) - P_x(t_1)\| + \int_0^T \|G(t_2, s) - G(t_1, s)\|\|\upsilon(s)\|ds \\
&\leq \frac{(t_2 - t_1)}{T + 2}\psi^*(R)\int_0^T \phi_g(s)ds + \frac{(t_2 - t_1)}{T + 2}\bar{\psi}(R)\int_0^T \phi_h(s)ds \\
&\quad + \psi(R)\|p\|_{L^\infty}\int_0^T \|G(t_2, s) - G(t_1, s)\|ds.
\end{aligned}$$

Therefore, $u \in Q$.

Step 3: N has a weakly sequentially closed graph.

Let $(x_n, y_n)_1^\infty$ be a sequence in $Q \times Q$ with $x_n(t) \to x(t)$ in (E, ω) for each $t \in J$, $y_n(t) \to y(t)$ in (E, ω) for each $t \in J$, and $y_n \in N(x_n)$ for $n \in \{1, 2, ...\}$. We shall show that $y \in Nx$. By the relation $y_n \in N(x_n)$, we mean that there exists $\upsilon_n \in S_{F,x_n}$ such that

$$y_n(t) = P_{x_n}(t) + \int_0^T G(t, s)\upsilon_n(s)ds.$$

We must show that there exists $\upsilon \in S_{F,x}$ such that, for each $t \in J$,

$$y(t) = P_x(t) + \int_0^T G(t, s)\upsilon(s)ds.$$

Since $F(\cdot, \cdot)$ has compact values, there exists a subsequence υ_{n_m} such that

$$\upsilon_{n_m}(\cdot) \to \upsilon(\cdot) \text{ in } (E, \omega) \text{ as } m \to \infty$$

and

$$\upsilon_{n_m}(t) \in F(t, x_n(t)) \text{ a.e. } t \in J.$$

Since $F(t, \cdot)$ has a weakly sequentially closed graph, $\upsilon \in F(t, x)$. The Lebesgue Dominated Convergence Theorem for the Pettis integral then implies that for each $\varphi \in E^*$,

$$\varphi(y_n(t)) = \varphi\left(P_{x_n}(t) + \int_0^T G(t, s)\upsilon_n(s)ds\right) \to \varphi\left(P_x(t) + \int_0^T G(t, s)\upsilon(s)ds\right)$$

i.e., $y_n(t) \to Nx(t)$ in (E, ω). We can repeat this for each $t \in J$, so $y(t) \in Nx(t)$.

Step 4: The implication (4) holds.

Now let V be a subset of Q such that $V = \overline{conv}(N(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{conv}(N(V) \cup \{0\})$ for all $t \in J$. Also, $NV(t) \subset NQ(t)$, for each $t \in J$, and is bounded in

$P(E)$. By $(H9)$ and the properties of the measure β, we have

$$\begin{aligned}
\beta(NV(t)) &= \beta\left\{P_x(t)+\int_0^T G(t,s)v(s)ds : v\in S_{F,x},\ x\in V,\ t\in J\right\}\\
&\le \beta\left\{P_x(t):\ x\in V,\ t\in J\right\}\\
&\quad+\beta\left\{\int_0^T G(t,s)v(s)ds : v\in S_{F,x},\ x\in V,\ t\in J\right\}\\
&\le \beta\left\{\frac{T+1-t}{T+2}\int_0^T g(s,x(s))ds+\frac{t+1}{T+2}\int_0^T h(s,x(s))ds : x\in V\right\}\\
&\quad+\beta\left\{\int_0^T G(t,s)v(s)ds : v(t)\in F(t,x(t)),\ x\in V,\ t\in J\right\}\\
&\le \int_0^T \frac{T+1-t}{T+2}\phi_g(s)\beta(V(s))ds+\int_0^T \frac{t+1}{T+2}\phi_h(s)\beta(V(s))ds\\
&\quad+\int_0^T \|G(t,s)\|p(s)\beta(V(s))ds\\
&\le \frac{T+1}{T+2}\int_0^T \phi_g(s)\beta(V(s))ds+\frac{T+1}{T+2}\int_0^T \phi_h(s)\beta(V(s))ds\\
&\quad+\int_0^T \|G(t,s)\|p(s)\beta(V(s))ds
\end{aligned}$$

for each $t\in J$. This means that

$$\|v\|_\infty \le \|v\|_\infty\left[\frac{T+1}{T+2}\int_0^T(\phi_g(s)+\phi_h(s))ds+\tilde{G}\int_0^T p(s)ds\right]$$

i.e.,

$$\|v\|_\infty\left[1-\frac{T+1}{T+2}\int_0^T(\phi_g(s)+\phi_h(s))ds+\tilde{G}\|p\|_\infty\right]\le 0.$$

By (12) it follows that $\|v\|_\infty = 0$. Thus, V is weakly relatively compact. Applying Theorem 2.1, we conclude that N has a fixed point that is a solution of the problem (1)–(3). □

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Quasilinearization Method Via Lower and Upper Solutions for Riemann–Liouville Fractional Differential Equations

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Abstract: Existence and comparison results of the linear and nonlinear Riemann–Liouville fractional differential equations of order q, $0 < q < 1$, are recalled and modified where necessary. Generalized quasilinearization method is developed for nonlinear fractional differential equations of order q, using upper and lower solutions. Quadratic convergence to the unique solution is proved via weighted sequences.

Keywords: *fractional differential equations; lower and upper solutions; quasilinearization method.*

Mathematics Subject Classification (2000): 34A34, 34A45.

1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as in engineering [9], chemistry [10, 17, 18], physics [3, 4, 11], and others [12, 13]. In the majority of the literature existence results for Riemann–Liouville fractional differential equations are proven by a fixed point method. Initially we will recall existence by lower and upper solution method, which is more comparable to our main results. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a solution to a needed solution. In this paper we construct such a method.

The iterative technique we manufacture is the method of quasilinearization for nonlinear Riemann–Liouville fractional differential equations of order q, $0 < q < 1$. This

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method was first developed in [1, 2, 16], the method we construct is more closely related to those found in [15], that is a generalized quasilinearization method via lower and upper solutions. This particular method is much like the monotone method in that we construct monotone sequences from lower and upper solutions of the original equation. Further, each iterate is the solution of the linear fractional differential equation, but unlike in the monotone method, these iterates are not of the form with constant coefficients. In the case of the Riemann–Liouville fractional derivative, the variable coefficient case complicates our method. Therefore, we will recall existence, comparison, and inequality results for this case, including a generalized Gronwall type inequality, which will be paramount to our main result. Further, we will present modifications to these results where pertinent to our work.

Further, in the construction of the quasilinearization method we require a much stronger hypothesis than the monotone iterative technique. We still require the existence of lower and upper solutions v, w such that $v \leq w$, but specifically we require the nonlinear function $f(t, x)$ to be convex (or concave) in x. Though this requirement may initially seem superfluous, with its application we are able to prove that the sequences we construct converge quadratically. Therefore, the sequences we construct may be more unwieldy, and the requirements more strict, than with the monotone method, but with this method the convergence is far faster. Further, with the assumption that f is convex automatically ensures that our solution is unique, which is not necessarily the case with the monotone method.

We note that this method has been studied in [8], but the authors have considered differential equations of the Caputo case. However the Caputo derivative only exists for C^1 functions. We do not make this assumption with the Riemann–Liouville derivative. In fact, the functions we consider generally have a singularity at the left-most endpoint, therefore they are only C^0 on a half open interval, with a special C_p property we will define below. One consequence of using the Riemann–Liouville derivative is that, in general, the sequences we construct, $\{\alpha_n\}, \{\beta_n\}$ do not converge uniformly to the unique solution, but the weighted sequences $\{t^p\alpha_n\}$, $\{t^p\beta_n\}$ converge uniformly and quadratically to $t^p x$, where x is the unique solution of the original equation and $p = 1 - q$.

Finally, we consider the case when f is not convex (nor concave), but there exists a function ϕ such that $f + \phi$ is convex. We construct the quasilinearization for this case and note that a function ϕ will always exist, therefore extending this method to any nonlinear fractional differential equation, provided f is C^2 in x. For more information on the method of quasilinearization via lower and upper solutions as it relates to ordinary differential equations, see [15].

2 Preliminary Results

In this section we consider results regarding the Riemann-Liouville (R-L) differential equations of order q, $0 < q < 1$. Specifically we recall existence and comparison results which will be used in our main result. In the next section we will apply these preliminary results to developing quasilinearization method for R-L fractional differential equations of order q. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1 Let $p = 1 - q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Remark 2.1 By the definition of C_p continuity and the properties of continuous functions it can be shown that the uniform limit of C_p functions is C_p, also $C_p(J, \mathbb{R})$ has a completeness property in that any uniformly Cauchy sequence of C_p functions converges uniformly to a C_p function. Further $C_p(J, \mathbb{R})$ is closed under continuous products, that is, if $x \in C_p(J, \mathbb{R})$ and $y \in C(J_0, \mathbb{R})$ then $xy \in C_p(J, \mathbb{R})$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2 Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q \phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q \phi(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{-q} \phi(s) ds,$$

and $I_t^q \phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q \phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi(s) ds.$$

Note that in cases where the initial value may be different, or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equation and is also of great importance in the study of the R-L derivative.

Definition 2.3 The Mittag–Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

Remark 2.2 We note that the C_p weighted Mittag–Leffler function

$$t^{q-1} E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{kq+q-1}}{\Gamma(kq+q)},$$

where λ is a constant, converges uniformly on J. This can be shown by using the fact that $E_{q,q}$ is entire and noting that there exists an $N > 0$ such that $nq + q - 1 > 0$ for all $n \geq N$. From here one can show that the sequence of partial sums of the above series is uniformly Cauchy.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.1 *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in \mathbb{R}^+$ can be found in [7]. Now we consider results for the nonhomogeneous linear R-L differential equation

$$D_t^q x(t) = y(t)x(t) + z(t) \tag{1}$$

with initial condition $\Gamma(q) t^p x(t)\big|_{t=0} = x^0$, where x^0 is a constant, $y \in C(J_0, \mathbb{R})$, and $z \in C_p(J, \mathbb{R})$.

Theorem 2.1 *If $y \in C(J_0, \mathbb{R})$ and $z \in C_p(J, \mathbb{R})$ then equation (1) has a unique solution $x \in C_p(J, \mathbb{R})$, given explicitly by*

$$x(t) = \sum_{k=0}^{\infty} \frac{x^0}{\Gamma(q)} T_y^k \left[t^{q-1}\right] + T_y^k \left[I_t^q z(t)\right],$$

which converges uniformly on J and where T_y is the operator defined by

$$T_y \phi(t) = I_t^q y(t) \phi(t).$$

Proof The proof of the homogeneous case, and that $t^p x(t)$ converges uniformly on J_0 can be found in [6], the refinement that $x(t)$ converges uniformly on J can be found in [5]. Note the nonhomogeneous case follows in exactly the same way as in [6]. Further in [5] it was assumed that $z \in C_p(J, \mathbb{R})$ such that $I_t^q z \in C(J_0, \mathbb{R})$, here we have relaxed this condition. The proof follows along the same lines as in [5] with appropriate modifications. That is, using that $z \in Cp$, and the fact that $E_{q,q}$ is entire, we can show the partial sums of the series x are uniformly Cauchy on J. That $x \in C_p(J, \mathbb{R})$ follows from applying Remark 2.1 and Lemma 2.1. Note that if $z(t) = 0$ for all $t \in J$ then we get that

$$x(t) = \frac{x^0}{\Gamma(q)} \sum_{k=0}^{\infty} T_y^k \left[t^{q-1}\right].$$

In many cases we may have an explicit form of y that may prove too unwieldy to place in a subscript. In this case we will use the following notation

$$\mathcal{E}(y, f) = \sum_{k=0}^{\infty} T_y^k \left[f\right],$$

and since the case where $f = t^{q-1}$ occurs so often we will define $\mathcal{E}$ with a single parameter to be this case. That is $\mathcal{E}(y) = \mathcal{E}(y, t^{q-1})$. Therefore the solution of (1) can be written as

$$x(t) = \frac{x^0}{\Gamma(q)} \mathcal{E}(y) + \mathcal{E}(y, I_t^q z). \tag{2}$$

Further, if y is identically a constant, say λ, it can be shown that (2) can be expressed as

$$x(t) = x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

This is the result discussed in [14], hence Theorem 2.1 generalizes the constant coefficient case, as expected.

Next we recall a comparison result we will utilize in our following results. Note this result is similar to the well known comparison result found in literature, as in [14], but we do not require the function to be Hölder continuous of order $\lambda > q$. We weaken this requirement because in our main result we will construct sequences from the solutions of linear R-L differential equations. As previously mentioned the solution to the linear equation with constant coefficient can be rewritten as

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + x^0 \sum_{k=1}^{\infty} \frac{\lambda^k t^{qk+q-1}}{\Gamma(qk+q)} + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) ds,$$

which is not Hölder continuous of any order due to the term containing t^{q-1}. Therefore we utilize the following result which weakens the Hölder continuity requirement, so that we can incorporate it in our main results.

Lemma 2.2 *Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t \in (0, t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \geq 0.$$

The proof of this lemma can be found in [7], along with further discussion as to why and how we weaken the Hölder continuous requirement of this known comparison result. We use this Lemma in the proof of the later main comparison result which will be paramount in the construction of the quasilinearization method. First we recall the nonlinear R-L fractional differential equation.

$$\begin{aligned} &D_t^q x = f(t, x), \\ &\Gamma(q) t^p x(t)\big|_{t=0} = x^0, \end{aligned} \tag{3}$$

where $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$. Note that a solution $x \in C_p(J, \mathbb{R})$ of (3) also satisfies the equivalent R-L integral equation

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds. \tag{4}$$

Thus if $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ then (3) is equivalent to (4). See [12, 14] for details. Now we will recall a Peano type existence theorem for equation (3).

Theorem 2.2 *Suppose $f \in C(R_0, \mathbb{R})$ and $|f(t, x)| \leq M$ on R_0, where*

$$R_0 = \{(t, x) \ : \ |t^p x(t) - x^0| \leq \eta, t \in J_0\}.$$

Then the solution of (3) exists on J.

This result is presented in [14], and in [7] it was proven that the solution can be extended to all of J, and the set R_0 was modified for our succeeding results regarding existence by method of upper and lower solutions. In the direction of this result we will consider the following comparison result, which will in turn yield a general Gronwall type inequality.

Theorem 2.3 *Let $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ and let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of (3), i.e.*

$$\begin{aligned} &D_t^q v \leq f(t, v), \\ &\Gamma(q) t^p v(t)\big|_{t=0} = v^0 \leq x^0, \end{aligned}$$

and

$$\begin{aligned} &D_t^q w \geq f(t, w), \\ &\Gamma(q) t^p w(t)\big|_{t=0} = w^0 \geq x^0. \end{aligned}$$

If f satisfies the following Lipschitz condition

$$f(t, x) - f(t, y) \leq L(x - y), \quad \text{when } x \geq y,$$

where $L > 0$, then $v(t) \leq w(t)$ on J.

The proof follows as in [14] with appropriate modifications, specifically we use Lemma 2.2 and do not require local Hölder continuity of order $\lambda > q$. Next we present a Gronwall type inequality for R-L fractional differential equations. A similar result in terms of fractional integral equations can be found in [6].

Theorem 2.4 *Let $v, z \in C_p(J, \mathbb{R})$ and $y \in C(J_0, \mathbb{R}^+)$, and suppose that*

$$D_t^q v \leq y(t)v(t) + z(t).$$

Then

$$v(t) \leq \frac{v^0}{\Gamma(q)}\mathcal{E}(y) + \mathcal{E}(y, I_t^q z).$$

The proof follows directly from Theorem 2.1 and Theorem 2.3. That is, since $y \geq 0$, $f(t, x) = yx + z$ satisfies the Lipschitz condition of Theorem 2.3 and letting x be the solution of (1) with $x^0 = v^0$ we obtain $v \leq x$. When y is identically a constant $\lambda \geq 0$, then we get the following Corollary.

Corollary 2.1 *Let $v, z \in C_p(J, \mathbb{R})$ and let $\lambda \geq 0$ be a constant, and suppose that*

$$D_t^q v \leq \lambda v(t) + z(t).$$

Then

$$v(t) \leq v^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

Now we will recall a result that gives us existence of a solution to (3) via lower and upper solutions.

Theorem 2.5 *Let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of (3) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as*

$$\Omega = \{(t, y) \,:\, t^p v(t) \leq y \leq t^p w(t), t \in J_0\}.$$

Then there exists a solution $x \in C_p(J, \mathbb{R})$ of (3) such that $v(t) \leq x(t) \leq w(t)$ on J.

The proof of this theorem can be found in [7]. We also note a final uniqueness result which is comparable to the analogous result for ordinary differential equations. As one might expect, if f satisfies the Lipschitz condition found in Theorem 2.3, then the solution x of (3) is unique. Further this result is proved in much the same way as in the case of ordinary differential equations, see [14] for more details. We mention this result here since it will be necessary in the construction of the quasilinearization method.

3 Method of Quasilinearization

In this section we develop the method of quasilinearization via lower and upper solutions. We consider three different cases, when the forcing function f is convex, concave in x, and can be made convex by the addition function ϕ. We construct monotone sequences such that the sequences of continuous extensions converge uniformly and monotonically to the continuous extension of the unique solution x of (3). Further, the rate convergence is quadratic.

Theorem 3.1 *Assume that*

(A1) $\alpha_0, \beta_0 \in C_p(J, \mathbb{R})$ *are lower and upper solutions of (3) respectively such that* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(A2) $f \in C(\Omega, \mathbb{R})$, $f_x(t,x) \geq 0$, $f_{xx}(t,x) \geq 0$ *exist and are continuous on* Ω, *where*

$$\Omega = \{(t,y) : \alpha_0(t) \leq y \leq \beta_0(t), t \in J_0\}.$$

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p\alpha_n$, *and* $t^p\beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

Proof First, by (A2) we have that f and f_x are nondecreasing in x on J_0, Lipschitz with respect to x on J_0, and

$$f(t,x) \geq f(t,y) + f_x(t,y)(x-y)$$

for any $(t,y) \in \Omega$. Further the function

$$g(t,x,y) = f(t,y) + f_x(t,y)(x-y)$$

is linear in x on J_0. Now we will construct the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. Let α_{n+1} be the unique solution of the Riemann–Liouville differential equation

$$\begin{aligned} &D_t^q \alpha_{n+1} = f(t,\alpha_n) + f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n), \\ &\Gamma(q) t^p \alpha_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{5}$$

for all $n \geq 0$, and where α_0 is the lower solution of (3) given in the hypothesis. Note that the above equation is of the form (1), therefore it has a unique solution by Theorem 2.1 provided $(t, \alpha_n) \in \Omega$, and therefore our sequence is well defined. Similarly, let β_{n+1} be the unique solution of

$$\begin{aligned} &D_t^q \beta_{n+1} = f(t,\beta_n) + f_x(t,\alpha_n)(\beta_{n+1} - \beta_n), \\ &\Gamma(q) t^p \beta_{n+1}(t)\big|_{t=0} = x^0. \end{aligned} \tag{6}$$

Now we will show that $\alpha_n \leq \beta_n$ for all $n \geq 0$. To do this first note that by hypothesis we have that $\alpha_0 \leq \beta_0$ on J, so letting this be our basis step, suppose that $\alpha_k \leq \beta_k$ is true up to some $k \geq 0$. Then we have

$$D_t^q \alpha_{k+1} = f(t,\alpha_k) + f_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k),$$

and by the consequences of (A2) we have that

$$D_t^q \beta_{k+1} \geq f(t,\alpha_k) + f_x(t,\alpha_k)(\beta_{k+1} - \alpha_k),$$

which by Theorem 2.3 gives us that $\alpha_{k+1} \leq \beta_{k+1}$ on J and thus by induction proves the claim.

Now we wish to show that that $\{\beta_n\}$ is monotone. To do so consider that

$$D_t^q \beta_1 \leq f(t,\beta_0) + f_x(t,\beta_0)(\beta_1 - \beta_0) \leq f(t,\beta_1),$$

which again, by Theorem 2.3 gives us that $\beta_1 \leq \beta_0$ on J. Now suppose $\beta_k \leq \beta_{k-1}$ up to some $k \geq 1$, then letting $\omega = \beta_{k+1} - \beta_k$, with $\omega^0 = 0$, by the consequences of (A2) and that $\alpha_n \leq \beta_n$ for all $n \geq 0$, we obtain

$$D_t^q \omega \leq [f_x(t, \beta_k) - f_x(t, \alpha_{k-1})](\beta_k - \beta_{k-1}) + f_x(t, \alpha_k)\omega \leq f_x(t, \alpha_k)\omega.$$

This implies by Theorem 2.4 that

$$\beta_{k+1} - \beta_k \leq \frac{\omega^0}{\Gamma(q)} \mathcal{E}(f_x(t, \alpha_k)) = 0,$$

thus proving, by induction, that $\{\beta_n\}$ is monotone. The proof that $\{\alpha_n\}$ is monotone follows by arguments similar to either of the previous induction proofs.

We now prove that

$$t^p \alpha_n \to t^p x \quad \text{and} \quad t^p \beta_n \to t^p x,$$

uniformly on J_0, and where x is the unique solution of (3). This result follows from an application of the Arzelà–Ascoli Theorem since for all $n \geq 0$ we have that

$$|t^p \alpha_n| \leq t^p |\alpha_n - \alpha_0| + t^p |\alpha_0| \leq t^p |\beta_0 - \alpha_0| + t^p |\alpha_0|,$$

implying that $\{t^p \alpha_n\}$ is uniformly bounded on J_0. That this sequence is equicontinuous is proved in a similar fashion to that found in [19]. We can prove a similar result for $\{t^p \beta_n\}$ as well. To show that both sequences converge to $t^p x$, suppose that $t^p \alpha_n$ instead converges uniformly to $t^p \alpha$, which gives us that α_n converges to α pointwise on J. Now consider the continuous extension of the integral form of α_{n+1},

$$t^p \alpha_{n+1} = \frac{x^0}{\Gamma(q)} + \frac{t^p}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big(f(s, \alpha_n) + f_x(s, \alpha_n)(\alpha_{n+1} - \alpha_n)\big) ds.$$

Applying the convergence properties outlined above we can show that the limit α satisfies

$$\alpha = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \alpha) ds$$

on J. Implying that $\alpha = x$, since x is the unique solution of (3). We note that $\{t^p \beta_n\}$ satisfies an analogous property.

Now we will prove that the sequences of continuous extensions $\{t^p \alpha_n\}$ and $\{t^p \beta_n\}$ converge quadratically. First we note that, since f is continuous on J_0, there exists a function F such that $f(t, x) = F(t, t^p x)$. Then we have that $f_{xx}(t, x) = t^{2p} F_{xx}(t, t^p x)$. Using this result, along with the mean value theorem we obtain

$$\begin{aligned}
D_t^q(x - \alpha_{n+1}) &= f(t, x) - f(t, \alpha_n) - f_x(t, \alpha_n)(\alpha_{n+1} - \alpha_n) \\
&= f_x(t, \xi)(x - \alpha_n) - f_x(t, \alpha_n)(\alpha_{n+1} - \alpha_n) \\
&\leq f_x(t, x)(x - \alpha_n) - f_x(t, \alpha_n)(\alpha_{n+1} - \alpha_n) \\
&= [f_x(t, x) - f_x(t, \alpha_n)](x - \alpha_n) + f_x(t, \alpha_n)(x - \alpha_{n+1}) \\
&= f_{xx}(t, \eta)(x - \alpha_n)^2 + f_x(t, \alpha_n)(x - \alpha_{n+1}) \\
&= F_{xx}(t, t^p \eta) t^{2p} (x - \alpha_n)^2 + f_x(t, \alpha_n)(x - \alpha_{n+1}) \\
&\leq N t^{2p} (x - \alpha_n)^2 + M(x - \alpha_{n+1}).
\end{aligned}$$

Here $\alpha_n \le \xi, \eta \le x$ on J, and N and M are bounds of F_{xx} and f_x respectively. Now by Corollary 2.1 and Remark 2.2 we have that

$$\begin{aligned}
t^p(x-\alpha_{n+1}) &\le t^p \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) N s^{2p} (x-\alpha_n)^2 ds \\
&\le t^p N \big\| t^p(x-\alpha_n) \big\|^2 \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) ds \\
&= t^p N \big\| t^p(x-\alpha_n) \big\|^2 \int_0^t \sum_{k=0}^{\infty} \frac{M^k (t-s)^{kq+q-1}}{\Gamma(qk+q)} ds \\
&= t^p N \big\| t^p(x-\alpha_n) \big\|^2 \sum_{k=0}^{\infty} \frac{M^k t^{kq+q}}{\Gamma(qk+q+1)} \\
&\le \frac{t^p N}{M} E_{q,1}(Mt^q) \big\| t^p(x-\alpha_n) \big\|^2.
\end{aligned}$$

Here $\|\cdot\|$ is the uniform norm on $C(J_0, \mathbb{R})$. Giving us that

$$\big\| t^p(x-\alpha_{n+1}) \big\| \le K \big\| t^p(x-\alpha_n) \big\|^2,$$

where $K = \frac{T^p N}{M} E_{q,1}(MT^q)$.

Now, letting $\rho_n = x - \alpha_n$ and $\omega_n = \beta_n - x$, showing that $\{t^p \beta_n\}$ converges quadratically follows with a similar argument, but in this case we get

$$D_t^q \omega_{n+1} \le F_{xx}(t,\sigma) t^{2p} [\omega_n + \rho_n] \omega_n + f_x(t, \alpha_n)(\omega_{n+1}) \le (N/2) t^{2p} (3\omega_n^2 + \rho_n^2) + M\omega_{n+1}.$$

Then from Corollary 2.1 we get

$$t^p \omega_{n+1} \le \frac{N t^p}{2M} E_{q,1}(Mt^q) \big\| t^{2p} (3\omega_n^2 + \rho_n^2) \big\|,$$

which finally implies that

$$\big\| \beta_{n+1} - x \big\| \le \frac{3K}{2} \big\| t^p(\beta_n - x) \big\|^2 + \frac{K}{2} \big\| t^p(x-\alpha_n) \big\|^2.$$

This concludes the proof.

A natural query is whether the results of Theorem 3.1 will still hold if f is concave as opposed to convex. The answer is affirmative, and we state the result below without the details of the proof.

Theorem 3.2 *Suppose (A1) of Theorem 3.1 holds. Further suppose that* $f \in C(\Omega, \mathbb{R})$, $f_x(t,x) \le 0$, $f_{xx}(t,x) \le 0$ *exist and are continuous on* Ω. *Then there exist monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p \alpha_n$, *and* $t^p \beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

We note that the proof of this theorem follows in the same lines as that of Theorem 3.1. The next case we consider is whether it is possible to construct the quasilinearization method when $f \in C^{0,2}(\Omega, \mathbb{R})$ is neither convex nor concave. As we will show, it is indeed possible provided we can find a function $\phi \in C^{0,2}(\Omega, \mathbb{R})$ such that $f + \phi$ is convex. We present this case as our final theorem.

Theorem 3.3 *Assume that*

(B1) $\alpha_0, \beta_0 \in C_p(J, \mathbb{R})$ *are lower and upper solutions of (3) respectively, such that* $\alpha_0 \leq \beta_0$ *on* J.

(B2) $f, \phi \in C^{0,2}(\Omega, \mathbb{R})$, $f_{xx} + \phi_{xx} \geq 0$ *and* $\phi_{xx} > 0$ *on* Ω, *where* Ω *is defined as in Theorem 3.1.*

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p \alpha_n$ *and* $t^p \beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

Proof Firstly, by consequences of (B2) we have that f is Lipschitz with respect to x. Further, since $f + \phi$ is convex we have that

$$F(t, x) \geq F(t, y) + F_x(t, y)(x - y), \tag{7}$$

where $F(t, x) = f(t, x) + \phi(t, x)$.

We construct the monotone sequences by letting α_{n+1} and β_{n+1} be the unique solutions of the linear R-L fractional differential equations,

$$\begin{aligned} &D_t^q \alpha_{n+1} = f(t, \alpha_n) + \big(F_x(t, \alpha_n) - \phi_x(t, \beta_n)\big)(\alpha_{n+1} - \alpha_n), \\ &\Gamma(q) t^p \alpha_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &D_t^q \beta_{n+1} = f(t, \beta_n) + \big(F_x(t, \alpha_n) - \phi_x(t, \beta_n)\big)(\beta_{n+1} - \beta_n), \\ &\Gamma(q) t^p \beta_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{9}$$

for all $n \geq 0$ and for $(t, \alpha_n), (t, \beta_n) \in \Omega$. Now we wish to show that $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ for all $n \geq 0$. First we will show that $\alpha_0 \leq \alpha_1$, to do so notice that

$$D_t^q \alpha_0 \leq f(t, \alpha_0) + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\alpha_0 - \alpha_0).$$

Therefore by Theorem 2.3 we have that $\alpha_0 \leq \alpha_1$ on J since $\alpha_0^0 \leq x^0$, and by a similar argument we also have that $\beta_1 \leq \beta_0$. Now we will show that $\alpha_1 \leq \beta_1$ on J. Note by consequences of (B2), that is (7), that ϕ_x is increasing in x, and by the application of the mean value theorem we can show that

$$\begin{aligned} D_t^q \beta_1 &\geq f(t, \alpha_0) + F_x(t, \alpha_0)(\beta_0 - \alpha_0) - [\phi(t, \beta_0) - \phi(t, \alpha_0)] \\ &\quad + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \beta_0) \\ &= f(t, \alpha_0) + F_x(\alpha_0)(\beta_0 - \alpha_0) - \phi_x(t, \xi)(\beta_0 - \alpha_0) \\ &\quad + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \beta_0) \\ &\geq f(t, \alpha_0) + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \alpha_0), \end{aligned}$$

where $\alpha_0 \leq \xi \leq \beta_0$. Therefore by Theorem 2.3 we have $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on J. Letting this be our basis step suppose $\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1}$ on J up to some $k \geq 1$, then by a similar process as when showing $\alpha_1 \le \beta_1$ we have that,

$$\begin{aligned} D_t^q \alpha_{k+1} &\ge f(t,\alpha_k) + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &\ge f(t,\alpha_{k-1}) - [\phi(t,\alpha_k) - \phi(t,\alpha_{k-1})] + F_x(t,\alpha_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &= f(t,\alpha_{k-1}) - \phi_x(t,\xi)(\alpha_k - \alpha_{k-1}) + F_x(t,\alpha_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &\ge f(t,\alpha_{k-1}) + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_{k-1}). \end{aligned}$$

Therefore by Theorem 2.3 we have that $\alpha_k \le \alpha_{k+1}$ on J, and by similar arguments we can show that $\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k$, which by induction implies that $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and $\alpha_n \le \beta_n$ for all $n \ge 0$. That $t^p\alpha_n$ and $t^p\beta_n$ converge uniformly to $t^p x$, where x is the unique solution of (3), is done in the same way as in Theorem 3.1. Now we will show that the sequences of continuous extensions converge quadratically on J_0. To do so, first note, as in Theorem 3.1 that there exist functions $G, \Phi \in C^{0,2}(\Omega, \mathbb{R})$ such that $G(t, t^p x) = F(t,x)$, and $\Phi(t, t^p x) = \phi(t,x)$, thus giving us that

$$F_{xx}(t,x) = t^{2p} G_{xx}(t, t^p x) \quad \text{and} \quad \phi_{xx}(t,x) = t^{2p}\Phi_{xx}(t,t^p x).$$

Now letting $\rho_{n+1} = x - \alpha_{n+1}$ and $\omega_{n+1} = \beta_{n+1} - x$, we have that

$$\begin{aligned} D_t^q \rho_{n+1} &= f(t,x) - [f(t,\alpha_n) + \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n)] \\ &= F(t,x) - F(t,\alpha_n) - \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n) \\ &\quad - [\phi(t,x) - \phi(t,\alpha_n)] \\ &= F_x(t,\xi_1)\rho_n - \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n)) - \phi_x(t,\xi_2)\rho_n \\ &\le [F_x(t,x) - F_x(t,\alpha_n)]\rho_n + (F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)\rho_{n+1} \\ &\quad + [\phi_x(t,\beta_n) - \phi_x(t,\alpha_n)]\rho_n \\ &\le F_{xx}(t,\eta_1)\rho_n^2 + f_x(t,\alpha_n)\rho_{n+1} + \phi_{xx}(t,\eta_2)(\omega_n + \rho_n)\rho_n \\ &= G_{xx}(t,t^p\eta_1)t^{2p}\rho_n^2 + f_x(t,\alpha_n)\rho_{n+1} + \Phi_{xx}(t,t^p\eta_2)t^{2p}(\omega_n+\rho_n)\rho_n \\ &\le N t^{2p}\rho_n^2 + M\rho_{n+1} + (L/2)t^{2p}(3\rho_n^2 + \omega_n^2), \end{aligned}$$

Where $\alpha_n \le \xi_1, \xi_2, \eta_1 \le x$, $\alpha_n \le \eta_2 \le x$, and where N, M, and L are bounds on G_{xx}, f_x, and Φ_{xx} respectively. Then by Corollary 2.1 and Remark 2.2 we have that

$$\begin{aligned} t^p\rho_{n+1} &\le t^p \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) s^{2p}\big[(N+3L/2)\rho_n^2 + (L/2)\omega_n^2\big] ds \\ &\le \frac{t^p}{M} E_{q,1}(Mt^q)\Big[(N+3L/2)\big\|t^p\rho_n\big\|^2 + (L/2)\big\|t^p\omega_n\big\|^2\Big]. \end{aligned}$$

Which finally gives us that

$$\big\|t^p(x-\alpha_{n+1})\big\| \le \frac{K}{2}(2N+3L)\big\|t^p(x-\alpha_n)\big\|^2 + \frac{KL}{2}\big\|t^p(\beta_n - x)\big\|^2,$$

where $K = \frac{T^p}{M}E_{q,1}(MT^q)$. Similarly, we can show that

$$\big\|t^p(\beta_{n+1}-x)\big\| \le \frac{K}{2}(3N+2L)\big\|t^p(\beta_n - x)\big\|^2 + \frac{KN}{2}\big\|t^p(x-\alpha_n)\big\|^2,$$

which finishes the proof.

This final case greatly extends the potential of the quasilinearization method. This is because for any function $f \in C^{0,2}(\Omega, \mathbb{R})$ we can always find a function $\phi \in C^{0,2}(\Omega, \mathbb{R})$ such that $f_{xx} + \phi_{xx} \geq 0$, and $\phi_{xx} > 0$. To show why this is true, suppose that f is not convex, then we can choose $A > 0$ such that

$$\min_{\Omega} \left\{ f_{xx}(t, x) \right\} = -A < 0.$$

Then we need only choose $\phi(t, x) = At^{2p}x^2$, to satisfy (B2). Further, since we can always find such a function we need not consider the case where f can be made concave by the sum of another function.

Remark 3.1 If we use lower and upper solutions one can extend the method of quasilinearization to forcing functions which are the sum of convex and concave functions as in [15]. This generalization will include all our results as special cases. However, this involves the study of linear fractional systems with variable coefficients. We will investigate this result elsewhere.

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Adaptive Regulation with Almost Disturbance Decoupling for Power Integrator Triangular Systems with Nonlinear Parametrization

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Abstract: The problem of almost disturbance decoupling for a class of nonlinear systems is considered. The controlled systems consist of a chain of power integrators perturbed by a lower-triangular vector field with nonlinear parametrization. By using the tool of adding a power integrator combined with the parameter separation technique, under a set of growth conditions a smooth adaptive controller is explicitly constructed to attenuate the influence of the disturbance on the output with an arbitrary degree of accuracy. The designed adaptive controller is in its minimum-order property, since the order of the dynamic compensator is equal to one. An illustrative example is given to verify the effectiveness of the proposed approach.

Keywords: *almost disturbance decoupling; smooth adaptive controller; adding a power integrator; nonlinear parametrization; parameter separation.*

Mathematics Subject Classification (2000): 93C10, 93C40.

1 Introduction

One of the main objectives in control theory is to suppress unknown disturbances. It will be ideal if the influence of disturbances on the output can be eliminated completely, or in other words, the disturbances are decoupled from the output. Unfortunately, in most practical situations it is impossible to achieve the exact disturbance decoupling. In this case, it is reasonable to aim at almost disturbance decoupling (ADD), which means that the influence of the disturbance on the output is attenuated to an arbitrary degree of accuracy via feedback control design. More precisely, the problem of ADD can be stated

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as: given a system and a prescribed positive scalar, find a feedback control law such that the resultant closed-loop system is stable and the gain between the exogenous input and the regulated output is less than or equal to the prescribed positive number. The start point of the problem of ADD on nonlinear systems is associated with the papers [1], [2] in the late 1980s. The performance of the ADD in [2] is characterized in terms of the L_∞-induced norm from the disturbance to the outputs, and the solution of the problem of ADD is explicitly constructed by applying singular perturbation methods. However, a drawback of the result in [2] is that internal stability, which is crucial for a meaningful application or a practical implementation, is not taken into account. Therefore, the internal stability of the closed-loop systems cannot be guaranteed even in the absence of the disturbance. This problem is solved later in [3]. By applying a recursive design technique, a global solution to the ADD problem with internal stability was presented for a chain of integrators perturbed by a lower triangular vector field. The result in [3] was later generalized to a larger class of nonlinear minimum phase systems in [4]. These two results in [3] and [4] were further extended to a class of nonminimum-phase nonlinear systems in [5]. The proposed approach in [5] required that the unstable part of the zero-dynamics was not affected by the disturbance. Such a restriction was relaxed in the results of [6] and [7]. The construction of control law in [7] is based on a recursive Lyapunov-based design approach. In the case of systems with vector relative degree $[1, 1, \cdots, 1]$, the ADD problem was tackled in [8] for a general minimum phase system subject to parameter uncertainty and with a controlled output that may be affected by the disturbance. In addition, the problem of ADD for general affine nonlinear systems was addressed in [9] for the case of state feedback and the solution was converted into the solution of the so-called Hamilton-Jacobi-Isaacs equation (HJIE). The global inverse L_2-gain design for a chain of integrators perturbed by lower triangular vector field was reported in [10]. For a class of multi-input multi-output nonlinear systems, the ADD problem was addressed in [11] for the systems with nested lower triangular structure, and the controller was explicitly constructed by applying the backstepping design technique.

If only the output information is available for the feedback design, only a few results are devoted to the ADD problem via output feedback in the existing literature. In [12], the problem of ADD via output feedback for general affine nonlinear systems was converted into the solution of Hamilton-Jacobi-Isaacs equation. In [13], a systematic design procedure to output feedback controller with the function of ADD was given for a class of systems with the nonlinear terms depending only on the output. For the nonlinear systems in the so-called output feedback form, in [14] the polynomial gain disturbance attenuation property was achieved via output feedback. For a class of nonlinear systems satisfying linear growth conditions, in [15] a linear dynamic output compensator attenuating the influence of the disturbance on the output was explicitly constructed by the feedback domination design. In the above-mentioned literature on the ADD problem, most of the considered nonlinear systems are feedback (partial) linearizable and/or linear in control input. Recently, the ADD problem was addressed in [16] for a class of inherently nonlinear systems. The class of the systems is in the form of a chain of power integrators perturbed by a lower-triangular vector field. Different from some previous results, the ADD problem is formulated in terms of $L_2 - L_{2p}$ gain for the inherently nonlinear systems, rather than the standard L_2-gain. The controller was explicitly constructed by applying the so-called technique of adding a power integrator developed in [17].

It is well-known that adaptive control is one of the effective ways to deal with control systems with parametric uncertainty [18]. When the ADD problem for nonlinear systems with unknown parameters is investigated, a natural idea is to design an adaptive control law to solve this problem. However, only a few results on adaptive regulation with almost adaptive decoupling for nonlinear systems are available in the existing literature. In [19], [20] and [21], adaptive controllers are designed to guarantee arbitrary disturbance attenuation on the output tracking error for smooth reference signals for uncertain systems with output depending nonlinearities. In [19] and [20], the disturbance enters linearly in the state space equation, while in [21] the disturbance enters nonlinearly. Very recently, in [22] the ADD problem was discussed for power integrator lower triangular nonlinear systems. The function of disturbance attenuation is characterized by $L_{2m} - L_{2mp}$ gain. The adaptive control law was explicitly constructed by employing the adaptive adding a power integrator technique proposed in [23]. However, the result in [22] is only applicable to the case where the unknown parameter enters linearly in the state space equation. In this paper we will deal with the almost disturbance decoupling problem for power integrator triangular systems with nonlinear parametrization. With the help of the parameter separation technique proposed in [24], a constructive solution that solves the ADD problem is derived by using the adaptive adding one power integrator. A key feature of our proposed adaptive controller with the function of disturbance attenuation is its minimum-order property, since the order of the dynamic compensator is equal to one.

It should be pointed out that some other problems have been investigated for nonlinear systems. In [25], by using a constructed Lyapunov function, the conditions of ultimate boundedness of solutions for a class of nonlinear systems were given. In [26], an original practical criterion of global stability analysis of nonlinear polynomial systems was proposed. In [27], as a generalization of Gronwalls inequality, generalized dynamic inequalities were introduced to the time scales scene. Then, linear systems with linear and nonlinear perturbations and their stability characteristics versus the unperturbed system were investigated.

For simplicity, throughout this paper we use $I[m, n]$ to denote the set $\{m, m+1, \cdots, n\}$ for two integers $m < n$. For a group of scalars x_i, $i \in I[1, j]$, we use $x_{[j]}$ to denote the vector $\begin{bmatrix} x_1 & x_2 & \cdots & x_j \end{bmatrix}^T$. $\|\cdot\|$ is used to denote the Euclidean norm of a vector.

2 Problem Formulation

We consider the following single-input single-output power integrator lower-triangular system with an unknown parameter vector θ:

$$\begin{cases} \dot{x}_i = x_{i+1}^{p_i} + f_i(x_{[i]}) + g_i(x_{[i]})w + \phi_i(x_{[i]}, \theta), \; i \in I[1, n-1], \\ \dot{x}_n = u^{p_n} + f_n(x_{[n]}) + g_n(x_{[n]})w + \phi_n(x_{[n]}, \theta), \\ y = h(x_1), \end{cases} \tag{1}$$

where $u \in \mathbb{R}$, $x = x_{[n]} \in \mathbb{R}^n$, $y \in \mathbb{R}$ and $w \in \mathbb{R}^s$ are the control input, system state, system output and disturbance signal, respectively; p_i, $i \in I[1, n]$, are positive integers and $f_i(\cdot), g_i(\cdot)$, $i \in I[1, n]$, and $h(\cdot)$, are smooth functions with $f_i(0) = 0$, $i \in I[1, n]$, and $h(0) = 0$; $\phi_i(\cdot)$, $i \in I[1, n]$ are continuous functions with $\phi_i(0, \theta) = 0$.

The objective of this paper is to design, under appropriate conditions, a smooth adaptive controller such that the closed-loop system is globally stable in the sense of Lyapunov, and the influence of the disturbance $w(t)$ on the output $y(t)$ is not greater than the prescribed level. To be specific, the following problem called the adaptive regulation with almost disturbance decoupling will be dealt with in this paper. In [28], some new results regarding the boundedness, stability and attractivity were provided for a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may contain time-dependent coefficients.

Adaptive Regulation with Almost Disturbance Decoupling (ARADD): Consider the power integrators with nonlinearly parameterized lower triangular structure (1). Given any real number $\gamma > 0$, find, if possible, a smooth adaptive controller

$$\begin{cases} \dot{\hat{\theta}} = \psi(x_{[n]}, \hat{\theta}), \ \psi(0,0) = 0, \\ u = u(x_{[n]}, \hat{\theta}), \ u(0,0) = 0, \end{cases} \tag{2}$$

such that the closed-loop system (1) – (2) satisfies the following:

1) when $w = 0$, the closed-loop system is globally stable in the sense of Lyapunov, and globally asymptotical regulation of the state is achieved, i.e., $\lim_{t\to\infty} x_{[n]}(t) = 0$.

2) for any disturbance $w \in L_2$, the response of the closed-loop system starting from the initial state $x(0) = 0$ is such that

$$\int_0^t |y(s)|^{2p_1} \, \mathrm{d}s \le \gamma^2 \int_0^t \|w(s)\|^2 \, \mathrm{d}s, \text{ for any } t \ge 0.$$

In order to solve the ARADD problem, the following assumptions are needed.

Assumption A1: $p_1 \ge p_2 \ge \cdots \ge p_n$ are odd integers.

Assumption A2: For any $i \in I[1,n]$,

$$\left|f_i(x_{[i]})\right| \le \alpha_i(x_{[i]}) \sum_{j=1}^{i} |x_j|^{p_i}, \tag{3}$$

where $\alpha_i(\cdot)$ is a nonnegative smooth function.

Assumption A3: For any $i \in I[1,n]$, $\left\|g_i(x_{[i]})\right\| \le \varphi_i(x_{[i]})$, where $\varphi_i(\cdot)$ is known bounding function that is nonnegative and smooth.

Assumption A4: For any $i \in I[1,n]$, $\left|\phi_i(x_{[i]},\theta)\right| \le \beta_i(x_{[i]},\theta) \sum_{j=1}^{i} |x_j|^{p_i}$, where $\beta_i(\cdot)$ is a nonnegative continuous function.

Before ending this section, we provide some useful lemmas. The first lemma is a slight extension of the well-known Young's inequality, and will be repeatedly used in the design of the adaptive controller. The proof can be found in [17].

Lemma 2.1 *For any positive integers m, n, and any real-valued function $\gamma(x,y) > 0$, the following inequality holds:*

$$|x|^m |y|^n \le \frac{m}{m+n}\gamma(x,y) |x|^{m+n} + \frac{n}{m+n}\gamma^{-m/n}(x,y) |y|^{m+n}.$$

By applying the above lemma, one can easily obtain the following conclusion [16]. This result will also play a vital role in the adding a power integrator design.

Lemma 2.2 *Let x, y and z, be real variables. Assume that $g_1 : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Then, for any positive integers m, n and real number $N > 0$, there exists a nonnegative smooth function $h_1 : \mathbb{R}^3 \to \mathbb{R}$ such that the following relation holds:*

$$|x^m [(y + xg_1(x,z))^n - (xg_1(x,z))^n]| \le \frac{|x|^{m+n}}{N} + |y|^{m+n} h_1(x,y,z).$$

The following lemma provides the parameter separation principle. It is this principle that enables us to deal with nonlinear parameterization. A constructive proof of the result can be found in [24].

Lemma 2.3 *For any real-valued continuous function $f(x,y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, there are smooth scalar functions $a(x) \geq 1$ and $b(y) \geq 1$, such that $|f(x,y)| \leq a(x)b(y)$.*

3 Global Adaptive Regulation

In this section we solve the problem of adaptive regulation with almost disturbance decoupling for the power integrator lower triangular system (1). Using the adding a power integrator technique as the design tool, we will explicitly construct a one-dimensional adaptive controller that solves the problem of ARADD with the help of the parameter separation technique provided in Lemma 2.3. Now we are ready to present the main result.

Theorem 3.1 *Under the condition of Assumptions A1 – A4, the ARADD problem for system (1) is solvable by a one-dimensional smooth adaptive controller*

$$\begin{cases} \dot{\hat{\Theta}} = \psi(x_{[n]}, \hat{\Theta}), \ \hat{\Theta} \in \mathbb{R}, \ \psi(0,0) = 0, \\ u = u(x_{[n]}, \hat{\Theta}), \ u(0,0) = 0. \end{cases} \tag{4}$$

Proof The proof is based on a feedback domination design approach which combines the technique of adding one power integrator [17, 23] with the parameter separation method [24]. The conclusion is obtained by applying mathematical induction method. Firstly, we need some preliminaries with the help of the parameter separation technique given in Lemma 2.3.

By Lemma 2.3, there exist two smooth functions $c_i(\theta) \geq 1$ and $\gamma_i(x_{[i]}) \geq 1$ satisfying

$$\beta_i(x_{[i]}, \theta) \leq \gamma_i(x_{[i]}) c_i(\theta).$$

Since θ is a constant vector, $c_i(\theta)$ is also a constant. Let $\Theta := \sum_{i=1}^n c_i(\theta)$ be a new unknown constant. Then Assumption A4 implies that there are smooth function $\gamma_i(x_{[i]}) \geq 1$ and an unknown constant $\Theta \geq 1$, such that

$$\left|\phi_i(x_{[i]}, \theta)\right| \leq \gamma_i(x_{[i]}) \Theta \sum\nolimits_{j=1}^{i} |x_j|^{p_i}. \tag{5}$$

Now we proceed to construct the smooth adaptive controller to solve the ARADD problem for the system (1).

Step 1: Define $\tilde{\Theta} = \Theta - \hat{\Theta}$, where $\hat{\Theta}(t)$ is the estimate of Θ to be designed later. Consider the Lyapunov function

$$V_1(x_1, \hat{\Theta}) = \frac{1}{p_1 + 1} x_1^{p_1+1} + \frac{1}{2}\tilde{\Theta}^2.$$

By (3), (5) and Lemma 2.1, there exist a smooth function $\rho_0(x_1) \geq 0$, such that for any $\beta > 0$

$$\begin{aligned}
& \dot{V}_1(x_1, \hat{\Theta}) + y^{2p_1} - \beta \|w\|^2 \\
= \; & x_1^{p_1}(x_2^{p_1} + f_1(x_1) + g_1(x_1)w + \phi_1(x_1, \theta)) - \dot{\hat{\Theta}}\tilde{\Theta} + y^{2p_1} - \beta \|w\|^2 \\
\leq \; & x_1^{p_1}x_2^{p_1} + x_1^{2p_1}\alpha_1(x_1) + |x_1^{p_1}|\,\varphi_1(x_1)\|w\| + x_1^{2p_1}\gamma_1(x_1)(\tilde{\Theta} + \hat{\Theta}) - \dot{\hat{\Theta}}\tilde{\Theta} \\
& + x_1^{2p_1}\rho_0(x_1) - \beta \|w\|^2 \\
\leq \; & x_1^{p_1}x_2^{p_1} + x_1^{2p_1}\alpha_1(x_1) + \frac{x_1^{2p_1}\varphi_1^2(x_1)}{4\beta} + x_1^{2p_1}\gamma_1(x_1)\hat{\Theta} \\
& + x_1^{2p_1}\rho_0(x_1) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\tilde{\Theta} \\
\leq \; & x_1^{p_1}x_2^{p_1} + x_1^{2p_1}\rho_1(x_1, \hat{\Theta}) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\tilde{\Theta},
\end{aligned}$$

where

$$\rho_1(x_1, \hat{\Theta}) = \alpha_1(x_1) + \gamma_1(x_1)\sqrt{\hat{\Theta}^2 + 1} + \frac{\varphi_1^2(x_1)}{4\beta} + \rho_0(x_1) \geq 0$$

and

$$\Psi_1(x_1, \hat{\Theta}) = x_1^{2p_1}\gamma_1(x_1) \geq 0.$$

It is easy to check that that the virtual controller

$$x_2^*(x_1, \hat{\Theta}) = -x_1\left[n + \rho_1(x_1, \hat{\Theta})\right]^{1/p_1} \tag{6}$$

satisfies

$$\dot{V}_1(x_1, \hat{\Theta}) + y^{2p_1} - \beta \|w\|^2 \leq -nx_1^{2p_1} + x_1^{p_1}(x_2^{p_1} - x_2^{*p_1}) + (\Psi_1(x_1, \hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta} + \eta_1\right) \tag{7}$$

with $\eta_1 = 0$. Moreover, the virtual control function $x_2^*(x_1, \hat{\Theta})$ is smooth due to the smooth nonnegativeness of functions $\alpha_1(x_1)$,$\gamma_1(x_1)$ and $\varphi_1(x_1)$. In addition

$$\left|\Psi_1(\xi_1, \hat{\Theta})\right| \leq |x_1|^{2p_1}\bar{\alpha}_1(\xi_1, \hat{\Theta}), \ \bar{\alpha}_1(\xi_1, \hat{\Theta}) = \gamma_1(\xi_1) \geq 0. \tag{8}$$

Step 2: Consider the (x_1, x_2)-subsystem of (1). For convenience, we let $x_1^* = 0$ in the sequential discussion. The change of coordinate

$$\xi_1 = x_1, \quad \xi_2 = x_2 - x_2^*(\xi_1, \hat{\Theta})$$

transforms the (x_1, x_2)-subsystem of (1) into

$$\begin{aligned}
\dot{\xi}_1 &= \delta_1(\xi_{[2]}, \hat{\Theta}) + \Phi_1(\xi_1, \hat{\Theta}, \theta) + G_1(\xi_1)w - \omega_1(\hat{\Theta})\dot{\hat{\Theta}}, \\
\dot{\xi}_2 &= x_3^{p_2} + \Delta_2(\xi_{[2]}, \hat{\Theta}) + \Phi_2(\xi_{[2]}, \hat{\Theta}, \theta) + G_2(\xi_{[2]})w - \omega_2(\xi_1, \hat{\Theta})\dot{\hat{\Theta}},
\end{aligned}$$

where

$$
\begin{aligned}
&\delta_1(\xi_{[2]},\hat{\Theta}) = (\xi_2 + x_2^*)^{p_1} + f_1(\xi_1),\\
&\Delta_2(\xi_{[2]},\hat{\Theta}) = f_2(\xi_{[2]} + x_{[2]}^*) - \tfrac{\partial x_2^*}{\partial \xi_1}\delta_1(\xi_{[2]},\hat{\Theta}),\\
&\Phi_1(\xi_1,\hat{\Theta},\theta) = \phi_1(\xi_1,\theta),\\
&\Phi_2(\xi_{[2]},\hat{\Theta},\theta) = \phi_2(\xi_{[2]} + x_{[2]}^*,\theta) - \tfrac{\partial x_2^*}{\partial \xi_1}\Phi_1(\xi_1,\hat{\Theta},\theta),\\
&G_1(\xi_1) = g_1(\xi_1),\\
&G_2(\xi_{[2]}) = g_2(\xi_{[2]} + x_{[2]}^*) - \tfrac{\partial x_2^*}{\partial \xi_1}G_1(\xi_1),\\
&\omega_1(\hat{\Theta}) = 0,\\
&\omega_2(\xi_1,\hat{\Theta}) = \tfrac{\partial x_2^*}{\partial \hat{\Theta}} - \tfrac{\partial x_2^*}{\partial \xi_1}\omega_1(\hat{\Theta}).
\end{aligned}
$$

Under the condition of Assumption A4, it follows from the relation (5) that

$$\left|\Phi_1(\xi_1,\hat{\Theta},\theta)\right| \le |x_1|^{p_1}\,\bar{\beta}_1(\xi_1,\hat{\Theta})\Theta,\ \bar{\beta}_1(\xi_1,\hat{\Theta}) = \gamma_1(\xi_1) \ge 0.$$

With this relation combined with (5), we have by applying Lemma 2.1

$$
\begin{aligned}
\left|\Phi_2(\xi_{[2]},\hat{\Theta},\theta)\right| &\le \left|\phi_2(\xi_{[2]} + x_{[2]}^*,\theta)\right| + \left|\frac{\partial x_2^*}{\partial \xi_1}\Phi_1(\xi_1,\theta)\right|\\
&\le \left(|\xi_1|^{p_2} + |\xi_2 + x_2^*|^{p_2}\right)\gamma_2(\xi_{[2]} + x_{[2]}^*)\Theta + \left|\frac{\partial x_2^*}{\partial \xi_1}\right| |x_1|^{p_1}\,\gamma_1(x_1)\Theta\\
&\le \left(|\xi_1|^{p_2} + |\xi_2|^{p_2}\right)\tilde{\beta}_2(\xi_{[2]},\hat{\Theta})\Theta + |\xi_1|^{p_1}\left|\frac{\partial x_2^*}{\partial \xi_1}\right|\gamma_1(\xi_1)\Theta
\end{aligned}
$$

for a smooth function $\tilde{\beta}_2(\cdot) \ge 0$. This inequality implies that

$$\left|\Phi_2(\xi_{[2]},\hat{\Theta},\theta)\right| \le \left(|\xi_1|^{p_2} + |\xi_2|^{p_2}\right)\bar{\beta}_2(\xi_{[2]},\hat{\Theta})\Theta \tag{9}$$

for a smooth function $\bar{\beta}_2(\cdot) \ge 0$, because $p_1 \ge p_2$. By similar ways, it is easy to show that the following two relations hold

$$
\begin{aligned}
&\left|\delta_1(\xi_{[2]},\hat{\Theta})\right| \le \left(|\xi_1|^{p_1} + |\xi_2|^{p_1}\right)\bar{\tau}_1(\xi_{[2]},\hat{\Theta}),\\
&\left|\Delta_2(\xi_{[2]},\hat{\Theta})\right| \le \left(|\xi_1|^{p_2} + (|\xi_2|^{p_2}\right)\tilde{\tau}_2(\xi_{[2]},\hat{\Theta}),
\end{aligned} \tag{10}
$$

for two nonnegative smooth functions $\bar{\tau}_1(\xi_{[2]},\hat{\Theta})$ and $\tilde{\tau}_2(\xi_{[2]},\hat{\Theta})$. By Assumption A3, it is known that there exist smooth nonnegative functions $\tilde{\varphi}_1(\xi_1)$ and $\tilde{\varphi}_2(\xi_{[2]})$, satisfying

$$
\begin{aligned}
&\|G_1(\xi_1)\| \le \varphi_1(\xi_1) = \tilde{\varphi}_1(\xi_1),\\
&\left\|G_2(\xi_{[2]})\right\| \le \left\|g_2(\xi_{[2]} + x_{[2]}^*) - \frac{\partial x_2^*}{\partial \xi_1}G_1(\xi_1)\right\| \le \tilde{\varphi}_2(\xi_{[2]}).
\end{aligned} \tag{11}
$$

Again using Lemma 2.1, we have from (9) and (10) that

$$
\begin{aligned}
&\left|\xi_2^{2p_1-p_2}\Phi_2(\xi_{[2]},\hat{\Theta},\theta)\right| \le \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\Theta\\
&\le \frac{1}{6}\xi_1^{2p_1} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\tilde{\Theta},
\end{aligned} \tag{12}
$$

$$\left|\xi_2^{2p_1-p_2}\Delta_2(\xi_{[2]},\hat{\Theta})\right| \le \frac{1}{6}\xi_1^{2p_1} + \xi_2^{2p_1}\check{\rho}_2(\xi_{[2]},\hat{\Theta}), \tag{13}$$

for some nonnegative smooth functions $\bar{\rho}_2(\xi_{[2]},\hat{\Theta})$ and $\check{\rho}_2(\xi_{[2]},\hat{\Theta})$. By applying Lemma 2.2, it is easy to show from (6) that

$$\left|\xi_1^{p_1}((\xi_2+x_2^*)^{p_1} - x_2^{*p_1})\right| \le \frac{\xi_1^{2p_1}}{6} + \xi_2^{2p_1}\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}), \tag{14}$$

for some nonnegative smooth function $\tilde{\rho}_2(\xi_{[2]},\hat{\Theta})$. Now, consider the Lyapunov function

$$V_2(\xi_{[2]},\hat{\Theta}) = V_1(x_1,\hat{\Theta}) + \frac{\xi_2^{2p_1-p_2+1}}{2p_1-p_2+1}$$

which is positive definite and radially unbounded. With the relations (7), (12), (13) and (14), a straightforward computation gives

$$\begin{aligned}
&\dot{V}_2(\xi_{[2]},\tilde{\Theta}) + y^{2p_1} - 2\beta\|w\|^2 \\
\le\ & -n\xi_1^{2p_1} + \xi_1^{p_1}((\xi_2+x_2^*)^{p_1} - x_2^{*p_1}) + (\Psi_1(x_1,\hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta}+\eta_1\right) \\
& +\xi_2^{2p_1-p_2}\left(x_3^{p_2} + \Delta_2(\xi_{[2]},\hat{\Theta}) + \Phi_2(\xi_{[2]},\hat{\Theta},\theta) + G_2(\xi_{[2]})w\right) \\
& -\xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta})\dot{\hat{\Theta}} - \beta\|w\|^2 \\
\le\ & -\left(n-\frac{1}{2}\right)\xi_1^{2p_1} + \xi_2^{2p_1}\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}) + (\Psi_1(x_1,\hat{\Theta}) - \dot{\hat{\Theta}})\left(\tilde{\Theta}+\eta_1\right) + \xi_2^{2p_1-p_2}G_2(\xi_{[2]})w \\
& +\xi_2^{2p_1-p_2}x_3^{p_2} + \xi_2^{2p_1}\check{\rho}_2(\xi_{[2]},\hat{\Theta}) + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} - \beta\|w\|^2 \\
& +\left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\tilde{\Theta} - \xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta})\dot{\hat{\Theta}} \\
=\ & -\left(n-\frac{1}{2}\right)\xi_1^{2p_1} + \xi_2^{2p_1-p_2}x_3^{p_2} + \xi_2^{2p_1}\left[\tilde{\rho}_2(\xi_{[2]},\hat{\Theta}) + \bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_2(\xi_{[2]},\hat{\Theta})\right] \\
& +\xi_2^{2p_1-p_2}G_2(\xi_{[2]})w - \beta\|w\|^2 + \left(\Psi_2(\xi_{[2]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta}+\eta_2(\xi_{[2]},\hat{\Theta})\right) + \Pi_2(\xi_{[2]},\hat{\Theta}),
\end{aligned} \tag{15}$$

where

$$\Psi_2(\xi_{[2]},\hat{\Theta}) = \Psi_1(\xi_1) + \frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta}),$$

$$\eta_2(\xi_{[2]},\hat{\Theta}) = \eta_1 + \xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta}),$$

$$\Pi_2(\xi_{[2]},\hat{\Theta}) = -\Psi_2(\xi_{[2]},\hat{\Theta})\xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta}) - \left[\frac{\xi_1^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_1^2)} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\right]\eta_1. \tag{16}$$

By applying Lemma 2.1 again, it is easy to derive from the relation (8) that

$$\left|\Psi_2(\xi_{[2]},\hat{\Theta})\right| \le (\xi_1^{2p_1} + \xi_2^{2p_1})\bar{\alpha}_2(\xi_{[2]},\hat{\Theta}) \tag{17}$$

for a smooth function $\bar{\alpha}_2(\xi_{[2]},\hat{\Theta}) \geq 0$. By the completion of square, it is easily derived from (11) that

$$\left\|\xi_2^{2p_1-p_2} G_2(\xi_{[2]})w\right\| \leq \left|\xi_2^{2p_1-p_2}\right| \tilde{\varphi}_2(\xi_{[2]}) \|w\| \leq \xi_2^{2p_1} \frac{\xi_2^{2p_1-2p_2}\tilde{\varphi}_2^2(\xi_{[2]})}{4\beta} + \beta \|w\|^2 . \quad (18)$$

In view of the relation (17), by using Lemma 2.1 it is easily obtained from (16) that the following relation holds

$$\begin{aligned} & \left|\Pi_2(\xi_{[2]},\hat{\Theta})\right| \quad (19) \\ \leq \; & (\xi_1^{2p_1}+\xi_2^{2p_1})\bar{\alpha}_2(\xi_{[2]},\hat{\Theta}) \left|\xi_2^{2p_1-p_2}\omega_2(\xi_1,\hat{\Theta})\right| + \frac{1}{6}\xi_1^{2p_1} + \xi_2^{2p_1}\bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\eta_1^2+1} \\ \leq \; & \frac{\xi_1^{2p_1}}{2} + \xi_2^{2p_1}\hat{\rho}_2(\xi_{[2]},\hat{\Theta}). \end{aligned}$$

for a nonnegative smooth functions $\hat{\rho}_2(\cdot)$. With the relations (14) – (19) in mind, it follows from (15) that

$$\begin{aligned} \dot{V}_2(\xi_{[2]},\hat{\Theta}) \leq \; & -(n-1)x_1^{2p_1} + \left(\Psi_2(\xi_{[2]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_2(\xi_{[2]},\hat{\Theta})\right) \\ & + \xi_2^{2p_1}\rho_2(\xi_{[2]},\hat{\Theta}) + \xi_2^{2p_1-p_2}\left(x_3^{p_2} - x_3^{*p_2}\right) + \xi_2^{2p_1-p_2}x_3^{*p_2}, \end{aligned}$$

where

$$\rho_2(\xi_{[2]},\hat{\Theta}) = \tilde{\rho}_2(\xi_{[2]},\hat{\Theta}) + \breve{\rho}_2(\xi_{[2]},\hat{\Theta}) + \bar{\rho}_2(\xi_{[2]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \hat{\rho}_2(\xi_{[2]},\hat{\Theta}) + \frac{\xi_2^{2p_1-2p_2}\tilde{\varphi}_2^2(\xi_{[2]})}{4\beta} \geq 0.$$

Choose

$$x_3^* = -\xi_2\left[n-1+\rho_2(\xi_{[2]},\hat{\Theta})\right]^{1/p_2}.$$

This smooth virtual controller will satisfy

$$\begin{aligned} \dot{V}_2(\xi_{[2]},\hat{\Theta}) + y^{2p_1} - 2\beta\|w\|^2 \leq \; & -(n-1)(\xi_1^{2p_1}+\xi_2^{2p_1}) + \xi_2^{2p_1-p_2}\left(x_3^{p_2} - x_3^{*p_2}\right) \\ & + \left(\Psi_2(\xi_{[2]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_2(\xi_{[2]},\hat{\Theta})\right). \end{aligned}$$

Inductive Step: Suppose for the system (1) with dimension k, there is a global change of coordinates $\xi_i = x_i - x_i^*(\xi_{[i-1]},\hat{\Theta})$, $i \in I[1,k]$, transforming (1) into the system

$$\begin{aligned} \dot{\xi}_1 &= \delta_1(\xi_{[2]},\hat{\Theta}) + \Phi_1(\xi_1,\hat{\Theta},\theta) + G_1(\xi_1)w - \omega_1(\hat{\Theta})\dot{\hat{\Theta}}, \\ & \cdots \quad (20) \\ \dot{\xi}_{k-1} &= \delta_{k-1}(\xi_{[k]},\hat{\Theta}) + \Phi_{k-1}(\xi_{[k-1]},\hat{\Theta},\theta) + G_{k-1}(\xi_{[k-1]})w - \omega_{k-1}(\xi_{[k-2]},\hat{\Theta})\dot{\hat{\Theta}}, \\ \dot{\xi}_k &= x_{k+1}^{p_k} + \Delta_k(\xi_{[k]},\hat{\Theta}) + \Phi_k(\xi_{[k]},\hat{\Theta},\theta) + G_k(\xi_{[k]})w - \omega_k(\xi_{[k-1]},\hat{\Theta})\dot{\hat{\Theta}}, \end{aligned}$$

where

$$x_i^* = -\xi_{i-1}\left[n-i+2+\rho_{i-1}(\xi_{[i-1]},\hat{\Theta})\right]^{1/p_{i-1}}, \; i \in I[2,k], \quad (21)$$

$$\left|\Phi_i(\xi_{[i]},\hat{\Theta},\theta)\right| \le \bar{\beta}_i(\xi_{[i]},\hat{\Theta})\Theta\sum_{j=1}^{i}|\xi_j|^{p_i}\ ,\ i\in I[1,k], \tag{22}$$

$$\left|\delta_i(\xi_{[i+1]},\hat{\Theta},\theta)\right| \le \bar{\tau}_i(\xi_{[i+1]},\hat{\Theta})\sum_{j=1}^{i}|\xi_j|^{p_i}\ ,\ i\in I[1,k-1], \tag{23}$$

$$\left|\Delta_k(\xi_{[k]},\hat{\Theta},\theta)\right| \le \tilde{\tau}_k(\xi_{[k]},\hat{\Theta})\sum_{j=1}^{k}|\xi_j|^{p_k}\ , \tag{24}$$

$$\left\|G_i(\xi_{[i]})\right\| \le \tilde{\varphi}_i(\xi_{[i]}), \tag{25}$$

for some nonnegative smooth functions $\bar{\beta}_i(\cdot)$, $\tilde{\varphi}_i(\cdot)$, $i\in I[1,k]$, $\tilde{\tau}_k(\cdot)$, and $\rho_i(\cdot)$, $\bar{\tau}_i(\cdot)$, $i\in I[1,k-1]$. Moreover, there is a virtual controller

$$x^*_{k+1}(x_{[k]},\hat{\Theta}) = -\xi_k\left[n-k+1+\rho_k(\xi_{[k]},\hat{\Theta})\right]^{1/p_k},\ \rho_k(\xi_{[k]},\hat{\Theta})\ge 0, \tag{26}$$

such that the closed-loop system (20) – (26) satisfies

$$\begin{aligned}\dot{V}_k(\xi_{[k]},\hat{\Theta}) + y^{2p_1} - k\beta\|w\|^2 \quad\le\quad & -(n-k+1)\sum_{i=1}^{k}\xi_i^{2p_1} + \xi_k^{2p_1-p_k}\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right) \\ & + \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right),\end{aligned} \tag{27}$$

where

$$V_k(\xi_{[k]},\hat{\Theta}) = \frac{1}{2}\tilde{\Theta}^2 + \sum_{i=1}^{k}\frac{\xi_i^{2p_1-p_i+1}}{2p_1-p_i+1},$$

is a positive definite and proper Lyapunov function. Moreover,

$$\left|\Psi_k(\xi_{[k]},\hat{\Theta})\right| \le \bar{\alpha}_k(\xi_{[k]},\hat{\Theta})\sum_{i=1}^{k}\xi_i^{2p_1}. \tag{28}$$

Then, in the case when the dimension of system (1) is equal to $k+1$, introduce the transformation $\xi_{k+1} = x_{k+1} - x^*_{k+1}(\xi_{[k]},\hat{\Theta})$. This, together with (26), leads to the augmented system

$$\begin{aligned}\dot{\xi}_1 &= \delta_1(\xi_{[2]},\hat{\Theta}) + \Phi_1(\xi_1,\hat{\Theta},\theta) + G_1(\xi_1) - \omega_1(\hat{\Theta})\dot{\hat{\Theta}}, \\ & \cdots \\ \dot{\xi}_k &= \delta_k(\xi_{[k+1]},\hat{\Theta}) + \Phi_k(\xi_{[k]},\hat{\Theta},\theta) + G_k(\xi_{[k]}) - \omega_k(\xi_{[k-1]},\hat{\Theta})\dot{\hat{\Theta}}, \\ \dot{\xi}_{k+1} &= x_{k+2}^{p_{k+1}} + \Delta_{k+1}(\xi_{[k+1]},\hat{\Theta}) + \Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta), \\ & +G_{k+1}(\xi_{[k+1]})w - \omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}},\end{aligned} \tag{29}$$

where

$$\omega_{k+1}(\xi_{[k]},\hat{\Theta}) = -\sum_{i=1}^{k}\frac{\partial x^*_{k+1}}{\partial\xi_i}\omega_i(\xi_{[i-1]},\hat{\Theta}) + \frac{\partial x^*_{k+1}}{\partial\hat{\Theta}},$$

$$\delta_k(\xi_{[k+1]},\hat{\Theta}) = \Delta_k(\xi_{[k+1]},\hat{\Theta}) + (\xi_{k+1} + x^*_{k+1})^{p_k},$$

$$\Delta_{k+1}(\xi_{[k+1]},\hat{\Theta}) = f_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i}\delta_i(\xi_{[i+1]},\hat{\Theta}),$$

$$\Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta) = \phi_{k+1}(\xi_{[k+1]} + x^*_{[k+1]},\theta) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i}\Phi_i(\xi_{[i]},\hat{\Theta},\theta),$$

$$G_{k+1}(\xi_{[k+1]}) = g_{k+1}(\xi_{[k]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i}G_i(\xi_{[i]}).$$

Under the condition of Assumption 4, the relation (5) holds. Combining this relation with the inductive assumption (22) and (21), and applying Lemma 2.1, we have

$$\begin{aligned}
\left|\Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta)\right| &\le \left|\phi_{k+1}(\xi_{[k+1]} + x^*_{[k+1]},\theta)\right| + \sum_{i=1}^{k}\left|\frac{\partial x^*_{k+1}}{\partial \xi_i}\right|\Phi_i(\xi_{[i]},\hat{\Theta},\theta) \\
&\le \gamma_{k+1}(\xi_{[k+1]} + x^*_{[k+1]})\Theta\sum_{i=1}^{k+1}|\xi_i + x^*_i|^{p_{k+1}} \\
&\quad + \sum_{i=1}^{k}\left[\left|\frac{\partial x^*_{k+1}}{\partial \xi_i}\right|\bar{\beta}_i(\xi_{[i]},\hat{\Theta})\Theta\sum_{j=1}^{i}|\xi_j|^{p_i}\right].
\end{aligned}$$

In view of the fact that and $p_1 \ge p_2 \ge \cdots \ge p_{k+1}$, there exists a smooth function $\bar{\beta}_{k+1}(\xi_{[k+1]},\hat{\Theta}) \ge 0$, such that

$$\left|\Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta)\right| \le \Theta\bar{\beta}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sum_{i=1}^{k+1}|\xi_i|^{p_{k+1}}. \tag{30}$$

Under the condition of Assumption A3, by applying the inductive assumption (25) and the smoothness of x^*_i, $i \in I[1,k]$, it is known that there is a smooth function $\tilde{\varphi}_{k+1}(\xi_{[k+1]})$ to satisfy

$$\left\|G_{k+1}(\xi_{[k+1]})\right\| \le \tilde{\varphi}_{k+1}(\xi_{[k+1]}). \tag{31}$$

According to the inductive assumptions (24) and (21), we can obtain by using Lemma 2.1 again

$$\begin{aligned}
\left|\delta_k(\xi_{[k+1]},\hat{\Theta})\right| &\le \left|\Delta_k(\xi_{[k]},\hat{\Theta})\right| + \left|(\xi_{k+1} + x^*_{k+1})^{p_k}\right| \\
&\le \tilde{\tau}_k(\xi_{[k]},\hat{\Theta})\sum_{i=1}^{k}|\xi_i|^{p_k} + \left|(\xi_{k+1} + x^*_{k+1})^{p_k}\right| \qquad (32) \\
&\le \bar{\tau}_k(\xi_{[k+1]},\hat{\Theta})\sum_{i=1}^{k+1}|\xi_i|^{p_k}
\end{aligned}$$

for some smooth function $\bar{\tau}_k(\xi_{[k+1]}, \hat{\Theta}) \geq 0$. According to Assumption 2 and by the relations (32) and (23), it can be derived that

$$\begin{aligned}
\left|\Delta_{k+1}(\xi_{[k+1]}, \hat{\Theta})\right| &\leq \left|f_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) - \sum_{i=1}^{k} \frac{\partial x^*_{k+1}}{\partial \xi_i} \delta_i(\xi_{[i+1]}, \hat{\Theta})\right| \\
&\leq \alpha_{k+1}(\xi_{[k+1]} + x^*_{[k+1]}) \sum_{i=1}^{k+1} |\xi_i + x_i^*|^{p_{k+1}} \\
&\quad + \sum_{i=1}^{k} \left[\left|\frac{\partial x^*_{k+1}}{\partial \xi_i}\right| \bar{\tau}_i(\xi_{[i+1]}, \hat{\Theta}) \sum_{j=1}^{i+1} |\xi_j|^{p_i}\right] \\
&\leq \tilde{\tau}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \sum_{j=1}^{k+1} |\xi_j|^{p_{k+1}}
\end{aligned} \tag{33}$$

for a smooth function $\tilde{\tau}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \geq 0$. Again using Lemma 2.1, we have from (30)

$$\begin{aligned}
&\left|\xi_{k+1}^{2p_1 - p_{k+1}} \Phi_{k+1}(\xi_{[k+1]}, \hat{\Theta}, \theta)\right| \\
\leq\ & \left[\frac{\sum_{i=1}^{k} \xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]}, \hat{\Theta}))} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})\right] \Theta \\
\leq\ & \frac{1}{6} \sum_{i=1}^{k} \xi_i^{2p_1} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \sqrt{\hat{\Theta}^2 + 1} && (34) \\
+\ & \left[\frac{\sum_{i=1}^{k} \xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]}, \hat{\Theta}))} + \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})\right] \tilde{\Theta}, && (35)
\end{aligned}$$

$$\left|\xi_{k+1}^{2p_1 - p_2} \Delta_{k+1}(\xi_{[k+1]}, \hat{\Theta})\right| \leq \frac{1}{6} \sum_{i=1}^{k} \xi_i^{2p_1} + \xi_{k+1}^{2p_1} \check{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}), \tag{36}$$

for some nonnegative smooth functions $\bar{\rho}_{k+1}(\xi_{[2]}, \hat{\Theta})$ and $\check{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})$. By applying Lemma 2.2, it is easy to show that

$$\left|\xi_k^{2p_1 - p_k}\left(\left(\xi_{k+1} + x^*_{k+1}\right)^{p_k} - x^{*p_k}_{k+1}\right)\right| \leq \frac{1}{6} \xi_k^{2p_1} + \xi_{k+1}^{2p_1} \tilde{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta}), \tag{37}$$

for some nonnegative smooth function $\tilde{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})$. Now consider the Lyapunov function

$$V_{k+1}(\xi_{[k+1]}, \hat{\Theta}) = V_k(\xi_{[k]}, \hat{\Theta}) + \frac{\xi_{k+1}^{2p_1 - p_{k+1} + 1}}{2p_1 - p_{k+1} + 1}.$$

With the relations (35), (36) and (37) and the inductive assumption (27), it is derived that the time derivative of V_{k+1} along the trajectories of system (29) satisfies

$$
\begin{aligned}
& \dot{V}_{k+1} + y^{2p_1} - (k+1)\beta \|w\|^2 \\
\leq\ & -(n-k+1)\sum_{i=1}^{k} \xi_i^{2p_1} + \xi_k^{2p_1-p_k}\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right) \\
+\ & \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right) \\
+\ & \xi_{k+1}^{2p_1-p_{k+1}}\left(x_{k+2}^{p_{k+1}} + \Delta_{k+1}(\xi_{[k+1]},\hat{\Theta}) + \Phi_{k+1}(\xi_{[k+1]},\hat{\Theta},\theta\right. \\
+\ & \left. G_{k+1}(\xi_{[k+1]})w - \omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}}\right) - \beta\|w\|^2 \\
\leq\ & -(n-k+1)\sum_{i=1}^{k} \xi_i^{2p_1} + \frac{1}{2}\sum_{i=1}^{k}\xi_i^{2p_1} + \left(\Psi_k(\xi_{[k]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_k(\xi_{[k]},\hat{\Theta})\right) \\
+\ & \xi_{k+1}^{2p_1-p_{k+1}} x_{k+2}^{p_{k+1}} + \xi_{k+1}^{2p_1}\left[\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right. \\
+\ & \left.\tilde{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right] - \xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta})\dot{\hat{\Theta}} \\
+\ & \left[\frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\xi_{[k]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right]\tilde{\Theta} \\
+\ & \xi_{k+1}^{2p_1-p_{k+1}} G_{k+1}(\xi_{[k+1]})w - \beta\|w\|^2
\end{aligned}
$$

$$
\begin{aligned}
=\ & -(n-k+\frac{1}{2})\sum_{i=1}^{k}\xi_i^{2p_1} + \xi_{k+1}^{2p_1}\left[\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sqrt{\hat{\Theta}^2+1} + \check{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right. && (38) \\
+\ & \left.\tilde{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right] + \xi_2^{2p_1-p_2}G_{k+1}(\xi_{[k+1]})w - \beta\|w\|^2 + \xi_{k+1}^{2p_1-p_{k+1}}x_{k+2}^{p_{k+1}} && (39) \\
& + \left(\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta}) - \dot{\hat{\Theta}}\right)\left(\tilde{\Theta} + \eta_{k+1}(\xi_{[k+1]},\hat{\Theta})\right) + \Pi_{k+1}(\xi_{[k+1]},\hat{\Theta}),
\end{aligned}
$$

where

$$
\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta}) = \Psi_k(\xi_{[k]},\hat{\Theta}) + \frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(x_{[k]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta}),
$$

$$
\eta_{k+1}(\xi_{[k+1]},\hat{\Theta}) = \eta_k(\xi_{[k]},\hat{\Theta}) + \xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta})
$$

$$
\begin{aligned}
\Pi_{k+1}(\xi_{[k+1]},\hat{\Theta}) \ =\ & -\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})\xi_{k+1}^{2p_1-p_{k+1}}\omega_{k+1}(\xi_{[k]},\hat{\Theta}) && (40) \\
-\ & \left[\frac{\sum_{i=1}^{k}\xi_i^{2p_1}}{3(1+\hat{\Theta}^2)(1+\eta_k^2(\xi_{[k+1]},\hat{\Theta}))} + \xi_{k+1}^{2p_1}\bar{\rho}_{k+1}(\xi_{[k+1]},\hat{\Theta})\right]\eta_k(\xi_{[k+1]},\hat{\Theta}).
\end{aligned}
$$

By using Lemma 2.1, it is easily derived from (28) that the following relation holds

$$
\left|\Psi_{k+1}(\xi_{[k+1]},\hat{\Theta})\right| \leq \bar{\alpha}_{k+1}(\xi_{[k+1]},\hat{\Theta})\sum_{i=1}^{k+1}|\xi_i|^{2p_1} \tag{41}
$$

for a smooth function $\bar{\alpha}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \geq 0$. In view of relation (41), we have

$$\begin{aligned}
& \Pi_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \\
\leq \;& \bar{\alpha}_{k+1}(\cdot) \left| \xi_{k+1}^{2p_1 - p_{k+1}} \omega_{k+1}(\cdot) \right| \sum_{i=1}^{k+1} |\xi_i|^{2p_1} + \frac{1}{6} \sum\nolimits_{i=1}^{k} \xi_i^{2p_1} \qquad (42) \\
+ \;& \xi_{k+1}^{2p_1} \bar{\rho}_{k+1}(\cdot) \sqrt{\eta_k^2(\cdot) + 1} \leq \frac{1}{2} \sum_{i=1}^{k} \xi_i^{2p_1} + \xi_{k+1}^{2p_1} \hat{\rho}_{k+1}(\xi_{[k+1]}, \hat{\Theta})
\end{aligned}$$

for a smooth function $\hat{\rho}_{k+1}(\cdot)$. On the other hand, it follows from (31) that

$$\begin{aligned}
\left\| \xi_2^{2p_1 - p_2} G_{k+1}(\xi_{[k+1]}) w \right\| & \leq \left| \xi_2^{2p_1 - p_2} \right| \tilde{\varphi}_{k+1}(\xi_{[k+1]}) \|w\| \\
& \leq \xi_2^{2p_1} \frac{\xi_2^{2p_1 - 2p_2} \tilde{\varphi}_{k+1}^2(\xi_{[k+1]})}{4\beta} + \beta \|w\|^2 . \qquad (43)
\end{aligned}$$

By substituting (42) and (43) into (39), it is clear that the following virtual controller

$$x_{k+2}^*(\xi_{[k+1]}, \hat{\Theta}) = -\xi_{k+1} \left[n - k + \rho_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \right]^{\frac{1}{p_{k+1}}}$$

with

$$\rho_{k+1}(\cdot) = \bar{\rho}_{k+1}(\cdot) \sqrt{\hat{\Theta}^2 + 1} + \check{\rho}_{k+1}(\cdot) + \tilde{\rho}_{k+1}(\cdot) + \hat{\rho}_{k+1}(\cdot) + \frac{\xi_2^{2p_1 - 2p_2} \tilde{\varphi}_{k+1}^2(\xi_{[k+1]})}{4\beta},$$

renders

$$\begin{aligned}
\dot{V}_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \quad \leq \quad & -(n-k) \sum_{i=1}^{k+1} |\xi_i|^{2p_1} + \xi_{k+1}^{2p_1 - p_{k+1}} \left[x_{k+2}^{p_{k+1}} - x_{k+2}^{*p_{k+1}} \right] \\
& + \left(\Psi_{k+1}(\xi_{[k+1]}, \hat{\Theta}) - \dot{\hat{\Theta}} \right) \left(\tilde{\Theta} + \eta_{k+1}(\xi_{[k+1]}, \hat{\Theta}) \right).
\end{aligned}$$

The aforementioned inductive argument shows that (27) holds for $k = n$. In fact, in the n-th step, one can construct explicitly a global change of coordinates $(\xi_1, \xi_2, \cdots, \xi_n)$, a positive-definite and proper Lyapunov function $V_n(\xi_{[n]}, \tilde{\Theta})$ and a smooth controller

$$u^*(\xi_{[n]}, \hat{\Theta}) = -\xi_n \left[1 + \rho_n(\xi_{[n]}, \hat{\Theta}) \right]^{1/p_n}$$

for some smooth functions $\rho_n(\cdot) \geq 0$ and $\Psi_{k+1}(\xi_{[k+1]}, \hat{\Theta})$, such that

$$\begin{aligned}
\dot{V}_n(\xi_{[n]}, \tilde{\Theta}) + y^{2p_1} - n\beta \|w\|^2 \quad \leq \quad & -\sum_{i=1}^{n} \xi_i^{2p_1} + \xi_n^{2p_1 - p_n} \left(u^{p_n} - u^{*p_n} \right) \\
& + \left(\Psi_n(\xi_{[n]}, \hat{\Theta}) - \dot{\hat{\Theta}} \right) \left(\tilde{\Theta} + \eta_n(\xi_{[n]}, \hat{\Theta}) \right).
\end{aligned}$$

Therefore, the one-dimensional smooth adaptive controller

$$\left\{ \begin{array}{l} \dot{\hat{\Theta}} = \Psi_n(\xi_{[n]}, \hat{\Theta}), \\ u = u^*(\xi_{[n]}, \hat{\Theta}), \end{array} \right. \qquad (44)$$

is such that

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) + y^{2p_1} - n\beta \|w\|^2 \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{45}$$

Set $\beta = \gamma^2/n$, we have

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) + y^{2p_1} - \gamma^2 \|w\|^2 \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{46}$$

When $w = 0$, it is derived that

$$\dot{V}_n(\xi_{[n]}, \hat{\Theta}) \leq -\sum_{i=1}^{n} \xi_i^{2p_1}. \tag{47}$$

According to the classical Lyapunov stability theory, it is known that the closed-loop system is global stable at the equilibrium $(\xi_{[n]}, \hat{\Theta}) = (0, 0)$. Since the Lyapunov function $V_n(\xi_{[n]}, \hat{\Theta})$ is positive definite and proper, it follows from (47) and La Salle's invariance principle that all the bounded trajectories of the closed-loop system approach the largest invariant set contained in $\left\{(\xi_{[n]}, \hat{\Theta}) : \dot{V}_n = 0\right\}$. Hence, $\lim_{t\to\infty} \xi_{[n]}(t) = 0$. This, combined with (21) with $k = n$, implies $\lim_{t\to\infty} x_{[n]}(t) = 0$. Moreover, note that $V_n(\cdot)$ is positive definite with $V_n(0) = 0$. It follows from (46) that

$$\int_0^t |y(s)|^{2p_1}\, \mathrm{d}s \leq \gamma^2 \int_0^t \|w\|^2\, \mathrm{d}s,\ \forall t \geq 0, \text{ when } x(0) = 0.$$

This completes the proof of the theorem.

The proof of Theorem 3.1 is constructive, thus the design procedure of the adaptive controller solving the ARADD problem is actually given. When $w = 0$ and $f_i(x_i) = 0$, $i \in I[1, n]$, it is easy to check that Theorem 3.1 recovers the global stabilization results obtained in [24]. In addition, for the case of linearly parameterized systems we have the following corollary from Theorem 3.1.

Corollary 3.1 *Consider the power integrator triangular system (1) in which* $\phi_i(x_{[i]}, \theta) = \phi_i(x_{[i]})\theta$. *If Assumptions A1–A3 hold and*

$$\phi_i(x_{[i]}) \leq \gamma_i(x_{[i]}) \sum\nolimits_{j=1}^{i} |x_j|^{p_i},\ i \in I[1, n],$$

then the ARADD problem is solvable by the one-dimensional smooth adaptive controller (4).

According to the result in [22], for linearly parameterized system (1) with s-dimensional unknown parameter θ, the designed adaptive controller is s-dimensional. However, the results presented in this paper indicate that the global adaptive regulation with almost disturbance decoupling for systems (1) is achievable by a smooth one-dimensional adaptive controller, no matter how big the number of unknown parameters is. This shows the minimum-order property of the proposed adaptive controller controller.

4 An Illustrative Example

Consider the following high-order planar nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^3 + \frac{\theta x_1^3}{1+(\sigma x_2)^2} + w, \\ \dot{x}_2 = u^3, \\ y = x_1, \end{cases},$$

where θ and σ are the unknown parameters and w is the disturbance. For this system, one has

$$p_1 = p_2 = 3, \quad f_1 = f_2 = 0, \quad g_1 = 1, g_2 = 0, \quad \phi_1(x) = \frac{\theta x_1^3}{1 + (\sigma x_2)^2}.$$

By letting

$$\alpha_1 = \alpha_2 = 0, \quad \varphi_1 = 1, \varphi_2 = 0, \quad \beta_1 = |\theta|,$$

it is easy to check that Assumptions A1-A4 are satisfied since

$$|\phi_1(x)| = \left| \frac{\theta x_1^3}{1 + (\sigma x_2)^2} \right| \leq |\theta| |x_1|^3.$$

In addition, it is easily obtained that $\Theta = |\theta|$ and $\gamma_1 = 1$.

Define $V_1 = \frac{1}{4}x_1^4 + \frac{1}{2}\tilde{\Theta}^2$. Then one has

$$\begin{aligned}
\dot{V}_1 &= x_1^3 \dot{x}_1 + \tilde{\Theta}\dot{\tilde{\Theta}} \\
&= x_1^3 \left(x_2^3 + \frac{\theta x_1^3}{1 + (\sigma x_2)^2} + w \right) - \tilde{\Theta}\dot{\hat{\Theta}} \\
&\leq x_1^3 x_2^3 + x_1^6 \left(\hat{\Theta} + \tilde{\Theta} \right) + |x_1|^3 |w| - \tilde{\Theta}\dot{\hat{\Theta}} \\
&\leq x_1^3 x_2^3 + x_1^6 \left(\hat{\Theta} + \tilde{\Theta} \right) + \frac{x_1^6}{4\beta} + \beta w^2 - \tilde{\Theta}\dot{\hat{\Theta}} \\
&\leq x_1^3 x_2^3 + x_1^6 \left(\sqrt{1 + \hat{\Theta}^2} + \frac{1}{4\beta} + 1 \right) - x_1^6 + \beta |w|^2 + \left(x_1^6 - \dot{\hat{\Theta}} \right) \tilde{\Theta}.
\end{aligned}$$

Let $\rho_1 = \sqrt{1 + \hat{\Theta}^2} + \frac{1}{4\beta} + 1$, $\Psi_1 = x_1^6$, $\rho_0 = 1$. Then one can obtian

$$\dot{V}_1 + y^6 - \beta w^2 \leq x_1^3 x_2^3 + x_1^6 \rho_1 + \left(\Psi_1 - \dot{\hat{\Theta}} \right) \tilde{\Theta}.$$

By choosing $x_2^* = -x_1 (2 + \rho_1)^{1/3}$, one can further obtain

$$\dot{V}_1 + y^6 - \beta w^2 \leq -2x_1^6 + x_1^3 \left(x_2^3 - x_2^{*3} \right) + \left(\Psi_1 - \dot{\hat{\Theta}} \right) \tilde{\Theta}.$$

Define $\xi_2 = x_2 - x_2^*$. Then it is derived that

$$\dot{\xi}_2 = \dot{x}_2 - \dot{x}_2^* = u^3 - \frac{\partial x_2^*}{\partial x_1}\dot{x}_1 - \frac{\partial x_2^*}{\partial \hat{\Theta}}\dot{\hat{\Theta}} = u^3 + \Delta_2 + \Phi_2 + G_2 w - \omega_2 \dot{\hat{\Theta}},$$

where

$$\Delta_2 = -\frac{\partial x_2^*}{\partial x_1} \left(\xi_2 + x_2^*\right)^3, \;\; \Phi_2 = -\frac{\partial x_2^*}{\partial x_1} \frac{\theta x_1^3}{1 + \left(\sigma x_2\right)^2}, \quad G_2 = -\frac{\partial x_2^*}{\partial x_1}, \; \omega_2 = -\frac{\partial x_2^*}{\partial \hat{\Theta}}.$$

By defining $V_2 = V_1 + \frac{1}{4}\xi_2^4$, one has

$$\begin{aligned}
&\dot{V}_2 + y^6 - 2\beta w^2 \\
&= \dot{V}_1 + y^6 - \beta w^2 - \beta w^2 + \xi_2^3 \dot{\xi}_2 \\
&\leq -2x_1^6 + x_1^3 \left(x_2^3 - x_2^{*3}\right) + \left(\Psi_1 - \dot{\hat{\Theta}}\right) \tilde{\Theta} - \beta w^2 \\
&+ \xi_2^3 u^3 + \xi_2^3 \Delta_2 + \xi_2^3 \Phi_2 + \xi_2^3 G_2 w - \xi_2^3 \omega_2 \dot{\hat{\Theta}} \\
&\leq -2x_1^6 + \left|x_1^3 \left(x_2^3 - x_2^{*3}\right)\right| + \left(\Psi_1 - \dot{\hat{\Theta}}\right) \tilde{\Theta} - \beta w^2 \\
&+ \xi_2^3 u^3 + \left|\xi_2^3 \Delta_2\right| + \left|\xi_2^3 \Phi_2\right| + \left|\xi_2^3 G_2 w\right| + \left|\xi_2^3 \omega_2 \dot{\hat{\Theta}}\right|.
\end{aligned}$$

Simple computations yield

$$\begin{aligned}
\left|\Delta_2\right| &= \left|\frac{\partial x_2^*}{\partial x_1} \left(\xi_2 + x_2^*\right)^3\right| \\
&= \left|\frac{\partial x_2^*}{\partial x_1}\right| \left|\xi_2^3 + x_2^{*3} + 3\xi_2^2 x_2^* + 3\xi_2 x_2^{*2}\right| \\
&\leq \left|\frac{\partial x_2^*}{\partial x_1}\right| \left(\left|\xi_2\right|^3 + \left|x_2^*\right|^3 + 3\left|\xi_2\right|^2 \left|x_2^*\right| + 3\left|\xi_2\right| \left|x_2^*\right|^2\right) \\
&\leq \left|\frac{\partial x_2^*}{\partial x_1}\right| \left(4\left|\xi_2\right|^3 + 4\left|x_2^*\right|^3\right) \\
&\leq 4\left(2 + \rho_1\right) \left|\frac{\partial x_2^*}{\partial x_1}\right| \left(\left|\xi_2\right|^3 + \left|x_1\right|^3\right),
\end{aligned}$$

$$\left|\Phi_2\right| = \left|\frac{\partial x_2^*}{\partial x_1}\right| \left|\frac{\theta x_1^3}{1 + \left(\sigma x_2\right)^2}\right| \leq \left|x_1\right|^3 \left|\frac{\partial x_2^*}{\partial x_1}\right| \Theta,$$

$$\begin{aligned}
\left|\Delta_2 \xi_2^3\right| &\leq 4\left(2 + \rho_1\right) \left|\frac{\partial x_2^*}{\partial x_1}\right| \left(\left|\xi_2\right|^3 + \left|x_1\right|^3\right) \left|\xi_2\right|^3 \\
&\leq 4\left(2 + \rho_1\right) \left|\frac{\partial x_2^*}{\partial x_1}\right| \xi_2^6 + 4\left(2 + \rho_1\right) \left|\frac{\partial x_2^*}{\partial x_1}\right| \left|x_1\right|^3 \left|\xi_2\right|^3 \\
&\leq \check{\rho}_2 \xi_2^6 + \frac{1}{6} x_1^6,
\end{aligned}$$

with

$$\check{\rho}_2 = 4\left(2 + \rho_1\right) \left|\frac{\partial x_2^*}{\partial x_1}\right| + 24\left(2 + \rho_1\right)^2 \left(\frac{\partial x_2^*}{\partial x_1}\right)^2,$$

$$
\begin{aligned}
\left|\Phi_2\xi_2^3\right| &\le |\xi_2|^3 |x_1|^3 \left|\frac{\partial x_2^*}{\partial x_1}\right| \Theta \\
&\le \left[\frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2 \left(1+\hat{\Theta}^2\right)\xi_2^6\right]\Theta \\
&\le \left(\frac{x_1^6}{6} + \bar{\rho}_2\sqrt{1+\hat{\Theta}^2}\xi_2^6\right) + \left[\frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2\xi_2^6\right]\tilde{\Theta},
\end{aligned}
$$

with

$$
\bar{\rho}_2 = \frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2 \left(1+\hat{\Theta}^2\right),
$$

$$
\begin{aligned}
&\left|x_1^3\left(x_2^3 - x_2^{*3}\right)\right| \\
&= \left|x_1^3\left[\left(\xi_2 + x_2^*\right)^3 - x_2^{*3}\right]\right| \\
&\le |x_1|^3 |\xi_2| \left(\frac{5}{2}\xi_2^2 + \frac{9}{2}x_2^{*2}\right) \\
&= \frac{5}{2}|x_1|^3 |\xi_2|^3 + \frac{9}{2}|x_1|^5 |\xi_2| \left(2+\rho_1\right)^{2/3} \\
&= \frac{1}{6}x_1^6 + \tilde{\rho}_2\xi_2^6,
\end{aligned}
$$

with

$$
\tilde{\rho}_2 = \frac{75}{4} + \frac{15}{64}9^5\left(2+\rho_1\right)^4,
$$

$$
\left|\xi_2^3 G_2 w\right| = \left|\xi_2^3\frac{\partial x_2^*}{\partial x_1}\right| |w| \le \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2 \xi_2^6 + \beta w^2.
$$

As a result, one has

$$
\begin{aligned}
&\dot{V}_2 + y^6 - 2\beta w^2 \\
\le& -\frac{3}{2}x_1^6 + \left[\tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2\sqrt{1+\hat{\Theta}^2} + \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\right]\xi_2^6 \\
&+ \left|\xi_2^3\frac{\partial x_2^*}{\partial\hat{\Theta}}\dot{\hat{\Theta}}\right| + u^3\xi_2^3 + \left(\Psi_2 - \dot{\hat{\Theta}}\right)\tilde{\Theta},
\end{aligned}
$$

with

$$
\Psi_2 = x_1^6 + \frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2\xi_2^6.
$$

Choose $\dot{\hat{\Theta}} = \Psi_2$. Then one has

$$\begin{aligned}
\left|\xi_2^3 \frac{\partial x_2^*}{\partial \hat{\Theta}} \dot{\hat{\Theta}}\right| &= \left|\xi_2^3\right| \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| \left(x_1^6 + \frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2 \xi_2^6 \right) \\
&\le \left(\frac{4}{3} \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |x_1|^3 \right) |x_1|^3 |\xi_2|^3 + \bar{\rho}_2 \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |\xi_2|^3 |\xi_2|^6 \\
&\le \frac{1}{2} \left(\frac{4}{3} \left|\frac{\partial x_2^*}{\partial \hat{\Theta}}\right| |x_1|^3 \right)^2 \xi_2^6 + \frac{1}{2} x_1^6 + \frac{1}{4} \bar{\rho}_2 \left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}} \right)^2 + 1 \right] \left(\xi_2^6 + 1\right) \xi_2^6 \\
&\le \hat{\rho}_2 \xi_2^6 + \frac{1}{2} x_1^6
\end{aligned}$$

with

$$\hat{\rho}_2 = \frac{8}{9} \left(\frac{\partial x_2^*}{\partial \hat{\Theta}} \right)^2 x_1^6 + \frac{1}{4} \bar{\rho}_2 \left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}} \right)^2 + 1 \right] \left(\xi_2^6 + 1\right).$$

Sequentially,

$$\dot{V}_2 + y^6 - 2\beta w^2 \le -x_1^6 + \left[\tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2 \sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta} \left(\frac{\partial x_2^*}{\partial x_1} \right)^2 \right] \xi_2^6 + u^3 \xi_2^3.$$

Choose $u = -\xi_2 \left[1+\rho_2\right]^{1/3}$ with $\rho_2 = \tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2 \sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta} \left(\frac{\partial x_2^*}{\partial x_1} \right)^2$. Then it is easily derived that $\dot{V}_2 + y^6 - 2\beta w^2 \le -x_1^6 - \xi_2^6$.

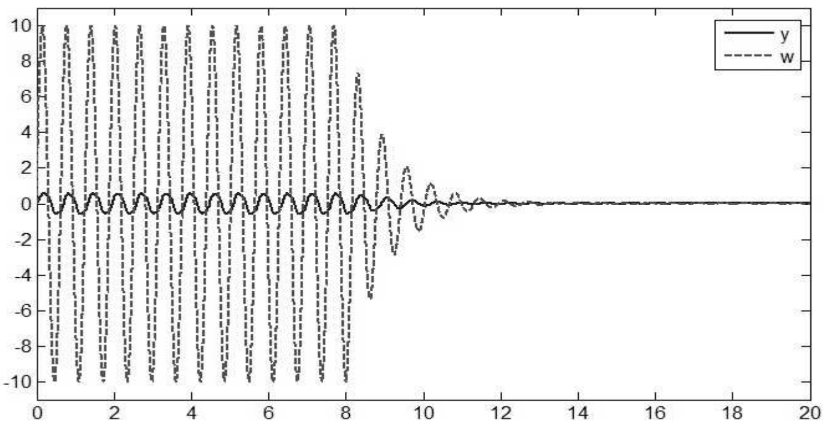

Figure 1: Disturbance signal and output response.

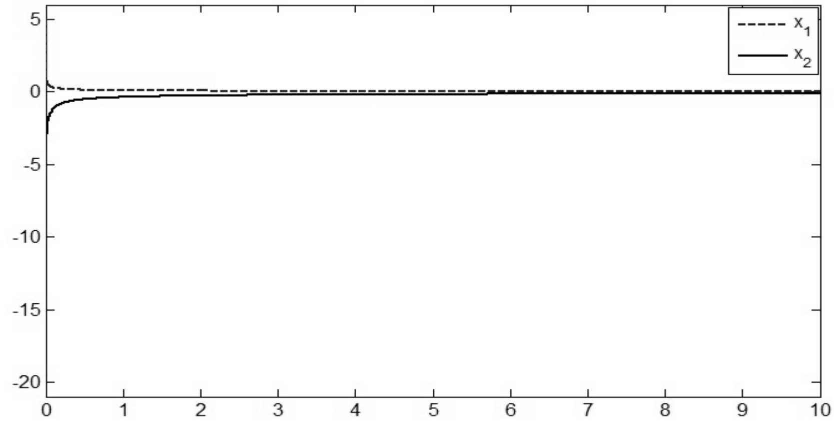

Figure 2: State response.

With the previous derivation, one can obtain the following control law

$$
\begin{aligned}
u &= -\xi_2 \left[1+\rho_2\right]^{1/3}, \\
\rho_2 &= \tilde{\rho}_2 + \check{\rho}_2 + \bar{\rho}_2 \sqrt{1+\hat{\Theta}^2} + \hat{\rho} + \frac{1}{4\beta}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2, \\
\hat{\rho}_2 &= \frac{8}{9}\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 x_1^6 + \frac{1}{4}\bar{\rho}_2\left[\left(\frac{\partial x_2^*}{\partial \hat{\Theta}}\right)^2 + 1\right]\left(\xi_2^6+1\right), \\
\bar{\rho}_2 &= \frac{3}{4}\left(\frac{\partial x_2^*}{\partial x_1}\right)^2\left(1+\hat{\Theta}^2\right), \\
\check{\rho}_2 &= 4\left(2+\rho_1\right)\left|\frac{\partial x_2^*}{\partial x_1}\right| + 24\left(2+\rho_1\right)^2\left(\frac{\partial x_2^*}{\partial x_1}\right)^2, \\
\tilde{\rho}_2 &= \frac{75}{4} + \frac{15}{64}9^5\left(2+\rho_1\right)^4,
\end{aligned}
$$

$$
\begin{aligned}
\rho_1 &= \sqrt{1+\hat{\Theta}^2} + \frac{1}{4\beta} + 1, \\
x_2^* &= -x_1\left(2+\rho_1\right)^{1/3}, \\
\frac{\partial x_2^*}{\partial x_1} &= -\left(2+\rho_1\right)^{1/3}, \\
\frac{\partial x_2^*}{\partial \hat{\Theta}} &= -\frac{x_1}{3}\left(2+\rho_1\right)^{-2/3}\left(1+\hat{\Theta}^2\right)^{-1/2}\hat{\Theta}, \\
\dot{\hat{\Theta}} &= x_1^6 + \frac{x_1^6}{3\left(1+\hat{\Theta}^2\right)} + \bar{\rho}_2\xi_2^6.
\end{aligned}
$$

Figures 1 and 2 give the simulation results of the resulstant closed-loop system under the obtained control law.

5 Conclusion

For the class of power integrator lower triangular systems with nonlinear parametrization, we formulated the problem of adaptive regulation with almost disturbance decoupling. Under a set of growth conditions, an explicit design of the adaptive smooth controller solving the ADD problem was provided. A significant feature of the obtained adaptive dynamical compensator is its minimum-order property. The results of this paper exploit a new application of the parameter separation technique proposed recently in [24].

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Exponentially Long Orbits in Boolean Networks with Exclusively Positive Interactions

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Abstract: The absence of negative feedback in Boolean networks tends to result in systems with relatively short orbits. We present a construction of N-dimensional Boolean networks that use only AND, OR, COPY gates and nevertheless have an exponentially large orbit (of size c^N for arbitrary $c < 2$). The construction is based on pseudorandom number generation algorithms. A previously obtained nontrivial upper bound on the orbit length under certain limitations on the number of outputs per node is shown to be optimal.

Keywords: *Boolean networks; monotone systems; gene networks; systems biology.*

Mathematics Subject Classification (2000): 06E99, 34C12, 39A33, 92B99, 94C10.

1 Introduction

The concept of a *Boolean network* was originally proposed in the late 1960's by Stuart Kauffman to model gene regulatory behavior at the cell level [13]. This type of modeling can sometimes capture the general dynamics of continuous systems in a simplified framework without the choice of specific nonlinearities or parameter values; see for instance [1]. Boolean networks are used in several other disciplines such as electrical engineering, computer science, and control theory, and analogous definitions are known under various names such as sequential dynamical systems [16] or Boolean difference equations [6].

An N-dimensional *Boolean dynamical system* or *Boolean network* (Π, g) consists of N variables $s_1, \dots, s_N$, each of which can have value 0 or 1 at any given time step t. The variables are updated according to $s_i(t+1) = g_i(s_1(t), \dots, s_N(t))$.

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Usually, individual update functions g_i depend only on very few of the variables. Let us say that s_j *is an input of* s_i and s_j *sends output to* g_i if there are two Boolean vectors s, s^* that differ only in variable s_j for which $g_i(s) \neq g_i(s^*)$. The input-output relation defines a digraph on the set of Boolean variables that is called the *connectivity* of the Boolean network. We call a Boolean network (Π, g) *a* K, M *network* if each of its update functions takes at most K inputs and each variable has at most M outputs.

A key problem in the study of Boolean networks is how the dynamics depends on the updating functions and the connectivity. This problem has been largely studied for so-called *random Boolean networks (RBNs)* where both the updating functions and the network connectivity are randomly drawn from a specified network distribution. This allows to make estimates on quantities such as the number of orbits and their length [2, 7, 14, 22]. Such estimates can be either obtained from simulation studies or analytically.

1.1 Cooperative Boolean networks

An important topic in the study of dynamical systems is the role of negative feedback. This notion is usually defined in terms of *negative feedback loops* that contain an odd number of negative interactions. *Monotone systems* are systems that contain only positive feedback loops. Here we study *cooperative systems,* that are systems in which there are no negative interactions between any two variables. In the context of networks with at most $K = 2$ inputs per variable, cooperativity is equivalent to the use of only the following update functions: constants, COPY, AND, OR. No negations are allowed [11].

Comparative simulation studies show that Boolean networks with no or only few negative feedback loops tend to have relatively shorter orbits [26]. The question naturally arises whether the assumption of cooperativity imposes nontrivial provable upper bounds on the length of orbits in Boolean networks. This question seems especially interesting for cooperative $K = 2$ Boolean networks, since $K = 2$ random Boolean networks tend to have much shorter orbits than RBNs drawn from distributions where $K > 2$ [2, 7, 14].

Since the state space of an N-dimensional Boolean network has size 2^N, a bound on the attractor length should be considered "nontrivial" if it scales like $o(2^N)$. In [15, 20], a simple example of a $K = 2$, $M = 2$ Boolean network is constructed with N variables and an orbit of length $2^{N-1} - 1$, which comprises exactly half of the state space. In contrast, the orbits of cooperative Boolean systems cannot comprise a fixed fraction of the state space [4, 11, 25], which already gives an upper bound that scales like $o(2^N)$. Upper bounds on orbit length that scale like $O(c^N)$ for some $c < 2$ were derived in [12] for $K = 2$, $M = 2$ cooperative networks under the assumption that at least a fixed positive fraction of update functions take exactly two inputs (see Theorem 3.1 below).

Theorem 1 of the preprint [12] shows that for any constant $1 < c < 2$ and all sufficiently large N there exists an N-dimensional $K = 2$, $M = 2$ cooperative Boolean network with at least one orbit of length $\geq c^N$. This shows that some such additional assumptions are needed for nontrivial upper bounds of type $O(c^N)$ on orbit length. Our main theorem here, Theorem 2.1, consists of a simplified proof of a similar result with a mild additional assumption as described below. Theorem 3.2 of Section 3 provides variations on Theorem 2.1 and shows, among other things, that one of the upper bounds on orbit length that was derived in [12] is sharp.

1.2 Additive Lagged-Fubini Generators

An additive lagged-Fubini generator (ALFG) is an algorithm to produce 'pseudorandom' numbers. It is the basis for some of the most widely used random number generators such as the Mersenne twister [19]. In its Boolean version, the ALFG consists simply of a sequence of Boolean values x_i satisfying the formula

$$x_i = x_{i-p} + x_{i-q} \bmod 2, \tag{1}$$

for all i, where $0 < q < p$ are fixed numbers. For particular choices of p and q, it has been shown that x_i can be periodic with maximal period $2^p - 1$ [17]. The sufficient and necessary condition for producing this orbit length is that the polynomial $x^p + x^q + 1$ is primitive modulo 2 [8]. Many such pairs (p, q) are known, including ones with values as large as $p = 6972593$ and $q = 3037958$. It is an open but widely believed conjecture that there are infinitely many such pairs, see for instance [8]. For a list of all admissible pairs (p, q) with $p \leq 1000$, see [27].

In [15, 20], the authors build a Boolean network which is easily seen to be equivalent to an ALFG, using a single loop of length p, and a single internal connection between two nodes q variables apart, as in Figure 1a. All update functions are equal to the simple copy function $f(a) = a$, except for one with two inputs, $f(a, b) = a + b \bmod 2 = a$ XOR b. The authors of these papers also point out that this reversible update function (Table 1) can be replaced by three canalyzing ones (see Section 3 for the definition of canalyzing functions). Yet even such an implementation would contain negations and its main feedback loop is negative. In particular, this network is not cooperative.

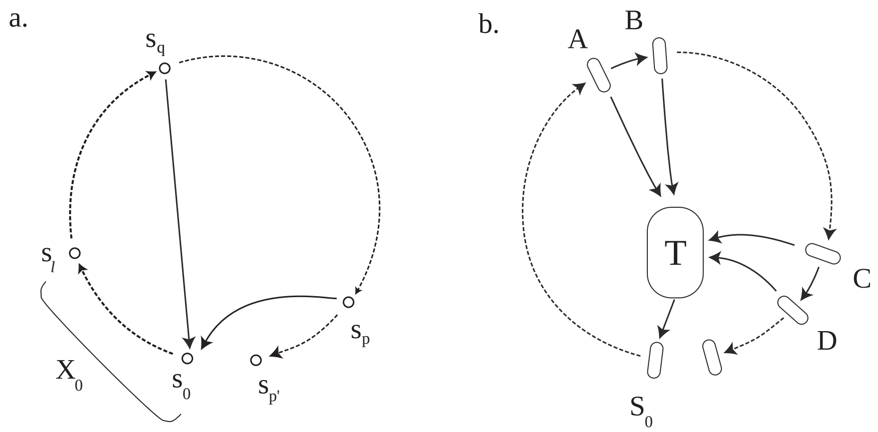

Figure 1: a. The original additive lagged-Fibonacci generator. A solid arrow represents a direct connection, and a dotted arrow is a chain of connected variables. For the purposes of our results, the variables are grouped into blocks of size ℓ such as shown for X_0, and additional nodes are added outside the main loop so that the total number of nodes is divisible by ℓ (see the dotted line joining x_s and $x_{s'}$). b. The cooperative network associated to a., where every variable corresponds to L different Boolean nodes. The inputs of the Boolean circuit T are $A = S_{\lfloor q/\ell \rfloor - m-1},\ B = S_{\lfloor q/\ell \rfloor - m},\ C = S_{\lfloor p/\ell \rfloor - m-1},\ D = S_{\lfloor p/\ell \rfloor - m}$.

2 Cooperative Boolean Networks with an Exponentially Long Orbit

Let us state the version of Theorem 1 of [12] that we are going to prove here.

Theorem 2.1 *Assume that there exist infinitely many positive integers $p > q$ such that (1) defines an AFLG with maximal period $2^p - 1$. Let $1 < c < 2$ be an arbitrary constant. Then for infinitely many N there exists an N-dimensional $K = 2$, $M = 2$ cooperative Boolean network with at least one orbit of length $\geq c^N$. In this network the only update functions are $s_i \vee s_j$, $s_i \wedge s_j$, and the copy function $f(s_i) = s_i$.*

Proof We start with an ALFG with maximal period $2^p - 1$ as in Figure 1a, with sufficiently large delays (p, q). In the Boolean setting we have the following equation:

$$s_0(t+1) = s_q(t) \text{ XOR } s_p(t) = s_0(t-q) \text{ XOR } s_0(t-p). \tag{2}$$

This network will be referred to as *the AFLG network.* It has an orbit with length $2^p - 1$ and two negative feedback loops as shown in Figure 1a.

For a given $c < 2$ we construct a Boolean network with N variables that uses only update functions AND, OR, COPY and has an orbit of length $\geq c^N$ states. Its dynamics closely mimics the one of the AFLG network. The idea is to group the variables $s_0, \ldots s_p$ into blocks of ℓ adjacent variables each, as in Figure 1b. The new network has as variables Boolean *vectors* $S_0, S_1, \ldots$, each with an even number L of bits. The values of L and ℓ will be chosen as in Lemma 2.2 below. Importantly, the values of each S_i are not arbitrary but chosen from the image of an injective function $\Gamma : \{0,1\}^\ell \to \{0,1\}^L$, i.e. they are thought of as coded sequences of ℓ bits. Additionally, the values of $\Gamma(s)$ are required to have exactly $L/2$ nonzero entries. Such a function Γ exists as long as $2^\ell \leq \binom{L}{L/2}$, which will be ensured by Lemma 2.2. In order to guarantee that the nodes in the ALFG can be divided in groups of ℓ, we introduce some additional nodes as in Figure 1a, which are however not part of the loop (see the legend of Figure 1 for details).

Notice that after ℓ time steps, the ALFG network has rotated the values of each group of ℓ variables into the next – except for the group $s_0, \ldots s_{\ell-1}$, whose values have been determined by a more complicated algorithm. The idea is that one time step in the new network will represent ℓ time steps in the ALFG. More precisely, each variable $S_i(t)$ is coding for the variables $X_i(\ell t) = (s_{i\ell}(\ell t), \ldots, s_{i\ell+\ell-1}(\ell t))$ at time ℓt, or $S_i(t) = \Gamma(X_i(\ell t))$.

In order to achieve this, we set $S_i(t+1) = S_{i-1}(t)$ for $i > 0$. As for the updating function of S_0, it is defined as the encoding Γ of $s_0, \ldots, s_\ell$ after iterating the ALFG for ℓ time steps. $S_0(t+1)$ is therefore a function G of the variables S_i encoding $s_{q-\ell+1}, \ldots, s_q$ and $s_{p-\ell+1}, \ldots, s_p$. In other words, given the arguments A, B, C, and D, one can compute S_0 at the next time step by first decoding them into their corresponding sequences of ℓ-vectors using the function Γ^{-1}, and assigning these values to the variables in ALFG, then iterating ALFG ℓ times, and finally encoding the resulting sequence X_0 using the function Γ.

The following technical lemma shows that this encoding function G can be implemented as a Boolean circuit (labeled T in Figure 1b) with only binary AND- and OR- and unary COPY gates and no negations. Such a Boolean circuit can be incorporated into our network without violating cooperativity or the commitment to build a $K = 2$, $M = 2$ network. The indegree and outdegree for a node of a Boolean circuits are defined analogously as for Boolean networks.

Our construction uses the fact that G is only used with arguments that have $L/2$ zeros and $L/2$ ones each. Given two Boolean $P-$vectors s, r, we say that $s \leq r$ if $s_i \leq r_i$

for all i. If either $s \leq r$ or $r \leq s$, the two vectors are called *comparable*. The following lemma will be applied to the proof of the main result for $P = 4L$.

Lemma 2.1 *Let $g : D \subseteq \{0,1\}^P \to \{0,1\}^L$ be an arbitrary function, defined on a domain D where no two elements are comparable. Then there exists a Boolean circuit B with input vector a of dimension P, and an output vector $b = (b_1, \dots, b_L)$, such that $b(t+m) = g(a)$, for some fixed delay m and any $a(t) \in D$. Furthermore, the circuit B uses only binary AND- and OR- and unary COPY gates and the indegree (outdegree) of every designated input (output) variable is 0.*

Proof The function g can be extended to a cooperative function h, i.e. one for which $s \leq r$ implies $h(s) \leq h(r)$, defined on all of $\{0,1\}^P$; see [11]. The result will follow from building a suitable Boolean circuit that computes the function h.

Consider a fixed component $h_i : \{0,1\}^P \to \{0,1\}$ of h. By the cooperativity of this function, one can write it in the normal form $h_i(s_1, \dots, s_P) = \Psi_1^i(s_1, \dots, s_P) \vee \dots \vee \Psi_{k_i}^i(s_1, \dots, s_P)$, where each Ψ_j^i is the conjunction of a number of variables, i.e., $\Psi_j^i(s_1, \dots, s_P) = s_{\alpha_{1i}} \wedge \dots \wedge s_{\alpha_{ji}}$. This suggests a way of computing h_i: define Boolean variables $\psi_j^i(t) := \Psi_j^i(s(t-1))$, and then let $h_i(t) := \psi_1^i(t-1) \vee \dots \vee \psi_{k_i}^i(t-1)$. Repeating this procedure for all components of h yields a Boolean circuit which computes h in $m = 2$ steps, and which is cooperative and has indegree (outdegree) 0 for every input (output).

In order to satisfy the condition that every node have in- and outdegree of at most 2, we need to modify this construction by introducing additional variables. First, note that the outdegree of every input s_i can be very large. One can define two additional variables which simply copy the value of $s_i(t)$, then four variables that copy the value of the previous two, etc. This procedure is repeated for each s_i so that at least as many copies of each variable are present as appear in the expressions of all ψ_j^i. A similar cascade can be used to define each ψ_j^i and h_i so that each indegree is at most two. If $\psi_i^j = s_{\alpha_1} \wedge s_{\alpha_2} \wedge s_{\alpha_3}$, say, then one can define $r_1(t) := s_{\alpha_1}(t-1)$, $r_2(t) := s_{\alpha_2}(t-1) \wedge s_{\alpha_3}(t-1)$, $\psi_i^j(t) := r_1(t-1) \wedge r_2(t-1)$. Similarly for longer disjunctions and each ψ_j^i and also similarly for h_i, in which case $\wedge$ is replaced by $\vee$ at each step. This produces a computation of h_i in m_i steps for each i. Finally, after introducing further additional variables at each component i if necessary to compensate for unequal lengths of the expressions for ψ_j^i, the Boolean vector $h(s_1, \dots, s_P)$ can be computed in exactly $m = \max(m_1, \dots, m_L)$ steps.

Notice that the function G is not computed by the Boolean circuit instantaneously, but after m steps. Since $S_i(t) = S_{i-m}(t-m)$, we correct for this by feeding the circuit an input which has been shifted back by m.

The new cooperative Boolean network has an orbit of length at least $(2^p - 1)/\ell$. Its dimension is

$$N = (p + \gamma + 1)L/\ell + T, \tag{3}$$

where $\gamma = p' - p < \ell$ reflects the need for dummy variables (see the legend of Figure 1) and T is the number of nodes involved in the Boolean circuit that computes G. Since T only depends on ℓ and L, not on p, the following lemma implies Theorem 2.1.

Lemma 2.2 *For arbitrary $1 < c < 2$ and sufficiently large p, there exist ℓ, N, and L as above such that*

$$\binom{L}{L/2} > 2^\ell, \quad \frac{2^p - 1}{\ell} > c^N.$$

Proof We prove first that there exist $L > 0$ even, an integer $\ell > 0$, and a real constant $\delta > 1$ such that

$$\binom{L}{L/2} > 2^{\ell} > c^{\delta L}. \tag{4}$$

Start by fixing an arbitrary $\delta > 1$ such that $c < c^{\delta} < 2$. The second inequality in (4) is equivalent to $L/\ell < \ln 2/(\delta \ln c)$. Let L be an even integer with $L = w\ell$, for some fixed $1 < w < \ln 2/(\delta \ln c)$. Since $\binom{L}{L/2} > \frac{2^L}{L+1} = \frac{2^{w\ell}}{w\ell+1}$ and $w > 1$, the first inequality in (4) is satisfied for sufficiently large ℓ.

It remains to show that $(2^p - 1)/\ell \geq c^N$ for some sufficiently large N as in (3). But since $c^{L/\ell} < 2^{1/\delta}$, expression (3) implies

$$\frac{c^{(p+\gamma+1)L/\ell+T}}{(2^p-1)/\ell} < \ell c^T \frac{2^{\frac{p+\gamma+1}{\delta}}}{2^p-1} < \ell c^T 2^{\frac{1}{\delta}(p+\gamma+1)-(p-1)}$$

which is < 1 for sufficiently large p.

It follows that for sufficiently large p, we can choose ℓ, L so that the system we constructed will contain an orbit of length at least c^N, as stated in Theorem 2.1.

3 Biased Update Functions and Long Orbits

The *bias* Λ of a Boolean function is the fraction of input vectors for which the function outputs 1. A Boolean function would be considered *biased* if $\Lambda \neq 0.5$. More specifically, let us say that an update function is *ε-biased* if $|\Lambda - 0.5| \geq \varepsilon$. Simulation studies of random Boolean networks indicate that networks with only strongly biased update functions tend to have shorter orbits than generic networks with a given number of inputs (see [23] and references therein). The following result from [12] gives a provable upper bound on the length of orbits in some ε-biased networks.

Theorem 3.1 *Let $\varepsilon, \alpha > 0$ and let K, M be positive integers. Then there exists a positive constant $c(\varepsilon, \alpha, K, M) < 2$ such that for all $c > c(\varepsilon, \alpha, K, M)$ and all sufficiently large N, the length of any orbit in any N-dimensional K-M Boolean network in which a proportion of least α of the update functions are ε-biased does not exceed c^N.*

In particular, $c(0.25, 1, 2, 2) \leq 10^{1/4}$.

To put the last sentence of Theorem 3.1 into perspective, consider Table 1 of Boolean functions with two inputs.

Thus for $K = 2$ a Boolean function is biased iff it is 0.25-biased iff it is in $\mathcal{C}_2 \cup \mathcal{F}$. The classes $\mathcal{C}_1$ and $\mathcal{C}_2$ constitute the *canalyzing* functions, in which a certain value of one of the inputs determines the function output [7]. Note the $\mathcal{C}_2$ contains the AND and the OR functions. We call a $K = 2$ Boolean system a *$\mathcal{C}_2$-network* if all its update functions are in the class $\mathcal{C}_2$. Notice that the last sentence of Theorem 3.1 gives an upper bound of $O(10^{N/4})$ for orbit lengths in N-dimensional $K = 2, M = 2$ $\mathcal{C}_2$-networks.

We can prove the following variants of Theorem 2.1 for $\mathcal{C}_2$-networks. Part (b) of Theorem 3.2 implies that the upper bound in the last sentence of Theorem 3.1 is sharp. The theorem does not require assumptions about the existence of AFLGs [12], but we state and prove it here in this form to emphasize the connection with the earlier construction.

In	$\mathcal{F}$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{R}$
0 0	1 0	0 1 0 1	1 0 0 0 0 1 1 1	1 0
0 1	1 0	0 1 1 0	0 1 0 0 1 0 1 1	0 1
1 0	1 0	1 0 0 1	0 0 1 0 1 1 0 1	0 1
1 1	1 0	1 0 1 0	0 0 0 1 1 1 1 0	1 0
cooperative	* *	* *	* *	
bias	1 0	$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}$	$\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{1}{4}\,\frac{3}{4}\,\frac{3}{4}\,\frac{3}{4}\,\frac{3}{4}$	$\frac{1}{2}\,\frac{1}{2}$

Table 1: The different Boolean update functions with $K = 2$ inputs (adapted from [7]).

s	1111	1110	1101	1100	1011	1010	0111	0101	0011	0000
$f(s)$	1111	1100	1010	1000	0101	0100	0011	0010	0001	0000
$h \circ f(s)$	1111	1110	1101	1100	1011	1010	0111	0101	0011	0000

Table 2: The values of f and $h \circ f$ on F.

Theorem 3.2 *Assume that there exist infinitely many positive integers $p > q$ such that (1) defines an AFLG with maximal period $2^p - 1$. Let c, c_1 be constants with $1 < c < 2$ and $1 < c_1 < 10^{1/4}$. Then for arbitrarily large N there exist cooperative Boolean networks with the following properties:*
(a) (Π, g) is a $\mathcal{C}_2$ network with at least one orbit of length $\geq c^N$,
(b) (Π, g) is a $\mathcal{C}_2$, $M = 2$ network with at least one orbit of length $\geq c_1^N$.

Proof For part (a), let (Σ, f) be a cooperative $K = 2$ Boolean network of dimension $N-2$ that contains an orbit of length c^N. We show how to turn (Σ, f) into a cooperative $\mathcal{C}_2$ Boolean network (Π, g) of dimension N. The update functions g_k for $k < N - 1$ of the new system are defined as follows:

If f_k is already in $\mathcal{C}_2$, then $g_k = f_k$.

If $f_k = s_{i_k}$, then $g_k = s_{i_k} \wedge s_{N-1}$.

If f_k is constant with value 1, then $g_k = s_{N-1} \vee s_N$; if f_k is constant with value 0, then $g_k = s_{N-1} \wedge s_N$.

Finally, we let $g_{N-1} = s_{N-1} \vee s_N$ and $g_N = s_{N-1} \wedge s_N$. Then (Π, g) is a cooperative $\mathcal{C}_2$-system. Now let $s \in \Sigma$ be a state in an orbit of length at least c^N of (Σ, f), and define a state $s^* \in \Pi$ by $s^* = [s_1, \ldots, s_{N-2}, 1, 0]$. Then the orbit of s^* in (Π, g) has the same length as the orbit of s in (Σ, f). This proves part (a).

For the proof of part (b), we need to implement the $\mathcal{C}_1$ functions that copy the value of one input variable by cooperative $\mathcal{C}_2$ functions in such a way that the overall dimension is not increased by more than a factor of $4 \log_{10} 2$.

Let us define Boolean vector functions f and h on four-dimensional Boolean vectors $s = (s_1, s_2, s_3, s_4)$ as follows:

$$f(s) = (s_1 \wedge s_2, s_1 \wedge s_3, s_2 \wedge s_4, s_3 \wedge s_4), \quad h(s) = (s_1 \vee s_2, s_1 \vee s_3, s_2 \vee s_4, s_3 \vee s_4).$$

Let $F = \{1111, 1110, 1101, 1100, 1011, 1010, 0111, 0101, 0011, 0000\}$ and $H = f(F)$.

Table 2 gives the values of $f, h \circ f$ on F.

As Table 2 shows, $h \circ f$ is the identity on F. It follows that h maps H onto F and $f \circ h$ is the identity on H.

Let L be a positive integer divisible by eight, and let $p := L/4$. Write $[L]$ as a disjoint union of blocks of four consecutive integers $i(1,r), i(2,r), i(3,r), i(4,r)$ for $r \in [p]$. Let $\vec{s}_r := (s_{i(1,r)}, s_{i(2,r)}, s_{i(3,r)}, s_{i(4,r)})$. Call a Boolean vector $s \in \{0,1\}^{[L]}$ *L-compliant* if

(a) $\vec{s}_r \in F$ for $1 \le r \le p/2$, $\vec{s}_r \in H$ for $p/2 < r \le p$, and

(b) s takes the value 1 exactly $L/2$ times.

Lemma 3.1 *Let $c_1 < 10^{1/4}$. Then there exist a positive integer ℓ and a positive integer L that is a multiple of eight such that $2^\ell > c_1^L$ and the number of L-compliant Boolean vectors is larger than 2^ℓ.*

Proof Let L be a positive integer that is an integer multiple of 8, and let V be the set of Boolean vectors $s \in \{0,1\}^L$ that satisfies condition (a) above. Since $|F| = |H| = 10$, it is clear that $|V| = 10^{L/4}$.

Let $|\vec{s}_r|$ denote the number of 1's in $\vec{s}_r$. For each $s \in V$ define the *signature of s* as $\sigma(s) = (\sigma_1(s), \ldots, \sigma_6(s))$, where
$\sigma_1(s) = |\{r : \ r \le p/2 \ \& \ |\vec{s}_r| = 4\}|$, and $\sigma_4(s) = |\{r : \ p/2 < r \ \& \ |\vec{s}_r| = 4\}|$,
$\sigma_2(s) = |\{r : \ r \le p/2 \ \& \ |\vec{s}_r| = 0\}|$, and $\sigma_5(s) = |\{r : \ p/2 < r \ \& \ |\vec{s}_r| = 0\}|$,
$\sigma_3(s) = |\{r : \ r \le p/2 \ \& \ |\vec{s}_r| = 3\}|$, and $\sigma_6(s) = |\{r : \ p/2 < r \ \& \ |\vec{s}_r| = 1\}|$.

Let $\sigma^{max} = (\frac{1}{16}, \frac{1}{16}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{4})$. Then the inequality

$$|\{s \in V : \sigma(s) = \sigma\}| \le |\{s \in V : \sigma(s) = \sigma^{max}\}| \tag{5}$$

for any possible signature σ. Moreover, observe that if $s \in V$ and $\sigma(s) = \sigma^{max}$, then s takes the value 1 exactly $L/2$ times, and hence s is L-compliant. Since the total number of possible signatures is bounded from above by $(L/4+1)^6$, it follows from (5) that the total number Q of L-compliant Boolean vectors satisfies the inequality

$$Q \ge \frac{10^{L/4}}{(L/4+1)^6}.$$

Notice that $\lim_{L\to\infty} \ L \ln 10^{1/4} - 6\ln(L/4+1) - L \ln c_1 = \infty$.
Thus for sufficiently large L we can find a positive integer ℓ with

$$L \ln 10^{1/4} - 6\ln(L/4+1) > \ell \ln 2 > L \ln c_1,$$

and the lemma follows.

Now fix $c_1 < 10^{1/4}$ and let L, ℓ be as in Lemma 3.1. Build an N-dimensional Boolean system (Π, g^-) as in the proof of Theorem 2.1, but with the following modifications for indices i where the value of S_i will just be copied to S_{i+2}:

We require that the values of S_i on the blocks S_i of length L are L-compliant vectors.

Instead of requiring $S_i(t+1) = S_{i+1}(t)$ and implementing this dynamics by $\mathcal{C}_1$ functions, we only require $S_i(t+2) = S_{i+2}(t)$ and implement this dynamics as follows: Let S_i be partitioned into blocks $b_{i,1}, \ldots, b_{i,L/4}$ of four Boolean values each, with $b_{i,r}(t) \in F$ for $r \le L/8$ and $b_{i,r} \in H$ for $L/8 < r \le L/4$. Define $b_{i,r}(t+1) = h(b_{i+1,r+L/8}(t))$ for $r \le L/8$ and $b_{i,r}(t+1) = f(b_{i+1,r-L/8}(t))$ for $L/8 < r \le L/4$.

This construction is possible by Lemma 3.1 and the observations on the functions f, h we made above, and the exact same argument as in the proof of Theorem 2.1 shows that one can choose initial states of (Π, g^-) that belong to an orbit of length $\ge c_2^N$, where

c_2 is a constant that depends only on L and ℓ and satisfies $c_1 < c_2 < 10^{1/4}$. It is also straightforward to verify that the resulting system is cooperative.

However, the system may not yet be a $\mathcal{C}_2$ system with $M = 2$, since the loops where copying occurs do not need to be of even length. So to ensure that we end up with an $M = 2$ system we may have at most two leftover sets S_i where the copying of some s_j needs to be implemented by $s_j \cap s_j^*$ using a few dummy variables s_j^*. Since the number of these 'leftover variables' is at most $2L$, this can be done without increasing N too much so that the resulting orbit will still have length $> c_1^N$ (see Appendix A of [12].)

4 Conclusion

Monotone and cooperative systems have been used as a modeling tool for gene regulatory systems, e.g. in [3, 5]. In the absence of negative feedback *continuous systems* converge generically towards an equilibrium under mild regularity hypotheses; see the work by Hirsch, Smith, Enciso, Mazco, and others [9, 10, 18, 24]. These generic convergence results have been generalized to the case of continuous monotone *maps*, in which case the generic iteration converges towards a periodic solution, with upper bounds for the maximum period [21]. In contrast, our results show that even very stringent conditions on *Boolean systems* with no negative interactions do not preclude very long orbits.

Our reasons for presenting a new proof of Theorem 1 in [12] are two-fold. First of all, the original construction given in [12] is somewhat difficult to read. We hope that the much simpler proof presented here will make the result more accessible to the mathematical community and will make the basic ideas that are common to both constructions more clearly visible. Second, as in [15, 20], this construction is based on additive lagged-Fubini generators (ALFG), which are the basis for the most commonly used pseudo-random number generators that are ubiquitous in applications from computer science and engineering. We hope that the proof presented here will highlight some important connections between number theory, computer science, and the study of Boolean networks and their applications, including the study of gene regulatory networks.

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Positive Solutions to an Nth Order Multi-point Boundary Value Problem on Time Scales

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Abstract: In this paper, we consider an nth order multi-point boundary value problem on time scales. We establish criteria for the existence of at least one or two positive solutions. We shall also obtain criteria which lead to nonexistence of positive solutions. Examples applying our results are also given.

Keywords: *positive solutions; fixed-point theorems; time scales; dynamic equations; cone.*

Mathematics Subject Classification (2000): 39A10.

1 Introduction

We are concerned with the following boundary value problem (BVP) on time scales $\mathbb{T}$:

$$\begin{cases} y^{\Delta^n}(t) + \lambda f(y^\sigma(t))) = 0, \quad t \in [a,b] \subset \mathbb{T}, \\ y^{\Delta^i}(a) = 0, \quad 0 \le i \le n-2, \\ \sum_{i=1}^m \alpha_i y^{\Delta^{n-2}}(\xi_i) = y^{\Delta^{n-2}}(\sigma(b)) \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f \in \mathcal{C}([0,\infty),[0,\infty))$, $n \ge 3$, $m \ge 1$ are integers, $a < \xi_1 < \xi_2 < ... < \xi_m < b$, $\alpha_i \in (0,+\infty)$ for $1 \le i \le m$ and $\sum_{i=1}^m \alpha_i < 1$.

We assume that $D = \sigma(b) - a - \sum_{i=1}^m \alpha_i(\xi_i - a) > 0$ and $\sigma(b)$ is right dense so that $\sigma^j(b) = \sigma(b)$ for $j \ge 1$.

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [10]. Some preliminary definitions and theorems on time scales can be found in the books [2, 3] which are excellent references for the calculus of time scales.

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Recently, existence results for positive solutions of second-order multi-point boundary value problems was studied by some authors [9, 11–16].

A few papers can be found in the literature on higher-order multi-point boundary value problems [4–7].

We were, in particular, motivated by [6, 7]. We study more general problem and we present results which guarantee the existence of at least one or two positive solutions and the nonexistence positive solutions. The methods discussed here are similar to earlier work [1].

This paper is organized as follows. Section 2 introduces some notation and several lemmas which play important roles in this paper. Section 3 gives nonexistence and multiplicity results for positive solutions to the BVP (1.1). In this article, the main tool is the following well-known Krasnosel'skii fixed point theorem in a cone [8].

Theorem 1.1 [8]. *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$$

be a completely continuous operator such that, either

(i) $\|Ay\| \leq \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \geq \|y\|$, $y \in P \cap \partial\Omega_2$; *or*

(ii) $\|Ay\| \geq \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \leq \|y\|$, $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries and Lemmas

Let $G_2(t,s)$ be the Green's function for the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) = 0, & t \in [a,b], \\ y(a) = 0, \\ \sum_{i=1}^m \alpha_i y(\xi_i) = y(\sigma(b)). \end{cases} \tag{2.1}$$

Then

$$G_2(t,s) = \begin{cases} \frac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^m \alpha_j(\xi_j-t)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(t-\sigma(s))}{\sigma(b)-a-\sum_{i=1}^m \alpha_i(\xi_i-a)}, \\ a \leq t \leq \sigma(b), \quad \xi_{i-1} \leq \sigma(s) \leq \min\{\xi_i, t\}, \quad i = \overline{1, m+1}, \\ \frac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^m \alpha_j(\xi_j-\sigma(s))]}{\sigma(b)-a-\sum_{i=1}^m \alpha_i(\xi_i-a)}, \\ a \leq t \leq \sigma(b), \quad \max\{\xi_{i-1}, t\} \leq \sigma(s) \leq \xi_i, \quad i = \overline{1, m+1}. \end{cases} \tag{2.2}$$

Lemma 2.1 *There exist a number $k \in (0,1)$ and a continuous function $\psi : [a,b] \to \mathbb{R}^+$ such that*

$$G_2(t,s) \leq \psi(s), \quad t \in [a, \sigma(b)], \ s \in [a,b],$$

and

$$G_2(t,s) \geq k\psi(s), \quad t \in [\xi_1, \sigma(b)], \ s \in [a,b],$$

where

$$\psi(s) = \tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

$$k = \min_{2\le i\le m}\{\tfrac{1}{\sigma(b)}\sum_{j=i}^{m}\alpha_j(\sigma(b)-\xi_j),\ \tfrac{\xi_1-a}{\sigma(s)-a}[1-\sum_{j=i}^{m}\alpha_j]\}. \qquad (2.3)$$

Proof Now, we will show that we may take $\psi(s) = \frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}$.
Upper bounds:

Case 1. Consider $a \le \sigma(s) \le \xi_1,\ \sigma(s) \le t$. Then

$$G_2(t,s) = \tfrac{\sigma(b)-t-\sum_{j=1}^{m}\alpha_j(\xi_j-t)}{D}(\sigma(s)-a) = \tfrac{\sigma(b)-\sum_{j=1}^{m}\alpha_j\xi_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a).$$

Since $\sum_{j=1}^{m}\alpha_j < 1$, the maximum occurs when $t=\sigma(s)$ and then

$$G_2(t,s) \le \tfrac{\sigma(b)-\sigma(s)+\sum_{j=1}^{m}\alpha_j(\sigma(s)-\xi_j)}{D}(\sigma(s)-a) \le \tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

since $\sum_{j=1}^{m}\alpha_j(\sigma(s)-\xi_j) \le 0$ for $\sigma(s)\le\xi_1$ and $\xi_j\in(a,b)$ with $a<\xi_1<\xi_2<...<\xi_{m-2}<b$.

Case 2. For $\xi_{r-1}\le t\le\xi_r,\ 2\le r\le m+1,\ \xi_{i-1}\le\sigma(s)\le\xi_i,\quad 2\le i\le r,\ \sigma(s)\le t$, we have

$$G_2(t,s) = \tfrac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-t)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(t-\sigma(s))}{D}$$

$$= \tfrac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{m}\alpha_j(t-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))(t-\sigma(s))}{D}$$

$$\le \tfrac{\sigma(b)-t+\sum_{j=1}^{m}\alpha_j(t-\sigma(s))}{D}(\sigma(s)-a)$$

$$\le \tfrac{\sigma(b)-\sigma(s)\sum_{j=1}^{m}\alpha_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a)$$

since $\sum_{j=i}^{m}\alpha_j(\sigma(s)-\xi_j)\le 0$ and $\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))\le 0$ for $\xi_{i-1}\le\sigma(s)\le\xi_i,\ 2\le i\le m+1$.

Since $\sum_{j=1}^{m}\alpha_j<1$, the maximum occurs when $t=\sigma(s)$ so

$$G_2(t,s) \le \frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}.$$

Case 3. For $\xi_{r-1}\le t\le\xi_r,\ 2\le r\le m,\ \xi_{i-1}\le\sigma(s)\le\xi_i,\quad r\le i\le m,\ t\le\sigma(s)$, we obtain

$$G_2(t,s) = \tfrac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^{m}a_j(\xi_j-\sigma(s))]}{D} \le \tfrac{(\sigma(b)-\sigma(s))(t-a)}{D} \le \tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

since $\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))\geq 0$ for $\xi_{i-1}\leq\sigma(s)\leq\xi_i,\ 2\leq i\leq m.$

Case 4. For $\xi_m\leq\sigma(s)\leq\sigma(b),\ \ t\leq\sigma(s)$, we clearly have

$$G_2(t,s)\leq\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}.$$

Lower bounds: We shall show that we may take an arbitrary interval $[\xi_1,\sigma(b)]\subset(a,\sigma(b)]$. We are looking for $\min\{G_2(t,s):t\in[\xi_1,\sigma(b)]\}$ as a function of s of the same form as the upper bound.

Case 1. Consider $0\leq\sigma(s)\leq\xi_1,\ \sigma(s)\leq t$, we get

$$G_2(t,s)=\tfrac{\sigma(b)-t-\sum_{j=1}^{m}\alpha_j(\xi_j-t)}{D}(\sigma(s)-a)=\tfrac{\sigma(b)-\sum_{j=1}^{m}\alpha_j\xi_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a).$$

Since $\sum_{j=1}^{m}\alpha_j<1$, the minimum occurs when $t=\sigma(b)$ and then

$$\begin{aligned}G_2(t,s)&\geq\tfrac{\sigma(b)-\sum_{j=1}^{m}\alpha_j\xi_j+\sigma(b)(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a)\\&>\tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}\tfrac{1}{\sigma(b)}\sum_{j=1}^{m}\alpha_j(\sigma(b)-\xi_j).\end{aligned}$$

Case 2. For $\xi_{r-1}\leq t\leq\xi_r,\ 2\leq r\leq m+1,\ \xi_{i-1}\leq\sigma(s)\leq\xi_i,\ \ 2\leq i\leq r,\ \sigma(s)\leq t$, we have

$$\begin{aligned}&G_2(t,s)\\&=\tfrac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{m}\alpha_j(t-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))(t-\sigma(s))}{D}\\&=\tfrac{1}{D}[t[(\sum_{j=1}^{m}\alpha_j-1)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))]+[\sigma(b)-\sigma(s)\sum_{j=1}^{m}\alpha_j\\&\qquad-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))](\sigma(s)-a)-\sigma(s)\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))].\end{aligned}$$

Since $(\sum_{j=1}^{m}\alpha_j-1)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))<0$, the minimum occurs when $t=\sigma(b)$, then

$$\begin{aligned}G_2(t,s)&\geq\tfrac{-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(b))(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(\sigma(b)-\sigma(s))}{D}\\&\geq\tfrac{1}{D}\sum_{j=i}^{m}\alpha_j(\sigma(b)-\xi_j)(\sigma(s)-a)\\&>\tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}\tfrac{1}{\sigma(b)}\sum_{j=i}^{m}\alpha_j(\sigma(b)-\xi_j).\end{aligned}$$

Case 3. For $\xi_{r-1}\leq t\leq\xi_r,\ 2\leq r\leq m,\ \xi_{i-1}\leq\sigma(s)\leq\xi_i,\ \ r\leq i\leq m,\ t\leq\sigma(s)$, we obtain

$$
\begin{aligned}
G_2(t,s) &= \tfrac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^m \alpha_j(\xi_j-\sigma(s))]}{D} \\
&= \tfrac{(t-a)[(\sigma(b)-\sigma(s))(1-\sum_{j=i}^m \alpha_j)-\sum_{j=i}^m \alpha_j(\xi_j-\sigma(b))]}{D} \\
&\geq \tfrac{(t-a)(\sigma(b)-\sigma(s))}{D}[1-\textstyle\sum_{j=i}^m \alpha_j] \\
&\geq \tfrac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D}[1-\textstyle\sum_{j=i}^m \alpha_j] \\
&= \tfrac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\tfrac{\xi_1-a}{\sigma(s)-a}[1-\textstyle\sum_{j=i}^m \alpha_j].
\end{aligned}
$$

Case 4. For $\xi_m \leq \sigma(s) \leq \sigma(b), \quad t \leq \sigma(s)$, we have

$$G_2(t,s) = \tfrac{(t-a)(\sigma(b)-\sigma(s))}{D} \geq \tfrac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D} = \tfrac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\tfrac{\xi_1-a}{\sigma(s)-a}.$$

Thus we can take

$$k = \min_{2\leq i\leq m}\{\tfrac{1}{\sigma(b)}\textstyle\sum_{j=i}^m \alpha_j(\sigma(b)-\xi_j),\ \tfrac{\xi_1-a}{\sigma(s)-a}[1-\textstyle\sum_{j=i}^m \alpha_j]\}. \quad \square$$

Lemma 2.2 *If y satisfies the boundary conditions*

$$\begin{cases} y^{\Delta^i}(a) = 0, \quad 0 \leq i \leq n-2, \\ \sum_{i=1}^m \alpha_i y^{\Delta^{n-2}}(\xi_i) = y^{\Delta^{n-2}}(\sigma(b)) \end{cases}$$

and

$$y^{\Delta^n}(t) \leq 0, \quad t \in [a,b],$$

then

$$y^{\Delta^{n-2}}(t) \geq 0.$$

Proof Let $P(t) = y^{\Delta^{n-2}}(t),\ t \in [a, \sigma(b)]$. Then we have

$$P^{\Delta^2}(t) \leq 0, \quad t \in [a,b]$$

$$P(a) = 0 \quad \text{and} \quad \textstyle\sum_{i=1}^m \alpha_i P(\xi_i) = P(\sigma(b)).$$

It must be true that $P(\sigma(b)) \geq 0$. To see this, assume to the contrary that $P(\sigma(b)) < 0$. Since $P(a) = 0$ and $P(t)$ is concave downward, we have

$$P(t) \geq \frac{t-a}{\sigma(b)-a}P(\sigma(b)), \quad t \in [a, \sigma(b)].$$

Therefore,

$$\sum_{i=1}^m \alpha_i P(\xi_i) - P(\sigma(b)) \geq \sum_{i=1}^m \alpha_i \frac{\xi_i - a}{\sigma(b)-a} P(\sigma(b)) - P(\sigma(b))$$

$$> \sum_{i=1}^m \alpha_i P(\sigma(b)) - P(\sigma(b))$$

$$> P(\sigma(b)) - P(\sigma(b)) = 0,$$

which is a contradiction.

Now, $P(a) = 0, \quad P(\sigma(b)) \geq 0$, and $P(t)$ is concave downward, so we have

$$P(t) = y^{\Delta^{n-2}}(t) \geq 0, \quad t \in [a, \sigma(b)].$$

This completes the proof of the lemma. □

Let $\mathbb{B}$ be the Banach space defined by

$$\mathbb{B} = \{y : y^{\Delta^n} \text{ is continuous on } [a,b],\ y^{\Delta^i}(a) = 0 \quad 0 \leq i \leq n-3\},$$

with the norm $\|y\| = \max_{t\in[a,\sigma(b)]} |y^{\Delta^{n-2}}(t)|$ and let

$$\mathcal{P} = \{y \in \mathbb{B} : y^{\Delta^{n-2}}(t) \geq 0, \quad \min_{t\in[\xi_1,\sigma(b)]} y^{\Delta^{n-2}}(t) \geq k\|y\| \},$$

where k is as in (2.3).

Solving the BVP (1.1) is equivalent to finding fixed points of the operator $L_\lambda : \mathcal{B} \to \mathcal{B}$ defined by

$$L_\lambda y(t) = \lambda \int_a^{\sigma(b)} G_n(t,s) f(y^\sigma(s)) \Delta s, \quad t \in [a, \sigma(b)]. \tag{2.4}$$

It can be verified that

$$G_2(t,s) = G_n^{\Delta^{n-2}}(t,s). \tag{2.5}$$

From (2.5), it follows that

$$(L_\lambda y)^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s)) \Delta s. \tag{2.6}$$

Solving the BVP (1.1) in $\mathbb{B}$ is equivalent to finding fixed points of the operator $L_\lambda^{\Delta^{n-2}}$ defined by (2.6).

Lemma 2.3 *The operator L_λ is completely continuous such that $L_\lambda(\mathcal{P}) \subset \mathcal{P}$.*

Proof From the continuity of $G_2(t,s)$ and $f(t)$ it follows that the operator L_λ defined by (2.4) is completely continuous in $\mathbb{B}$. By Lemma 2.1, Lemma 2.2, and definition of $\mathcal{P}$, we get $L_\lambda \mathcal{P} \subset \mathcal{P}$. □

3 Existence of Positive Solutions

Now we are ready to establish a few sufficient conditions for the existence of at least one or two positive solutions and the nonexistence of positive solutions of (1.1).

Now we define

$$l^0 = \lim_{\|u\|\to 0} \frac{f(u)}{\|u\|}, \quad l^\infty = \lim_{\|u\|\to \infty} \frac{f(u)}{\|u\|}.$$

Theorem 3.1 *For each* λ, *satisfying*

$$\frac{1}{kl^\infty \int_a^{\sigma(b)} \psi(s)\Delta s} < \lambda < \frac{1}{l^0 \int_a^{\sigma(b)} \psi(s)\Delta s}, \tag{3.1}$$

there exists at least one positive solution of (1.1).

Proof Let λ be given as in (3.1). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{k(l^\infty - \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s} \le \lambda \le \frac{1}{(l^0 + \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s}.$$

Now, turning to l^0, there exists an $p > 0$ such that $f(y) \le (l^0 + \epsilon)\|y\|$ for $0 < \|y\| \le p$. So, for $y \in \mathcal{P}$ with $\|y\| = p$, we have from the fact that $0 \le G_2(t,s) \le \psi(s)$ for $t \in [a, \sigma(b)]$, $s \in [a,b]$,

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\le \lambda \int_a^{\sigma(b)} \psi(s) f(y^\sigma(s))\Delta s \\ &\le \lambda (l^0 + \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s \|y\| \\ &\le \|y\| = p.\end{aligned}$$

Next, considering l^∞, there exists $\hat{q} > 0$ such that $f(y) \ge (l^\infty - \epsilon)\|y\|$ for $\|y\| \ge \hat{q}$. Let $q = \max\{2p, \hat{q}\}$. Then for $y \in \mathcal{P}$ with $\|y\| = q$, and $t \in [\xi_1, \sigma(b)]$ we get

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\ge \lambda k \int_a^{\sigma(b)} \psi(s)\Delta s (l^\infty - \epsilon)\|y\| \\ &\ge \|y\| = q.\end{aligned}$$

By Theorem 1.1, L_λ has a fixed point y such that $p \le \|y\| \le q$. The proof is complete. □

Theorem 3.2 *For each* λ *satisfying*

$$\frac{1}{kl^0 \int_a^{\sigma(b)} \psi(s)\Delta s} < \lambda < \frac{1}{l^\infty \int_a^{\sigma(b)} \psi(s)\Delta s}, \tag{3.2}$$

there exists at least one positive solution of (1.1).

Proof Let λ be given as in (3.2), and choose let $\epsilon > 0$ such that

$$\frac{1}{k(l^0-\epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s} \le \lambda \le \frac{1}{(l^\infty+\epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s}.$$

Beginning with l^0, there exists an $p > 0$ such that $f(y) \ge (l^0 - \epsilon)\|y\|$ for $0 < \|y\| \le p$. So, for $y \in \mathcal{P}$ with $\|y\| = p$, and $t \in [\xi_1, \sigma(b)]$ we have

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\ge \lambda k \int_a^{\sigma(b)} \psi(s) f(y^\sigma(s))\Delta s \\ &\ge \lambda k (l^0-\epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s \|y\| \\ &\ge \|y\| = p.\end{aligned}$$

It remains to consider l^∞. There exists $\hat{q} > 0$ such that $f(y) \le (l^\infty + \epsilon)\|y\|$ for $\|y\| \ge \hat{q}$. There are two cases:

For case (a), suppose $N > 0$ is such that $f(y) \le N$, for all $0 \le y < \infty$. Let $q = \max\{2p, \lambda N \int_a^{\sigma(b)} \psi(s)\Delta s\}$. Then $y \in \mathcal{P}$ and $\|y\| = q$, we have

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\le \lambda N \int_a^{\sigma(b)} \psi(s)\Delta s \\ &\le \|y\| = q.\end{aligned}$$

For case (b), let $g(h) := \max\{f(y) : 0 \le y^{\Delta^{n-2}} \le h\}$. The function g is nondecreasing and $\lim_{h\to\infty} g(h) = \infty$. Choose $q = \max\{2p, \hat{q}\}$ so that $g(q) \ge g(h)$ for $0 \le h \le q$. For $y \in \mathcal{P}$ and $\|y\| = q$, we have

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\le \lambda g(q) \int_a^{\sigma(b)} \psi(s)\Delta s\end{aligned}$$

$$\leq \lambda(l^\infty + \epsilon) q \int_a^{\sigma(b)} \psi(s)\Delta s$$

$$\leq \|y\| = q.$$

By Theorem 1.1, L_λ has a fixed point y such that $p \leq \|y\| \leq q$. The proof is complete. □

In the rest of the paper we assume that $f(y) > 0$ on $\mathbb{R}^+$. Set

$$A = \int_a^{\sigma(b)} \psi(s)\Delta s.$$

Theorem 3.3 *If either* $l^0 = \infty$ *or* $l^\infty = \infty$, *then for all* $0 < \lambda \leq \lambda_0$, *where*

$$\lambda_0 := \tfrac{1}{A} \sup_{r>0} \tfrac{r}{\max_{0<\|u\|\leq r} f(u)}, \tag{3.3}$$

(1.1) *has at least one positive solution.*

(b) *If either* $l^0 = 0$ *or* $l^\infty = 0$, *then for all* $\lambda \geq \lambda_0$, *where*

$$\lambda_0 := \frac{1}{A} \inf_{r>0} \frac{r}{\min_{0<\|u\|\leq r} f(u)},$$

(1.1) *has at least one positive solution.*

Proof We now prove the part (a) of Theorem 3.3. By (3.3), there exists $r > 0$ such that

$$\lambda_0 = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0<\|u\|\leq r} f(u)}.$$

If $\|y\| = r$, it follows that

$$\|L_\lambda y\| = \max_{t\in[a,\sigma(b)]} |(L_\lambda y)^{\Delta^{n-2}}(t)| \leq \lambda_0 \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \leq r.$$

So for all $0 < \lambda \leq \lambda_0$ we have

$$\|L_\lambda y\| \leq \|y\|.$$

Fix $\lambda \leq \lambda_0$. Choose $R > 0$ sufficiently large so that

$$\lambda R k \int_a^{\sigma(b)} \psi(s)\Delta s \geq 1. \tag{3.4}$$

Since $l^0 = \infty$, there is $p > 0$ such that

$$\frac{f(y)}{\|y\|} \geq R$$

for $t \in [a, \sigma(b)],\ 0 < \|y\| \leq p$. Hence we have that

$$f(y) \geq R\|y\|$$

for $t \in [a, \sigma(b)],\ 0 < \|y\| \leq p$. For $y \in \mathcal{P},\ \|y\| = p$ and $t \in [\xi_1, \sigma(b)]$, we get

$$(L_\lambda y)^{\Delta^{n-2}}(t) \geq \lambda Rk \int_a^{\sigma(b)} \psi(s)\Delta s\|y\| \geq \|y\| = p$$

by (3.4). By Theorem 1.1, L_λ has a fixed point y such that $\min\{p, r\} \leq \|y\| \leq \max\{p, r\}$.

Next, we use the assumption that $l^\infty = \infty$. Since $l^\infty = \infty$ there is a $q > 0$ such that

$$\frac{f(y)}{\|y\|} \geq R$$

for $\|y\| \geq q$ and R is chosen so that (3.4) holds. It follows that

$$f(y) \geq R\|y\|$$

for $\|y\| \geq q$.

For $y \in \mathcal{P},\ \|y\| = q$ and $t \in [\xi_1, \sigma(b)]$, we have

$$(L_\lambda y)^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s$$

$$\geq \lambda Rk \int_a^{\sigma(b)} \psi(s)\Delta s\|y\|$$

$$\geq q = \|y\|$$

by (3.4). By Theorem 3.1, then L_λ has a fixed point y such that $\min\{q, r\} \leq \|y\| \leq \max\{q, r\}$. This completes the proof of part (a). Part (b) holds in an analogous way. □

Theorem 3.4 *a) If $l^0 = l^\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$, (1.1) has two positive solutions.*
b) If $l^0 = l^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, (1.1) has two positive solutions.

Now, we give a nonexistence result as follows.

Theorem 3.5 (a) *If there is a constant $c > 0$ such that $f(y) \geq c\|y\|$, then there is a $\lambda_0 > 0$ such that (1.1) has no positive solutions for $\lambda \geq \lambda_0$.*
(b) *If there is a constant $c > 0$ such that $f(y) \leq c\|y\|$, then there is a $\lambda_0 > 0$ such that (1.1) has no positive solutions for $0 < \lambda \leq \lambda_0$.*

Proof We now prove the part (a) of this theorem. Assume there is a constant $c > 0$ such that $f(y) \geq c\|y\|$. Assume $y(t)$ is a solution of the BVP (1.1). We will show that for λ sufficiently large this leads to a contradiction. We have for $t \in [\xi_1, \sigma(b)]$,

$$y^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \geq ck\lambda_0 \int_a^{\sigma(b)} \psi(s)\Delta s \|y\|.$$

If we pick λ_0 sufficiently large so that $ck\lambda_0 \int_a^{\sigma(b)} \psi(s)\Delta s > 1$ for all $\lambda \geq \lambda_0$, then we have $y^{\Delta^{n-2}} > \|y\|$ which is a contradiction. The proof of part (b) is similar. □

Example 3.1 We illustrate Theorem 3.2 with specific time scale $\mathbb{T} = \{\frac{1}{2^n} : n \in \mathbb{N}_0\} \cup \{0\} \cup [1,5]$.

Consider the system:

$$\begin{cases} y^{\Delta^n}(t) + \lambda f(y^\sigma(t)) = 0, & t \in [0,1/2] \subset \mathbb{T}, \\ y^{\Delta^i}(0) = 0, \quad 0 \leq i \leq n-2, & \\ 1/3y(1/4) + 1/5y(1/8) + 1/10y(1/64) = y(5), & \end{cases} \tag{3.5}$$

where $f = 1 + \sqrt{y},\ \alpha_1 = 1/3,\ \alpha_2 = 1/5, \alpha_3 = 1/10,\ a = 0,\ b = 1/2,\ \xi_1 = 1/4,\ \xi_2 = 1/8,\ \xi_3 = 1/64,\ n \geq 3$.

Since $f = 1 + \sqrt{y}$, we have

$$l^0 = \infty \qquad l^\infty = 0.$$

We get $\psi(s) = \frac{4096}{1967}s(1-2s),\ \int_0^1 \psi(s)\Delta s = \frac{4096}{41307}$. Therefore the assumptions of Theorem 3.2 are satisfied. By Theorem 3.2, for all $\lambda \in (0,\infty)$, (3.5) has at least one positive solution.

Example 3.2 We illustrate Theorem 3.3 with specific time scale $\mathbb{T} = \{\frac{n}{3} : n \in \mathbb{N}\} \cup [7/3,5]$.

Consider the system:

$$\begin{cases} y^{\Delta^3}(t) + \lambda f(y^\sigma(t)) = 0, & t \in [1,2], \\ y(1) = y^\Delta(1) = 0, & \\ 1/2y(4/3) + 1/3y(5/3) = y(7/3), & \end{cases} \tag{3.6}$$

where $f = e^y,\ \alpha_1 = 1/2,\ \alpha_2 = 1/3,\ a = 1,\ b = 2,\ \xi_1 = 4/3,\ \xi_2 = 5/3$.

Hence $l^\infty = \infty$. Since

$$A = \int_1^{7/3} \psi(s)\Delta s = \frac{60}{153}, \quad \sup_{r>0} \frac{r}{\max_{0<\|y\|\leq r} e^y} = \sup_{r>0} \frac{r}{e^r} = \frac{1}{e},$$

we have

$$\lambda_0 = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0<\|y\|\leq r} f(y)} = \frac{153}{60} e^{-1}.$$

So, by Theorem 3.3, for all $\lambda \in (0, \frac{153}{60}e^{-1}]$, (3.6) has one positive solution.

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A Common Fixed Point Theorem for a Sequence of Self Maps in Cone Metric Spaces

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Abstract: In this paper, we obtain a new common fixed point theorem by using a new contractive condition in cone metric spaces. Our result generalizes and extends well known result in complete metric spaces.

Keywords: *cone metric spaces; common fixed point; sequence; normal.*

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1 Introduction

The study of fixed points of functions satisfying certain contractive conditions has been at the center of vigorous research activity, for example see [1]–[5] and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, see [6]–[10]. Recently, Huang and Zhang [11] have replaced the real numbers by ordering Banach space and define cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [12]–[16]. Choudhury [17] introduced mutually contractive sequence of self maps and proved a fixed point theorem. The purpose of this paper is to obtain a new common fixed point theorem by using a new contractive condition in cone metric spaces. Our result generalizes and extends many known results in metric spaces.

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Consistent with Huang and Zhang [11], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).

Definition 1.1 [11] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.1 [11] Let $E = R^2, P = \{(x, y) \in E | x, y \geq 0\} \subset R^2, X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.2 [11] Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N, d(x_n, x_m) \ll c$;

(f) a Convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is know that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. The limit of a convergent sequence is unique provided that P is a normal cone with normal constant K[11].

Lemma 1.1 [*11*] *Let* (X, d) *be a cone metric space,* P *be a normal cone with normal constant* K*. Let* $\{x_n\}$ *be a sequence in* X*. Then,* $\{x_n\}$ *is a Cauchy sequence if and only if* $d(x_n, x_m) \to \theta (n, m \to \infty)$.

Lemma 1.2 [*11*] *Let* (X, d) *be a cone metric space,* P *be a normal cone with normal constant* K*. Let* $\{x_n\}$ *be a sequence in* X*. If* $\{x_n\}$ *converges to* x *and* $\{x_n\}$ *converges to* y*, then* $x = y$*. That is the limit of* $\{x_n\}$ *is unique.*

Definition 1.3 Let (X, d) be a cone metric space. A sequence $\{T_i\}_{i=1}^{\infty}$ of self-mappings on a complete cone metric space is said to be mutually contractive if for all $i, j = 1, 2, \cdots,$ with $i \neq j$,

$$d(T_i x, T_j y) \leq kd(x, y) \text{ for all } x, y \in X \text{ with } x \neq y,$$

where $k \in (0, 1)$ is a constant.

2 Main Result

Theorem 2.1 *Let (X, d) be a complete cone metric space. P be a normal cone with normal constant K. $\{T_i\}_{i=1}^{\infty}$ be a sequence of self-mappings on X such that*
(1) T_i is continuous for all $i, j = 1, 2, \cdots$;
(2) $\{T_i\}_{i=1}^{\infty}$ is mutually contractive;
(3) $T_iT_j = T_jT_i$ for all $i, j = 1, 2, \cdots$.
Then the sequence $\{T_n\}_n$ has a unique common fixed point in X.

Proof Let x_0 be an arbitrary point in X. We construct a sequence $\{x_n\} \subset X$ as follows:

$$x_1 = T_1x_0, x_2 = T_2x_1, \cdots, x_n = T_nx_{n-1}, \cdots$$

Then the following cases may arise:.

Case I: If no terms of $\{x_n\}$ are equal. Then, using (2), we get:

$$d(x_n, x_{n+1}) = d(T_nx_{n-1}, T_{n+1}x_n) \le kd(x_{n-1}, x_n).$$

By repeated application of above inequalities, we get

$$d(x_n, x_{n+1}) = d(T_nx_{n-1}, T_{n+1}x_n) \le k^n d(x_0, x_1).$$

So for $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &\le (k^{n-1} + \cdots + k^m)d(x_0, x_1) \le \frac{k^m}{1-k} d(x_0, x_1) \end{aligned}$$

We get $\|d(x_n, x_m)\| \le \frac{k^m}{1-k} K\|d(x_0, x_1)\|$. This implies $d(x_n, x_m) \to \theta(n, m \to \infty)$. Hence x_n is a Cauchy sequence by Lemma 1.1. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*(n \to \infty)$. Now, we prove that x^* is a fixed point of T_i.

Since two consecutive terms of $\{x_n\}$ are unequal, for an arbitrary integer $i > 0$ and $c \gg \theta$, we can find n such that $x^* \ne x_{n-1}, n > i$,

$$d(x^*, x_n) < c, \quad \text{and} \quad d(x^*, x_{n-1}) < c.$$

Then, we get

$$\begin{aligned} d(x^*, T_ix^*) &\le d(x^*, x_n) + d(x_n, T_ix^*) \\ &= d(x^*, x_n) + d(T_nx_{n-1}, T_ix^*) \\ &\le d(x^*, x_n) + kd(x_{n-1}, x^*). \end{aligned}$$

Thus, $\|d(x^*, T_ix^*)\| \le K(\|d(x^*, x_n)\| + k\|d(x_{n-1}, x^*)\|) \to 0$ since $c \gg \theta$ is arbitrary. Hence $\|d(x^*, T_ix^*)\| = 0$. This implies $x^* = T_ix^*$. So, x^* is a fixed point of T_i.

Case II: If $x_i = x_{i-1}$ for some positive integer i. Then $x_{i-1} = T_ix_{i-1}$. Let $x^* = x_{i-1}$, that is, $x^* = T_ix^*$, $x^* \ne T_jx^*$ and further assume that $x^* \ne T_j^n x^*$ for all $n = 1, 2, \cdots$.

Thus, we get

$$d(x^*, T_j^2x^*) = d(T_ix^*, T_j(T_jx^*)) \le kd(x^*, T_jx^*).$$

Similarly,

$$d(x^*, T_j^3 x^*) \leq k^2 d(x^*, T_j x^*).$$

Consequently,

$$d(x^*, T_j^n x^*) \leq k^{n-1} d(x^*, T_j x^*) \quad \text{for all} \quad n = 2, 3, \cdots.$$

We get $\|d(x^*, T_j^n x^*)\| \leq k^{n-1} K \|d(x^*, T_j x^*)\|$. This implies $d(x^*, T_j^n x^*) \to \theta$ as $n \to \infty$, that is

$$T_j^n x^* \to x^* \quad \text{as} \quad n \to \infty.$$

Since T_i is continuous, we get

$$T_j(T_j^n x^*) = T_j^{n+1} x^* \to T_j x^* \quad \text{as} \quad n \to \infty.$$

In the view of Lemma 1.2, we have $x^* = T_j x^*, j = 1, 2, \cdots$. This is a contradiction, so $x^* = T_j^l x^*$ for some l.

Let l be the smallest integer with this property. Then, we get

$$x^* \neq T_j^m x^* \quad \text{for some} \quad m = 1, 2, \cdots, l-1.$$

Thus,

$$\begin{aligned} d(x^*, T_j^{l-1} x^*) &= d(T_i x^*, T_j(T_j^{l-2} x^*)) \leq k d(x^*, T_j^{l-2} x^*) \\ &= k d(T_i x^*, T_j(T_j^{l-3} x^*)) \leq k^2 d(x^*, T_j^{l-3} x^*) \leq \cdots \leq k^{l-2} d(x^*, T_j x^*), \end{aligned}$$

hence $x^*, T_j x^*, T_j^2 x^*, \cdots, T_j^{k-1} x^*$ are all distinct. Therefore,

$$\begin{aligned} d(x^*, T_j x^*) &= d(T_j^l x^*, T_j(T_i x^*)) = d(T_j(T_j^{l-1} x^*), T_i(T_j x^*)) \\ &\leq k d(T_j^{l-1} x^*, T_j x^*) = k d(T_j(T_j^{l-2} x^*), T_i(T_j x^*)) \\ &\leq k^2 d(T_j^{l-2} x^*, T_j x^*) \leq \cdots \leq k^{l-2} d(T_j^2 x^*, T_j x^*) \\ &= k^{l-2} d(T_j^2(T_i x^*), T_j x^*) = k^{l-2} d(T_i(T_j^2 x^*), T_j x^*) \\ &\leq k^{l-1} d(T_j^2 x^*, x^*) = k^{l-1} d(T_j(T_j x^*), T_i x^*) \leq k^l d(T_j x^*, x^*). \end{aligned}$$

Hence $\|d(x^*, T_j x^*)\| = 0$ and $x^* = T_j x^*$ for all $j = 1, 2, \cdots$.

To show uniqueness, assume y^* is another common fixed point of T_i, then

$$d(x^*, y^*) = d(T_i(x^*), T_j(y^*)) \leq k d(x^*, y^*).$$

Hence $\|d(x^*, y^*)\| = 0$ and $x^* = y^*$, that is, x^* is a unique common fixed point of the sequence $\{T_n\}_n$. $\square$

Remark 2.1 Let us remark that in Theorem 2.1, setting $E = R, P = [0, +\infty), \|x\| = |x|, x \in E$, we get the well know result in complete metric space.

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Cone Inequalities and Stability of Dynamical Systems

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Abstract: The paper is devoted to working out new methods for stability analysis of equilibrium states of nonlinear dynamic systems in a partially ordered space. The concerned classes of differential systems are described by operator inequalities and inclusions using the notion of derivative with respect to a cone of nonlinear operator. Sufficient stability conditions of equilibrium states are formulated for sets of nonlinear and pseudolinear systems with the interval and polyhedral types operator coefficients. More general result is presented in the form of comparison principle for a finite set of differential systems.

Keywords: *dynamic system; pseudolinear system; monotone system; positive system; Lyapunov stability; cone inequality; partially ordered space.*

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1 Introduction

Stability analysis for dynamic systems with parameter or functional uncertainties is one of the fundamental issues in system and control theory. The applied researches employ continuous and discrete models of dynamic objects whose states possess certain properties with respect to a cone in the phase space (positivity, monotonicity, cooperativity, etc.). For example, these properties can be determined very often by using a cone of nonnegative vectors, a cone of symmetric nonnegatively definite matrices, an ellipsoidal cone, etc. Many important advances have been achieved on the basis of the operator theory in partially ordered spaces (see, e.g., [1–8]). In addition, classes of positive and monotone systems arise in stability theory as systems of comparison [7, 9–11].

We study generalized classes of positive and monotone dynamic systems with respect to a cone and give characterization for such systems by means of operator inequalities and

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inclusions. We formulate analogs of the Lyapunov theorem on the stability of equilibrium states of nonlinear autonomous differential systems with respect to the first approximation using the notion of derivative of nonlinear operator with respect to a cone. Finally, we propose general technique for comparison of a set of differential systems and formulate robust stability conditions for some families of nonlinear, pseudolinear and linear systems in terms of the cone and operator inequalities.

2 Definitions and Auxiliary Facts

A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a *wedge* if $\alpha\mathcal{K} + \beta\mathcal{K} \subseteq \mathcal{K}$ $\forall\alpha, \beta \geq 0$. A wedge $\mathcal{K}$ with *edge* $\mathcal{K} \cap -\mathcal{K} = \{0\}$ is a *cone*. A space with a wedge is partially ordered: $X \overset{\mathcal{K}}{\leq} Y \Leftrightarrow Y - X \in \mathcal{K}$. A *solid* cone contains nonempty sets of *interior* points $\operatorname{int}\mathcal{K}$ and *boundary* $\partial\mathcal{K}$. A cone $\mathcal{K}$ is *normal* if $0 \overset{\mathcal{K}}{\leq} X \overset{\mathcal{K}}{\leq} Y$ implies $\|X\| \leq \nu\|Y\|$, where ν is a universal constant. The least of these numbers ν is the *normality constant* of $\mathcal{K}$. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$, then the cone $\mathcal{K}$ is *reproducing*. A reproducing cone $\mathcal{K}$ is *non-flat*, i.e. $X = X_+ - X_-$ and $X_\pm \in \mathcal{K}$ imply $\|X_\pm\| \leq \mu\|X\|$, where μ is a universal constant. The *dual* cone $\mathcal{K}^*$ consists of linear nonnegative functionals. Moreover,

$$\mathcal{K} = \{X \in \mathcal{E} : \varphi(X) \geq 0, \forall\varphi \in \mathcal{K}^*\}, \quad \mathcal{K}^* = \{\varphi \in \mathcal{E}^* : \varphi(X) \geq 0, \forall X \in \mathcal{K}\},$$

$$\operatorname{int}\mathcal{K} = \{X \in \mathcal{K} \colon \varphi(X) > 0, \forall\varphi \neq 0 \in \mathcal{K}^*\}, \quad \partial\mathcal{K} = \{X \in \mathcal{K} \colon \exists\varphi \neq 0 \in \mathcal{K}^*, \varphi(X) = 0\}.$$

A functional $\varphi \in \mathcal{E}^*$ is *uniformly positive* if $\varphi(X) \geq \gamma\|X\|$ for some $\gamma > 0$ and $\forall\, X \in \mathcal{K}$.

A *convex shell* of $X_1, \ldots, X_n \in \mathcal{E}$ is defined by

$$\operatorname{Co}\{X_1, \ldots, X_n\} = \Big\{X : X = \sum_{i=1}^{n} \alpha_i X_i, \ \sum_{i=1}^{n} \alpha_i = 1, \ \alpha_i \geq 0, \ i = \overline{1, n}\Big\}.$$

A set $\mathcal{D} \subset \mathcal{E}$ is $\mathcal{K}$*–convex* if $X \overset{\mathcal{K}}{\leq} Y$ implies $\operatorname{Co}\{X, Y\} \subseteq \mathcal{D}$ for $X, Y \in \mathcal{D}$.

Let $\mathcal{E}(\mathcal{E}_1)$ be a Banach space with a cone $\mathcal{K}(\mathcal{K}_1)$. An operator $M : \mathcal{E} \to \mathcal{E}_1$ is positive if $M\mathcal{K} \subseteq \mathcal{K}_1$. The operator is *monotone* if $X \overset{\mathcal{K}}{\leq} Y \Rightarrow MX \overset{\mathcal{K}_1}{\leq} MY$. The operator inequality $M_2 \trianglerighteq M_1$ means that $M_2 - M_1$ is positive. A linear invertible operator M is *positive invertible* if $\mathcal{K}_1 \subseteq M\mathcal{K}$. Since $(M^{-1})^* = (M^*)^{-1}$, positive invertibility of M leads to positive invertibility of M^*. If $\mathcal{K}_1$ is a normal reproducing cone and $M_1 \trianglelefteq M \trianglelefteq M_2$, then positive invertibility of M_1 and M_2 yields positive invertibility of M, furthermore $M_2^{-1} \trianglelefteq M^{-1} \trianglelefteq M_1^{-1}$ [1]. An operator $M : \mathcal{E} \to \mathcal{E}$ is called *positive-off-diagonal*, if $X \in \mathcal{K}$ and $\varphi \in \mathcal{K}^*$ with $\varphi(X) = 0$ imply $\varphi(MX) \geq 0$. Obviously, if $M \trianglerighteq \alpha I$ for a certain real α, where I is the identity operator, then M is positive-off-diagonal. The inverse statement holds under certain additional conditions with $\alpha \leq -\nu\mu\|M\|$, where ν and μ are normality and non-flatness constants of M, respectively [4].

A linear operator of the form $M = L - P$, $P\mathcal{K} \subseteq \mathcal{K}_1 \subseteq L\mathcal{K}$, with a normal reproducing cone $\mathcal{K}_1$ is positive invertible if and only if $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the operator pencil of $T(\lambda) = P - \lambda L$. If $\mathcal{K}_1$ is solid, then $\rho(T) < 1 \Leftrightarrow M\mathcal{K} \cap \operatorname{int}\mathcal{K}_1 \neq \emptyset$ [7].

A linear bounded operator $F'(X)$ is called the Gâteaux derivative of a nonlinear operator $F(X)$ at X, if $\lim_{\varepsilon\to 0} \varepsilon^{-1}\big[F(X + \varepsilon H) - F(X)\big] = F'(X)H$ exists in the sense of strong convergence. If this relation holds only for $H \in \mathcal{K}$, then F' is *the Gâteaux derivative of F with respect to a cone* $\mathcal{K}$ [13]. The Fréchet derivative F' with respect to $\mathcal{K}$ is defined by $F(X + H) - F(X) = F'(X)H + o(\|H\|)$, $H \in \mathcal{K}$. The Fréchet derivative is also the Gâteaux derivative. If the Gâteaux derivative is continuous in a neighborhood of X, then it is the Fréchet derivative. We denote the Gâteaux and Fréchet derivatives with respect to $\mathcal{K}$ and $-\mathcal{K}$ by $F'_+(X)$ and $F'_-(X)$, respectively. If $F'(X)$ exists, then $F'_+(X) = F'_-(X) = F'(X)$.

3 Classes of Dynamic Systems in a Partially Ordered Space

Assume that a dynamic system $\mathcal{S}$ operates in a certain domain $\mathcal{D}$ of a Banach space $\mathcal{E}$ and its states are defined by

$$X_t = E(X_\tau, \tau, t) \in \mathcal{E}, \quad \tau \in \Upsilon, \; t \in \Upsilon_\tau, \tag{1}$$

where E is an operator of the transition from initial state X_τ to state X_t and such that

$$E(X,\tau,\tau) = X, \quad E\big(E(X,\tau,t),t,s\big) = E(X,\tau,s), \quad t \in \Upsilon_\tau, \quad s \in \Upsilon_t,$$

$\Upsilon \subseteq R^1$ is an ordered set of indices, $\Upsilon_\tau = \{t \in \Upsilon : t \geq \tau\}$. The system is *continuous, discrete* or *hybrid* subject to the structure of Υ. Note that $E(\cdot,\tau,\tau) \equiv I$ is the identity operator. If $E(\Theta,\tau,t) \equiv \Theta$, then $X_t \equiv \Theta$ is the *equilibrium state* of $\mathcal{S}$. We shall consider only the isolated equilibrium states of dynamic systems.

Let $\mathcal{K}_t$ be a constant or time-varying set in $\mathcal{E}$. If $E(\mathcal{K}_\tau,\tau,t) \subseteq \mathcal{K}_t$ for $t \in \Upsilon_\tau$, then $\mathcal{K}_t$ is an *invariant set* of system $\mathcal{S}$. The system is *positive* with respect to an invariant cone $\mathcal{K}_t$. System $\mathcal{S}$ is *monotone* with respect to a cone $\mathcal{K}_t$ if

$$X_\tau \overset{\mathcal{K}_\tau}{\leq} Y_\tau \quad \Rightarrow \quad X_t = E(X_\tau,\tau,t) \overset{\mathcal{K}_t}{\leq} Y_t = E(Y_\tau,\tau,t) \tag{2}$$

for any $\tau \in \Upsilon$ and $t \in \Upsilon_\tau$. A positive (monotone) dynamic system $\mathcal{S}$ is defined by a positive (monotone) operator E with respect to $\mathcal{K}_t$. Denote the classes of monotone and positive systems with respect to $\pm\mathcal{K}_t$ by $\mathcal{M}$ and $\mathcal{M}_0^\pm$, respectively.

Consider the sets

$$\mathcal{K}_t^+(\Theta) = \big\{X \in \mathcal{E} : X \overset{\mathcal{K}_t}{\geq} \Theta\big\}, \quad \mathcal{K}_t^-(\Theta) = \big\{X \in \mathcal{E} : X \overset{\mathcal{K}_t}{\leq} \Theta\big\},$$

where $\Theta \in \mathcal{E}$, $\mathcal{K}_t$ is a cone. For the class of systems with invariant sets $\mathcal{K}_t^\pm(\Theta)$, we use the notation $\mathcal{M}_0^\pm(\Theta)$. Denote the classes of systems which posses the property (2) with $Y_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^-(\Theta)$ and $Y_\tau \in \mathcal{K}_\tau^-(\Theta)$ by $\mathcal{M}_1^+(\Theta)$, $\mathcal{M}_2^+(\Theta)$, $\mathcal{M}_1^-(\Theta)$ and $\mathcal{M}_2^-(\Theta)$, respectively. It is obvious that

$$\mathcal{M} \subseteq \mathcal{M}_1^\pm(\Theta) \subseteq \mathcal{M}_2^\pm(\Theta), \quad \mathcal{M} \subseteq \mathcal{M}_1(\Theta) \subseteq \mathcal{M}_2(\Theta),$$

where $\mathcal{M}_1(\Theta) = \mathcal{M}_1^+(\Theta) \cap \mathcal{M}_1^-(\Theta)$, $\mathcal{M}_2(\Theta) = \mathcal{M}_2^+(\Theta) \cap \mathcal{M}_2^-(\Theta)$. A system of $\mathcal{M}_2^\pm(\Theta)$ is monotone in $\mathcal{K}_t^\pm(\Theta)$. Every system of $\mathcal{M}_2^+(\Theta)$, $\mathcal{M}_2^-(\Theta)$ or $\mathcal{M}_2(\Theta)$ with the equilibrium state $X_t \equiv \Theta$ belongs to $\mathcal{M}_0^+(\Theta)$, $\mathcal{M}_0^-(\Theta)$ or $\mathcal{M}_0(\Theta) = \mathcal{M}_0^+(\Theta) \cap \mathcal{M}_0^-(\Theta)$, respectively.

We describe the classes of systems $\mathcal{S}$ introduced above via the inclusions

$$E'_\pm(X,\tau,t)\,\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad X \in \mathcal{D}, \; \tau \in \Upsilon, \; t \in \Upsilon_\tau, \tag{3}$$

where $E'_\pm(X,\tau,t)$ are the Gâteaux derivatives of $E(X,\tau,t)$ with respect to $\pm\mathcal{K}_\tau$.

Lemma 3.1 *Suppose that $E(X,\tau,t)$ is Gâteaux differentiable with respect to $\pm\mathcal{K}_\tau$ in a $\mathcal{K}_\tau$-convex domain $\mathcal{D}$ for $\tau \in \Upsilon$, $t \in \Upsilon_\tau$. Then:* (i) *$\mathcal{S} \in \mathcal{M}$ if and only if one of the inclusions* (3) *holds;* (ii) *$\mathcal{S} \in \mathcal{M}_0^\pm(\Theta)$ if $E(\Theta,\tau,t) - \Theta \in \pm\mathcal{K}_t$ and* (3) *holds for $X \in \mathcal{K}_\tau^\pm(\Theta)$;* (iii) *$\mathcal{S} \in \mathcal{M}_2^\pm(\Theta)$ if and only if* (3) *holds for $X \in \mathcal{K}_\tau^\pm(\Theta)$.*

Proof The necessity assertions (i)–(iii) are obtained by using the definitions of the corresponding classes of systems $\mathcal{S}$ and the Gâteaux derivatives

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[E(X+\varepsilon H,\tau,t)-E(X,\tau,t)\big]=E'_{\pm}(X,\tau,t)H,\quad X\in\mathcal{D},\quad H\in\mathcal{K}_{\tau}^{\pm}.$$

The sufficiency assertions (i)–(iii) follow from the Lagrange type formula:

$$\varphi\big(E(X+H,\tau,t)-E(X,\tau,t)\big)=\varphi\big(E'_{\pm}(Z,\tau,t)H\big),$$

where $\varphi\in\mathcal{E}^*$, $Z=X+\mu H\in\mathrm{Co}\{X,X+H\}$, $0<\mu<1$, X and $X+H$ are arbitrary points of a certain convex set. For this purpose, we use only functionals $\varphi\in\pm\mathcal{K}_t^*$ and the property of $\mathcal{K}_\tau$–convexity of $\mathcal{D}$. Moreover, $Z=(1-\mu)X+\mu(X+H)\in\mathcal{D}$ for $X\in\mathcal{D}$ and $H\in\pm\mathcal{K}_\tau$. □

Consider the nonlinear differential system

$$\dot{X}=F(X,t),\quad t\geq\tau\geq 0,\tag{4}$$

where F is a continuous operator function that guarantees the existence and uniqueness of the continuously differentiable solution $X(t)=E(X_\tau,\tau,t)$ for any $\tau\geq 0$, $X_\tau\in\mathcal{D}$. Let $\mathcal{K}_t$ be a cone in the phase space $\mathcal{E}$. For example, the Lyapunov transformation $\mathcal{K}_t=L(t)\mathcal{K}$ of a given cone $\mathcal{K}$ is a cone also. In this case, we can study the solutions (4) in the form $X(t)=L(t)Z(t)$ by means of a constant cone $\mathcal{K}$ instead of $\mathcal{K}_t$ in a phase space of the transformed system

$$\dot{Z}=L^{-1}(t)F(L(t)Z,t)-L^{-1}(t)\dot{L}(t).$$

For $t\geq 0$, we introduce the following conditions:

$$X\overset{\mathcal{K}_t}{\geq}\Theta,\ \varphi\in\mathcal{K}_t^*,\ \varphi(X-\Theta)=0\ \ \Rightarrow\ \ \varphi\left(F(X,t)\right)\geq 0,\tag{5}$$

$$X\overset{\mathcal{K}_t}{\leq}Y,\ \varphi\in\mathcal{K}_t^*,\ \varphi(X-Y)=0\ \ \Rightarrow\ \ \varphi\left(F(X,t)-F(Y,t)\right)\leq 0.\tag{6}$$

Let $\mathcal{F}_0^{\pm}(\Theta)$ denote the classes of operator functions F satisfying (5) with respect to $\pm\mathcal{K}_t$. Let $\mathcal{F}$ be a class of operator functions satisfying (6). We also define the classes of operator functions $\mathcal{F}_1^{+}(\Theta)$, $\mathcal{F}_2^{+}(\Theta)$, $\mathcal{F}_1^{-}(\Theta)$ and $\mathcal{F}_2^{-}(\Theta)$, that possess property (6) with $Y\in\mathcal{K}_t^{+}(\Theta)$, $X\in\mathcal{K}_t^{+}(\Theta)$, $X\in\mathcal{K}_t^{-}(\Theta)$ and $Y\in\mathcal{K}_t^{-}(\Theta)$, respectively. Denote $\mathcal{F}_k(\Theta)=\mathcal{F}_k^{+}(\Theta)\cap\mathcal{F}_k^{-}(\Theta)$, $k=0,1,2$. It is obvious that $\mathcal{F}\subseteq\mathcal{F}_1^{\pm}(\Theta)\subseteq\mathcal{F}_2^{\pm}(\Theta)$.

Lemma 3.2 [8] *Let $\mathcal{K}_t$ be a solid cone possessing the extension property*

$$0\leq\tau<t\ \ \Rightarrow\ \ \mathcal{K}_\tau\subseteq\mathcal{K}_t.\tag{7}$$

Then: (i) *system* (4) *is monotone with respect to $\mathcal{K}_t$ if $F\in\mathcal{F}$;* (ii) *system* (4) *belongs to $\mathcal{M}_0^{\pm}(\Theta)$ if $F\in\mathcal{F}_0^{\pm}(\Theta)$;* (iii) *system* (4) *belongs to $\mathcal{M}_0^{\pm}(\Theta)\cap\mathcal{M}_k^{\pm}(\Theta)$ if $F\in\mathcal{F}_k^{\pm}(\Theta)$, $k=1,2$;* (iv) *system* (4) *belongs to $\mathcal{M}_k^{\pm}(\Theta)$ if $F(\Theta,t)\in\pm\mathcal{K}_t$ and $F\in\mathcal{F}_k^{\pm}(\Theta)$, $k=1,2$.*

Note that the cone inequality

$$F(X,t)\overset{\mathcal{K}_t}{\geq}\alpha_{+}(X,t)\,(X-\Theta),\quad X-\Theta\in\partial\mathcal{K}_t,\quad t\geq 0,$$

where $\alpha_\pm(X,t)$ are scalar functions, yields $F \in \mathcal{F}_0^+(\Theta)$. Analogously, if

$$F(X,t) - F(Y,t) \overset{\mathcal{K}_t}{\leq} \beta(X,Y,t)\,(X-Y), \quad Y - X \in \partial\mathcal{K}_t, \quad t \geq 0,$$

where $\beta(X,Y,t)$ is a scalar function, then $F \in \mathcal{F}$. If this condition holds for $Y \in \mathcal{K}_t^+(\Theta)$, $X \in \mathcal{K}_t^+(\Theta)$, $X \in \mathcal{K}_t^-(\Theta)$ and $Y \in \mathcal{K}_t^-(\Theta)$, then $F \in \mathcal{F}_1^+(\Theta)$, $F \in \mathcal{F}_2^+(\Theta)$, $F \in \mathcal{F}_1^-(\Theta)$ and $F \in \mathcal{F}_2^-(\Theta)$, respectively.

We can describe the classes of operator function $\mathcal{F}$, $\mathcal{F}_0^\pm(\Theta)$ and $\mathcal{F}_2^\pm(\Theta)$ by means of the following operator inequalities generated by $\mathcal{K}_t$:

$$F'_\pm(X,t) \trianglerighteq \beta_\pm(X,t)I, \quad X \in \mathcal{D}, \quad t \geq 0, \tag{8}$$

where $\beta_\pm(X,t)$ are scalar function. These inequalities ensure that $F'_\pm(X,t)$ are positive-off-diagonal with respect to $\mathcal{K}_t$ for $X \in \mathcal{D}$ and $t \geq 0$. In view of Lemma 3.2, we have the following characterization of the introduced classes of differential systems (4).

Lemma 3.3 *Suppose that the operator $F(X,t)$ is Gâteaux differentiable with respect to $\pm\mathcal{K}_t$ in the $\mathcal{K}_t$-convex domain $\mathcal{D}$ for $t \geq 0$. Then:* (i) *$F \in \mathcal{F}$ if one of the operator inequalities* (8) *holds;* (ii) *$F \in \mathcal{F}_0^\pm(\Theta)$ if $F(\Theta,t) \in \pm\mathcal{K}_t$ and* (8) *holds for $X \in \mathcal{K}_t^\pm(\Theta)$;* (iii) *$F \in \mathcal{F}_2^\pm(\Theta)$ if* (8) *holds for $X \in \mathcal{K}_t^\pm(\Theta)$.*

Proof The assertions (i)–(iii) of Lemma 3.3 are obtained by using the Lagrange type formula:

$$\varphi\big(F(X+H,t) - F(X,t)\big) = \varphi\big(F'_\pm(Z,t)H\big), \quad H \in \pm\mathcal{K}_t, \ \varphi \in \pm\mathcal{K}_t^*,$$

where $Z = X + \mu H \in \mathrm{Co}\{X, X+H\}$, $0 < \mu < 1$. If $F'_\pm(Z,t)H$ is continuous, then

$$F(X+H,t) - F(X,t) = \int_0^1 F'_\pm(X+\mu H,t)H\,d\mu, \quad H \in \pm\mathcal{K}_t. \ \square$$

Let's introduce some classes of operator functions which are used in the theory of comparison systems. We write $F \in \overline{\mathcal{F}}$, if one can establish a correspondence between solutions of (4) and solutions of the differential inequalities $\dot{Z} \overset{\mathcal{K}_t}{\leq} F(Z,t)$ such that

$$Z(\tau) \overset{\mathcal{K}_\tau}{\leq} X(\tau) \ \Rightarrow \ Z(t) \overset{\mathcal{K}_t}{\leq} X(t), \quad t > \tau \geq 0.$$

In addition, if $X(\tau) \in \mathcal{K}_\tau^+(\Theta)$ ($Z(\tau) \in \mathcal{K}_\tau^+(\Theta)$), then $F \in \overline{\mathcal{F}}_1(\Theta)$ ($F \in \overline{\mathcal{F}}_2(\Theta)$). Similarly, we introduce the classes $\underline{\mathcal{F}}$, $\underline{\mathcal{F}}_1(\Theta)$ and $\underline{\mathcal{F}}_2(\Theta)$ by using $-\mathcal{K}_t$ instead of $\mathcal{K}_t$. It is obvious that $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}}_1(\Theta) \subseteq \overline{\mathcal{F}}_2(\Theta)$ and $\underline{\mathcal{F}} \subseteq \underline{\mathcal{F}}_1(\Theta) \subseteq \underline{\mathcal{F}}_2(\Theta)$.

If $F \in \overline{\mathcal{F}} \cup \underline{\mathcal{F}}$, then system (4) is monotone with respect to $\mathcal{K}_t$. If $F \in \overline{\mathcal{F}}$ and $F(\Theta,t) \in \mathcal{K}_t$ ($F \in \underline{\mathcal{F}}$ and $F(\Theta,t) \in -\mathcal{K}_t$), then system (4) belongs to $\mathcal{M}_0^+(\Theta)$ ($\mathcal{M}_0^-(\Theta)$).

Lemma 3.4 *Under the conditions of Lemma 3.2, we have:* (i) *$\mathcal{F} \subseteq \overline{\mathcal{F}} \cap \underline{\mathcal{F}}$;*
(ii) *$\mathcal{F}_k^+(\Theta) \cap \mathcal{F}_0^+(\Theta) \subseteq \overline{\mathcal{F}}_k(\Theta)$, $\mathcal{F}_k^-(\Theta) \cap \mathcal{F}_0^-(\Theta) \subseteq \underline{\mathcal{F}}_k(\Theta)$, $k = 1, 2$.*

By analogy, we can introduce and study classes of difference systems in a Banach space $\mathcal{E}$ with respect to a cone $\mathcal{K}_t \subset \mathcal{E}$ (see [12]).

4 Stability of Equilibrium States of Autonomous Systems

Definition 4.1 The equilibrium state $X_t \equiv \Theta$ of system $\mathcal{S}$ is *stable in* $\mathcal{K}_t^+(\Theta)$ if, for any $\varepsilon > 0$ and $\tau \in \Upsilon$, there exists $\delta > 0$ such that $X_\tau \in \mathcal{S}_\delta(\tau) \Rightarrow X_t \in \mathcal{S}_\varepsilon(t)$ for $t \in \Upsilon_\tau$, where $\mathcal{S}_\varepsilon(t) = \{X \in \mathcal{K}_t^+(\Theta) : \|X - \Theta\| \leq \varepsilon\}$. If, for a certain $\delta > 0$, $X_\tau \in \mathcal{S}_\delta(\tau) \Rightarrow \|X_t - \Theta\| \to 0$ as $t \to \infty$, then the state $X_t \equiv \Theta$ is *asymptotically stable in* $\mathcal{K}_t^+(\Theta)$.

Lemma 4.1 [8] *Let $\mathcal{K}_t$ be a normal reproducing cone. The state $X \equiv \Theta$ of system $\mathcal{S} \in \mathcal{M}_1(\Theta)$ is Lyapunov stable (asymptotically stable) if and only if it is stable (asymptotically stable) in $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$.*

At first, we formulate known results for linear systems. Let $\mathcal{K} \subset \mathcal{E}$ be a normal reproducing cone. Positive system $\dot{X} = AX$ with a linear bounded operator $A : \mathcal{E} \to \mathcal{E}$ is exponentially stable if and only if $-A$ is positive invertible. If $\mathcal{K} \subseteq (\gamma I - A)\mathcal{K}$ for $\gamma \geq 0$, then the system is exponentially stable and positive with respect to $\mathcal{K}$ [14]. Moreover, the system is exponentially stable if $\mathcal{K} \subset -A\mathcal{K} \cap (\gamma_0 I - A)\mathcal{K}$ for a certain $\gamma_0 > [\rho^2(A) - r^2(A)]/[2r(A)]$, where $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, $r(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$ [15].

Now we formulate the asymptotic stability conditions for an isolated equilibrium state of nonlinear autonomous system in terms of positive invertible operators.

Theorem 4.1 *Let $\mathcal{K}$ be a normal reproducing cone. The state $X \equiv \Theta$ of system*

$$\dot{X} = F(X), \quad F(\Theta) = 0, \quad t \geq 0, \tag{9}$$

is Lyapunov asymptotically stable if one of the following conditions holds:
(a) $F \in \mathcal{F}_0^+(\Theta) \cup \mathcal{F}_0^-(\Theta)$, *there exists the Fréchet derivative $F'(\Theta)$, and $-F'(\Theta)$ is positive invertible:*

$$\mathcal{K} \subseteq -F'(\Theta)\mathcal{K}. \tag{10}$$

(b) $F \in \mathcal{F}_1(\Theta)$, *there exist the Fréchet derivatives $F'_\pm(\Theta)$ with respect to $\pm\mathcal{K}$, and $-F'_\pm(\Theta)$ are positive invertible:*

$$\mathcal{K} \subseteq -F'_+(\Theta)\mathcal{K} \cap F'_-(\Theta)\mathcal{K}. \tag{11}$$

Proof (a) For $X = \Theta + H$, system (9) is represented as follows:

$$\dot{H} = F'(\Theta)H + R(\Theta, H), \quad R(\Theta, H) = o(\|H\|), \quad H \in \mathcal{E}.$$

In order to use the Lyapunov theorem on stability with respect to the first approximation, we establish the asymptotic stability of the linear system

$$\dot{H} = F'(\Theta)H. \tag{12}$$

System (12) is positive with respect to $\mathcal{K}$ and $-\mathcal{K}$. Indeed, using the relations

$$F(\Theta + \varepsilon H) = \varepsilon F'(\Theta)H + R(\Theta, \varepsilon H), \quad \frac{R(\Theta, \varepsilon H)}{\varepsilon\|H\|} \underset{\varepsilon \to 0}{\to} 0,$$

and the fact that $F \in \mathcal{F}_0^+(\Theta) \cup \mathcal{F}_0^-(\Theta)$, we have

$$H \in \pm\mathcal{K}, \ \varphi \in \pm\mathcal{K}^*, \ \varphi(H) = 0 \quad \Rightarrow \quad \frac{\varphi(F'(\Theta)H)}{\|H\|} + \frac{\varphi(R(\Theta, \varepsilon H))}{\varepsilon\|H\|} \geq 0.$$

This implies that $\varphi(F'(\Theta)H) \geq 0$, i.e. the positivity conditions of system (12) are satisfied (see Lemma 3.2). In view of (10), system (12) is exponentially stable. Moreover, the state $X \equiv \Theta$ of original system (9) is Lyapunov asymptotically stable.

(b) If $F \in \mathcal{F}_1(\Theta)$, then system (9) belongs to $\mathcal{M}_1(\Theta)$ and has the invariant sets $\mathcal{K}^{\pm}(\Theta)$. For $X = \Theta + H \in \mathcal{K}^{\pm}(\Theta)$, we have the systems

$$\dot{H} = F'_{\pm}(\Theta)H + R_{\pm}(\Theta, H), \quad R_{\pm}(\Theta, H) = o(\|H\|), \quad H \in \pm\mathcal{K}.$$

According to Lemma 4.1, the asymptotic stability in $\mathcal{K}$ and $-\mathcal{K}$ of the zero state $H \equiv 0$ of the systems yields the Lyapunov asymptotic stability of the state $X \equiv \Theta$ of original system (9). The linear systems $\dot{H} = F'_{\pm}(\Theta)H$ are positive with respect to $\mathcal{K}$ and $-\mathcal{K}$ and exponentially stable (see above). Therefore, the state $X \equiv \Theta$ of system (9) is Lyapunov asymptotically stable. □

Note that, in the case of a solid cone $\mathcal{K}$, conditions (10) and (11) are equivalent to consistency of the corresponding systems of cone inequalities:

$$H \overset{\mathcal{K}}{\geq} 0, \quad F'(\Theta)H \overset{\mathcal{K}}{<} 0, \tag{13}$$

$$H_- \overset{\mathcal{K}}{\leq} 0 \overset{\mathcal{K}}{\leq} H_+, \quad F'_+(\Theta)H_+ \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F'_-(\Theta)H_-. \tag{14}$$

Conjecture 4.1 *Let system* (9) *belong to* $\mathcal{M}_1(\Theta)$ *with respect to a normal solid cone* $\mathcal{K}$ *and let the following cone inequalities be feasible:*

$$X_- \overset{\mathcal{K}}{\leq} \Theta \overset{\mathcal{K}}{\leq} X_+, \quad F(X_+) \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F(X_-). \tag{15}$$

Then the state $X \equiv \Theta$ *of system* (9) *is Lyapunov asymptotically stable.*

Consider the pseudolinear differential system

$$\dot{X} = A(X)X, \quad t \geq 0, \tag{16}$$

where A is a continuous operator function with the values $A(X)$ that are assumed to be linear bounded operators in $\mathcal{E}$. The Gâteaux (Fréchet) derivatives and Gâteaux (Fréchet) derivatives with respect to $\pm\mathcal{K}$ of $F(X) = A(X)X$ have the form

$$F'(X) = A(X) + B(X), \quad B(X)H = [A'(X)H]X,$$

$$F'_{\pm}(X) = A(X) + B_{\pm}(X), \quad B_{\pm}(X)H = [A'_{\pm}(X)H]X,$$

where $A'(X)$ and $A'_{\pm}(X)$ are the Gâteaux (Fréchet) derivatives of $A(X)$, the values $B(X)$ and $B_{\pm}(X)$ are linear operators in $\mathcal{E}$. Since $F'(0) = F'_{\pm}(0) = A(0)$, we have the following corollary of Theorem 4.1.

Corollary 4.1 *Let one of the following off-diagonal positivity type constraints hold:*

$$A(X) \trianglerighteq \alpha_{\pm}(X)I, \quad X \in \pm\partial\mathcal{K},$$

$$A(X) + B(X) \trianglerighteq \beta(X)I, \quad X \in \pm\mathcal{K},$$

$$A(X) + B_{\pm}(X) \trianglerighteq \beta_{\pm}(X)I, \quad X \in \mathcal{D},$$

where $\mathcal{K}$ *is a solid cone,* $\alpha_{\pm}(X)$*,* $\beta(X)$ *and* $\beta_{\pm}(X)$ *are scalar functions. Then the zero state* $X \equiv 0$ *of system* (9) *is Lyapunov asymptotically stable if the following system of cone inequalities is feasible:*

$$H \overset{\mathcal{K}}{\geq} 0, \quad A(0)H \overset{\mathcal{K}}{<} 0. \tag{17}$$

Similarly, we can formulate the asymptotic stability conditions of the isolated equilibrium states for some classes of autonomous nonlinear and pseudolinear difference system (see [8]).

Example 4.1 Consider the pseudolinear system

$$\dot{x} = A(x)x, \quad A(x) = \mathrm{diag}\{d - Cx\}, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{18}$$

where $d \in \mathbb{R}^n$ is a vector, C is an invertible $n \times n$ matrix, $\mathrm{diag}\{\cdot\}$ denotes the diagonal $n \times n$ matrix generated by n vector components. This system is the Kolmogorov type model describing the dynamics of growth and interaction of n populations. There are two equilibrium states $\theta_0 = 0$ and $\theta_1 = C^{-1}d$.

The diagonal matrix $A(x)$ for any $x \in \mathbb{R}^n$ is positive-off-diagonal with respect to the cones $\pm\mathcal{K}$, where $\mathcal{K} = \mathbb{R}^n_+$. Therefore, (18) is positive with respect to $\pm\mathcal{K}$ and the asymptotic stability condition (17) of the state $x \equiv \theta_0$ is reduced to the inequality $d \overset{\mathcal{K}}{<} 0$.

Fréchet derivative of the vector function $F(x) = A(x)x$ has the form $F'(x) = A(x) + B(x)$, where $B(x) = -\mathrm{diag}\{x\}\,C$. The matrix $F'(x)$ is positive-off-diagonal for $x - \theta_1 \in \pm\partial\mathcal{K}$ if $B_1 = \mathrm{diag}\{\theta_1\}\,C$ is negative-off-diagonal. By virtue of Lemmas 3.2 and 3.3, system (18) belongs to $\mathcal{M}_0^{\pm}(\theta_1)$. Moreover, according to Theorem 4.1, the state $x \equiv \theta_1$ of the system is asymptotically stable if B_1 is a M-matrix, i.e. $B_1^{-1} \trianglerighteq 0$ and B_1 is negative-off-diagonal.

5 Comparison Principle for a Set of Differential Systems

Consider a set of independent systems of the type (4):

$$\mathcal{S}_i: \quad \dot{X}_i = F_i(X_i, t), \quad X_i \in \mathcal{E}_i, \quad t \geq 0, \ i = \overline{1, s}. \tag{19}$$

For simplicity, we denote $X = (X_1, \dots, X_s)$, $F(X, t) = (F_1(X_1, t), \dots, F_s(X_s, t))$, $\mathcal{E} = \mathcal{E}_1 \times \dots \times \mathcal{E}_s$ and rewrite (19) as

$$\dot{X} = F(X, t), \quad X \in \mathcal{E}, \quad t \geq 0. \tag{20}$$

Let $\mathcal{X}$ be a space with a wedge $\mathcal{W}_t$, and let $W : \mathcal{E} \times [0, \infty) \to \mathcal{X}$ be a continuous operator function together with its partial derivatives and not everywhere positive with respect to $\mathcal{W}_t$.

Definition 5.1 Systems (19) are called *comparable* if $W(X(t), t) \in \mathcal{W}_t$ whenever $W(X(\tau), \tau) \in \mathcal{W}_\tau$ for $t > \tau \geq 0$. Simultaneously, W is the *operator of comparison* of systems (19).

Theorem 5.1 *Let $\mathcal{W}_t$ be a solid cone satisfying* (7). *Then systems* (19) *are comparable if and only if*

$$W(X, t) \in \mathcal{W}_t, \ \ \varphi \in \mathcal{W}_t^*, \ \ \varphi(W(X, t)) = 0 \ \Rightarrow \ \varphi(D_t W(X, t)) \geq 0, \quad t \geq 0, \tag{21}$$

where D_t is the operator of differentiation along solutions of (20).

Proof We construct an invariant set of (20) in the form $\mathcal{I}_t = \{X \in \mathcal{E} : W(X, t) \in \mathcal{W}_t\}$. The operator of differentiation along solutions of the system is defined as

$$D_t W(X, t) = W'_X(X, t)\, F(X, t) + W'_t(X, t),$$

where $W_t'(X,t)$ is the strong time derivative and $W_X'(X,t)$ is the Gâteaux derivative.

Let $X(t)$ satisfy (20), $X(\tau) \in \mathcal{I}_\tau$ and $X(\xi) \in \partial\mathcal{I}_\xi$ for some $\xi \geq \tau$. Then

$$\int_\xi^t D_s W(X(s),s)ds = W(X(t),t) - W(X(\xi),\xi)$$

and $\varphi(W(X(\xi),\xi)) = 0$ for some $\varphi \neq 0 \in \mathcal{W}_\xi^*$. For $\varepsilon > 0$ and $Y \in \operatorname{int}\mathcal{W}_\xi$, we define a neighbourhood of $\mathcal{I}_t$ in the form $\mathcal{I}_t^\varepsilon = \{X \in \mathcal{E} : W_\varepsilon(X,t) \in \mathcal{W}_t\}$, where

$$W_\varepsilon(X,t) = W(X,t) + \varepsilon \arctan(t-\xi)\, Y.$$

It is obvious that $\mathcal{I}_t \subset \mathcal{I}_t^\varepsilon$, and $\mathcal{I}_t^\varepsilon \to \mathcal{I}_t$ as $\varepsilon \to 0$, $t \geq \xi$. Since $\varphi(Y) > 0$, according to (21), for some $\delta > 0$, we have

$$\varphi\left(D_t W_\varepsilon(X(t),t)\right) = \varphi\left(D_t W(X(t),t)\right) + \frac{\varepsilon}{1+(t-\xi)^2}\,\varphi(Y) > 0, \quad \xi \leq t \leq \xi+\delta,$$

$$\int_\xi^{\xi+\delta} \varphi\left(D_t W_\varepsilon(X(t),t)\right) dt = \varphi(W_\varepsilon(X(\xi+\delta),\xi+\delta)) > 0.$$

It means that the trajectory $X(t)$ at $t = \xi$ cannot leave $\mathcal{I}_\xi^\varepsilon$, i.e. $W_\varepsilon(X(t),t) \in \mathcal{W}_\xi$ for $\xi \leq t \leq \xi+\delta$. Otherwise $\varphi(W_\varepsilon(X(\xi),\xi)) = 0$ and $\varphi(W_\varepsilon(X(\xi+\delta),\xi+\delta)) < 0$ for some $\varphi \in \mathcal{W}_\xi^*$ and $\delta > 0$. According to (7), we have $X(t) \in \mathcal{I}_t^\varepsilon$ for $\xi \leq t \leq \xi+\delta$. By virtue of the closedness of $\mathcal{W}_t$, we get $W_\varepsilon(X(t),t) \to W(X(t),t) \in \mathcal{W}_t$ as $\varepsilon \to 0$, $\xi \leq t \leq \xi+\delta$. Thus, $\mathcal{I}_t$ is an invariant set of system (20).

The converse statement follows from the Lagrange type relation:

$$\varphi(W(X(\xi+\delta),\xi+\delta)) - \varphi(W(X(\xi),\xi)) = \delta\,\varphi(D_\zeta W(X(\zeta),\zeta)),$$

where $\zeta \in (\xi,\xi+\delta)$. If $\varphi(W(X(\xi),\xi)) = 0$ and $X(\xi+\delta) \in \mathcal{I}_{\xi+\delta}$, then it is necessary that the inequality $\varphi(D_\xi W(X(\xi),\xi)) \geq 0$ holds for sufficiently small $\delta > 0$. □

Note that (21) holds if

$$D_t W(X,t) \overset{\mathcal{W}_t}{\geq} \alpha(X,t)W(X,t), \quad X \in \partial\mathcal{I}_t, \quad t \geq 0,$$

where $\alpha(X,t)$ is a certain scalar function.

Now, we formulate known results of comparison for two and three systems with the zero equilibrium states. In some cases, these results can be established as corollaries of Theorem 5.1. In phase spaces of the comparison systems, we shall use normal reproducing cones with bounded normality constants. Consider the following cases.

Case 1. Let $s = 2$, $F_1(\Theta,t) \equiv 0$, $F_2(\Omega,t) \equiv 0$ and $W(X,t) = X_2 - V(X_1,t)$, where $V : \mathcal{E}_1 \times [0,\infty) \to \mathcal{E}_2$ is a continuous and everywhere positive operator function with respect to a normal reproducing cone $\mathcal{K}_t \subset \mathcal{E}_2$. If

$$D_t V(X_1,t) \overset{\mathcal{K}_t}{\leq} F_2(V(X_1,t),t), \quad F_2 \in \overline{\mathcal{F}}_2(\Omega), \quad t \geq 0, \tag{22}$$

then $\mathcal{S}_2$ is an *upper comparison system* for system $\mathcal{S}_1$ in the sense that:

$$\Omega \overset{\mathcal{K}_\tau}{\leq} V(X_1(\tau),\tau) \overset{\mathcal{K}_\tau}{\leq} X_2(\tau) \;\Rightarrow\; \Omega \overset{\mathcal{K}_t}{\leq} V(X_1(t),t) \overset{\mathcal{K}_t}{\leq} X_2(t), \;\; t > \tau \geq 0.$$

Thus, systems (19) are comparable with the operator of comparison W.

We assume that an operator V has the following additional properties:

$$V(\Theta, t) \equiv \Omega, \quad \|V(X,t) - \Omega\| \geq v(X) > 0, \;\; X \neq \Theta, \quad v(\Theta) = 0, \quad t \geq 0, \tag{23}$$

where v is a continuous function such that $\|X - \Theta\| \leq \|Y - \Theta\|$ whenever $v(X) \leq v(Y)$.

Theorem 5.2 *Let an everywhere positive operator V satisfy* (22) *and* (23). *Then the solution $X_1 \equiv \Theta$ of $\mathcal{S}_1$ is Lyapunov stable (asymptotically stable) if the solution $X_2 \equiv \Omega$ of $\mathcal{S}_2$ is stable (asymptotically stable) in $\mathcal{K}_t^+(\Omega)$.*

Case 2. Let $s = 3$, $F_1(\Omega, t) \equiv F_3(\Omega, t) \equiv 0$, $F_2(\Theta, t) \equiv 0$, $\mathcal{E}_1 = \mathcal{E}_3$ and $W(X,t) = [V(X_2,t) - X_1, X_3 - V(X_2,t)]$, where $V : \mathcal{E}_2 \times [0,\infty) \to \mathcal{E}_1$ is a continuous operator function and $\mathcal{K}_t \subset \mathcal{E}_1$ is a normal reproducing cone. If

$$F_1(V(X_2,t),t) \overset{\mathcal{K}_t}{\leq} D_t V(X_2,t) \overset{\mathcal{K}_t}{\leq} F_3(V(X_2,t),t), \quad F_1 \in \underline{\mathcal{F}}_1(\Omega), \; F_3 \in \overline{\mathcal{F}}_1(\Omega), \tag{24}$$

then, for $X_1(\tau) \in \mathcal{K}_\tau^-(\Omega)$ and $X_3(\tau) \in \mathcal{K}_\tau^+(\Omega)$, we have

$$X_1(\tau) \overset{\mathcal{K}_\tau}{\leq} V(X_2(\tau),\tau) \overset{\mathcal{K}_\tau}{\leq} X_3(\tau) \;\Rightarrow\; X_1(t) \overset{\mathcal{K}_t}{\leq} V(X_2(t),t) \overset{\mathcal{K}_t}{\leq} X_3(t), \quad t > \tau \geq 0. \tag{25}$$

It means that three systems (19) are comparable with the operator of comparison W and cone $\mathcal{W}_t = \mathcal{K}_t \times \mathcal{K}_t$. Then $\mathcal{S}_1$ ($\mathcal{S}_3$) is a *lower (upper) comparison system* for $\mathcal{S}_2$.

Theorem 5.3 *Let V satisfy* (23) *and* (24). *Then the solution $X_2 \equiv \Theta$ of system $\mathcal{S}_2$ is Lyapunov stable (asymptotically stable) if the solution $X_1 \equiv \Omega$ of $\mathcal{S}_1$ and the solution $X_3 \equiv \Omega$ of $\mathcal{S}_3$ are stable (asymptotically stable) in $\mathcal{K}_t^-(\Omega)$ and $\mathcal{K}_t^+(\Omega)$, respectively.*

Proof Since $\mathcal{K}_t$ is reproducing and non-flat, we have

$$V(X_2(\tau),\tau) - \Omega = U_+ - U_-, \quad \|U_\pm\| \leq \gamma \, \|V(X_2(\tau),\tau) - \Omega\|, \quad U_\pm \in \mathcal{K}_\tau,$$

where $\gamma > 0$ is a universal constant. Let $X_1(t)$ and $X_3(t)$ be the solutions of systems $\mathcal{S}_1$ and $\mathcal{S}_3$ with the initial conditions $X_1(\tau) = \Omega - U_- \in \mathcal{K}_\tau^-(\Omega)$ and $X_3(\tau) = \Omega + U_+ \in \mathcal{K}_\tau^+(\Omega)$, respectively. Then $X_1(t) \in \mathcal{K}_t^-(\Omega)$, $X_3(t) \in \mathcal{K}_t^+(\Omega)$ and

$$\|X_1(\tau) - \Omega\| \leq \gamma \, \|V(X_2(\tau),\tau) - \Omega\|, \quad \|X_3(\tau) - \Omega\| \leq \gamma \, \|V(X_2(\tau),\tau) - \Omega\|.$$

By virtue of (25) and the normality of $\mathcal{K}_t$, we get

$$\|V(X_2(t),t) - \Omega\| \leq \alpha \|X_1(t) - \Omega\| + \beta \|X_3(t) - \Omega\|, \quad t \geq \tau.$$

where $\alpha > 0$ and $\beta > 0$ depend on the normality constant of $\mathcal{K}_t$.

It follows from (23) and the continuity of $V(X,t)$ that, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\|X_2(t) - \Theta\| \leq \varepsilon$ whenever $\|V(X_2(t),t) - \Omega\| \leq \delta_0$ for $t \geq \tau$.

Now we use the stability properties of the solution $X_1 \equiv \Omega$ of $\mathcal{S}_1$ and the solution $X_3 \equiv \Omega$ of $\mathcal{S}_3$ in $\mathcal{K}_t^-(\Omega)$ and $\mathcal{K}_t^+(\Omega)$, respectively. We choose $\delta_\pm > 0$ so that the inequalities $\|X_1(\tau) - \Omega\| \leq \delta_-$ and $\|X_3(\tau) - \Omega\| \leq \delta_+$ yield the corresponding inequalities $\|X_1(t) - \Omega\| \leq \delta_0/(2\alpha)$ and $\|X_3(t) - \Omega\| \leq \delta_0/(2\beta)$ for $t \geq \tau$.

Finally, we choose $\delta > 0$ so that

$$\|X_2(\tau) - \Theta\| \leq \delta \;\Rightarrow\; \|V(X_2(\tau),\tau) - \Omega\| \leq \min\{\delta_-, \delta_+\}/\gamma.$$

Then, according to the arguments presented above, we get $\|X_2(t) - \Theta\| \leq \varepsilon$ for $t > \tau$, i.e., the solution $X_2 \equiv \Theta$ of system $\mathcal{S}_2$ is Lyapunov stable. In this case, $X_2(t) \to \Theta$ if $X_1(t) \to \Omega$ and $X_3(t) \to \Omega$ as $t \to \infty$. □

The proofs of Theorems 5.2 and 5.3 are analogous.

Case 3. Let $s \geq 2$. The *arrangement* problems for systems (19) can be formulated in the form of a general comparison problem using the block operator

$$W(X,t) = \left[\ V_2(X_2,t) - V_1(X_1,t), \quad \dots \quad, V_s(X_s,t) - V_{s-1}(X_{s-1},t)\ \right].$$

If $\mathcal{S}_i$ are comparable with $\mathcal{W}_t = \mathcal{K}_t \times \cdots \times \mathcal{K}_t$, where $\mathcal{K}_t$ is a wedge in $\mathcal{X}_1$, then

$$V_1(X_1(t),t) \overset{\mathcal{K}_t}{\leq} \cdots \overset{\mathcal{K}_t}{\leq} V_s(X_s(t),t), \quad t > \tau \geq 0,$$

provided that this ordering takes place at an arbitrary initial time $t = \tau$. In particular, if $V_i(X_i,t) = \|X_i\|_{\mathcal{E}_i}$, then the solutions of comparable systems (19) are ordered by norms:

$$\|X_1(\tau)\|_{\mathcal{E}_1} \leq \cdots \leq \|X_s(\tau)\|_{\mathcal{E}_s} \ \Rightarrow\ \|X_1(t)\|_{\mathcal{E}_1} \leq \cdots \leq \|X_s(t)\|_{\mathcal{E}_s}, \quad t > \tau \geq 0.$$

Example 5.1 Consider a set of pseudolinear systems

$$\dot{x}_i = A_i(x_i,t)x_i, \quad x_i \in \mathbb{C}^{n_i}, \quad t \geq 0, \quad i = \overline{1,s}, \tag{26}$$

where $A_i(x_i,t)$ are continuous $n_i \times n_i$ matrices. We specify an operator of comparison of the systems with respect to the cone $\mathcal{W} = \mathbb{R}_+^{s-1}$:

$$W(X,t) = \left[\ x_2^*Q_2x_2 - x_1^*Q_1x_1, \quad \dots \quad, x_s^*Q_sx_s - x_{s-1}^*Q_{s-1}x_{s-1}\ \right],$$

where $Q_i(t) = Q_i^*(t) > 0$ are Hermitian positive definite matrices. Then

$$D_tW(X,t) = \left[\ x_2^*H_2x_2 - x_1^*H_1x_1, \quad \dots \quad, x_s^*H_sx_s - x_{s-1}^*H_{s-1}x_{s-1}\ \right],$$

where $H_i(x_i,t) = A_i^*(x_i,t)Q_i(t) + Q_i(t)A_i(x_i,t) + \dot{Q}_i(t)$, $i = \overline{1,s}$. Using Theorem 5.1 and the two-sided estimations

$$[\lambda_{\min}(H_i - \lambda Q_i) - \alpha]\, x_i^*Q_ix_i \leq x_i^*(H_i - \alpha Q_i)x_i \leq [\lambda_{\max}(H_i - \lambda Q_i) - \alpha]\, x_i^*Q_ix_i,$$

one can establish that the solutions of (26) are ordered in the form $x_1(t)^*Q_1(t)x_1(t) \leq \cdots \leq x_s(t)^*Q_s(t)x_s(t)$, $t \geq \tau \geq 0$, if the following relations hold:

$$\lambda_{\max}(H_i - \lambda Q_i) \leq \lambda_{\min}(H_{i+1} - \lambda Q_{i+1}), \quad i = \overline{1,s-1}.$$

In particular, in the case $Q_i \equiv I$, the inequalities $\lambda_{\max}(A_i^* + A_i) \leq \lambda_{\min}(A_{i+1}^* + A_{i+1})$, $i = \overline{1,s-1}$, ensure the ordering of systems (26) with respect to the Hermitian norm. Here, for matrix pencils and Hermitian matrices, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues, respectively.

6 Robust Stability Analysis of Differential Systems

Consider the family of differential systems

$$\dot{X} = F(X,t), \quad F(\Theta,t) \equiv 0, \quad t \geq 0, \tag{27}$$

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{E}, \quad t \geq 0, \tag{28}$$

under the conditions of existence and uniqueness of solutions $X(t)$, $t > \tau \geq 0$, in a Banach space $\mathcal{E}$ contained a normal reproducing cone $\mathcal{K}_t$ with a bounded normality constant. We isolate the extreme systems

$$\dot{\underline{X}} = \underline{F}(\underline{X},t), \quad \underline{F}(\Theta,t) \equiv 0, \quad t \geq 0, \tag{29}$$

$$\dot{\overline{X}} = \overline{F}(\overline{X},t), \quad \overline{F}(\Theta,t) \equiv 0, \quad t \geq 0. \tag{30}$$

If $\underline{F} \in \underline{\mathcal{F}}_1(\Theta)$ and $\overline{F} \in \overline{\mathcal{F}}_1(\Theta)$, then, for $\underline{X}(\tau) \in \mathcal{K}_\tau^-(\Theta)$, $\overline{X}(\tau) \in \mathcal{K}_\tau^+(\Theta)$, we have

$$\underline{X}(\tau) \overset{\mathcal{K}_\tau}{\leq} X(\tau) \overset{\mathcal{K}_\tau}{\leq} \overline{X}(\tau) \;\Rightarrow\; \underline{X}(t) \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t), \quad t > \tau \geq 0.$$

In this case, (29) ((30)) is a lower (upper) comparison system for any system (27), (28).

Assumed that $V(X,t) \equiv X$ and $\Theta = \Omega$ in Theorem 5.3, we have the following result.

Theorem 6.1 *Let $\underline{F} \in \underline{\mathcal{F}}_1(\Theta)$ and $\overline{F} \in \overline{\mathcal{F}}_1(\Theta)$. Then the solution $X \equiv \Theta$ of any system* (27), (28) *is Lyapunov stable (asymptotically stable), if the solution $\underline{X} \equiv \Theta$ of* (29) *and the solution $\overline{X} \equiv \Theta$ of* (30) *are stable (asymptotically stable) in $\mathcal{K}_t^-(\Theta)$ and $\mathcal{K}_t^+(\Theta)$, respectively.*

Now, we consider instead of (28) the conditions

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{31}$$

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0. \tag{32}$$

Theorem 6.2 *Let $\mathcal{K}_t$ be a normal solid cone satisfying* (7)*. If* (31) *holds with $\underline{F} \in \mathcal{F}_0^+(\Theta)$ and $\overline{F} \in \mathcal{F}_2^+(\Theta)$, then stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $\overline{X} \equiv \Theta$ of* (30) *involves stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $X \equiv \Theta$ of any system* (27), (31)*. By analogy, if* (32) *holds with $\underline{F} \in \mathcal{F}_2^-(\Theta)$ and $\overline{F} \in \mathcal{F}_0^-(\Theta)$, then stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $\underline{X} \equiv \Theta$ of* (29) *involves stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $X \equiv \Theta$ of any system* (27), (32).

Note that under the conditions of Theorem 6.2, we have $\Theta \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t)$ and $\underline{X}(t) \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \Theta$ for $t > \tau \geq 0$ as soon as these inequalities hold at $t = \tau$ (see Section 3). If (31) holds, then $\underline{F} \in \mathcal{F}_0^+(\Theta)$ implies $F \in \mathcal{F}_0^+(\Theta)$. Similarly, if (32) holds, then $\overline{F} \in \mathcal{F}_0^-(\Theta)$ implies $F \in \mathcal{F}_0^-(\Theta)$.

Consider the pseudolinear system

$$\dot{X} = A(X,t)X, \quad t \geq 0, \tag{33}$$

with the isolated equilibrium state $X \equiv \Theta$ under one of the following conditions:

$$\underline{A}(X,t) \trianglelefteq A(X,t) \trianglelefteq \overline{A}(X,t), \quad X \overset{\mathcal{K}_t}{\geq} \Theta \overset{\mathcal{K}_t}{\geq} 0, \quad t \geq 0, \tag{34}$$

$$\underline{A}(X,t) \trianglelefteq A(X,t) \trianglelefteq \overline{A}(X,t), \quad X \overset{\mathcal{K}_t}{\leq} \Theta \overset{\mathcal{K}_t}{\leq} 0, \quad t \geq 0. \tag{35}$$

The values of continuous operator functions $A(X,t)$, $\underline{A}(X,t)$ and $\overline{A}(X,t)$ are linear bounded operators in $\mathcal{E}$. If $X \equiv \Theta$ is an equilibrium state of the system, then either $\Theta = 0$ or $\Theta \neq 0$ and $\Theta \in \ker A(\Theta,t)$ at $t \geq 0$. The extreme systems

$$\dot{\underline{X}} = \underline{A}(\underline{X},t)\underline{X}, \quad t \geq 0, \tag{36}$$

$$\dot{\overline{X}} = \overline{A}(\overline{X},t)\overline{X}, \quad t \geq 0, \tag{37}$$

have the equilibrium states $\underline{X} \equiv \Theta$ and $\overline{X} \equiv \Theta$, respectively. So, we formulate the corollaries of Theorem 6.2 and Lemma 3.3 using the following constraints:

$$\underline{A}(X,t) + \underline{B}_+(X,t) \trianglerighteq \underline{\beta}_+(X,t)I, \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{38}$$

$$\overline{A}(X,t) + \overline{B}_+(X,t) \trianglerighteq \overline{\beta}_+(X,t)I, \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{39}$$

$$\underline{A}(X,t) + \underline{B}_-(X,t) \trianglerighteq \underline{\beta}_-(X,t)I, \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0, \tag{40}$$

$$\overline{A}(X,t) + \overline{B}_-(X,t) \trianglerighteq \overline{\beta}_-(X,t)I, \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0, \tag{41}$$

where $\underline{B}_\pm(X,t)H = [\underline{A}'_\pm(X,t)H]X$, $\overline{B}_\pm(X,t)H = [\overline{A}'_\pm(X,t)H]X$, $\underline{A}'_\pm(X,t)$ and $\overline{A}'_\pm(X,t)$ are the Gâteaux (Fréchet) derivatives with respect to $\pm\mathcal{K}_t$, $\underline{\beta}_\pm(X,t)$ and $\overline{\beta}_\pm(X,t)$ are scalar functions.

Corollary 6.1 *Let $\mathcal{K}_t$ be a normal solid cone satisfying* (7). *If* (38) *and* (39) *hold, then stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $\overline{X} \equiv \Theta$ of* (37) *involves stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $X \equiv \Theta$ of any system* (33), (34). *By analogy, if* (40) *and* (41) *hold, then stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $\underline{X} \equiv \Theta$ of* (36) *involves stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $X \equiv \Theta$ of any system* (33), (35).

Note that in Corollary 6.1, we can use the constraints

$$\underline{A}(X,t) \trianglerighteq \underline{\alpha}_+(X,t)I, \quad \underline{A}(X,t)\Theta \overset{\mathcal{K}_t}{\geq} 0, \quad X - \Theta \in \partial\mathcal{K}_t, \quad t \geq 0, \tag{42}$$

$$\overline{A}(X,t) \trianglerighteq \overline{\alpha}_-(X,t)I, \quad \overline{A}(X,t)\Theta \overset{\mathcal{K}_t}{\leq} 0, \quad \Theta - X \in \partial\mathcal{K}_t, \quad t \geq 0, \tag{43}$$

instead of (38) and (41), respectively.

Example 6.1 Consider the family of pseudolinear systems

$$\dot{x} = A(x,t)x, \quad \underline{A}(x,t) \trianglelefteq A(x,t) \trianglelefteq \overline{A}(x), \quad x \in \mathbb{R}_+^n, \quad t \geq 0, \tag{44}$$

where $\underline{A}(x,t) = \underline{A}_0(t) + \sum_{j=1}^n x_j \underline{A}_j(t)$, $\overline{A}(x) = \overline{A}_0 + \sum_{j=1}^n x_j \overline{A}_j$, $\underline{A}_i(t) \trianglelefteq \overline{A}_i$, $\underline{A}_i(t) = \|\underline{a}_{ks}^{(i)}(t)\|_{k,s=1}^n$ and $\overline{A}_i = \|\overline{a}_{ks}^{(i)}\|_{k,s=1}^n$ are $n \times n$ matrices, $i = \overline{0,n}$. Here $\mathcal{K} = \mathbb{R}_+^n$ is a cone of nonnegative vectors and $\trianglelefteq$ denotes the elementwise matrix inequality.

The Gâteaux (Fréchet) derivative of $F(x,t) = A(x,t)x$ has the form

$$F'(x,t) = A(x,t) + B(x,t), \quad B(x,t) = \left[(\partial A/\partial x_1)\, x, \ldots, (\partial A/\partial x_n)\, x\right],$$

So, for $\underline{F}(x,t) = \underline{A}(x,t)x$ and $\overline{F}(x) = \overline{A}(x)x$, we have

$$\underline{F}'(x,t) = \underline{F}'_\pm(x,t) = \underline{A}_0(t) + \textstyle\sum_{j=1}^n x_j \underline{B}_j(t), \quad \underline{B}_j(t) = \|\underline{a}_{ks}^{(j)}(t) + \underline{a}_{kj}^{(s)}(t)\|_{k,s=1}^n,$$

$$\overline{F}'(x) = \overline{F}'_\pm(x) = \overline{A}_0 + \textstyle\sum_{j=1}^n x_j \overline{B}_j, \quad \overline{B}_j = \|\overline{a}_{ks}^{(j)} + \overline{a}_{kj}^{(s)}\|_{k,s=1}^n.$$

Then conditions (38) and (42) with $\Theta = 0$ are reduced to the form

$$\underline{a}_{ks}^{(0)}(t) \geq 0, \quad \underline{a}_{ks}^{(j)}(t) + \underline{a}_{kj}^{(s)}(t) \geq 0, \quad k \neq s, \quad t \geq 0, \quad j = \overline{1,n};$$
$$\underline{a}_{ks}^{(i)}(t) \geq 0, \quad k \neq s, \quad t \geq 0, \quad i = \overline{0,n},$$

respectively. If one of these conditions holds and in addition

$$\overline{A}_0^{-1} \trianglelefteq 0, \quad \overline{a}_{ks}^{(0)} \geq 0, \quad \overline{a}_{ks}^{(j)} + \overline{a}_{kj}^{(s)} \geq 0, \quad k \neq s, \quad j = \overline{1,n},$$

then according to Corollary 6.1 the zero equilibrium state of any system (44) is asymptotically stable in $\mathcal{K}$ (see also assertion (a) of Theorem 4.1 and Corollary 4.1).

Consider the parameter family of autonomous pseudolinear systems

$$\dot{X} = A(X,p)X, \quad A(X,p) = \sum_{i=1}^{s} p_i A_i(X), \;\; X \in \mathcal{E}, \;\; t \geq 0, \tag{45}$$

where $p = [p_1, \ldots, p_s]^\top \in \mathbb{R}_+^s$ is a vector of nonnegative scalar parameters. The values of operator functions $A_i(X)$ and $A(X,p)$ are linear bounded operators in $\mathcal{E}$.

Corollary 6.2 *Let all the operators $A_i(X)$ satisfy one of the off-diagonal positivity type constraints of Corollary 4.1 with a normal solid cone $\mathcal{K}$, and the system of cone inequalities $H \overset{\mathcal{K}}{\geq} 0$ and $A_i(0)H \overset{\mathcal{K}}{<} 0$ for $i = \overline{1,s}$ is feasible. Then the zero solution $X \equiv 0$ of any system* (45) *for $p \in \mathbb{R}_+^s$ is Lyapunov asymptotically stable.*

Consider the family of linear differential systems

$$\dot{X} = A(t)X, \quad \underline{A}(t) \trianglelefteq A(t) \trianglelefteq \overline{A}(t), \quad t \geq 0, \tag{46}$$

where the inequality $\trianglelefteq$ between linear operators is generated by a normal reproducing cone $\mathcal{K}_t$. In (46), we isolate the extreme systems:

$$\dot{X} = \underline{A}(t)X, \tag{47}$$

$$\dot{X} = \overline{A}(t)X. \tag{48}$$

Theorem 6.3 *Any system* (46) *is positive with respect to $\mathcal{K}_t$ if*

$$e^{\underline{A}(\theta)\delta}\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad t \geq \theta \geq \tau \geq 0, \quad t - \tau \geq \delta \geq 0. \tag{49}$$

Moreover, if system (48) *is asymptotically stable, then any positive system* (46) *is asymptotically stable.*

Proof Note that $\mathcal{K}_t$ in (49) with $\delta = 0$ satisfies (7). The evolutional and exponential operators of system (46) are connected by [16]

$$E(t,\tau) = \lim_{n\to\infty} \left[e^{A(\vartheta_n)h_n} \ldots e^{A(\vartheta_1)h_n}\right], \quad e^{A(\vartheta)h} = \lim_{n\to\infty} \left[E(\vartheta, \vartheta - h/n)\right]^n,$$

where $\vartheta_k \in [t_k, t_{k+1}]$, $t_k = \tau + kh_n$, $h_n = (t-\tau)/n$, $k = \overline{0,n}$, $t \geq \tau$, $\vartheta \geq 0$, $h \geq 0$. Therefore, (49) ensures positivity of (46). In the case of a constant cone, the inverse statement holds also. If $A(t)\mathcal{K}_t \subseteq \mathcal{K}_t$ and (7) hold, then

$$e^{A(\vartheta)h}\mathcal{K}_\tau = \sum_{k=0}^{\infty}(h^k/k!)A^k(\vartheta)\,\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad \tau \leq \vartheta \leq t, \quad 0 \leq h \leq t - \tau.$$

Let $A(t) = A_1(t) + A_2(t)$ and $e^{A_s(\vartheta)h}\mathcal{K}_\tau \subseteq \mathcal{K}_t$, $s = 1, 2$. Then

$$e^{A(\vartheta)h} = \lim_{n\to\infty} 2^{-n}\left[e^{A_1(\vartheta)\frac{h}{n}}e^{A_2(\vartheta)\frac{h}{n}} + e^{A_2(\vartheta)\frac{h}{n}}e^{A_1(\vartheta)\frac{h}{n}}\right]^n, \quad e^{A(\vartheta)h}\mathcal{K}_\tau \subseteq \mathcal{K}_t,$$

and consequently $E(t,\tau)\mathcal{K}_\tau \subseteq \mathcal{K}_t$. Assuming that $A_1(t) = \underline{A}(t)$ and $A_2(t) = A(t) - \underline{A}(t)$, we have the positivity of any system (46) with respect to $\mathcal{K}_t$.

Let $X(t)$ and $\overline{X}(t)$ be the solutions of (46) and (48) with initial conditions $X(\tau) = X_\tau$ and $\overline{X}(\tau) = \overline{X}_\tau$, respectively. Since $0 \overset{\mathcal{K}_\tau}{\leq} X_\tau \overset{\mathcal{K}_\tau}{\leq} \overline{X}_\tau$ implies $0 \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t)$, $t \geq \tau \geq 0$, and $\mathcal{K}_t$ is normal, the asymptotic stability of system (48) ensures the asymptotic stability in $\pm\mathcal{K}_t$ of any positive system (46). Moreover, if $\mathcal{K}_t$ is reproducing, then any system (46) is Lyapunov asymptotically stable. □

Remark 6.1 Note that any system (46) is positive with respect to $\mathcal{K}_t$ if the operator inequality $\underline{A}(t) \trianglerighteq \alpha(t)I$ holds for some scalar function $\alpha(t)$ (see Corollary 6.1 and the notation below). This inequality ensures (49) subject to (7). Indeed,

$$e^{\underline{A}(\vartheta)\delta} = e^{\alpha(\vartheta)\delta}\, e^{[\underline{A}(\vartheta)-\alpha(\vartheta)I]\delta}, \quad e^{\underline{A}(\vartheta)\delta}\mathcal{K}_\tau = \sum_{k=0}^{\infty}(\delta^k/k!)[\underline{A}(\vartheta) - \alpha(\vartheta)I]^k\,\mathcal{K}_\tau \subseteq \mathcal{K}_\vartheta \subseteq \mathcal{K}_t.$$

Example 6.2 Consider the family of linear systems

$$\dot{x} = A(t)x, \quad \underline{A}(t) \trianglelefteq A(t) \trianglelefteq \overline{A}, \;\; \overline{A}^{-1} \trianglelefteq 0, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \tag{50}$$

where $\underline{A}(t)$ is a matrix function with nonnegative off-diagonal entries, $-\overline{A}$ is an M-matrix and $\trianglelefteq$ denotes the elementwise matrix inequality. The system $\dot{x} = \underline{A}(t)x$ is positive with respect to the cone $\mathbb{R}^n_+$, and the system $\dot{x} = \overline{A}x$ is asymptotically stable. Thus, any system (50) is asymptotically stable and positive with respect to $\mathbb{R}^n_+$.

Example 6.3 Consider the family of linear systems in a matrix space $\mathbb{C}^{n\times n}$

$$\dot{X} = M(t)X, \quad \underline{M}(t) \trianglelefteq M(t) \trianglelefteq \overline{M}(t), \quad X \in \mathbb{C}^{n\times n}, \quad t \geq 0, \tag{51}$$

where $\underline{M}(t)X = A^*(t)X + XA(t)$, $\overline{M}(t)X = A^*(t)X + XA(t) + \sum\limits_{i=1}^{s} B^*(t)XB(t)$, $\trianglelefteq$ is an operator inequality generated by the cone of Hermitian positive semidefinite matrices $\mathbb{K}_n$. Since $e^{\underline{M}(\vartheta)\delta}X = e^{A^*(\vartheta)\delta}Xe^{A(\vartheta)\delta}$, the Lyapunov equation

$$\dot{X} = A^*(t)X + XA(t)$$

and any system (51) are positive with respect to $\mathbb{K}_n$. If the system

$$\dot{X} = A^*(t)X + XA(t) + \sum_{i=1}^{s} B^*(t)XB(t) \tag{52}$$

is asymptotically stable, then any system (51) is positive and asymptotically stable. Autonomous system of the type (52) is asymptotically stable, if the linear matrix inequality

$$A^*X + XA + \sum_{i=1}^{s} B^*XB < 0$$

has a solution $X = X^* > 0$.

Note that the matrix differential equation (52) is known as the second-moment equation for the Itô stochastic system. This equation is positive and monotone with respect to $\mathbb{K}_n$.

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Periodic and Subharmonic Solutions for a Class of Noncoercive Superquadratic Hamiltonian Systems

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Abstract: Some existence theorems are obtained for periodic and subharmonic solutions to noncoercive first order Hamiltonian systems and to similar second order Hamiltonian systems, when the Hamiltonian satisfies a superquadratic condition and need not satisfy the global Ambrosetti–Rabinowitz condition. For the resolution, we use minimax methods in critical point theory, especially a Local Linking Theorem and a Generalized Mountain Pass Theorem.

Keywords: *Hamiltonian systems; periodic solutions; subharmonics; critical points.*

Mathematics Subject Classification (2000): 34C25, 34A34, 37J45, 35Q40.

1 Introduction

Consider the nonautonomous first order Hamiltonian systems

$$J\dot{x} - u^*A(t)u(x) + u^*G'(t, u(x)) = 0, \tag{1.1}$$

where $u : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^m$ $(1 \leq m \leq 2N)$ is a linear operator, A is a continuous $T-$periodic function $(T > 0)$ from $\mathbb{R}$ into the space of symmetric $(m \times m)$-matrices, $G : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function, $T-$ periodic in the first variable, differentiable with respect to the second variable and its derivative $G'(t,x) = \frac{\partial G}{\partial x}(t,x)$ is continuous, and J is the standard symplectic matrix:

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

When $A(t) = 0$ for all $t \in \mathbb{R}$, $m = 2N$ and $u = id_{\mathbb{R}^{2N}}$, Rabinowitz has proved in [7] the existence of periodic solutions for (1.1) under some suitable conditions, in particular the following superquadratic condition:

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There exist two constants $\mu > 2$ and $r > 0$ such that for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^{2N}$, $|x| \geq r$

$$0 < \mu G(t,x) \leq G'(t,x)\, x, \tag{1.2}$$

where $x.y$ denotes the standard inner product of x, y in $\mathbb{R}^{2N}$ and $|.|$ denotes the corresponding norm. Since then, condition (1.2) has been used extensively in the literature, see [2–7,9,10]. If $m = 2N$, $u = id_{\mathbb{R}^{2N}}$ and G satisfies the superquadratic condition (1.2), the existence of nontrivial periodic solutions for the Hamiltonian systems (1.1), was studied by Li-Szulkin in [3] when A is a constant symmetric $(2N \times 2N)$ matrix, and by Li-Willem when $A(t)$ is a continuous periodic map from $\mathbb{R}$ into the space of symmetric $(2N \times 2N)$ matrices, not necessary constant. In [10], the author has studied the same problem as in [4] in the general case when u is not necessary the identity.

By remarking that the condition (1.2) does not cover some superquadratic nonlinearity like

$$G(t,y) = |y|^2 \left[ln(1 + |x|^p)\right]^q, \; p, q > 1, \tag{1.3}$$

the author has studied, recently in [11], the existence of nontrivial periodic solution for (1.1) when the function G satisfies some superquadratic conditions which cover the cases as in (1.3). In particular, the author has assumed that the function G satisfies the two following assumptions:

there exist constants $1 < \alpha < 2$ and $a > 0$ such that

$$|G'(t,y)| \leq a(|y|^\alpha + 1), \; \forall (t,y) \in \mathbb{R} \times \mathbb{R}^m; \tag{1.4}$$

there exist constants $\beta > \frac{1}{2-\alpha}$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \geq b\,|y|^\beta, \; \forall t \in \mathbb{R}, \; \forall\, |y| \geq r. \tag{1.5}$$

Consider the function G defined in $\mathbb{R} \times \mathbb{R}^m$ by

$$G(t,y) = \left|cos(\frac{2\pi}{T}t)\right| |y|^{\alpha+1} + |y|^2\, ln(1 + |y|^2), \tag{1.6}$$

where $\frac{3}{2} < \alpha < 2$. A simple computation shows that G neither satisfies the condition (1.2), nor (1.5). In section 3, we will extend the ranges of α and β and obtain the existence of nontrivial $T-$ periodic solutions of (1.1) under some superquadratic conditions covering the cases as in (1.6). For the resolution, we shall use a Local Linking Theorem.

The existence of subharmonic solutions for (1.1), i.e. of distinct $kT-$periodic solutions of (1.1), has been investigated in [2,6,9] when $A(t) = 0$ for all $t \in \mathbb{R}$, $m = 2N$, $u = id_{\mathbb{R}^{2N}}$ and G satisfies the condition (1.2). In section 4, we are interested in the existence of infinitely many subharmonic solutions of the Hamiltonian systems (1.1) when $A(t) = 0$ for all $t \in \mathbb{R}$, u is not necessary the identity and the function G satisfies some superquadratic conditions covering the cases as in (1.6). The main obstacle in obtaining such solutions is the fact that any $T-$periodic solution is also $kT-$periodic. For the resolution, we shall use the minimax methods in critical point theory, specially, a Generalized Mountain Pass Theorem.

2 Preliminaries

We will recall here some basic results needed in the proof of our next results.

2.1 Linking theorem [4]

Let X be a real Banach space with a direct sum decomposition

$$X = X^1 \oplus X^2.$$

Consider two sequences of subspaces

$$X_0^1 \subset X_1^1 \subset \cdots \subset X^1, \; X_0^2 \subset X_1^2 \subset \cdots \subset X^2$$

such that

$$X^j = \overline{\cup_{n\in\mathbb{N}} X_n^j}, \; j = 1, 2.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we denote by X_α the space

$$X_{\alpha_1}^1 \oplus X_{\alpha_2}^2.$$

Let us recall that

$$\alpha \leq \beta \Leftrightarrow \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2.$$

A sequence $(\alpha_n) \subset \mathbb{N}^2$ is admissible if, for every $\alpha \in \mathbb{N}^2$, there exists $m \in \mathbb{N}$ such that

$$n \geq m \Rightarrow \alpha_n \geq \alpha.$$

For every function $f : X \longrightarrow \mathbb{R}$, we denote by f_α the function f restricted to the space X_α.

Definition 2.1 Let $f \in C^1(X, \mathbb{R})$. The function f satisfies the $(PS)^*$ condition if every sequence (x_{α_n}) such that (α_n) is admissible and

$$x_{\alpha_n} \in X_{\alpha_n}, \; \sup_{n\in\mathbb{N}} f(x_{\alpha_n}) < \infty, \; f'_{\alpha_n}(x_{\alpha_n}) \longrightarrow 0,$$

possesses a subsequence which converges to a critical point of f.

Definition 2.2 The function $f \in C^1(X, \mathbb{R})$ has a local linking at 0, with respect to (X^1, X^2) if, for some $r > 0$,

$$f(x) \geq 0, \; x \in X^1, \; ||x|| \leq r,$$

$$f(x) \leq 0, \; x \in X^2, \; ||x|| \leq r.$$

Remark 2.1 If f has a local linking at 0, then 0 is a critical point of f.

Theorem 2.1 *Suppose that $f \in C^1(X, \mathbb{R})$ satisfies the following assumptions*
a) f has a local linking at 0 and $X^1 \neq \{0\}$,
b) f satisfies the $(PS)^*$ condition,
c) f maps bounded sets into bounded sets,
d) for every $m \in \mathbb{N}$, $f(x) \longrightarrow -\infty$ as $\|x\| \longrightarrow +\infty$, $x \in X_m^1 \oplus X^2$.
Then f has at least two critical points.

2.2 Generalized Mountain Pass Theorem

Let X be a real Banach space. We shall say that $f \in C^1(X,\mathbb{R})$ satisfies the Cerami-condition (C) if every sequence (x_n) in X satisfying

$$(f(x_n)) \text{ is bounded and } \|f'(x_n)\|\,(1+\|x_n\|) \to 0 \text{ as } n \to \infty$$

possesses a convergent subsequence.

As shown in [1], a deformation lemma can be proved with the weaker condition (C) replacing the used (PS) condition, and it turns out that the Generalized Mountain Pass Theorem holds true under condition (C). We then have:

Theorem 2.2 *Let X be a real Hilbert space with inner product $< .,. >$. Suppose $X = X^1 \oplus X^2$ and $f \in C^1(X,\mathbb{R})$ satisfies the Cerami-condition (C) and the following conditions:*
a) $f(x) = \frac{1}{2} < P^+x - P^-x, x > +b(x)$, where $P^+ : E \longrightarrow E^+$ and $P^- : E \longrightarrow E^-$ are the orthogonal projections and b' is compact,
b) there exist constants $m,\ \rho > 0$, such that

$$f(x) \geq m,\ \forall x \in \partial B_\rho \cap X^1,$$

c) there exist $e \in \partial B_1 \cap X^1$ and constants $r_1,\ r_2 > 0$ such that

$$f(x) \leq 0,\ \forall x \in \partial Q,$$

where

$$Q = \{se/0 \leq s \leq r_1\} \oplus \left\{x \in X^2/\, \|x\| \leq r_2\right\}.$$

Then f possesses a critical value $c \geq m$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{x\in Q} f(h(x)),$$

where

$$\Gamma = \left\{h \in C(\overline{Q}, E)/h = id \text{ on } \partial Q\right\}.$$

3 Existence of Periodic Solutions

Let $u : \mathbb{R}^{2N} \to \mathbb{R}^m$ $(1 \leq m \leq 2N)$ be a nontrivial linear operator with adjoint u^*, A be a continuous $T-$periodic function $(T > 0)$ from $\mathbb{R}$ into the space of symmetric $(m \times m)$-matrices and $G : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $(t,y) \to G(t,y)$ be a continuous function, $T-$ periodic in the first variable, differentiable with respect to the second variable and its derivative $G'(t,y) = \frac{\partial G}{\partial y}(t,y)$ is continuous. Consider the noncoercive Hamiltonian systems

$$(HS) \qquad J\dot{x} - u^*A(t)u(x) + u^*G'(t,u(x)) = 0.$$

We are interested in the existence of nontrivial $T-$periodic solutions for (HS).
Consider the following assumptions

$$(G_0) \qquad JKer\,u \subset Ker\,u.$$

(G_1) $$G(t,y) = o(|y|^2) \ as \ |y| \to 0, \ uniformly \ in \ t \in \mathbb{R}.$$

(G_2) $$\lim_{|y|\to\infty} \frac{G(t,y)}{|y|^2} = +\infty, \ uniformly \ in \ t \in \mathbb{R}.$$

(G_3) There exist constants $\alpha > 1$ and $a > 0$ such that

$$|G'(t,y)| \leq a(|y|^\alpha + 1), \ \forall \ t \in \mathbb{R}, \ \forall y \in \mathbb{R}^m.$$

(G_4) There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \geq b\,|y|^\beta, \ \forall \ t \in \mathbb{R}, \ \forall\,|y| \geq r.$$

(G_5) There exists a constant $\delta > 0$ such that either

(i) $$G(t,y) \geq 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \leq \delta,$$

or

$$G(t,y) \leq 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \leq \delta.$$

Our first main result in this section is the following:

Theorem 3.1 *Assume conditions* $(G_0) - (G_4)$ *hold. If* 0 *is an eigenvalue of* $J\frac{d}{dt} - u^*Au$, *assume also* (G_5). *Then the system* (HS) *possesses at least one nontrivial* $T-$*periodic solution.*

Example 3.1 Let $p, q > 1$ be two real numbers. The function

$$G(t,y) = |y|^2 \, [ln(1 + |x|^p)]^q$$

satisfies $(G_1) - (G_5)$. The linear map $u : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by $u(p,q) = p$ satisfies (G_0). Let $A(t) = Id_N$. Therefore for all $T > 0$, the corresponding Hamiltonian system (HS) possesses at least a nontrivial $T-$periodic solution.

Remark 3.1 Observe that if x is a periodic solution of (HS) then $y(t) = x(-t)$ is a periodic solution of

$$J\dot{y}(t) + u^*A(-t)u(y) - u^*G'(-t, u(y)) = 0.$$

Hence, it is easy to see that we obtain the same result of Theorem 3.1 if we replace assumptions (G_2) and (G_4) respectively by the following ones

$$\lim_{|y|\to\infty} \frac{G(t,y)}{|y|^2} = -\infty, \ uniformly \ in \ t \in \mathbb{R}.$$

There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$G'(t,y).y - 2G(t,y) \leq -b\,|y|^\beta, \ \forall \ t \in \mathbb{R}, \ \forall\,|y| \geq r.$$

Now consider the noncoercive second order Hamiltonian systems

$$(NS) \qquad \ddot{x} - u^* A(t)u(x) + u^* W'(t, u(x)) = 0$$

where $u : \mathbb{R}^N \longrightarrow \mathbb{R}^m$, $(1 \le m \le N)$ is a linear operator with adjoint u^*, $A(t)$ is a symmetric $m \times m$ matrix, continuous and $T-$periodic, $W : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function $T-$periodic in the first variable and continuously differentiable with respect to the second variable. Consider the following assumptions:

$$(W_1) \qquad W(t, y) = o(|y|^2) \ as \ |y| \to 0, \ uniformly \ in \ t \in \mathbb{R}.$$

$$(W_2) \qquad \lim_{|y| \to \infty} \frac{W(t, y)}{|y|^2} = +\infty, \ uniformly \ \ in \ t \in \mathbb{R}.$$

(W_3) There exist constants $\alpha > 1$ and $a > 0$ such that

$$|W'(t, y)| \le a(|y|^\alpha + 1), \ \forall \ t \in \mathbb{R}, \ \forall y \in \mathbb{R}^m.$$

(W_4) There exist constants $\beta > \alpha$, $b > 0$ and $r > 0$ such that

$$W'(t, y).y - 2W(t, y) \ge b\,|y|^\beta \,, \ \forall t \in \mathbb{R}, \ \forall\,|y| \ge r.$$

(W_5) There exists a constant $\delta > 0$ such that either

$$(i) \qquad W(t, y) \ge 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \le \delta,$$

or

$$(ii) \qquad W(t, y) \le 0, \ \forall t \in \mathbb{R}, \ \forall\,|y| \le \delta.$$

Our second main result in this section is the following:

Theorem 3.2 *Assume conditions* $(W_1) - (W_4)$ *hold. If* 0 *is an eigenvalue of* $J\frac{d^2}{dt^2} - u^* A u$, *assume also* (W_5). *Then the system* (NS) *possesses at least one non-trivial* $T-$*periodic solution.*

3.1 Proof of Theorem 3.1

Let $S^1 = \mathbb{R}/T\mathbb{Z}$ and $E = H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$ be the Sobolev space of $T-$periodic functions with inner product $< .,. >_{H^{1/2}}$ and norm $\|.\|_{H^{\frac{1}{2}}}$ defined by

$$< x, y >_{H^{\frac{1}{2}}} = \hat{x}_0.\hat{y}_0 + \pi \sum_{k \in \mathbb{Z}} |k|\,\hat{x}_k.\hat{y}_k$$

and

$$\|x\|_{H^{\frac{1}{2}}} = \left(|\hat{x}_0|^2 + \pi \sum_{k \in \mathbb{Z}} |k|\,|\hat{x}_k|^2 \right)^{\frac{1}{2}}$$

for $x, y \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$, where

$$x(t) \cong \sum_{k \in \mathbb{Z}} exp(J\frac{2k\pi t}{T})\hat{x}_k, \ \hat{x}_k \in \mathbb{R}^{2N},$$

and

$$y(t) \cong \sum_{k\in\mathbb{Z}} exp(J\frac{2k\pi t}{T})\hat{y}_k,\ \hat{y}_k \in \mathbb{R}^{2N}.$$

Consider the closed subspace of $H^{1/2}(S^1, \mathbb{R}^{2N})$

$$X = \left\{ x \in H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})/x(t) \in (Ker\, u)^{\perp}\ a.e. \right\}.$$

It is well known that the space X is compactly embedded in $L^s(S^1, \mathbb{R}^{2N})$ for every $s \in [1, \infty[$ (see [5]) and as a consequence there exists a constant $\gamma_s > 0$ such that

$$\|x\|_{L^s} \le \gamma_s \|x\|_{H^{\frac{1}{2}}},\ \forall x \in X. \tag{3.1}$$

Define on X the bilinear form

$$B(x, y) = -\frac{1}{2}\int_0^T [J\dot{x}.y - A(t)u(x).u(y)]dt.$$

Let X^+ (resp.X^-) be the positive (resp. negative) space corresponding to the spectral decomposition of B in X and $X^0 = \ker B$. Then $X = X^+ \oplus X^- \oplus X^0$. In fact it is not difficult to check that X^+, X^- and X^0 are mutually orthogonal in $L^2(S^1, \mathbb{R}^{2N})$. Denote Q the quadratic form associated to B:

$$Q(x) = -\frac{1}{2}\int_0^T [J\dot{x}.x - A(t)u(x).u(x)]dt.$$

We prove (see [11]) that there exists a constant $\nu > 0$ such that

$$Q(x) \ge \nu \|x\|^2,\ \forall x \in X^+, \tag{3.2}$$

$$Q(x) \le -\nu \|x\|^2,\ \forall x \in X^-. \tag{3.3}$$

Now, since X^0 is of finite dimension, there exists a constant $a_1 > 0$ such that

$$\|x\|_{H^{\frac{1}{2}}} \le a_1 \|x\|_{L^2}, \forall x \in X^0. \tag{3.4}$$

We deduce from (3.2), (3.3) and (3.4) that the following expression

$$\|x\|^2 = \left\|x^+ + x^- + x^0\right\|^2 = Q(x^+) - Q(x^-) + \left|x^0\right|^2_{L^2} \tag{3.5}$$

where $x^i \in X^i,\ i = +, -, 0$, is an equivalent norm on X, which will be considered in the following. Therefore we deduce from (3.1) that for all $s \in [1, \infty[$, there exists a constant $\mu_s > 0$ such that

$$\|x\|_{L^s} \le \mu_s \|x\|, \forall x \in X. \tag{3.6}$$

If zero is not an eigenvalue of $J\frac{d}{dt} - u^*Au$, we take

$$X^1 = X^+,\ X^2 = X^-.$$

If zero is an eigenvalue of $J\frac{d}{dt} - u^*Au$, we take

$$X^1 = X^+ \oplus X^0,\ X^2 = X^-,\ if\ G(t, y) \le 0\ for\ |y| \le \delta,$$

$$X^1 = X^+,\ X^2 = X^- \oplus X^0,\ if\ G(t,y) \geq 0\ for\ |y| \leq \delta.$$

In the following, we will assume that zero is an eigenvalue of $J\frac{d}{dt} - u^*Au$ and

$$G(t,y) \leq 0,\ for\ |y| \leq \delta. \tag{3.7}$$

The other cases are similar.

Define a functional f in X by

$$f(x) = -\frac{1}{2}\int_0^T [J\dot{x}.x - A(t)u(x).u(x)]dt - \int_0^T G(t,u(x))dt.$$

It is easy to see that there exist two constants $m, M > 0$ such that

$$m|x| \leq |u(x)| \leq M|x|,\ \forall x \in (Ker\, u)^{\perp}. \tag{3.8}$$

Combine this with $(G_3)(ii)$, there are two constants $c, d > 0$ such that

$$|u^*G'(t,u(x))| \leq c|x|^{\beta} + d,\ \forall t \in \mathbb{R}, \forall x \in (Ker\, u)^{\perp}. \tag{3.9}$$

Therefore, we conclude that $f \in C^1(X,\mathbb{R})$ and maps bounded sets into bounded sets. Now, let us choose Hilbertian basis $(e_n)_{n\in\geq 1}$ for X^1 and $(e_n)_{n\leq -1}$ for X^2. Define

$$X_n^1 = space(e_1, ..., e_n),\ n \geq 1$$

$$X_n^2 = space(e_{-1}, ..., e_{-n}),\ n \geq 1$$

$$X^j = \overline{\bigcup_{n\geq 1} X_n^j},\ j = 1, 2.$$

We will proceed by successive lemmas.

Lemma 3.1 *The functional f satisfies the $(PS)^*$ condition.*

Proof Consider a sequence (x_{α_n}) such that (α_n) is admissible and

$$x_{\alpha_n} \in X_{\alpha_n},\ c = \sup_{n\in\mathbb{N}} f(x_{\alpha_n}) < \infty,\ f'_{\alpha_n}(x_{\alpha_n}) \to 0\ as\ n \to \infty. \tag{3.10}$$

We claim that (x_{α_n}) is bounded. Suppose by contradiction that (x_{α_n}) is not bounded, then going, if necessary, to a subsequence, we can assume that $\|x_{\alpha_n}\| \to \infty\ as\ n \to \infty$. By (G_4) and (3.8) there exists a constant $c_1 > 0$ such that for all $t \in \mathbb{R}$ and for all $x \in (Ker\, u)^{\perp}$

$$G'(t,u(x)).u(x) - 2G(t,u(x)) \geq b\,|x|^{\beta} - c_1. \tag{3.11}$$

Therefore, by noting $x_{\alpha_n} = x_n$ and $f_{\alpha_n} = f_n$, we have

$$-f'_n(x_n).x_n + 2f(x_n) = \int_0^T [G'(t,u(x_n)).u(x_n) - 2G(t,u(x_n))]dt$$

$$\geq b\int_0^T |x_n|^{\beta} - c_1T.$$

Combining this with (3.6), we obtain

$$\frac{\int_0^T |x_n|^\beta \, dt}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.12}$$

Let $x_n = x_n^+ + x_n^- + x_n^0 \in X^+ \oplus X^- \oplus X^0$. By (G_3), Hölder's inequality, (3.6) and (3.8), we have

$$\begin{aligned}
f_n'(x_n).x_n^+ &= \left\|x_n^+\right\|^2 - \int_0^T G'(t, u(x_n)).u(x_n^+)dt \\
&\geq \left\|x_n^+\right\|^2 - \int_0^T |G'(t, u(x_n))| \left|u(x_n^+)\right| dt \\
&\geq \left\|x_n^+\right\|^2 - a\int_0^T (|u(x_n)|^\alpha + 1) \left|u(x_n^+)\right| dt \\
&\geq \left\|x_n^+\right\|^2 - a\int_0^T (M^\alpha |x_n|^\alpha + 1)M \left|x_n^+\right| dt \\
&\geq \left\|x_n^+\right\|^2 - aM^{\alpha+1}[\int_0^T (|x_n|^\alpha)^{\frac{\beta}{\alpha}} dt]^{\frac{\alpha}{\beta}} [\int_0^T \left|x_n^+\right|^{\frac{\beta}{\beta-\alpha}} dt]^{\frac{\beta-\alpha}{\beta}} - aM \left\|x_n^+\right\|_{L^1} \\
&\geq \left\|x_n^+\right\|^2 - aM^{\alpha+1}\mu_{\beta\beta-\alpha} \left\|x_n^+\right\|_{L^\beta}^\alpha \left\|x_n^+\right\| - aM\mu_1 \left\|x_n^+\right\|
\end{aligned}$$

for all integer $n \in \mathbb{N}$, which implies that

$$\left\|x_n^+\right\| \leq \|f_n'(x_n)\| + c_2 \|x_n\|_{L^\beta}^\alpha + c_3, \ \forall n \in \mathbb{N}, \tag{3.13}$$

where c_2, c_3 are two constants. Since $1 < \alpha < \beta$, we deduce from (3.12) and (3.13) that

$$\frac{\|x_n^+\|}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.14}$$

Similarly

$$\frac{\|x_n^-\|}{\|x_n\|} \longrightarrow 0, \ as \ n \longrightarrow \infty. \tag{3.15}$$

By (G_4) and (3.8), there exist two constants $c_4, \ c_5 > 0$ such that

$$G'(t, u(y)).u(y) - 2G(t, u(y)) \geq c_4 |y| - c_5, \ \forall (t, y) \in \mathbb{R} \times (Ker\, u)^\perp, \tag{3.16}$$

which implies

$$\begin{aligned}
2f(x_n) - f_n'(x_n).x_n &= \int_0^T [G'(t, u(x_n)).u(x_n) - 2G(t, u(x_n))]dt \\
&\geq \int_0^T [c_4 |x_n| - c_5]dt \\
&\geq \int_0^T [c_4 \left|x_n^0\right| - c_4 \left|x_n^+\right| - c_4 \left|x_n^-\right| - c_5]dt.
\end{aligned} \tag{3.17}$$

Moreover, it follows from the equivalence of the norms on the finite dimensional subspace X^0 that there exists a positive constant d such that

$$\|x\| \leq d \|x\|_{L^1}, \ \forall x \in X^0. \tag{3.18}$$

Combining (3.6), (3.17) and (3.18) we obtain

$$2f(x_n) - f'_n(x_n).x_n \geq c_4\frac{1}{d}\left\|x_n^0\right\| - c_4\mu_1\left\|x_n^+\right\| - c_4\mu_1\left\|x_n^-\right\| - c_5T. \tag{3.19}$$

Therefore, by (3.14), (3.15) and (3.19), we have

$$\frac{\left\|x_n^0\right\|}{\|x_n\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.20}$$

We deduce from (3.14), (3.15) and (3.20) that

$$1 = \frac{\|x_n\|}{\|x_n\|} \leq \frac{\left\|x_n^0\right\| + \|x_n^-\| + \|x_n^+\|}{\|x_n\|} \longrightarrow 0 \text{ as } n \longrightarrow \infty, \tag{3.21}$$

which is a contradiction. So (x_n) must be bounded. Since the space X is closed in the reflexive space $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2N})$, then X is also reflexive and the sequence (x_n) possesses a subsequence (x_{n_k}) weakly convergent to a point x. Note that

$$Q(x_{n_k}^+ - x^+) = (f'_{n_k}(x_{n_k}) - f'(x)).(x_{n_k}^+ - x^+) \tag{3.22}$$

$$+\int_0^T [G'(t, u(x_{n_k})) - G'(t, u(x))].[u(x_{n_k}^+) - u(x^+)]dt$$

which implies that $x_{n_k}^+ \longrightarrow x^+$ in X. Similarly, $x_{n_k}^- \longrightarrow x$ in X. It follows that $x_{n_k} \longrightarrow x$ in X and $f'(x) = 0$. So f satisfies the $(PS)^*$ condition. The proof of Lemma 3.1 is complete.

Lemma 3.2 *The functional f satisfies the local linking condition at zero.*

Proof By assumption (G_3) and (3.8), there exists a constant $b_1 > 0$ such that

$$|G(t, u(x))| \leq b_1(|x|^{\alpha+1} + |x|), \ \forall t \in \mathbb{R}, \ \forall x \in (Ker\, u)^{\perp}. \tag{3.23}$$

Assumption (G_1) and (3.8) imply that for any $\epsilon > 0$, there exists a constant $R > 0$ such that

$$|G(t, u(x))| \leq \epsilon\,|x|^2, \ \forall t \in \mathbb{R}, \ \forall\,|x| \leq R. \tag{3.24}$$

Combining (3.23) with (3.24), we obtain

$$|G(t, u(x))| \leq (\epsilon\,|x|^2 + M_1\,|x|^{\alpha+1}), \ \forall t \in \mathbb{R}, \ \forall x \in (Keru)^{\perp} \tag{3.25}$$

where $M_1 = b_1(1 + R^{\alpha})$. Hence we obtain by (3.6)

$$\left|\int_0^T G(t, u(x))dt\right| \leq \epsilon\mu_2^2\,\|x\|^2 + M_1\mu_{\alpha+1}^{\alpha+1}\,\|x\|^{\alpha+1}. \tag{3.26}$$

So for all $x \in X^2 = X^-$

$$f(x) \leq -\|x\|^2 + \epsilon\mu_2^2\,\|x\|^2 + M_1\mu_{\alpha+1}^{\alpha+1}\,\|x\|^{\alpha+1}. \tag{3.27}$$

Since $\alpha > 1$ and ϵ is arbitrary, we deduce that there exists a constant $r > 0$ small enough such that

$$f(x) \leq 0, \ \forall x \in X^2, \ \|x\| \leq r. \tag{3.28}$$

Now, let $\eta > 0$ be such that

$$\forall\, x \in (Ker\, u)^{\perp},\ |x| \le \eta \Rightarrow |u(x)| \le \delta \qquad (3.29),$$

where δ is introduced in (G_5). Since X^0 is a finite dimensional space, there exists a constant $\rho > 0$ such that

$$\|x\|_{\infty} \le \rho \|x\|\,,\ \forall x \in X^0. \qquad (3.30)$$

Let $x = x^0 + x^+ \in X^1 = X^0 \oplus X^+$ such that $\|x\| \le \frac{\eta}{2\rho}$ and set

$$I = \left\{ t \in [0,T] / \left|x^+(t)\right| \le \frac{\eta}{2} \right\}.$$

On I, we have by (3.30)

$$|x(t)| \le \left|x^0(t)\right| + \left|x^+(t)\right| \le \left\|x^0\right\|_{\infty} + \frac{\eta}{2} \le \eta,$$

hence, by (3.7) and (3.29)

$$\int_0^T G(t, u(x))dt \le 0. \qquad (3.31)$$

On $[0,T] \mid I$, we have also by (3.30)

$$|x(t)| \le \left|x^0(t)\right| + \left|x^+(t)\right| \le \rho \left\|x^0\right\| + \left|x^+(t)\right| \le \frac{\eta}{2} + \left|x^+(t)\right| \le 2\left|x^+(t)\right|.$$

Hence, by (3.6) and (3.25), we obtain

$$\left| \int_{[0,T]|I} G(t, u(x))dt \right| \le 4\epsilon\mu_2^2 \left\|x^+\right\|^2 + 2^{\alpha+1} M_1 \mu_{\alpha+1}^{\alpha+1} \left\|x^+\right\|^{\alpha+1}.$$

Therefore, we have

$$f(x) \ge \left\|x^+\right\|^2 - 4\epsilon\mu_2^2 \left\|x^+\right\|^2 - 2^{\alpha+1} M_1 \mu_{\alpha+1}^{\alpha+1} \left\|x^+\right\|^{\alpha+1} - \int_I G(t, u(x))dt. \qquad (3.32)$$

Since $\alpha > 1$, we deduce from (3.31) and (3.32), by taking ϵ small enough, that there exists a constant $0 < r < \frac{\eta}{2\rho}$ such that

$$f(x) \ge 0,\ \forall\, x \in X^1,\ \|x\| \le r. \qquad (3.33)$$

Properties (3.28) and (3.33) show that f satisfies the local linking condition at zero which completes the proof of Lemma 3.2.

Lemma 3.3 *For each $m \in \mathbb{N}$, $f(x) \longrightarrow -\infty$ as $\|x\| \longrightarrow \infty$, $x \in X_m^1 \oplus X^2$.*

Proof For $x = x^+ + x^0 + x^- \in X_m^1 \oplus X^2$, we have

$$f(x) = \left\|x^+\right\|^2 - \left\|x^-\right\|^2 - \int_0^T G(t, u(x^+ + x^0 + x^-))dt. \qquad (3.34)$$

Since X_m^1 is of finite dimension, there exists a positive constant γ_1 such that

$$\left\|x^+ + x^0\right\| \le \gamma_1 \left\|x^+ + x^0\right\|_{L^2},\ \forall x = x^+ + x^0 \in X_m^1. \qquad (3.35)$$

On the other hand, by assumption $(G_2)(i)$ and (3.8), there exists a constant $c_6 > 0$ such that

$$-G(t,u(x)) \leq -2\gamma_1 |x|^2 + c_6,\ \forall\ x \in (Ker\,u)^{\perp}. \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we obtain for $x = x^+ + x^0 + x^- \in X_m^1 \oplus X^2$

$$f(x) \leq \left\|x^+\right\|^2 - \left\|x^-\right\|^2 - 2\gamma_1[\left\|x^+\right\|_{L^2}^2 + \left\|x^-\right\|_{L^2}^2 + \left\|x^0\right\|_{L^2}^2] + c_6 T$$

$$\leq -\left\|x^-\right\|^2 - \left\|x^+\right\|^2 - 2\left\|x^0\right\|^2 + c_6 T$$

which concludes the proof of Lemma 3.3.

We deduce from the previous lemmas that the functional f satisfies all the assumptions of the Local Linking Theorem and hence the functional f possesses at least two distinct critical points on X. Therefore the Hamiltonian system (HS) has at least one non trivial $T-$periodic solution.

3.2 Proof of Theorem 3.2

We consider only the case when 0 is an eigenvalue of $-\frac{d^2}{dt^2} + u^*Au$ and

$$W(t,y) \leq 0,\ \forall t \in \mathbb{R},\ \forall |y| \leq \delta. \tag{3.37}$$

The other cases are similar and simpler.

We shall apply the Local Linking Theorem to the functional

$$f(x) = \frac{1}{2}\int_0^T [|\dot{x}|^2 + A(t)u(x).u(x)]dt - \int_0^T W(t,u(x))dt$$

defined on the following closed subspace X of $H^1(S^1,\mathbb{R}^N)$

$$X = \{x \in H^1(S^1,\mathbb{R}^N)/x(t) \in (Ker\,u)^{\perp} a.e.\}$$

where $H^1(S^1,\mathbb{R}^N)$ is the space of $T-$ periodic absolutely continuous vector functions from S^1 into $\mathbb{R}^N$ whose first derivatives have square integrable norm. The inner product on $H^1(S^1,\mathbb{R}^N)$ is given by

$$< u, v >_{H^1} = \int_0^T [u(t).v(t) + \dot{u}(t).\dot{v}(t)]dt.$$

The functional f is continuously differentiable on X and maps bounded sets into bounded sets. Moreover the critical points of f correspond to the $T-$ periodic solutions of the system (NS) (see [9]).

Let X^+ (resp. X^-) be the positive (resp. negative) space corresponding to the spectral decomposition of $-\frac{d^2}{dt^2} + u^*Au$ in X and $X^0 = Ker(-\frac{d^2}{dt^2} + u^*Au)$. Let $X^2 = X^-$ and $X^1 = X^0 \oplus X^+$ and choose a Hilbertian basis $(e_n)_{n\geq 0}$ for X^1. Define

$$X_n^1 = span(e_0, e_1, ..., e_n), n \in \mathbb{N},$$

$$X_n^2 = X^2, n \in \mathbb{N}.$$

It is well known that X^0, X^2 are of finite dimensional.
As in the proof of Theorem 3.1, we prove by using assumptions (W_3), (W_4) that f

satisfies the $(PS)^*$ condition and by using assumptions (W_1), (W_3) that f satisfies the local linking at zero. Assumption (W_2) implies that f satisfies assertion $d)$ of the Local Linking Theorem. Consequently the functional f satisfies all the Local Linking Theorem assumptions and then it has at least two critical points. Therefore the system (NS) possesses a nontrivial $T-$ periodic solution.

4 Subharmonic Solutions

Let u, u^* and G be defined as in Section 3, we are interested in the existence of infinitely many subharmonic solutions of the Hamiltonian systems

$$(HS) \qquad J\dot{x} + u^* G'(t, u(x)) = 0,$$

i.e. of distinct $kT-$ periodic solutions of (HS).
Let $\alpha > 1$ be as in (G_3) and consider the following assumptions:
(G_4') There exist constants $\beta > \alpha - 1$, $b > 0$ and $r > 0$ such that

$$G'(t, y).y - 2G(t, y) \geq b\,|y|^{\beta}, \ \forall t \in \mathbb{R}, \ \forall \ |y| \geq r.$$

$$(G_5') \qquad G(t, y) \geq 0, \ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Our main result in this section is

Theorem 4.1 *Assume $(G_0) - (G_3)$, (G_4') and (G_5') hold. Then the Hamiltonian system (HS) possesses infinitely many subharmonic solutions.*

Example 4.1 Let $\frac{3}{2} \leq \alpha < 2$ be a real number. The function

$$G(t, y) = \left| cos(\frac{2\pi}{T} t) \right| |y|^{\alpha+1} + |y|^2 \, ln(1 + |y|^2)$$

satisfies $(G_1) - (G_3)$, (G_4') and (G_5'). The linear map $u : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ defined by $u(p, q) = p$ satisfies (G_0). Therefore the corresponding Hamiltonian system (HS) possesses infinitely many subharmonic solutions.

Remark 4.1 We obtain the same result if we replace assumptions (G_2), (G_4') and (G_5') respectively by

$$\lim_{|y| \to \infty} \frac{G(t, y)}{|y|^2} = -\infty, \ uniformly \ in \ t \in \mathbb{R}.$$

There exist constants $\beta > \alpha - 1$, $b > 0$ and $r > 0$ such that

$$G'(t, y).y - 2G(t, y) \leq -b\,|y|^{\beta}, \ \forall t \in \mathbb{R}, \ \forall \ |y| \geq r,$$

$$G(t, y) \leq 0, \ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Proof of Theorem 4.1. Choose $k \in \mathbb{N}$. By making the change of variables $s = k^{-1}t$, (HS) transforms to

$$(HS_k) \qquad J\dot{y} + k u^* G'(ks, y(s)) = 0.$$

Hence, finding $kT-$periodic solutions of (HS) is equivalent to finding $T-$periodic solutions of (HS_k). Let X be the space introduced in section 3 and consider the functional f_k defined over X by

$$f_k(y) = -\frac{1}{2}\int_0^T J\dot{y}.yds - k\int_0^T G(ks, u(y(s)))ds.$$

The assumptions of Theorem 4.1 imply that f_k is continuously differentiable in X and critical points of f_k are $T-$periodic solutions of (HS_k). Let X^+, X^- and X^0 be respectively the positive, negative and null subspaces of X corresponding to the spectral decomposition of the quadratic form

$$Q(y) = -\frac{1}{2}\int_0^T J\dot{y}.yds.$$

Then $X = X^- \oplus X^0 \oplus X^+$ and as in Section 3, we consider the equivalent norm on X given by

$$\|y\|^2 = Q(y^+) - Q(y^-) + \left|y^0\right|^2,$$

where $y = y^- + y^0 + y^+ \in X = X^- \oplus X^0 \oplus X^+$. Then we have

$$f_k(y) = \left\|y^+\right\|^2 - \left\|y^-\right\|^2 - k\int_0^T G(ks, u(y))ds. \tag{4.1}$$

We will apply the Generalized Mountain Pass Theorem to the functional f_k over X with $X^1 = X^+$ and $X^2 = X^0 \oplus X^-$. We will proceed by successive lemmas.

Lemma 4.1 *The functional f_k satisfies the Cerami's condition (C).*

Proof Let (y_n) be a sequence such that $(f_k(y_n))$ is bounded from above and $\|f_k'(y_n)\|(1 + \|y_n\|) \longrightarrow 0$ as $n \longrightarrow \infty$. We claim that (y_n) is a bounded sequence in X. For otherwise, going if necessary to a subsequence, we can assume that $\|y_n\| \longrightarrow \infty$ *as* $n \longrightarrow \infty$.

By (G_4') and (3.8), there is a constant $c > 0$ such that

$$G'(ks, u(y)).u(y) - 2G(ks, u(y)) \geq b\,|y|^\beta - c,\ \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m \tag{4.2}$$

which implies with that

$$2f_k(y_n) - f_k'(y_n).y_n = k\int_0^T [G'(ks, u(y_n)).u(y_n) - 2G(ks, u(y_n))]ds$$

$$\geq k[b\int_0^T |y_n|^\beta\, ds - cT].$$

Hence for a given $k \in \mathbb{N}$, we get

$$\int_0^T |y_n|^\beta\, ds \leq c_1 \tag{4.3}$$

for all integer n and some positive constant $c_1 > 0$.

Now, let $y_n = y_n^- + y_n^0 + y_n^+ \in X^- \oplus X^0 \oplus X^+$ and set

$$p = \frac{2\beta+1}{2\alpha-1} > 1 \ and \ q = \frac{p}{p-1} = \frac{2\beta+1}{2\beta+1-\alpha}. \tag{4.4}$$

It follows from Höder's inequality, (3.1) and (4.3) that

$$\begin{aligned}
\int_0^T |y_n|^\alpha \left|y_n^+\right| ds &= \int_0^T |y_n|^{\frac{\beta}{p}} |y_n|^{\alpha-\frac{\beta}{p}} \left|y_n^+\right| ds \\
&\le [\int_0^T (|y_n|^{\frac{\beta}{p}})^p ds]^{\frac{1}{p}} [\int_0^T (|y_n|^{\alpha-\frac{\beta}{p}} \left|y_n^+\right|)^q ds]^{\frac{1}{q}} \\
&\le [\int_0^T (|y_n|^\beta) ds]^{\frac{1}{p}} [\int_0^T (|y_n|^{\alpha-\frac{\beta}{p}})^{2q} ds]^{\frac{1}{2q}} [\int_0^T \left|y_n^+\right|^{2q} ds]^{\frac{1}{2q}} \\
&\le [\int_0^T |y_n|^\beta ds]^{\frac{1}{p}} \|y_n\|^{\frac{\beta+\alpha}{2\beta+1}}_{L^{\frac{\beta+\alpha}{\beta+1-\alpha}}} \left\|y_n^+\right\|_{L^{2q}} \\
&\le c_1^{\frac{1}{p}} \gamma^{\frac{\beta+\alpha}{2\beta+1}}_{\frac{\beta+\alpha}{\beta+1-\alpha}} \gamma_{2q} \|y_n\|^{\frac{\beta+\alpha}{2\beta+1}} \left\|y_n^+\right\|
\end{aligned} \tag{4.5}$$

for all integer n. By (G_3), (3.1), (3.8), (4.3) and (4.5), we have

$$\begin{aligned}
f_k'(y_n).y_n^+ &= \left\|y_n^+\right\|^2 - k\int_0^T G'(ks, u(y_n)).u(y_n^+) ds \\
&\ge \left\|y_n^+\right\|^2 - k\int_0^T |G'(ks, u(y_n))| \left|u(y_n^+)\right| ds \\
&\ge \left\|y_n^+\right\|^2 - ka\int_0^T (|u(y_n))|^\alpha + 1) \left|u(y_n^+)\right| ds \\
&\ge \left\|y_n^+\right\|^2 - kaM^{\alpha+1}\int_0^T (|y_n|^\alpha \left|y_n^+\right| ds - kaM\int_0^T \left|y_n^+\right| ds \\
&\ge \left\|y_n^+\right\|^2 - kaM^{\alpha+1} c_1^{\frac{1}{p}} \gamma^{\frac{\beta+\alpha}{2\beta+1}}_{\frac{\beta+\alpha}{\beta+1-\alpha}} \gamma_{2q} (\|y_n\|^{\frac{\beta+\alpha}{2\beta+1}} \left\|y_n^+\right\| - kaM\gamma_1 \left\|y_n^+\right\|
\end{aligned}$$

for all integer n. Noting that $\frac{\beta+\alpha}{2\beta+1} < 1$, one sees

$$\frac{\|y_n^+\|}{\|y_n\|} \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty. \tag{4.6}$$

Similarly for y_n^-, we have

$$\frac{\|y_n^-\|}{\|y_n\|} \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty. \tag{4.7}$$

On the other hand, since X^0 is of finite dimension, there exists a constant $\gamma > 0$ such that

$$\|y\| \le \gamma^2 \|y\|_{L^2}, \ \forall y \in X^0. \tag{4.8}$$

Therefore by Hölder's inequality, (3.1) and (4.8) we have

$$\frac{1}{\gamma^2} \left\|y_m^0\right\|^2 \le \int_0^T \left|y_m^0\right|^2 ds \le \int_0^T |y_n|^{\frac{\beta}{\beta+1}} |y_n|^{\frac{\beta+2}{\beta+1}} ds$$

$$\leq [\int_0^T |y_n|^\beta \, ds]^{\frac{1}{\beta+1}} [\int_0^T |y_n|^{\frac{\beta+2}{\beta}} \, ds]^{\frac{\beta}{\beta+1}}$$

$$\leq (c_1)^{\frac{1}{\beta+1}} (\gamma_{\frac{\beta+2}{\beta}})^{\frac{\beta+2}{\beta+1}} \|y_n\|^{\frac{\beta+2}{\beta+1}} . \tag{4.9}$$

Since $\frac{\beta+2}{\beta+1} < 2$, we deduce from (4.9)

$$\frac{\|y_n^0\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{4.10}$$

Hence by (4.6), (4.7) and (4.10) we have

$$1 = \frac{\|y_n\|}{\|y_n\|} \leq \frac{\|y_n^0\| + \|y_n^-\| + \|y_n^+\|}{\|y_n\|} \longrightarrow 0 \quad as \quad n \longrightarrow \infty, \tag{4.11}$$

which is a contradiction. Therefore (y_n) must be bounded. Then by a standard argument, (y_n) has a convergent subsequence, which shows that f_k satisfies the Cerami's condition.

Lemma 4.2 *There exist constants $m > 0$ and $\alpha > 0$ such that*

$$f_k(y) \geq m, \ \forall y \in \partial B_\rho \cap X^1. \tag{4.12}$$

Proof As in (3.26), for all $\epsilon > 0$, there exists a constant $M_1 > 0$ such that

$$\left| \int_0^T G(ks, u(y)) ds \right| \leq \epsilon \gamma_2^2 \|y\|^2 + M_1 \gamma_{\alpha+1}^{\alpha+1} \|y\|^{\alpha+1}, \forall y \in X. \tag{4.13}$$

Now for all $x \in X^1 = X^+$, we have by (4.13)

$$f_k(y) = \frac{1}{2} \|y\|^2 - k \int_0^T G(ks, u(y)) ds$$

$$\geq \frac{1}{2} \|y\|^2 - k\epsilon C^2 \|y\|^2 - kC^{\alpha+1} M_1 \|y\|^{\alpha+1},$$

where $C = sup(1, \gamma_2, \gamma_{\alpha+1})$. So letting $\epsilon = \frac{1}{4kC^2}$ and $\rho = \frac{1}{8}(kM_1 C^{\alpha+1})^{-\frac{1}{\alpha-1}}$, we have

$$f_k(y) \geq \frac{1}{4}\rho^2 - kM_1 (C\rho)^{\alpha+1} = \frac{1}{8}\rho^2 = m > 0 \tag{4.15}$$

for $y \in X^1$ with $\|y\| = \rho$.

Lemma 4.3 *There exist $e \in X^1$ and two constants $r_1, r_2 > 0$ such that*

$$f_k(y) \leq 0, \ \forall y \in \partial Q, \tag{4.16}$$

where

$$Q = \{se / 0 \leq s \leq r_1\} \oplus \left\{ y \in X^2 / \|y\| \leq r_2 \right\}.$$

Proof Let $e \in X^1$ with $\|e\| = 1$. By (G_2) and (3.8), there exists a constant $M_2 > 0$ such that

$$G(ks, u(y)) \geq \gamma^2 |y|^2 - M_2, \ \forall t \in \mathbb{R}, \ \forall y \in (Ker\, u)^{\perp}, \tag{4.17}$$

where γ is the constant given by (4.8). It follows from (4.8) and (4.17) that for all $s > 0$ and $y \in X^2 = X^0 \oplus X^-$

$$\begin{aligned} f_k(se + y) &= \frac{1}{2}s^2 - \frac{1}{2}\left\|y^-\right\|^2 - k\int_0^T G(ks, u(se + y))ds \\ &\leq \frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\gamma^2 \|se + y\|_{L^2}^2 + kM_2T \\ &\leq \frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\gamma^2(s^2 \|e\|_{L^2}^2 + \|_-\|_{L^2}^2 + \|y_0\|_{L^2}^2) + kM_2T \\ &\leq \frac{1}{2}s^2 - ks^2 - \frac{1}{2}\left\|y^-\right\|^2 - \|y_0\|_{L^2}^2 + kM_2T. \end{aligned} \tag{4.18}$$

Let

$$r_1 = \frac{\sqrt{2kM_2T}}{2k - 1}, \ r_2 = \sqrt{2kM_2T},$$

it is clear from (4.18) that

$$f_k(se + y) \leq 0 \ either \ s \geq r_1 \ or \ \|y\| \geq r_2. \tag{4.19}$$

Let

$$Q = \{se/0 \leq s \leq r_1\} \oplus \left\{y \in X^2/\, \|y\| \leq r_2\right\}. \tag{4.20}$$

Then we have $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_1 = \left\{y \in X^0 \oplus X^-/\, \|y\| \leq r_2\right\}, \quad Q_2 = r_1 e \oplus \left\{y \in X^0 \oplus X^-/\, \|y\| \leq r_2\right\},$$

$$Q_3 = \{se/0 \leq s \leq r_1\} \oplus \left\{y \in X^0 \oplus X^-/\, \|y\| = r_2\right\}.$$

By (4.19), one has

$$f_k(y) \leq 0, \ \forall y \in Q_2 \cup Q_3.$$

It follows from $(G_5)(i)$ that $f_k(y) \leq 0$ for all $y \in X^0 \oplus X^-$, which implies that

$$f_k(y) \leq 0, \ \forall y \in Q_1.$$

Hence we obtain (4.16). The proof of Lemma 4.3 is complete.

By Lemma 4.1-3, we conclude that the functional f_k satisfies all the assumptions of the Generalized Mountain Pass Theorem. Therefore for a given $k \in \mathbb{N}$, there exists a critical point $y_k \in X$ of f_k such that $f_k(y_k) > 0$.

Finally, we claim that the system (HS) has infinitely many subharmonic solutions. Note that $y_1(ks)$ satisfies (HS_k), in fact

$$\frac{d}{ds}(y_1(ks)) = k\frac{dy_1}{ds}(ks) = kJu^*G'(ks, y_1(ks)).$$

If $y_k(s) = y_1(ks)$, it is easy to check that

$$c_k = f_k(y_k) = kf_1(y_1) = kc_1. \tag{4.21}$$

Since $c_1 = f_1(y_1) > 0$, one has that $c_k \longrightarrow +\infty$ as $k \longrightarrow \infty$. Noting that

$$c_k \leq \sup_{y\in Q} f_k(y) = \sup_{y\in Q}[\frac{1}{2}(s^2 - \left\|y^-\right\|^2) - k\int_0^T G(ks, u(y)ds] \leq \frac{1}{2}r_1^2 \leq M_2T, \tag{4.22}$$

where Q is defined as in (4.20). Combining (4.21) with (4.22) yields a contradiction as $k \longrightarrow \infty$. Therefore the sequence (c_k) of critical values is bounded and there is a $k_1 \in \mathbb{N}$ such that $y_k(s) \neq y_1(ks)$ for all $k \geq k_1$.

Now, consider the $T-$periodic function $G_1(t,x) = k_1 G(k_1 t, x)$. By the same technicals as in the previous steps, we prove that the following Hamiltonian system

$$J\frac{dz}{ds} + jG_1^{'}(js, u(z)) = 0 \tag{3.23}$$

possesses a sequence of nonzero $T-$periodic solutions (z_j) such that there exists an integer k_2 satisfying $z_j(s) \neq z_1(js)$ for all $j \geq k_2$. Moreover, from the form of (3.23) and the corresponding variational problem we have

$$z_j(s) = y_{jk_1}(s) \ and \ y_{jk_1}(s) \neq y_1(jk_1 s) \ for \ all \ j \geq k_2.$$

By repeating this reasoning infinitely, we obtain a sequence $x_1(t) = y_1(t)$, $x_{k_1}(t) = y_{k_1}(\frac{t}{k_1})$, $x_{k_1k_2}(t) = y_{k_1k_2}(\frac{t}{k_1k_2})$, ... of distincts nonzero solutions of the system (HS) with x_l is $lT-$ periodic. The proof of Theorem 4.1 is complete.

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