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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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Periodic Solutions of Singular Integral Equations

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Abstract: We consider a scalar integral equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds$$

in which $C(t,s)$ has a singularity at $t = s$. There are periodic assumptions on a, C, and g. First we prove a fixed point theorem of the Krasnoselskii–Schaefer type. We then construct a Liapunov functional which allows us to satisfy the conditions of the fixed point theorem and to prove that there is a periodic solution.

Keywords: *integral equations; fixed point theorems; periodic solutions; Liapunov functionals.*

Mathematics Subject Classification (2000): 45D05, 45D20, 45M15.

1 Introduction

We consider a scalar integral equation

$$x(t) = a(t) - \int_{-\infty}^{t} C(t,s)g(s,x(s))ds \tag{1}$$

for which there is a $T > 0$ so that

$$a(t+T) = a(t),\ g(t+T,x) = g(t,x),\ C(t+T,s+T) = C(t,s) \tag{2}$$

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for all $t \in \Re$ and $s < t$ with a and g continuous. We denote by $(\mathcal{P}_T, \|\cdot\|)$ the Banach space of continuous T-periodic functions.

If g is Lipschitz and if C is small enough then a contraction mapping will yield a periodic solution. If C is convex then Liapunov arguments will produce *a priori* bounds. Under compactness conditions, Schaefer's fixed point theorem will yield a periodic solution. A collection of such results are found in Burton [7]. A recent n-dimensional result is given in [17].

In this paper we ask that g satisfies

$$|g(t,x) - g(t,y)| \leq K|x-y| \tag{3}$$

for all $x, y \in \Re$ and some $K > 0$, while C satisfies a truncated convexity condition, but has a significant singularity at $t = s$. We derive a set of conditions measuring the magnitude of the singularity that will still permit proof of the existence of a periodic solution using a combination Krasnoselskii–Schaefer fixed point theorem which we will prove in Section 2.

2 A Fixed Point Theorem

In this section, we will prove a fixed point theorem of Krasnoselskii-Schaefer type in which the mapping function has the form $Px = Bx + Ax$ with A being compact and $(I-B)^{-1}$ continuous on an appropriate subset M of a Banach space S. The theorem resembles that of Burton–Kirk [6] without having a λ term in B. See [8, 10, 11, 13, 14, 15] for work on Krasnoselskii and Schaefer theorems and their extended forms.

Since P is the sum of two operators, it is in general a non-self map; that is, P may not necessarily map a closed convex subset M of S into itself. To prove the existence of a fixed point of P, we apply topological degree theory or transversality method by constructing a homotopy U_λ on M with $U_1 = P$. It is assumed that $U_\lambda(\phi) = U(\lambda, \phi)$ is a continuous mapping of $[0,1] \times M$ into a compact subset of S. In many applications, U_0 is a constant map sending M to a point $p \in M/\partial M$. In this case, U_0 is an "essential" map. If $U_\lambda(\phi)$ is fixed point free on ∂M for all $\lambda \in (0,1)$, then $U_1(\phi)$ is essential having a fixed point property in M (Granas and Dugundji [9, p.120-123]). This fact is often written in the form of Leray–Schauder principle or its nonlinear alternatives which states that either

(A_1) U_1 has a fixed point in M or

(A_2) there exists $x \in \partial M$ and $\lambda \in (0,1)$ with $x = U_\lambda(x)$

(see [1, p. 48], [9, p. 123], [15, p. 28], [16]).

Theorem 2.1 *Let $(S, \|\cdot\|)$ be a Banach space, $A, B : S \to S$ such that A is continuous with A mapping bounded sets into compact sets, $(I-B)^{-1}$ exists and is continuous on $(I-B)S$ with $\lambda A(M) \subset (I-B)S$ for each closed convex subset $M \subset S$ and $\lambda \in [0,1]$. Then either*

(i) $x = Bx + \lambda Ax$ has a solution in S for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Proof Since $\lambda A(M) \subset (I-B)S$, we have $0 \in (I-B)S$. If $x^* = (I-B)^{-1}(0)$, then x^* is the unique fixed point of B. For each positive integer n, define a closed and bounded set

$$M_n = \{x \in S : \|x\| \le n\}.$$

We choose n sufficiently large so that $x^* \in M_n/\partial M_n$. Now $(I-B)^{-1}$ exists and is continuous on $(I-B)S$. Since A is continuous with A mapping M_n into a compact set, so is $(I-B)^{-1}(\lambda A)$ for each $\lambda \in [0,1]$. Define $U : [0,1] \times M_n \to S$ by

$$U(\lambda, \phi) = (I-B)^{-1}(\lambda A\phi).$$

Then $U_\lambda(\phi) = U(\lambda, \phi)$ is a continuous mapping of $[0,1] \times M_n$ into a compact subset of S. Indeed, set $\Gamma = \{\lambda A\phi : \lambda \in [0,1],\ \phi \in M_n\}$ and let $\{(\lambda_k, \phi_k)\}$ be a sequence in $[0,1] \times M_n$. We may assume that $\lambda_k \to \lambda_0 \in [0,1]$ as $k \to \infty$. Since AM_n is contained in a compact subset of S, there exists a convergent subsequence $\{A\phi_{k_j}\}$ of $\{A\phi_k\}$. Now $\{\lambda_{k_j} A\phi_{k_j}\}$ converges in S. This implies that Γ is pre-compact, and so is $(I-B)^{-1}\Gamma$. Observe that for all $\phi \in M_n$,

$$U_0(\phi) = (I-B)^{-1}(0) = x^*$$

is a constant map. Moreover, $x^* \in M_n/\partial M_n$. By the statement of nonlinear alternatives (A_1) and (A_2) above, either U_1 has a fixed point in M_n or there exists $x_n \in \partial M_n$ such that $x_n = U_\lambda(x_n)$ for some $\lambda \in (0,1)$. This implies that either $x = Bx + Ax$ has a solution in M_n or there exists $x_n \in \partial M_n$ with $x_n = Bx_n + \lambda Ax_n$ for some $\lambda \in (0,1)$. In the later case, we have $\|x_n\| = n$. Thus, if (i) does not hold, then $\|x_n\| \to \infty$ as $n \to \infty$ and (ii) must hold. This completes the proof.

Remark 2.1 It is clear that if B is a contraction mapping with contraction constant $0 < \alpha < 1$, then $(I-B)^{-1}$ exists and is continuous on S. Many generalized or nonlinear contractions satisfy this condition (see [2, 3, 8, 11, 12, 13]).

3 Technical Conditions

We now introduce the conditions which will produce the *a priori* bound needed in the fixed point theorem, as well as the required compactness. The kernel, $C(t,s)$, can have a singularity at $t = s$, but we ask that there exists a fixed $\epsilon > 0$ so that

$$C(t,s) \ge 0,\ C_s(t,s) \ge 0,\ C_t(t,s) \le 0,\ C_{st}(t,s) \le 0 \tag{4}$$

provided that

$$-\infty < s \le t - \epsilon,\ t < \infty. \tag{5}$$

Moreover, if $x \in \mathcal{P}_T$, then

$$\int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds \quad \text{and} \quad \int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds \quad \text{are continuous.} \tag{6}$$

The ϵ will play a central role. First, assume that there is a $\eta < 1$ with

$$K\int_{t-\epsilon}^{t} |C(t,s)|ds \le \eta,\ t \in \Re. \tag{7}$$

Next, there are positive constants α and β with $2\alpha+\beta<2$ so that both

$$\int_s^{s+\epsilon}[\epsilon C_s(u,u-\epsilon)+C(u,u-\epsilon)+|C(u,s)|]du<\alpha,\ s\in\Re \tag{8}$$

and

$$C(t,t-\epsilon)\epsilon+\int_{t-\epsilon}^{t}|C(t,s)|ds<\beta,\ t\in\Re. \tag{9}$$

The work here is motivated by and is an extension of [4]. Relations (7)–(9) specify the strength of the singularity. For a "mild" singularity such as $C(t,s)=[t-s]^{-p}$, $0<p<1$, then (4), (5), (7)–(9) are satisfied for any $K>0$ when it is allowed that ϵ can be taken sufficiently small. But (6) would fail. The following function satisfies (4)-(9) with $0<\epsilon\le 1$ and an appropriate constant $k>0$

$$C(t,s)=\frac{k}{(t-s)(1+|\ln(t-s)-\ln\epsilon|)^2}.$$

We now define for $0\le\lambda\le 1$ a companion equation to (1)

$$x(t)=\lambda\left[a(t)-\int_{-\infty}^{t-\epsilon}C(t,s)g(s,x(s))ds\right]-\int_{t-\epsilon}^{t}C(t,s)g(s,x(s))ds. \tag{1_λ}$$

The mappings $A,B:\mathcal{P}_T\to\mathcal{P}_T$ mentioned in the theorem are defined by $\phi\in\mathcal{P}_T$ which implies that

$$(A\phi)(t):=a(t)-\int_{-\infty}^{t-\epsilon}C(t,s)g(s,\phi(s))ds \tag{10}$$

and

$$(B\phi)(t):=-\int_{t-\epsilon}^{t}C(t,s)g(s,\phi(s))ds. \tag{11}$$

By (6), if $\phi\in\mathcal{P}_T$ then ϕ is continuous so these integrals are continuous functions. To see that $A\phi,B\phi\in\mathcal{P}_T$ we note that

$$\begin{aligned}(A\phi)(t+T)&=a(t+T)-\int_{-\infty}^{t+T-\epsilon}C(t+T,s)g(s,\phi(s))ds\\&=a(t)-\int_{-\infty}^{t-\epsilon}C(t+T,s+T)g(s+T,\phi(s+T))ds=(A\phi)(t)\end{aligned}$$

while

$$\begin{aligned}(B\phi)(t+T)&=-\int_{t+T-\epsilon}^{t+T}C(t+T,s)g(s,\phi(s))ds\\&=-\int_{t-\epsilon}^{t}C(t+T,s+T)g(s+T,\phi(s+T))ds=(B\phi)(t).\end{aligned}$$

Moreover, by (3) and (7), B is a contraction.

4 A Liapunov Functional

We begin with the assumption that there is an $L > 0$ with

$$xg(t,x) \geq 0 \text{ for } |x| \geq L \tag{12}$$

and that

$$\lim_{s\to-\infty}(t-s)C(t,s) = 0 \text{ for fixed } t. \tag{13}$$

Then define a Liapunov functional by

$$V(t,\epsilon) = \lambda\int_{-\infty}^{t-\epsilon} C_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds. \tag{14}$$

This Liapunov functional in the continuous case with finite delay was recently discussed in [5].

Lemma 4.1 *If $x \in \mathcal{P}_T$ solves (1_λ) then $V'(t,\epsilon)$ satisfies*

$$\begin{aligned} V'(t,\epsilon) \leq{}& \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ &+ 2g(t,x)\left[\lambda C(t,t-\epsilon)\int_{t-\epsilon}^t g(v,x(v))dv - \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\right] \\ &+ 2g(t,x)[\lambda a(t) - x(t)]. \end{aligned} \tag{15}$$

Proof Taking into account that $C_{st} \leq 0$ we have

$$\begin{aligned} V'(t,\epsilon) \leq{}& \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ &+ 2\lambda g(t,x)\int_{-\infty}^{t-\epsilon} C_s(t,s)\int_s^t g(v,x(v))dvds. \end{aligned}$$

If we integrate the last term by parts and use (13) in the lower limiting evaluation, keeping in mind that x is bounded, we obtain

$$\begin{aligned} V'(t,\epsilon) \leq{}& \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ &+ 2\lambda g(t,x)\left[C(t,s)\int_s^t g(v,x(v))dv\Big|_{-\infty}^{t-\epsilon} + \int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds\right] \\ ={}& \lambda C_s(t,t-\epsilon)\left(\int_{t-\epsilon}^t g(v,x(v))dv\right)^2 \\ &+ 2\lambda g(t,x)\left[C(t,t-\epsilon)\int_{t-\epsilon}^t g(v,x(v))dv\right] \\ &+ 2g(t,x)\left[\lambda\int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds + \int_{t-\epsilon}^t C(t,s)g(s,x(s))ds\right] \\ &- 2g(t,x)\int_{t-\epsilon}^t C(t,s)g(s,x(s))ds. \end{aligned}$$

Using (1_λ) in the next-to-last term yields (15).

We will integrate (15) to relate $g(t,x(t))$ to $a(t)$ and then use that relation in a lower bound on the Liapunov functional to obtain the *a priori* bound. We now obtain that lower bound.

Lemma 4.2 *For any $q>0$, if $x \in \mathcal{P}_T$ solves (1_λ), then*

$$
\begin{aligned}
(x(t)-\lambda a(t))^2 &\le 2(1+q^{-1})\int_{-\infty}^{t-\epsilon} C_s(t,s)ds\, V(t,\epsilon)\\
&\quad + 2(1+q^{-1})\epsilon C^2(t,t-\epsilon)\int_{t-\epsilon}^{t} g^2(s,x(s))ds\\
&\quad + (1+q)\bigg(\int_{t-\epsilon}^{t}|C(t,s)|ds\bigg)^2\bigg(K\|x\| + \sup_{0\le u\le T}|g(u,0)|\bigg)^2. \qquad (16)
\end{aligned}
$$

Proof Let $q>0$ be fixed and define $H=(1+\lambda q)\bigg(\int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds\bigg)^2$ so that from (1_λ) we obtain

$$
\begin{aligned}
&(x(t)-\lambda a(t))^2 = \bigg(\lambda\int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds + \int_{t-\epsilon}^{t} C(t,s)g(s,x(s))ds\bigg)^2\\
&\le \lambda(1+q^{-1})\bigg(\int_{-\infty}^{t-\epsilon} C(t,s)g(s,x(s))ds\bigg)^2 + H\\
&= \lambda(1+q^{-1})\bigg(-C(t,s)\int_s^t g(u,x(u))du\bigg|_{-\infty}^{t-\epsilon}\\
&\quad + \int_{-\infty}^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\bigg)^2 + H\\
&\text{(using (13) and } x\in\mathcal{P}_T)\\
&= \lambda(1+q^{-1})\bigg(-C(t,t-\epsilon)\int_{t-\epsilon}^{t} g(u,x(u))du\\
&\quad + \int_{-\infty}^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\bigg)^2 + H\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\bigg(\int_{t-\epsilon}^{t} g(u,x(u))du\bigg)^2\\
&\quad + 2(1+q^{-1})\bigg(\int_{-\infty}^{t-\epsilon} C_s(t,s)\int_s^t g(u,x(u))duds\bigg)^2 + H\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t} g^2(u,x(u))du + H\\
&\quad + 2(1+q^{-1})\int_{-\infty}^{t-\epsilon} C_s(t,s)ds\int_{-\infty}^{t-\epsilon} C_s(t,s)\bigg(\int_s^t g(u,x(u))du\bigg)^2 ds\\
&\le 2\lambda(1+q^{-1})C^2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t} g^2(u,x(u))du\\
&\quad + 2(1+q^{-1})\int_{-\infty}^{t-\epsilon} C_s(t,s)ds\, V(t,\epsilon)
\end{aligned}
$$

$$+ (1+q)\left(\int_{t-\epsilon}^{t} |C(t,s)|ds\right)^2 \left(K\|x\| + \sup_{0\le u\le T} |g(u,0)\|\right)^2,$$

as required.

Lemma 4.3 *If*

$$|g(t,x)| \le |x| \text{ for } |x| \ge L, \tag{17}$$

where L *is defined in* (12)*, then for any* $\gamma > 0$ *there is an* $M > 0$ *such that for any solution of* (1_λ) *in* $\mathcal{P}_T$ *we have*

$$\begin{aligned} V'(t,\epsilon) &\le Ma^2(t) + [\gamma+\beta-2]g^2(t,x(t)) + M \\ &\quad + \int_{t-\epsilon}^{t} [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]g^2(s,x(s))ds. \end{aligned} \tag{18}$$

Proof By Cauchy inequality, for any $\gamma > 0$, there is an $M > 0$ such that

$$2g(t,x)a(t) \le \gamma g^2(t,x) + Ma^2(t).$$

By (17), we may choose M so large that

$$-2g(t,x)x \le -2g^2(t,x) + M$$

for all $t \ge 0$ and $x \in \Re$. Now from (15) we have

$$\begin{aligned} V'(t,\epsilon) &\le \gamma g^2(t,x) + Ma^2(t) \\ &\quad - 2g^2(t,x) + M + C_s(t,t-\epsilon)\epsilon \int_{t-\epsilon}^{t} g^2(v,x(v))dv \\ &\quad + C(t,t-\epsilon)\int_{t-\epsilon}^{t} [g^2(t,x(t)) + g^2(v,x(v))]dv \\ &\quad + \int_{t-\epsilon}^{t} |C(t,s)|[g^2(t,x(t)) + g^2(s,x(s))]ds \\ &= Ma^2(t) + g^2(t,x)\left[\gamma - 2 + \epsilon C(t,t-\epsilon) + \int_{t-\epsilon}^{t} |C(t,s)|ds\right] + M \\ &\quad + \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + |C(t,s)|]g^2(s,x(s))ds \\ &\quad \text{by (9)} \\ &\le Ma^2(t) + g^2(t,x)[\gamma+\beta-2] + M \\ &\quad + \int_{t-\epsilon}^{t} [\epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon) + |C(t,s)|]g^2(s,x(s))ds, \end{aligned}$$

as required.

Lemma 4.4 *If* (17) *holds, if* $\epsilon \le T$*, and if* γ *is small enough then there is a* $\mu > 0$ *so that if* x *solves* (1_λ) *and* $x \in \mathcal{P}_T$ *then*

$$\int_0^T g^2(s,x(s))ds \le (M/\mu)\int_0^T a^2(s)ds + TM/\mu. \tag{19}$$

Proof We are going to integrate (18) from 0 to T and note that $0 = V(T,\epsilon) - V(0,\epsilon)$. First, we estimate the integral of the last term in (18) as follows. We have

$$\begin{aligned}
&\int_0^T \int_{t-\epsilon}^t [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]g^2(s,x(s))dsdt \\
&\le \int_{-\epsilon}^T \int_s^{s+\epsilon} [|C(t,s)| + \epsilon C_s(t,t-\epsilon) + C(t,t-\epsilon)]dtg^2(s,x(s))ds \\
&\le \alpha \int_{-\epsilon}^T g^2(s,x(s))ds \le 2\alpha \int_0^T g^2(s,x(s))ds.
\end{aligned}$$

With this information we now integrate (18) and obtain

$$\begin{aligned}
0 = V(T,\epsilon) - V(0,\epsilon) &\le M\int_0^T a^2(s)ds + TM \\
&\quad + \int_0^T [\gamma - 2 + \beta + 2\alpha]g^2(s,x(s))ds \\
&\le M\int_0^T a^2(s)ds - \mu\int_0^T g^2(s,x(s))ds + TM
\end{aligned}$$

since $\beta + 2\alpha < 2$ and γ can be made as small as we please.

Lemma 4.5 *Let the conditions of Lemma* 4.4 *hold and suppose there is a* $Q > 0$ *with*

$$\int_{-\infty}^{t-\epsilon} C_s(t,s))(t+T-s)^2 ds \le Q. \tag{20}$$

Then there is a $Q^* > 0$ *with* $V(t,\epsilon) \le Q^*$.

Proof We have

$$\begin{aligned}
V(t,\epsilon) &= \int_{-\infty}^{t-\epsilon} C_s(t,s)\bigg(\int_s^t g(u,x(u))du\bigg)^2 ds \\
&\le \int_{-\infty}^{t-\epsilon} C_s(t,s)(t-s)\int_s^t g^2(u,x(u))duds \\
&\le \int_{-\infty}^{t-\epsilon} C_s(t,s)(t-s)\bigg[\int_s^{t+T}(M/\mu)a^2(u)du + (t-s+T)TM/\mu\bigg]ds \\
&\le \int_{-\infty}^{t-\epsilon} C_s(t,s)(t+T-s)^2 ds[(M/\mu)\|a^2\| + TM/\mu]
\end{aligned}$$

from which the result follows.

Lemma 4.6 *Let the conditions of Lemma* 4.5 *hold. Then there exists a constant* $J > 0$ *such that* $\|x\| < J$ *whenever* x *is* T*-periodic solution of* (1_λ) *for* $0 < \lambda \le 1$.

Proof By (9) and (13), we have

$$\int_{-\infty}^{t-\epsilon} C_s(t,s)ds = C(t,t-\epsilon) \le \beta/\epsilon.$$

If $x \in \mathcal{P}_T$ solves (1_λ), then (19) holds, and by Lemma 4.5, $V(t, \epsilon) \le Q^*$. Now taking into account that (7) holds with $\eta < 1$, we obtain from (16) that

$$(x(t)-\lambda a(t))^2 \le 2(1+q^{-1})(\beta/\epsilon)Q^* + 2(1+q^{-1})(\beta^2/\epsilon)TM(\|a^2\|+1)/\mu + (1+q)(\eta\|x\|+\beta g^*)^2,$$

where $g^* = \|g(t,0)\|$. Since $\eta < 1$, we may choose $q > 0$ small enough so that $(1+q)\eta^2 < 1$, and hence, there exists $J > 0$ such that $\|x\| < J$. The proof is complete.

5 Continuity and Compactness

We select part of (10) and define the mapping $U : \mathcal{P}_T \to \mathcal{P}_T$ by $\phi \in \mathcal{P}_T$ which implies that

$$(U\phi)(t) = \int_{-\infty}^{t-\epsilon} C(t,s)g(s,\phi(s))ds. \tag{21}$$

Then U is well defined on P_T by (6). By a change of variable we have

$$(U\phi)(t) = \int_{-\infty}^{t} C(t,s-\epsilon)g(s-\epsilon,\phi(s-\epsilon))ds$$

with a fully convex kernel.

Lemma 5.1 *Suppose that* $\int_{-\infty}^{t-\epsilon}[|C(t,s)|+|C_t(t,s)|]ds$ *is bounded for all* $t \in \Re$. *Then* U *is continuous on* P_T *and for each* $J > 0$, $\Gamma = \{U(\phi) : \phi \in \mathcal{P}_T|, \|\phi\| \le J\}$ *is uniformly bounded and equicontinuous.*

Proof First, there is a J^* such that $\phi \in \Gamma$ implies that $|g(t,\phi(t))| \le J^*$ and there is a C^* with

$$\int_{-\infty}^{t-\epsilon}[|C(t,s)|+|C_t(t,s)|]ds \le C^*, \quad t \in \Re. \tag{22}$$

It is clear that $U\phi \in P_T$ by (6) and the argument following (10). We now show that U is continuous on P_T. If $\tilde{\phi}$, $\phi \in P_T$, then

$$\begin{aligned} |U(\phi)(t) - U(\tilde{\phi})(t)| &= \left| \int_{-\infty}^{t-\epsilon} C(t,s)g(s,\phi(s))ds - \int_{-\infty}^{t-\epsilon} C(t,s)g(s,\tilde{\phi}(s))ds \right| \\ &= \left| \int_{-\infty}^{t-s} C(t,s)\left[g(s,\phi(s)) - g(s,\tilde{\phi}(s))\right] ds \right|. \end{aligned} \tag{23}$$

Since g is uniformly continuous on $[0,T] \times \{x \in R : |x| \le \|\tilde{\phi}\| + 1\}$, for any $\epsilon > 0$, there exists $0 < \delta < 1$ such that $\|\phi - \tilde{\phi}\| < \delta$ implies $|g(s,\phi(s)) - g(s,\tilde{\phi}(s))| < \varepsilon$ for all $s \in [0,T]$. It follows from (23) that $\|U(\phi) - U(\tilde{\phi})\| \le \epsilon C^*$. Thus, F is continuous on P_T.

Next, for an arbitrary $\phi \in \Gamma$ we have

$$\frac{d}{dt}(U\phi)(t) = C(t,t-\epsilon)g(t-\epsilon,\phi(t-\epsilon)) + \int_{-\infty}^{t-\epsilon} C_t(t,s)g(s,x(s))ds.$$

and this derivative is bounded by

$$C(t,t-\epsilon)J^* + J^* \int_{-\infty}^{t-\epsilon} |C_t(t,s)|ds \le J^* \sup_{0\le t\le T} \|C(t,t-\epsilon)\| + J^*C^*.$$

This implies that Γ is equicontinuous. The uniform boundedness of Γ follows from the inequality

$$|U(\phi)(t)| \leq \int_{-\infty}^{t-\epsilon} |C(t,s)||g(s,\phi(s))|ds \leq J^*C^*.$$

6 Periodic Solutions

We will show the existence of T-periodic solutions of (1) by applying Theorem 2.1. By (10) and (11), we see that $x \in P_T$ is a solution of (1_λ) if and only if it is a fixed point of $B + \lambda A$.

Theorem 6.1 *If* (2)-(9), (12), (13), (17), (20), *and* (22) *hold with* $\epsilon \leq T$, *then* (1) *has a* T*-periodic solution.*

Proof Let the mappings A and B be defined in (10) and (11) with $S = P_T$. Then B is a contraction mapping with contraction constant η, and hence, $(I - B)^{-1}$ exists and is continuous on $(I - B)S = S$. By Lemma 5.1 and the Ascoli–Arzela theorem, we see that A is continuous and maps bounded sets into compact sets. It is also clear that $\lambda A(M) \subset (I - B)S$ for each closed convex subset $M \subset S$ and $\lambda \in [0, 1]$. Now by Lemma 4.6, the set of solutions to $x = Bx + \lambda Ax$ is bounded. Therefore, the alternative (i) of Theorem 2.1 must hold; that is, $B + A$ has a fixed point in P_T which is a T-periodic solution of (1).

Remark 6.1 Observe that the continuity of $C(t,s)$ with respect to s for $t-\epsilon < s < t$ is not required for fixed t. One may readily verify that the function $C(t,s)$ defined by $C(t,s) = k(t-s)^{-p}$ for $t - s \geq \epsilon$ and $C(t,s) = (t-s)^{-q}$ for $0 < t-s < \epsilon$ with $p > 2, 0 < q < 1, 0 < \epsilon \leq 1, k > 0$ satisfy all conditions of Theorem 6.1 for an appropriately chosen constant k.

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Analysis of an In-host Model for HIV Dynamics with Saturation Effect and Discrete Time Delay

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Abstract: We present an in-host HIV/AIDS model with saturation effect and a discrete time delay. It is shown that infection is endemic when $\mathcal{R}_0 > 1$ but dies out when $\mathcal{R}_0 < 1$. The switching phenomenon for the stable equilibria is observed when a discrete time delay is incorporated. The parameters that can control the disease transmission are also discussed. Numerical simulations are carried out to verify and support the analytical results and illustrate possible behavior scenarios of the model.

Keywords: *HIV/AIDS; stability; delay; switching.*

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1 Introduction

Throughout the ages and despite all medical and sanitary progress humankind has severely been afflicted by infectious diseases. The spread of human immune virus (HIV) is alarming today and becomes a global crisis of the modern era. No other disease engenders as much fear, revulsion, despair and utter helplessness as acquired immunodeficiency syndrome (AIDS). In a survey carried out in 2009, it was noted that about 33.3 million people are living with HIV/AIDS and 2.6 million people have newly been infected during this year only. Further, in this 2009 the number of AIDS-related deaths is estimated as 1.8 million [1]. The sexually active and risk groups such as truck drivers, commercial sex workers, bathhouse customers, and drinkers are known to play a central role in HIV population dynamics.

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HIV infection typically begins when an HIV particle containing two copies of the HIV RNA encounters a cell with a surface molecule called cluster designation 4 (CD4). Although these CD4+ T cells appear to be the main targets of HIV, other immune system cells with and without CD4 molecules on their surfaces are infected as well. Among these cells, monocytes and macrophages act as reservoirs of HIV by harboring a large amount of the virus without being killed. CD4+ T cells also serve as important reservoirs of HIV; a small proportion of these cells harbor HIV in a stable and inactive form. Normal immune processes may activate these cells, which leads to the production of new HIV virions [2]. HIV causes AIDS by destroying a type of white blood cells (T cells or CD4 cells) that the immune system must have to fight infection. AIDS is the final stage of HIV infection. It can take about 5 to 15 years for a person infected with HIV, even without treatment, to reach this stage [3]. In brief, HIV carries copies of its DNA and inserts this into the host cell's (mainly CD4+ T cells) DNA. The host cell after being stimulated to reproduce, it reproduces copies of HIV virus. Further the count of CD4+ T cells is a primary indicator used to measure progression of HIV infection. Chronic HIV infection causes gradual depletion of the CD4+ T cells' pool, and thus progressively compromises the host's immune response to opportunistic infections, leading to AIDS. Three main stages of disease progression after HIV virus is introduced into the body are as follows: the first one is the initial transient — a relatively short period of time when both the T cell population and the virus population increase greatly. This is followed by the second stage, clinical latency — a period of time when there are extremely large numbers of virus and T cells undergoing incredible dynamics, the overall result of which is an appearance of latency (disease steady state). The AIDS stage follows finally, and it is characterized by a drop in T cells to a very low number (or zero) and the virus grows without any bound and leads to death. In particular cell-cell fusions also have an important pathogenic role in vivo [4].

Wodarz and Nowak [5] showed through a diversity threshold model that evolution of virus can drive disease progression and also destruct the immune system. They also pointed out that mathematical models may be used to correlate the long-term immunological control of HIV and designing of therapy that convert a progressing patient into a state of long-term non-progression. Culshaw and Ruan [6] modified the model proposed by Perelson *et al.* [7] by introducing discrete time delay and studied the effect of time delay on the stability of equilibria. Further, Nelson and Perelson [8] developed and studied a set of models that include intercellular delays, combination antiretroviral therapy and the dynamics of both infected and uninfected T cells. The role of drug efficacy was highlighted along with general stability results of non-linear delay differential equation while Bachar and Dorfmayr [9] modeled the latent period and the delayed onset of positive treatment effects in the patients and carried out stability analysis of the system with numerical simulations depending on the size of the treatment-induced delay. On the other hand Banks and Bortz [10] studied cellular HIV infection models by using sensitivity methodology for non-linear delay system and carried out a typical sensitivity investigation. Zhou *et al.* [11] investigated the dynamics of a model of HIV infection of CD4+ T-cells with cure rate and obtained threshold conditions on $\mathcal{R}_0$ for persistence and periodic solutions. Mukandavire *et al.* [12] analyzed a mathematical model for HIV/AIDS with time delay due to incubation period and remarked that prolonged incubation period due to medical interventions may yield higher HIV/AIDS prevalence whereas Pastore [13] studied an HIV model incorporating mutation and discussed the effects of a virus attack on the human immune system in the presence of HIV infection and the break down of the immune system. Stilianakis and Schenzle [14] studied an intra-host dynamics of HIV-1 infection by incorporating the effect of the permanently increasing susceptibility of CD4+T cell clones and suggested that the HIV evolutionary speed plays a crucial role in the progression of disease. Li and Shu investigated an in-host viral model with intracellular delay [15] and observed that for $\mathcal{R}_0 > 1$, the infection persists and the chronic-infection equilibrium is locally as well as globally asymptotically stable. They further stated that without cell division no sustained oscillations regime exists even if in the presence of intracellular delays.

The interaction between HIV and the human immune system is a highly dynamic and multifactorial process and as a result it is essential to base therapeutic interventions on a more solid theoretical ground than it has been the case until now. Previous studies considered different aspects on models of HIV/AIDS, namely, effect of mutation, cellular HIV infection, inter-cellular delays, delay due to incubation period only to mention a few. To the best of our knowledge, none of the studies considered the saturation effects and latent class. For in-host models of HIV/AIDS to be more realistic, the saturation effects should be incorporated together with the effect of delay on the latently infected class. Actually saturation effect is applicable because of the presence of large number of virions. Hence we incorporate both these effects into the model system and our interest is to explore the effects of various parameters involved in the development of infection using analytic and numerical methods. The main thrust of the paper is to highlight the effect of delay and also the role of the rate of production of new virions.

The paper is organized as follows: in Section 2 we present the mathematical model and assumptions made in the formulation. Conditions for boundedness and existence of equilibria of the model are derived in Section 3. The basic reproductive number, $\mathcal{R}_0$, is also computed in this section. The local stability behaviour of the infection-free and endemic equilibria of the model in the absence of delay is discussed in Section 4 where global stability behaviour of the endemic equilibrium is also studied. In Section 5, stability switching behaviour is addressed. A brief discussion rounds up the paper in Section 6 with numerical simulations.

2 Mathematical Model

In [16] we note that some cells after being infected by the HIV, enter a latent class. Although these cells do not produce new virions while in this class, they are reactivated later to do so. On basis of these views, here we formulate an in-host HIV model with a latent infected class and incorporate a discrete time delay along with saturation effect.

The relationship between the virus and the uninfected cells is similar to the relationship between predator and prey in ecological problem and with this analogy βX is the functional response of the viruses to the uninfected cells. Further, we assume that the function that describes the rate at which uninfected cells are produced by the host is a decreasing function of virions. When the numbers of virions tend to zero then the uninfected cells are produced at a constant rate. Thus one can infer that virions affect the production of uninfected cells by the host. In other words uninfected cells are produced by the organism at the rate $\frac{c}{k+V}$ which depends on the number of virions in an organism. This is analogous to assuming that not all newborn cells are uninfected. Then infected cells and latent cells are produced by the organism at certain rates (vertical transmission of HIV/AIDS). Consequently, we consider uninfected cells being produced by the host at the rate $\frac{c}{k+V}$.

The following system of differential equations specifies the model

$$\begin{aligned}
\frac{dV}{dt} &= aY_1 - bV,\\
\frac{dX}{dt} &= \frac{c}{k+V} - dX - \beta XV,\\
\frac{dY_1}{dt} &= q_1\beta XV - f_1Y_1 + \delta Y_2\left(t-\tau\right),\\
\frac{dY_2}{dt} &= q_2\beta XV - f_2Y_2 - \delta Y_2,
\end{aligned} \tag{1}$$

where $V(t), X(t), Y_1(t), Y_2(t)$ represent the number of virions, number of uninfected target cells, number of productive infected cells and number of latent infected cells respectively at any time, in a host.

The virus is replicated by the infected cells, so its rate of production, a is assumed to be proportional to Y_1. Virions die at a specific rate b. The uninfected cells are produced by the host at a specific rate $\frac{c}{k+V}$. They die at a rate d, and become infected by the virus at a specific rate βV, entering Y_1 class and Y_2 class respectively,in proportions. A proportion q_1 of the infected cells become productively infected while the remaining proportion, $q_2 = (1-q_1)$ become latently infected. Productive infected cells and latent infected cells die at specific rates $f_1 = e_1 + d$ and $f_2 = e_2 + d$, respectively, where d is the natural death rate, e_1 and e_2 are the additional death rates due to infection. Only the Y_1 cells produce virions, and Y_2 cells move to the Y_1 class at a per capita rate δ. Further, $\tau\,(0 < \tau < \infty)$ is the delay due to the formation of productive infected class from the latent infected class. The parameter c is a constant and k is the half saturation constant.

3 Boundedness and Equilibria

In this section we first show that the solutions of model system (1) are bounded.

Lemma 3.1 *If $a < f_1$ then the solutions of model system (1) are bounded.*

Proof Define the function $U = V + X + Y_1 + Y_2$. Now

$$\dot{U} < \frac{c}{k} - bV - dX + (a - f_1)Y_1 - f_2Y_2.$$

For each $\lambda > 0$ the following inequality is fulfilled:

$$\dot{U} + \lambda U \leq \frac{c}{k} - (b-\lambda)V - (d-\lambda)X - (f_1 - a - \lambda))Y_1 - (f_2 - \lambda)Y_2.$$

If we choose $\lambda < \min\{b, d, f_1 - a, f_2\}$, then right hand side is bounded $\forall (V, X, Y_1, Y_2) \in \mathbb{R}^4_+$. Thus, $\dot{U} + \lambda U \leq \frac{c}{k}$. Applying a theorem on differential inequality we have

$$0 \leq U \leq \frac{c}{k\lambda} + \frac{1}{e^{\lambda t}} U(V(0), X(0), Y_1(0), Y_2(0))$$

and $0 \leq U \leq \frac{c}{k\lambda}$ for $t \to 0$. Thus, all solutions of system (1) enter the region

$$B = \{(V(t), X(t), Y_1(t), Y_2(t)) : U \leq \frac{c}{k\lambda} + \epsilon, \forall \epsilon > 0\}.$$

The assumption $a < f_1$ indicates that to keep the population under control, the production rate of virions must be below the specific death rate of productive infected cells. The system has two equilibrium points given by:

(1) $E_1(0, \frac{c}{kd}, 0, 0)$, (2) $E_2(\frac{aY_1^*}{b}, \frac{b^2c}{(kb+aY_1^*)(bd+a\beta Y_1^*)}, Y_1^*, \frac{f_1q_2Y_1^*}{f_2q_1+\delta q_1+\delta q_2})$, provided

$a > \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$, $Y_1^* = \frac{1}{2a^2\beta}\left[-ad(d+k\beta)+\sqrt{a^2b^2(d-k\beta)^2+\frac{4a^3bc\beta^2(f_2q_1+\delta q_1+\delta q_2)}{f_1(f_2+\delta)}}\right]$.

Latent infected cells Y_2, become productive infected cells Y_1, at a rate δ after a period of time $\frac{1}{\delta+f_2}$. Hence, adding contributions from cells $Y_1$and Y_2 cells, the basic reproductive number becomes $\mathcal{R}_0 = \frac{\beta ac}{bdkf_1}(q_1+q_2\frac{\delta}{\delta+f_2})$. The inequality $\mathcal{R}_0 > 1$ represents the same threshold condition as the expression $a > \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$. Hence E_2 exists only when $\mathcal{R}_0 > 1$.

4 Stability Analysis without Delay

In this section we investigate the local stability characteristics of the infection-free equilibrium point, E_1 and endemic equlibrium point, E_2 of the system. Global stability of E_2 is also discussed.

4.1 Local stability analysis

The Jacobian matrix of model system (1) is as follows:

$$J = \begin{pmatrix} -b & 0 & a & 0 \\ -\beta X - \frac{c}{(k+V)2} & -(d+\beta V) & 0 & 0 \\ q_1\beta X & q_1\beta V & -f_1 & \delta \\ q_2\beta X & q_2\beta V & 0 & -(f_2+\delta) \end{pmatrix}.$$

Theorem 4.1 *The infection-free equilibrium E_1 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof The characteristic equation of the Jacobian matrix of model system (1) at E_1 is $\lambda^3 + A\lambda^2 + B\lambda + C = 0$, where

$$A = b+f_1+f_2+\delta,\ B = (b+f_1)(f_2+\delta)+bf_1-\frac{ac\beta q_1}{kd},\ C = (\delta+f_2)(bf_1-\frac{ac\beta q_1}{kd})-\frac{ac\beta\delta q_2}{kd}.$$

Now $C > 0$ implies that $a < \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$. Again if $a < \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$ then $AB - C > 0$.

Further the inequality $\mathcal{R}_0 < 1$ represents the same threshold condition as the expression $a < \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$. Hence, the result follows by Routh-Hurwitz criterion for the equilibrium point E_1.

Theorem 4.2 *The equilibrium point E_2 is locally asymptotically stable if $D_i > 0$, for $i = 1, 2, 3, 4$; where $D_1 = P, D_2 = PQ - R, D_3 = P(QR - PS) - P^2$ and $D_4 = SD_3$.*

Proof The characteristic equation of the Jacobian matrix of model system (1) at E_2 is given by

$$\lambda^4 + P\lambda^3 + Q\lambda^2 + R\lambda + S = 0,$$

where $P = b + d + \delta + f_1 + f_2 + \beta V^*$,
$Q = bf_1 + (b + f_1)(d + \beta V^* + f_2 + \delta) + (d + \beta V^*)(f_2 + \delta) - aq_1\beta X^*$,
$R = bf_1(d+\beta V^* + f_2 + \delta) + (b + f_1)(d+\beta V^*)(f_2 + \delta) + aq_1\beta V^*\{\beta X^* + \frac{c}{(k+V^*)^2}\} - a(d + \beta V^*)q_1\beta X^* - aq_1\beta X^*(f_2 + \delta) - aq_2\delta\beta X^*$,
$S = a\{q_1\beta V^*(f_2 + \delta) + q_2\delta\beta V^*\}\{\beta X^* + \frac{c}{(k+V^*)^2}\} - a\delta q_2\beta X^*(d + \beta V^*) - aq_1\beta X^*(f_2 + \delta)(d + \beta V^*) + bf_1(d + \beta V^*)(f_2 + \delta)$.

Hence, by Routh–Hurwitz criterion E_2 is locally asymptotically stable if $D_i > 0$, for $i = 1, 2, 3, 4$; where $D_1 = P, D_2 = PQ - R, D_3 = P(QR - PS) - P^2$ and $D_4 = SD_3$.

4.2 Global stability analysis of the endemic equilibrium

We now show that the endemic equilibrium point $E_2(V^*, X^*, Y_1^*, Y_2^*)$ is globally asymptotically stable in the set B as its domain of attraction under certain conditions as follows. Define

$$W(V, X, Y_1, Y_2) = \frac{1}{2}(V - V^*)^2 + \frac{1}{2}(X - X^*)^2 + \frac{1}{2}(Y_1 - Y_1^*)^2 + \frac{1}{2}(Y_2 - Y_2^*)^2.$$

The time derivative of W along the solution of model system (1) is

$$\begin{aligned}
\dot{W} &= (V - V^*)\dot{V} + (X - X^*)\dot{X} + (Y_1 - Y_1^*)\dot{Y}_1 + (Y_2 - Y_2^*)\dot{Y}_2 \\
&= (V - V^*)(aY_1 - bV) + (X - X^*)(\frac{c}{k + V} - dX - \beta XV) \\
&+(Y_1 - Y_1^*)(q_1\beta XV - f_1Y_1 + \delta Y_2) + (Y_2 - Y_2^*)(q_2\beta XV - f_2Y_2 - \delta Y_2) \\
&\le -b(V - V^*)^2 - (d + \beta V^*)(X - X^*)^2 - f_1(Y_1 - Y_1^*)^2 - (f_2 + \delta)(Y_2 - Y_2^*)^2 \\
&+\frac{c}{k}(\frac{1}{k + V^*} + \frac{\beta}{d})\,|V - V^*|\,|X - X^*| + (a + \frac{q_1\beta c}{dk})\,|V - V^*|\,|Y_1 - Y_1^*| \\
&+\frac{q_2\beta c}{dk}\,|V - V^*|\,|Y_2 - Y_2^*| + q_1\beta V^*\,|X - X^*||Y_1 - Y_1^*| + q_2\beta V^*\,|X - X^*|\,|Y_2 - Y_2^*| \\
&+\delta\,|Y_1 - Y_1^*|\,|Y_2 - Y_2^*| \\
&= -a_{11}(V - V^*)^2 - a_{22}(X - X^*)^2 - a_{33}(Y_1 - Y_1^*)^2 - a_{44}(Y_2 - Y_2^*)^2 \\
&+2a_{12}\,|V - V^*|\,|X - X^*| + 2a_{13}\,|V - V^*|\,|Y_1 - Y_1^*| + 2a_{14}\,|V - V^*|\,|Y_2 - Y_2^*| \\
&+2a_{23}\,|X - X^*|\,|Y_1 - Y_1^*| + 2a_{24}\,|X - X^*|\,|Y_2 - Y_2^*| + 2a_{34}\,|Y_1 - Y_1^*|\,|Y_2 - Y_2^*| \\
&= -X^TMX,
\end{aligned} \tag{2}$$

where $X^T = \{|V - V^*|\,, |X - X^*|\,, |Y_1 - Y_1^*|\,, |Y_2 - Y_2^*|\}$ and $M = [a_{ij}]_{4\times4}$. Elements of the matrix M are given by: $a_{11} = b$, $a_{22} = d + \beta V^*$, $a_{33} = f_1$, $a_{44} = f_2 + \delta$, $a_{12} = a_{21} = -\frac{c}{2k}\left(\frac{1}{V^*+k} + \frac{\beta}{d}\right)$, $a_{13} = a_{31} = -\frac{1}{2}(a + \frac{q_1\beta c}{dk})$,$a_{14} = a_{41} = -\frac{q_2\beta c}{2dk}$,$a_{23} = a_{32} = -\frac{q_1\beta V^*}{2}$,$a_{24} = a_{42} = -\frac{q_2\beta V^*}{2}$, $a_{34} = a_{43} = -\frac{\delta}{2}$.

Here, M is positive definite if the following inequalities

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} > 0$$

hold simultaneously.

Theorem 4.3 *Suppose $a < f_1$, E_2 is globally asymptotically stable, if M is positive definite where $M = [a_{ij}]_{4\times 4}$.*

Proof Since B is a global attractor we may restrict our attention to solutions initiating in $\overset{\circ}{B}$. From the above inequalities, the right hand side of equation (2), which is considered as a quadratic form in the variables $|V - V^*|, |X - X^*|, |Y_1 - Y_1^*|, |Y_2 - Y_2^*|$is negative definite for $(V, X, Y_1, Y_2) \in\overset{\circ}{B}$. Hence $\dot{W}(V, X, Y_1, Y_2)$ is negative definite about E_2 and consequently $W(V, X, Y_1, Y_2)$ is a Lyapunov function for $(V, X, Y_1, Y_2) \in\overset{\circ}{B}$. This completes the proof.

5 Stability Analysis with Delay

In this section, dynamical behaviour of the system near the equilibrium points E_1 and E_2 are discussed in the presence of delay.

5.1 Local stability analysis

Before stating the theorems we require the following result in Kuang [17]. For a scalar differential equation

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} X(t) + \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} X(t-\tau) = 0, \quad a_n \neq 0, \quad n \geq m.$$

The characteristic equation takes the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \qquad P(\lambda) = \sum_{k=0}^{n} a_k\lambda^k, \quad Q(\lambda) = \sum_{k=0}^{m} b_k\lambda^k. \tag{3}$$

Theorem 5.1 *Consider equation (3), where $P(\lambda)$ and $Q(\lambda)$ are analytic functions in $Re\lambda > 0$ and satisfy the following conditions:*

(i) *$P(\lambda)$and $Q(\lambda)$ have no common imaginary root;*

(ii) *$\bar{P}(-iy) = P(iy)$, $\bar{Q}(-iy) = Q(iy)$ for real y; '-' denotes complex conjugate;*

(iii) *$P(0) + Q(0) \neq 0$;*

(iv) $\limsup\,[|Q(\lambda)/P(\lambda)| : |\lambda| \to \infty, Re\lambda \geq 0] < 1$;

(v) *$F(y) = |P(iy)|^2 - |Q(iy)|^2$ for real y has at most a finite number of real zeros.*

Then the following statements are true:

(a) *If $F(y) = 0$ has no positive roots, then no stability switch may occur;*

(b) *If $F(y) = 0$ has at least one positive root and each of them is simple, then as τ increases, a finite number of stability switches may occur, and eventually the considered equation becomes unstable.*

Now we state and prove our results.

Theorem 5.2 *Stability switches occur or do not occur near the equilibrium point E_1 as τ increases when $\mathcal{R}_0 > 1$ or $\mathcal{R}_0 < 1$ respectively.*

Proof The characteristic equation of the system with delay at E_1 is given by

$$\lambda^4 + \epsilon_1\lambda^3 + \eta_1\lambda^2 + \mu_1\lambda + \omega_1 + \zeta_1\lambda e^{-\lambda\tau} + \rho_1 e^{-\lambda\tau} = 0,$$

where $\epsilon_1 = b + d + f_1 + f_2 + \delta$, $\eta_1 = bd + (b+d)(f_1 + f_2 + \delta) + f_1(f_2 + \delta) - \frac{aq_1\beta c}{kd}$,
$\mu_1 = bd(f_1 + f_2 + \delta) + f_1(b+d)(f_2 + \delta) - \frac{aq_1\beta c}{kd}(d + f_2 + \delta)$,
$\omega_1 = (f_2 + \delta)(bdf_1 - \frac{aq_1\beta c}{kd}), \zeta_1 = -\frac{aq_2\delta\beta c}{kd}$ and $\rho_1 = -\frac{aq_2\delta\beta c}{k}$.

Again this equation is of the form

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0,$$

where $P(\lambda) = \lambda^4 + \epsilon_1\lambda^3 + \eta_1\lambda^2 + \mu_1\lambda + \omega_1$ and $Q(\lambda) = \zeta_1\lambda + \rho_1$. Clearly $P(\lambda)$ and $Q(\lambda)$ have no common imaginary root. Obviously $\bar{P}(-iy) = P(iy)$, $\bar{Q}(-iy) = Q(iy)$ for real y. Also $P(0) + Q(0) \neq 0$. Now, $\limsup[|Q(\lambda)/P(\lambda)| : |\lambda| \to \infty, Re\lambda \geq 0] < 1$,
$F(y) = |P(iy)|^2 - |Q(iy)|^2$
$= y^8 + (\epsilon_1^2 - 2\eta_1)y^6 + (\eta_1^2 + 2\omega_1 - 2\epsilon_1\mu_1)y^4 + (\mu_1^2 - \zeta_1^2 - 2\eta_1\omega_1)y^2 + (\omega_1^2 - \rho_1^2)$.

Putting $y^2 = z$ we get

$$z^4 + (\epsilon_1^2 - 2\eta_1)z^3 + (\eta_1^2 + 2\omega_1 - 2\epsilon_1\mu_1)z^2 + (\mu_1^2 - \zeta_1^2 - 2\eta_1\omega_1)z + (\omega_1^2 - \rho_1^2) = 0.$$

We have $(\omega_1^2 - \rho_1^2) > 0$ which implies that $a < \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$. Consequently, $F(y) = 0$ has a positive root when $a > \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$, which is simple. Further when $a < \frac{bdkf_1(f_2+\delta)}{\beta c(f_2q_1+\delta q_1+\delta q_2)}$, $F(y) = 0$ does not have a positive root. The result follows by the application of Theorem 5.1.

Theorem 5.3 *The endemic equilibrium E_2 remains stable if $\sigma > 1$ and switches from its stability to instability if $\sigma < 1$, where*

$$\sigma = \frac{(f_2 + \delta)^2[bf_1(d + \beta V^*) + aq_1\beta[\frac{cV^*}{(k+V^*)^2} - dX^*]]^2}{[aq_2\beta\delta[\frac{cV^*}{(k+V^*)^2} - dX^*]]^2}.$$

Proof Proceeding along the lines of proof of Theorem 5.2 we obtained the result.

6 Numerical Simulations and Discussion

We modeled the interaction inside the body between the HIV virus and uninfected target cells. A virus particle (or virion) does absolutely nothing on its own. Virion hijacks the machinery of the cell for its own replication when it gets entry to the host cell. It then leaves the cell, and the process is repeated. In this way our immune system loses its control over our body. In this study βX is the functional response of the virus to the infected cell. Saturation effect due to virions and the effect of time delay due to production of new virions from the latent infected class to the productive infected class are also considered. The current study does not consider the effects of immune response but this will be considered elsewhere. We now explain the dynamical behavior of the model using hypothetical set of parameter values for different situations and if experimental data are available, one can give more insight of the dynamics of our model. All numerical simulations are generated using MATLAB® (The Mathworks, Inc., Version 7.10.0.499, R2010a).

Figure 1 demonstrates that infection free equilibrium exists and is locally asymptotically stable as shown in Theorem 4.1. The parameter values used are: $a = 0.5$ per month; $b = 1$ per month; $c = 10$ per month; $k = 1$; $d = 1$ per month; $\beta = 0.2$ per month; $q_1 = 0.3$; $f_1 = 1$ per month; $\delta = 2$ per month; $q_2 = 0.7$; $f_2 = 0.1$ per month. Here $\mathcal{R}_0 = 0.97$. In other words infection dies out in this situation.

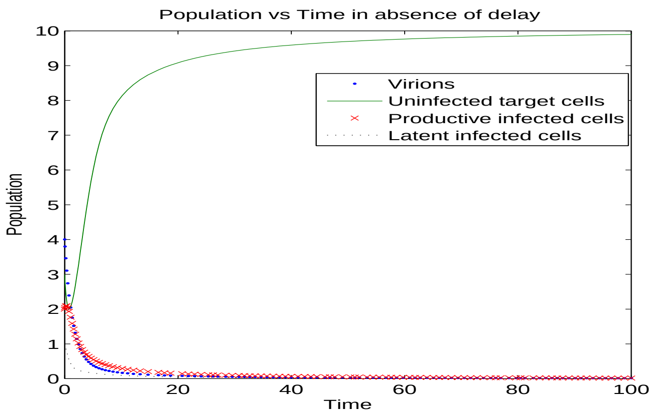

Figure 1: The figure shows that the infection-free equilibrium is locally asymptotically stable.

Existence of the endemic equilibrium is shown in Figure 2. Conditions for local asymptotic stability of this equilibrium are obtained in Theorem 4.2. Figure 2 is generated with the choice of the parameter values $a = 5$ per month; $b = 1$ per month; $c = 10$ per month; $k = 1$; $d = 1$ per month; $\beta = 0.2$ per month; $q_1 = 0.3$; $f_1 = 1$ per month; $\delta = 2$ per month; $q_2 = 0.7$; $f_2 = 0.1$ per month. It is important to note that in this case $\mathcal{R}_0 = 9.7$. Hence infection is endemic in nature and prevails in the human body.

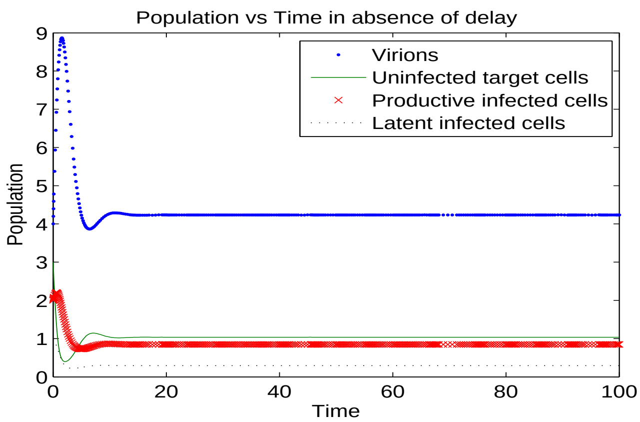

Figure 2: The figure demonstrates that the endemic equilibrium is locally asymptotically stable.

With the choice of parameter values $a = .5$ per month; $b = 1$ per month; $c = 10$ per month; $k = 1$; $d = 1$ per month; $\beta = 0.2$ per month; $q_1 = 0.3$; $f_1 = 1$ per month; $\delta = 2$ per month; $q_2 = 0.7$; $f_2 = 0.1$ per month and $\tau = 18$ months, we note from Figure 3 that infection-free equilibrium exists and is locally asymptotically stable without any stability switching as shown in Theorem 5.2. This implies no possibility of infection occurs. Further in this case $\mathcal{R}_0 = 0.97$.

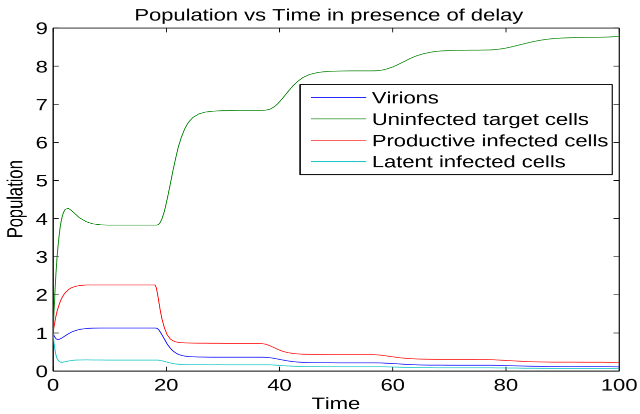

Figure 3: The figure depicts that the infection-free equilibrium remains stable in the presence of delay

With the following choice of parameter values: $a = 5$ per month; $b = 1$ per month; $c = 10$ per month; $k = 1$; $d = 1$ per month; $\beta = 0.2$ per month; $q_1 = 0.3$; $f_1 = 1$ per month; $\delta = 2$ per month; $q_2 = 0.7$; $f_2 = 0.1$ per month and $\tau = 18$ months, Figure 4 is obtained. From this set of values we get $\sigma > 1$ and $\mathcal{R}_0 = 9.7$. The figure shows that the system remains asymptotically stable through slight oscillations. Again by increasing τ no sustained oscillations are observed for the system. Biologically the disease prevails within the human body with slight ups and downs.

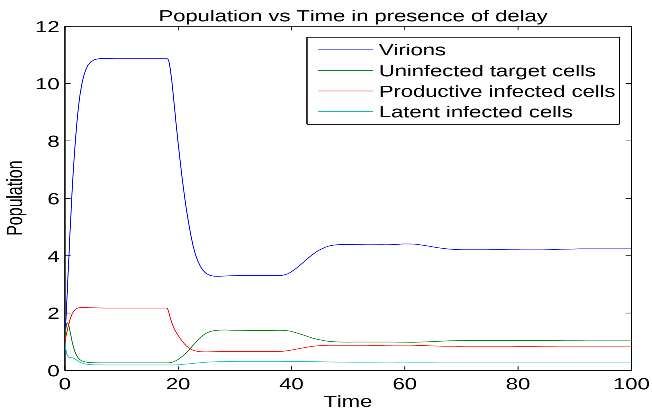

Figure 4: The figure shows that the endemic equilibrium, in the presence of delay, ultimately remains stable when $\sigma > 1$.

Figure 5 is obtained by using the following parameter values: $a = 5$ per month; $b = 1$ per month; $c = 10$ per month; $k = 1$; $d = 1$ per month; $\beta = 200$ per month; $q_1 = 0.3$; $f_1 = 1$ per month; $\delta = 2$ per month; $q_2 = 0.7$; $f_2 = 0.1$ per month and $\tau = 18$ months. This set of values of the parameters gives $\sigma < 1$. This figure depicts that the system switches from its stability to instability to stability etc. in the presence of delay. On the basis of Figure 5, we may interpret biologically that the disease spreads randomly with unusual manner within the individual. It is important to note that $\mathcal{R}_0 \gg 1$ in this situation and β plays a vital role.

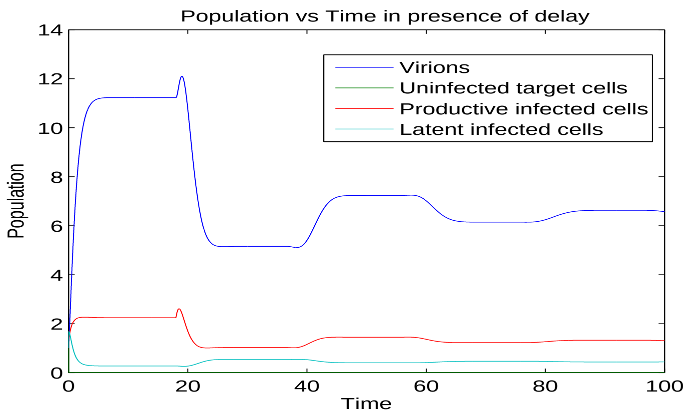

Figure 5: The figure shows that the endemic equilibrium becomes unstable in the presence of delay when $\sigma < 1$.

From the analysis and numerical simulations we observe that endemic establishment of the infection occurs for $\mathcal{R}_0 > 1$ whereas the infection dies out when $\mathcal{R}_0 < 1$. Again, if the rate of production of virus, a, is dominated by the specific death rate of productive infected cells, f_1, then the population cannot be explored although infection remains there. In brief, from the analysis we observed that the rate of production of virus through replication by infected cell has an important role over the stability of the system. Thus, we may reduce HIV infection that leads to AIDS by controlling the rate of production of virus through replication. It is important to note that delay has destabilizing effect on the system in the presence of latent class. Hence the latent class has a major role on the dynamics of the system which is clear from our analytical findings and numerical simulations.Saturation effects give more intricate dynamics also. Further β, δ and q_1 are also the key parameters of the system.Hence, in order to restore the outbreak of the disease, we have to take some control measures on these parameters with great care.

A definite AIDS cure is still under research. The current model can be extended by incorporating immune response, age structure and other modifications. We hope that some interesting results will be found in near future to save us from this fatal disease.

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Existence of a Regular Solution to Quasilinear Implicit Integrodifferential Equations in Banach Space

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Abstract: In the present work, we establish first the existence of a unique local mild solution using contraction mapping theorem and after that the existence of a local classical solution to a class of quasilinear implicit integrodifferential equations in a Banach space. Finally, we demonstrate one application of the results established.

Keywords: *quasilinear evolution equation; mild solution; classical solution; contraction mapping theorem; C_0-semigroups.*

Mathematics Subject Classification (2000): 34G20, 35D10, 35L55.

1 Introduction

Let X and Y be two real Banach spaces such that the embedding $Y \hookrightarrow X$ is dense and continuous. Consider the following quasilinear implicit integrodifferential equation in X

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{1}$$

where $0 < T < \infty$, $A(t, u)$ is a linear operator in X for each u in an open subset W of X, G is a nonlinear Volterra integral operator defined from $C(J, X)$ into $C(J, X)$ where $J = [0, T]$ and the nonlinear map f is defined from $J \times W \times W$ into X. We follow the approach of T. Kato [13, 16, 17] to establish the existence of a unique *classical solution* to (1) under the assumptions (H1)-(H8) to be stated in the next section.

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Crandall and Souganidis [4] have used a different method to prove the existence, uniqueness and continuous dependence of a continuously differentiable solution to the quasilinear evolution equation

$$\frac{du(t)}{dt} + A(u(t))u(t) = 0, \quad 0 < t \leq T, \quad u(0) = u_0,$$

under similar assumptions considered by T. Kato [16].

T. Kato [16] has proved two general theorems on the nonhomogeneous quasilinear evolution equation

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = f(t, u(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{2}$$

one on the existence and uniqueness, and the other on the continuous dependence of a solution on the initial data. Also, he has shown that these theorems are applicable to the different kinds of quasilinear differential equations such as Korteweg-de Vries equation, Navier-Stokes equation and Euler equation, equations for compressible fluids, magnetohydrodynamics equations, coupled Maxwell and Dirac equations etc. The results in [16] are based on the theory of linear 'hyperbolic' equation which was developed in [14, 15].

Murphy [19] constructed a family of approximate solutions to the homogeneous quasilinear evolution equation

$$\frac{du(t)}{dt} + A(t, u(t))u(t) = 0, \quad 0 < t \leq T, \quad u(0) = u_0. \tag{3}$$

He showed that the approximate solution converges to a "limit solution" and this "limit solution" becomes a unique solution to (3) under certain additional assumptions. [12] has extended the result in [19] to the nonhomogeneous equation (2) under slightly more general conditions than those of [16].

In [2], Bahuguna had used the method of lines (also known as Rothe's method) and the techniques of Crandall and Souganidis [4] to prove the existence, uniqueness and continuous dependence of a *strong solution* to the quasilinear explicit integrodifferential equaton

$$\frac{du(t)}{dt} + A(u(t))u(t) = K(u)(t) + f(t), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in a Banach space X whose dual X^* is assumed to be uniformly convex under the additional assumption of compactness on the embedding of Y in X and where K is the nonlinear Volterra operator. Using technique of [2], Bahuguna and Shukla [3] have established similar results for the quasilinear implicit integrodifferential equation

$$\frac{du(t)}{dt} + A(u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in Banach spaces. Further, using same technique of papers [2] and [3], Dubey [5] has established the similar result for the equation (1).

For the application of analytic semigroups to related quasilinear evolution equations we refer to Amann [1], Lunardy [18] while for the applications of fixed point theorems the reader may refer to Kartsatos [9, 10], Kartsatos and Parrott [11] and references cited therein.

Dubey [6] has established the local existence and uniqueness of a classical solution of an abstract second order integrodifferential equation in a Banach space by using the theory of analytic semigroups and contraction mapping theorem . The continuation of classical solution, the maximal interval of the existence and the global existence of the classical solution have been also studied. Pandey, Ujlayan and Bahuguna [8]considered an abstract semilinear hyperbolic integrodifferential equation and used the theory of resolvent operators to establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. Under some additional conditions on the nonlinear maps they also proved the existence of a classical solution.

The plan of the paper is as follows. In the second section, we collect some preliminaries, notations and some results which easily follow from the hypotheses. In the third section, first, we establish the existence of a unique local mild solution using contraction mapping theorem and also the existence of a local classical solution to (1). Finally, in the last section, we demonstrate one application of the results established in earlier sections.

2 Preliminaries

Let X and Y be as in the earlier section. The norm in any Banach space Z is denoted by $\|.\|_Z$. $\bar{B}_Z(r, z_0)$ is the closure of the open ball $B_Z(r, z_0) = \{z \in Z \mid \|z - z_0\|_Z < r\}$ with radius r and center at z_0 in the Banach space Z. The space of all bounded linear operators from a Banach space X to a Banach space Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. Let J denote the interval $[0, T]$. The space $C^m(J, Z)$ represents the space of all m-times continuously differentiable functions defined from J into Z, $m = 1, 2, ...$; endowed with the supremum norm

$$\|u\|_{C^m(J,Z)} = \sum_{1 \le i \le m} \sup_{t \in J} \|u^{(i)}(t)\|, \quad u \in C^m(J, Z),$$

where $u^{(i)}$ denotes the ith derivative of u with $u^{(0)} = u$. Let W be a subset of X. A family $\{A(t, w) : (t, w) \in J \times W\}$, of infinitesimal generators of C_0-semigroups $S_{t,w}(s), s \ge 0$ on X is called stable if there exists real numbers $M \ge 1$ and ω, known as *stability constants*, such that

$$\rho(A(t, w)) \supset (\omega, \infty) \quad \text{for} \quad (t, w) \in J \times W,$$

where $\rho(A(t, w))$ is the resolvent set of $A(t, w)$ and

$$\left\| \prod_{j=1}^{k} R(\lambda; A(t_j, w_j)) \right\|_{B(X)} \le M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega$$

and every finite sequence

$$0 \le t_1 \le t_2 \le .... \le t_k \le T, \quad w_j \in W, \quad 1 \le j \le k.$$

For a linear operator S in X, by the part $\tilde{S}$ of S in a subspace Y of X, we mean a linear operator $\tilde{S}$ with domain $D(\tilde{S}) = \{x \in D(S) \cap Y \mid Sx \in Y\}$ and values $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$.

Let $S_{t,w}(s)$, $s \ge 0$, be the C_0-semigroup generated by $A(t, w)$, $(t, w) \in J \times W$. A subset Y of X called $A(t, w)$-admissible if Y is an invariant subspace of operator $S_{t,w}(s)$, $s \ge 0$, and the restriction of $S_{t,w}(s)$ to Y is a C_0-semigroup in Y.

For more details of the above mentioned notions, one may refer to the chapters 5 and 6 in Pazy [7]. On the family of operators $\{A(t,w) : (t,w) \in J \times W\}$, we make the same assumptions $(\tilde{H}1)$-$(\tilde{H}4)$ considered in §6.6.4 in Pazy [7] for the homogeneous quasilinear evolution equation, as restated below.
(H1) The family $\{A(t,w) : \ (t,w) \in J \times W\}$ is stable.
(H2) Y is $A(t,w)$-admissible for $(t,w) \in J \times W$ and the family $\{\tilde{A}(t,w) : \ (t,w) \in J \times W\}$ of the parts of $A(t,w)$ in Y is stable in Y.
(H3) For $(t,w) \in J \times W$, $D(A(t,w)) \supset Y$, $A(t,w)$ is a bounded linear operator from Y to X, and the map $t \mapsto A(t,w)$ is continuous in $B(Y,X)$ with associated norm $\|.\|_{Y \to X}$ for every $w \in W$.
(H4) There is a positive constant L_A such that

$$\|A(t,w_1) - A(t,w_2)\|_{Y \to X} \le L_A \|w_1 - w_2\|_X$$

for every $w_1, w_2 \in W$ and $0 \le t \le T$.

A two parameter family of bounded linear operators $U(t,s)$, $0 \le s \le t \le T$, on X is called an *evolution system* if the following two conditions are satisfied:

(i) $U(s,s) = I$ and $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$.

(ii) The map $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

If $u \in C(J,X)$ has values in W and the family $\{A(t,w) : (t,w) \in J \times W\}$ of the operators satisfies the assumptions(H1)-(H4) then there exists a unique evolution system $U_u(t,s)$ in X satisfying

$$\text{(i)} \quad \|U_u(t,s)\|_X \le M e^{\omega(t-s)} \tag{4}$$

for $0 \le s \le t \le T$, where M and ω are the stability constants;

$$\text{(ii)} \quad \frac{\partial^+}{\partial t} U_u(t,s) w|_{t=s} = A(s,u(s))w \tag{5}$$

for $w \in Y$ and $0 \le s \le T$;

$$\text{(iii)} \quad \frac{\partial}{\partial s} U_u(t,s) w = -U_u(t,s) A(s,u(s)) w \tag{6}$$

for $\omega \in Y$ and $0 \le s \le T$.

Further, there exists a positive constant C_0 such that for every $u, v \in C(J,X)$ with values in W and for every $y \in Y$, we have

$$\|U_u(t,s)y - U_v(t,s)y\|_X \le C_0 \|y\|_Y \int_s^t \|u(\tau) - v(\tau)\|_X d\tau. \tag{7}$$

For details of the above mentioned results, one may refer to Theorem 6.4.3 and Lemma 6.4.4 in Pazy [7].

We further assume that
(H5) For every $u \in C(J,X)$ satisfying $u(t) \in W$ for $t \in J$, we have

$$U_u(t,s)Y \subset Y, \quad \text{for} \quad t,s \in J \quad \text{and} \quad s \le t$$

and $U_u(t,s)$ is strongly continuous in Y for $s,t \in J$ and $s \le t$.
(H6) Closed convex subsets of Y are also closed in X.
(H7) The nonlinear map $G : C(J,X) \to C(J,X)$ satisfy the following:

(a) For all $u, v \in \bar{B}_{C(J,X)}(\tilde{u_0}, r)$,

$$\|G(u) - G(v)\|_{C(J,X)} \leq \mu_G(r)\|u - v\|_{C(J,X)},$$

where $\mu_G(r)$ is a nonnegative nondecreasing function and $\tilde{u_0} \in C(J, X)$ is defined by $\tilde{u}_0 = u_0$ for all $t \in J$.

(b) The subspace $C(J, Y)$ of space $C(J, X)$ is an invariant subspace of the map G, i.e. the map $G : C(J, Y) \to C(J, Y)$ satisfies

$$\|G(u)(t)\|_Y \leq \lambda_G(r) \quad \text{for} \quad u \in \bar{B}_Y(u_0, r),$$

where $\lambda_G(r)$ is a nonnegative nondecreasing function. In particular, we may take operator G as a Volterra operator defined by

$$G(u)(t) = \int_0^t a(t - s)k(s, u(s))ds,$$

where a is a real valued continuous function defined on J and k is defined on $J \times Y$ into Y and $\|k(t, w)\|_Y \leq C_k$ for every $(t, w) \in J \times Y$. Clearly, the map G satisfies (b).
(H8) The nonlinear map $f : J \times W \times W \to X$ satisfies

(a) For $(t, u, v) \in J \times (W \cap Y) \times (W \cap Y)$ and $f(t, u, v) \in Y$, we have

$$\|f(t, u, v)\|_Y \leq \lambda_f(r)$$

for all $(t, u, v) \in J \times W \times W$ with $\|u\|_Y + \|v\|_Y \leq r$, where $\lambda_f(r)$ is a nonnegative nondecreasing function.

(b) In both $Z = X,\ Y$, the map f satisfies the Lipschitz like condition

$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\|_Z \leq \mu_f(r)[|\phi(t_1) - \phi(t_2)| + \|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z],$$

for all $t_1, t_2 \in [0, T]$ and all $u_i, v_i \in \bar{B}_Y(u_0, r)$, $i = 1, 2$, where ϕ is a real-valued continuous function of bounded variation on $[0, T]$ and $\mu_f(r)$ is a nonnegative nondecreasing function.

By a *mild solution* to (1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ with values in W satisfying the integral equation

$$u(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s)f(s, u(s), G(u)(s))ds, \quad t \in J. \tag{8}$$

By the *Classical solution* u to (1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ such that $u(t) \in Y \cap W$ for $t \in (0, T]$, $u \in C^1((0, T], X)$ and satisfies (1) in X. If there exists a T' with $0 < T' \leq T$ and a function $u \in C(J', X)$, where $J' = [0, T']$ such that u is a mild (classical) solution to (1) on J', then u is called a *local mild (classical)* solution to (1).

3 Main Result

In this section, we prove the following result.

Theorem 3.1 *Suppose that $u_0 \in Y$ and the family $\{A(t, w)\}$ of linear operators for $t \in J = [0, T]$ and $w \in W = \{y \in Y : \|y - u_0\|_Y \leq r\}$, for fixed $r > 0$, satisfy the assumptions (H1)-(H6) and $A(t, w)u_0 \in Y$ satisfies*

$$\|A(t, w)u_0\|_Y \leq C_A \tag{9}$$

for all $(t, w) \in J \times W$.

Further, suppose that the nonlinear maps G *and* f *satisfy (H7) and (H8), respectively. Then, there exists a unique local classical solution to (1).*

Proof First, we establish the existence of a unique local mild solution to (1). We note that from assumption (H5), it follows that

$$\|U_u(t,s)\|_{B(Y)} \leq C_1 \tag{10}$$

for $s \leq t$, $s, t \in J$ and every $u \in C(J, X)$ with values in W. We choose

$$T_0 = \min\left\{T, \frac{r}{2C_A C_1}, \frac{r}{2C_1\lambda_f(R_1)}, \frac{1}{2P}\right\}, \tag{11}$$

where

$$P = C_0\|u_0\|_Y + Me^{\omega T}\mu_f(R_1)(1+\mu_G(r)) + C_0\lambda_f(R_1)T$$

and

$$R_1 = r + \|u_0\|_Y + \lambda_G(r).$$

Let S be the subset of $C(J_0, X)$ defined by

$$S = \{u \in C(J_0, X) \mid u(0) = u_0, \text{ and } u(t) \in W \text{ for } t \in J_0\},$$

where $J_0 = [0, T_0]$. From (H6), it follows that S is a closed convex subset of $C(J_0, X)$. Next, we define a mapping $F : S \to S$ by

$$Fu(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds \tag{12}$$

and check that F is well defined. Clearly, $Fu(0) = u_0$, $Fu \in C(J_0, X)$ and (H5) implies that $Fu(t) \in Y$ for $t \in J_0$. It remains to show that $\|Fu(t) - u_0\|_Y \leq r$ for $t \in J_0$. Now,

$$Fu(t) - u_0 = U_u(t,0)u_0 - u_0 + \int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds. \tag{13}$$

Integrating (6) in X from 0 to t, we get

$$U_u(t,0)u_0 - u_0 = \int_0^t U_u(t,\tau)A(\tau, u(\tau))u_0 d\tau. \tag{14}$$

Using (9) and (10) in (14), we obtain

$$\|U_u(t,0)u_0 - u_0\|_Y \leq C_1 C_A T_0 \leq \frac{r}{2}. \tag{15}$$

Further, using (10) and (H8), we get

$$\left\|\int_0^t U_u(t,s)f(s, u(s), G(u)(s))ds\right\|_Y \leq C_1\lambda_f(R_1)T_0 \leq \frac{r}{2}, \tag{16}$$

since $\|u(s)\|_Y + \|G(u)(s)\|_Y \leq R_1$. Using (15) and (16) in (13), we see that F is well defined. For $u, v \in S$, we have

$$\begin{aligned} Fu(t) - Fv(t) &= (U_u(t,0) - U_v(t,0))u_0 \\ &\quad + \int_0^t [U_u(t,s)f(s, u(s), G(u)(s)) - U_v(t,s)f(s, v(s), G(v)(s))]ds \\ &= T_1 + T_2, \end{aligned} \tag{17}$$

where T_1 and T_2 represent the first and second terms on the right hand side of (17). We use (7) to obtain $\|T_1\|_X \le C_0\|u_0\|_Y T_0\|u-v\|_{C(J_0,X)}$. Further, from (H7), (H8) and (7) it follows that

$$\begin{aligned}\|T_2\|_X &\le \left\|\int_0^t U_u(t,s)[f(s,u(s),G(u)(s)) - f(s,v(s),G(v)(s))]ds\right\|_X \\ &\quad + \left\|\int_0^t [U_u(t,s)-U_v(t,s)]f(s,v(s),G(v)(s))ds\right\|_X \\ &\le [Me^{\omega T}\mu_f(R_1)(1+\mu_G(r)) + C_0\lambda_f(R_1)T]T_0\|u-v\|_{C(J_0,X)}.\end{aligned}$$

Also,

$$\begin{aligned}\|f(s,u(s),G(u)(s)) &- f(s,v,(s),G(v)(s)\|_X \\ &\le \mu_f(R_1)[\|u(s)-v(s)\|_X + \|G(v)(s)-G(u)(s)\|_X] \\ &\le \mu_f(R_1)[\|u-v\|_{C(J_0,X)} + \|G(u)-G(v)\|_{C(J_0,X)}] \\ &\le \mu_f(R_1)(1+\mu_G(r))\|u-v\|_{C(J_0,X)}.\end{aligned}$$

Hence, from (17), we have

$$\|Fu - Fv\|_{C(J_0,X)} \le PT_0\|u-v\|_{C(J_0,X)} \le \frac{1}{2}\|u-v\|_{C(J_0,X)}.$$

This shows that, F is a contraction map from S to S. Since S is closed in X, by the contraction mapping theorem, F has a unique fixed point $u \in S$ which is the local mild solution to (1).

Now, we show that $u \in C(J_0, Y)$. For $s, t \in J_0$ with $s \le t$, we have

$$\begin{aligned}u(t) - u(s) &= (U_u(t,0) - U_u(s,0))u_0 \\ &\quad + \int_0^s (U_u(t,\eta) - U_u(s,\eta))f(\eta,u(\eta),G(u)(\eta))d\eta \\ &\quad + \int_s^t U_u(t,\eta)f(\eta,u(\eta),G(u)(\eta))d\eta.\end{aligned}$$

Since $U_u(t,s)$ is strongly continuous in Y for $s,t \in J$ and $s \le t$. So, for every $\epsilon > 0$, there exist δ_1, $\delta_2 > 0$ such that

$$t,s \in J_0 \quad \text{with} \quad |t-s| \le \delta_1 \quad \Rightarrow \quad \|U_u(t,0) - U_u(s,0)\|_{B(Y)} \le \frac{\epsilon}{3\|u_0\|_Y}$$

and

$$t,s \in J_0 \quad \text{with} \quad |t-s| \le \delta_2 \quad \Rightarrow \quad \|U_u(t,\eta) - U_u(s,\eta)\|_{B(Y)} \le \frac{\epsilon}{3\lambda_f(R_1)T_0}.$$

Let $\delta = \min\{\delta_1, \delta_2, \frac{\epsilon}{3C_1\lambda_f(R_1)}\}$. Then, for $s,t \in J_0$

$$|t-s| \le \delta \Rightarrow \|u(t) - u(s)\|_Y \le \epsilon.$$

Thus, $u \in C(J_0, Y)$.

Consider the following linear evolution equation

$$\frac{dv(t)}{dt} + B(t)v(t) = h(t), \quad 0 < t \le T_0, \quad v(0) = u_0, \tag{18}$$

where $B(t) = A(t, u(t))$ and $h(t) = f(t, u(t), G(u)(t))$ for $t \in J_0$ and u being the unique fixed point of F in S. We note that $B(t)$ satisfies (H1)-(H3) of §5.5.3 in Pazy [7].

We have to prove that $h \in C(J_0, Y)$. For $s, t \in J_0$ (we assume without loss of generality that $s \leq t$), we have

$$\begin{aligned} \|h(t) - h(s)\|_Y &= \|f(t, u(t), G(u)(t)) - f(s, u(s), G(u)(s))\|_Y \\ &\leq \mu_f(R_1)[|\phi(t) - \phi(s)| + \|u(t) - u(s)\|_Y + \|G(u)(t) - G(u)(s)\|_Y]. \end{aligned}$$

As ϕ is a continuous function of bounded variation on J, $u \in C(J_0, Y)$ and $G(u) \in C(J_0, Y)$ for $u \in C(J_0, Y)$. So, for every $\epsilon > 0$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_1 \quad \Rightarrow \quad |\phi(t) - \phi(s)| \leq \frac{\epsilon}{3\mu_f(R_1)},$$

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_2 \quad \Rightarrow \quad \|u(t) - u(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)}$$

and

$$t, s \in J_0 \quad \text{with} \quad |t - s| \leq \delta_3 \quad \Rightarrow \quad \|G(u)(t) - G(u)(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, for $s, t \in J_0$, we have: $|t - s| \leq \delta \Rightarrow \|h(t) - h(s)\|_Y \leq \epsilon$. Thus, $h \in C(J_0, Y)$. Theorem 5.5.2 in Pazy [7] implies that there exists a unique function $v \in C(J_0, Y)$ such that $v \in C^1(J_0/\{0\}, X)$ satisfying (18) in X and v is given by

$$v(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s) f(s, u(s), G(u)(s)) ds, \quad t \in J_0,$$

where $U_u(t, s)$, $0 \leq s \leq t \leq T_0$ is the evolution system generated by the family $\{A(t, u(t))\}$, $t \in J_0$, of linear operators in X. The uniqueness of v implies that $v \equiv u$ on J_0 and hence u is a unique local classical solution to (1). This completes the proof.

4 Application

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the differential operator

$$A(t, x, u; D)w = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(t, x, u(t, x))\frac{\partial w}{\partial x_j}\right) + c(t, x, u(t, x))w,$$

where $a_{ij}(t, x, u(t, x))$ and $c(t, x, u(t, x))$ are real valued functions defined on $I \times \overline{\Omega} \times \mathbb{R}$ and $I = [0, T]$, $0 < T < \infty$. We assume that $a_{ij} \in C[I \times \overline{\Omega} \times W, \mathbb{R}]$, where $W = C^{2l+1}(I \times \overline{\Omega}, \mathbb{R})$ with $1/2 < l < 1$, $a_{ij} = a_{ji}$, $(1 \leq i, j \leq n)$ and there exists some $\delta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t, x, u(t, x)) q_i q_j \geq \delta |q|^2, \quad q = (q_1, ..., q_n) \in \mathbb{R}$$

holds for each $(t, x, u(t, x)) \in I \times \overline{\Omega} \times \mathbb{R}$.

Consider the partial integrodifferential equation

$$\frac{\partial u(t,x)}{\partial t} + A(t,x,u;D)u(t,x) = f(t,x,u(t,x),K(u)(t,x)), \ (t,x) \in (0,T] \times \Omega \tag{19}$$

with boundary condition

$$u(t,x) = 0 \quad \text{for} \quad (t,x) \in (0,T] \times \partial\Omega$$

and initial condition

$$u(0,x) = u_0(x) \quad \text{for} \quad x \in \Omega,$$

where

$$K(u)(t,x) = \int_0^t a(t-s)k(s,x,u(s,x),\nabla u(s,x))ds,$$

$$\nabla = (D_1, D_2, ..., D_n), \quad D_i = \frac{\partial}{\partial x_i},$$

the function a is a real valued continuous function of bounded variation in $\mathbb{R}$ and the function $f(t,x,u,v)$ is also a real valued continuous function defined on $I \times \overline{\Omega} \times W \times W$ and for every $t_0 > 0$, $r_0 > 0$ there exists $L_0 > 0$ such that if $\|u_1\| \le r_0$, $\|u_2\| \le r_0$, then

$$\|f(t,x,u_1,v_1) - f(s,x,u_2,v_2\| \le L_0[|\psi(t) - \psi(s)| + \|u_1 - u_2\| + \|v_1 - v_2\|]$$

for $x \in \Omega$, $u_i, v_i \in W$, $i = 1,2$ and ψ is a real valued continuous function of bounded variation. $u : I \times \Omega \to \mathbb{R}$ is unknown function and u_0 is its initial value.

Further, we assume that $k : [0,\infty) \times \Omega \times W \times W \to \mathbb{R}$ is continuous and for every $t_0 > 0$, $r_0 > 0$ there exists $M_0 > 0$ such that if $\|u\| \le r_0$, $\|v\| \le r_0$, then

$$\|k(t,x,u,\xi) - k(t,x,v,\eta)\| \le M_0[\|u - v\| + \|\xi - \eta\|]$$

for all $0 \le t \le t_0$, $x \in \Omega$ and $u, v, \xi, \eta \in W$.

Let $\frac{n}{2l-1} < p < \infty$ and $X = L^p(\Omega)$ with the usual norm

$$\|u\|_p = \left[\int_\Omega |u|^p dx\right]^{1/p},$$

then integrodifferential equation (19) can be reformulated as abstract integrodifferential equation (1) in Banach space X, where

$$A(t,u)w = A(t,x,u;D)w$$

with domain

$$D(A(t,u)) = \{w \in W_p^2(\Omega) : w(t,x) = 0, (t,x) \in (0,T] \times \partial\Omega\}$$

and

$$f(t,u,G(u))(x) = f(t,x,u(t,x),K(u)(t,x)).$$

We note that the assumptions (H1)-(H8) are satisfied thus we may apply the result of the earlier section to guarantee the existence of unique classical solution of (19).

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Stability Analysis of Phase Synchronization in Coupled Chaotic Systems Presented by Fractional Differential Equations

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Abstract: In this paper, we have considered phase synchronizations in coupled chaotic systems presented by fractional differential equations. This synchronization occurs when some eigenvalues of the matrix found in the linear approximation of difference evolutional equation between coupled chaotic systems have zero real parts. Here, we have used nonlinear feedback function for synchronization. We have also demonstrated some numerical examples to show the accuracy of our analytical stability in some coupled chaotic fractional differential equations.

Keywords: *chaos; synchronization; fractional differential equations.*

Mathematics Subject Classification (2000): 34H10, 34D06, 34A08.

1 Introduction

As Pecora and Carroll have shown [1] in coupled chaotic systems, a complete synchronization occurs if the difference between various states of synchronized systems converges to zero. They have also shown that, synchronization stability depends upon the signs of the conditional Lyapunov exponents. That is, if all of the Lyapunov exponents of the response system under the action of the driver are negative, then there is a complete and stable synchronization between the drive and response systems. Stability of the synchronization can also be verified using the Jacobian matrix in the linearized system, where the linearized system represents the state difference between the drive and response chaotic systems [2]. Following this stability analysis and despite the theory of stability analysis in dynamical system, if this Jacobian matrix is of full rank and all of

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its eigenvalues are negative, then the system will converge to zero and yield complete synchronization. However, phase synchronization occurs when this Jacobian matrix has some zero eigenvalues. In this case, the difference between various states of synchronized systems may not necessarily converge to the zero, but will stay less than or equal to a constant.

Recently, fractional differential equations (FDEs) have been utilized to study dynamical systems in general, chaos, and synchronization in particular [3]–[7]. It is well-known that FDEs are useful because of their non-local nature, whereas for integer order (classical) differential equations that this property is the local one. Although the theory of fractional calculus is a 300-year-old topic which can trace back to Leibniz, Riemann, Liouville, Grnwald and Letnikov, the applications of fractional calculus to physics and engineering are just a recent focus of interest [8, 9]. Many systems are known to display fractional order dynamics, such as viscoelastic system [10], colored noise, dielectric polarization [11], electrode-electrolyte polarization [12] and electromagnetic wave [13], the control of fractional order dynamic systems [14] and so on. The main goal of this paper is to discuss the stability analysis of phase synchronization in coupled chaotic systems presented by FDEs. To do this, after some primarily definitions in the next section we implement the nonlinear coupling feedback function method for some coupled chaotic FDEs to discuse synchronization and phase synchronization in section 3. We also present two criteria for phase synchronization in both coupled chaotic ODE and FDE systems. Then in section 4, we illustrate the numerical results of two coupled chaotic systems in the form of FDEs in which the phase synchronizations and their convergences exist.

2 Preliminaries

In this section, we present some basic definitions and properties [8, 15].

2.1 Fractional Calculus

Definition 2.1 A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.

Definition 2.2 Let $f \in C_\mu$ and $\mu \geq 1$, then the (left-sided) Riemann–Liouville integral of order $\alpha, \alpha > 0$, is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s.$$

Definition 2.3 The (left-sided) Caputo fractional derivative of $f, f \in C^m_{-1}$ with order $\alpha > 0$ and $m \in \mathbb{N} \cup 0$, is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha} = D_*^\alpha f(t) = \begin{cases} [I^{m-\alpha} \frac{d^m}{dt^m} f(t)], & m-1 < \alpha \leq m, \quad m \in \mathbb{N}, \\ \frac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

2.2 Numerical method for solving FDEs

Recently, the approximate numerical techniques for FDEs have been developed in literature, which are numerically stable and can be applied to both linear and nonlinear FDEs. Diethelm et al. [16] presented a PECE (predict, evaluate, correct, evaluate) type method for numerical solution of FDEs with Caputo derivatives, which is a generalization of the classical one-step Adams–Bashforth–Moulton algorithm for first order ordinary differential equations.

The fractional Predictor–Corrector (PC) algorithm is based on the analytical property that the following FDE

$$D^{\alpha}y(t) = f(t, y(t)), \quad 0 \le t \le T,$$
$$y^{(k)}(0) = y_0^{(k)}, k = 0, 1, ..., m-1 \ (m = \lceil \alpha \rceil)$$

is equivalent to the Volterra integral equation [16]

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds.$$

Now, set $h = T/N, t_n = nh, n = 0, 1, ..., N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$ and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)),$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & if \ \ j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j-1)^{\alpha+1}, & if \ \ 1 \le j \le n, \\ 1, & if \ \ j = n+1, \end{cases}$$

and

$$y_h^p(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)),$$

in which $b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha})$. Therefore, the estimation error of this approximation is $\max_{j=0,1,\dots,N} |y(t_j) - y_h(t_j)| = O(h^p)$, where $p = \min(2, 1+\alpha)$.

3 Phase Synchronization in Fractional Order Dynamical Systems

Here, we use the nonlinear coupling feedback function method introduced by Ali and Fang [25] to couple two chaotic FDEs. Using this method on the FDE $\mathbf{D}^{\alpha}\mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t))$, we suppose the vector-valued function $\mathbf{F}(\mathbf{x}(t))$ is decomposed into linear, $\mathbf{L}(\mathbf{x}(t))$, and non-linear, $\mathbf{N}(\mathbf{x}(t))$, components. That is,

$$\mathbf{F}(\mathbf{x}(t)) = \mathbf{L}(\mathbf{x}(t)) - \mathbf{N}(\mathbf{x}(t)). \tag{1}$$

Now consider two chaotic FDEs systems whose associated vector functions are decomposed as in (1) and coupled by using the non-linear parts of their vector functions as follows:

$$\mathbf{D}^{\alpha}\mathbf{x_1}(t) = \mathbf{L}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_1}(t)) + s[\mathbf{N}(\mathbf{x_1}(t)) - \mathbf{N}(\mathbf{x_2}(t))], \tag{2}$$
$$\mathbf{D}^{\alpha}\mathbf{x_2}(t) = \mathbf{L}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_2}(t)) + s[\mathbf{N}(\mathbf{x_2}(t)) - \mathbf{N}(\mathbf{x_1}(t))]. \tag{3}$$

Here, systems (2) and (3) serve as drive and response systems, respectively, and s measures the strength of their coupling. In a manner analogous to integer order differential equations, the stability of the synchronization in this fractional situation can be studied by using the evolutional equation of the difference between systems (2) and (3). This equation is determined by the linear approximation

$$\mathbf{D}^{\alpha}\mathbf{e}(t) = \left[\mathbf{L} + (2s-1)\frac{\partial \mathbf{N}}{\partial \mathbf{x}}\right]\mathbf{e}(t), \tag{4}$$

where $\mathbf{e}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. It is well-known from linear stability theory in dynamical systems that if $\alpha = 1$ and $s = 0.5$, then the stability type of the zero equilibrium in Eq. (4) reflects the stability type of the synchronization between the two chaotic systems and depends upon the signs of the real parts of the eigenvalues $\mathbf{L}$ [6]. However, in the case $0 < \alpha < 1$ and $s = 0.5$ we cannot use this stability criterion, instead we can use the following Matignon's theorem [18].

Theorem 3.1 *The linearized system of fractional differential equations,* $\mathbf{D}^{\alpha}\mathbf{x}(t) = \mathbf{L}(\mathbf{x}(t))$*, is asymptotically stable if and only if* $|\arg(spec(\mathbf{L}))| > \alpha\pi/2$.

We recall that in the case of phase synchronization the error $\mathbf{e}(t)$ converges to a constant or remains bounded by a constant. So, by just some modification on Theorem 1, we can analyse the convergence of phase synchronization.

Theorem 3.2 *Define* $\mathbf{E}(t) = \mathbf{e}(t) - \mathbf{c}$ *and let* $s = 0.5$*. Then the linear system of fractional differential equations* $\mathbf{D}^{\alpha}\mathbf{E}(t) = \mathbf{L}(\mathbf{E}(t))$ *is asymptotically stable if and only if* $|\arg(spec(\mathbf{L}))| > \alpha\pi/2$*. In this case, the vector* $\mathbf{e}(t)$ *converges to* $\mathbf{c}$ *at the rate* $t^{-\alpha}$.

Note that stability exists if and only if either asymptotic stability exists or those eigenvalues which satisfy $|\arg(spec(\mathbf{L}))| = \alpha\pi/2$ have geometric multiplicity one.

4 Numerical Results

To see our assertion in above analytical justification for the phase synchronization in FDEs, we first consider the diffusionless Lorenz chaotic system presented by FDEs

$$\begin{cases} D^{\alpha}x = -x - y, \\ D^{\alpha}y = -xz, \\ D^{\alpha}z = -xy + r. \end{cases} \tag{5}$$

This system is chaotic for $\alpha = 1$ and $r \in (0, 5)$ [19]. With the same value of r, system (5) remains chaotic for $0.88 < \alpha < 1$. Now using the nonlinear coupling feedback function method, drive and response systems can be presented by

$$\begin{cases} D^{\alpha}x_1 = -x_1 - y_1, \\ D^{\alpha}y_1 = -x_1z_1 + s(x_1z_1 - x_2z_2), \\ D^{\alpha}z_1 = x_1y_1 + r + s(x_2y_2 - x_1y_1), \\ D^{\alpha}x_2 = -x_2 - y_2, \\ D^{\alpha}y_2 = -x_2z_2 + s(x_2z_2 - x_1z_1), \\ D^{\alpha}z_2 = x_2y_2 + r + s(x_1y_1 - x_2y_2). \end{cases} \tag{6}$$

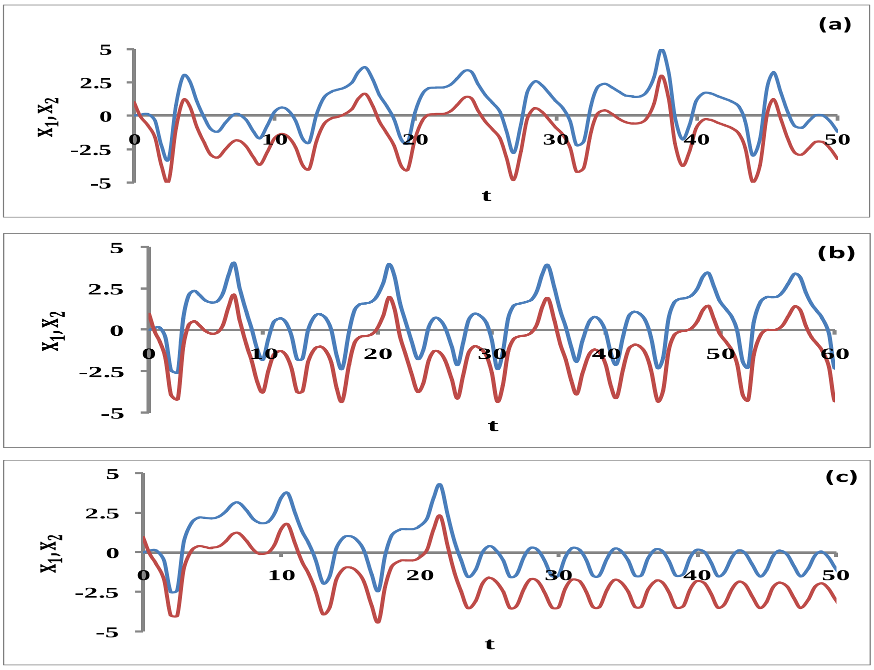

Figure 1: Phase synchronization in (x_1, x_2) plane for $\alpha = 0.95$ in (a), $\alpha = 0.91$ in (b) and $\alpha = 0.9$ in (c).

Here matrix $\mathbf{L}$ in error linear approximation (4) will be

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As we can see, the eigenvalues of matrix $\mathbf{L}$ are -1 and zero with multiplicity 2. So the condition for phase synchronization exists. In addition, it is easy to see that the condition in Theorem 2 is also satisfied for the convergence of this phase synchronization. Now using the PC method described in Section 2 to approximate the solutions of system (6), with $s = 0.5$, the results are illustrated in Figures 1 for different values of α. As we can see in Figure 1-c the phase synchronization exits, but the chaotic solution is merging to the limit cycle. This is because of the derivatives order $\alpha = 0.9$ which affects the system and changes its chaotic solution to the limit cycle.

As the next example, we introduce a new chaotic system in 4-dimensional space as follows.

$$\begin{cases} D^\alpha x = -ax - by + w, \\ D^\alpha y = -cy - axz, \\ D^\alpha z = -z + axy + d, \\ D^\alpha w = -fw - exz. \end{cases} \tag{7}$$

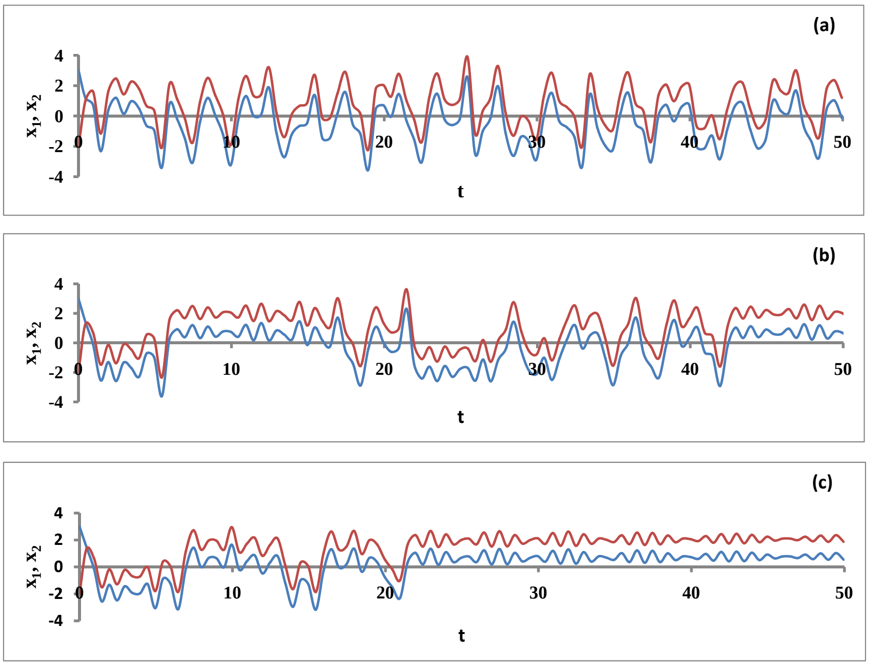

Figure 2: Phase synchronization in (x_1, x_2) plane for $\alpha = 0.95$ in (a), $\alpha = 0.9$ in (b) and $\alpha = 0.89$ in (c).

This system is chaotic for the parameters values $a = 3$, $b = 2$, $c = 1$, $d = 15$, $e = 0.2$ and $f = 1$. The system will remain chaotic for $0.92 \leq \alpha < 1$. Using nonlinear coupling feedback function method, system (7) is coupled as follows

$$\begin{cases} D^\alpha x_1 = -ax_1 - by_1 + w_1, \\ D^\alpha y_1 = -cy_1 - axz_1 + sa(x_1z_1 - x_2z_2), \\ D^\alpha z_1 = -z_1 + ax_1y_1 + d + sa(x_2y_2 - x_1y_1), \\ D^\alpha w_1 = -fw_1 - ex_1z_1 + se(x_1z_1 - x_2z_2), \\ D^\alpha x_2 = -ax_2 - by_2 + w_2, \\ D^\alpha y_2 = -cy_2 - axz_2 + sa(x_2z_2 - x_1z_1), \\ D^\alpha z_2 = -z_2 + ax_2y_2 + d + sa(x_1y_1 - x_2y_2), \\ D^\alpha w_2 = -fw_2 - ex_2z_2 + se(x_2z_2 - x_1z_1). \end{cases} \tag{8}$$

For this system matrix $\mathbf{L}$ in error linear approximation (4) will be

$$\begin{pmatrix} -a & -b & 0 & 1 \\ 0 & -c & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -f \end{pmatrix}$$

and its eigenvalues are $-a$, $-c$, -1 and $-f$. Some of the values for these parameters in which the phase synchronization happens are $a = 3$,$b = 2$, and $c = f = 0$. Obviously, the convergence criterion in Theorem 2 is satisfied here for system (8). Again, using the PC method to approximate the solutions of this system, with $s = 0.5$, the results are illustrated in Figures 2 for different values of α. Here, in Figure 2-c the phase synchronization exits, but the chaotic solution will change to the limit cycle. This change is again the affect of the derivatives order $\alpha = 0.89$, which turns the chaotic solution into the limit cycle.

5 Conclusions

As we discussed in this article, phase synchronization is a rare phenomenon, which occurs in some coupled chaotic systems. Direct stability criterion of the dynamical system cannot be applied for the convergence of phase synchronization. However, as we discussed in Theorem 1 and 2, these criteria can be adapted somehow in which we can apply for the convergence of phase synchronization in either ODE or FDE coupled chaotic systems. The illustrated diffusionless Lorenz system in Example 1 and the new 4-dimensional system in Example 2 showed our assertion for existence and stated convergence criterion.

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Self Recurrent Wavelet Neural Network Based Direct Adaptive Backstepping Control for a Class of Uncertain Non-Affine Nonlinear Systems

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Abstract: This paper proposes an adaptive backstepping control strategy for a class of uncertain non affine systems using self recurrent neural networks. To assure the stable tracking of nonlinear non affine system, it is first converted to an affine like form and subsequently a wavelet based adaptive backstepping controller is developed. Self recurrent wavelet neural network (SRWNN) is used to approximate the uncertainties present in the system as well as to compensate the highly dynamic nonlinearities inserted by these uncertainties in the control terms. In addition robust control terms are also designed to attenuate the approximation error due to SRWNN. Based on the Lyapunov theory, the online adaptation laws and stability of the closed loop system are verified. A numerical example is provided to verify the effectiveness of theoretical development.

Keywords: *non-affine systems; self recurrent wavelet networks; backstepping control; adaptive control; Lyapunov analysis.*

Mathematics Subject Classification (2000): 49J35, 34A34, 92C20.

1 Introduction

Over last few years, several efforts on the development of adaptive control strategies for uncertain nonlinear systems have been cited in the literature. In these cases the common assumption was that the system is affine in input [1, 2]. However the development of control strategies is still an active area of research.

To deal with the non affine systems, two control strategies are cited in the literature. One is based on the dynamic inversion satisfying the assumptions of Tikhonov theorem

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from the singular perturbation theory. Other is based on the conversion of non affine system to an affine like form by applying a suitable transformation and designing the controller for the later form by implicit function theorem [3]–[7].

Backstepping is a recursive design methodology where some appropriate functions of state variables as pseudocontrol inputs for lower dimension subsystems of the overall system are derived. Each backstepping stage results in a new pseudocontrol design, expressed in terms of the pseudocontrol designs from preceding design stages. When the procedure is terminated, a feedback design for the true control input results, which achieves the original design objective by virtue of a final Lyapunov function, which is formed by summing up the Lyapunov functions associated with each individual design stage. Thus, the backstepping control approach is capable of keeping the robustness properties with respect to the uncertainties [13]–[15]. Via adaptive backstepping this methodology can be effectively extended to non linear systems with unmodelled dynamics [1].

Based on the concept of transformation of non affine systems into affine like form, some researchers have proposed adaptive backstepping based control schemes for non affine uncertain systems [7].

Employment of neural network (NN) as an approximation tool in adaptive control strategies has greatly relaxed the assumptions on linear parameterized nonlinearities and thereby broadens the class of the uncertain nonlinear systems which can be effectively dealt by adaptive controllers [8]. However there are certain difficulties associated with NN based controller. The basis functions are generally not orthogonal or redundant; i.e., the network representation is not unique and is probably not the most efficient one. Furthermore, the convergence of neural networks may not be guaranteed. Even when it exhibits a good convergence rate, the training procedure may still be trapped in some local minima depending on the initial settings. Wavelet neural networks are feed-forward neural networks using wavelets as activation function. Due to their space and frequency localization properties, the learning capability of WNN is superior to conventional neural networks. Training algorithms for WNN converge in smaller number of iterations than for conventional neural networks. These WNN combines the capability of artificial neural network for learning ability and capability of wavelet decomposition for identification ability. Thus WNN based control systems can achieve better control performance than NN based control systems [9, 10]. The feedforward structure of the conventional WNN limits the applicability of these networks only to static environmental conditions. These networks are not very effective under the frequently changing operating conditions and dynamic properties as they can not adapt rapidly under such circumstances. To overcome this problem, a feedback mechanism is inserted in conventional WNN giving rise to either output recurrent WNN (ORWNN) or self recurrent WNN (SRWNN). These recurrent networks combines the properties of recurrency with the convergence properties of WNN to solve the complex control problems [11, 12].

This paper deals with the designing of a backstepping based adaptive tracking controller for a class of uncertain non affine systems. SRWNN are used for approximating the system uncertainty as well as to compensate the nonlinearities arising in the controller terms due to these uncertainties.

For the class of the system under consideration the backstepping control terms contain the system nonlinearities as well as their derivatives of various orders. Consideration of these derivatives while deriving the controller terms results in numerically untraceable solution, whereas if these derivatives are neglected, it results in approximate backstepping.

In this work such derivative terms are approximated by using SRWNN, thereby reducing the mathematical complexities as well as improving the accuracy of the controller strategy.

The paper is organized as follows: Section 2 deals with the system preliminaries, system description is given in Section 3. SRWNN based backstepping controller designing aspects are discussed in Section 4. Effectiveness of the proposed strategy is illustrated through an example in Section 5 while Section 6 concludes the paper.

2 System Preliminaries

2.0.1 Self recurrent wavelet neural network

Wavelet network is a type of building block for function approximation. The building block is obtained by translating and dilating the mother wavelet function. SRWNN is modified form of WNN composed of a self feedback wavelon layer as shown in Figure 1. Due to the self feedback layer the wavelon layer can store the past information of the network, thereby capturing the dynamic response of the system. This modification allows SRWNN to approximate dynamic nonlinearities with high degree of accuracy. This makes SRWNN more suitable tool for the adaptive control strategies as compared to conventional WNN.

Output of an n dimensional SRWNN with m wavelet nodes is

$$f = \sum_{i=1}^{m} \alpha_i \varphi_i \left(\theta_i, \bar{\varphi}_i, x, w_i, c_i\right), \tag{1}$$

where φ_i is the i^{th} wavelet node given by

$$\varphi_i \left(\theta_i, \bar{\varphi}_i, x, w_i, c_i\right) = \prod_{j=1}^{n} \varphi_{ij}(\theta_{ij}, \bar{\varphi}_{ij}, x, w_{ij}, c_{ij}), \tag{2}$$

where φ_{ij} is the j^{th} wavelon of i^{th} wavelet node. $x = [x_1, x_2, \ldots, x_n]^T$ is the vector of the states of the system and act as external input vector the SRWNN, whereas $\bar{\varphi}_i = [\bar{\varphi}_{i1}, \bar{\varphi}_{i2}, \ldots, \bar{\varphi}_{in}]$ is the previous value vector of the wavelon constituting the i^{th} wavelet node This vector serves as the memory element and stores the previous information of the network, and acts as the feedback input for the respective wavelon. $\theta_i = [\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}]$ is the weight vector of the feedback input. Whereas $w_i = [w_{i1}, w_{i2}, \ldots, w_{in}]$ and $c_i = [c_{i1}, c_{i2}, \ldots, c_{in}]$ are dilate and translate vectors respectively. The net input applied to the wavelet network is given by $z_i = [x_1 + \theta_{i1}{\varphi'}_{i1}, x_2 + \theta_{i2}{\varphi'}_{i2}, \ldots, x_n + \theta_{in}{\varphi'}_{in}]^T$.

Now (1) can be rewritten as

$$f = \alpha^T \varphi\left(x, \theta, \bar{\varphi}, w, c\right), \tag{3}$$

where $w = [w_1, w_2, ..., w_m]^T \in R^{mxn}$ and $c = [c_1, c_{2,}..., c_m]^T \in R^{mxn}$ are dilation and translation parameters respectively; $\alpha = [\alpha_1, \alpha_2, .., \alpha_m]^T \in R^m$ and $\theta = [\theta_1, \theta_2, .., \theta_m]^T \in R^{nxm}$ are the output and feedback weights respectively. $\bar{\varphi} = [\bar{\varphi}_1, \bar{\varphi}_2 .., \bar{\varphi}_m]^T \in R^{nxm}$ is the feedback input vector of SRWNN.

Let f^* be the optimal function approximation using an ideal wavelet approximator then

$$f = f^* + \Delta = \alpha^{*T} \varphi^* + \Delta, \tag{4}$$

where $\varphi^* = \varphi(x, \theta^*, \bar{\varphi}, w^*, c^*)$ and $\alpha^*, w^*, c^*, \theta^*$ are the optimal parameter vectors of α, w, c, θ respectively and Δ denotes the approximation error and is assumed to be bounded by $|\Delta| \le \Delta^*$, in which Δ^* is a positive constant. Optimal parameter vectors needed for the best approximation of the function are difficult to determine so define an estimate function as

$$\hat{f} = \hat{\alpha}^T \hat{\varphi}, \tag{5}$$

where $\hat{\varphi} = \varphi\left(x, \hat{w}, \hat{c}, \hat{\theta}, \bar{\varphi}\right)and\hat{\alpha}, \hat{w}, \hat{c}, \hat{\theta}$ are the estimates of $\alpha^*, w^*, c^*, \theta^*$ respectively. Define the estimation error as

$$\tilde{f} = f - \hat{f} = f^* - \hat{f} + \Delta = \alpha^T \tilde{\varphi} + \hat{\alpha}^T \tilde{\varphi} + \tilde{\alpha}^T \hat{\varphi} + \Delta, \tag{6}$$

where $\tilde{\alpha} = \alpha^* - \hat{\alpha}$, $\tilde{\varphi} = \varphi^* - \hat{\varphi}$.

By properly selecting the number of nodes, the estimation error $\tilde{f}$ can be made arbitrarily small on the compact set so that the bound $\left\|\tilde{f}\right\| = \tilde{f}_m$ holds for all $x \in \Re$.

Using Taylor expansion linearization technique to transform the nonlinear function into a partially linear form as a step towards the derivation of online tuning laws for the wavelet parameters to achieve the favorable estimation of system dynamics [1]

$$\tilde{\varphi} = A^T \tilde{w} + B^T \tilde{c} + C^T \tilde{\theta} + h, \tag{7}$$

where $\tilde{w} = w^* - \hat{w}, \tilde{c} = c^* - \hat{c}, \tilde{\theta} = \theta^* - \hat{\theta}$ and h are the vectors of higher order terms and

$$\begin{aligned} A &= \left[\tfrac{d\varphi_1}{dw}, \tfrac{d\varphi_2}{dw}, ..., \tfrac{d\varphi_m}{dw}\right]\Big|_{w=\hat{w}}, \\ B &= \left[\tfrac{d\varphi_1}{dc}, \tfrac{d\varphi_2}{dc}, ..., \tfrac{d\varphi_m}{dc}\right]\Big|_{c=\hat{c}}, \\ C &= \left[\tfrac{d\varphi_1}{d\theta}, \tfrac{d\varphi_2}{d\theta}, ..., \tfrac{d\varphi_m}{d\theta}\right]\Big|_{\theta=\hat{\theta}}, \end{aligned}$$

with

$$\begin{aligned} \tfrac{d\hat{\varphi}_i}{dw} &= \left[0, ..., 0, \tfrac{d\hat{\varphi}_i}{dw_{1i}}, \tfrac{d\hat{\varphi}_i}{dw_{2i}}, ..., \tfrac{d\hat{\varphi}_i}{dw_{ni}}, 0...0\right]^T, \\ \tfrac{d\hat{\varphi}_i}{dc} &= \left[0, ..., 0, \tfrac{d\hat{\varphi}_i}{dc_{1i}}, \tfrac{d\hat{\varphi}_i}{dc_{2i}}, ..., \tfrac{d\hat{\varphi}_i}{dc_{ni}}, 0...0\right]^T, \\ \tfrac{d\hat{\varphi}_i}{d\theta} &= \left[0, ..., 0, \tfrac{d\hat{\varphi}_i}{d\theta_{1i}}, \tfrac{d\hat{\varphi}_i}{d\theta_{2i}}, ..., \tfrac{d\hat{\varphi}_i}{d\theta_{ni}}, 0...0\right]^T. \end{aligned}$$

Substituting (7) into (6), we have

$$\tilde{f} = \left(\tilde{\alpha}^T \left(\hat{\varphi} - \mathrm{A}_1^T \hat{w} - B_1^T \hat{c} - C^T \hat{\theta}\right) + \tilde{w}^T A \hat{\alpha} + \tilde{c}^T B \hat{\alpha} + \tilde{\theta}^T C \hat{\alpha} + \varepsilon\right), \tag{8}$$

where ε is the uncertain term.

3 System Description

Consider a non affine system of the form

$$\begin{aligned} &\dot{x}_1 = x_2 + \phi_1(x, u), \\ &\dot{x}_2 = x_3 + \phi_2(x, u), \\ &\vdots \\ &\dot{x}_n = \phi_n(x, u), \\ &y = x_1, \end{aligned} \tag{9}$$

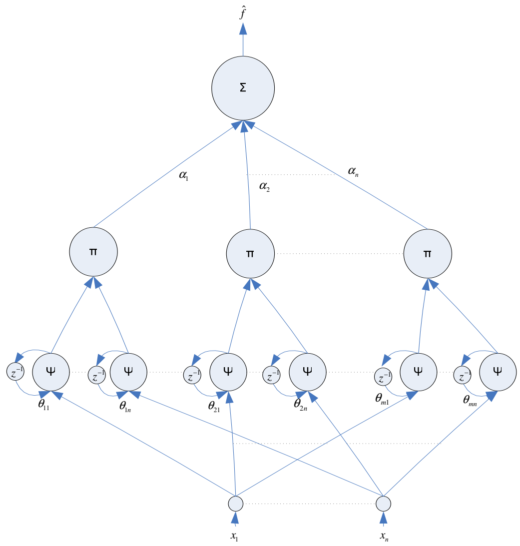

Figure 1: Self recurrent wavelet network.

where $x = [x_1, x_2, ..., x_n]^T$, u, y are state variable, control input and output respectively. $\phi = [\phi_1, \phi_2, ..., \phi_n]^T : \Re^{n+1} \to \Re^n$ are smooth unknown, nonlinear functions of state variables and input.

Applying the transformation the system (9) can be converted to an affine like form and can be rewritten as [7]

$$\begin{aligned}
&\dot{x}_1 = x_2 + \phi_1(x, u),\\
&\dot{x}_2 = x_3 + \phi_2(x, u),\\
&\vdots,\\
&\dot{x}_n = \phi_n(x, u) = u + (\phi_n(x, u) - u) = u + f(x, u),\\
&y = x_1.
\end{aligned} \tag{10}$$

The objective is to formulate a state feedback control law to achieve the desired tracking performance. The control law is formulated using the transformed system (10). Let $\bar{y}_d = [y_d, \dot{y}_d, \dots, \overset{n-1}{y_d}]^T$ be the vector of desired tracking trajectory. Following assumptions are taken for the systems under consideration.

Assumption 3.1 1. Desired trajectory $y_d(t)$ is assumed to be smooth, continuous C^n and available for measurement.

2. The nonlinear function $\phi_n(x, u)$ satisfies: $\left|\frac{\partial}{\partial u}\phi_n(x, u)\right| \geq \beta \geq 0$, which ensures the controllability of the system.

In the next section the SRWNN based adaptive control strategy for (10) is discussed.

4 SRWNN Based Adaptive Backstepping Controller Design

Define the state tracking error vector $e(t)$ as $e(t) = [x_1 - y_d, x_2 - \dot{y}_d, \ldots, x_n - \overset{n-1}{y}]^T$. So the error system of (10) becomes

$$\dot{e}_1 = e_2 + \phi_1(x,u), \tag{11}$$

$$\dot{e}_2 = e_3 + \phi_2(x,u), \tag{12}$$

$$\vdots$$

$$\dot{e}_n = u + f(x,u) - \overset{n}{y_d}. \tag{13}$$

Considering subsystem (11), let e_{2d} be the desired value of the e_2 required to stabilize (11), $e_{2d} = -k_1 e_1 - \hat{\phi}_1 + e_{2dr}$, where $k_1 > 0$, $\hat{\phi}_1$ is the SRWNN approximation of ϕ_1. e_{2dr} is the robust term used to attenuate the uncertainties introduced by the SRWNN. The online tuning laws for the wavelet parameters are:

$$\begin{array}{l} \dot{\hat{\alpha}}_1 = -\dot{\tilde{\alpha}}_1 = \beta_{11} e_1 (\hat{\varphi}_1 - A_1^T \hat{w}_1 - B_1^T \hat{c}_1 - C_1^T \hat{\theta}_1), \\ \dot{\hat{w}}_1 = -\dot{\tilde{w}}_1 = \beta_{12} e_1 A \hat{\alpha}_1, \\ \dot{\hat{c}}_1 = -\dot{\tilde{c}}_1 = \beta_{13} e_1 B_1 \hat{\alpha}_1, \\ \dot{\hat{\theta}}_1 = -\dot{\tilde{\theta}}_1 = \beta_{14} e_1 C_1 \hat{\alpha}_1. \end{array} \tag{14}$$

And the robust control term is defined as

$$e_{2dr} = -\frac{(\rho_1^2 + 1) e_1}{2\rho_1^2}, \tag{15}$$

where ρ_1 is the prescribed attenuation, β_{11}, β_{12}, β_{13} and β_{14} are the positive learning rates. Similarly the pseudo controller design for recursive i^{th} subsystem is given by

$$e_{(i+1)d} = (-\delta_i - k_i(e_i - e_{id}) - (e_{i-1} - e_{(i-1)d}) + e_{(i+1)dr}), \tag{16}$$

where $k_i > 0$ and δ_i is the approximation of $\phi_i - \dot{e}_{id}$. The term $\dot{e}_{id}$ contains the higher order derivatives of previous pseudo controller terms which in turn consist of state variables, input and their derivatives. Presence of all such terms makes it highly dynamic in nature and hence SRWNN is the most appropriate tool for the approximation if such highly dynamic nonlinear term. $e_{(i+1)dr}$ is the robust term used to attenuate the uncertainties introduced by the SRWNN. The online tuning laws for the wavelet parameters are:

$$\begin{array}{l} \dot{\hat{\alpha}}_i = -\dot{\tilde{\alpha}}_i = \beta_{i1} (e_i - e_{id}) (\hat{\varphi}_i - A_i^T \hat{w}_i - B_i^T \hat{c}_i - C_i^T \hat{\theta}_i), \\ \dot{\hat{w}}_i = -\dot{\tilde{w}}_i = \beta_{i2} (e_i - e_{id}) A_i \hat{\alpha}_i, \\ \dot{\hat{c}}_i = -\dot{\tilde{c}}_i = \beta_{i3} (e_i - e_{id}) B_i \hat{\alpha}_i, \\ \dot{\hat{\theta}}_i = -\dot{\tilde{\theta}}_i = \beta_{i4} (e_i - e_{id}) C_i \hat{\alpha}_i. \end{array} \tag{17}$$

And the robust control term is defined as

$$e_{idr} = -\frac{(\rho_i^2 + 1)(e_i - e_{id})}{2\rho_i^2}, \tag{18}$$

where ρ_i is the prescribed attenuation, β_{i1}, β_{i2}, β_{i3} and β_{i4} are the positive learning rates. Proceeding in the same manner the control law for the overall system is defined as

$$u = (-\delta_n - k_n(e_n - e_{nd}) - (e_{n-1} - e_{(n-1)d}) + u_r + \overset{n}{y_d}), \tag{19}$$

where $k_n > 0$ and δ_n is the approximation of $f - \dot{e}_{nd}$. u_r is the robust term used to attenuate the uncertainties introduced by the SRWNN. The online tuning laws for the wavelet parameters are:

$$\begin{aligned}
\dot{\hat{\alpha}}_n &= -\dot{\tilde{\alpha}}_n = \beta_{n1}(e_n - e_{nd})(\hat{\varphi}_n - A_n^T \hat{w}_n - B_n^T \hat{c}_n - C_n^T \hat{\theta}_n),\\
\dot{\hat{w}}_n &= -\dot{\tilde{w}}_n = \beta_{n2}(e_n - e_{nd})A_n \hat{\alpha}_n,\\
\dot{\hat{c}}_n &= -\dot{\tilde{c}}_n = \beta_{n3}(e_n - e_{nd})B_n \hat{\alpha}_n,\\
\dot{\hat{\theta}}_n &= -\dot{\tilde{\theta}}_n = \beta_{n4}(e_n - e_{nd})C_n \hat{\alpha}_n.
\end{aligned} \tag{20}$$

And the robust control term is defined as

$$u_r = -\frac{(\rho_n^2 + 1)(e_n - e_{nd})}{2\rho_n^2}, \tag{21}$$

where ρ_n is the prescribed attenuation, β_{n1}, β_{n2}, β_{n3} and β_{n4} are the positive learning rates.

5 Simulation Results

Simulation is performed to verify the effectiveness of proposed SRWNN based backstepping control strategy. Consider a system of the form

$$\begin{aligned}
\dot{x}_1 &= x_2 + 0.1x_1^2,\\
\dot{x}_2 &= \tfrac{u^3}{3} + \sin u + ux_1^2 + 0.5x_1^4,\\
y &= x_1.
\end{aligned} \tag{22}$$

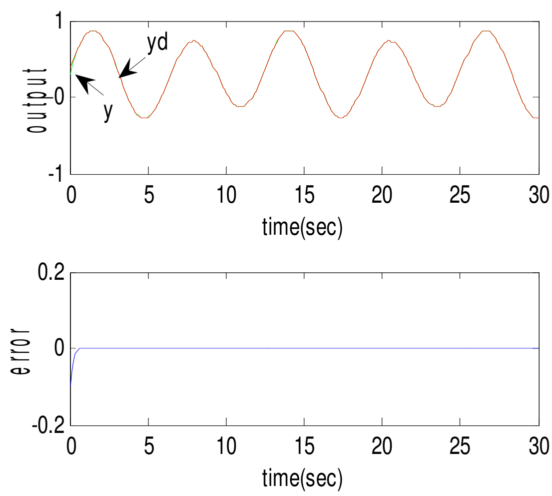

Figure 2: System output and tracking error.

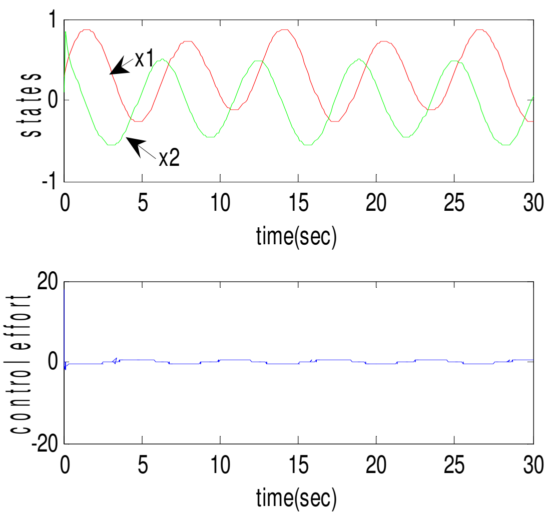

Figure 3: States of the system and control signal.

System belongs to the class of uncertain non affine systems defined by (9) with $n = 2$. The proposed controller strategy is applied to this system with an objective to solve the tracking problem of system.

The desired trajectory is taken as $y_d = 0.5 \sin t + 0.1 \cos \frac{t}{2} + 0.3$. Initial conditions are taken as $[0.3, 0.3]^T$. Attenuation level for the robust control terms is taken as 0.01. Controller parameters are taken as $k_1 = 10, k_2 = 10$. Two self recurrent wavelet networks with Mexican hat as the mother wavelet are used for approximating the unknown system dynamics. Wavelet parameters for these wavelet networks are tuned online using the proposed adaptation laws, initial conditions for all the wavelet parameters are set to zero. Simulation results are shown in Figure 2 and Figure 3. As observed from the figures, system response tracks the desired trajectory rapidly.

6 Conclusion

A SRWNN based adaptive backstepping control strategy is proposed for solving the tracking control problem for a class of non affine systems with unknown system dynamics. Self recurrent adaptive wavelet networks are used for approximating the unknown system dynamics of the system. Adaptation laws are developed for online tuning of the wavelet parameters. The theoretical analysis is validated by the simulation results.

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Existence of Almost Automorphic Solutions of Neutral Functional Differential Equation

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Abstract: In this work we use the theory of evolution semigroup of bounded linear operators and fixed point theorem to establish the existence and uniqueness of a mild solution of a neutral functional differential equation in a Banach space.

Keywords: *almost automorphic function; evolution semigroup; neutral functional differential equation; mild solution.*

Mathematics Subject Classification (2000): 37L05, 34K06, 34A12.

1 Introduction

In 1964, S. Bochner introduced almost automorphic functions in one of his landmark paper [10]. Almost automorphic functions are more general than almost periodic functions. Many authors had established the almost periodic solution of differential equations in abstract spaces ([8, 9, 13, 15], etc.). The theory has been generalized by many authors for almost automorphic solutions ([11, 12, 14], etc.). Goldstein [14] has considered the following differential equation in a Banach space X

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$

where A generates an exponentially stable C_0- semigroup and f be a jointly continuous function and shown the existence of almost automorphic solution of the problem if f is almost automorphic.

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These days, functional differential equations have been of very great interest, for many mathematicians. Bahuguna [1] studied a class of partial functional differential equtations and its application to population dynamics. Analytical techniques of semigroup theory have been applied in [2], [3] and [4], which we are also going to use in this paper.

Bahuguna and Muslim [5] also considered the second order history valued delay differential equations [4] and used evolution equations and semigroup theory to find approximation of a solution. Recently, D.N. Pandey, A. Ujlayan and D. Bahuguna [6] proved existence and uniqueness of a hyperbolic integrodifferential equation with a nonlocal condition.

Abbas and Bahuguna [7] considered the following nonautonomous neutral functional differential equations

$$\frac{d}{dt}(x(t) - F_1(t, x(t-g(t)))) = A(t)x(t) + F_2(t, x(t), x(t-g(t))), \tag{2}$$

where $A(t)$ generates an exponentially stable evolution systems and g is a continuous function. The authors have shown the existence of an almost periodic mild solutions using Kransnoselskii's fixed point theorem and theory of evolution operator. They also assumed the well known Acquistapace–Terreni conditions which ensure the existence of evolution family.

In the present work we study the existence of an almost automorphic solution of equation (2) using the evolution semigroup and the Banach fixed point approach.

2 Preliminaries

Let X be a complex Banach space endowed with the norm $\|.\|_X$. $\mathbb{N}$, $\mathbb{R}$ *and* $\mathbb{C}$ stand for Natural, Real and Complex numbers respectively. Let $B(X)$ be a Banach space of all bounded linear operators from X to itself; endowed with norm $\|.\|_{B(X)}$ given by

$$\|L\|_{B(X)} = \sup\{\|Lx\|_X : x \in X \text{ and } \|x\|_X \leq 1\}.$$

Now, we will recall certain definitions to be used subsequently in this paper.

Definition 2.1 A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence $\{s_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f(t+\tau_n) = g(t)$ and $\lim_{n\to\infty} g(t-\tau_n) = f(t)$ for all $t \in \mathbb{R}$.

We denote by $AA(X)$ the set of all such functions.

Definition 2.2 A continuous function $f : \mathbb{R}\times X \to X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in Y$, where Y is any bounded subset of X.

Equivalently, for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ we can extract a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $g(t, x) = \lim_{n\to\infty} f(t+\tau_n, x)$ is well defined for all $t \in \mathbb{R}$ and for all $x \in Y$ and $f(t, x) = \lim_{n\to\infty} g(t-\tau_n, x)$ is well defined for all $t \in \mathbb{R}$ and for all $x \in Y$.

Lemma 2.1 $(AA(X), \|.\|_{AA(X)})$ *is a Banach space with supremum norm, given by* $\|f\|_{AA(X)} = \sup_{t\in\mathbb{R}} \|f(t)\|$.

Lemma 2.2 *If* $f : \mathbb{R} \to X$ *is almost automorphic, then* f *is bounded.*

For the proof of the above two lemmas, we refer to [12].

Lemma 2.3 *Suppose $\mathbb{Z}$ and $\mathbb{W}$ are Banach spaces. Let $F : \mathbb{R} \times \mathbb{Z} \to \mathbb{W}$ be an almost automorphic function in $t \in \mathbb{R}$, for each $z \in \mathbb{Z}$ and assume that F satisfies Lipschitz condition in z uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to \mathbb{Z}$ be an almost automorphic function, then the function $\Phi : \mathbb{R} \to \mathbb{W}$, defined by $\Phi(t) = f(t, \phi(t))$ is almost automorphic.*

In [18], Acquistapace and Terreni gave conditions on $A(t)$, $t \in \mathbb{R}$, which ensure the existence of unique evolution family $\{U(t, s) : t \geq s > -\infty\}$ on X, such that

$$u(t) = U(t, 0)u(0) + \int_0^t U(t, \xi)f(\xi)d\xi,$$

where $u(t)$ satisfies

$$\frac{du(t)}{dt} = A(t)u(t) + f(t), \quad t \in \mathbb{R}.$$

Lemma 2.4 *ATC (Acquistapace–Terreni condition). Let*

$$S_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \theta\} \cup \{0\} \subset \rho(A(t)), \quad \theta \in (\frac{\pi}{2}, \pi).$$

If there exist a constant K_0 and a set of real numbers $\alpha_1, ..., \alpha_k, \beta_1, ...\beta_k$ with $0 \leq \beta_i < \alpha_i \leq 2, i = 1, 2, ...k$, such that

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\|_{B(X)} \leq K_0 \ \Sigma_{i=1}^k (t - \alpha)^{\alpha_i} |\lambda_i|^{\beta_i - 1}$$

for $t, s \in \mathbb{R}$ and $\lambda \in S_\theta \backslash \{0\}$ and there exists constant $M \geq 0$ such that

$$\|(\lambda - A(t))^{-1}\| \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in S_\theta,$$

then there exists a unique evolution family $\{U(t, s) : t \geq s > -\infty\}$ on X.

These conditions resulting from Theorem 2.3 of [17] are known as "Acquistapace–Terreni conditions".

Definition 2.3 A mild solution of (2) is a continuous function $x : \mathbb{R} \to X$, satisfying

$$\begin{aligned} x(t) - F_1(t, x(t - g(t)) &= U(t, s)(x(s) - F_1(s, x(s - g(s)))) \\ &\quad + \int_a^t U(t, \xi)F_2(\xi, x(\xi), x(\xi - g(\xi)))d\xi \qquad (3) \end{aligned}$$

for $t \geq s$ all $s \in \mathbb{R}$.

Note: We say, an evolution family $\{U(t, s)\}_{t \geq s > -\infty}$ is exponentially stable, if $\exists\, M \geq 1$ and $\delta > 0$ such that $\|U(t, s)\| \leq Me^{-\delta(t-s)}$ for $t \geq s$. When $s \to -\infty$ the above equation takes the form

$$x(t) = F_1(t, x(t - g(t))) + \int_{-\infty}^t U(t, \xi)F_2(\xi, x(\xi), x(\xi - g(\xi)))d\xi.$$

Assumptions:

($\mathbf{C}_1$) : $F_1(t,x), F_2(t,x,y)$ are almost automorphic.

($\mathbf{C}_2$) : F_1 and F_2 are Lipschitz continuous that is there exist positive numbers $L_{F_1}(t)$ and $L_{F_2}(t)$ such that

$$\|F_1(t,x) - F_1(t,y)\| \leq L_{F_1}(t)\|x-y\|_{AA(X)},$$

$$\|F_2(t,x,u) - F_2(t,y,v)\| \leq L_{F_2}(t)(\|x-y\|_{AA(X)} + \|u-v\|_{AA(X)}).$$

($\mathbf{C}_3$) : $\{U(t,s) : t \geq s\}$ is an exponentially stable evolution family on X.

($\mathbf{C}_4$) : For every sequence $\{s_n\}$ of real numbers there exists a subsequence $\{\tau_n\}$ and for any fixed $s \in \mathbb{R}$, $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, it follows that

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\| \leq \epsilon e^{-\frac{\delta}{2}(t-s)}$$

and

$$\|U(t-\tau_n, s-\tau_n) - U(t,s)\| \leq \epsilon e^{-\frac{\delta}{2}(t-s)} \quad \text{for all } t \geq s \in \mathbb{R}.$$

3 Almost Automorphic Solution

We define the mapping F by

$$(Fx)(t) = F_1(t, x(t-g(t))) + \int_{-\infty}^{t} U(t,s)F_2(s, x(s), x(s)-g(s)))ds.$$

Lemma 3.1 *For $x(.) \in AA(X)$, we have Fx is also almost automorphic.*

Proof Since F_1 is Lipschitz and $F_1 \in AA(\mathbb{R}, X)$; by Lemma 2.3, we have

$$F_1(t, x(t-g(t))) = K(t) \in AA(X).$$

By (C_2), we have $F_2(., x(.), y(.)) \in AA(\mathbb{R} \times X \times X, X)$, also we have assumed that F_2 is Lipschitz with respect to both variables x and y, further using the fact that $X \times X$ is Banach space; hence from Lemma 2.3, one can easily see that $F_2(., x(.), y(.)) \in AA(X)$.

Next, we define $F_2(t, x(t), y(t)) = H(t)$, where $H(.) \in AA(X)$. Now we show that

$$\begin{aligned}
\|Fx\|_{AA(X)} &< \infty, \\
\|Fx(t)\|_X &\leq \|K(t)\|_X + \int_{-\infty}^{t} \|U(t,s)\| \|F_2(s, x(s), x(s-g(s)))\|_X ds \\
&\leq M_1 + \int_{-\infty}^{t} Me^{-\delta(t-s)} \|H(s)\|_X ds \\
&\leq M_1 + M_2 \frac{M}{\delta} < \infty. \quad \text{where } \sup_{t\in\mathbb{R}} \|H(t)\| = M_2.
\end{aligned}$$

Thus, we have shown that Fx is bounded.

Now, we show that $(Fx)(t)$ is almost automorphic with respect to $t \in \mathbb{R}$. Since $H(.) \in AA(X)$ for all sequence $\{s_n\}$ of real numbers, there exists a subsequence $\{\tau_n\}$ such that

$(H_1)\ :\ h(t) = \lim_{n\to\infty} H(t+\tau_n)$ is well defined for all $t \in \mathbb{R}$.

$(H_2)\ :\ H(t) = \lim_{n\to\infty} h(t-\tau_n)$ is well defined for all $t \in \mathbb{R}$.

As we are going to use Lebesgue dominated convergence theorem to show that $(Fx)(t+\tau_n) \to (Gx)(t)$ as $n \to \infty$; we need to show $|Fx(t+\tau_n)| < l(t)$ for all $n \in \mathbb{N}$; where l is some integrable function. Consider

$$\begin{aligned}(Fx)(t+\tau_n) &= F_1(t+\tau_n, x(t+\tau_n - g(t+\tau_n)) \\ &\quad + \int_{-\infty}^{t+\tau_n} U(t+\tau_n, s)F_2(s, x(s), x(s-g(s)))ds. \\ &= F_1(t+\tau_n, x(t+\tau_n - g(t+\tau_n)) \\ &\quad + \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)F_2(s+\tau_n, x(s+\tau_n), x(s+\tau_n - g(s+\tau_n)))ds.\end{aligned}$$

Taking the norm on both sides, we have

$$\begin{aligned}\|(Fx)(t+\tau_n)\| &\leq \|K\|_{AA(X)} + \int_{-\infty}^{t} \|U(t+\tau_n, s+\tau_n)\|\|H(s+\tau_n)\|ds \\ &\leq M_1 + \frac{M_2 M}{\delta} \quad (\|H\| \leq M_2).\end{aligned}$$

By (H_1), for any fixed $s \in \mathbb{R}, \epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$ we have

$$\|H(s+\tau_n) - h(s)\| \leq \epsilon.$$

In addition by (C_4) for s and ϵ as above there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\| < \epsilon e^{\frac{-\delta}{2}(t-s)}.$$

Let $N = \max\{N_1, N_2\}$, then

$$\begin{aligned}&\|U(t+\tau_n, s+\tau_n)H(s+\tau_n) - U(t,s)h(s)\| \\ &\leq \|U(t+\tau_n, s+\tau_n) - U(t,s)\|\|H(s+\tau_n)\| + \|U(t,s)\|\|H(s+\tau_n) - h(s)\| \\ &\leq M_2 \epsilon e^{\frac{-\delta}{2}(t-s)} + M\epsilon e^{\frac{-\delta}{2}(t-s)}\end{aligned}$$

$$\Rightarrow U(t+\tau_n, s+\tau_n)H(s+\tau_n) \to U(t,s)h(s)$$

as $n \to \infty$ for all fixed $s \in \mathbb{R}$ and $t \geq s$. Since $K(.) \in AA(X)$, for any sequence $\{s_n\}$ of real numbers there exists a subsequence $\{\tau_n\}$ such that

$$\lim_{n\to\infty} K(t+\tau_n) = k(t), \quad \lim_{n\to\infty} k(t-\tau_n) = K(t).$$

Thus, we have $K(t+\tau_n) \to k(t)$ as $n \to \infty$. By Lebesgue dominated convergence theorem we get $(Fx)(t+\tau_n) \to Gx(t)$ as $n \to \infty$. In a similar way we can show that $(Gx)(t-\tau_n) \to (Fx)(t)$ as $n \to \infty$ for all $t \in \mathbb{R} \quad \Rightarrow Fx \in AA(X)$.

Theorem 3.1 *Let $x(.)$ be an almost automorphic function and F_1, F_2 and $U(t,s)$ satisfy all conditions from (C_1) to (C_4). Then equation (2) has unique almost automorphic mild solution, whenever $(L_{F_1} + 2L_{F_2}\frac{M}{\delta}) < 1$.*

Proof It follows by Lemma 3.1, that $Fx \in AA(X)$, whenever x does. Let us assume that

$$L_{F_1} = \sup_{t\in\mathbb{R}} L_{F_1}(t), \quad L_{F_2} = \sup_{t\in\mathbb{R}} L_{F_2}(t).$$

For $x, y \in AA(X)$, we have:

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\| \\
&\leq \|F_1(t, x(t - g(t))) - F_1(t, y(t - g(t)))\| \\
&+ \int_{-\infty}^{t} \|U(t,s)F_2(s, x(s), x(s - g(s))) - U(t,s)F_2(s, y(s), y(s - g(s)))\|ds \\
&\leq L_{F_1}(s)\|x - y\| \\
&+ L_{F_2}(s)\{\|x(s) - y(s)\| + \|x(s - g(s)) - y(s - g(s))\|\} \int_{-\infty}^{t} Me^{-\delta(t-s)}ds \\
&\leq L_{F_1}\|x - y\|_{AA(X)} + 2L_{F_2}\|x - y\|_{AA(X)} \int_{-\infty}^{t} Me^{-\delta(t-s)}ds \\
&\leq L_{F_1}\|x - y\|_{AA(X)} + 2L_{F_2}\frac{M}{\delta}.
\end{aligned}$$

By Banach contraction principle, F has a unique fixed point $x \in AA(X)$ such that $Fx = x$.

Fixing $s \in \mathbb{R}$, we have

$$x(t) = F_1(t, x(t - g(t))) + \int_{-\infty}^{t} U(t,s)F_2(s, x(s), x(s - g(s)))ds.$$

Since $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s$, let

$$x(\xi) = F_1(\xi, x(\xi - g(\xi))) + \int_{-\infty}^{\xi} U(\xi,s)F_2(s, x(s), x(s - g(s)))ds$$

so

$$U(t,\xi)x(\xi) = U(t,\xi)F_1(\xi, x(\xi - g(\xi))) + \int_{-\infty}^{\xi} U(t,s)F_2(s, x(s), x(s - g(s)))ds.$$

For $t \geq \xi$,

$$\begin{aligned}
\int_{\xi}^{t} U(t,s)F_2(s, x(s), x(s - g(s)))ds &= \int_{-\infty}^{t} U(t,s)F_2(s, x(s), x(s - g(s)))ds \\
&\quad - \int_{-\infty}^{\xi} U(t,s)F_2(s, x(s), x(s - g(s)))ds \\
&= x(t) - U(t,\xi)x(\xi) - F_1(t, x(t - g(t))) \\
&\quad + U(t,\xi)F_1(\xi, x(\xi - g(\xi))).
\end{aligned}$$

Hence we get

$$\begin{aligned}
x(t) &= F_1(t, x(t - g(t))) - U(t,\xi)F_1(\xi, x(\xi - g(\xi))) \\
&\quad + U(t,\xi)x(\xi) + \int_{\xi}^{t} U(t,s)F_2(s, x(s), x(s - g(s)))ds. \qquad (4)
\end{aligned}$$

Remark 3.1 Consider the following differential equation

$$\frac{d}{dt}(x(t) - F_1(t, x(t-g(t)))) = A(t)x(t) + F_2(t, x(t), \int_{-\infty}^{t} G(t-s)f(s,x(s))ds), \tag{5}$$

where $G \in L^1(\mathbb{R})$ and f is almost automophic, Lipschitz with respect to second variable. Now $f \in AA(\mathbb{R} \times X, X)$ and f is Lipshitz by Lemma 2.3, we have $f \in AA(X)$.
Let $f(t, x(t)) = \psi(t)$.

If we can show $\int_{-\infty}^{t} G(t-s)f(s,x(s))$ is almost automorphic, then as a consequence of the above theorem, equation (5) has a unique almost automorphic solution.
As ψ is almost automorphic for every sequence of real numbers $\{t_n\}$ there exists a subsequence $\{\tau_n\}$ such that $\lim_{n\to\infty} \psi(t+\tau_n) = \psi_1(t)$ is well defined for all $t \in \mathbb{R}$ and $\psi(t) = \lim_{n\to\infty} \psi_1(t-\tau_n)$ is well defined for all $t \in \mathbb{R}$.

Consider

$$\begin{aligned}
&\| \int_{-\infty}^{t+\tau_n} G(t+\tau_n - s)\psi(s)ds - \int_{-\infty}^{t} G(t-s)\psi_1(s)ds\| \\
&= \| \int_{-\infty}^{t} G(t-s)\psi(s+\tau_n)ds - \int_{-\infty}^{t} G(t-s)\psi_1(s)ds\| \\
&\le (\|\psi(s+\tau_n) - \psi_1(s)\|) \int_{-\infty}^{t} |G(t-s)|ds \\
&\le M'(\|\psi(s+\tau_n) - \psi_1(s)\|)
\end{aligned}$$

for some $M' < \infty \to 0$ as $n \to \infty$. Thus, $\int_{-\infty}^{t} G(t-s)\psi(s)ds$ is almost automorphic and we have the result.

4 Example

Consider the following equation

$$u'' + (\varepsilon_2 u^2 + 1)u' + u = \varepsilon_1 \frac{d}{dt}\big(\sin\big(\frac{1}{\sin t + \sin\sqrt{2}t}\big)u^2(t-g(t))\big) - \varepsilon_2(\cos t + \cos\sqrt{2}t).$$

Let $u = u_1$ and $u_1' = u_2$, then we can write the above equation in matrix form as follows

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{d}{dt}F_1(t, U(t-g(t))) + F_2(t, U(t), U(t-g(t))),$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$F_1(t, U(t-g(t))) = \begin{pmatrix} 0 \\ \sin(\frac{1}{\sin t + \sin\sqrt{2}t})u_1^2 \end{pmatrix},$$

$$F_2(t, U(t), U(t-g(t))) = \begin{pmatrix} 0 \\ \varepsilon_2(\cos t + \cos\sqrt{2}t) - \varepsilon_2 u_1^2 u_2 \end{pmatrix}.$$

This is of the form (2). Thus we can apply our results to ensure the existence and uniqueness of almost automorphic solutions.

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Quantum Dynamics of a Nonlinear Kicked Oscillator

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Abstract: The quantum dynamics of a nonlinear kicked oscillator is studied by a recently proposed complex non-hermitian Hamiltonian technique. It is shown that the probability density and the energy function display either a growing or a decaying exponential time dependence characteristic of absorption or dissipation. It is furthermore shown that though a decrease in the kicking period increases the diffusive motion leading to the ballistic spreading, increasing its value does not apparently favour any localization. The anharmonicity also enhances the dissipative dynamics but with time gives rise to energy crossings typical of a quantum chaos. The variation in the spatial periodicity of the delta-function kicking however exhibits a more complex behaviour showing diffusive character to super-diffusion leading to ballistic motion on the one side and the quantum localization on the other.

Keywords: *nonlinear kicked oscillator; quantum diffusion; dissipation; localization*

Mathematics Subject Classification (2000): 35Q72, 81Q50, 37L50.

1 Introduction

Recent years have witnessed a flurry of investigations in the area of quantum chaos and dynamical quantum localization and in this context the quantum dynamics of area preserving maps has attracted a particular attention [1, 2]. The kicked harmonic oscillator is an example that belongs to this class and has been studied quite extensively in the last two decades [3, 4, 5, 6]. The kicked harmonic oscillator however has generated some renewed interest in recent times for it simulates some interesting low-dimensional systems like quantum wires, semiconductor superlattices [7] or trapped ions [8] periodically

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kicked by intense monochromatic short-time pulses like laser light. The effect of Gross–Pitaevskii nonlinearity in the kicked oscillator has been recently studied by Artuso and Rebuzzini [9] and they have shown that the qualitative features depend strongly on the parameters of the system.

The dynamics of a kicked oscillator system can be determined by studying the generalized master equation for the probability distribution or by calculating the propagator and thereby constructing the time-dependent wave function using the propagator as the kernel for the time development. Of course the probability density obtained from these methods has to be consistent with the time-dependent Schrödinger equation. Several techniques have been proposed in recent years to deal with the nonlinear differential equations with varying degree of success and rigour. Kovalev et al. have developed the method of oriental manifold to study the geometric properties of nonlinear differential systems with control [10]. Stability of systems with linear and nonlinear perturbations has been studied by Jeffrey J. DaCunha [11]. Practical stability and controllability for a class of nonlinear discrete systems with time delay have been investigated by Su et al. [12].

Recently Liboff and Porter [13] have proposed a novel technique to study the energy absorption and dissipation in quantum systems by introducing a complex non-Hermitian term in the Hamiltonian. The purpose of the present paper is to apply this technique to study the dynamics of a kicked oscillator problem. Since the trapping potential of the real ion traps or the ion-ion potential in a quantum wire may not be strictly harmonic, we introduce the anharmonicity in the problem and study its effect on the oscillator dynamics. We also include a spatial periodicity in the kicking term. It must however be mentioned that the nonlinearity considered here is different from that studied in [9]. We consider the oscillator to be anharmonic, while in [9] the potential is nonlinear in the wavefunction itself and therefore the problem of [9] requires a self-consistent solution. Our problem will be important for a quantum wire like a carbon nanotube with an anharmonic confining potential and periodically kicked by a laser wave. We find, as expected from [13], that the probability density and the energy of a kicked nonlinear oscillator exhibit as a function of time a growing or a decaying behaviour depending on the sign of the coefficient that gives the kicking strength. We observe that as the kicking period is decreased, the motion of the nonlinear oscillator becomes more and more diffusive finally reaching the ballistic regime. However, interestingly enough, increasing the kicking period does not apparently yield any localization. The anharmonicity is also found to favour the dissipative dynamics and gives rise to crossing of the energy curves characteristic of quantum chaos. We furthermore show that the spatial periodicity of the kicking potential seems to be a very sensitive parameter the variation of which can lead to a variety of features ranging from a ballistic motion through classical diffusion to dynamical localization. In what follows we shall first briefly discuss the method of Liboff and Porter [13] and then apply it to the problem of a nonlinear kicked oscillator.

2 General Formalism of Liboff and Porter

Consider a complex Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(x) + i\hbar\alpha(x,t) = H_0 + i\hbar\alpha(x,t), \tag{1}$$

where $\alpha(x,t)$ can be written as a product of the space-part and the time-part and the unperturbed Hamiltonian satisfies the time-independent Schrödinger equation

$$H_0 u_n(x) = E_n^{(0)} u_n(x). \tag{2}$$

We can therefore write

$$[H_0 + i\hbar\alpha(x,t)]u_n(x) = E_n u_n(x) = [E_n^{(0)} + i\hbar\alpha(x,t)]u_n(x). \tag{3}$$

The time-dependent Schrödinger equation for the problem is given by

$$i\hbar\partial\Psi/\partial t = H\Psi = [H_0 + i\hbar\alpha(x,t)]\Psi, \tag{4}$$

where $\Psi(t)$ can be formally written as

$$\Psi(t) = exp\ [-\frac{i}{\hbar}\int_0^t d\lambda H(\lambda)]\Psi(0), \tag{5}$$

where $\Psi(0)$ is the initial state function given by

$$\Psi(0) = \sum_n a_n u_n. \tag{6}$$

The expectation value of the Hamiltonian at $t = 0$ yields

$$< \Psi(0)|H|\Psi(0) >= \sum_n |a_n|^2 E_n^{(0)} = E_0, \tag{7}$$

where the expansion coefficients may be fixed from the knowledge of the initial configuration of the state of the system. Substituting (6) in (5) and using the eigenvalue equation (3), we get

$$\Psi(t) = e^{g(t)} \sum_n a_n u_n e^{-\frac{i}{\hbar}E_n^{(0)}t}, \tag{8}$$

where

$$g(t) = \int_0^t dt\alpha(t), \tag{9}$$

so that the real part of the energy expectation value at time t ($E(t)$) is given by

$$E(t) = Re < \Psi(t)|H|\Psi(t) >= E_0 e^{\ 2\int_0^t d\lambda\alpha(\lambda)}. \tag{10}$$

3 Kicked Harmonic Oscillator

We shall now employ this formalism to a kicked nonlinear oscillator for which we write

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4 \tag{11}$$

and choose $\alpha(x,t)$ as

$$\alpha(x,t) = -\varepsilon < \cos(kx) > \sum_{s=1}^{N} \delta(t - Ts), \tag{12}$$

where ϵ gives the measure of the δ-function kicking, k measures the spatial periodicity of the kicking potential, T is the kicking period, N is the number of kicks and $< \cos(kx) >\equiv \kappa$ is the expectation value of $\cos(kx)$ taken with respect to the eigenstate of the effective harmonic oscillator. We assume that the initial state is prepared in the n-th excited state of the linear oscillator and incorporate the quartic term using a mean-field approximation so that the unperturbed potential can be written as an effective harmonic oscillator with a new frequency

$$\tilde{\omega} = [\omega^2 + \frac{2\lambda}{m} < x^2 >]^{1/2}, \tag{13}$$

where

$$< x^2 >= \int_{-\infty}^{\infty} u_n^{(0)} x^2 u_n^{(0)} dx. \tag{14}$$

According to our initial configuration,

$$< x^2 >= (n + \frac{1}{2}), \tag{15}$$

and hence

$$\tilde{\omega} = [1 + 2\lambda(n + \frac{1}{2})]^{1/2}, \tag{16}$$

where we have assumed $m = \hbar = \omega = 1$. A simple calculation shows that κ can be obtained as

$$\kappa = e^{-\beta^2} L_n(\beta^2), \tag{17}$$

where $\beta = (1/2\tilde{\omega})^{1/2} k$ and $L_n(x)$ is the Laguerre polynomial of order n and is given by

$$L_n(x) = \sum_{m=0}^{n} (-1)^m \frac{n!\beta^{2m}}{(m!)^2 (n-m)!}. \tag{18}$$

The time-dependent energy $E(t)$ is then finally obtained as

$$E(t) = E_0 P(t), \tag{19}$$

where $P(t)$ is the temporal probability that the system would be found in the state $\Psi(t)$ at time t and is given by

$$P(t) =< \Psi(t)|\Psi(t) >= e^{-2\varepsilon\kappa\phi(t)}, \tag{20}$$

where

$$\phi(t) = \sum_{s=1}^{N} \int_0^t \delta(\lambda - sT) d\lambda. \tag{21}$$

One can immediately see that $\phi(t)$ is equal to the number of s values for which s is less than t/T. We would like to point out here that in [10] the value of $\phi(t)$ has been determined erroneously. In fact the definition of $\phi(t)$ in [10] violates causality. We obtain

$$P_l[lT \le t < (l+1)T] = e^{-2\varepsilon\kappa l}, \tag{22}$$

where $l = 0, 1, 2, \dots,$ and consequently the time-dependent energy of the n-th excited state of the nonlinear oscillator reads

$$E_n[lT \le t < (l+1)T] = (n + \frac{1}{2}) e^{-2\varepsilon\kappa l}. \tag{23}$$

4 Numerical Results and Discussion

In Figure 1, we show the behaviour of the energy of the first excited state as a function of time. In (12), we have chosen the sign of the kicking term as negative in order that the dynamics becomes dissipative. We show the energy dissipation with time for three values of the kicking strength ε. One can observe that the delta-function kicking results in instantaneous dissipations in the energy at the kicking times and remains constant in between two successive kicks leading to a stair-case like structure. As the kicking strength increases, the dissipation of course becomes more and more rapid.

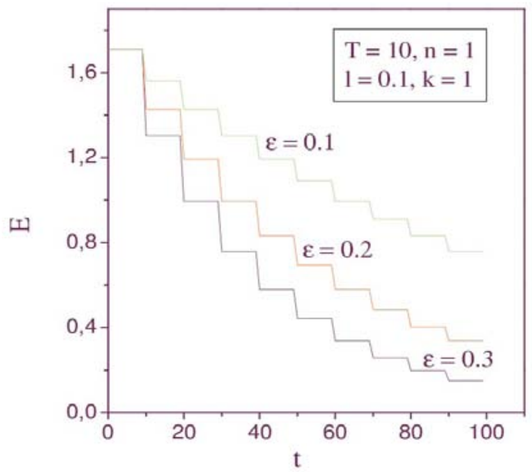

Figure 1: The first excited state energy E as a function of time for three values of the kicking strength ε.

In Figure 2 we compare the dissipation for the first two excited states ($n = 1$ and $n = 2$) of the nonlinear oscillator. The dissipation behaviour of the second excited state is almost similar to the first one, except that the decay rate is faster for the second excited state than that for the first excited state. In Figure 3 we study the energy-time behaviour for the first excited state for a few values of the nonlinearity parameter λ. We find that the energy, as expected for the present system, is initially larger for a larger value of λ but the decay is interestingly faster for larger λ-values. This leads to an interesting crossing of the energies at a long enough time which may be attributed to a dynamics akin to quantum chaos.

The variation of the dissipative dynamics as we change the kicking period T is studied in Figure 4. It is clearly evident that the dissipation becomes more and more rapid as the kicking period decreases so much so that for very low values of T, the dynamics is essentially ballistic, while for very large values of T the energy spreading is more or less diffusive. However we do not observe any localization here.

From (17) and (22) one can note that neither the temporal probability $P(t)$ nor the energy E(t) is a monotonically increasing or decreasing function of k. In fact the dependence of E on k is quite interesting which we show in Figure 5 where we have plotted the energy for the first two excited states as a function of k for three values of time t. One can see that the first excited state energy has two maxima, lying symmetrically

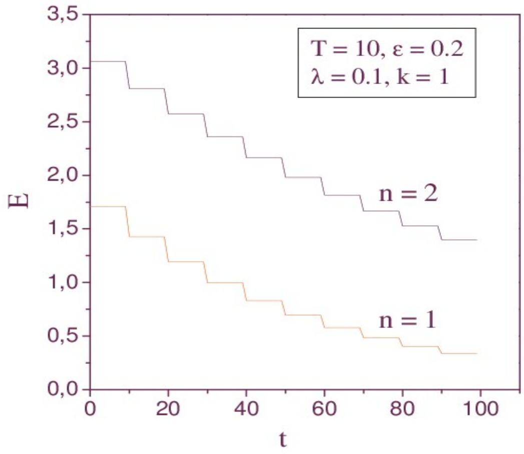

Figure 2: The decay of the energies of first two excited states $n = 1$ and $n = 2$ with time.

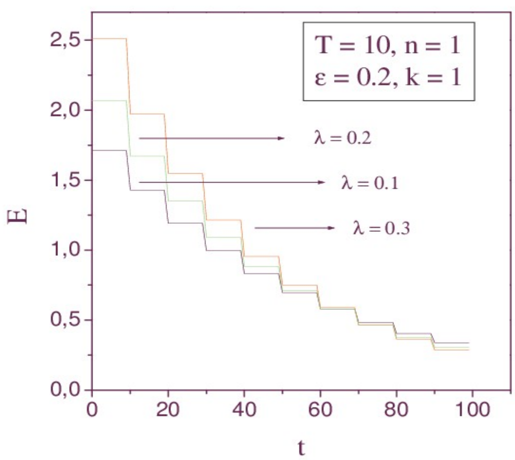

Figure 3: The dissipation of the first excited state with time for three values of the nonlinearity parameter λ.

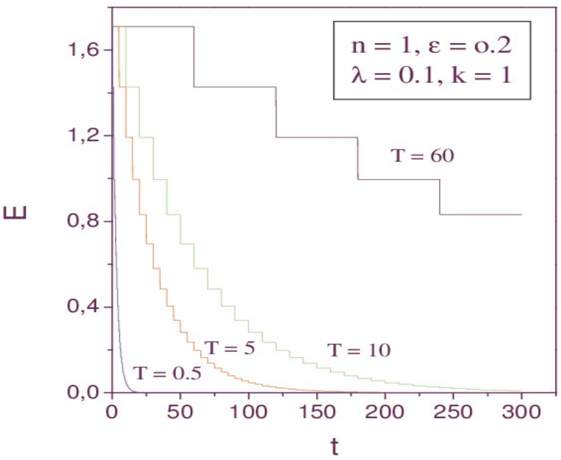

Figure 4: The dissipation of the first excited state with time for four values of the kicking period T.

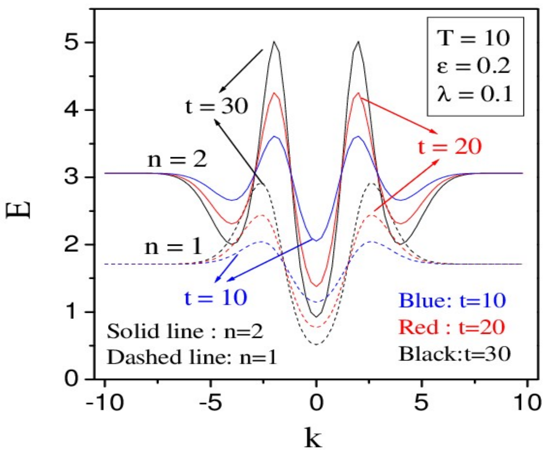

Figure 5: The variation of the time-dependent energy of the first two excited states as a function of the spatial periodicity k of the kicking potential for three values of time t.

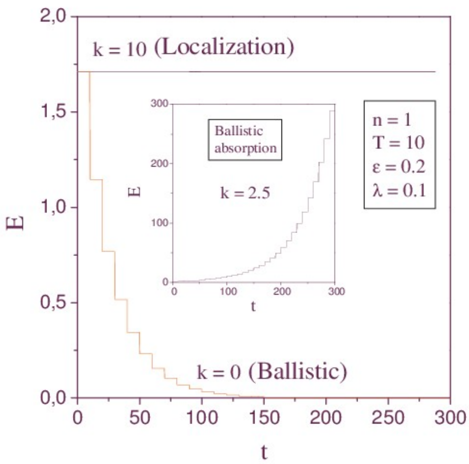

Figure 6: The variation of the energy of the first excited state as a function of time t for $k = 0$ and $k = 10$. The inset shows the behaviour for $k = 2.5$.

about $k = 0$ where it has a central minimum and the energy saturates to a constant after k reaches some critical value. Interestingly, however, the locations of the maxima and minima on the k-line do not change with time. For $n = 2$, however, two additional (secondary) minima develop symmetrically on either side of $k = 0$ and the maxima get shifted outward on both sides of $k = 0$, while the central minima still lie at $k = 0$.

The maxima-minima structure seems to have an interesting bearing on the dynamical behaviour of the system and throws up a variety of possibilities for different ranges of the k-values. For $n = 1$, with the parameter values we have chosen, the maxima occurs at around $k = \pm 2.5$. The central minimum for all the cases however occur at $k = 0$ as has already pointed out. This value of $k = 0$ gives dissipative dynamics in the ballistic regime as shown in Figure 6. It is quite clear from Figure 5 that for large values of k the system would exhibit localization. We have confirmed this behaviour by plotting E as a function of t for $k = 10$ in Figure 6. Interestingly however, $k = 2.5$ for the ground state corresponds to ballistic absorption. In the inset of Figure 6 we display this behaviour.

In Figure 7 we show in more detail the dissipative behaviour of the system for different values of k. For the value of k close to 1.48, one can observe that the dynamics is more or less diffusive and for lower values of k it becomes more and more super-diffusive and finally reaches almost the ballistic limit, while for about $k = 1.51$, the system shows a dynamical localization.

5 Conclusion

In conclusion, we have studied the dynamics of a nonlinear oscillator kicked by a time-periodic δ-function potential that has a spatial periodicity of the cosine-form using a complex nonhermitian Hamiltonian technique recently proposed by Liboff and Porter [13]. We have observed that the system can exhibit exponential growth or decay depending on the sign of the kicking term and the value of the spatial periodicity parameter. In particular, we have studied the case of dissipation and have shown that it increases

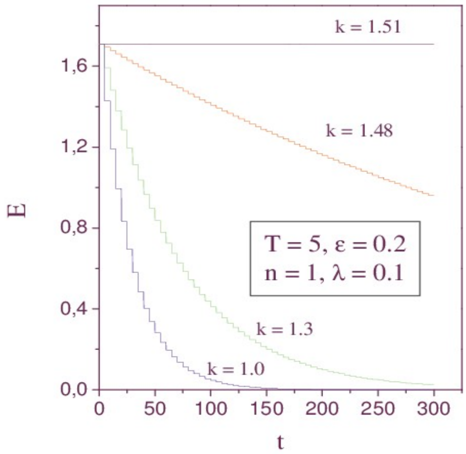

Figure 7: The variation of the energy of the first excited state as a function of time t for $k = 1.0$, $k = 1.3$, $k = 1.48$ and $k = 1.51$.

with increasing kicking strength. The dissipation however takes place instantaneously with kicking and energy remains constant between two successive kicking leading to a stair-case structure. As a function of the nonlinearity parameter, there occurs a crossing of the energy curves that seems to characterize the onset of a quantum chaos. However, the system never shows any indication of localization for any value of the kicking period that we have considered. Rather it exhibits a more and more diffusive behaviour as the kicking period decreases reaching finally the ballistic limit. Most interestingly, the dynamics of the system seems to depend quite sensitively on the spatial periodicity parameter, the variation of which gives rise to a variety of rich phenomena ranging from diffusive to super-diffusive behaviour to ballistic spreading on the one side and to dynamical quantum localization on the other.

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Analysis of Periodic Nonautonomous Inhomogeneous Systems

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Abstract: This paper addresses the analysis of a class of near-periodic systems in which the dynamics can be described by a set of nonlinear differential equations with no known equilibrium solution. Linear models are developed by performing a power series expansion about a time-periodic reference motion. The result is a nonautonomous, inhomogeneous system consisting of a set of parametrically excited linear differential equations with time-periodic forcing excitations. The method of linearization assures that the time period of the parametric and forcing excitations is the same.

Floquet theory is used to address the stability of the homogeneous parametrically excited system. However, the linear system is inhomogeneous due to the forcing excitation. A modification of Floquet theory allows the use of Floquet multipliers or characteristic exponents to analytically examine the transitory and steady-state behavior of the inhomogeneous system.

Keywords: *dynamical; Floquet theory; linear control.*

Mathematics Subject Classification (2000): 34A34, 93C05.

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1 Introduction

There are several methods available to determine the long-term behavior of a system that can be described by a set of nonlinear differential equations

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}). \tag{1}$$

The most obvious method to investigate the behavior of a nonlinear system is to numerically solve the differential equations subject to specified initial conditions. One example would be the ordinary differential equations solvers in MATLAB [19]. Modern computational capability makes the issue of computer processor and memory limitations virtually irrelevant. However, numerical simulations can not always provide the definitive label of *stability.* The solutions may be virtually identical for a system that is slightly asymptotically stable, neutrally stable, or slightly unstable unless the simulation is carried out for a very long time.

Stability theorems generally address stability about an equilibrium or about a known solution. Stability in the sense of Lyapunov [17, 18] requires that, for motion about an equilibrium, the system output be dependent upon the magnitude of the initial conditions. Similar theorems loosen the requirement for stability about an equilibrium and address stability about a known solution to the system [5].

The direct method of Lyapunov uses a Lyapunov function, $v(x)$ to directly assess the stability of the differential equations in question without having to determine a first variation [11, 15]. Furthermore, the converse of the theorem is also true. If the equilibrium is stable, then the function $v(x)$ exists. However, there is no "prescription" for determining an appropriate Lyapunov function. The function can be difficult to determine, particularly for a complex system.

The strength of the nonlinearity of a system determines whether it is periodic, quasi-periodic or chaotic. Poincaré introduced the concept of a *phase-space* where all possible motions of a system are represented by a family of trajectories [7]. The degree to which a system is chaotic is determined by the sensitivity of the trajectories to initial conditions or perturbations, where small changes can cause widely diverging outcomes. The sensitivity to initial conditions can be quantified by a Lyapunov exponent. In general, Lyapunov exponents can not be found analytically and require the use of numerical methods [8].

A Poincaré Section maps the intersection of a dynamical orbit in state-space with a one-dimension lower subspace (phase-space) that is transverse to the flow of the orbit. While a Poincaré map can aid in determining stability [4, 10, 16], it is essentially a schematic for presenting the results of a numerical simulation at discrete time periods. If the period of a solution is many times the fundamental sample period used for the map, it may require simulation for a long time before the repetition appears.

The existence of many linear analysis tools justifies the attempt to linearize the system of (1). In general, a power series expansion about an equilibrium results in a reduced equation known as the first variation or first approximation of (1) with respect to the equilibrium condition [11]. The result is a constant coefficient linear system given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{2}$$

Classical linear analysis methods for determining the stability of this linear system are well known [2, 6].

If the known solution used for linearization is nonautonomous, the matrix $\mathbf{A}$ is time varying. In general, linear analysis methods can be applied to time-varying systems with some modification such as canonical transformations or substitution of dynamic eigenvalues [13, 14, 23]. The stability of homogeneous linear equations with a time-periodic $\mathbf{A}$ matrix can be assessed by applying methods such as Hill's method of infinite determinants [20, 22] or Floquet–Lyapunov theory [3, 11, 26]. Floquet theory can also be used to determine if an inhomogeneous system has periodic solutions [3]. For time-periodic systems with time-periodic forcing functions, the literature typically addresses this form of steady-state behavior only [12, 24]. If the homogeneous equation exhibits asymptotic stability then the forced oscillations tend toward a periodic steady-state [3, 11].

This paper presents an extension of Floquet theory to a system which has no equilibrium or known solution. Equations are linearized about a time-periodic motion which closely approximates the nonlinear behavior. The behavior is almost periodic (a dynamical system that appears to almost retrace an orbit through phase space [1]). The result is a time-periodic linear system driven by a time-periodic forcing excitation having the same time period T, as the coefficients of $\mathbf{A}(t)$. The extension applies to the general case and is not limited to asymptotic behavior. Based on Floquet multipliers, the stability of the inhomogeneous system can be analyzed and performance metrics analogous to classical control theory settling time can be determined. The theoretical development is validated using a spinning pendulum.

2 Nonautonomous Inhomogeneous Systems from a Nonequilibrium Reference

The autonomous nonlinear differential equations that describe the motion of interest are given by

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}(\mathbf{z}), \\ \mathbf{f}(\mathbf{z}) &\neq \mathbf{0}, \end{aligned} \tag{3}$$

which have no known solution or equilibrium to be used as a reference for analyzing stability. Additionally, the solution is unknown except through numerical integration. However, the motion is known to be almost periodic and, in some type of limit behavior, periodic.

Lacking a traditional equilibrium or solution, the nominal periodic motion of the system will be used as a reference condition $\mathbf{z}_R$, and a series expansion is performed

$$\begin{aligned} \dot{\mathbf{z}}_R + \delta\dot{\mathbf{z}} &= \mathbf{f}(\mathbf{z}_R) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right]_R \delta\mathbf{z}, \\ \dot{\mathbf{z}}_R &\neq \mathbf{f}(\mathbf{z}_R). \end{aligned} \tag{4}$$

The reference condition is not a solution to (3) and cannot be eliminated from the equations, resulting in a linear system containing a forcing excitation of the form

$$\begin{aligned} \delta\dot{\mathbf{z}} &= \mathbf{A}(t)\delta\mathbf{z} + \mathbf{g}(t), \\ \mathbf{A}(t) &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{z}}\right]_R, \\ \mathbf{g}(t) &= \mathbf{f}(\mathbf{z}_R) - \dot{\mathbf{z}}_{\mathbf{R}}, \end{aligned} \tag{5}$$

with a time-varying $\mathbf{A}$ matrix and a time-varying forcing excitation $\mathbf{g}$. By selection of the reference condition as time-periodic, the linear system of (5) has the following properties

$$\begin{aligned} \mathbf{A}(t) &= \mathbf{A}(t+T), \\ \mathbf{g}(t) &= \mathbf{g}(t+T), \end{aligned} \tag{6}$$

where T is the period common to both the matrix $\mathbf{A}$ and the vector $\mathbf{g}$.

3 Floquet Theory

Linear homogeneous differential equations with time-periodic coefficients given by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{7}$$

where $\mathbf{A}$ is time-periodic

$$\mathbf{A}(t) = \mathbf{A}(t+T) \tag{8}$$

and T is the time period can be assessed by applying Floquet–Lyapunov theory as given in Theorem 3.1[20, 26].

Theorem 3.1 *(Floquet–Lyapunov theorem). Any fundamental matrix $\mathbf{X}(t)$ of equation (7) with T-periodic coefficients is expressible in the form*

$$\mathbf{X}(t) = \mathbf{F}(t)e^{\mathbf{K}t}, \tag{9}$$

where $\mathbf{F}(t)$ is a nonsingular continuous T-periodic $n \times n$ matrix-function whose derivative is an integrable piecewise-continuous function, and $\mathbf{K}$ is some constant matrix.

Given that $\mathbf{F}(t)$ is time-periodic, the stability of the trivial solution to (7) depends entirely upon the eigenvalues of the matrix $\mathbf{K}$. The eigenvalues of $\mathbf{K}$ are known as the Floquet characteristic exponents, ε, and can be found by first determining the eigenvalues of $\mathbf{X}(T)$, known as the Floquet multipliers, σ. The matrix $\mathbf{X}(T)$, called the monodromy matrix, is the fundamental set of solutions to (7) when $t = T$ and with initial conditions of $\mathbf{X}(0) = \mathbf{I}$. The monodromy matrix can be determined numerically or through other means such as a multiple parameter perturbation method [25]. The characteristic exponents are then determined by

$$\epsilon = \frac{\ln \sigma}{T}. \tag{10}$$

Table 1 summarizes properties of solutions corresponding to the properties of the characteristic exponents and multipliers.

4 Extended Floquet Theory

Floquet theory does not address stability of the inhomogeneous system described by (5) where the forcing excitation $\mathbf{g}(t)$ is present. However, the T-periodic nature of $\mathbf{g}(t)$ allows for an extension to the theory. The solution to the inhomogeneous system of (5) can be expressed in terms of $\mathbf{X}(T)$ as follows [26]

$$\mathbf{z}(t) = \mathbf{X}(t)\left[\mathbf{x}(0) + \int_0^t \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{11}$$

Property of Solutions	Characteristic Exponents, ε	Multipliers, σ
Lyapunov Stability	Real parts nonpositive: zero or pure imaginary ϵ (if present) are semisimple eigenvalues of $\mathbf{K}$	Inside or on unit circle. Latter case corresponds to semisimple eigenvalues of $\mathbf{K}$
Asymptotic Stability	Real parts negative	Inside unit circle
Instability	At least one characteristic exponent with positive real part or a pure imaginary (or zero) exponent that is not semisimple	At least one multipier either outside the unit circle or on the unit circle and not semisimple

Table 1: Properties of solutions of systems with periodic coefficients.

Given that, according to Floquet theory, the monodromy matrix satisfies the following identity at time $t = t + T$

$$\mathbf{X}(t+T) \equiv \mathbf{X}(t)\mathbf{X}(T), \tag{12}$$

the following theorem for the solution to (11) after n time periods $\mathbf{z}(nT)$ can be established.

Theorem 4.1 *The solution to (11) after n time periods, where n is an integer, is given by*

$$\mathbf{z}(nT) = \mathbf{X}(T)^n\mathbf{x}(0) + \left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right] \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau. \tag{13}$$

Proof At $t = T$, the solution to (11) becomes

$$\mathbf{z}(T) = \mathbf{X}(T)\left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{14}$$

Extending (14) to two time periods $2T$, yields

$$\mathbf{z}(2T) = \mathbf{X}(2T)\left[\mathbf{x}(0) + \int_0^{2T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{15}$$

Expanding the term inside the integer and applying the identity of (12) yield

$$\mathbf{z}(2T) = \mathbf{X}(T)^2\left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_T^{2T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{16}$$

Applying the variable change $U = \tau - T$, $dU = d\tau$ to equation (16) results in

$$\mathbf{z}(2T) = \mathbf{X}(T)^2\left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_0^T \mathbf{X}(U+T)^{-1}\mathbf{g}(U+T)dU\right]. \tag{17}$$

Once again applying the identity from (12) and substitution of the time-periodic properties of (6) we get

$$\mathbf{z}(2T) = \mathbf{X}(T)^2 \left[\mathbf{x}(0) + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_0^T \mathbf{X}(T)^{-1}\mathbf{X}(U)^{-1}\mathbf{g}(U)dU \right]. \tag{18}$$

Finally, (18) can be reduced to

$$\mathbf{z}(2T) = \mathbf{X}(T)^2\mathbf{x}(0) + \left[\mathbf{X}(T)^2 + \mathbf{X}(T)\right] \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau. \tag{19}$$

Repeated application of the identity from (12) and substitution of (6) leads to a solution to (11) after n time periods nT

$$\mathbf{z}(nT) = \underbrace{\mathbf{X}(T)^n\mathbf{x}(0)}_{Homogeneous} + \underbrace{\underbrace{\left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right]}_{Summation} \underbrace{\int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau}_{Integral,\ \mathbf{\Lambda}}}_{Inhomogeneous}. \tag{20}$$

Note that the behavior of (20) as n approaches to infinity can be predicted strictly based on knowledge of the response during the first time period T. The steady-state behavior can be evaluated by examining each term in (20) as n approaches infinity. The homogeneous and inhomogeneous terms will be evaluated separately in the following subsections.

4.1 Homogeneous behavior

The behavior as n increases to infinity of the homogeneous term in (20) is dependent upon the Floquet multipliers of the monodromy matrix $\mathbf{X}(T)$. The behavior is given in Table 2 for various properties of the magnitude of the largest Floquet multiplier $\rho[\mathbf{X}(T)]$.

The Limits of Powers Theorem [21], given in Theorem 4.2, guarantees the existence of $\lim_{n\to\infty} \mathbf{X}(T)^n$ for Properties 1 and 2.

Theorem 4.2 *(Limits of Powers Theorem). For* $\mathbf{X} \in C^{k\times k}$*,* $\lim_{n\to\infty} \mathbf{X}^n$ *exists if and only if* $\rho[\mathbf{X}] < 1$ *or* $\rho[\mathbf{X}] = 1$*, where* 1 *is the only eigenvalue on the unit circle and is semisimple.*

When it exists $\lim_{n\to\infty} \mathbf{X}^n =$ *the projector onto* $N(\mathbf{I} - \mathbf{A})$ *along* $R(\mathbf{I} - \mathbf{A})$*, where* N *is the null space and* R *is the range space.*

Property 2 is of particular interest. According to Floquet theory, as summarized in Table 1, a system with the largest multiplier(s) identically equal to one (semisimple) exhibits stability in the sense of Lyapunov, not asymptotic stability as in Property 1. Therefore, the $\lim_{n\to\infty} \mathbf{X}(T)^n$ exists but is not necessarily zero. The concept of Cesaro summability [21], given in Theorem 4.3, yields additional information about the value of the limit for Property 2.

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	$\mathbf{X}(T)^n$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Converges to 0
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Converges to $\mathbf{G}$
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Nonconvergent Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues, not semisimple	Divergent
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Divergent

Table 2: Properties of the homogeneous term of $\mathbf{z}(nT)$.

Theorem 4.3 *(Cesaro summability).*

For $\mathbf{X} \in C^{k\times k}$, $\mathbf{X}$ is Cesaro summable if and only if

$\rho[\mathbf{X}] < 1$ or $\rho[\mathbf{X}] = 1$ with each eigenvalue on the unit circle being semisimple.

When it exists the Cesaro limit

$$\lim_{n\to\infty} \frac{\mathbf{I}+\mathbf{X}+\ldots+\mathbf{X}^{n-1}}{n} = \mathbf{G}$$

is the projector onto $N(\mathbf{I}-\mathbf{A})$ along $R(\mathbf{I}-\mathbf{A})$, exactly the same as the ordinary limit described above in the Limits of Powers Theorem, had it existed.

$\mathbf{G} \neq 0$ if and only if 1 is an eigenvalue of $\mathbf{X}$, in which case $\mathbf{G}$ is the spectral projector associated with an eigenvalue of 1.

Note that the existence of the $\lim_{n\to\infty} \mathbf{X}^n$ implies that the Cesaro sum $\mathbf{G}$ exists and they have the same value. However, the existence of $\mathbf{G}$ does not imply the existence of $\lim_{n\to\infty} \mathbf{X}^n$. The Cesaro sum also exists when the largest Floquet multiplier magnitude is equal to one. In other words, the multiplier is not identically one, but has both real and imaginary parts with magnitude equal to one. This is the case for Property 3. The Cesaro sum $\mathbf{G}$ exists and $\mathbf{G} = 0$, but $\lim_{n\to\infty} \mathbf{X}(T)^n$ does not exist. The Cesaro sum is essentially the mean value of $\mathbf{X}(T)^n$ as n increases to infinity, indicting that $\mathbf{X}(T)^n$ oscillates with both positive and negative values around a mean of zero. Therefore, the homogeneous portion of the solution to $\mathbf{z}(nT)$ does not converge but remains bounded and oscillates indefinitely. As predicted by Table 1, Property 3 also exhibits stability in the sense of Lyapunov.

For Properties 4 and 5, the $\lim_{n\to\infty} \mathbf{X}(T)^n$ does not exist and also the Cesaro Sum does not exist. Therefore, the solution for $\mathbf{z}(nT)$ diverges. This result is in accordance with Floquet theory which predicts instability.

The results shown in Table 2 are completely consistent with the behavior predicted by Floquet theory in Table 1. This is not surprising as the homogeneous term of (20) is the solution to the time-periodic system in (7) at discrete multiples of the time period T.

4.2 Inhomogeneous behavior

The inhomogeneous term in (20) consists of the product of a summation and an integral term. The integral term, $\mathbf{\Lambda}$, is a definite integral over the time span of zero to T and is therefore a constant vector. The convergent or divergent behavior of the inhomogeneous term will be determined by the summation term and the Floquet multipliers of $\mathbf{X}(T)$. This result is given in Table 3.

For Floquet multipliers with magnitude less than one, as in Property 1, the convergence characteristics of the summation $[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$ are given by the Neumann series [21], shown in Theorem 4.4.

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	$[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Converges to $[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]$
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Unbounded
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Nonconvergent Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues, not semisimple	Unbounded
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Unbounded

Table 3: Properties of the summation term of $\mathbf{z}(nT)$.

Theorem 4.4 *(Neumann series). For $\mathbf{X} \in C^{k \times k}$, the following statements are equivalent:*

the Neumann series $\mathbf{I} + \mathbf{X} + \mathbf{X}^2 + \dots$ converges;

$\rho[\mathbf{X}] < 1$*;*

$\lim_{n\to\infty} \mathbf{X}^n = 0$*;*

In which case $[\mathbf{I} - \mathbf{X}]^{-1}$ exists and $\sum_{n=0}^{\infty} \mathbf{X}^n = [\mathbf{I} - \mathbf{X}]^{-1}$.

Although $\lim_{n\to\infty} \mathbf{X}^n$ does exist for Property 2, according to Cesaro summability, the limit is a non-zero constant $\mathbf{G}$. In the limit, the summation term $[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)]$ becomes a diverging algebraic series increasing by $\mathbf{G}$ with each additional term. Therefore, the summation term diverges.

For Property 3 with semisimple Floquet multipliers on the unit circle but not identically one, the Cesaro sum $\mathbf{G} = 0$. However, as mentioned in the previous section, the $\lim_{n\to\infty} \mathbf{X}(T)^n$ does not exist. $\mathbf{X}(T)^n$ oscillates with both positive and negative values around a mean of zero. Therefore, the summation term, oscillates around some constant value.

For Properties 4 and 5, the summation term is unbounded.

4.3 Stability of the inhomogeneous system

Table 4 shows how the addition of a forcing term affects the steady-state solution of the inhomogeneous system. With all Floquet multipliers of $\mathbf{X}(T)$ less than one, as for

Property	$\rho[\mathbf{X}(T)]$ Mag. of largest Floquet Multiplier	Floquet $\mathbf{x}(nT)$	Inhomogeneous $\mathbf{z}(nT)$
1	$\rho[\mathbf{X}(T)] < 1$ Semisimple	Asymptotic Stability	Bounded
2	$\rho[\mathbf{X}(T)] = 1$ One is the only multiplier on the unit circle and is semisimple	Lyapunov Stability	Unbounded
3	$\rho[\mathbf{X}(T)] = 1$ Multipliers, other than one, on the unit circle are semisimple	Lyapunov Stability	Bounded
4	$\rho[\mathbf{X}(T)] = 1$ Multiple eigenvalues not semisimple	Unstable	Unbounded
5	$\rho[\mathbf{X}(T)] > 1$ Multipliers outside the unit circle	Unstable	Unbounded

Table 4: Homogeneous vs inhomogeneous properties of $\mathbf{z}(nT)$.

Property 1, $\mathbf{z}(nT)$ converges to a nonzero value instead of to zero (asymptotic stability) for the homogeneous system. The solution converges to

$$\lim_{n\ to\infty} \mathbf{z}(nT) = [\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \mathbf{\Lambda}. \tag{21}$$

For Property 2, with Floquet multipliers of $\mathbf{X}(T)$ identically equal to one (semisimple), $\mathbf{z}(nT)$ is driven from Lyapunov stable to unbounded with the addition of the forcing excitation. The summation in the homogeneous term is unbounded, causing the solution to diverge. The solution in the limit is given by

$$\lim_{n\ to\infty} \mathbf{z}(nT) = \mathbf{x}(0) + \left[\mathbf{X}(T)^n + ... + \mathbf{X}(T)^2 + \mathbf{X}(T)\right]\ \mathbf{\Lambda}. \tag{22}$$

For Property 3, if the largest Floquet multiplier of $\mathbf{X}(T)$ has magnitude equal to one, is semisimple, but is not identically one, then the Lyapunov stable homogeneous system remains bounded with the addition of the forcing term. Neither term in equation 20 converges to a limit, indicating oscillation within some finite bound. The basic behavior of the system has not changed with the addition of a forcing term.

For Properties 4 and 5, both the homogeneous and inhomogeneous terms of equation 20 diverge and $\mathbf{z}(nT)$ is unbounded. The basic behavior of the system has not changed with the addition of a forcing term.

To summarize, there are two instances where the addition of the forcing excitation changes the fundamental behavior of the system. First, for Property 1, $\mathbf{z}(nT)$ converges to a nonzero steady-state instead of to zero for the homogeneous system. Second, for Property 2, $\mathbf{z}(nT)$ is driven from Lyapunov stable to unbounded with the addition of the forcing excitation.

Lyapunov stability [17, 18] presupposes that motion is analyzed with respect to an equilibrium or rest condition. As explained earlier, the system of interest has no equilibrium, and the nonlinear differential equations are linearized about a time-varying reference condition. Therefore, when evaluating the steady-state behavior of $\mathbf{z}(nT)$ in Table 4, the results are with respect to the reference behavior $\mathbf{z}_R(t)$. In relevant literature, a forced time-periodic system with or without an equilibrium is termed stable or asymptotically stable according to the Floquet multipliers, and the steady-state behavior is time-periodic [12, 24]. However, Lyapunov stability requires that the solution can be made arbitrarily small by changing the value of the initial conditions. For Properties 1 and 3, the steady-state solution is not dependent only on the initial conditions. For this reason, Table 3 utilizes the terms *bounded* or *unbounded* to refer to $\mathbf{z}(nT)$ as opposed to *stable* or *unstable*.

4.4 Transient behavior

A linear homogeneous differential equation given by

$$\ddot{x} + B\dot{x} + Cx = 0, \tag{23}$$

where B and C are constants can be expressed as a set of first-order equations in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{24}$$

where $\mathbf{A}$ is a constant coefficient matrix. In classical control theory, the transient behavior can be determined by the eigenvalues λ of the $\mathbf{A}$ matrix if the eigenvalues are a complex pair with real parts less than zero [2, 5, 6].

$$\lambda = \sigma \pm j\omega_d, \ \sigma < 0. \tag{25}$$

If so, the solution to (23) is expressed as

$$\mathbf{x}(t) = e^{\sigma t}(\mathbf{C_1} sin\omega_d t + \mathbf{C_2} cos\omega_d t) = \mathbf{C}e^{\sigma t}(sin\omega_d t + \boldsymbol{\phi}), \tag{26}$$

where $\mathbf{C}$, $\mathbf{C_1}$ and $\mathbf{C_2}$ are vector constants determined by the initial conditions $\mathbf{x}(0)$, $\sigma = -\zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The parameter ζ is the damping ratio of the second-order system, ω_n is the natural frequency and $\boldsymbol{\phi}$ are phase angles. The exponential term $\mathbf{C}e^{\sigma t}$ defines a decaying *envelope* that determines the rate at which the sinusoidal oscillations decrease to zero with time. A transient characteristic is the time constant

$$T_c = \frac{1}{\sigma}, \tag{27}$$

which is the time at which the exponential decreases to 37 percent of the initial value. A related characteristic is the settling time

$$T_s = \frac{number\ of\ time\ constants}{\sigma} \tag{28}$$

which is the time at which the exponential decreases to a desired absolute percent of the initial value. For example, the settling time to within 2 percent is approximately 4 time constants. The time constant and settling time are characteristics that can be extended to the homogeneous portion of (20).

The assumption is made that the homogeneous portion of (20) is second-order with complex-conjugate Floquet characteristic exponents with negative real parts (Floquet multipliers will lie inside the unit circle). According to Table 2, $\mathbf{X}(T)^n$ will converge to zero. At each multiple of n, the matrix $\mathbf{A}(nT)$ has the same constant value. Therefore, the homogeneous solution, $\mathbf{x}(nT) = \mathbf{X}(T)^n\mathbf{x}(0)$ at each multiple of the time period is identical to the solution of a constant coefficient system and $\mathbf{x}(nT)$ will lie along a damped sinusoid given by

$$\tilde{\mathbf{x}}(t) = e^{\sigma t}(\mathbf{C_1} sin\omega_d t + \mathbf{C_2} cos\omega_d t) = \mathbf{C}e^{\sigma t}(sin\omega_d t + \boldsymbol{\phi}). \tag{29}$$

Therefore, the homogeneous solution, $\mathbf{x}(nT)$, will also converge within the exponential envelope $\mathbf{C}e^{\sigma t}$. The classical control theory concepts of time constant and settling time can be directly applied to the homogeneous portion of $\mathbf{z}(nT)$. The number of integer time periods to reach the required settling time is given by

$$n_s = \frac{number\ of\ time\ constants}{\sigma T} = \frac{T_s}{T}, \tag{30}$$

where n_s can be rounded to the next higher integer and guarantee that $\mathbf{x}(nT)$ is equal to (or less than) the required percent of its maximum value

$$\frac{x(nT)}{C} \leq e^{\sigma n_s T}. \tag{31}$$

The inhomogeneous portion of $\mathbf{z}(t)$ can be shown to be time-periodic. Consider the solution $\mathbf{z}(t + nT)$, where $0 < t < T$. The inhomogeneous part of the solution $\mathbf{z}_i(t + nT)$ is given by

$$\mathbf{z}_i(t + nT) = \mathbf{X}(t + nT) \int_0^{t+nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau \tag{32}$$

which can be written as

$$\mathbf{z}_i(t + nT) = \mathbf{X}(t)\mathbf{X}(nT) \left[\int_0^{nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_{nT}^{t+nT} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{33}$$

Using the definition of a Neumann series (Theorem 4.4) and a procedure similar to that of Theorem 4.1, (33) converges to

$$\mathbf{z}_i(t + nT) = \mathbf{X}(t) \left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^{t} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{34}$$

Assuming that (34) is time-periodic, then the equation is also a steady-state solution given by

$$\mathbf{z}_{ss}(t) = \mathbf{X}(t) \left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\boldsymbol{\Lambda} + \int_0^{t} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{35}$$

If the assumption is true then

$$\mathbf{z}_{ss}(t) = \mathbf{z}_{ss}(t + T) = \mathbf{X}(t + T) \left[[\mathbf{I} - \mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^{t+T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right]. \tag{36}$$

Expanding the integral term yields

$$\mathbf{z}_{ss}(t+T) =$$

$$\mathbf{X}(t)\mathbf{X}(T)\left[[\mathbf{I}-\mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} + \int_0^T \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \int_T^{t+T} \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right] \quad (37)$$

which, again using the procedure of Theorem 4.1, reduces back to

$$\mathbf{z}_{ss}(t) = \mathbf{z}_{ss}(t+T) = \mathbf{X}(t)\left[[\mathbf{I}-\mathbf{X}(T)]^{-1}[\mathbf{X}(T)]\ \boldsymbol{\Lambda} \int_0^t \mathbf{X}(\tau)^{-1}\mathbf{g}(\tau)d\tau\right] \quad (38)$$

proving the assumption that $\mathbf{z}_{ss}(t)$ is time-periodic is true.

The steady-state solution can be determined by integration over a single time period. Since the inhomogeneous portion of the solution to $\mathbf{z}(t)$ is time-periodic and, therefore, contains no "transient" terms, the entire solution converges to the steady-state with the settling time characteristics of the homogeneous system described above. As mentioned earlier, given that the initial conditions are the initial vector of the periodic solution, the entire solution is time-periodic [3, 11].

5 Spinning Pendulum Example

The system to be analyzed, as shown in Figure 1 is a mass attached to a fixed point by a rigid tether. The pendulum is spinning in a gravitational field. The nonlinear equations of motion are given by

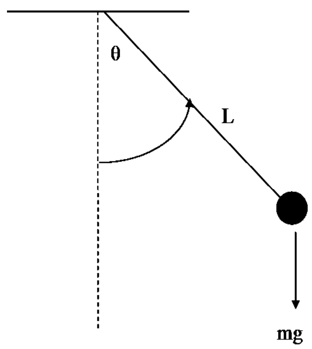

Figure 1: Pendulum spinning in a constant gravity field.

$$\ddot{\theta} = \frac{-g}{L}\sin\theta. \quad (39)$$

The reference condition chosen is the limiting behavior for $g << L$ which is a constant-rate spin

$$\begin{bmatrix} \theta \\ \dot{\theta}_0 \end{bmatrix}_R = \begin{bmatrix} \theta_0 \\ \theta_0 + \dot{\theta}_0 t \end{bmatrix}. \quad (40)$$

The linearized equations of motion are given by

$$\begin{bmatrix} \dot{\delta\theta} \\ \ddot{\delta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\theta & 0 \end{bmatrix}_R \begin{bmatrix} \delta\theta \\ \dot{\delta\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-g}{L}\sin\theta \end{bmatrix}_R, \tag{41}$$

where g is the acceleration due to gravity and L is the length of the tether. The ratio g/L is set to 0.1 for this example. The reference condition is $\theta_0 = -100\ deg$ and $\dot{\theta}_0 = \pi/3\ rad/s$. The time period for both the parametric and the forced excitation is $T = 2\pi/\dot{\theta}_0 = 6\ s$.

The monodromy matrix $\mathbf{X}(T)$ is found by numerical simulation of the homogeneous portion of (41) with unity initial conditions for one time period. This results in the following Floquet multipliers

$$\boldsymbol{\sigma} = 0.918 \pm 0.397i, \quad |\boldsymbol{\sigma}| = 1. \tag{42}$$

The pendulum system has a pair of complex Floquet multipliers with magnitude equal to one. Therefore the system exhibits Property 3 from Tables 2, 3 and 4. The homogeneous portion of the system is Lyapunov stable (see Table 2) as shown in Figure 2 where $\delta\theta$ is plotted for 30 time periods. The inhomogeneous system response is nonconvergent but bounded (see Table 3) by some finite value as shown in Figure 3. Figures 2 and 3 show both the result of a numerical simulation for 30 time periods of (41) and also the response, $\mathbf{z}(nT)$, calculated from (20) at each time period.

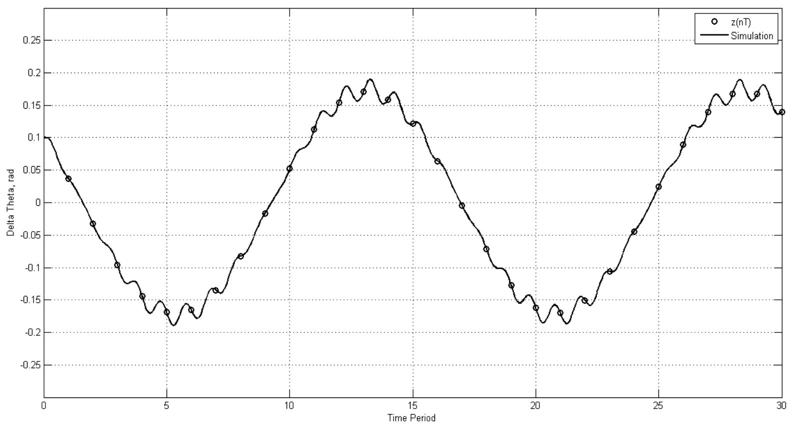

Figure 2: Homogeneous response to a small perturbation.

If a negative feedback controller with a proportional gain, K_p, and derivative gain, K_d, is applied to the pendulum system, the linear system of (41) becomes

$$\begin{bmatrix} \dot{\delta\theta} \\ \ddot{\delta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\theta - K_p & -K_d \end{bmatrix}_R \begin{bmatrix} \delta\theta \\ \dot{\delta\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-g}{L}\sin\theta \end{bmatrix}_R. \tag{43}$$

With gains of $K_p = K_d = 0.06$ and an initial condition of $\delta\theta = 0.3\ rad$, the Floquet multipliers become

$$\sigma = 0.034 \pm 0.835i, \quad |\sigma| = 0.835, \tag{44}$$

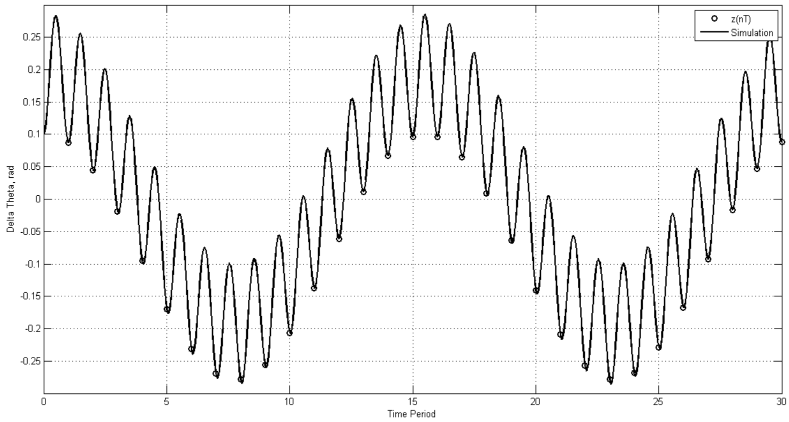

Figure 3: Inhomogeneous response to a small perturbation.

which are now a complex pair that lie inside the unit circle. The homogeneous system with negative feedback has become asymptotically stable as shown in Figure 4. As shown in Figure 5, the inhomogeneous system response is bounded and asymptotically approaches a time-periodic steady-state response.

Using (10) the corresponding Floquet characteristic exponents for (44) can be calculated

$$\epsilon = -0.030 \pm 0.255i \tag{45}$$

resulting in a time constant, $T_c = 33.3\ s = 5.6\ \textit{time periods}$ and a settling time to 2 percent, $T_s = 133.3\ s = 22.2\ \textit{time periods}$. The number of integer time periods for $\mathbf{z}(t)$ to settle to 2 percent is therefore $n_s = 23$. The homogeneous response in Figure 4 confirms that these results show good agreement with the simulated output. The solution for $\mathbf{z}(t)$ in Figure 5 shows the same settling time to the periodic steady-state.

6 Conclusions

When a near-periodic system is linearized about a time-periodic reference motion, the result is a linear parametrically excited system with a periodic forcing function. The solution to the system has been derived at each integer time period which requires knowledge of the system for the first time period only. The behavior of the homogeneous and inhomogeneous portions of the response can be predicted by using the Floquet characteristic exponents or multipliers. By adding a forcing excitation, the general behavior predicted by Floquet theory for the homogeneous system changed only for the case of semisimple multipliers that are identically equal to one. The presence of the forcing excitation caused the solution to diverge.

The classical control theory concept of settling time has been extended to the forced parametrically excited system. The homogeneous solution at each linear time period is the solution to a constant coefficient system which converges to zero at an exponential rate which can be determined from the Floquet characteristic exponents. It has been

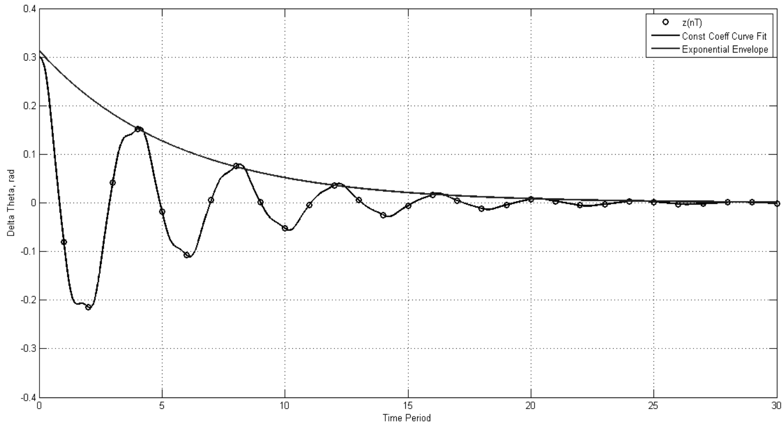

Figure 4: Homogeneous response to a small perturbation (controlled system).

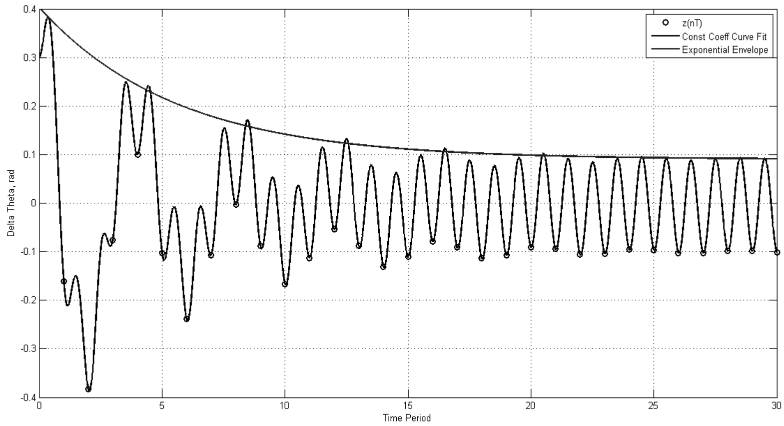

Figure 5: Inhomogeneous response to a small perturbation (controlled system).

shown that the entire inhomogeneous solution converges to a steady-state at the same exponential rate.

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Passive Delayed Static Output Feedback Control for a Class of T-S Fuzzy Systems

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Abstract: This paper investigates the problem of passive delayed static output feedback control for a class of fuzzy systems. The system is described by a state-space Takagi–Sugeno (T-S) fuzzy model with additive delays and interval parameter uncertainties. The aim is to design a fuzzy delayed static output feedback controller which ensures the closed-loop system is passive for all admissible uncertainties. In terms of linear matrix inequalities, a delay-dependent condition for the solvability of the above passive control problem is presented. A simulation example is provided to illustrate the effectiveness of the proposed design approach.

Keywords: *passive control; static output feedback; additive delays; T-S fuzzy models; interval parameter uncertainties.*

Mathematics Subject Classification (2000): 93C42, 93D09, 93D15.

1 Introduction

It is known that Takagi-Sugeno (T-S) fuzzy model, which is described by IF-THEN rules, provides an effective way to represent complex nonlinear systems in terms of fuzzy sets linear sub-systems [1, 13]. Time delays are commonly encountered in various engineering systems. Considerable attention has been paid to the stability analysis and synthesis for T-S fuzzy systems with time delays [12, 16], these results can be classified into two categories, namely, delay independent and delay dependent results. In most of these works, the state vector has a single delay. In this paper, we consider a class of T-S

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fuzzy systems with additive time-varying delays with totally different properties. Such a system model is suitable in the analysis of networked control systems [11].

Recently, passive control has attracted lots of attention among control community [9, 14]. For example, some results on passive control for T-S fuzzy systems were obtained for discrete- and continuous-time systems in [1] and [7], respectively. However, many papers deal with state feedback controllers. In practical applications, state variables may not be measured for many nonlinear systems. So it is meaningful to control a system via output feedback controllers; static output feedback control strategy is simple in controller structures, compared with dynamic output feedback control strategy. The problem of static output feedback controller design for discrete-time T-S fuzzy systems was considered in [2, 3], while for continuous-time T-S fuzzy systems, the static output feedback controller design problem was investigated in [4]. It is worth mentioning that, the delayed feedback control approach has attracted much attention over the past several years [8]. One can get rid of the need for explicitly determining any information about the underlying dynamics other than the period of the desired orbit, by using time delay in the feedback loop [6, 10]. However, to the best of our knowledge, the problem of passive delayed static output feedback control for continuous-time T-S fuzzy systems with interval parameter uncertainties and additive delays has not been solved.

In this paper, we consider the passive delayed static output feedback control problem for a class of fuzzy systems with uncertain parameters and delays. The purpose is to design a full-order fuzzy delayed static output feedback controller such that the resulting closed-loop system is passive irrespective of the parameter uncertainties. A sufficient condition for the solvability of this problem is proposed and an explicit expression of a desired static output feedback controller is also given.

Notation: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). I and 0 denote the identity and the zero matrix with appropriate dimensions. $*$ is used as an ellipsis for terms induced by symmetry. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $Sym(X)$ denotes the expression $X + X^T$.

2 Main Results

The Takagi-Sugeno (T-S) fuzzy dynamic model is described by fuzzy IF-THEN rules, which locally represent linear input-output relations of nonlinear systems. A continuous-time T-S fuzzy model with additive delays and interval parameter uncertainties can be described by

Plant Rule i: IF $s_1(t)$ is μ_{i1} and ... and $s_p(t)$ is μ_{ip}, THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + A_{di} x(t - \tau_1(t) - \tau_2(t)) + B_i u(t) + D_{1i} w(t), && (1) \\ y(t) &= C_i x(t), && (2) \\ z(t) &= E_i x(t) + D_{2i} w(t) + E_{1i} u(t), && (3) \\ x(t) &= \phi(t) \quad \forall t \in [-\bar{\tau}_{12}, 0], \quad i = 1, 2, \ldots, r, && (4) \end{aligned}$$

where μ_{ij} is the fuzzy set and r is the number of IF-THEN rules; $s_1(t)$, ..., $s_p(t)$ are the premise variables. Throughout this paper, it is assumed that the premise variables do not depend on control variables; $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the control input; $y(t) \in \mathbb{R}^s$ is the measured output; $z(t) \in \mathbb{R}^q$ is the controlled output; $w(t) \in \mathbb{R}^p$ is the noise signal; $\tau(t)$ is the time delays in state either constant or time varying satisfying $0 \le \tau_i(t) \le \bar{\tau}_i,\ 0 \le \dot{\tau}_i(t) \le d_i,\ i = 1, 2$, where $\bar{\tau}_i$ and d_i are constants. For simplicity, set

$$\bar{\tau}_{12} = \bar{\tau}_1 + \bar{\tau}_2,\ d_{12} = d_1 + d_2.$$

For all $1 \le p, q \le n,\ 1 \le k \le n_B$ with $\underline{A}_i = [\underline{a}_i^{pq}],\ \overline{A}_i = [\overline{a}_i^{pq}],\ \underline{A}_{di} = [\underline{a}_{di}^{pq}],\ \overline{A}_{di} = [\overline{a}_{di}^{pq}],\ \underline{B}_i = [\underline{b}_i^{pk}],\ \overline{B}_i = [\overline{b}_i^{pk}]$, we define the following interval uncertain matrix sets:

$$\begin{aligned}
\mathcal{A}_i &= \{[a_i^{pq}]_{n\times n} : \underline{a}_i^{pq} \le a_i^{pq} \le \overline{a}_i^{pq}, 1 \le p, q \le n\},\\
\mathcal{A}_{di} &= \{[a_{di}^{pq}]_{n\times n} : \underline{a}_{di}^{pq} \le a_{di}^{pq} \le \overline{a}_{di}^{pq}, 1 \le p, q \le n\},\\
\mathcal{B}_i &= \{[b_i^{pk}]_{n\times n} : \underline{b}_i^{pk} \le b_i^{pk} \le \overline{b}_i^{pk}, 1 \le p \le n, 1 \le k \le n_B\},
\end{aligned}$$

and let $A_i \in \mathcal{A}_i,\ A_{di} \in \mathcal{A}_{di},\ B_i \in \mathcal{B}_i$, for $i = 1, 2, \dots, r$.

Now, let

$$\begin{aligned}
A_{0i} &= \frac{1}{2}(\underline{A}_i + \overline{A}_i),\ \Delta A_i = \frac{1}{2}(\overline{A}_i - \underline{A}_i),\ A_{d0i} = \frac{1}{2}(\underline{A}_{di} + \overline{A}_{di}),\\
\Delta A_{di} &= \frac{1}{2}(\overline{A}_{di} - \underline{A}_{di}),\ B_{0i} = \frac{1}{2}(\underline{B}_i + \overline{B}_i),\ \Delta B_i = \frac{1}{2}(\overline{B}_i - \underline{B}_i).
\end{aligned}$$

Then $A_i,\ A_{di}$ and B_i in (1) can be rewritten as

$$\begin{aligned}
A_i &= A_{0i} + \sum_{p,q=1}^{n} e_p \left|g_{a_i}^{pq}\right| e_q^T,\ A_{di} = A_{d0i} + \sum_{p,q=1}^{n} e_p \left|g_{a_{di}}^{pq}\right| e_q^T,\\
B_i &= B_{0i} + \sum_{p=1}^{n}\sum_{k=1}^{n_B} e_p \left|g_{b_i}^{pk}\right| e_k^T,
\end{aligned}$$

where $\sum_{p,q=1}^{n} e_p \left|g_{a_i}^{pq}\right| e_q^T,\ \sum_{p,q=1}^{n} e_p \left|g_{a_{di}}^{pq}\right| e_q^T$, and $\sum_{p=1}^{n}\sum_{k=1}^{n_B} e_p \left|g_{b_i}^{pk}\right| e_k^T$ denote the interval parameter uncertainties; $e_p, e_q \in R^n$ and $e_k \in R^{n_B}$ are the column vectors with pth, qth, kth element to be 1 and others to be 0; $g_{a_i}^{pq},\ g_{a_{di}}^{pq}$, and $g_{b_i}^{pk}$ are variant parameters satisfying $\left|g_{a_i}^{pq}\right| \le \Delta a_i^{pq},\ \left|g_{a_{di}}^{pq}\right| \le \Delta a_{di}^{pq}$, and $\left|g_{b_i}^{pk}\right| \le \Delta b_i^{pk}$, respectively.

Then the final output of the fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(s(t))[A_i x(t) + A_{di}x(t - \tau_1(t) - \tau_2(t)) + B_i u(t) + D_{1i}w(t)], \tag{5}$$

$$y(t) = \sum_{i=1}^{r} h_i(s(t))[C_i x(t)], \tag{6}$$

$$z(t) = \sum_{i=1}^{r} h_i(s(t))[E_i x(t) + D_{3i}w(t) + E_{1i}u(t)], \tag{7}$$

where

$$\begin{aligned}
h_i(s(t)) &= \frac{\varpi_i(s(t))}{\sum_{i=1}^{r}\varpi_i(s(t))}, \quad \varpi_i(s(t)) = \prod_{j=1}^{p}\mu_{ij}(s_j(t)),\\
s(t) &= [s_1(t) \quad s_2(t) \quad \cdots \quad s_p(t)],
\end{aligned}$$

in which $\mu_{ij}(s_j(t))$ is the grade of membership of $s_j(t)$ in μ_{ij}. Then, it can be seen that

$$\begin{aligned} h_i(s(t)) &\geq 0, \quad i = 1, \cdots, r, \\ \sum_{i=1}^{r} h_i(s(t)) &= 1, \quad \forall t. \end{aligned} \tag{8}$$

Now, by the parallel distributed compensation, we consider the following full-order fuzzy delayed static output feedback controller for the fuzzy system (5)–(7):

$$u(t) = \sum_{i=1}^{r} h_i(s(t))[K_i y(t-\tau)], \tag{9}$$

where K_i is matrix to be determined later and τ is a given scalar.

From (5)–(7) and (9), the closed-loop system can be obtained as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t))[A_i x(t) + B_i K_j C_l x(t-\tau) \\ &\quad + A_{di} x(t - \tau_1(t) - \tau_2(t)) + D_{1i} w(t)], \qquad (10) \\ z(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))[E_i x(t) + E_{1i} K_j C_l x(t-\tau) + D_{3i} w(t)]. \qquad (11) \end{aligned}$$

As a performance measure for T-S fuzzy system (10)–(11), the definition of passivity is as follows:

Definition 2.1 [5] The system (10)–(11) is called passive if there exists a scalar $\gamma \geq 0$ such that

$$2\int_0^t \omega(t)^T z(t)dt \geq -\gamma \int_0^t \omega(s)^T \omega(s)ds \tag{12}$$

for all $t \geq 0$ and for all solutions of (10)–(11) with $x_0 = 0$, where γ is some constant which depends on the initial condition of the system.

2.1 Passivity analysis

We first give the following results which will be used in the proof of our main results.

Lemma 2.1 [15] *Given matrices $\mathcal{X} = \mathcal{X}^T$, $\mathcal{D}$, $\mathcal{Z}$ and $\mathcal{R} = \mathcal{R}^T > 0$ of appropriate dimensions, we have*

$$\mathcal{X} + \mathcal{D}\mathcal{F}\mathcal{Z} + \mathcal{Z}^T\mathcal{F}^T\mathcal{D}^T < 0$$

for all $\mathcal{F}$ satisfying $\mathcal{F}^T\mathcal{F} \leq \mathcal{R}$ if and only if there exists a scalar $\epsilon > 0$ such that

$$\mathcal{X} + \epsilon\mathcal{D}\mathcal{D}^T + \epsilon^{-1}\mathcal{Z}^T\mathcal{R}\mathcal{Z} < 0.$$

Theorem 2.1 *Consider the closed-loop fuzzy system in (5)–(7) with interval parameter uncertainties and additive delays. Suppose that the controller gain matrices in (9) are known. Given positive scalars γ, d_1, d_{12}, $\bar{\tau}_1$, $\bar{\tau}_2$ and $\bar{\tau}_{12}$, if there exist matrices $P > 0$, $Q > 0$, $R_i > 0$, $Z_i > 0$, $M_i > 0$, S, and positive scalars ε_{1ijpq}, ε_{2ijpq}, ε_{3ijpq}, ε_{4ijpq}, ε_{5ijpk}, ε_{6ijpk}, for $p, q = 1, \ldots, n$ and $k = 1, \ldots, n_B$, such that the following linear matrix inequalities (LMIs) hold:*

$$\begin{aligned} \Psi_{iil} &< 0,\ i,l=1,\dots,r, && (13)\\ \Psi_{ijl}+\Psi_{jil} &< 0,\ 1\le i<j\le r, l=1,\dots,r, && (14)\\ M_k &< Z_k,\ k=1,2,3, && (15) \end{aligned}$$

where

$$\Psi_{ijl}=\begin{bmatrix} \Omega_{ijl} & \Omega_s & \Omega_s & \Delta_{a_i}^{pq}\Omega_e & \Delta_{a_{di}}^{pq}\Omega_e & \Delta_{b_i}^{pk}\Omega_e\\ * & -\Omega_{\varepsilon 1ij}-\Omega_{\varepsilon 2ij} & 0 & 0 & 0 & 0\\ * & * & -\Omega_{\varepsilon 5ijB} & 0 & 0 & 0\\ * & * & * & -\Omega_{\varepsilon 3ij} & 0 & 0\\ * & * & * & * & -\Omega_{\varepsilon 4ij} & 0\\ * & * & * & * & * & -\Omega_{\varepsilon 6ijB} \end{bmatrix},$$

$$\begin{aligned} \Omega_{ijl} &= \Phi_{0ijl}+\sum_{p,q=1}^{n}\Big[(\varepsilon_{1ijpq}\Delta_{a_i}^{pq^2}+\varepsilon_{3ijpq})W_{eq1}^TW_{eq1}+(\varepsilon_{2ijpq}\Delta_{a_{di}}^{pq^2}+\varepsilon_{4ijpq})W_{eq2}^TW_{eq2}\Big]\\ &\quad+\sum_{p=1}^{n}\sum_{k=1}^{n_B}\Big(\varepsilon_{5ijpk}\Delta_{b_i}^{pk^2}+\varepsilon_{6ijpk}\Big)W_{ek1}^TW_{ek1}, \end{aligned}$$

$$\begin{aligned} W_{eq1} &= \begin{bmatrix} e_q^T & 0_{1,(m+3)n}\end{bmatrix},\ W_{ek1}=\begin{bmatrix} 0_{1,n} & e_k^TK_jC_l & 0_{1,(m+2)n}\end{bmatrix},\\ W_{eq2} &= \begin{bmatrix} 0_{1,3n} & e_q^T & 0_{1,mn}\end{bmatrix}, \Omega_s=\begin{bmatrix} \bar{S}e_1 & \cdots & \bar{S}e_n\end{bmatrix},\\ \Omega_e &= \begin{bmatrix} \bar{e}_1S^T & \cdots & \bar{e}_nS^T\end{bmatrix},\ \bar{e}_pS=\begin{bmatrix} 0_{1,mn} & e_p^TS^T & 0_{1,3n}\end{bmatrix},\\ \bar{S}e_p &= \begin{bmatrix} Se_p\\ 0_{(m+3)n,1}\end{bmatrix},\ \Omega_{\varepsilon nij}=\begin{bmatrix} \varepsilon_{nij11} & 0 & 0\\ * & \ddots & 0\\ * & * & \varepsilon_{nijnn}\end{bmatrix},\\ \Omega_{\varepsilon mijB} &= \begin{bmatrix} \varepsilon_{mij11} & 0 & 0\\ * & \ddots & 0\\ * & * & \varepsilon_{mijnn_B}\end{bmatrix}, \Phi_{0ijl}=\begin{bmatrix} \Sigma_{01ijl} & \Sigma_{02ijl} & \Sigma_4\\ * & \Sigma_{03i} & \Sigma_5\\ * & * & -\Sigma_6\end{bmatrix},\\ \Sigma_{01ijl} &= \begin{bmatrix} Q_{A0i} & SB_{0i}K_jC_l+L_{12}^T+L_{32}^T & L_{21}-L_{11}+L_{13}^T+L_{33}^T\\ * & -Q & L_{22}-L_{12}\\ * & * & (d_1-1)R_1+R_2+Sym\{L_{23}-L_{13}\}\end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \Sigma_{02ijl} &= \begin{bmatrix} SA_{d0i}+\hat{L}_{21} & SD_{1i}-E_i^T+L_{15}^T+L_{35}^T & P-S+A_{0i}^TS^T+L_{16}^T+L_{36}^T\\ -L_{22}-L_{32} & -C_l^TK_j^TE_{1i}^T & C_l^TK_j^TB_{0i}^TS^T\\ \hat{L}_{23} & L_{25}^T-L_{15}^T & L_{26}^T-L_{16}^T\end{bmatrix},\\ \Sigma_{03i} &= \begin{bmatrix} \begin{matrix}(d_{12}-1)(R_2+R_3)\\ +Sym\{-L_{24}-L_{34}\}\end{matrix} & -L_{25}^T-L_{35}^T & A_{d0i}^TS^T-L_{26}^T-L_{36}^T\\ * & -D_{3i}-D_{3i}^T-\gamma & D_{1i}^TS^T\\ * & * & \hat{Z}_s\end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
\Sigma_4 &= \begin{bmatrix} \bar{\tau}_1 L_{11} & \bar{\tau}_2 L_{21} & \bar{\tau}_{12} L_{31} \\ \bar{\tau}_1 L_{12} & \bar{\tau}_2 L_{22} & \bar{\tau}_{12} L_{32} \\ \bar{\tau}_1 L_{13} & \bar{\tau}_2 L_{23} & \bar{\tau}_{12} L_{33} \end{bmatrix}, \ \Sigma_5 = \begin{bmatrix} \bar{\tau}_1 L_{14} & \bar{\tau}_2 L_{24} & \bar{\tau}_{12} L_{34} \\ \bar{\tau}_1 L_{15} & \bar{\tau}_2 L_{25} & \bar{\tau}_{12} L_{35} \\ \bar{\tau}_1 L_{16} & \bar{\tau}_2 L_{26} & \bar{\tau}_{12} L_{36} \end{bmatrix}, \\
\Sigma_6 &= diag(\bar{\tau}_1 M_1, \bar{\tau}_2 M_2, \bar{\tau}_{12} M_3), \\
Q_{A0i} &= SA_{0i} + A_{0i}^T S^T + Q + R_1 + R_3 + Sym\{L_{11} + L_{31}\}, \\
\hat{L}_{21} &= -L_{21} - L_{31} + L_{14}^T + L_{34}^T, \ \hat{L}_{23} = -L_{23} - L_{33} + L_{24}^T - L_{14}^T, \\
\hat{Z}_s &= \bar{\tau}_1 Z_1 + \bar{\tau}_2 Z_2 + \bar{\tau}_{12} Z_3 - S - S^T.
\end{aligned}$$

Then the closed-loop system (10)–(11) is passive.

Proof For system (10)-(11), we define the following Lyapunov functional candidate:

$$V(t) = x(t)^T P x(t) + V_1(t) + V_2(t) + V_3(t), \tag{16}$$

where

$$\begin{aligned}
V_1(t) &= \int_{t-\tau}^{t} x(s)^T Q x(s) ds, \\
V_2(t) &= \int_{t-\tau_1(t)}^{t} x(s)^T R_1 x(s) ds + \int_{t-\tau_1(t)-\tau_2(t)}^{t-\tau_1(t)} x(s)^T R_2 x(s) ds \\
&\quad + \int_{t-\tau_1(t)-\tau_2(t)}^{t} x(s)^T R_3 x(s) ds, \\
V_3(t) &= \int_{t-\bar{\tau}_1}^{t} d\theta \int_{\theta}^{t} \dot{x}(s)^T Z_1 \dot{x}(s) ds + \int_{t-\bar{\tau}_{12}}^{t-\bar{\tau}_1} d\theta \int_{\theta}^{t} \dot{x}(s)^T Z_2 \dot{x}(s) ds \\
&\quad + \int_{t-\bar{\tau}_{12}}^{t} d\theta \int_{\theta}^{t} \dot{x}(s)^T Z_3 \dot{x}(s) ds.
\end{aligned}$$

The time derivative of $V(t)$ is given by

$$\dot{V}(t) = 2x(t)^T P \dot{x}(t) + \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t),$$

where

$$\dot{V}_1(t) = x(t)^T Q x(t) - x(t-\tau)^T Q x(t-\tau), \tag{17}$$

$$\begin{aligned}
\dot{V}_2(t) &= x(t)^T (R_1 + R_3) x(t) - (1 - \dot{\tau}_1(t)) x(t-\tau_1(t))^T (R_1 - R_2) x(t-\tau_1(t)) \\
&\quad -(1 - \dot{\tau}_1(t) - \dot{\tau}_2(t)) x(t-\tau_1(t)-\tau_2(t))^T (R_2 + R_3) x(t-\tau_1(t)-\tau_2(t)) \\
&\le x(t)^T (R_1 + R_3) x(t) - x(t-\tau_1(t))^T [(1-d_1) R_1 - R_2] x(t-\tau_1(t)) \\
&\quad -(1-d_{12}) x(t-\tau_1(t)-\tau_2(t))^T (R_2 + R_3) x(t-\tau_1(t)-\tau_2(t)),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\dot{V}_3(t) &= \dot{x}(t)^T (\bar{\tau}_1 Z_1 + \bar{\tau}_2 Z_2 + \bar{\tau}_{12} Z_3) \dot{x}(t) - \int_{t-\bar{\tau}_1}^{t} \dot{x}(s)^T Z_1 \dot{x}(s) ds \\
&\quad - \int_{t-\bar{\tau}_{12}}^{t-\bar{\tau}_1} \dot{x}(s)^T Z_2 \dot{x}(s) ds - \int_{t-\bar{\tau}_{12}}^{t} \dot{x}(s)^T Z_3 \dot{x}(s) ds \\
&\le \dot{x}(t)^T (\bar{\tau}_1 Z_1 + \bar{\tau}_2 Z_2 + \bar{\tau}_{12} Z_3) \dot{x}(t) - \int_{t-\tau_1(t)}^{t} \dot{x}(s)^T Z_1 \dot{x}(s) ds \\
&\quad - \int_{t-\tau_1(t)-\tau_2(t)}^{t-\tau_1(t)} \dot{x}(s)^T Z_2 \dot{x}(s) ds - \int_{t-\tau_1(t)-\tau_2(t)}^{t} \dot{x}(s)^T Z_3 \dot{x}(s) ds.
\end{aligned} \tag{19}$$

By the Newton–Leibniz formula, for any appropriately dimensioned matrices $L_i,\ i = 1, 2, 3$, we have the following equations

$$\begin{aligned}
\Lambda_1 &= 2\xi(t)^T L_1[x(t) - x(t-\tau_1(t)) - \int_{t-\tau_1(t)}^{t} \dot{x}(s)ds] = 0, \qquad (20)\\
\Lambda_2 &= 2\xi(t)^T L_2[x(t-\tau_1(t)) - x(t-\tau_1(t)-\tau_2(t)) - \int_{t-\tau_1(t)-\tau_2(t)}^{t-\tau_1(t)} \dot{x}(s)ds] = 0, \qquad (21)\\
\Lambda_3 &= 2\xi(t)^T L_3[x(t) - x(t-\tau_1(t)-\tau_2(t)) - \int_{t-\tau_1(t)-\tau_2(t)}^{t} \dot{x}(s)ds] = 0, \qquad (22)
\end{aligned}$$

where

$$\xi(t) = [x(t)^T \quad x(t-\tau)^T \quad x(t-\tau_1(t))^T \quad x(t-\tau_{12}(t))^T \quad w(t)^T \quad \dot{x}(t)^T]^T,$$

On the other hand, for matrices $Z_j = Z_j^T,\ j = 1, 2, 3,\ M_i = M_i^T,\ i = 1, 2, 3$, which satisfy

$$M_1 < Z_1,\ M_2 < Z_2,\ M_3 < Z_3,$$

one can get the following inequalities:

$$\begin{aligned}
\Upsilon_1 &= \tau_1(t)\xi(t)^T L_1 M_1^{-1} L_1^T \xi(t) - \int_{t-\tau_1(t)}^{t} \xi(t)^T L_1 Z_1^{-1} L_1^T \xi(t)ds > 0, \qquad (23)\\
\Upsilon_2 &= \tau_1(t)\xi(t)^T L_1 M_1^{-1} L_1^T \xi(t) - \int_{t-\tau_1(t)}^{t} \xi(t)^T L_1 Z_1^{-1} L_1^T \xi(t)ds > 0, \qquad (24)\\
\Upsilon_3 &= \tau_1(t)\xi(t)^T L_1 M_1^{-1} L_1^T \xi(t) - \int_{t-\tau_1(t)}^{t} \xi(t)^T L_1 Z_1^{-1} L_1^T \xi(t)ds > 0. \qquad (25)
\end{aligned}$$

It then follows from (17)-(25) that

$$\begin{aligned}
\dot{V}(t) \leq\ & \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t))\{\xi^T(t)\Theta_{ijl}\xi(t) + \bar{\tau}_1\xi^T(t)L_1M_1^{-1}L_1^T\xi(t)\\
&+\bar{\tau}_2\xi^T(t)L_2M_2^{-1}L_2^T\xi(t) + \bar{\tau}_{12}\xi^T(t)L_3M_3^{-1}L_3^T\xi(t)\\
&-\int_{t-\tau_1(t)}^{t} [\xi^T(t)L_1 + \dot{x}(s)^T Z_1]Z_1^{-1}[L_1^T\xi(t) + Z_1\dot{x}(s)]ds\\
&-\int_{t-\tau_{12}(t)}^{t-\tau_1(t)} [\xi^T(t)L_2 + \dot{x}(s)^T Z_2]Z_2^{-1}[L_2^T\xi(t) + Z_2\dot{x}(s)]ds\\
&-\int_{t-\tau_{12}(t)}^{t} [\xi^T(t)L_3 + \dot{x}(s)^T Z_3]Z_3^{-1}[L_3^T\xi(t) + Z_3\dot{x}(s)]ds, \qquad (26)
\end{aligned}$$

where

$$\Theta_{ijl} = \begin{bmatrix} \Sigma_{1ijl} & \Sigma_{2ijl} \\ * & \Sigma_{3i} \end{bmatrix}, \ \bar{S} = [S^T \ 0 \ \cdots \ 0 \ S^T]^T,$$

$$\Sigma_{1ijl} = \begin{bmatrix} Q_{Ai} & SB_iK_jC_l + L_{12}^T + L_{32}^T & L_{21} - L_{11} + L_{13}^T + L_{33}^T \\ * & -Q & L_{22} - L_{12} \\ * & * & (d_1 - 1)R_1 + R_2 + Sym\{L_{23} - L_{13}\} \end{bmatrix},$$

$$\Sigma_{2ijl} = \begin{bmatrix} SA_{di} + \hat{L}_{21} & SD_{1i} + L_{15}^T + L_{35}^T & P - S + A_i^T S^T + L_{16}^T + L_{36}^T \\ -L_{22} - L_{32} & 0 & C_l^T K_j^T B_i^T S^T \\ \hat{L}_{23} & L_{25}^T - L_{15}^T & L_{26}^T - L_{16}^T \end{bmatrix},$$

$$\Sigma_{3i} = \begin{bmatrix} \begin{array}{c} (d_{12} - 1)(R_2 + R_3) \\ +Sym\{-L_{24} - L_{34}\} \end{array} & -L_{25}^T - L_{35}^T & A_{di}^T S^T - L_{26}^T - L_{36}^T \\ * & 0 & D_{1i}^T S^T \\ * & * & \hat{Z}_s \end{bmatrix}.$$

Since $Z_j > 0, \ j = 1, 2, 3$, the last three terms in (26) are all less than 0. From this, one can obtain

$$\begin{aligned} & \dot{V}(t) - 2z(t)^T\omega(t) - \gamma\omega(t)^T\omega(t) \\ \leq \ & \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t))\{\xi^T(t)\Phi_{ijl}\xi(t)\}, \end{aligned} \tag{27}$$

where

$$\Phi_{ijl} = \begin{bmatrix} \Sigma_{1ijl} & \hat{\Sigma}_{2ijl} & \Sigma_4 \\ * & \hat{\Sigma}_{2ijl} & \Sigma_5 \\ * & * & -\Sigma_6 \end{bmatrix},$$

$$\hat{\Sigma}_{2ijl} = \begin{bmatrix} SA_{di} + \hat{L}_{21} & SD_{1i} - E_i^T + L_{15}^T + L_{35}^T & P - S + A_i^T S^T + L_{16}^T + L_{36}^T \\ -L_{22} - L_{32} & -C_l^T K_j^T E_{1i}^T & C_l^T K_j^T B_i^T S^T \\ \hat{L}_{23} & L_{25}^T - L_{15}^T & L_{26}^T - L_{16}^T \end{bmatrix},$$

$$\hat{\Sigma}_{3i} = \begin{bmatrix} \begin{array}{c} (d_{12} - 1)(R_2 + R_3) \\ +Sym\{-L_{24} - L_{34}\} \end{array} & -L_{25}^T - L_{35}^T & A_{di}^T S^T - L_{26}^T - L_{36}^T \\ * & -D_{3i} - D_{3i}^T - \gamma & D_{1i}^T S^T \\ * & * & +\hat{Z}_s \end{bmatrix}.$$

Replacing A_i, A_{di}, B_i, in Φ_{ijl} of the inequality in (27) with $A_i = A_{0i} + \sum_{p,q=1}^{n} e_p \left|g_{a_i}^{pq}\right| e_q^T$, $A_{di} = A_{d0i} + \sum_{p,q=1}^{n} e_p \left|g_{a_{di}}^{pq}\right| e_q^T$, and $B_i = B_{0i} + \sum_{p=1}^{n}\sum_{k=1}^{n_B} e_p \left|g_{b_i}^{pk}\right| e_k^T$, respectively, we have

$$\begin{aligned} & \dot{V}(t) - 2z(t)^T\omega(t) - \gamma\omega(t)^T\omega(t) \\ \leq \ & \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t))\{\xi^T(t)\Psi_{ijl}\xi(t)\} \\ = \ & \sum_{i=1}^{r}\sum_{l=1}^{r} h_i^2(s(t))h_l(s(t))\xi^T(t)\Psi_{iil}\xi(t) \\ & +2\sum_{i=1, i<j}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t))\xi(t)^T\frac{\Psi_{ijl} + \Psi_{jil}}{2}\xi(t). \end{aligned}$$

From (13) and (14), we can obtain that $\Psi_{iil} < 0,\ \Psi_{ijl} + \Psi_{jil} < 0$. Then we can get (12). This completes the proof. □

2.2 Delayed static output-feedback controller design

Now, we are in a position to present a solution to the passive delayed static output feedback controller design problem.

Theorem 2.2 *Consider the closed-loop fuzzy system in (5)–(7) with interval parameter uncertainties and additive delays. Given positive scalars $d_1,\ d_{12},\ \bar{\tau}_1,\ \bar{\tau}_2$ and $\bar{\tau}_{12}$, and let $\gamma > 0$ be a prescribed constant scalar. The passive control problem is solvable if there exist matrices $\tilde{P} > 0,\ \tilde{Q} > 0,\ \tilde{R}_i > 0,\ \tilde{Z}_i > 0,\ \tilde{M}_i > 0,\ X$, and positive scalars $\hat{\varepsilon}_{1ijpq},\ \hat{\varepsilon}_{2ijpq},\ \hat{\varepsilon}_{3ijpq},\ \hat{\varepsilon}_{4ijpq},\ \hat{\varepsilon}_{5ijpk},\ \hat{\varepsilon}_{6ijpk}$, for $p, q = 1, \ldots, n$ and $k = 1, \ldots, n_B$, such that the following LMIs hold:*

$$\begin{aligned} \mathcal{J}_{iil} &< 0,\ i, l = 1, \ldots, r, &(28)\\ \mathcal{J}_{ijl} + \mathcal{J}_{jil} &< 0,\ 1 \le i < j \le r,\ l = 1, \ldots r, &(29)\\ \tilde{M}_k &< \tilde{Z}_k,\ k = 1, 2, 3, &(30) \end{aligned}$$

where

$$\mathcal{J}_{ijl} = \begin{bmatrix} \mathcal{H}_{ijl} & \Omega_{eq1} & \Omega_{eq2} & \hat{W}_{eq1}^T & \hat{W}_{eq2}^T & \Omega_{ek1} & \hat{W}_{ek1}^T \\ * & -\hat{\Omega}_{\varepsilon 1ij} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\hat{\Omega}_{\varepsilon 2ij} & 0 & 0 & 0 & 0 \\ * & * & * & -\hat{\Omega}_{\varepsilon 3ij} & 0 & 0 & 0 \\ * & * & * & * & -\hat{\Omega}_{\varepsilon 4ij} & 0 & 0 \\ * & * & * & * & * & -\hat{\Omega}_{\varepsilon 5ijB} & 0 \\ * & * & * & * & * & * & -\hat{\Omega}_{\varepsilon 6ijB} \end{bmatrix},$$

$$\begin{aligned} \mathcal{H}_{ijl} &= \mathcal{F}_{0ijl} + \sum_{p,q=1}^{n} \Big[(\hat{\varepsilon}_{1ijpq} + \hat{\varepsilon}_{2ijpq}) \bar{e}_p \bar{e}_p^T + (\varepsilon_{3ijpq} \Delta_{a_i}^{pq^2} + \varepsilon_{4ijpq} \Delta_{a_{di}}^{pq^2}) \hat{e}_p^T \hat{e}_p \Big] \\ &\quad + \sum_{p=1}^{n} \sum_{k=1}^{n_B} \Big(\varepsilon_{5ijpk} \bar{e}_p \bar{e}_p^T + \varepsilon_{6ijpk} \Delta_{b_i}^{pk^2} \hat{e}_p^T \hat{e}_p \Big), \\ \hat{W}_{eq1} &= \begin{bmatrix} e_q^T X & 0_{1,(m+3)n} \end{bmatrix},\ \hat{W}_{ek1} = \begin{bmatrix} 0_{1,n} & e_k^T N_j C_l & 0_{1,(m+2)n} \end{bmatrix}, \\ \hat{W}_{eq2} &= \begin{bmatrix} 0_{1,3n} & e_q^T X & 0_{1,mn} \end{bmatrix}, \Omega_{eq1} = \begin{bmatrix} \Delta_{a_i}^{11} \hat{W}_{eq1}^T & \cdots & \Delta_{a_i}^{nn} \hat{W}_{eq1}^T \end{bmatrix}, \\ \Omega_{eq2} &= \begin{bmatrix} \Delta_{a_{di}}^{11} \hat{W}_{eq2}^T & \cdots & \Delta_{a_{di}}^{nn} \hat{W}_{eq2}^T \end{bmatrix},\ \Omega_{ek1} = \begin{bmatrix} \Delta_{b_i}^{11} \hat{W}_{ek1}^T & \cdots & \Delta_{b_i}^{nn} \hat{W}_{ek1}^T \end{bmatrix}, \\ \bar{e}_p &= \begin{bmatrix} e_p \\ 0_{(m+3)n,1} \end{bmatrix},\ \hat{e}_p = \begin{bmatrix} 0_{1,mn} & e_p^T & 0_{1,3n} \end{bmatrix}, \\ \hat{\Omega}_{\varepsilon nij} &= \begin{bmatrix} \hat{\varepsilon}_{nij11} & 0 & 0 \\ * & \ddots & 0 \\ * & * & \hat{\varepsilon}_{nijnn} \end{bmatrix},\ \hat{\Omega}_{\varepsilon mijB} = \begin{bmatrix} \hat{\varepsilon}_{mij11} & 0 & 0 \\ * & \ddots & 0 \\ * & * & \hat{\varepsilon}_{mijnn_B} \end{bmatrix}, \\ \mathcal{F}_{0ijl} &= \begin{bmatrix} \mathcal{G}_{01ijl} & \mathcal{G}_{02ijl} & \mathcal{G}_4 \\ * & \mathcal{G}_{03i} & \mathcal{G}_5 \\ * & * & -\mathcal{G}_6 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{01ijl} &= \begin{bmatrix} \tilde{Q}_{A0i} & B_{0i}N_jC_l + \tilde{L}_{12}^T + \tilde{L}_{32}^T & \tilde{L}_{21} - \tilde{L}_{11} + \tilde{L}_{13}^T + \tilde{L}_{33}^T \\ * & -\tilde{Q} & \tilde{L}_{22} - \tilde{L}_{12} \\ * & * & (d_1-1)\tilde{R}_1 + \tilde{R}_2 + Sym\{\tilde{L}_{23} - \tilde{L}_{13}\} \end{bmatrix}, \\
\mathcal{G}_{02ijl} &= \begin{bmatrix} A_{d0i}X + \vec{L}_{21} & D_{1i} - X^TE_i^T + \tilde{L}_{15}^T + \tilde{L}_{35}^T & \tilde{P} - X + X^TA_{0i}^T + \tilde{L}_{16}^T + \tilde{L}_{36}^T \\ -\tilde{L}_{22} - \tilde{L}_{32} & -C_l^TN_j^TE_{1i}^T & C_l^TN_j^TB_{0i}^T \\ \vec{L}_{23} & \tilde{L}_{25}^T - \tilde{L}_{15}^T & \tilde{L}_{26}^T - \tilde{L}_{16}^T \end{bmatrix}, \\
\mathcal{G}_{03i} &= \begin{bmatrix} \begin{array}{l}(d_{12}-1)(\tilde{R}_2 + \tilde{R}_3) \\ +Sym\{-\tilde{L}_{24} - \tilde{L}_{34}\}\end{array} & -\tilde{L}_{25}^T - \tilde{L}_{35}^T & X^TA_{d0i}^T - \tilde{L}_{26}^T - \tilde{L}_{36}^T \\ * & -D_{3i} - D_{3i}^T - \gamma & D_{1i}^T \\ * & * & \vec{Z}_x \end{bmatrix}, \\
\mathcal{G}_4 &= \begin{bmatrix} \bar{\tau}_1L_{11} & \bar{\tau}_2L_{21} & \bar{\tau}_{12}L_{31} \\ \bar{\tau}_1L_{12} & \bar{\tau}_2L_{22} & \bar{\tau}_{12}L_{32} \\ \bar{\tau}_1L_{13} & \bar{\tau}_2L_{23} & \bar{\tau}_{12}L_{33} \end{bmatrix}, \ \mathcal{G}_5 = \begin{bmatrix} \bar{\tau}_1\tilde{L}_{14} & \bar{\tau}_2\tilde{L}_{24} & \bar{\tau}_{12}\tilde{L}_{34} \\ \bar{\tau}_1\tilde{L}_{15} & \bar{\tau}_2\tilde{L}_{25} & \bar{\tau}_{12}\tilde{L}_{35} \\ \bar{\tau}_1\tilde{L}_{16} & \bar{\tau}_2\tilde{L}_{26} & \bar{\tau}_{12}\tilde{L}_{36} \end{bmatrix}, \\
\mathcal{G}_6 &= diag(\bar{\tau}_1\tilde{M}_1, \bar{\tau}_2\tilde{M}_2, \bar{\tau}_{12}\tilde{M}_3), \\
\tilde{Q}_{A0i} &= A_{0i}X + X^TA_{0i}^T + \tilde{Q} + \tilde{R}_1 + \tilde{R}_3 + Sym\{\tilde{L}_{11} + \tilde{L}_{31}\}, \\
\vec{L}_{21} &= -\tilde{L}_{21} - \tilde{L}_{31} + \tilde{L}_{14}^T + \tilde{L}_{34}^T, \\
\vec{L}_{23} &= -\tilde{L}_{23} - \tilde{L}_{33} + \tilde{L}_{24}^T - \tilde{L}_{14}^T, \\
\vec{Z}_x &= \bar{\tau}_1\tilde{Z}_1 + \bar{\tau}_2\tilde{Z}_2 + \bar{\tau}_{12}\tilde{Z}_3 - X - X^T,
\end{aligned}$$

and the following equality constraint satisfied

$$MC_l = C_lX. \tag{31}$$

Furthermore, a desired passive delayed static output feedback controller is given in the form (9) with parameters as follows:

$$K_i = N_iM^{-1}, \ 1 \le i \le r. \tag{32}$$

Proof Suppose there exist matrices $\tilde{Q},\ \tilde{P},\ \tilde{R}_i,\ \tilde{Z}_j,\ \tilde{M}_i,\ i,j = 1,2,3$, and X satisfying (28)-(31). Applying the Schur complement formula to (28) results in

$$\begin{aligned}
&\mathcal{F}_{0ijl} + \sum_{p,q=1}^{n}\Big[(\hat{\varepsilon}_{1ijpq} + \hat{\varepsilon}_{2ijpq})\bar{e}_p\bar{e}_p^T + (\varepsilon_{3ijpq}\Delta_{a_i}^{pq^2} + \varepsilon_{4ijpq}\Delta_{a_{di}}^{pq^2})\hat{e}_p^T\hat{e}_p \\
&+\hat{W}_{eq1}^T\hat{\Omega}_{\varepsilon1ij}^{-1}\Delta_{a_i}^{pq^2}\hat{W}_{eq1} + \hat{W}_{eq2}^T\hat{\Omega}_{\varepsilon2ij}^{-1}\Delta_{a_{di}}^{pq^2}\hat{W}_{eq2} + \hat{W}_{eq1}^T\hat{\Omega}_{\varepsilon3ij}^{-1}\hat{W}_{eq1} + \hat{W}_{eq2}^T\hat{\Omega}_{\varepsilon4ij}^{-1}\hat{W}_{eq2}\Big] \\
&+\sum_{p=1}^{n}\sum_{k=1}^{n_B}(\varepsilon_{5ijpk}\bar{e}_p\bar{e}_p^T + \varepsilon_{6ijpk}\Delta_{b_i}^{pk^2}\hat{e}_p^T\hat{e}_p + \hat{W}_{ek1}^T\hat{\Omega}_{\varepsilon5ijB}^{-1}\Delta_{b_i}^{pk^2}\hat{W}_{ek1} \\
&+\hat{W}_{ek1}^T\hat{\Omega}_{\varepsilon6ijB}^{-1}\hat{W}_{ek1}) < 0,
\end{aligned}$$

then, by Lemma 2.1, it is easy to have

$$\begin{aligned}
\mathcal{F}_{0ijl} \quad &+ \quad sym\{\sum_{p,q=1}^{n}[\bar{e}_p \mid f_{A_i}^{pq} \mid \hat{W}_{eq1} + \bar{e}_p \mid f_{A_{di}}^{pq} \mid \hat{W}_{eq2} + \hat{W}_{eq1}^T \mid f_{A_i}^{pq} \mid^T \hat{e}_p \\
&+ \quad \hat{W}_{eq2}^T \mid f_{A_{di}}^{pq} \mid^T \hat{e}_p] + \sum_{p=1}^{n}\sum_{k=1}^{n_B}[\bar{e}_p \mid f_{B_i}^{pk} \mid \hat{W}_{ek1} + \hat{W}_{ek1}^T \mid f_{B_i}^{pk} \mid^T \hat{e}_p]\} < 0.
\end{aligned}$$

Replacing $A_i = A_{0i} + \sum_{p,q=1}^{n} e_p \left| g_{a_i}^{pq} \right| e_q^T$, $A_{di} = A_{d0i} + \sum_{p,q=1}^{n} e_p \left| g_{a_{di}}^{pq} \right| e_q^T$, and $B_i = B_{0i} + \sum_{p=1}^{n} \sum_{k=1}^{n_B} e_p \left| g_{b_i}^{pk} \right| e_k^T$, in the proceeding inequality with A_i, A_{di}, B_i, respectively, we can get

$$\begin{bmatrix} \mathcal{G}_{1ijl} & \mathcal{G}_{2ijl} & \mathcal{G}_4 \\ * & \mathcal{G}_{3i} & \mathcal{G}_5 \\ * & * & -\mathcal{G}_6 \end{bmatrix} < 0, \tag{33}$$

where

$$\begin{aligned}
\mathcal{G}_{1ijl} &= \begin{bmatrix} \tilde{Q}_{Ai} & B_i N_j C_l + \tilde{L}_{12}^T + \tilde{L}_{32}^T & \tilde{L}_{21} - \tilde{L}_{11} + \tilde{L}_{13}^T + \tilde{L}_{33}^T \\ * & -\tilde{Q} & \tilde{L}_{22} - \tilde{L}_{12} \\ * & * & (d_1 - 1)\tilde{R}_1 + \tilde{R}_2 + Sym\{\tilde{L}_{23} - \tilde{L}_{13}\} \end{bmatrix}, \\
\mathcal{G}_{2ijl} &= \begin{bmatrix} A_{di}X + \vec{L}_{21} & D_{1i} - X^T E_i^T + \tilde{L}_{15}^T + \tilde{L}_{35}^T & \tilde{P} - X + X^T A_i^T + \tilde{L}_{16}^T + \tilde{L}_{36}^T \\ -\tilde{L}_{22} - \tilde{L}_{32} & -C_l^T N_j^T E_{1i}^T & C_l^T N_j^T B_{0i}^T \\ \vec{L}_{23} & \tilde{L}_{25}^T - \tilde{L}_{15}^T & \tilde{L}_{26}^T - \tilde{L}_{16}^T \end{bmatrix}, \\
\mathcal{G}_{3i} &= \begin{bmatrix} \begin{matrix} (d_{12} - 1)(\tilde{R}_2 + \tilde{R}_3) \\ +Sym\{-\tilde{L}_{24} - \tilde{L}_{34}\} \end{matrix} & -\tilde{L}_{25}^T - \tilde{L}_{35}^T & X^T A_{di}^T - \tilde{L}_{26}^T - \tilde{L}_{36}^T \\ * & -D_{3i} - D_{3i}^T - \gamma & D_{1i}^T \\ * & * & \vec{Z}_x \end{bmatrix}.
\end{aligned}$$

Suppose there exits a nonsingular matrix S satisfying

$$S = X^{-T}.$$

Without loss of generality, we can define

$$\begin{aligned}
\tilde{L}_{1j} &= X^T L_{1j} X,\ \tilde{L}_{2j} = X^T L_{2j} X,\ \tilde{L}_{3j} = X^T L_{3j} X,\ j = 1, 2, 3, 4, 6, \\
\tilde{Q} &= X^T Q X,\ \tilde{P} = X^T P X,\ \tilde{R}_i = X^T R_i X,\ \tilde{Z}_j = X^T Z_j X,\ \tilde{M}_i = X^T M_i X, \\
\tilde{L}_{i5} &= L_{i5} X,\ i, j = 1, 2, 3.
\end{aligned}$$

Now pre- and post- multiplying the LMIs in (33) by $diag(S, S, S, S, I, S, \cdots, S)$ and $diag(S^T, S^T, S^T, S^T, I,\ S^T, \cdots, S^T)$, respectively, then we have $\Phi_{ijl} < 0$, which, by Schur complement can be converted to $\Psi_{ijl} < 0$. Following the similar procedure, we can get $\Psi_{ijl} + \Psi_{jil} < 0$ from the inequality in (29). Pre- and post- multiplying the LMI in (30) with S and S^T, we can get (15). Thus, we obtain (13)-(15) in Theorem 2.1. Finally, by Theorem 2.1, the closed-loop system in (10)-(11) is passive. The proof is completed. □

Remark 2.1 It is observed from Theorem 2.2 that the static output feedback controller design is the feasibility problem of LMIs (28)-(30) with equality constraint (31). However, this kind of problem has been solved in [19] via genetic algorithms and in [18] via the LMI-based algorithms, which can be easily implemented with polynomial running time. Hence, in this paper, we will convert the equality constraint problem to the LMI problem [4].

3 An Illustrative Example

In this section, we provide an example to illustrate the passive delayed static output feedback controller design approach developed in this paper.

The uncertain Takagi–Sugeno (T-S) fuzzy system considered in this example is with two rules with the following parameters:

$$\begin{aligned}
A_{01} &= \begin{bmatrix} -5 & 0.2 \\ 0 & 0.01 \end{bmatrix},\ A_{d01} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix},\ B_{01} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 1 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},\ D_{31} = 0.2,\ E_1 = \begin{bmatrix} -0.4 & 0.1 \end{bmatrix}, \\
E_{11} &= \begin{bmatrix} 0.1 & 0.2 \end{bmatrix},\ A_{02} = \begin{bmatrix} -6 & 0 \\ 0.1 & 0.05 \end{bmatrix},\ A_{d02} = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \\
B_{02} &= \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix},\ C_2 = C_1,\ D_{12} = D_{11},\ D_{32} = 0.3,\ E_2 = \begin{bmatrix} 0.1 & 0.4 \end{bmatrix}, \\
E_{12} &= \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},\ \Delta a_i^{pq} = 0.001I,\ \Delta a_{di}^{pq} = 0.002I,\ \Delta b_i^{pk} = 0.001I.
\end{aligned}$$

The membership functions are chosen as:

$$h_1(x_1(t)) = \begin{cases} \frac{1}{3}, & for\ x_1 < -1, \\ \frac{2}{3} + \frac{1}{3}x_1, & for\ |x_1| \le 1, \\ 1, & for\ x_1 > 1. \end{cases} \quad h_2(x_1(t)) = 1 - h_1(x_1(t)).$$

In this example, given $\tau = 0.1,\ \bar{\tau}_1 = 0.4$, we have the maximum of $\bar{\tau}_2 = 3$; while given $\tau = 0.1,\ \bar{\tau}_2 = 0.1$, we have the maximum of $\bar{\tau}_1 = 3$.

In order to design a fuzzy passive static output feedback controller for the T-S model, we first choose

$$\tau = 0.1,\ \bar{\tau}_1 = 0.1,\ \bar{\tau}_2 = 0.1,\ d_1 = 0.4,\ d_{12} = 0.8,\ \gamma = 0.5$$

and the initial condition is $x(0) = \begin{bmatrix} 0.1 & -0.8 \end{bmatrix}^T$, the disturbance input $w(t)$ is assumed to be

$$w(t) = \frac{1}{t+0.1},\ t \ge 0.$$

Then, solving the LMIs in (28)–(30) and (31), we obtain the solution as follows:

$$N_1 = \begin{bmatrix} 0.1060 & -0.0538 \\ -0.0488 & 0.0746 \end{bmatrix},\ N_2 = \begin{bmatrix} 0.4842 & -0.2166 \\ -0.1829 & 0.1025 \end{bmatrix},$$

and the fuzzy delayed static output feedback controller gains are given by

$$K_1 = \begin{bmatrix} 2.2558 & -0.3342 \\ -0.6589 & 0.2232 \end{bmatrix},\ K_2 = \begin{bmatrix} 10.8439 & -2.4939 \\ -3.9293 & 0.9533 \end{bmatrix}.$$

With the static output feedback fuzzy controller, the simulation result of the state response of the nonlinear system are given in Figure 1. From the simulation result, it can be seen the designed fuzzy output feedback controller is effective.

4 Conclusion

The problem of passive delayed static output feedback control for uncertain Takagi–Sugeno fuzzy systems with interval parameters and additive delays has been studied. In terms of linear matrix inequalities, a sufficient delay-dependent condition for the existence of a full-order fuzzy delayed static output feedback controller, which guarantees the closed-loop system is passive, has been obtained. An example has been provided to show the effectiveness of the proposed method.

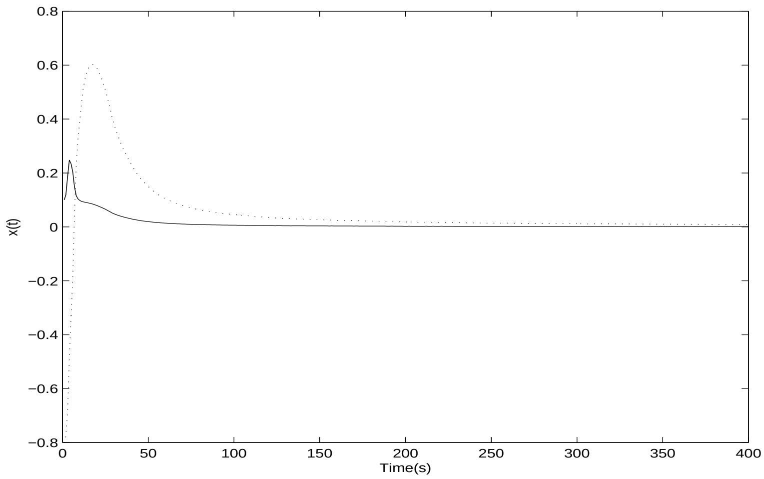

Figure 1: State response $x_1(t)$ (—) and $x_2(t)$ $(\cdots)$.

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Existence and Uniqueness for Nonlinear Multi-variables Fractional Differential Equations

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Abstract: The existence and uniqueness of solutions of nonlinear multi-variables fractional differential equations have been investigated. Using Schauder fixed points theorems and Global contraction mapping theory, we obtain two results concerning the existence and uniqueness of solutions respectively. Moreover, our results are more general than in [8].

Keywords: *existence and uniqueness; nonlinear multi-variables fractional differential equations; Schauder fixed points theorems.*

Mathematics Subject Classification (2000): 35G99, 35A05.

1 Introduction

In recent years, interest has increased concerning the fractional differential equations [5, 13, 26]. Most of works are devoted to the solvability of linear fractional equations in terms of special functions [1, 7] and to problems of analyticity in the complex domain [6]. There are also some studies on the solution of nonlinear differential equations [8]–[11] and [20]. D. Delbosco argues nonlinear fractional equation [11]. E. Ahmed has investigated the fractional-order Lotka–Volterra predator-prey system [20]. Very few contributions exist, as far as we know, concerning nonlinear multi-variables fractional equations of the form

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$$\begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases}$$

where $0 < s_i < 1$, and $i = 1, \cdots, n$ and ${}_0D_t^{s_i}$ is the standard Riemann–Liouville fractional derivative, considered in $\mathbb{R}^+$ or in an interval $(0, a)$, with $a > 0$.

Fractional-order calculus will play an important role in mechatronic and biological systems. It has been found that the behavior of many physical systems can be properly described by using the fractional order system theory. For example, heat conduction, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, visco-elastic systems, quantum evolution of complex systems, quantitative finance and diffusion wave are among the known dynamical systems that were modeled using fractional order equations. In fact, real world processes generally or most likely are nonlinear multi-variable fractional order systems. In the last 6 years, considerable attention has also been paid to obtain analytical existence conditions for nonlinear fractional order systems [27]. Our aim is to analyze uniqueness conditions further more. The paper is organized as follows. In Section 2 we recall the definitions of fractional integral and derivative and related basic properties used in the text. Section 3 contains results for solutions which are continuous at the origin. Conclusions are given in Section 4.

2 Definitions and Preliminary Results

The definitions and the results of the fractional calculus reported below are not exhaustive but rather oriented to the subject of this paper. For the proofs, which are omitted, we refer the reader to Miller and Ross [7] or other texts on basic fractional calculus.

Definition 2.1 The uniform formula of a fractional integral with $\alpha \in (0, 1)$ is defined as

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau)\, d\tau, $$

where $f(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$, is an arbitrary integrable function, ${}_0D_t^{-\alpha}$ is the fractional integral of order α on $[0, t]$, $t > 0$ and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number, the Riemann–Liouville fractional derivative is defined as

$$ {}_0D_t^p f(t) = \tfrac{d^{[p]+1}}{dt^{[p]+1}} \left[{}_0D_t^{-[p]-p+1} f(t) \right]. $$

The following properties are some of the main ones of the fractional derivatives and integrals [12, 18, 20].

Property 1. ${}_0D_t^p t^\upsilon = \frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-p)} t^{\upsilon - p}$, where $p \in \mathbb{R}$, $\upsilon > -1$.

Property 2. ${}_0D_t^p \left({}_0D_t^q f(t)\right) = {}_0D_t^{p+q} f(t)$, where $p < 0, q < 0$.

Property 3. ${}_0D_t^p \left({}_0D_t^{-p} f(t)\right) = f(t)$, where $p \in \mathbb{R}^2$, $t > 0$.

Property 4. ${}_0D_t^{-\alpha} f(0) = 0$, where $f \in C[0,a]$, $\alpha \in (0,1)$.

Property 5. ${}_0D_t^{p} f(t) = 0$, where $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\alpha \in (0,1)$, then $f(x) = cx^{p-1}$, $c \in \mathbb{R}$.

Proposition 2.1 *Assume that* $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ *with a fractional derivative order* $0 < \alpha < 1$ *that belongs* $C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. *Then*

$$ {}_aD_t^{-\alpha}\left({}_aD_t^{\alpha} f(t)\right) = f(t) + cx^{\alpha-1} $$

for some $\mathrm{c} \in \mathbb{R}$.

When the function $f(t)$ is in $C(\mathbb{R}^+)$, then $c = 0$.

In all the definitions and results of this section the set $\mathbb{R}^+$ can be substituted by the intervals $(0,a)$ or $(0,a]$, $a > 0$. For simplicity, in the next sections we shall often limit arguments to the choice $a = 1$. A more precise analysis of the operators ${}_0D_t^{-\alpha}, {}_0D_t^{\alpha}$ can be given in the frame of the spaces $C_r(\mathbb{R}^+)$, $r > 0$, of all functions $f \in C\left(\mathbb{R}_0^+\right)$ such that $x^r\ f \in C\left(\mathbb{R}_0^+\right)$.

Let $0 < \alpha < 1$; if $f \in C\left(\mathbb{R}_0^+\right)$ with $r < \alpha$, then ${}_0D_t^{-\alpha} f \in C\left(\mathbb{R}_0^+\right)$ with ${}_0D_t^{-\alpha} f(0) = 0$. If $f \in C_\alpha(\mathbb{R}^+)$, then ${}_0D_t^{-\alpha} f$ is bounded at the origin if $f \in C\left(\mathbb{R}_0^+\right)$. With $\alpha < r < 1$, then we may expect ${}_0D_t^{-\alpha} f$ to be unbounded at the origin. Concerning Proposition 2.1, the last part can now be stated more precisely. If $f \in C_r\left(\mathbb{R}_0^+\right)$ with $r < 1 - \alpha$ and ${}_0D_t^{\alpha} f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, then ${}_0D_t^{-\alpha}\left({}_0D_t^{\alpha} f(t)\right) = f(t)$.

3 Existence and Uniqueness

Consider the fractional differential equations

$$ \begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases} \tag{1} $$

where $0 < s_i < 1$, and $i = 1, \cdots, n$ and $f_i : [0,a] \times \mathbb{R} \to \mathbb{R}, 0 < a < +\infty$ are given functions, continuous in $(0,a) \times \mathbb{R}$.

We introduce the following definition of a solution for (1).

Definition 3.1 Let $C^*[0,a]$ be the class of continuous column vector $U(t) = (u_1(t), u_2(t), \cdots u_n(t))$ whose components $u_1(t), u_2(t), \cdots u_n(t) \in C[0,a]$ the class of continuous functions on the interval $[0,a]$. The norm of $U \in C^*[0,a]$ is given by

$$ \|U\| = \max_{1 \le i \le n} \left\{ \sup_{0 \le x \le a} u(x) \right\}. $$

Definition 3.2 By a solution of the fractional differential equations (1) we mean a column vector $U \in C^*[0,a]$. This vector satisfies (1).

Remark 3.1 We may apply the results of Section 2, in particular Proposition 2.1 and the subsequent remarks, to reduce(1) to integral equations. In fact, if $U(t) = (u_1(t), u_2(t), \cdots u_n(t)) \in C^*[0,a]$ or more generally $U \in C_r^*[0,a]$, with $r < 1 - s$,where $s = \min\limits_{1\le i\le n} \{s_i\}$, and further assumptions guarantee $f_i(t, u_1(t), u_2(t), \cdots u_n(t)) \in C[0,a] \cap L^1[0,a]$, then equations (1) are equivalent to the integral equations

$$\begin{cases} u_1(t) = {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_i(t) = {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_n(t) = {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)). \end{cases} \tag{2}$$

Such a reduction will be systematically used in this section. We first present Schauder fixed point theorem. It can be easily proved [25].

Theorem 3.1 *Let* E *be a closed bounded convex subset of a normed space* X*. If $f : E \to E$ is a compact map such that $f(E)$ is contained in* E*, then there is an x in E such that $f(x) = x$.*

Then, we give a local existence theorem.

Theorem 3.2 *Let $0 < s_i < 1$, $i = 1, \cdots, n$,$s = \min\limits_{1\le i\le n} \{s_i\}$, $0 \le \sigma < s < 1$ and $f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \in C(0,1]$. Assume that $t^\sigma f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \in C(0,1]$ $t^\sigma f_i(t, u_1(t), u_2(t), \cdots u_n(t)) \in C(0,1]$. Then the fractional differential equations*

$$\begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases} \tag{3}$$

have a least continuous solution $U \in C^*[0,1]$, for a suitable $\delta \le 1$.

Proof According to Remark 3.1, we are reduced to consider the following nonlinear integral equations

$$\begin{cases} u_1(t) = {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_i(t) = {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_n(t) = {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)). \end{cases}$$

Let $T : C^*[0,1] \to C^*[0,1]$ be the operator defined as

$$(TU)(t) = \begin{pmatrix} {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)) \end{pmatrix}^T.$$

We claim that the operator T is compact. Indeed, the operator is the composition of two simple operators in this way

$$T = A \circ N,$$

where

$$(NU)(t) = \begin{pmatrix} t^{\sigma} f_1(t, U(t)) \\ \vdots \\ t^{\sigma} f_i(t, U(t)) \\ \vdots \\ t^{\sigma} f_n(t, U(t)) \end{pmatrix}^T$$

is a continuous and bounded operator (Nemytskii operator) and

$$(AV)(t) = \begin{pmatrix} \frac{1}{\Gamma(s_1)} \int_0^t (t-\tau)^{s_1-1} \tau^{-\sigma} v_1(\tau, U(\tau))\, d\tau \\ \vdots \\ \frac{1}{\Gamma(s_i)} \int_0^t (t-\tau)^{s_i-1} \tau^{-\sigma} v_i(\tau, U(\tau))\, d \\ \vdots \\ \frac{1}{\Gamma(s_n)} \int_0^t (t-\tau)^{s_n-1} \tau^{-\sigma} v_n(\tau, U(\tau))\, d \end{pmatrix}^T$$

is a compact operator, since $s - \sigma > 0$ as for example in [5].

Moreover, from Section 2, we have for $0 < t \leq \delta \leq 1$.

$$(AV)(t) \prec \begin{pmatrix} |Av_1(t, U(t))| \\ \vdots \\ |Av_i(t, U(t))| \\ \vdots \\ |Av_n(t, U(t))| \end{pmatrix}^T$$

$$\prec \begin{pmatrix} \sup\limits_{0 \leq t \leq \delta} |v_1(t, U(t))| \frac{1}{\Gamma(s_1)} \int_0^t (t-\tau)^{s_1-1} \tau^{-\sigma} v_1(\tau, U(\tau))\, d\tau \\ \vdots \\ \sup\limits_{0 \leq t \leq \delta} |v_i(t, U(t))| \frac{1}{\Gamma(s_i)} \int_0^t (t-\tau)^{s_i-1} \tau^{-\sigma} v_i(\tau, U(\tau))\, d\tau \\ \vdots \\ \sup\limits_{0 \leq t \leq \delta} |v_n(t, U(t))| \frac{1}{\Gamma(s_n)} \int_0^t (t-\tau)^{s_n-1} \tau^{-\sigma} v_n(\tau, U(\tau))\, d\tau \end{pmatrix}^T$$

$$\prec \begin{pmatrix} \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_1)} \delta^{s_1-\sigma} \sup\limits_{0 \leq t \leq \delta} |v_1(t, U(t))| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} \delta^{s_i-\sigma} \sup\limits_{0 \leq t \leq \delta} |v_i(t, U(t))| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_n)} \delta^{s_n-\sigma} \sup\limits_{0 \leq t \leq \delta} |v_n(t, U(t))| \tau \end{pmatrix}^T.$$

Let

$$\varepsilon = \max_{1\leq i\leq n}\left\{\frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)}\delta^{s_i-\sigma}\right\},$$

therefore, taking the norms in $C^*[0,\delta]$,

$$\|AV\| \leq \varepsilon\,\|V\|,$$

where we may assume $\varepsilon > 0$ as small as we want by shrinking $\delta > 0$.

Now fix B_r as a domain of the operator T, where $B_r = \{V \in C^*[0,\delta] : \|V\| < r\}$, which is a convex, bounded, and closed subset of the Banach space $C^*[0,\delta]$.

For δ sufficiently small, we have $T(B_r) \subseteq B_r$ The Schauder fixed point theorem assures that operator T has at least one fixed point and then (3) has at least one continuous solution. U defined on $C^*[0,\delta]$, where $\delta \leq 1$.

Example 3.1 Observe that we cannot expect uniqueness for such solutions, in general. Consider for example the equations

$$\begin{cases} {}_0D_t^{1/2}u_1 = \frac{3\mathrm{T}(3/4)}{\Gamma(1/4)}u_2^{1/2}, \\ {}_0D_t^{1/4}u_2 = \frac{2\mathrm{T}(1/2)}{\Gamma(1/4)}u_1^{1/3}, \end{cases}$$

which admit the two solutions $(0,0)$ and $\left(x^{3/4}, x^{1/2}\right)$.

The following theorem shows that uniqueness and global existence can be obtained under an uniform Lipschitz-type assumption.

Theorem 3.3 *Let* $0 < s_i < 1$, $i = 1,\cdots,n$, $s = \min\limits_{1\leq i\leq n}\{s_i\}$, $0 \leq \sigma < s < 1$ *and* $F(t,U) = (f_1(t,U), f_2(t), \cdots f_n(t)) \in C^*_\sigma[0,1]$. *Assume further*

$$\|F(t,U) - F(t,V)\| \leq \frac{L}{t^\sigma}\|U - V\| \tag{4}$$

for some positive constant L *independent of* $U, V \in \mathbb{R}^n$, $t \in (0,1]$. *Then the fractional differential equations (3) have a unique solution* $U \in C^*[0,1]$.

Proof As in the proof of Theorem 3.3, we are reduced to studying the operator

$$(TU)(t) = \begin{pmatrix} {}_0D_t^{-s_1}f_1(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_i}f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_n}f_n(t, u_1(t), u_2(t), \cdots, u_n(t)) \end{pmatrix}^T$$

which is well defined and continuous as a map $T : C^*[0,1] \to C^*[0,1]$, in the view of the assumption of continuity on $t^\sigma f_i(t)$. Let us define the k iterates of the operator T as is standard

$$T^1 = T, T^K = T \circ T^{K-1}.$$

It will be sufficient to prove that T^K is a contraction operator for K being sufficiently larger. Actually, we have for $U, V \in C^*[0,1]$.

$$\left\|T^k U(t) - T^k V(t)\right\| \le \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)} x^{k(s^*-\sigma)} \|U-V\|, \tag{5}$$

where the constant H depends only on $s^* \in \{s_i\}$ and σ, in fact

$$TU(t) - TV(t) = \begin{pmatrix} {}_0D_t^{-s_1}(f_1(t,U) - v_1(t,U)) \\ \vdots \\ {}_0D_t^{-s_i}(f_i(t,U) - v_i(t,U)) \\ \vdots \\ {}_0D_t^{-s_n}(f_n(t,U) - v_n(t,U)) \end{pmatrix}^T \prec \begin{pmatrix} \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_1)} x^{s_1-\sigma}\|U-V\| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} x^{s_i-\sigma}\|U-V\| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_n)} x^{s_n-\sigma}\|U-V\| \end{pmatrix}^T.$$

Let

$$\frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s^*)} x^{s_*-\sigma} = \max_{1\le i\le n}\left\{\sup_{0\le x\le 1}\left|\frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} x^{s_i-\sigma}\right|\right\},$$

therefore (5) is proved for $k=1$, if $H \ge \Gamma(1-\sigma)$. Assuming by induction that(5) is valid for k, we obtain similarly

$$\begin{aligned} & T^{k+1}U(t) - T^{k+1}V(t) \\ & \prec \begin{pmatrix} \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\Gamma(s_1)} \|U-V\| \int_a^t (t-\tau)^{s_1} \tau^{k(s_1-\sigma)-\sigma} d\tau \\ \vdots \\ \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\Gamma(s_i)} \|U-V\| \int_a^t (t-\tau)^{s_i} \tau^{k(s_i-\sigma)-\sigma} d\tau \\ \vdots \\ \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\Gamma(s_n)} \|U-V\| \int_a^t (t-\tau)^{s_n} \tau^{k(s_n-\sigma)-\sigma} d\tau \end{pmatrix}^T \\ & \prec \begin{pmatrix} \frac{\Gamma(k(s_1-\sigma)-\sigma)H^k L^{k+1}}{\Gamma(k(s^*-\sigma)+1)\Gamma((k+1)(s_1-\sigma)+1)} \|U-V\| t^{(k+1)(s_1-\sigma)} \\ \vdots \\ \frac{\Gamma(k(s_i-\sigma)-\sigma)H^k L^{k+1}}{\Gamma(k(s^*-\sigma)+1)\Gamma((k+1)(s_i-\sigma)+1)} \|U-V\| t^{(k+1)(s_i-\sigma)} \\ \vdots \\ \frac{\Gamma(k(s_n-\sigma)-\sigma)H^k L^{k+1}}{\Gamma(k(s^*-\sigma)+1)\Gamma((k+1)(s_n-\sigma)+1)} \|U-V\| t^{(k+1)(s_n-\sigma)} \end{pmatrix}^T \end{aligned}$$

and then (5) is proved for $k+1$, if H is given by

$$H = \max_k H_k,\ H_k = \max_{1\le i\le n}\left\{\frac{\Gamma(k(s_i-\sigma)-\sigma)}{\Gamma(k(s_i-\sigma)+1)}\right\}. \tag{6}$$

Note that (6) defines actually a finite H, since $H_k \le 1$, for $k \ge (1+\sigma)/(s-\sigma)$, Taking k sufficiently large in (6), we have, say, $(HL)^k \Big/ \Gamma(k(s^*-\sigma)+1) \le 1/2$. and therefore $\left\|T^k U(t) - T^k V(t)\right\| \le \frac{1}{2}\|U-V\|$ which gives the proof.

Similarly, the existence and uniqueness for initial value problem of nonlinear multi-variables fractional differential equations also can be proved. In particular, for one-dimensional case ${}_0D_t^s u(t) = f(t,u)$, we obtain identical results in [8].

Example 3.2 Consider for example the equations

$$\begin{cases} {}_0D_t^{1/2}u_1 = u_2, \\ {}_0D_t^{1/4}u_2 = u_1, \end{cases}$$

which admit a unique solution $(0,0)$, defined on $[0,1]$. Since it suffices (4) for $L = 2, \sigma = 1/5$.

Example 3.3 Consider he following general nonlinear system

$${}_aD_t^{\alpha}y(t) + N(y(t)) = g(t), \quad t \in [0,1],$$

where N represents a nonlinear operator with $N(0) = 0$, g(t) is a function with respect to t.

For $L = \max_y(N'(y)), \sigma = 0$, where $F(t,U) = g(t) - N(U)$, we have

$$\|F(t,U) - F(t,V)\| = |N(U) - N(V)| \leq \frac{L}{t^{\sigma}} \|U - V\|,$$

i.e. the system admits a unique solution $(0,0)$ defined on $[0,1]$.

4 Conclusion

In this paper, we prove existence and uniqueness theorems for some classes of nonlinear multi-variables fractional differential equations. It extends the original results for fractional differential equations and provides convenience for our further work on nonlinear multi-variable fractional equations.

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