Volume 11, Number 1, 2011 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 11 Number 1 2011

CONTENTS

 ISSN 1562-8353 Printed in Ukraine
To receive contents and abstracts by e-mail, visit our Website at: **http://www.e-ndst.kiev.ua**

NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 11, No. 1, 2011

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

InforMath Publishing Group
http://www.e-ndst.kiev.ua

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. Nonlinear Dynamics and Systems Theory (ND&ST) is an international journal devoted to publishing peer-refereed, high quality, original papers, brief notes and review articles focusing on nonlinear dynamics and systems theory and their practical applications in engineering, physical and life sciences. Submission of a manuscript is a representation that the submission has been approved by all of the authors and by the institution where the work was carried out. It also represents that the manuscript has not been previously published, has not been copyrighted, is not being submitted for publication elsewhere, and that the authors have agreed that the copyright in the article shall be assigned exclusively to InforMath Publishing Group by signing a transfer of copyright form. Before submission, the authors should visit the website:

http://www.e-ndst.kiev.ua

for information on the preparation of accepted manuscripts. Please download the archive Sample_NDST.zip containing example of article file (you can edit only the file Samplefilename.tex).

(2) Manuscript and Correspondence. Manuscripts should be in English and must meet common standards of usage and grammar. To submit a paper, send by e-mail a file in PDF format directly to

Professor A.A. Martynyuk, Institute of Mechanics,
Nesterov str.3, 03057, MSP 680, Kiev-57, Ukraine
e-mail: anmart@stability.kiev.ua; center@inmech.kiev.ua

or to one of the Regional Editors or to a member of the Editorial Board. Final version of the manuscript must typeset using LaTex program which is prepared in accordance with the style file of the Journal. Manuscript texts should contain the title of the article, name(s) of the author(s) and complete affiliations. Each article requires an abstract not exceeding 150 words. Formulas and citations should not be included in the abstract. AMS subject classifications and key words must be included in all accepted papers. Each article requires a running head (abbreviated form of the title) of no more than 30 characters. The sizes for regular papers, survey articles, brief notes, letters to editors and book reviews are: (i) 10-14 pages for regular papers, (ii) up to 24 pages for survey articles, and (iii) 2-3 pages for brief notes, letters to the editor and book reviews.

(3) Tables, Graphs and Illustrations. Each figure must be of a quality suitable for direct reproduction and must include a caption. Drawings should include all relevant details and should be drawn professionally in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach the electronic file of the artwork (preferably in PCX format).

(4) References. References should be listed alphabetically and numbered, typed and punctuated according to the following examples. Each entry must be cited in the text in form of author(s) together with the number of the referred article or in the form of the number of the referred article alone.

Journal: [1] Poincare, H. Title of the article. *Title of the Journal* **Vol. l** (No.l), Year, Pages. [Language]

Book: [2] Liapunov, A.M. *Title of the book.* Name of the Publishers, Town, Year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book.* (Eds.). Name of the Publishers, Town, Year, Pages. [Language]

(5) Proofs and Sample Copy. Proofs sent to authors should be returned to the Editorial Office with corrections within three days after receipt. The corresponding author will receive a sample copy of the issue of the Journal for which his/her paper is published.

(6) Editorial Policy. Every submission will undergo a stringent peer review process. An editor will be assigned to handle the review process of the paper. He/she will secure at least two reviewers' reports. The decision on acceptance, rejection or acceptance subject to revision will be made based on these reviewers' reports and the editor's own reading of the paper.

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 11 Number 1 2011

CONTENTS

Founded by A.A. Martynyuk in 2001.
Registered in Ukraine Number: KB 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

AIMS AND SCOPE

Nonlinear Dynamics and Systems Theory is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed.

PUBLICATION AND SUBSCRIPTION INFORMATION

Nonlinear Dynamics and Systems Theory will have 4 issues in 2011, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher (mailto:anmart@stability.kiev.ua), SWETS Information Services B.V. (mailto:Operation-Academic@nl.swets.com), EBSCO Information Services (mailto:journals@ebsco.com), or website of the Journal: http://e-ndst.kiev.ua. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

ABSTRACTING AND INDEXING SERVICES

Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.

Nonlinear Dynamics and Systems Theory, 11 (1) (2011) 1–5

PERSONAGE IN SCIENCE

Academician A.A. Martynyuk

to the 70th Birthday Anniversary

J.H. Dshalalow [1*], N.A. Izobov [2] and S.N. Vassilyev [3]

[1] *Department of Applied Mathematics Florida Institute of Technology, Melbourne, Fl, 32901, USA*
[2] *Institute of Mathematics NAS of Byelorussia, Surganova Str., 11, Minsk, Byelorussia*
[3] *Institute of Control Problems Russian Academy of Sciences, Profsoyuznaja Str., 65, Moscow, 117997, Russia*

On March 6, 2011, a Member of the National Academy of Sciences of Ukraine, Habilitation Doctor and Ph.D. of physical and mathematical sciences, Professor Anatoly Andreevich Martynyuk turns 70. The Editorial Board of the International Scientific Journal "Nonlinear Dynamics and Systems Theory" congratulates him on this occasion and wishes him a great health and new significant achievements in his scientific endeavors.

1 Brief Outline of Martynyuk's Life

March 6, 2011 marked the 70th Birthday of the very prominent scientist in area of theoretical mechanics and mathematics Anatoliy Andreyevich Martynyuk who made a significant contribution to the development of motion stability theories and their applications.

A.A. Martynyuk was born in the family of a railway mechanics, Andrey Gerasimovich Martynyuk, who lived in the Cherkassky region of UkrSSR (since 1991 Ukraine). After graduating from Physical and Mathematical Department of B. Khmelnitsky State University of Cherkassy he entered the post-graduate school supervised by Professor A.N. Golubentsev at the Institute of Mechanics of Academy of Science of Ukr.SSR (now the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine) and focused on problems of finite stability on a given time interval. Three years later Martynyuk conducted research and published several papers with a new approach for an estimation of finite stability of motion on a given time interval (practical stability). The above problems were treated by such renowned scientists as N.D. Moiseyev, N.G. Chetayev and others. In spite of this challenge, Martynyuk obtained new qualitative results.

* Corresponding author: mailto:eugene@fit.edu

Along with these investigations and those included in his Doctoral Thesis (which he defended in 1973), Martynyuk developed foundations of motion stability theory for large scale dynamical systems with non-asymptotically stable subsystems. In 1978, he founded the Department of Processes Stability at the Institute of Mechanics of Academy of Sciences of Ukr.SSR, and he has been its Head ever since. Among other activities at the Department of Processes Stability, Martynyuk chaired the Seminar "Stability Theory and its Applications" during which various new results obtained by Ukrainian and European scholars were discussed. In 2009, Martynyuk was elected the Full Member of the National Academy of Science of Ukraine. In 2008, he was awarded a National Prize in the field of Science and Technology.

The main scientific results obtained by Martynyuk were as follows:

- nonclassical motion stability theories (technical and practical stability in the whole);
- applications of integral inequalities in qualitative theory of differential equations;
- the development of a comparison method in nonlinear mechanics;
- stability analysis of large scale systems under structural perturbations;
- topological dynamics (the method of limiting equations);
- the development of matrix-valued Lyapunov functions;
- stability of uncertain and fuzzy dynamical systems;
- stability theory of dynamical equations on a time scale;
- mathematical problems of population dynamics.

Alongside his prolific scientific research, Martynyuk has been involved in very intense scientific-organizational and publishing activities. In 1982, he organized publication of the book "Lectures on Theoretical Mechanics" by A.M. Lyapunov. He also made an ample amount of work as the editor of the International Series of Scientific Monographs "Stability and Control: Theory, Methods and Applications" copyrighted by the Gordon and Breach Science Publishers (Great Britain). From 1992 to 2002 they published 22 volumes in the Series which gained in a worldwide recognition.

In 2001 Martynyuk founded the International Academic Journal of "Nonlinear Dynamics and Systems Theory" and its online version at `http://e-ndst.kiev.ua` and he has been its chief editor.

In 2006 he established a new International Series of Scientific Monographs, Textbooks and Lecture Courses entitled "Stability, Oscillations and Optimization of Systems" at the Cambridge Scientific Publishers and is serving as its chief editor. By now, 4 volumes of the Series have appeared.

Martynyuk is an editorial board member of three Russian-language journals: the International Journals of "Applied Mechanics", "Nonlinear Oscillations" and "Electronic Modeling" and two English-language journals: "Journal of Applied Mathematics and Stochastic Analysis" (USA) and "Differential Equations and Dynamical Systems" (India).

Martynyuk has been a major advisor to 23 Candidates (Ph.D.) and 3 Doctors (Habilitation) in Physical and Mathematical Sciences who are now employed and are being successful workers in various countries of the former Soviet Union. He is a member of several Scientific Councils on awarding scientific degrees of Candidate (Ph.D.) and Doctor (Habilitation) of Physical and Mathematical Sciences. Martynyuk is a vice-president of the National Committee on Theoretical and Applied Mechanics of Ukraine.

2 List of Monographs and Books by A.A. Martynyuk

I. *Technical Stability in Dynamics.* Tekhnika, Kiev, 1973. [Russian]

II. *Motion Stability of Composite Systems.* Naukova Dumka, Kiev, 1975. [Russian]

III. *Integral Inequalities and Stability of Motion.* Naukova Dumka, Kiev, 1979. (with R. Gutowski). [Russian]

IV. *Dynamics and Motion Stability of Wheeled Transporting Vehicles.* Tekhnika, Kiev, 1981. (with L.G. Lobas and N.V. Nikitina). [Russian]

V. *Practical Stability of Motion.* Naukova Dumka, Kiev, 1983. [Russian]

VI. *Large Scale Systems Stability under Structural and Singular Perturbations.* Naukova Dumka, Kiev, 1984. (with Ly.T. Grujić and M. Ribbens-Pavella). [Russian]

VII. *Large-Scale Systems Stability under Structural and Singular Perturbations.* Springer-Verlag, Berlin, 1987. (with Ly.T. Grujić and M. Ribbens–Pavella).

VIII. *Stability Analysis of Nonlinear Systems.* Marcel Dekker, New York, 1989. (with V. Lakshmikantham and S. Leela).

IX. *Stability of Motion: Method of Integral Inequalities.* Naukova Dumka, Kiev, 1989. (with V. Lakshmikantham and S. Leela). [Russian]

X. *Practical Stability of Nonlinear Systems.* World Scientific, Singapore, 1990. (with V. Lakshmikantham and S. Leela).

XI. *Stability of Motion: Method of Limiting Equations.* Naukova Dumka, Kiev, 1990. (with J. Kato and A.A. Shestakov). [Russian]

XII. *Stability of Motion: Method of Comparison.* Naukova Dumka, Kiev, 1991. (with V. Lakshmikantham and S. Leela). [Russian]

XIII. *Some Problems of Mechanics of Nonautonomous Systems.* Mathematical Institute of SANU, Beograd–Kiev, 1992. (with V.A. Vujicić). [Russian]

XIV. *Stability Analysis: Nonlinear Mechanics Equations.* Gordon and Breach Science Publishers, Amsterdam, 1995.

XV. *Stability of Motion of Nonautonomous Systems: Method of Limiting Equations.* Gordon and Breach Science Publishers, Amsterdam, 1996. (with J. Kato and A.A. Shestakov).

XVI. *Advances in Nonlinear Dynamics.* Gordon and Breach Science Publishers, Amsterdam, 1997. (with S. Sivasundaram).

XVII. *Stability by Liapunov's Matrix Function Method with Applications.* Marcel Dekker, New York, 1998.

XVIII. *Theory of Practical Stability with Applications.* Harbin Institute of Technology, Harbin, 1999. (with Sun Zhen qi). [Chinese]

XIX. *Qualitative Methods in Nonlinear Dynamics: Novel Approaches to Liapunov's Matrix Function.* Marcel Dekker, New York, 2002.

XX. *Stability and Stabilization of Nonlinear Systems with Random Structures.* Taylor & Francis, London and New York, 2002. (with I.Ya. Kats).

XXI. *Advances in Stability Theory at the End of the 20th Century.* (Ed.: A.A. Martynyuk). Taylor & Francis, London and New York, 2003.

XXII. *Theory of Practical Stability with Applications.* Second Edition, Revised and Expanded. Chinese Academy of Sciences Publishing Company, Beijing, 2003. (with Sun Zhen qi). [Chinese]

XXIII. *Qualitative Analysis of Nonlinear Systems with Small Parameter.* Chinese Academy of Sciences Publishing Company, Beijing, 2006 (with Sun Zhen qi). [Chinese]

XXIV. *Stability of Motion: The Role of Multicomponent Liapunov's Functions*, Cambridge Scientific Publishers, London, 2007.

XXV. *Advances in Chaotic Dynamics and Applications.* (Eds.: C. Cruz–Hernndez and A.A. Martynyuk). Cambridge: Cambridge Scientific Publishers, 2010.

3 List of Personal Papers by A.A. Martynyuk (Continued)*

130. To the theory of direct Liapunov's method. *Dokl. Acad. Nauk* **406** (3) (2006) 309–312. [Russian]

131. On stability of set trajectories of nonlinear dynamics. *Dokl. Acad. Nauk* **414** (3) (2007) 299–303. [Russian]

132. On polydynamics of nonlinear systems on time scales. *Dokl. Acad. Nauk* **414** (4) (2007) 455–458. [Russian]

133. Stability analysis of large-scale functional differential systems. *Ukr. Math. Journ.* **59** (3) (2007) 87–98. [Russian]

134. General problem on polydynamics on time scales *Dokl. Nats. Acad. Nauk Ukr.* (1) (2008) 7–13. [Russian]

135. On comparison principle for matrix differential equations. *Dokl. Nats. Acad. Nauk Ukr.* (12) (2008) 28–33. [Russian]

* The titles 1–129 were published in the Journal *Nonlinear Dynamics and Systems Theory*, Vol. 6, issue 1, 2006.

136. On exponential stability dynamic systems on time scales. *Dokl. Acad. Nauk* **421** (4) (2008) 312–317. [Russian]

137. An exploration of polydynamics on nonlinear equations on time scales. *ICIC Express Letters* **2** (2) (2008) 155–160.

138. Novel trends in the theory of direct Liapunov method. In: *Advances in Nonlinear Analysis: Theory, Methods and Applications* (Eds.: S. Sivasundaram at al.). Cambridge Scientific Publishers, Cambridge, 2008, 221–232.

139. Criterion of uniform stability of nonlinear systems in the hole. *Dokl. Nats. Acad. Nauk Ukr.* (1) (2009) 35–39. [Russian]

140. Comparison principle for a set differential equation with robust causal operator. *Dokl. Acad. Nauk* **427** (6) (2009) 750-753. [Russian]

141. On instability solutions of dynamic equations on time scales. *Dokl. Nats. Acad. Nauk of Ukraine* (10) (2009) 21–26. [Russian]

142. On direct Liapunov method for equations with fractional derivative. *Dokl. Nats. Acad. Nauk Ukr.* (3) (2010) 33–34. [Russian]

143. On a mathematical model of world dynamics and sustainable development. *Dokl. Nats. Acad. Nauk Ukr.* (7) (2010) 16–21. [Russian]

144. Exponential stability on time scales under structural perturbations. *Dokl. Nats. Acad. Nauk of Ukraine* (9) (2010) 24–29. [Russian]

145. On stability of the set impulsive equations. *Dokl. Acad. Nauk* **436** (5) (2011) 593–596. [Russian]

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 7–52

Stability in the Models of Real World Phenomena

A.A. Martunyuk

S.P. Timoshenko Institute of Mechanics of NAS of Ukraine,
Nesterov str., 3, Kyiv, 03057, Ukraine

Received: March 29, 2010; Revised: January 25, 2011

Abstract: In this paper we consider several examples of real world models to illustrate the general methods of stability analysis of nonlinear systems developed recently in the Department of Processes Stability of S.P. Timoshenko Institute of Mechanics of NAS of Ukraine.

Keywords: *robot dynamics and control; neural networks on time scales; lasers; dynamic economic models; fuzzy control; scalar and vector Lyapunov functions.*

Mathematics Subject Classification (2000): 70E60, 92B20, 78A60, 91B62, 93C42, 93D30.

1 Introduction

In this paper, we offer several examples of real world models to illustrate the general methods of stability analysis developed in the books [8, 16, 17].

Section 2 deals with the motion stability problem of robot motion whose mathematical model takes into account the dynamics of the environment interacting with the robot. We apply here some integral inequalities from Chapter 1 of the book [8].

In Section 3, we consider neural networks on time scales and introduce the study of the stability problem in this new direction.

In Section 4, we consider a problem of stability of regular synchronous generator of optical connected lasers.

Section 5 presents models from economics and using the method of vector Lyapunov functions proves that a market tends to some given evolution independent of initial conditions.

Finally in Section 6, we analyze a model of impulsive Takagi–Sugeno systems with application to the mathematical model in population growth under the impulsive control.

* Corresponding author: mailto:center@inmech.kiev.ua

2 Stability of a Robot Interacting with a Dynamic Environment

A dynamic robot model is described by the differential equation

$$H(q)\ddot{q} + h(q,\dot{q}) = \tau + J^T(q)F, \tag{2.1}$$

where q, $\dot{q}$, $\ddot{q} \in R^n$ are the vectors of the generalized coordinates, velocities and accelerations of the robot; $H(q)$ is the positive definite matrix of inertia moments of manipulator mechanisms; $h(q,\dot{q})$ is the n-dimensional nonlinear vector function which takes into consideration centrifugal, Coriolis and gravitational moments; $\tau = \tau(t)$ is the n-dimensional vector on input (control); $J^T(q)$ is the $n \times m$ Jacobi matrix associated with the motion velocity of control robot devices and its generalized coordinates; $F(t)$ is the n-dimensional vector of generalized forces or generalized forces and moments acting on the executive robot device due to the dynamic environment.

Under the condition when the environment does not admit eigen "motions" independent of the motion of the executive robot organs, the mathematical model of environment is described by the nonlinear vector equation

$$M(s)\ddot{s} + L(s,\dot{s}) = -F, \tag{2.2}$$

$$s = \varphi(q), \tag{2.3}$$

where s is the vector of the environment motions; $\varphi(q)$ is the vector function connecting the coordinates s and q. Note that in the case of traditional hybrid control, the environment plays the role of kinematic limitation and the relationship (2.3) becomes

$$\varphi(q) = 0. \tag{2.4}$$

Under certain assumptions the equation (2.2) may be represented as

$$M(q)\ddot{q} + L(q,\dot{q}) = -S^T(q)F, \tag{2.5}$$

where $M(q)$ is the $n \times m$ non-singular matrix; $L(q,\dot{q})$ is the nonlinear n-dimensional vector function; $S^T(q)$ is the $n \times m$ matrix of the n rank.

Thus the equation set (2.1), (2.5) represents the closed mathematical model of the robot interacting with the environment.

Let $q_p(t)$ be the n-dimensional vector of the program value of the generalized coordinates, $\dot{q}_p(t)$ be the n-dimensional vector of the program value of the generalized velocities, $F_p(t)$ be the m-dimensional vector of forces corresponding to the program values of the generalized coordinates and velocities. The program values of force $F_p(t)$ and those of functions $q_p(t)$, $\dot{q}_p(t)$, $\ddot{q}_p(t)$ cannot be arbitrary and should satisfy the relationship $F_p \equiv \Phi(q_p(t), \dot{q}_p(t), \ddot{q}_p(t))$ where $\Phi \in C(R^n \times R^n \times R^n, R^m)$. The connected system of equations (2.1), (2.5) can easily be reduced to the form

$$M(q)\ddot{q} - M(q_p) + L(q,\dot{q}) - L(q_p,\dot{q}_p) + \left[S^T(q) - S^T(q_p)\right]F_p = -S^T(q)(F - F_p). \tag{2.6}$$

The n-dimensional vector of deviations of the program trajectory from real one is designated by y. Then the equation (2.6) becomes

$$\ddot{y} + K(t,y,\dot{y}) = -M^{-1}(y + q_p)S^T(y + q_p)(F - F_p), \tag{2.7}$$

where

$$\begin{aligned}K(t,y,\dot y) &= M^{-1}(y+q_p)\Big\{L(y+q_p,\dot y+\dot q_p)-\\ &- L(q,\dot q_p)+\big[M(y+q_p)-M(q_p)\big]\ddot q_p+\\ &+\big[S^T(y+q_p)-S^T(q_p)\big]F_p\Big\}.\end{aligned}$$

The equation set (2.7) is transformed to the following form

$$\frac{dx}{dt}=A(t)x+\alpha(t,x)+\beta(t,x)\mu(t),$$

where

$$x=(x_1,x_2)^T,\quad x_1=y,\quad x_2=\dot y,\quad A(t)=\begin{pmatrix} O_n & I_n\\ -\left.\dfrac{\partial K}{\partial y}\right|_{(y,\dot y)=(0,0)} & -\left.\dfrac{\partial K}{\partial \dot y}\right|_{(y,\dot y)=(0,0)}\end{pmatrix},$$

O_n and I_n are the respective zero and unit matrices of dimension n,

$$\alpha(t,x)=\begin{pmatrix}0\\ -\alpha_0(t,x_1,x_2)\end{pmatrix},\qquad \alpha_0(y,\dot y,t)=o(\|y\|^2+\|\dot y\|^2)^{1/2},$$
$$\beta(t,x)=\begin{pmatrix}0\\ -M^{-1}(x_1+q_p)S^T(x_1+q_p)\end{pmatrix},\qquad \mu(t)=F(t)-F_p(t).$$

Within the general statement, the problem of choosing the program forces $F_p(t)$ is associated with studying the solutions of the differential equation

$$\frac{d\mu}{dt}=Q(\mu),$$

where $Q\in C(R^m,R^m)$, $\mu(t)=F(t)-F_p(t)$, $\mu(t_0)=0$ and $Q(0)=0$.

Thus the problem of stability of the robot motion interacting with a dynamic environment results in the need to analyze the solutions of the systems of equations

$$\frac{dx}{dt}=A(t)x+\alpha(t,x)+\beta(t,x)\mu(t),\quad x(t_0)=x_0, \tag{2.8}$$
$$\frac{d\mu}{dt}=Q(\mu),\quad \mu(t_0)=\mu_0 \tag{2.9}$$

under certain assumptions of functions specifying the action of dynamic environment on the robot.

Consider the independent equation (2.9) which specifies the influence of dynamic environment on the executive organ of the robot. From (2.9) it follows that

$$\mu(t)=\mu_0+\int\limits_{t_0}^{t}Q(\mu(s))\,ds,\quad t\ge t_0. \tag{2.10}$$

The term in the equation (2.8) which specifies the action of environment on the robot is designated by $u(t,x)=\beta(t,x)\mu(t)$ for $(t,x)\in R_+\times D$, $D=\{x\colon \|x\|<H\}$, H is sufficiently small, the function $u(t,x)$ satisfies the inequality

$$\|u(t,x)\|\le p(t), \tag{2.11}$$

where $p(t)$ is the function integrable over any finite time interval. With

$$\mu(t) = F(t) - F_p(t) \tag{2.12}$$

and (2.10) representing the deviation of program value of the force $F_p(t)$ from the force $F(t)$ acting due to the dynamic environment, the action of environment on the robot may be estimated by the function $p(t)$. We introduce the designations

$$p_0 = \sup_{t\geq 0} p(t), \quad p_1 = \sup_{t\geq 0} \int\limits_t^{t+1} p(s)\,ds, \quad p_2 = \sup_{t\geq 0} \left(\int\limits_t^{t+1} p^2(s)\,ds \right)^{1/2}.$$

Further the following definition will be used.

Definition 2.1 Let for any $\varepsilon > 0$ the values $\Delta > 0$ and $\delta > 0$ be those for which the inequality $\|x(t)\| < \varepsilon$ occurs for solving the equation (2.8) with $t \geq 0$ if $\|x(0)\| < \delta$ and one of the following conditions is satisfied

(1) $p_0 \leq \Delta$;

(2) $p_1 \leq \Delta$;

(3) $p_2 \leq \Delta$.

Here we consider that robot motion is:

(a) *stable with limited action* of environment on the robot (case 1);

(b) *stable with limited, on the average, action* of environment on the robot (case 2);

(c) *stable with limited, on the quadratic average, action* of environment on the robot (case 3).

It is of interest to consider the action of environment on the robot when the limiting relationship $\|u(t,x)\| \to 0$, $t \to \infty$ is uniformly satisfied over x with sufficiently low values $\|x\|$. This corresponds to the choice of τ control in (2.1) when $F(t) \to F_p(t)$, $t \to \infty$.

In the case when H in the estimate of the domain D is not small ($H < \infty$) and consequently, the large neighborhood of the equilibrium state of the robot-mechanical system is considered, the estimate

$$\|u(t,x)\| \leq \lambda(t)\|x\| \tag{2.13}$$

should be taken instead of (2.11), where $\lambda(t)$ is the integrable function such that

$$\int\limits_0^\infty \lambda(s)\,ds < +\infty. \tag{2.14}$$

Let us make the following assumptions on the equations (2.8) and (2.9):

I. The fundamental matrix $X(t)$ of solutions of the first approximations of the system (2.8) satisfies the inequality

$$\|X(t)X^{-1}(s)\| \leq Ne^{-\gamma(t-s)}, \tag{2.15}$$

where N and γ are positive constants independent of t_0. Note that the condition (2.15) guarantees the exponential stability of the zero solution of

$$\frac{dx}{dt} = A(t)\,x. \tag{2.16}$$

II. The vector function $\alpha(t,x)$ in (2.8) satisfies the following condition: for each $L > 0$ the values $D = D(L)$ and $T = T(L)$ exist, such that

$$\|\alpha(t,x)\| \le L\|x\| \tag{2.17}$$

with $\|x\| \le D$ and $t \ge T$.

III. The influence of the vector function of dynamic environment on robot satisfies the condition $\|u(t,x)\| \to 0$ with $t \to \infty$, uniformly over x with sufficiently small values $\|x\|$.

Theorem 2.1 *The equations* (2.8) *and* (2.9) *of the robot movement interacting with the environment are assumed to be those where the conditions I–III are satisfied. Then there exists $t_0 \in R_+$ such that any movement $x(t;t_0,x_0)$ of the robot simulated by the system* (2.8) *will approach to zero with $t \to \infty$ and sufficiently small values $\|x(t_0)\|$.*

Proof When the condition I is satisfied, the value L in (2.17) is defined by the formula $L = \gamma(2N)^{-1}$:

$$\|\alpha(t,x)\| \le \frac{\gamma}{2N}\,\|x\|, \quad t \ge T. \tag{2.18}$$

From the condition III it follows that $\sigma > 0$ exists such that

$$\|u(t,x)\| \le \sigma < \frac{\gamma - NL}{2N}\,\delta, \quad t \ge T. \tag{2.19}$$

For a certain $t_0 \in R_+$ with $t \ge t_0$ we have

$$x(t) = W(t,t_0)x_0 + \int_{t_0}^{t} W(t,\tau)\big[\alpha(\tau,x(\tau)) + u(\tau,x(\tau))\big]\,ds. \tag{2.20}$$

With the estimates (2.15), (2.17)–(2.19) we find from (2.20)

$$\begin{aligned} \|x(t)\| \le Ne^{-\gamma(t-t_0)}\|x_0\| + NL\int_{t_0}^{t} e^{-\gamma(t-s)}\|x(s)\|\,ds \\ + N\int_{t_0}^{t} e^{-\gamma(t-s)}\|u(s,x(s))\|\,ds. \end{aligned} \tag{2.21}$$

Let us designate $M(t) = \max\limits_{t_0 \le s \le t} \|x(s)\|$ and represent (2.21) as

$$M(t) \le N\|x_0\| + \frac{NL}{\gamma}\,M(t) + \frac{N\sigma}{\gamma} \le \frac{N\gamma}{\gamma - NL}\,\|x_0\| + \frac{N\sigma}{\alpha - NL}\,.$$

Since $2N\sigma(\beta - NL)^{-1} < \delta$, then $M(t) < \delta$ with all $t \geq t_0$ as soon as

$$\|x_0\| < \frac{\delta(\beta - NL)}{4N\beta} < \frac{\delta}{4N}\,.$$

Set $\Lambda = \lim\limits_{t\to\infty} \sup \|x(t)\|$. It is evident that $0 \leq \Lambda \leq \delta < +\infty$ and the sequence $\{t_j\}$, $j = 1, 2, \dots$ exists such that the limiting relationship $\|x(t_j)\| \to \Lambda$ is valid with $t_j \to +\infty$, $j \to +\infty$.

From the inequality (2.21) we obtain

$$\begin{aligned}\|x(t_j)\| \leq N\|x_0\|e^{-\gamma(t_j-t_0)} &+ NL \int\limits_{t_0}^{t_j/2} e^{-\gamma(t_j-s)}\|x(s)\|\,ds \\ &+ NL \int\limits_{t_j/2}^{t_j} e^{-\gamma(t_j-s)}\|x(s)\|\,ds + N \int\limits_{t_0}^{t_j/2} e^{-\gamma(t_j-s)}\|u(s,x(s))\|\,ds \\ &+ N \int\limits_{t_j/2}^{t_j} e^{-\gamma(t_j-s)}\|u(s,x(s))\|\,ds.\end{aligned}$$

For a given $\eta > 0$ there exists J_η such that $\|x(t_j)\| > \Lambda - \eta$ for all $j \geq J_\eta$ and $\|x(t)\| < \Lambda + \eta$ with $t \geq t_j/2$. Consequently, with $j \geq J_\eta$ we find

$$\begin{aligned}\Lambda - \eta \leq N\|x_0\|e^{-\gamma(t_j-t_0)} + \frac{NL\delta}{\gamma}\,e^{-\frac{1}{2}\beta t_j} + \frac{NL(\Lambda+\eta)}{\gamma} + \frac{N\sigma}{\gamma}\,e^{-\frac{1}{2}\beta t_j} \\ + \frac{NL}{\gamma} \max_{\frac{1}{2}t_j \leq s \leq t_j} \|u(t, x(s))\|.\end{aligned}$$

Thus $\Lambda - \eta \leq \dfrac{NL(\Lambda+\eta)}{\gamma}$ is obtained as $j \to +\infty$. Since $NL\beta^{-1} < 1/2$, we have $\Lambda < 3\eta$. It follows from arbitrariness of η that $\Lambda = 0$. With the definition of Λ, we may conclude that the motion $x(t)$ at vanishing interactions of robot with the environment tends to the equilibrium state corresponding to the zero solution of (2.8).

Further we study the motion of the robot interacting with dynamic environment under the conditions (2.13) and (2.14). For providing sufficient stability conditions the following Lemma is needed.

Lemma 2.1 *Let γ be the positive constant and the function $\lambda(t) \in C(R_+, R_+)$ be such that*

$$\int\limits_0^\infty \lambda(s)\,ds < +\infty \quad \text{or} \quad \lim_{t\to+\infty} \lambda(t) = 0.$$

Then

$$\lim_{t\to+\infty} e^{-\gamma t} \int\limits_0^t e^{\gamma s}\lambda(s)\,ds = 0.$$

Proof Let us first prove the case when $\lambda(t)$ is integrable. For the given $\varepsilon > 0$ we choose t to be large enough that

$$\int_{t/2}^{\infty} \lambda(s)\,ds < \frac{\varepsilon}{2}\,, \qquad e^{-\gamma t}\int_{0}^{t/2} \lambda(s)\,ds < \frac{\varepsilon}{2}\,.$$

Then

$$e^{-\gamma t}\int_{0}^{t/2} e^{\gamma s}\lambda(s)\,ds \le e^{-\gamma \frac{t}{2}}\int_{0}^{t/2} \lambda(s)\,ds < \frac{\varepsilon}{2}\,,$$

$$e^{-\gamma t}\int_{t/2}^{0} e^{\gamma s}\lambda(s)\,ds \le \int_{t/2}^{t} \lambda(s)\,ds \le \int_{t/2}^{\infty} \lambda(s)\,ds < \frac{\varepsilon}{2}\,.$$

Consequently,

$$e^{-\gamma t}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds < \varepsilon$$

with a sufficiently large t. Therefore

$$\lim_{t\to\infty} e^{-\gamma t}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds = 0.$$

Consider the case $\lambda(t) \to 0$ with $t \to +\infty$. If $\int_{0}^{\infty} e^{\gamma s}\lambda(s)\,ds < +\infty$ the statement of Lemma 2.1 is evident. On the other hand, using the L'Hospital rule we obtain

$$\lim_{t\to+\infty} \frac{1}{e^{\gamma t}}\int_{0}^{t} e^{\gamma s}\lambda(s)\,ds = \lim_{t\to\infty}\frac{\lambda(t)}{\gamma} = 0.$$

Lemma 2.2 *Let the function $u(t)$ be continuous and non-negative and satisfy the inequality*

$$u(t) \le c + \int_{0}^{t} \bigl[ku(s) + \lambda(s)\bigr]\,ds, \quad t \ge 0,$$

where c and k are non-negative constants. Then

$$u(t) \le ckt + \int_{0}^{t} e^{k(t-s)}\lambda(s)\,ds, \quad t \ge 0.$$

Proof of this lemma follows by the standard method developed in the theory of integral inequalities.

Theorem 2.2 *The equations* (2.8), (2.9) *of robot movement interacting with the environment are supposed to be such that*

(1) *the condition I is satisfied;*

(2) *for any* $\varepsilon > 0$ *there exists* $L = L(\varepsilon) > 0$ *such that* $\|\alpha(t,x)\| \le L(\varepsilon)\|x\|$ *with* $\|x\| \le \delta$, $t \ge 0$*;*

(3) *the vector function* $u(t,x) = \beta(t,x)\mu(t)$ *satisfies the estimate*

$$\|u(t,x)\| \le \sigma < \frac{\gamma - NL}{2N}\,\delta, \quad t \ge 0.$$

Then any robot motion beginning in the domain $\{x \in R^{2n}\colon \|x(0)\| < \delta/(2N)\}$ *will remain in the domain* $\{x \in R^{2n}\colon \|x\| < \delta/2\}$ *for all* $t \ge 0$*.*

Proof From the inequality (2.21) we obtain

$$u(t) \le N\|x_0\| + \int\limits_0^t \left[NL(\varepsilon)u(s) + Ne^{\gamma s}\|u(s,x(s))\|\right] ds, \tag{2.22}$$

where $u(t) = e^{\gamma t}\|x(t)\|$. Applying Lemma 2.2 to (2.22) we find

$$u(t) \le e^{NLt}\left[N\|x_0\| + \int\limits_0^t Ne^{\gamma s}\|u(s,x(s))\|e^{-NLs}\,ds\right]$$

and consequently,

$$\|x(t)\| \le N\|x_0\|e^{-(\gamma-NL)t} + Ne^{-(\gamma-NL)t}\int\limits_0^t \|u(s,x(s))\|e^{(\gamma-NL)s}\,ds. \tag{2.23}$$

From the inequality $\|x_0\| \le \delta/(2N)$ it follows that the first summand in (2.23) will be smaller than $\delta/2$ for all $t \ge 0$. From condition 3 of Theorem 2.2 it follows that

$$\frac{\delta(\gamma - NL)}{2}\,e^{-(\gamma-NL)t}\int\limits_0^t e^{(\gamma-NL)s}\,ds \le \frac{\delta}{2}\,. \tag{2.24}$$

Consequently, from (2.24) we obtain

$$\begin{aligned}\|x(t)\| &\le N\|x_0\|e^{-(\gamma-NL)t} + \frac{\delta}{2}\left(1 - e^{-(\gamma-NL)t}\right)\\ &\le \frac{\delta}{2}\,e^{-(\gamma-NL)t} + \frac{\delta}{2}\left(1 - e^{-(\gamma-NL)t}\right) = \frac{\delta}{2}\end{aligned}$$

for all $t \ge 0$.

The proof is complete.

Remark 2.1 From Theorem 2.2 it follows that if $\|u(t,x)\| \to 0$ or $\int\limits_0^\infty \|u(s,x(s))\|\,ds < \infty$, the robot motion tends to the equilibrium state as $t \to +\infty$.

Case A. Consider the interactions of the robot with dynamic environment when functions $\beta(t,x)\mu(t)$ satisfy the estimate

$$\|\beta(t,x)\mu(t)\| \le \lambda(t) \tag{2.25}$$

for $\|x\| \le r$, $r > 0$, $t \ge 0$ and

$$G(t) = \int_t^{t+1} \lambda(s)\,ds \to 0 \tag{2.26}$$

as $t \to \infty$.

It is evident that the condition (2.26) will be satisfied if $\lambda(t) \to 0$ with $t \to +\infty$ or $\int_0^\infty \lambda(s)\,ds < +\infty$. It is shown (see [26]) that the function $\lambda(t)$ may be determined as follows:

$$\lambda(t) = \begin{cases} 1 & \text{with} \quad t = 3n, \\ 0 & \text{with} \quad 3n + \frac{1}{n} \le t \le 3(n+1) - \frac{1}{n+1}, \\ 0 & \text{with} \quad 0 \le t \le 2. \end{cases}$$

The robot motion under the conditions (2.25) and (2.26) is described by the following statement.

Theorem 2.3 *Let us assume that the equations* (2.8) , (2.9) *of the perturbed motion of the robot interacting with environment are such that*

(1) *for the equations of the first approximation* (2.16) *the condition I is satisfied;*

(2) *for the vector function* $\alpha(t,x)$ *nonlinearities with any* $L > 0$, $\delta = \delta(L) > 0$ *and* $\tau = \tau(L) > 0$ *exist such that* $\|\alpha(t,x)\| \le L\|x\|$ *with* $\|x\| \le \delta$ *and* $t \ge \tau$*;*

(3) *for the arbitrary solution* $\mu(t)$ *of the relationships* (2.10) *and* (2.12) *which determine the quality of unsteady response to the robot interaction with environment, the conditions* (2.25) *and* (2.26) *are satisfied.*

Then the time $\tau^* \ge 0$ *and the domain* $S_\delta = \{x \in R^{2n} : \|x\| < \delta,\ \delta > 0\}$ *will be found, such that the robot motion starting in the domain* S_δ *at any time moment* $t_0 \ge \tau^*$*, will approach the equilibrium state, i.e.* $\|x(t)\| \to 0$ *at* $t \to +\infty$.

Proof When the condition (1) is satisfied the Cauchy matrix $W(t,s)$ of the linear approximation (2.16) of the system (2.8) satisfies the condition $\|W(t,t_0)\| \le Ne^{-\gamma(t-t_0)}$ at all $t \ge t_0$. Let $0 < L < \min\{(\gamma/N), r\}$. By the condition (2), $\tau(L)$ and $\delta(L)$ can be chosen such that $\tau(L) \ge 1$ and $\delta(L) \le L$. Besides, let $\tau^* \ge \tau(L)$ be such that with $t \ge \tau^*$ the estimate

$$\int_1^t e^{-(\gamma-NL)(t-s)}\lambda(s)\,ds < \frac{\delta(L)}{2N} = \delta_1 \tag{2.27}$$

is valid.

It is easy to show that for all $t \ge t_0 \ge 1$ the inequality

$$\int_{t_0}^t \lambda(s)\,ds \le \int_{t_0-1}^t G(s)\,ds$$

is satisfied. Thus for any $k > 0$ the estimate

$$\int\limits_{t_0}^{t} e^{ks}\lambda(s)\,ds \le \int\limits_{t_0-1}^{t} e^{k(s+1)}\left[\int\limits_{s}^{s+1}\lambda(u)\,du\right]ds = \int\limits_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds \tag{2.28}$$

is valid

With (2.28) we obtain

$$e^{-kt}\int\limits_{t_0}^{t} e^{ks}\lambda(s)\,ds \le e^{-kt}\int\limits_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds. \tag{2.29}$$

Applying the L'Hospital rule to the expression in the right side of the inequality (2.29) we can find

$$\lim_{t\to\infty} e^{-kt}\int\limits_{t_0-1}^{t} e^{k(s+1)}G(s)\,ds = 0$$

whence it follows that the inequality (2.27) is justified. Then let $t_0 \ge \tau^*$ and $\|x(t_0)\| < \delta_1 = \dfrac{\delta(L)}{2N}$. From the equality (2.20) we obtain

$$\|x(t)\| \le N\delta_1 e^{-\gamma(t-t_0)} + N\int\limits_{t_0}^{t} e^{-\gamma(t-s)}\big[L\|x(s)\| + \lambda(s)\big]\,ds,$$

thus

$$\|x(t)\|e^{\gamma t} \le N\delta_1 e^{\gamma t_0} + \int\limits_{t_0}^{t}\big[NL\|x(s)\|e^{\gamma s} + Ne^{\gamma s}\lambda(s)\big]\,ds. \tag{2.30}$$

Let us designate $\|x(t)\|e^{\gamma t} = w(t)$ and use Lemma 2.2 for the inequality (2.30). It is easy to see that

$$w(t) \le N\delta_1 e^{\gamma t_0} e^{NL(t-t_0)} + \int\limits_{t_0}^{t} e^{NL(t-s)}Ne^{\gamma s}\lambda(s)\,ds,$$

or

$$\|x(t)\| \le N\delta_1 e^{-(\gamma-NL)(t-t_0)} + N\int\limits_{t_0}^{t} e^{-(\gamma-NL)(t-s)}\lambda(s)\,ds.$$

Then by the condition (2.27) we find

$$\|x(t)\| \le N\delta_1 + N\int\limits_{t_0}^{t} e^{-(\gamma-NL)(t-s)}\lambda(s)\,ds < \frac{\delta}{2} + N\delta_1 = \delta.$$

Thus, $\|x(t)\| < \delta$ for all $t \ge t_0$ and the limiting relationship $\|x(t)\| \to 0$ is satisfied for $t \to +\infty$. The statement is proved.

Case B. Three conditions for the vector function $\beta(t,x)\mu(t)$, $\mu(t) = F(t) - F_p(t)$ will be taken into consideration which specify the robot interacting with the dynamic environment. The following estimate of the function of transient process in (2.8) is needed.

Lemma 2.3 *Let the conditions be satisfied for the equations of perturbed motion* (2.8)*:*

(1) *the Cauchy matrix* $W(t,t_0)$ *of the equations of the first approximation* (2.16) *satisfies the condition I;*

(2) *for the vector function of nonlinearities* $\alpha(t,x)$ *with each* $L > 0$*, a certain value* $H = H(L) > 0$ *exists such that* $\|\alpha(t,x)\| \le L\|x\|$ *for* $\|x\| \le H$ *and* $t \ge 0$*;*

(3) *for any function* $\mu(t)$*, satisfying the relationships* (2.10) *and* (2.12) *the estimation holds for all* $\|x\| < H$ *and* $t \ge 0$*.*

Then for sufficiently small initial perturbations $x_0 = x(0)$ *and* $\mu(0) = F(0) - F_p(0)$ *the transient process in* (2.8) *satisfies the estimate*

$$\|x(t)\| \le N(\Phi_1(t) + \Phi_2(t)), \tag{2.31}$$

where

$$\Phi_1(t) = e^{-\varkappa t}\|x_0\|, \quad x_0 = x(0),$$
$$\Phi_2(t) = e^{-\varkappa t}\int\limits_0^t e^{\varkappa s} p(s)\,ds, \quad \varkappa = \beta - NL.$$

The estimate (2.31) follows from Lemma 6.1 of Barbashin [1], p. 185, where the function $\Phi_2(t)$ is shown to satisfy one of the following inequalities for all $t \ge 0$

$$\Phi_2(t) \le p_0\varkappa^{-1}, \quad \Phi_2(t) \le p_1 e^{\varkappa}(1-e^{-\varkappa})^{-1}, \quad \Phi_2(t) \le p_2(1-e^{-\varkappa})^{-1}\left(\frac{e^{2\varkappa}-1}{2\varkappa}\right)^{1/2}.$$

The sufficient conditions which provide the stability of motion of the robot interacting with the environment are given in the following statement.

Theorem 2.4 *The equations of perturbed motion of the robot interacting with the environment are supposed to be such that*

(1) *for the equations of the first approximation* (2.16) *the condition I is satisfied;*

(2) *for the vector function of nonlinearities* $\alpha(t,x)$ *with any* $L > 0$, $\delta = \delta(L) > 0$ *exists such that* $\|\alpha(t,x)\| \le L\|x\|$ *with* $\|x\| \le H$ *and* $t \ge 0$*;*

(3) *for any function* $\mu(t)$ *which satisfies the relationships* (2.10) *and* (2.12) *for all* $\|x\| \le H$ *and* $t \ge 0$ *the estimate* (2.11) *and one of the inequalities are satisfied*

$$p_0 < \frac{\Delta}{2N}\,\varkappa, \tag{2.32}$$

$$p_1 < \frac{\Delta}{2N}\,e^{-\varkappa}\big(1-e^{-\varkappa}\big), \tag{2.33}$$

$$p_2 < \frac{\Delta}{2N}\left(\frac{2\varkappa}{e^{2\varkappa}-1}\right)^{1/2}\big(1-e^{-\varkappa}\big). \tag{2.34}$$

Then under any initial condition

$$x_0 = x(0), \quad \mu(0) = F(0) - F_p(0) \tag{2.35}$$

for which $\|x_0\| < \Delta(2N)^{-1}$*, the transient process of the system* (2.8) *satisfies the estimate*

$$\|x(t)\| \le N(\Phi_1(t) + \Phi_2(t)) \tag{2.36}$$

for all $t \ge 0$ *and* $\|x(t)\| < \Delta$.

The ***Proof*** of Theorem 2.4 is based on the estimate of the transient process (2.31). Under the initial conditions (2.35) when $\|x_0\| < \Delta(2N)^{-1}$, the estimate $\Phi_1(t) < \Delta(2N)^{-1}$ is valid for the function $\Phi_1(t)$ for all $t \ge 0$. When the conditions (2.32)–(2.34) are satisfied the estimate $\Phi_2(t) < \Delta(2N)^{-1}$ is valid for the function $\Phi_2(t)$. Therefore it follows from (2.36) that $\|x(t)\| < \Delta$ for all $t \ge 0$. The proof of Theorem is complete.

Next we will show that the motion of the robot interacting with the environment medium may be dissipative under appropriate limitation on the initial state x_0 and the function $\mu(t) = F(t) - F_p(t)$.

Theorem 2.5 *Let us suppose that for the equation* (2.8) *of perturbed motion of robot interacting with environment*

(1) *the conditions (1)–(2) of the Theorem 2.4 hold;*

(2) *the inequalities*

$$p_0 < \rho \frac{\Delta}{N} \varkappa, \tag{2.37}$$

$$p_1 < \rho \frac{\Delta}{N} e^{-\varkappa}(1 - e^{-\varkappa}), \tag{2.38}$$

$$p_2 < \rho \frac{\Delta}{N} \left(\frac{2\varkappa}{e^{2\varkappa} - 1}\right)^{1/2} (1 - e^{-\varkappa}), \tag{2.39}$$

are satisfied in Δ*-neighborhood of the state* $x = 0$*, i.e. with all* $(x \ne 0) \in \{x : \|x\| < \Delta\}$ *where* $0 < \rho < 1,\ 0 < \delta < \Delta(2N)^{-1}$.

Then the positive number $\tau \in R_+$ *exists such that for* $t > \tau$ *and* $\|x_0\| < \delta$ *the transient process in* (2.8) *satisfies the estimate* $\|x(t)\| < \delta$ *for all* $t \ge \tau$.

Proof Consider the estimate (2.36). Then choose $\tau > \frac{1}{\varkappa} \ln N(1 - p)^{-1}$ and the estimate for the functions $\Phi_1(t)$ and $\Phi_2(t)$ can be obtained. For $t > \tau$ we have $\Phi_1(t) = e^{-\varkappa t}\|x_0\| < (1 - p)\delta N^{-1}$ for all $t \ge \tau$. When at least one of the conditions (2.37)–(2.39) is satisfied, $\Phi_2(t) < \rho\delta N^{-1}$ is obtained for all $t \ge \tau$. It follows from the estimate (2.36) that the transient process in the system will be damping, i.e. $\|x(t)\| < \delta$ for all $t > \tau$.

Further the equations of the perturbed motion (2.8) will be considered under the following assumptions:

I$'$. The matrix $A(t)$, the vector function of nonlinearity $\alpha(t, x)$ and the vector function $\beta(t, x)\mu(t)$ where $\mu(t) = F(t) - F_p(t)$ are continuous and periodic with respect to t. The period of these functions are supposed to be common, for example, unity.

II′. As above, the assumption I is preserved for the case of periodic matrix $A(t)$, i.e.

$$\|W(t,s)\| \le Ne^{-\gamma(t-s)}, \tag{2.40}$$

where $W(t,s) = X(t)X^{-1}(s)$.

III′. The vector function of nonlinearities $\alpha(t,x)$ in the domain $\|x\| < H$, $t \ge 0$ satisfies the Lipschitz condition

$$\|\alpha(t,x) - \alpha(t,y)\| \le K\|x - y\|. \tag{2.41}$$

IV′. The constants N, γ, K in the inequalities (2.40), (2.41) satisfy the inequality $\varkappa^* = \gamma - NK > 0$.

Theorem 2.6 *Suppose that for the equations of perturbed motion* (2.8) *for the robot interacting with the environment, the conditions I′–IV′ are satisfied. Besides, one of the conditions* (2.37)–(2.39) *is satisfied. Then in the domain* $\|x\| < H(2N)^{-1}$ *the periodic robot motion* $z(t)$ *is possible, and for any other motion* $x(t)$ *of the robot, which is started in the domain* $\|x(0)\| \le H(2N)^{-1}$*, the limiting relationship* $\|x(t) - z(t)\| \to 0$ *with* $t \to +\infty$ *is valid, i.e. the periodic robot motion is asymptotically stable.*

The ***Proof*** of this theorem is based on the principle of contracting mappings and Theorem 6.4 in Barbashin [1].

3 Stability Analysis of Neural Networks on Time Scales

In this section we consider stability of a *neural network on time scale* [6] the dynamics of which is described by equation of the type

$$x^{\Delta}(t) = -Bx(t) + TS(x(t)) + J, \quad t \in [0, +\infty), \tag{3.1}$$

whose solution $x(t; t_0, x_0)$ for $t = t_0$ takes the value x_0, i.e.

$$x(t_0; t_0, x_0) = x_0, \quad t_0 \in [0, +\infty), \quad x_0 \in \mathbb{R}^n, \tag{3.2}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is an arbitrary time scale, $0 \in \mathbb{T}$, $\sup \mathbb{T} = +\infty$. In system (3.1) $x^{\Delta}(t)$ is a Δ-derivative on time scale $\mathbb{T}$, $x \in \mathbb{R}^n$ characterizes the state of neurons, $T = \{t_{ij}\} \in \mathbb{R}^{n\times n}$, the components t_{ij} describe the interaction between the i-th and j-th neurons, $S\colon \mathbb{R}^n \to \mathbb{R}^n$, $S(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, the function s_i describes response of the i-th neuron, $B \in \mathbb{R}^{n\times n}$, $B = \operatorname{diag}\{b_i\}$, $b_i > 0$, $i = 1, 2, \dots, n$, $J \in \mathbb{R}^n$ is a constant external input vector.

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = d/dt$ and the initial problem (3.1)–(3.2) is equivalent to the initial problem for a continuous *Hopfild type neural network*

$$\frac{dx(t)}{dt} = -Bx(t) + TS(x(t)) + J, \quad t \ge 0, \tag{3.3}$$

$$x(t_0; t_0, x_0) = x_0, \quad t_0 \ge 0, \quad x_0 \in \mathbb{R}^n. \tag{3.4}$$

If $\mathbb{T} = \mathbb{N}_0$, then $x^{\Delta}(k) = x(k+1) - x(k) = \Delta x(k)$ and the initial problem (3.1)–(3.2) is equivalent to

$$\Delta x(k) = -Bx(k) + TS(x(k)) + J, \quad t \in \mathbb{N}_0, \tag{3.5}$$

$$x(k_0; k_0, x_0) = x_0, \quad k_0 \in \mathbb{N}_0, \quad x_0 \in \mathbb{R}^n. \tag{3.6}$$

We assume relative to system (3.1) the following.

S$_1$. The vector-function $f(x) = -Bx + TS(x) + J$ is regressive.

S$_2$. There exist positive constants $M_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(u)| \le M_i$ for all $u \in \mathbb{R}$.

S$_3$. There exist positive constants $L_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(u) - s_i(v)| \le L_i|u - v|$ for all $u, v \in \mathbb{R}$.

S$_4$. Granularity function of the time scale $\mathbb{T}$ $0 < \mu(t) \in \mathcal{M}$ for all $t \in [0, +\infty)$, where $\mathcal{M} \subset \mathbb{R}$ is a compact set.

We recall that the matrix $A \in \mathbb{R}^{n\times n}$ is called *M-matrix* if its all non-diagonal elements are non-positive and all principle minors are positive.

We denote by $r = \Big(\sum_{i=1}^n \big(\sum_{j=1}^n M_j|T_{ij}| + |J_i|\big)^2/b_i^2\Big)^{1/2}$ and $\Lambda = \operatorname{diag}\{L_i\} \in \mathbb{R}^{n\times n}$ and prove the following assertion.

Theorem 3.1 *If for system* (3.1) *conditions* S$_1$–S$_4$ *are satisfied then there exists an equilibrium state* $x(t) = x^*$ *of system* (3.1) *and moreover,* $\|x^*\| \le r$. *Besides, if the matrix* $B\Lambda^{-1} - |T|$ *is an M-matrix, this equilibrium state is unique.*

Proof For the state $x(t) = x^*$ to be the equilibrium state of system (3.1) it is necessary and sufficient that

$$-Bx^* + TS(x^*) + J = 0$$

or

$$x^* = B^{-1}(TS(x^*) + J).$$

Consider the mapping $h\colon \mathbb{R}^n \to \mathbb{R}^n$, $h(x) = (h_1(x), h_2(x), \dots, h_n(x))^{\mathrm{T}}$,

$$h_i(x) = \frac{1}{b_i}\biggl(\sum_{j=1}^n T_{ij}s_j(x_j) + J_i\biggr), \quad i = 1, 2, \dots, n.$$

Since

$$\|h(x)\| \le \biggl(\sum_{i=1}^n \frac{1}{b_i^2}\biggl(\sum_{j=1}^n M_j|T_{ij}| + |J_i|\biggr)^2\biggr)^{1/2} = r,$$

the continuous mapping h carries the convex compact set $B_r = \{x \in \mathbb{R}^n : \|x\| \le r\}$ onto itself. The Schauder principle implies that the mapping h possesses a fixed point x^* which is the equilibrium state of system (3.1).

Besides, if the matrix $B\Lambda^{-1} - |T|$ is an M-matrix, the mapping $H\colon \mathbb{R}^n \to \mathbb{R}^n$

$$H(x) = -Bx + TS(x) + J$$

is a homeomorphism (see [28]). This implies uniqueness of the equilibrium state of system (3.1). The theorem is proved.

Let x^* be the equilibrium state of system (3.1). We perform the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (3.1)–(3.2) as

$$y^{\triangle}(t) = -By(t) + TG(y(t)), \quad t \in [0, +\infty), \tag{3.7}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in [0, +\infty), \quad y_0 \in \mathbb{R}^n, \tag{3.8}$$

where $y \in \mathbb{R}^n$, $G\colon \mathbb{R}^n \to \mathbb{R}^n$, $G(y) = (g_1(y_1), g_2(y_2), \dots, g_n(y_n))^{\mathrm{T}}$, $G(y(t)) = S(y(t) + x^*) - S(x^*)$.

If for system (3.1) assumptions S_1–S_4 are valid, then for system (3.7) the following assertions hold true.

G_1. The vector-function $g(y) = -By + TG(y)$ is regressive.

G_2. For all $u \in \mathbb{R}$ $|g_i(u)| \le 2M_i$, $i = 1, 2, \dots, n$.

G_3. For all $u, v \in \mathbb{R}$ $|g_i(u) - g_i(v)| \le L_i|u - v|$, $i = 1, 2, \dots, n$.

G_4. $G(0) = 0$.

Note that under conditions G_1–G_4 there exists a unique solution of problem (3.7)–(3.8).

Designate by $\underline{b} = \min\{b_i\}$, $\overline{b} = \max\{b_i\}$, $L = \max\{L_i\}$.

Theorem 3.2 *For system* (3.1) *assume that assumptions* S_1–S_4 *are valid on time scale* $\mathbb{T}$ *and there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in [0, +\infty)$. *If the inequality*

$$2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2 \ge 0,$$

is satisfied, the equilibrium state $x(t) = x^*$ *of system* (3.1) *is uniformly asymptotically stable.*

Proof It is clear that the behavior of solution $x(t)$ of system (3.1) in the neighborhood of the equilibrium state x^* is equivalent to the behavior of solution $y(t)$ of system (3.7) in the neighborhood of zero. For the proof we shall apply the Lyapunov function $V(y) = y^{\mathrm{T}}y$. If $y(t)$ is Δ-differentiable at the point $t \in \mathbb{T}^k$, for the derivative of function $V(y(t))$ we have the expression

$$\begin{aligned} V^{\Delta}(y(t)) &= (y^{\mathrm{T}}(t)\, y(t))^{\Delta} = y^{\mathrm{T}}(t)\, y^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}\, y(\sigma(t)) \\ &= y^{\mathrm{T}}(t)\, y^{\Delta}(t) + [y^{\Delta}(t)]^{\mathrm{T}}[y(t) + \mu(t) y^{\Delta}(t)]. \end{aligned}$$

The derivative of V along solutions of system (3.7) is given by

$$\begin{aligned} V^{\Delta}(y(t))|_{(3.7)} &= 2y^{\mathrm{T}}(t)\, y^{\Delta}(t) + \mu(t)[y^{\Delta}(t)]^{\mathrm{T}} y^{\Delta}(t) \\ &= 2y^{\mathrm{T}}(t)[-By(t) + TG(y(t))] + \mu(t)\| - By(t) + TG(y(t))\|^2 \\ &\le -2\lambda_m(B)\|y(t)\|^2 + 2\|y(t)\|\|T\|\|G(y(t))\| + \mu^*(\|B\|\|y(t)\| + \|T\|\|G(y(t))\|)^2 \\ &= -2\underline{b}\,\|y(t)\|^2 + 2\|T\|\|G(y(t))\|\|y(t)\| + \mu^*(\overline{b}\,\|y(t)\| + \|T\|\|G(y(t))\|)^2. \end{aligned}$$

We shall estimate separately the term $\|G(y(t))\|$:

$$\|G(y(t))\| \le \left(\sum_{i=1}^{n} L_i^2 y_i^2(t)\right)^{1/2} \le \max_i\{L_i\}\left(\sum_{i=1}^{n} y_i^2(t)\right)^{1/2} = L\,\|y(t)\|.$$

As a result we have

$$\begin{aligned} V^{\Delta}(y(t))|_{(3.7)} &\le -2\,\underline{b}\,\|y(t)\|^2 + 2L\|T\|\|y(t)\|^2 + \mu^*\left(\overline{b}\,\|y(t)\| + L\|T\|\|y(t)\|\right)^2 \\ &= -\left(2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2\right)\|y(t)\|^2. \end{aligned}$$

Therefore, the equilibrium state $y(t) = 0$ of system (3.7) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Lemma 3.1 *Assume that $g_i \in C^2(\mathbb{R})$, $g_i(0) = 0$, $i = 1, 2, \dots, n$, and constants $K_i > 0$, $i = 1, 2, \dots, n$, exist so that $|g_i''(u)| \le K_i$ for all $u \in \mathbb{R}$. Then the vector-function $G(y)$ can be represented as*

$$G(y) = Hy + G_2(y),$$

where $H = \operatorname{diag}\{g_i'(0)\} \in \mathbb{R}^{n\times n}$, $G_2\colon \mathbb{R}^n \to \mathbb{R}^n$ and the estimate

$$\|G_2(y)\| \le K\|y\|^2, \tag{3.9}$$

holds true, where $K = \max_i\{K_i\}/2$.

Proof We decompose functions $g_i(y_i)$ by the Maclaurin formula

$$g_i(y_i) = g_i'(0)y_i + 1/2\, g_i''(\theta_i y_i)y_i^2, \quad \theta_i \in (0,1).$$

Then

$$G(y) = \begin{pmatrix} g_1'(0)y_1 + 1/2\, g_1''(\theta_1 y_1)y_1^2 \\ g_2'(0)y_2 + 1/2\, g_2''(\theta_2 y_2)y_2^2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ g_n'(0)y_n + 1/2\, g_n''(\theta_n y_n)y_n^2 \end{pmatrix} = Hy + G_2(y),$$

where $G_2(y) = \frac{1}{2}\operatorname{diag}\{g_i''(\theta_i y_i)\}z$, $z = (y_1^2, y_2^2, \dots, y_n^2)^{\mathrm{T}}$.

$$\begin{aligned}\|G_2(y)\| &= \frac{1}{2}\left(\sum_{i=1}^n (g_i''(\theta_i y_i))^2 y_i^4\right)^{1/2} \le K\left(\sum_{i=1}^n y_i^4\right)^{1/2} \\ &\le K\left(\sum_{i=1}^n y_i^4 + \sum_{k\ne j} y_k^2 y_j^2\right)^{1/2} = K\sum_{i=1}^n y_i^2 = K\|y\|^2.\end{aligned}$$

Theorem 3.3 *Let the following conditions be satisfied*

(1) *for system (3.1) on time scale $\mathbb{T}$ assumptions S_1–S_4 are valid;*

(2) *functions $s_i \in C^2(\mathbb{R})$ and there exist constants $K_i > 0$ such that $|s_i''(u)| \le K_i$ for all $u \in \mathbb{R}$, $i = 1, 2, \dots, n$;*

(3) *there exists a constant $\mu^* \in \mathcal{M}$ such that $\mu(t) \le \mu^*$ for all $t \in [0, +\infty)$;*

(4) *there exists a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that the inequality $\lambda_M(PB_1 + B_1^T P) + \mu^*\|P\|\|B_1\|^2 < 0$ holds true, where $B_1 = -B + TH$, $H = \operatorname{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n}$.*

Then the equilibrium state $x(t) = x^*$ of system (3.1) is uniformly asymptotically stable.

Proof We use the function $V(y) = y^{\mathrm{T}}Py$. For the derivative of function V along solutions of system (3.7) we have

$$\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} &= y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(\sigma(t)) \\
&= y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(t) + \mu(t)y^{\Delta}(t)^{\mathrm{T}}Py^{\Delta}(t) \\
&= y^{\mathrm{T}}(t)P\big[B_1y(t) + TG_2(y(t))\big] + \big[B_1y(t) + TG_2(y(t))\big]^{\mathrm{T}}Py(t) \\
&\quad + \mu(t)\big[B_1y(t) + TG_2(y(t))\big]^{\mathrm{T}}P\big[B_1y(t) + TG_2(y(t))\big] \\
&\le y^{\mathrm{T}}(t)\big[PB_1 + B_1^{\mathrm{T}}P\big]y(t) + 2y^{\mathrm{T}}(t)PTG_2(y(t)) + \mu(t)\|P\|\|B_1y(t) + TG_2(y(t))\|^2 \\
&\le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2 + 2\|P\|\|T\|\|G_2(y(t))\|\|y(t)\| \\
&\quad + \mu(t)\|P\|\|G_2(y(t))\|^2\|T\|^2 + 2\mu(t)\|P\|\|B_1\|\|T\|\|G_2(y(t))\|\|y(t)\|.
\end{aligned}$$

Using inequality (3.9) and condition (3) of Theorem 3.3 we get

$$\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} &\le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|P\|\|B_1\|^2\big)\|y(t)\|^2 \\
&\quad + 2K\|P\|\|T\|\|y(t)\|^3 + 2\mu^*K\|P\|\|B_1\|\|T\|\|y(t)\|^3 + \mu^*K^2\|P\|\|T\|^2\|y(t)\|^4.
\end{aligned}$$

Designate

$$\begin{aligned}
a &= -\big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|B_1\|\|P\|^2\big) > 0, \\
\psi(\|y\|) &= a\|y\|^2, \\
m(\psi) &= 2a^{-\frac{1}{3}}K\|P\|\|T\|\,(1 + \mu^*\|B_1\|)\,\psi^{\frac{1}{3}} + \mu^*a^{-2}K^2\|P\|\|T\|^2\psi.
\end{aligned}$$

For the derivative of function V along solutions of system (3.7) we obtain the inequality

$$V^{\Delta}(y(t))|_{(3.7)} \le -\psi(\|y\|) + m(\psi(\|y\|)).$$

Since the function $\psi \in K$-class, $\lim\limits_{\psi\to 0} m(\psi) = 0$ and therefore, the equilibrium state $y(t) = 0$ of system (3.7) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x(t) = x^*$ of system (3.1).

We define the function

$$\beta_k(t) = \begin{cases} \mu^{-1}(t)\log|1 + \mu(t)k(t)|, & \text{if} \quad \mu(t) > 0, \\ k(t), & \text{if} \quad \mu(t) = 0, \end{cases}$$

where $k \in \mathcal{R},\ t \in [0, +\infty)$.

Theorem 3.4 *Let the following conditions be satisfied*

(1) *for system* (3.1) *assumptions* $\mathrm{S}_1 - \mathrm{S}_3$ *hold true.*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(u)| \le K_i$ *for all* $u \in \mathbb{R},\ i = 1, 2, \dots, n.$

(3) *there exist a positive definite symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *and a constant* $M > 0$ *such that* $|1 + \mu(t)A(t)| \ge M$ *for all* $t \in [0, +\infty)$*, where* $B_1 = -B + TH,\ H = \operatorname{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n},\ A(t) = \lambda_M(PB_1 + B_1^TP) + \mu(t)\|P\|\|B_1\|^2.$

Then, if

(a) $\lim_{t\to+\infty}\sup\beta_A(t) = q < 0$, *the equilibrium state* $x(t) = x^*$ *of system* (3.1) *is exponentially stable;*

(b) $\sup\{\beta_A(t)\colon\ t\in[0,+\infty)\} = \overline{q} < 0$, *the equilibrium state* $x(t) = x^*$ *of system* (3.1) *is uniformly exponentially stable.*

Proof We shall apply function $V(y) = y^{\mathrm{T}}Py$ and for the derivative of function V along solutions of system (3.7) we shall use the expression obtained in the previous theorem

$$
\begin{aligned}
V^{\Delta}(y(t))|_{(3.7)} &\le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2\\
&\quad+2\|P\|\|T\|\|G_2(y(t))\|\|y(t)\|\\
&\quad+2\mu(t)\|P\|\|B_1\|\|T\|\|G_2(y(t))\|\|y(t)\| \quad +\mu(t)\|P\|\|G(y(t))\|^2\|T\|^2\\
&\le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2\big)\|y(t)\|^2+\big(2K\|P\|\|T\|\|y(t)\|\\
&\quad+2\mu(t)K\|P\|\|B_1\|\|T\|\|y(t)\|+\mu(t)K^2\|P\|\|T\|^2|y(t)\|^2\big)\|y(t)\|^2\\
&= A(t)\|y(t)\|^2+\Phi(t,V(y)),
\end{aligned}
$$

where

$$\Phi(t,V) = \big(2K\|P\|\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\|T\|^2V\big)V.$$

Consider the set $\mathcal{T} = \{t\in[0,+\infty):\ \mu(t)\ne 0\}$. If there exists $\sup\mathcal{T} < +\infty$ then there exists $t_1\in[0,+\infty)$ such that $\mu(t) = 0$ for all $t\in[t_1,+\infty)$. If the set $\mathcal{T}$ is not bounded, the condition $\lim_{t\to+\infty}\sup\beta_A(t) = q < 0$ implies that there exists a sufficiently large $t_2\in[0,+\infty)\cap\mathcal{T}$ such that for all $t\in[t_2,+\infty)\cap\mathcal{T}$ inequality $\beta_A(t) < 0$ holds true. This yields for all $t\in[t_2,+\infty)\cap\mathcal{T}$ the inequality

$$\log\big|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)\big| < 0.$$

Then

$$
\begin{gathered}
\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)-1 < 1,\\
\|P\|\|B_1\|^2\mu^2(t)+\lambda_M(PB_1+B_1^{\mathrm{T}}P)\mu(t)-2 \le 0.
\end{gathered}
$$

Since $D = \lambda_M(PB_1+B_1^{\mathrm{T}}P)^2+8\|P\|\|B_1\|^2 \ge 0$, we obtain the estimate $\mu(t)\le\mu_1$ for all $t\in[t_2,+\infty)\cap\mathcal{T}$, where $\mu_1 = (-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D})/2\|P\|\|B_1\|^2 \ge 0$. Hence, one can conclude that $\mu(t)\le\mu_1$ for all $t\in[t_3,+\infty)$, $t_3 = \max\{t_1,t_2\}$. If $t\in[0,\rho(t_3)]\cap\mathbb{T}$ then $\mu(t)\le t_3$. This implies the estimate $\mu(t)\le\mu^* = \max\{\mu_1,t_3\}$ for all $t\in[0,+\infty)$. Since

$$
\begin{aligned}
\frac{\Phi(t,V)}{V} &= 2K\|P\|\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\|T\|^2V\\
&\le 2\|P\|K\|T\|(1+\mu^*\|B_1\|)\sqrt{V}+\mu^*K^2\|P\|\|T\|^2V,
\end{aligned}
$$

we get $\Phi(t,V)/V\to 0$ for $V\to 0$ uniformly in t. According to Theorem 2 from the paper [20] we conclude that the equilibrium state $y(t) = 0$ of system (3.7) is exponentially stable. This is equivalent to the exponential stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Now we shall prove the second part of the theorem. Condition $\sup\{\beta_A : t \in [0,+\infty)\} = \overline{q} < 0$ for $t \in \mathcal{T}$ implies

$$\log|1 + \mu(t)(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\|B_1\|^2)| \le \mu(t)\overline{q} < 0$$

for all $t \in \mathcal{T}$. Hence, we get

$$\mu(t) \le \frac{-\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \sqrt{D}}{2\|P\|\|B_1\|^2} = \mu^*, \quad \mu^* \ge 0, \quad t \in \mathcal{T}.$$

That is $\mu(t) \le \mu^*$ for all $t \in [0,+\infty)$. Then, similar to the above, we have $\Phi(t,V)/V \to 0$ for $V \to 0$ uniformly in t.

Therefore, all conditions of Theorem 2 from the paper [20] are satisfied and the equilibrium state $y = 0$ of system (3.7) is uniformly exponentially stable. This is equivalent to the uniform exponential stability of the equilibrium state $x(t) = x^*$ of system (3.1).

Remark 3.1 Consider the scale $\mathbb{T} = \mathbb{N}_0$ $(\mu(t) \equiv 1)$. In this case system of equations (3.1) is equivalent to system (3.5) and the condition of uniform asymptotic stability of the equilibrium state of system (3.1) established in Theorem 3.2 for $\mu^* = 1$ becomes

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 \ge 0.$$

This result coincides completely with the following result for discrete system (3.5).

Theorem 3.5 *For neural discrete system* (3.5) *let assumptions* $\mathrm{S}_2 - \mathrm{S}_3$ *be satisfied. Then the equilibrium state* $x(t) = x^*$ *of system* (3.5) *is uniformly asymptotically stable, provided that*

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 \ge 0.$$

Proof Consider function $y(k) = x(k) - x^*$ and rewrite equations (3.5) as

$$y(k+1) = (-B + I)y(k) + TG(x(k)), \quad k \in \mathbb{N}_0, \tag{3.10}$$

where I is an identity $n \times n$-matrix and for the first difference of function $V(y) = y^{\mathrm{T}}y$ we get the estimate

$$\begin{aligned}
\Delta V(y(k))|_{(3.10)} &= y^{\mathrm{T}}(k+1)y(k+1) - y^{\mathrm{T}}(k)y(k) \\
&= [(-B+I)y(k) + TG(y(k))]^{\mathrm{T}}[(-B+I)y(k) + TG(y(k))] - y^{\mathrm{T}}(k)y(k) \\
&= y^{\mathrm{T}}(k)B^{\mathrm{T}}By(k) - 2y^{\mathrm{T}}(k)B^{\mathrm{T}}y(k) - 2y(k)^{\mathrm{T}}BTG(y(k)) \\
&\quad + 2y^{\mathrm{T}}(k)TG(y(k)) + G^{\mathrm{T}}(y(k))T^{\mathrm{T}}TG(y(k)) \\
&\le \|B\|^2\|y(k)\|^2 - 2\lambda_m(B)\|y(k)\|^2 + 2L\|B\|\|T\|\|y(k)\|^2 \\
&\quad + 2L\|T\|\|y(k)\|^2 + \|T\|^2\|G(y(k))\|^2 \\
&\le \left[\overline{b}^2 - 2\underline{b} + 2L\overline{b}\|T\| + 2L\|T\| + \|T\|^2L^2\right]\|(y(k))\|^2 \\
&= -\left[2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2\right]\|(y(k))\|^2 \le 0.
\end{aligned}$$

This yields the assertion of the theorem.

Theorem 3.6 *Let assumption* S_3 *be fulfilled. If for every fixed* $t \in \mathbb{T}$ *the matrix* $C = (I - \mu(t)B)\Lambda^{-1} - \mu(t)|T|$ *is an* M*-matrix, the function* $f(x) = -Bx + TS(x) + J$ *is regressive.*

Proof We fix $t \in \mathbb{T}$ and consider the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$R(x) = x + \mu(t)f(t,x) = (I - \mu(t)B)x + \mu(t)TS(x) + \mu(t)J.$$

Designate by $\widetilde{B} = (I - \mu(t)B)$, $\widetilde{T} = \mu(t)T$ and $\widetilde{J} = \mu(t)J$. Then we get

$$R(x) = \widetilde{B}x + \widetilde{T}S(x) + \widetilde{J}.$$

Since the matrix $C = \widetilde{B}\Lambda^{-1} - |\widetilde{T}|$ is an M-matrix, the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (see [28]). Hence follows the reversibility of the mapping $R(x)$ which is equivalent to the reversibility of the operator $I + \mu(t)f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$.

Example 3.1 On the time scale

$$\mathbb{P}_{1,b} = \bigcup_{j=0}^{\infty}[j(1+b), j(1+b)+1], \quad b > 0,$$

we consider a neural network

$$\begin{aligned} x_1^{\Delta} &= -b_1x_1 + t_{11}s_1(x_2) + t_{12}s_2(x_2) + i_1, \\ x_2^{\Delta} &= -b_2x_1 + t_{21}s_1(x_1) + t_{22}s_2(x_2) + i_2, \end{aligned} \tag{3.11}$$

where $x_1, x_2 \in \mathbb{R}$,

$$b_1 = b_2 = 1, \quad T = \begin{pmatrix} 0.1 & -0.5 \\ 0.5 & 0.1 \end{pmatrix}, \qquad s_1(u) = s_2(u) = \tanh u.$$

For the time scale $\mathbb{P}_{1,b}$ the granularity function

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{j=0}^{\infty}[j(1+b), j(1+b)+1), \\ b, & t \in \bigcup_{j=0}^{\infty}\{j(1+b)+1\}. \end{cases}$$

We take matrix $P = \operatorname{diag}\{0.5, 0.5\}$ and write out all the functions and constants mentioned in the conditions of Theorem 3.4

$$M_1 = M_2 = L_1 = L_2 = 1, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$K_1 = K_2 = 8\left|e^{\frac{2+\sqrt{3}}{2}} - e^{-\frac{2+\sqrt{3}}{2}}\right| \Big/ \left(e^{\frac{2+\sqrt{3}}{2}} + e^{-\frac{2+\sqrt{3}}{2}}\right)^3,$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -0.9 & -0.5 \\ 0.5 & -0.9 \end{pmatrix}, \quad C = \begin{pmatrix} 1-1.1b & -0.5b \\ -0.5b & 1-1.1b \end{pmatrix},$$

$$\lambda_M(PB_1 + B_1^{\mathrm{T}}P) = -0.9, \quad A(t) = -0.9 + 0.53\,b, \quad \|B_1\|^2 = 1.06,$$

$$\beta_A(t) = \begin{cases} b^{-1}\log|1 + b(-0.9 + 0.53\,b)|, & t \in \bigcup\limits_{j=0}^{\infty}\{j(1+b)+1\}, \\ -0.9 + 0.53\,b, & t \in \bigcup\limits_{j=0}^{\infty}[j(1+b), j(1+b)+1). \end{cases}$$

The regressivity condition has the inequalities

$$\begin{cases} 1 - 1.1\,b > 0, \\ (1-1.1b)^2 - 0.25\,b^2 > 0, \end{cases}$$

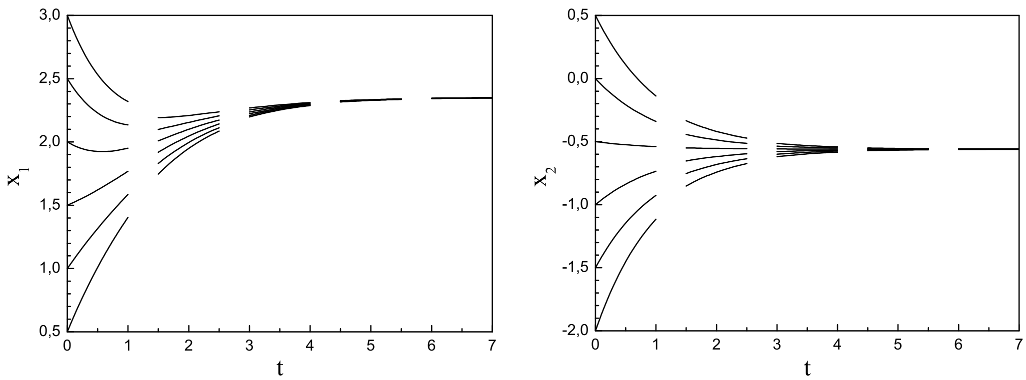

Figure 1. Dependence of functions $x_1(t)$ and $x_2(t)$ on time t obtained by numerical solution of system of equations (3.11). The first figure is drawn for the initial values: $x_2(0) = 1$ and $x_1(0) = 0.5;\ 1;\ 1.5;\ 2;\ 2.5;\ 3$. The second figure is drawn for the initial values: $x_1(0) = 2$ and $x_2(0) = -2;\ -1.5;\ -1;\ -0.5;\ 0;\ 0.5$.

which yields $b < 0.625$.

Since $1 + b(-0.9 + 0.53\,b) \geq 1 + b_0(-0.9 + 0.53\,b_0)$, $b_0 = 0.9/(2 \cdot 0.53)$ for any b, we can take for the constant M the following value: $M = 1 + b_0(-0.9 + 0.53\,b_0) \simeq 0.61$.

For $b < 1.69$ the system of inequalities

$$\begin{cases} M \leq |1 + b(-0.9 + 0.53\,b)| < 1, \\ -0.9 + 0.53\,b < 0 \end{cases}$$

is satisfied. This implies that

$$\sup_t \beta_A(t) = \max\{b^{-1} \log|1 + b(-0.9 + 0.53\,b)|, -0.9 + 0.53\} < 0.$$

Since the matrix

$$B\Lambda^{-1} - |T| = \begin{pmatrix} 0.9 & -0.5 \\ -0.5 & 0.9 \end{pmatrix}$$

is an M-matrix, for $0 < b < 0.625$ system (3.11) possesses a unique equilibrium state for any $i_1, i_2 \in \mathbb{R}$ and this equilibrium state is uniformly exponentially stable.

We shall consider a model example for this problem. We take the following values of the constants: $i_1 = 2$, $i_2 = -1$, $b = 0.5$. The result of numerical solution of system (3.11) is shown on Figures 1. It is seen from the Figures 1, for arbitrary chosen initial conditions $x_1(0) = 0.5 \div 3$ and $x_2(0) = -2 \div 0.5$, the functions $x_1(t)$ and $x_2(t)$ approach asymptotically with time t to the equilibrium state $x_1^* \simeq 2.35$, $x_2^* \simeq -0.56$.

4 Stability of Regular Synchronous Generation of Optically Coupled Lasers

This section deals with the stability with respect to linear approximation of some periodic solutions to a system of nonlinear differential equations. This system describes some experimental realization of a "chaotic" CO_2-laser with a 100 per cent depth-modulated periodic pumping by alternate current (see [10, 11]).

Variation of the factor of strengthening g and amplitude E of a synchronized field of *two optically coupled lasers* is described by the simplest model

$$\begin{aligned} \tau\dot{g} &= g_0(t) - g(1+E^2), \\ \dot{E} &= (g - \widetilde{g}_{\mathrm{th}})E/2, \end{aligned} \tag{4.1}$$

where τ is an efficient time of relaxation of the active medium ($\tau \gg 1$), $g_0(t) = A(1+\sin\omega t)$ is a $(2\pi/\omega)$-periodic pumping, $\widetilde{g}_{\mathrm{th}} = g_{\mathrm{th}} + 2M(1-\sqrt{1-(\Delta/M)^2})$ is a threshold coefficient of strengthening. Here g_{th} means threshold strengthening, M is a real positive coupling factor, Δ is a value of resonance eigenfrequency detuning (further on — detuning). For the problems considered below the difference of real medium kinetics of CO_2-from the model one is not of essential importance (see [11]).

The mode of phase synchronization, for which the field amplitudes of both lasers are equal at any moment and the phase is constant and depends on detuning, is realized under the condition $|\Delta| < M$. Moreover, the dynamics of two coupled lasers coincides with the dynamics of one equivalent laser whose threshold grows with the growth of detuning. In the mode of synchronous generation (for a fixed M) the growth of detuning corresponds to lessening of the parameter $A/\widetilde{g}_{\mathrm{th}}$ for the equivalent laser. Due to the complex bifurcation diagram of the laser with periodic pumping this results in generation of both chaotic and regular signals.

Designate by $(g_T(t), E_T(t))^T$, $t \in [t_0', \infty) = T_0$, $t_0' \ge 0$, T-periodic solution of system of equations (4.1) with the initial condition

$$g(t_0') = g_0', \quad E(t_0') = E_0' \tag{4.2}$$

and define variables y_1 and y_2 of the perturbed motion of system (4.1) as

$$y_1 = g - g_T(t), \quad y_2 = E - E_T(t).$$

Then the perturbed equations of motion(4.1) are

$$\begin{aligned} \tau\dot{y}_1 &= -(1+E_T^2(t))y_1 - 2g_T(t)E_T(t)y_2 - 2E_T(t)y_1y_2 - g_T(t)y_2^2 - y_1y_2^2, \\ 2\dot{y}_2 &= E_T(t)y_1 + (g_T(t) - \widetilde{g}_{\mathrm{th}})y_2 + y_1y_2. \end{aligned} \tag{4.3}$$

For the linear approximation of system (4.3) (designated as (4.3′)) we construct an auxiliary matrix-valued function [12, 13]

$$U(t,y_1,y_2) = \begin{bmatrix} p_{20}y_1^2 & p_{11}(t)y_1y_2 \\ p_{11}(t)y_1y_2 & p_{02}y_2^2 \end{bmatrix},$$

where p_{20} and p_{02} are finite positive constants, $p_{11}(t) \in C^1(\mathbb{R},\mathbb{R})$, and a scalar Lyapunov function

$$v(t,y,\eta) = \eta^T U(t,y_1,y_2)\eta, \tag{4.4}$$

where $y = (y_1,y_2)^T$ and $\eta = (\eta_1,\eta_2)^T > 0$.

Total time derivative of function (4.4) found by virtue of linear approximation of system (4.3) is

$$\begin{aligned} \left.\frac{dv}{dt}\right|_{(4.3')} &= (-2\eta_1^2 p_{20}(1+E_T^2(t))/\tau + \eta_1\eta_2 p_{11T}(t)E_T(t))y_1^2 \\ &\quad + (\eta_2^2 p_{02}(g_T(t) - \widetilde{g}_{\mathrm{th}}) - 4\eta_1\eta_2 p_{11T}(t)g_T(t)E_T(t)/\tau)y_2^2 \\ &= s_{20}(t)y_1^2 + s_{02}(t)y_2^2, \end{aligned}$$

if $p_{11T}(t)$ is assumed to be a T-periodic solution of the linear differential equation

$$\begin{aligned} \dot{p}_{11} = &((1+E_T^2(t))/\tau - g_T(t) + \widetilde{g}_{\text{th}})p_{11} \\ &+ (2\eta_1 p_{20} g_T(t)/(\tau\eta_2) - \eta_2 p_{02}/(2\eta_1))E_T(t). \end{aligned} \tag{4.5}$$

Conditions of *uniform asymptotic stability of T-periodic solution* of system of equations (4.3) (noncritical case) are established in the form of a system of inequalities

$$\begin{aligned} &p_{20}p_{02} - p_{11T}^2(t) > 0, \\ &s_{20}(t) < 0, \\ &s_{02}(t) < 0 \quad \text{for all} \quad t \in [t', t'+T], \quad t' \in T_0. \end{aligned} \tag{4.6}$$

Thus, the problem on asymptotic stability of some signals of the equivalent CO_2-laser is reduced to the problem of finding T-periodic solutions to nonlinear non-stationary initial problem (4.1)–(4.2) and linear inhomogeneous equation (4.5) with periodic coefficients and the initial condition

$$p_{11}(t_0') = p_{110}'. \tag{4.7}$$

This, in its turn, involves preliminary study of the problem on the domain where equations (4.1) and (4.5) form T-system (see [25]) and establishing existence conditions for the corresponding T-periodic solutions passing through the point (g_0', E_0', p_{110}') at the initial instant t_0'.

We set

$$T = k(2\pi/\omega), \tag{4.8}$$

where k is a positive integer, and define the domain $D \subset \mathbb{R}^3$ which singles out T-system, by the inequalities

$$D: \quad |g| \le g_{\max}, \quad |E| \le E_{\max}, \quad |p_{11}| \le p_{11\,\max}. \tag{4.9}$$

We introduce the vector $M = (M_1, M_2)^T$ and the scalar M_3 which bounds for all $t \in T_0$ and $(g, E, p_{11}) \in D$ the absolute values of the corresponding right-hand sides of equations (4.1) and (4.5) (further on f_1, f_2 and f_3):

$$\begin{aligned} &M_1 = (2A + g_{\max}(1+E_{\max}^2))/\tau, \\ &M_2 = (g_{\max} + \widetilde{g}_{\text{th}})E_{\max}/2, \\ &M_3 = ((1+E_{\max}^2)/\tau + g_{\max} + \widetilde{g}_{\text{th}})p_{11\,\max} + (2\eta_1 p_{20} g_{\max}/(\tau\eta_2) + \eta_2 p_{02}/(2\eta_1))E_{\max}. \end{aligned} \tag{4.10}$$

Continuous vector function $f = (f_1, f_2)^T$ periodic in t with the period T satisfies in $T_0 \times [-g_{\max}, g_{\max}] \times [-E_{\max}, E_{\max}]$ the Lipschitz condition with the matrix

$$K = \begin{bmatrix} (1+E_{\max}^2)/\tau & 2g_{\max}E_{\max}/\tau \\ E_{\max}/2 & (g_{\max} + \widetilde{g}_{\text{th}})/2 \end{bmatrix},$$

and the scalar continuous periodic function f_3 in $T_0 \times [-p_{11\,\max}, p_{11\,\max}]$ with the constant

$$K_3 = (1+E_{\max}^2)/\tau + g_{\max} + \widetilde{g}_{\text{th}}.$$

Following the definition of T-system and relating with vector-function $(f^T, f_3)^T$ and domain D the nonempty set D_f of points $\mathbb{R}^3$ contained in D together with its $\frac{T}{2}(M^T, M_3)^T$-neighborhood the conditions defining T-system are obtained in the form of a system of inequalities

$$2g_{\max} - TM_1 > 0, \quad 2E_{\max} - TM_2 > 0, \quad 2p_{11\max} - TM_3 > 0,$$
$$\frac{T}{\pi}\frac{K_{11}+K_{22}+\sqrt{(K_{11}-K_{22})^2+4K_{12}K_{21}}}{2} < 1, \quad \frac{T}{\pi}K_3 < 1.$$

Moreover, it is also assumed that the initial value (g_0', E_0', p_{110}') belongs to D_f.

The immediate construction of the desired T-periodic solutions is achieved, for example, by the method of trigonometric collocations by a numerical-analytical scheme. To this end, we assume that the values of functions $f_j(t, g, E, p_{11})$, $j = 1, 2, 3$, calculated basing on the m-th approximation to the desired periodic solution coincide in $N = 2r+1$ collocation points $t_i = i\frac{T}{N}$, $i = 0, 1, \dots, 2r$, with the values of the trigonometric polynomials

$$f_j^m = \alpha_{j0}^m + \sum_{l=1}^{r}(\alpha_{jl}^m \cos l\Omega t + \beta_{jl}^m \sin l\Omega t), \tag{4.11}$$

where $\Omega = 2\pi/T$. Then the vectors of the coefficients

$$f_j^{m\Gamma} = (\alpha_{j0}^m, \alpha_{j1}^m, \beta_{j1}^m, \dots, \alpha_{jr}^m, \beta_{jr}^m)^T \tag{4.12}$$

of trigonometric polynomials (4.11) are expressed via the respective vectors of values of these polynomials

$$f_j^{mM} = (f_j(t_i, g^m(t_i), E^m(t_i), p_{11}^m(t_i)))_{i=0}^{2r}$$

with the help of the matrix

$$\Gamma = [\Gamma_{pq}]_{p,q=1}^{N},$$

where

$$\Gamma_{pq} = \begin{cases} \frac{1}{N}, & p = 1, \\ \frac{2}{N}\cos\left(p(q-1)\frac{\pi}{N}\right), & p = 2, 4, \dots, 2r, \\ \frac{2}{N}\sin\left((p-1)(q-1)\frac{\pi}{N}\right), & p = 3, 5, \dots, N, \end{cases}$$

and

$$f_j^{m\Gamma} = \Gamma f_j^{mM}.$$

By introducing into consideration the $N \times N$-two-diagonal matrix

$$\mu^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\frac{1}{\Omega} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\Omega} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\Omega} & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\Omega} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -\frac{1}{r\Omega} \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{r\Omega} & 0 \end{bmatrix}$$

and N-dimensional vectors

$$z_j^{m\Gamma} = \left(\alpha_{j0}^{m\prime} + \sum_{l=1}^{r}(\alpha_{jl}^{m\prime}\cos l\Omega t_0' + \beta_{jl}^{m\prime}\sin l\Omega t_0'), 0, \dots, 0\right)^{\mathrm{T}},$$

where

$$(\alpha_{j0}^{m\prime}, \alpha_{j1}^{m\prime}, \beta_{j1}^{m\prime}, \dots, \alpha_{jr}^{m\prime}, \beta_{jr}^{m\prime})^T = \mu^1 f_j^{m\Gamma},$$

we obtain the vectors of the coefficients of $(m+1)$-th "trigonometric" approximation to the desired T-periodic solution in the form

$$\begin{aligned} g^{m+1,\Gamma} &= g^{0\Gamma} + \mu^1 f_1^{m\Gamma} - z_1^{m\Gamma}, \\ E^{m+1,\Gamma} &= E^{0\Gamma} + \mu^1 f_2^{m\Gamma} - z_2^{m\Gamma}, \\ p_{11}^{m+1,\Gamma} &= p_{11}^{0\Gamma} + \mu^1 f_3^{m\Gamma} - z_3^{m\Gamma}, \end{aligned}$$

where $g^{0\Gamma}$, $E^{0\Gamma}$ and $p_{11}^{0\Gamma}$ are the vectors of the coefficients of appropriate zero approximations.

The form of the zero approximation $(g^0(t), E^0(t))^{\mathrm{T}}$ and the vector of the initial values at the collocation points and the initial vector of the coefficients of the right-hand sides f_1, f_2 of equations (4.1) respectively are taken based on solution of system (4.1) linearized by the equation for g

$$\begin{aligned} g^0(t) &= C_g e^{-\frac{t}{\tau}} + \frac{A}{1+\omega^2\tau^2}(\sin\omega t - \omega\tau\cos\omega t) + A, \qquad g^0(t_0') = g_0', \\ E^0(t) &= C_E \exp\bigg\{\frac{1}{2}\bigg(- C_g\tau e^{-\frac{t}{\tau}} + (A - \widetilde{g}_{\mathrm{th}})t \\ &\quad - \frac{A}{1+\omega^2\tau^2}\bigg(\frac{1}{\omega}\cos\omega t + \tau\sin\omega t\bigg)\bigg)\bigg\}, \qquad E^0(t_0') = E_0', \end{aligned}$$

where constants C_g and C_E are defined univalently. We take solution of the corresponding homogeneous initial problem (4.5), (4.7) as $p_{11}^0(t)$, assuming T-periodic functions to be known

$$g_T(t) \approx g^m(t) = \sum_{j=-r}^{r} g_j^m e^{i\Omega jt}, \quad E_T(t) \approx E^m(t) = \sum_{j=-r}^{r} E_j^m e^{i\Omega jt},$$

where $g_j^m = (\alpha_{gj}^m - i\beta_{gj}^m)/2$, $g_{-j}^m = \overline{g_j^m}$, $E_j^m = (\alpha_{Ej}^m - i\beta_{Ej}^m)/2$, $E_{-j}^m = \overline{E_j^m}$, and $\alpha_{gj}^m, \beta_{gj}^m$ and $\alpha_{Ej}^m, \beta_{Ej}^m$ stand for coefficients (4.12) of the corresponding trigonometric series (4.11). Then

$$\begin{aligned} p_{11}^0(t) &= C_{p_{11}} \exp\bigg\{\bigg(\bigg(1 + \sum_j E_j^m E_{-j}^m\bigg)t + \sum_j \sum_{s\neq -j} \frac{E_j^m E_s^m}{i\Omega(j+s)} e^{i\Omega(j+s)t}\bigg)/\tau \\ &\quad + (\widetilde{g}_{\mathrm{th}} - g_0^m)t - \sum_{j\neq 0} \frac{g_j^m}{i\Omega j} e^{i\Omega jt}\bigg\}, \quad p_{11}^0(t_0') = p_{110}'. \end{aligned}$$

The control of convergence of the described iteration process of finding T-periodic solution is performed by comparing with a pre-given accuracy ε_1 the difference between the vectors of coefficients of the m-th and $(m+1)$-th trigonometric approximations for $g_T(t), E_T(t), p_{11T}(t)$ with zero-vector, and by comparing with a pre-given accuracy ε_2 the mean values of functions $f_j(t, g^m(t), E^m(t), p_{11}^m(t))$, taken over a period, with zero. The latter condition is necessary and sufficient (see [25]) for the existence of periodic solutions of the period T passing through the point $(g_0', E_0', p_{110}') \in D_f$ for $t = t_0'$ and is an indicator of a good choice of the values k (see (4.8)), $g_{\max}, E_{\max}, p_{11\,\max}$ (see (4.9)), $t_0', g_0', E_0', p_{110}'$ (see (4.2), (4.7)), $p_{20}, p_{02}, \eta_1, \eta_2$ (see (4.10)) and the parameter values of the system under consideration.

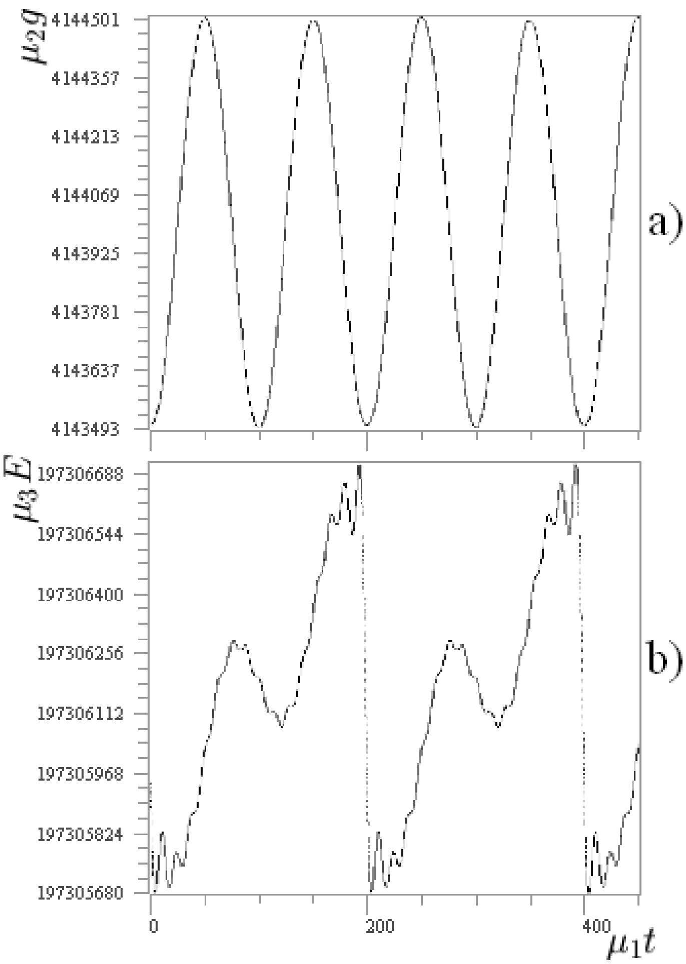

Figure 2. Graphs of $(4\pi/\omega)$-periodic functions $g_T(t)$ and $E_T(t)$.

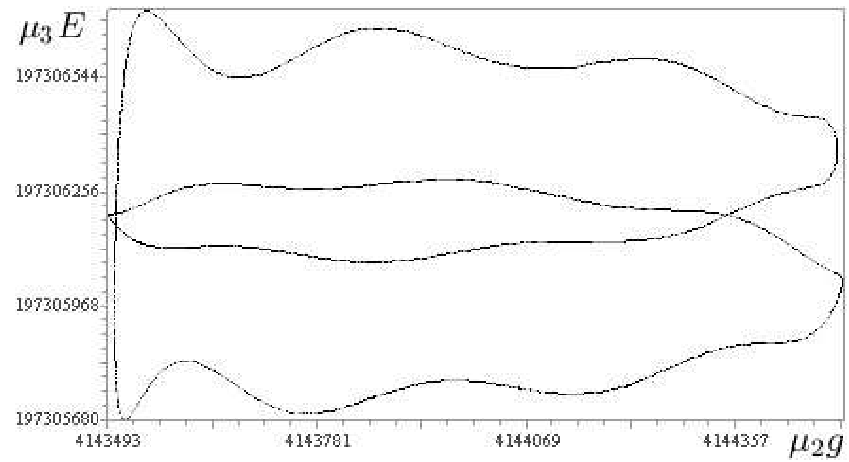

Figure 3. Phase trajectory corresponding to $(4\pi/\omega)$-periodic solution of $(g_T(t), E_T(t))^{\mathrm{T}}$.

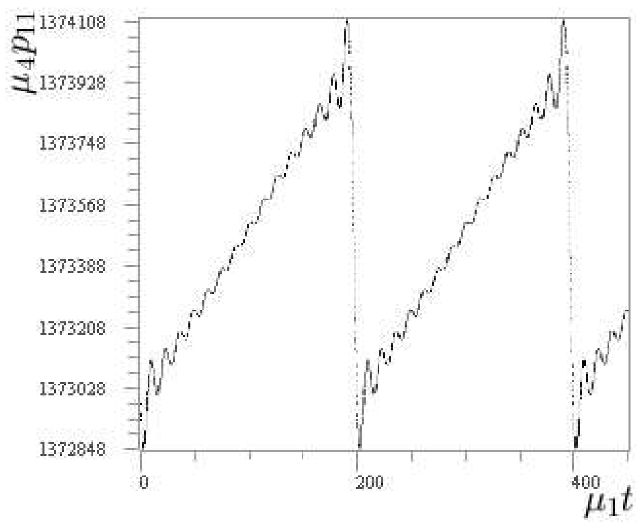

Figure 4. Graph of $(4\pi/\omega)$-periodic function $p_{11T}(t)$.

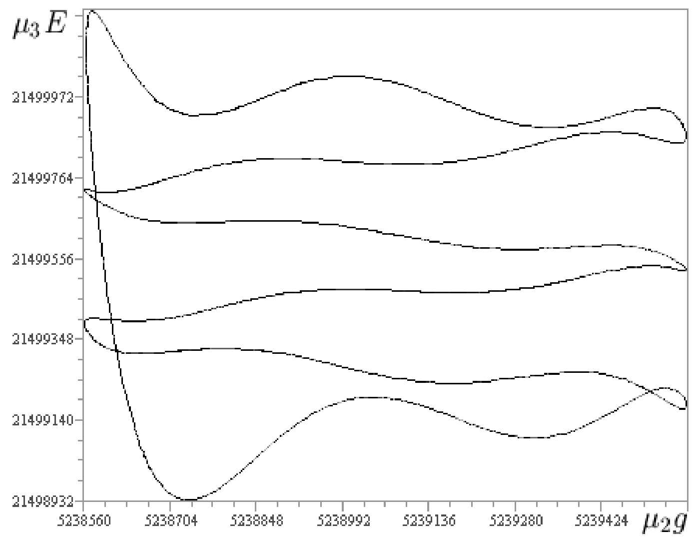

Figure 5. Phase trajectory corresponding to $(6\pi/\omega)$-periodic solution of $(g_T(t), E_T(t))^{\mathrm{T}}$.

For the values $\omega = 40.96241$, $A = 0.39856$, $g_{\text{th}} = 0.4$, $M = 0.1$, $\tau = 400$, $\Delta = 0.001$, $t_0' = 0$, $g_{\max} = 4.77321$, $g_0' = 0.40001$, $E_{\max} = 9.27539$, $E_0' = 0.03817$ with the use of $N = 31$ collocation points during 5 iterations ($\varepsilon_1 = 10^{-25}$, $\varepsilon_2 = 10^{-5}$) a periodic solution was constructed for the initial problem (4.1), (4.2) with duplication of the period ($k = 2$). The corresponding graphs are shown in Figures 2 and 3. Uniform asymptotic stability of the corresponding zero solution of system (4.3) is established during 4 iterations by constructing with the same accuracy the periodic function $p_{11T}(t)$ (Figure 4) satisfying conditions (4.6). Here $\eta_1 = 8.01158$, $\eta_2 = 4.38394$, $p_{20} = 2.97746$, $p_{02} = 0.14038$, $p_{11\max} = 4.65370$, $p_{110}' = 0.42974$ and $\mu_1 = 651$, $\mu_2 = 10358490$, $\mu_3 = 5168882277$, $\mu_4 = 3194900$ are the scale multiplies.

Uniform asymptotically stable signal with triple period with respect to the pumping period (Figure 5) is investigated in the same way for the parameters changed as compared with the previous example $\omega = 52.116990$, $A = 0.399742$, $g_{\max} = 0.904412$, $g_0' = 0.399723$, $E_{\max} = 8.306199$, $E_0' = 0.002538$, $\varepsilon_2 = 10^{-6}$, $\eta_1 = 8.776919$, $\eta_2 = 0.385523$, $p_{20} = 2.070760$, $p_{02} = 8.392944$, $p_{11\max} = 8.833734$, $p_{110}' = 0.083020$, $\mu_2 = 13105471$, $\mu_3 = 8472130609$.

The considered examples demonstrate the possibility of parallel solution of some problems on the spectrum and structure of collective mode as well as their stability and competition between the mode of composed resonator. The method of constructing an auxiliary function pointed out in the context of matrix-valued Lyapunov functions allows to calculate stability domains of some periodic signals of coupled lasers with periodic pumping in the regime of synchronous generation.

5 Models of World Dynamics and Sustainable Development

The *Forrester model of world dynamics* (see [5, 23]) is constructed in terms of the approach developed in the investigation of complex systems with nonlinear feedbacks. In the modeling of world dynamics the following global processes are taken into account:

(i) quick growth of the world population;

(ii) industrialization and the related production growth;

(iii) restricted food resources;

(vi) growth of industrial wastes;

(v) shortage of natural resources.

The main variables in the Forrester model are:

(1) population P (further on the designation X_1 is used);

(2) capital stocks K (X_2);

(3) stock ratio in agricultural industry X (X_3);

(4) level of environmental pollution Z (X_4);

(5) quantity of nonrenewable natural resources R (X_5).

Factors through which the variables $X_1, \dots, X_5$, effect one another are:

- relative number of population P_p (population normed to its number in 1970);
- specific stocks K_p;
- level of living standard C;
- relative level of meals F;
- normed value of specific stocks in agricultural industry X_p;
- relative pollution Z_s;
- ratio of the resources left R_R.

In addition to the enumerated factors Forrester also considers the notion of "quality of living" Q. This factor depends on the variables P_p, C, F and Z_s: $Q = Q_C Q_F Q_P Q_Z$.

For the variables P, K, X, Z, R interpreted as system equations, the equations of the type

$$\frac{dy}{dt} = y^+ - y^-, \tag{5.1}$$

are written, where y^+ is a positive rate of velocity growth of the variable; y^- is a negative rate of velocity diminishing of the variable y. In a simplified form the *world dynamics equations* are

$$\begin{aligned} \frac{dP}{dt} &= P(B - D), & \frac{dZ}{dt} &= Z_+ - T_Z^{-1} Z, \\ \frac{dK}{dt} &= K_+ - T_K^{-1} K, & \frac{dR}{dt} &= -R_-, \\ \frac{dX}{dt} &= X_+ - T_X^{-1} X, \end{aligned} \tag{5.2}$$

where B is a birth rate, D is a death rate, K_+ is a velocity of capital stocks production, X_+ is an increment of the ratio of agricultural industry stocks, Z_+ is a velocity of pollution generation, T_Z is a characteristic time of natural decay of pollutants, and R_- is a velocity of resource consumption.

Mathematical analysis of model (5.2) reveals the existence of stationary and quasi-stationary solutions which are interpreted as a "global equilibrium" and a "stable society".

Let a "nation" N (a totality of international organizations) form the public opinion about global processes occurring on a certain level of the system. The measure of the change of the public opinion $\chi(t)$ will be modeled on each system by the equation (see [18])

$$\frac{d^2\chi}{dt^2} + m^2\chi = 0, \quad \chi'(t_0) = \chi_0', \quad \chi(t_0) = \chi_0. \tag{5.3}$$

Here the value m is a function of variables (1)–(5) at times $t = t_0$. Moreover, for the system levels the equations of (5.1) type are written

$$\frac{dy}{dt} = y^+ - y^- + b(t), \tag{5.4}$$

where the "discontent" function $b(t)$ is as follows

$$b(t) = ge^{\pm\alpha|\chi(t)|}, \quad \alpha = \text{const} > 0. \tag{5.5}$$

Here g is a factor of "discontent" reflecting the change of the "level of living standard" of the countries involved into world dynamics. Correlation (5.5) models the increase (decrease) of discontent of the current global processes depending on changes of the measure of the public opinion.

Thus, the Forrester model (5.1)–(5.2) is generalized by the equations

$$\begin{aligned}
\frac{dX_1}{dt} &= X_1(B - D) + g_1e^{\pm\alpha|\chi(t)|},\\
\frac{dX_2}{dt} &= K_+ - T_K^{-1}X_2 + g_2e^{\pm\alpha|\chi(t)|},\\
\frac{dX_3}{dt} &= X_+ - T_X^{-1}X_3 + g_3e^{\pm\alpha|\chi(t)|},\\
\frac{dX_4}{dt} &= Z_+ - T_Z^{-1}X_4 + g_4e^{\pm\alpha|\chi(t)|},\\
\frac{dX_5}{dt} &= -R_- + g_5e^{\pm\alpha|\chi(t)|},\\
&\frac{d^2\chi}{dt^2} + m^2\chi = 0,
\end{aligned} \tag{5.6}$$

where $g_1, \dots, g_5$ are the discontent factors on the corresponding level of the system.

It is proposed to describe *general nonlinear model of world dynamics* by a system of differential equations of the type

$$\frac{dX_i}{dt} = W_i(X) + g_ie^{\pm\alpha|\chi(t)|}, \tag{5.7}$$

$$\frac{d^2\chi}{dt^2} + m^2\chi = 0, \quad i = 1, 2, \dots, N. \tag{5.8}$$

Here $X = (X_1, \dots, X_5, \dots, X_N) \subseteq S(H)$, where $X_1, \dots, X_5$ are the Forrester variables and $X_{5+1}, \dots, X_n$ are some other variables involved into the world dynamics equations, $W_i\colon S(H) \to R_+^N$ is a vector-function with the components describing the variation of parameters on the appropriate system level. It is assumed that the solution $(X^T(t), \chi(t))^T$ of system of coupled equations (5.7)–(5.8) exists for all $t \geq t_0$ with the initial conditions $(X_0^T, \chi_0', \chi_0)^{\mathrm{T}} \in \mathrm{int}(R_+^N,\, R \times R)$.

Assume that the system of nonlinear equations

$$\begin{gathered} W_1(X) + g_1 e^{\pm\alpha|\chi(t)|} = 0, \\ \dots\dots\dots\dots\dots \\ W_N(X) + g_N e^{\pm\alpha|\chi(t)|} = 0 \end{gathered}$$

possesses a quasistationary solution $X_n(t) = (X_{1n}(t),\, \dots,\, X_{nN}(t))^T$ for any bounded function $\chi(t)$ being a solution of equation (5.8). Moreover, the Lyapunov substitution

$$Y(t) = X(t) - X_n(t)$$

brings system of equations (5.7) to the form

$$\frac{dY}{dt} = Y(t, Y), \tag{5.9}$$

where $Y(t, Y) = W(Y + X_n(t)) + Ge^{\pm|\chi(t)|} - (W(X_n(t)) + Ge^{\pm|\chi(t)|})$. It is clear that $Y(t, 0) = 0$ for all $t \geq 0$. System (5.9) is a system of perturbed equations of world dynamics.

The problem of sustainable development is associated with the analysis of solution $Y = 0$ of equation(5.9). The stability analysis of solutions will be carried out with respect to two measures H_0 and H taking the values from the sets

$$\begin{gathered} \Phi = \{H \in C(R_+ \times R^N,\, R_+)\colon \inf_{(t,Y)} H(t, Y) = 0\}; \\ \Phi_0 = \{H \in \{\Phi\colon \inf_{Y} H(t, Y) = 0 \;\text{ for every }\; t \in R_+\}. \end{gathered}$$

We need the following definition.

Definition 5.1 The world dynamics (5.7)–(5.8) has *sustainable development with respect to two measures* if for every $\varepsilon > 0$ and $t_0 \in R_+$ there exists a positive function $\delta(t_0, \varepsilon) > 0$ continuous in t_0 for every ε such that the condition $H_0(t_0, Y_0) < \delta$ implies the estimate $H(t, Y(t)) < \varepsilon$ for all $t \geq t_0$ for any bounded solution $\chi(t)$ of equation (5.8).

Note that if system (5.7) having no zero solution ($W(0, \chi(t)) \neq 0$ for $X = 0$) and has the nominal solution $X_n(t)$ then the measures H_0 and H can be taken as follows: $H(t, X) = H_0(t, X) = \|X - X_n(t)\|$, where $\|\cdot\|$ is an Euclidean norm of the vector X. If it is of interest to study stability of the development in the Forrester variables, the measures H_0 and H are taken as: $H(t, X) = \|X - X_n(t)\|_s$, $1 \leq s \leq 5$, and $H_0(t, X) = \|X - X_n(t)\|$. This corresponds to stability analysis of system (5.7) in two measures with respect to a part of variables.

For system (5.9) assume that the elements $u_{ij}(t, Y)$ of the matrix-valued function

$$U(t, Y) = [u_{ij}(t, Y)], \quad i, j = 1, \dots, m, \quad m < N,$$

are constructed, where $u_{ii} \in C(R_+ \times R^N,\, R_+)$ and $u_{ij} \in C(R_+\times\, R^N,\, R)$ for $(i \neq j) \in [1, m]$. The function

$$V(t, Y, w) = w^T U(t, Y)w, \quad w \in R^m, \tag{5.10}$$

is considered together with the function

$$D^+V(t, Y, w) = w^T D^+U(t, Y)w, \tag{5.11}$$

where $D^+U(t, Y)$ is the upper right Dini derivative calculated element-wise for the matrix-valued function $U(t, Y)$.

Conditions of the sustainable development in two measures (H_0, H) are established in the following result.

Theorem 5.1 *Let the functions in equations of global dynamics* (5.7)–(5.8) *be defined and continuous in the domain of values* $(t, Y, \chi) \in R_+ \times \mathcal{S} \times D$. *If, moreover,*

(1) *measures* H_0 *and* H *are of class* Φ*;*

(2) *function* (5.10) *satisfies the condition* $V(t, Y, w) \in C(R_+\times\, \mathcal{S} \times R^m,\, R_+)$ *and is locally Lipchitz in* Y*;*

(3) *function* $V(t, Y, w)$ *satisfies the estimates*

(a) $a(H(t, Y)) \leq V(t, Y, w) \leq b(t, H_0(t, Y))$ *for all* $(t, Y, w) \in S(h, H) \times R^m$ *or*

(b) $a(H(t, Y)) \leq V(t, Y, w) \leq c(H_0(t, Y))$

where $a, c \in K$*–class and* $b \in CK$*–class of comparison functions;*

(4) *there exists a matrix-valued function* $\Theta(Y, w)$*,* $\Theta \in C(R^N \times R^m,\ R^{m\times m})$ *and* $\Theta(0, w) = 0$ *for all* $(w \neq 0) \in R^m$ *such that*

$$D^+V(t, Y, w) \leq \mathrm{e}^T \widehat{\Theta}(Y, w)\mathrm{e}$$

for all $(t, Y, w) \in \mathcal{S} \times R^m$*, where* $\mathrm{e} = (1, 1, \dots, 1)^T \in R^m$*,* $\mathcal{S} \subset (R^N \times R_+)$*,* $\widehat{\Theta}(Y, w) = \frac{1}{2}\left(\Theta(Y, w) + \Theta^T(Y, w)\right)$ *for any bounded solution* $\chi(t)$ *of equation* (5.8).

Then

(a) *world dynamics* (5.7)–(5.8) *has sustainable development with respect to two measures if the matrix* $\widehat{\Theta}(Y, w)$ *is negative semi-definite, the measure* H *is continuous with respect to the measure* H_0 *and condition (3)(a) is satisfied;*

(b) *world dynamics* (5.7)–(5.8) *has uniformly sustainable development with respect to two measures if the matrix* $\widehat{\Theta}(Y, w)$ *is negative semi-definite, the measure* H *is uniformly continuous with respect to the measure* H_0 *and condition (3)(b) is satisfied.*

Proof We note that function $V(t, Y, w)$ determined by formula (5.10) is a scalar pseudo-quadratic form with respect to $w \in R^m$. Therefore, the property of definite sign of function (5.10) with respect to the measure H does not require the H-sign-definiteness of the elements $u_{ij}(t, x)$ of matrix $U(t, Y)$. First we shall prove assertion (a) of Theorem 5.1. Conditions (1), (2), and (3a) imply that the function $V(t, Y, w)$ is weakly

H_0-decreasing. Thus, for $t_0 \in R$, $(t_0 \in R_+)$ there exists a constant $\Delta_0 = \Delta_0(t_0) > 0$ such that for $H_0(t_0, x_0) < \Delta_0$ the inequality

$$V(t_0, Y_0, w) \le b(t_0, H_0(t_0, Y_0)) \tag{5.12}$$

holds true.

Also, condition (3a) implies that there exists a $\Delta_1 \in (0, H)$ such that

$$a(H(t,x)) \le V(t,x,w) \quad \text{for} \quad H(t,x) \le \Delta_1. \tag{5.13}$$

The fact that the measure H is continuous with respect to the measure H_0 implies that there exist a function $\varphi \in CK$ and a constant $\Delta_2 = \Delta_2(t_0) > 0$ such that

$$H(t_0, Y_0) \le \varphi(t_0, H_0(t_0, Y_0)) \quad \text{for} \quad H_0(t_0, Y_0) < \Delta_2, \tag{5.14}$$

where Δ_2 is taken so that

$$\varphi(t_0, \Delta_2) < \Delta_1. \tag{5.15}$$

Let $\varepsilon \in (0, \Delta_0)$ and $t_0 \in R$ $(t_0 \in \mathcal{T}_\tau)$ be given. Since the functions $a \in K$ and $b \in CK$, given ε and t_0, one can choose $\Delta_3 = \Delta_3(t_0, \varepsilon) > 0$ so that

$$b(t_0, \Delta_3) < a(\varepsilon). \tag{5.16}$$

We take $\delta(t_0) = \min(\Delta_1, \Delta_2, \Delta_3)$. Conditions (5.12)–(5.16) imply that for $H_0(t_0, Y_0) < \delta$ the inequalities

$$a(H(t_0, Y_0)) \le V(t_0, Y_0, w) \le b(t_0, H_0(t_0, Y_0)) < a(\varepsilon)$$

are fulfilled. From this we get

$$H(t_0, Y_0) < \varepsilon.$$

Let $Y(t; t_0, Y_0) = Y(t)$ be a solution of system (5.9) with the initial conditions for which $H_0(t_0, Y_0) < \delta$. We shall make sure that under conditions of Theorem 5.1 the estimate

$$H(t, Y(t)) < \varepsilon \quad \text{for all} \quad t \ge t_0$$

holds true. Assume that there exists a $t_1 \ge t_0$ such that

$$H(t_1, Y(t)) = \varepsilon \quad \text{and} \quad H(t, Y(t)) < \varepsilon, \quad t \in [t_0, t_1),$$

for solution $Y(t; t_0, Y_0)$ with the initial conditions $H_0(t_0, Y_0) < \delta$.

Condition (4) and the fact that the matrix $\widehat{\Theta}(Y, w)$ is negative semi-definite in the domain $\mathcal{S}$ imply that the roots $\lambda_i = \lambda_i(Y, w)$ of the equation

$$\det[\widehat{\Theta}(Y, w) - \lambda E] = 0$$

satisfy the condition $\lambda_i(Y, w) \le 0$, $i = 1, 2, \dots, m$, in the domain S. Therefore,

$$D^+V(t, Y, w) \le e^T \widehat{\Theta}(Y, w) e \le 0$$

and for all $t \in [t_0, t_1]$ the sequence of inequalities

$$a(\varepsilon) = a(H(t_1, Y(t_1))) \le V(t, Y, w) \le V(t_0, Y_0, w) \le b(t_0, H_0(t_0, Y_0)) < a(\varepsilon)$$

is satisfied.

The contradiction obtained disproves the assumption that $t_1 \in [t_0, +\infty)$. Thus, system (5.7)–(5.8) is (H_0, H)-stable.

Assertion (b) of Theorem 5.1 is proved in the same way. Besides, it is taken into account that condition (3)(b) is satisfied and the measure H is uniformly continuous with respect to the measure H_0, the value δ can be taken independent of $t_0 \in R$ $(t_0 \in R_+)$. Hence the uniform (H_0, H)-stability of system (5.7)–(5.8) follows.

Note that the construction of a suitable function (5.10) in terms of the matrix function $U(t, Y)$ is essentially simplified because the elements $u_{ij}(t, Y)$ can be associated with the world dynamics equations on a certain system level.

6 Stability Analysis of Takagi–Sugeno Impulsive Systems

6.1 General results

Consider the impulsive fuzzy dynamic model of Takagi–Sugeno. Given the properly defined input variables and membership functions, the T-S fuzzy rules for a multivariable system considered herein are of the form:

$$R^i,\ i = \overline{1, r}\colon\ \text{if}\ \ z_1(t)\ \text{is}\ M_{i1}\ \ \text{and} \ldots \text{and}\ \ z_n(t)\ \text{is}\ M_{in},\ \ \text{then}$$

$$\begin{cases} \dfrac{dx(t)}{dt} = A_i x(t), & t \neq \tau_k, \\ x(t^+) = B_i x(t), & t = \tau_k, \quad k = 1, 2, \ldots (k \in \mathbb{N}), \\ x(t_0^+) = x_0, \end{cases} \tag{6.1}$$

where $x(t) = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is the state vector, $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$ is the premise variable vector associated with the systems states and inputs, $x(t^+)$ is the right value of $x(t)$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n}$ are the system matrices, $M_{ij}(\cdot)$ are the membership functions of the fuzzy sets M_{ij} and r is the number of fuzzy rules. We suppose that B_i are non-singular matrices and $0 < \theta_1 \leq \tau_{k+1} - \tau_k \leq \theta_2 < \infty$.

We also suppose that at the moments of impulsive effects $\{\tau_k\}$ the solution $x(t)$ is left continuous, i.e., $x(\tau_k^-) = x(\tau_k)$.

The state equation can be defined as follows

$$\begin{cases} \dfrac{dx(t)}{dt} = \sum\limits_{i=1}^{r} \mu_i(z(t)) A_i x(t), & t \neq \tau_k, \\ x(t^+) = \sum\limits_{i=1}^{r} \mu_i(z(t)) B_i x(t), & t = \tau_k, \quad k \in \mathbb{N}, \\ x(t_0^+) = x_0, \end{cases} \tag{6.2}$$

where

$$\mu_i(z) = \frac{\omega_i(z)}{\sum_{i=1}^{r} \omega_i(z)} \quad \text{with} \quad \omega_i(z) = \prod_{j=1}^{n} M_{ij}(z_j).$$

Clearly $\sum\limits_{i=1}^{r} \mu_i(z) = 1$ and $\mu_i(z) \geq 0$, $i = \overline{1, r}$. Next, without loss of generality we take $z = x$.

The stability analysis in the sense of Lyapunov of zero solution $x = 0$ of system (6.2) is the aim of this section. Before the main results, the following assumption is made regarding the T-S fuzzy system (6.2).

Assumption 6.1 There exist $\gamma > 0$ and $\varepsilon > 0$ such that the functions $\mu_i(x)$ for system (6.2) satisfy the inequality $\|D_x^+\mu_i(x)\| \le \gamma\|x\|^{-1+\varepsilon}$, $i = \overline{1,r}$.

In this assumption $D_x^+\mu_i(x)$ denotes the upper Dini derivative of $\mu_i(x)$, i.e.

$$D_x^+\mu_i(x) = \limsup\{(\mu_i(x(t+\Delta)) - \mu_i(x(t)))/\Delta\colon\ \Delta \to 0\}.$$

Remark 6.1 It should be noted that Assumption 3 admits unique existence of solutions for system (6.2).

Let $\mathcal{E}$ denote the space of symmetric $n\times n$-matrices with scalar product $(X,Y) = \mathrm{tr}(XY)$ and corresponding norm $\|X\| = \sqrt{(X,X)}$, where $\mathrm{tr}(\cdot)$ denotes the trace of corresponding matrix. Let $K \subset \mathcal{E}$ be a cone of positive semi-definite symmetric matrices. Next we will define the following linear operators $\mathfrak{F}_i X = A_i^T X + XA_i$, $\mathfrak{B}_{ij}X = B_i^T XB_j$, for all $X \in \mathcal{E}$, $i,j = \overline{1,r}$.

Several theorems are first proved to demonstrate that if certain hypotheses are satisfied, the stability of the above nonlinear system can be obtained using the direct Lyapunov method. It is shown that stability conditions can be formulated in terms of Linear Matrix Inequalities.

Theorem 6.1 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1,\theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\left(\frac{1}{2}(\mathfrak{B}_{ji} + \mathfrak{B}_{ij}) - I + \sum_{k=1}^{p-1}\frac{(-1)^{k+1}(\mathfrak{F}_i)^k\theta^k}{k!}\right)X < 0, \quad i,j = \overline{1,r}, \tag{6.3}$$

$$(-1)^p(\mathfrak{F}_i)^p X \ge 0. \tag{6.4}$$

Before we prove Theorem 6.1 we have the following remark.

Remark 6.2 It should be noted that

(1) $(\mathfrak{F}_i)^p X = \mathfrak{F}_{i_1}\mathfrak{F}_{i_2}\dots\mathfrak{F}_{i_p}X$, where $i_1 = i$, $i_2 = j$, $i_1,\dots,i_p = \overline{1,r}$;

(2) for $i_1,\dots,i_p = \overline{1,r}$

$$\left(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\right)^p X = \sum_{i_p=1}^{r}\cdots\sum_{i_1=1}^{r}\mu_{i_p}(x)\dots\mu_{i_1}(x)\mathfrak{F}_{i_1}\mathfrak{F}_{i_2}\dots\mathfrak{F}_{i_p}X.$$

Proof Choose the Lyapunov function namely from class V_0, $V(t,x) = x^T P(t,x)x$, where

$$P(t,x) = \begin{cases} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)}X - \int\limits_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)}ds Q, & \text{for} \quad t \in (\tau_k,\tau_{k+1}],\\ X, & \text{for} \quad t = \tau_{k+1}^+. \end{cases}$$

Q and X are symmetric positive definite $n\times n$-matrices. Later we shall show that $P(t,x) \overset{K}{>} 0$ in some neighborhood of the origin. First let us consider the derivative of

$V(t,x)$ with respect to time. If $t \neq \tau_k$, then we have

$$D_t^+ V(t,x)\big|_{(6.2)} = x^T \sum_{i=1}^{r} \mu_i(x)(A_i^T P(t,x) + P(t,x)A_i)x + x^T D_t^+ P(t,x)x$$
$$= x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x + x^T D_t^+ P(t,x)x,$$

where

$$D_t^+ P(t,x)\big|_{(6.2)} = e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} \Big(-\sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-\tau_k) - \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i \Big) X$$
$$- \int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} \Big(-\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i - \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) \Big) dsQ - Q$$
$$= -\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i \Big(e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} X - \int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} ds\, Q \Big)$$
$$- e^{-\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} \times \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k)$$
$$+ \int_{\tau_k}^{t} e^{-\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q - Q$$
$$= -\sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t) - e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k)$$
$$+ \int_{\tau_k}^{t} e^{-\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q - Q.$$

Hence, for the derivative $D_t^+ V(t,x)\big|_{(6.2)}$, we have the estimates:

$$D_t^+ V(t,x)\big|_{(6.2)} = x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x - x^T \sum_{i=1}^{r} \mu_i(x)\mathfrak{F}_i P(t,x)x$$
$$- x^T Q x - x^T \Big[e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i X(t-\tau_k) \Big] x$$
$$+ x^T \Big[\int_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} \sum_{i=1}^{r} D_x^+ \mu_i(x) \frac{dx}{dt} \mathfrak{F}_i(t-s) ds\, Q \Big] x$$
$$\leq -\lambda_{\min}(Q)\|x\|^2$$
$$+ \theta_2 e^{\sum\limits_{i=1}^{r}\mu_i(x)\|\mathfrak{F}_i\|\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\| \, \|\mathfrak{F}_i\| \, \|X\| \left\| \frac{dx}{dt} \right\| \|x\|^2$$
$$+ \theta_2^2 e^{\sum\limits_{i=1}^{r}\mu_i(x)\|\mathfrak{F}_i\|\theta_2} \sum_{i=1}^{r} \|D_x^+ \mu_i(x)\| \, \|\mathfrak{F}_i\| \, \|Q\| \left\| \frac{dx}{dt} \right\| \|x\|^2,$$

where $\lambda_{\min}(\cdot) > 0$ is the minimal eigenvalue of corresponding matrix. Denote by $a = \max\limits_{i=\overline{1,r}} \|A_i\|$. Then since $\|\mathfrak{F}_i X\| \le \|A_i^T X + XA_i\| \le 2\|A_i\|\,\|X\|$ we get $\|\mathfrak{F}_i\| \le 2\|A_i\| \le 2a$, $i = \overline{1,r}$. It is also clear that

$$\Big\|\frac{dx}{dt}\Big\| \le \sum_{i=1}^{r} \mu_i(x)\|A_i\|\,\|x\| \le a\|x\|.$$

Hence the following inequality is fulfilled

$$\begin{aligned} D_t^+ V(t,x)\big|_{(6.2)} &\le -\lambda_{\min}(Q)\|x\|^2 + 2a^2\theta_2 e^{2a\theta_2} \sum_{i=1}^{r} \|D_x^+\mu_i(x)\|\,\|X\|\,\|x\|^3 \\ &+ 2a^2\theta_2^2 e^{2a\theta_2} \sum_{i=1}^{r} \|D_x^+\mu_i(x)\|\,\|Q\|\,\|x\|^3 \\ &\le \Big(-\lambda_{\min}(Q) + 2a^2 r\theta_2\gamma e^{2a\theta_2}\big(\,\|X\| + \theta_2\|Q\|\,\big)\|x\|^{\varepsilon}\Big)\|x\|^2. \end{aligned}$$

Therefore $D_t^+ V(t,x)\big|_{(6.2)} < 0$ for all x from the ball $\|x\| < R$, where

$$R = \left(\frac{\lambda_{\min}(Q)}{2a^2 r\theta_2\gamma e^{2a\theta_2}\big(\,\|X\| + \theta_2\|Q\|\,\big)}\right)^{1/\varepsilon}.$$

Consider the difference $\Delta V = V(t^+, x(t^+)) - V(t,x)$:

$$\begin{aligned} \Delta V\big|_{(6.2)} &= x^T(t^+)P(t^+)x(t^+) - x^T(t)P(t)x(t) = x^T(t^+)Xx(t^+) \\ &- x^T\Bigg(e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-\tau_{k-1})}X - \int\limits_{\tau_{k-1}}^{\tau_k} e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-s)}ds\,Q\Bigg)x \\ &\le x^T \sum_{j=1}^{r}\sum_{i=1}^{r} \mu_j(x)\mu_i(x) B_j^T X B_i x - x^T e^{-\sum\limits_{i=1}^{r}\mu_i(x(\tau_k))\mathfrak{F}_i(\tau_k-\tau_{k-1})} Xx \\ &+ x^T \int\limits_{0}^{\theta_2} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i y} dy\,Qx, \end{aligned}$$

where $y = \tau_k - s$.

Next we shall prove the following inequality

$$e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})}X \overset{K}{\ge} \left(I - \sum_{k=1}^{p-1} \frac{(-1)^{k+1}\left(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\right)^k(\tau_k-\tau_{k-1})^k}{k!}\right)X. \qquad (6.5)$$

Let us choose an arbitrary element $\Phi \in K^* = K$ and consider an expansion in a Maclaurin series of the scalar function

$$\begin{aligned} \psi_\Phi(h) = \mathrm{tr}\Bigg(\Phi\Bigg(&e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})h}X - X \\ &+ \sum_{k=1}^{p-1} \frac{(-1)^{k+1}\left(\sum_{i=1}^{r}\mu_i(x)\mathfrak{F}_i\right)^k(\tau_k-\tau_{k-1})^k h^k}{k!}X\Bigg)\Bigg), \end{aligned}$$

$h \geq 0$, restricting p-order terms

$$\psi_\Phi(h) = \psi_\Phi(0) + \psi'_\Phi(0)h + \dots + \frac{\psi_\Phi^{(p-1)}(0)h^{p-1}}{(p-1)!} + \frac{\psi_\Phi^{(p)}(\xi)h^p}{p!}, \quad \xi \in (0,h).$$

Let $h = 1$, then since $\psi_\Phi(0) = \psi'_\Phi(0) = \dots = \psi_\Phi^{(p-1)}(0) = 0$, we get $\psi_\Phi(1) = \dfrac{\psi_\Phi^{(p)}(\xi)}{p!}$, where

$$\psi_\Phi^{(p)}(\xi) = \operatorname{tr}\biggl(\Phi\biggl((-1)^p\biggl(\sum_{i=1}^r \mu_i(x)\mathfrak{F}_i(\tau_k - \tau_{k-1})\biggr)^p e^{-\sum\limits_{i=1}^r \mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})\xi} X\biggr)\biggr).$$

Inequality (6.4) and positivity of operator $e^{-\sum\limits_{i=1}^r \mu_i(x)\mathfrak{F}_i(\tau_k-\tau_{k-1})\xi}$ give estimate $\psi_\Phi^{(p)}(\xi) \geq 0$. Thus $\psi_\Phi(1) \geq 0$ for all $\Phi \in K^*$ and therefore inequality (6.5) is satisfied.

Consider the function

$$f_x(\theta_2) = x^T \int_0^{\theta_2} e^{-\sum\limits_{i=1}^r \mu_i(x)\mathfrak{F}_i y} dy\, Qx.$$

By Lagrange theorem we have

$$f_x(\theta_2) = f'_x(\zeta)\theta_2 = x^T\theta_2 e^{-\sum\limits_{i=1}^r \mu_i(x)\mathfrak{F}_i\zeta} Qx,$$

where $\zeta \in (0,\theta_2)$ and therefore

$$\|f_x(\theta_2)\| \leq \|x\|^2 e^{\sum\limits_{i=1}^r \mu_i(x)\|\mathfrak{F}_i\|\theta_2} \|Q\|\theta_2 \leq \theta_2 e^{2a\theta_2}\|Q\|\|x\|^2. \tag{6.6}$$

Inequalities (6.3), (6.5), (6.6) yield

$$\begin{aligned}
\Delta V\big|_{(6.2)} &\leq -x^T \sum_{i_1=1}^r \cdots \sum_{i_{p-1}=1}^r \mu_{i_1}(x)\dots\mu_{i_{p-1}}(x) Q_{i_1 i_2 \dots i_{p-1}} x + \theta_2 e^{2a\theta_2}\|Q\|\|x\|^2 \\
&\leq -\sum_{i_{p-1}=1}^r \cdots \sum_{i_1=1}^r \mu_{i_{p-1}}(x)\dots\mu_{i_1}(x)\lambda_{\min}(Q_{i_1 i_2\dots i_{p-1}})\|x\|^2 \\
&\quad + \theta_2 e^{2a\theta_2}\|Q\|\,\|x\|^2 \leq \bigl(-\lambda^* + \theta_2 e^{2a\theta_2}\|Q\|\bigr)\|x\|^2,
\end{aligned}$$

where $Q_{i_1 i_2 \dots i_{p-1}}$ are positive definite matrices,

$$\lambda^* = \min \lambda_{\min}(Q_{i_1 i_2 \dots i_{p-1}}), \quad i_1, \dots, i_{p-1} = \overline{1,r}.$$

It is clear that $\Delta V\big|_{(6.2)} \leq 0$ if $\|Q\| \leq \dfrac{\lambda^*}{\theta_2} e^{-2a\theta_2}$ (we can choose, for example, $Q = \dfrac{\lambda^*}{2\sqrt{n}\theta_2} e^{-2a\theta_2} I$).

Next we shall show that $P(t,x) \overset{K}{>} 0$ for all $t \in \mathbb{R}$ i.e., $V(t,x)$ is a positive definite function. Since $V(t,x)$ is decreasing function, we have for $\|x\| < R$ and $t \in [\tau_k, \tau_{k+1})$, $k \in \mathbb{N}$

$$\begin{aligned} x^T P(t,x)x &\geq x^T(\tau_{k+1})P(\tau_{k+1}, x(\tau_{k+1}))x(\tau_{k+1}) \\ &\geq x^T(\tau_{k+1}^+)P(\tau_{k+1}^+, x(\tau_{k+1}^+))x(\tau_k^+) \geq \lambda_{\min}(X)\|x(\tau_{k+1}^+)\|^2 > 0. \end{aligned}$$

As a result, we have $V(t,x) > 0$, $D_t^+ V(t,x)\big|_{(6.2)} < 0$ and $\Delta V\big|_{(6.2)} \leq 0$ for all $\|x\| < R$.

Therefore the zero solution of impulsive Takagi–Sugeno fuzzy system (6.2) is asymptotically stable. This completes the proof of Theorem 6.1.

Let p be fixed then we shall name the LMIs (6.3)–(6.4) by p-order stability conditions of system (6.2).

Next we shall formulate 2-nd order stability conditions of system (6.2).

Corollary 6.1 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\begin{aligned} \frac{1}{2}(B_j^T X B_i + B_i^T X B_j) - X + (A_j^T X + X A_j)\theta < 0, \qquad & i,j = \overline{1,r}, \\ A_i^T A_j^T X + X A_j A_i + A_j^T X A_i + A_i^T X A_j \geq 0, \qquad & i,j = \overline{1,r}. \end{aligned}$$

Suppose that fuzzy system (6.2) is such that $A_1 = A_2 = \cdots = A_n = A$. Then we have the following 4-th order stability conditions.

Corollary 6.2 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite matrix X such that*

$$\begin{aligned} &\frac{1}{2}\,(B_j^T X B_i + B_i^T X B_j) - X + (A^T X + XA)\theta \\ &\quad - \frac{1}{2}(\,(A^T)^2 X + 2A^T X A + X A^2\,)\theta^2 + \frac{1}{6}\theta^3\big((A^T)^3 X \\ &\quad + 3((A^T)^2 X A + A^T X A^2) + X A^3\big) < 0, \quad i,j = \overline{1,r}, \end{aligned} \tag{6.7}$$

$$(A^T)^4 X + 4((A^T)^3 X A + A^T X A^3) + 6(A^T)^2 X A^2 + X A^4 \geq 0. \tag{6.8}$$

Example 6.1 Let us consider the impulsive system (6.2) with the following system matrices

$$A_1 = A_2 = A = \begin{pmatrix} -2 & 0.5 \\ 0.4 & 0.1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1.1 & 0.1 \\ 0.2 & 0.2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.2 & 0.15 \\ 0.1 & 0.3 \end{pmatrix}.$$

Let the period of control action $\theta_1 = \theta_2 = \theta = 0.12$ and suppose that Assumption 6.1 holds. Then it is easy to check that matrix $X = \begin{pmatrix} 0.0756 & 0.0102 \\ 0.0102 & 0.3261 \end{pmatrix}$ satisfies LMIs (6.7), (6.8). Therefore by Corollary 6.2 the zero solution $x = 0$ of the considered fuzzy system is asymptotically stable.

Remark 6.3 It is easy to verify that 2-nd order stability conditions are not available to discuss stability analysis of the above fuzzy system.

Remark 6.4 It should be noted that it is impossible to take stability analysis of fuzzy system from Example 6.1 via paper [29] because the discrete components (matrices B_1 and B_2) are unstable and stability conditions from the paper are neglected. Note that matrix A is also unstable. So, our stability conditions are available to investigate the impulsive T-S fuzzy system in which continuous and discrete components may be all unstable.

Let $p \geq 2$ and $G_{i_1 i_2 \cdots_{p-1}}$ be positive definite matrices. Consider the following matrix equations for $i_1, \dots, i_{p-1} = \overline{1, r}$

$$\left(\frac{1}{2}(\mathfrak{B}_{ji} + \mathfrak{B}_{ji}) - I + \sum_{k=1}^{p-1} \frac{(-1)^{k+1} (\mathfrak{F}_i)^k \theta^k}{k!}\right) X = -G_{i_1 i_2 \dots i_{p-1}}. \tag{6.9}$$

Similarly to Theorem 6.1 we have the following result.

Theorem 6.2 *Under Assumption 6.1 the equilibrium state $x = 0$ of fuzzy system (6.2) is asymptotically stable if for all $\theta \in [\theta_1, \theta_2]$ there exists a common symmetric positive definite solution X of (6.9) such that the following inequality is fulfilled*

$$e^{2a\theta} \frac{(2a\theta)^p}{p!} < \frac{\lambda^*}{\|X\|},$$

where $a = \max\limits_{i=\overline{1,r}} \|A_i\|$, $\lambda^* = \min \lambda_{\min}(G_{i_1 i_2 \dots i_{p-1}})$ for $i_1, \dots, i_{p-1} = \overline{1, r}$.

Next, we state the following assumption.

Assumption 6.2 *There exist $R_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ and $\varepsilon > 0$ such that the functions $\mu_i(x)$, $i = \overline{1, r}$, satisfy the inequality*

$$\|D_x^+ \mu_i(x)\| \leq \begin{cases} \gamma_1 \|x\|^{-1+\varepsilon}, & \textit{for} \quad \|x\| \leq R_0, \\ \gamma_2 \|x\|^{-1-\varepsilon}, & \textit{for} \quad \|x\| \geq R_0. \end{cases}$$

Taking into account Assumption 6.2 we can establish the following.

Theorem 6.3 *Let in Assumption 6.2 constants γ_1, γ_2, R_0 be such that*

$$\gamma_1 \gamma_2 < \frac{\lambda^2_{\min}(Q)}{4a^4 r^2 \theta_2^2 e^{4a\theta_2} (\,\|X\| + \theta_2 \|Q\|\,)^2}$$

and

$$\left(\frac{\lambda_{\min}(Q)}{2a^2 r \theta_2 \gamma_2 e^{2a\theta_2} \big(\,\|X\| + \theta_2 \|Q\|\,\big)}\right)^{-1/\varepsilon} < R_0,$$

$$R_0 < \left(\frac{\lambda_{\min}(Q)}{2a^2 r \theta_2 \gamma_1 e^{2a\theta_2} \big(\,\|X\| + \theta_2 \|Q\|\,\big)}\right)^{1/\varepsilon},$$

where $a = \max\limits_{i=\overline{1,r}} \|A_i\|$, Q is a symmetric positive definite $n \times n$- matrix and X is a common symmetric positive definite matrix such that conditions (6.3), (6.4) of Theorem 6.1 hold. Then the zero solution of impulsive fuzzy system (6.2) is globally asymptotically stable.

Proof Choose for a candidate the Lyapunov function from class V_0, $V(t,x) = x^T P(t,x)x$, where

$$P(t,x) = \begin{cases} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-\tau_k)} X - \int\limits_{\tau_k}^{t} e^{-\sum\limits_{i=1}^{r}\mu_i(x)\mathfrak{F}_i(t-s)} ds\, Q, & \text{for} \quad t \in (\tau_k, \tau_{k+1}], \\ X, & \text{for} \quad t = \tau_{k+1}^{+}, \end{cases}$$

where Q and X are symmetric positive definite $n \times n$-matrices. Let us consider the derivative of $V(t,x)$ with respect to time (notice that $V(t,x)$ is radially unbounded function). If $t \neq \tau_k$ then we have two cases:

(1) if $\|x\| \leq R_0$ then similar to the proof of Theorem 6.1 we get

$$D_t^+ V(t,x)\big|_{(6.2)} \leq \big(-\lambda_{\min}(Q) + 2a^2 r\theta_2 e^{2a\theta_2}\gamma_1\big(\|X\| + \theta_2\|Q\|\big)\|x\|^{\varepsilon}\big)\|x\|^2.$$

Clearly $D_t^+ V(t,x)\big|_{(6.2)} < 0$ by conditions of Theorem 6.3;

(2) if $\|x\| \geq R_0$ then by analogy we get

$$D_t^+ V(t,x)\big|_{(6.2)} \leq \big(-\lambda_{\min}(Q) + 2a^2 r\theta_2 e^{2a\theta_2}\gamma_2\big(\|X\| + \theta_2\|Q\|\big)\|x\|^{-\varepsilon}\big)\|x\|^2.$$

Clearly $D_t^+ V(t,x)\big|_{(6.2)} < 0$ by conditions of Theorem 6.3. Thus we have showed that $D_t^+ V(t,x)\big|_{(6.2)} < 0$ for all $x \in \mathbb{R}^n$.

Similar to the proof of Theorem 6.1 we can show (taking into account the conditions of Theorem 6.3) that $\Delta V\big|_{(6.2)} = V(t^+, x(t^+)) - V(t,x) \leq 0$ and $V(t,x) > 0$. Therefore the zero solution of impulsive Takagi–Sugeno fuzzy system (6.2) is globally asymptotically stable.

Remark 6.5 In spite of advantages of LMI method, the existence of solution that satisfies the sufficient conditions is not guaranteed. This happens when the number of fuzzy rules is increased or too many system's matrices are imposed.

Remark 6.6 The result of this section can be utilized on chaotic, inverted pendulum, biological, electrical dynamical systems etc. Moreover in practice it is enough to verify(using, for example Matlab LMI toolbox) 2-nd order or 4-th order stability conditions.

6.2 Impulsive Fuzzy Control for Ecological Prey–Predator Community

It is well-known that control problem is an important task for mathematical theory of artificial ecosystems. Impulsive control of such systems is more favorable due to seasonal functioning of this type of systems. Some problem of impulsive control for homotypical model has been considered in the paper [15]. But for practice it is suitable to consider models with fuzzy impulsive control because it is almost impossible to accurately measure the biomass of one or another biological species but possible to roughly estimate those.

Consider a Lotka–Volterra type prey-predator model (with interspecific competition among preys) whose evolution is described by the following equations

$$\begin{aligned} \frac{dN_1}{dt} &= \alpha N_1 - \beta N_1 N_2 - \gamma N_1^2, \\ \frac{dN_2}{dt} &= -mN_2 + s\beta N_1 N_2, \end{aligned} \tag{6.10}$$

where $N_1(t)$ is the biomass of preys, $N_2(t)$ is the biomass of predators, α is the growth rate of the preys, m is the death rate of the predators, γ is the rate of the interspecific competition among preys, β is the per-head attack rate of the predators, and s is the efficiency of converting preys to predators.

Suppose that the ecosystem is controlled via regulation of the number of species at certain fixed moments of time(impulsive control) $\theta, 2\theta, \dots, k\theta, \dots$ and the regulation is reduced either to elimination or fulmination of the representatives of species. Taking into account these assumptions we have to add the regulator equations to the system of the evolution as

$$\begin{aligned}
&\Delta N_1 = u_1(N_1, N_2),\\
&\Delta N_2 = u_2(N_1, N_2), \quad t = k\theta, \quad k \in \mathbb{N},
\end{aligned}$$

where u_1, u_2 are feedback functions, θ is a period of control action.

Under these assumptions the equations of closed controlled ecosystem become

$$\begin{aligned}
&\frac{dN_1}{dt} = \alpha N_1 - \beta N_1 N_2 - \gamma N_1^2,\\
&\frac{dN_2}{dt} = -mN_2 - s\beta N_1 N_2, \quad t \neq k\theta,\\
&\Delta N_1 = u_1(N_1, N_2),\\
&\Delta N_2 = u_2(N_1, N_2), \quad t = k\theta, \quad k \in \mathbb{N}.
\end{aligned} \tag{6.11}$$

Besides the trivial equilibrium state, equation (6.10) has also the positive asymptotically stable states

$$N_1^* = \frac{m}{s\beta}, \quad N_2^* = \frac{s\alpha\beta - m\gamma}{s\beta^2}.$$

It is clear that if the number of preys is much greater than the equilibrium ones then some amount of preys is eliminated and vice versa. Analogous situation occurs with the predators. Thus, the impulsive fuzzy controls are designed regarding the rules:

if $N_i \ll N_i^*$, then $u_i(N_1, N_2) = \psi_i(N_i^* - N_i), \quad \psi_i > 0, \qquad i = 1, 2;$
if $N_i \gg N_i^*$, then $u_i(N_1, N_2) = \chi_i(N_i^* - N_i), \quad \chi_i \in (0, 1), \; i = 1, 2,$

where ψ_i are the fulmination rates, χ_i are the elimination rates.

The fuzzy relation $x \gg y$ ("x is much larger than y") can be formalized using the following membership function

$$\omega(x, y) = \begin{cases} \dfrac{1}{1 + 1/(x-y)^2}, & \text{if } x > y,\\ 0, & \text{if } x \le y. \end{cases}$$

Next, we define the variables of disturbance of motion $x_1(t) = N_1(t) - N_1^*$, $x_2(t) = N_2(t) - N_2^*$. Then the equations for system (6.11) become (using linearization):

$$\begin{cases}
\dfrac{dx_1}{dt} = -\dfrac{m\gamma}{s\beta} x_1 - \dfrac{m}{s},\\
\dfrac{dx_2}{dt} = -\dfrac{\alpha\beta s - m\gamma}{\beta} x_1, \quad t \neq k\theta,\\
\Delta x_1 = u_1,\\
\Delta x_2 = u_2, \quad t = k\theta.
\end{cases} \tag{6.12}$$

The Takagi–Sugeno fuzzy model (6.1) of system (6.12) is specified by the following four rules:

R^1: if $N_1 \ll N_1^*$ and $N_2 \ll N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_1 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^2: if $N_1 \ll N_1^*$ and $N_2 \gg N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_2 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^3: if $N_1 \gg N_1^*$ and $N_2 \gg N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_3 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

R^4: if $N_1 \gg N_1^*$ and $N_2 \ll N_2^*$, then

$$\begin{cases} \dfrac{dx(t)}{dt} = Ax(t), & t \neq k\theta, \\ x(t^+) = B_4 x, & t = k\theta, \\ x(t_0^+) = x_0. \end{cases}$$

It is obvious that Assumption 6.1 holds for membership function $\omega(x, y)$. Using Corollary 6.1 the stability analysis of nontrivial equilibrium position for ecosystem is reduced to checking the existence of symmetric positive definite matrix X such that the following LMIs hold true:

$$\begin{gathered} \frac{1}{2}(B_i^T X B_j + B_j^T X B_i) - X + (A^T X + XA)\theta < 0, \quad i, j = \overline{1, 4}, \\ (A^T)^2 X + 2A^T X A + X A^2 \geq 0. \end{gathered} \tag{6.13}$$

Matrices A, B_1, B_2, B_3 and B_4 are as follows

$$A = \begin{pmatrix} -\dfrac{m\gamma}{s\beta} & -\dfrac{m}{s} \\ \dfrac{\alpha\beta s - m\gamma}{\beta} & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1-\psi_1 & 0 \\ 0 & 1-\psi_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1-\psi_1 & 0 \\ 0 & 1-\chi_2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 1-\chi_1 & 0 \\ 0 & 1-\chi_2 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1-\chi_1 & 0 \\ 0 & 1-\psi_2 \end{pmatrix}.$$

Next we consider the stability analysis of the obtained Takagi–Sugeno fuzzy model for ecosystem's evolution with the following parameters: $\alpha = 4$, $\gamma = 0.3$, $\beta = 0.5$, $m = 1.2$, $s = 0.4$, $\theta = 0.5$ and the parameters of impulsive control: $\psi_1 = 0.9$, $\psi_2 = 0.5$, $\chi_1 = 0.99$, $\chi_2 = 0.6$.

It is easy to check that matrix $X = \begin{pmatrix} 1.7427 & 1.8779 \\ 1.8779 & 8.2018 \end{pmatrix}$ satisfies inequalities (6.13). Therefore by Corollary 6.1 the equilibrium state of ecological system is asymptotically stable (see Figure 6).

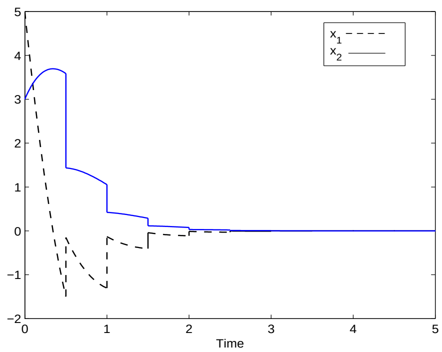

Figure 6. Evolution of $x_1(t)$ and $x_2(t)$ (stable result).

Let us change the parameters of impulsive control: $\psi_1 = 6$, $\psi_2 = 4$, $\chi_1 = 0.6$, $\chi_2 = 0.2$. In this case the solution of LMIs (6.13) is infeasible and computer simulation gives an unstable result (see Figure 7).

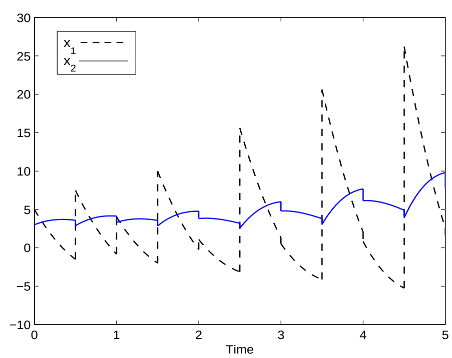

Figure 7. Evolution of $x_1(t)$ and $x_2(t)$ (unstable result).

Based on the well-known Lyapunov direct method, sufficient conditions have been derived to guarantee the asymptotic stability and globally asymptotic stability of the equilibrium point of impulsive T-S fuzzy systems. It is shown that these sufficient conditions are expressed easily as a set of LMIs. It is also concluded that the obtained stability conditions allow to investigate the impulsive T-S fuzzy system in which continuous and discrete components may be all unstable.

7 Conclusion

The results given in Section 2 are adopted from Martynyuk and Chernienko [21]. The model of robot interacting with dynamic environment is due to DeLuca and Manes [2]. It should be noted that the importance of studying the problem of stability of motion of a robot interacting with a dynamic environment was discussed in contemporary literature.

The contents of Section 3 are essentially new (see Martynyuk and Lukyanova [22]). For continuous neural networks see Hopfield [7], Wang and Michel [27], etc. and for discrete-time neural networks see Michel, Farrel and Sun [24], etc.

Section 4 is based on the results by Lila and Martynyuk [12, 13]. We note that the approach proposed for stability analysis of periodic solutions of system (4.1) can be extended for the cases where the presence of phase of coefficient of optic constraint between the lasers exists, neutral stability in linear approximation and in the study of dynamics of many-modulus systems.

In Section 5 the model (5.2) is taken from the monograph by Forrester [5]. The model (5.6) is a new. Theorem 5.1 is taken from Martynuyuk [19]. Some other models of the world dynamics are in Egorov *et al.* [4], Levashov [9], etc.

Section 6 is adapted from Denysenko, Martynyuk and Slyn'ko [3].

Acknowledgements

I would like to thank Professors S. Leela and V. Lakshmikantham for their careful reading the text and advice, which has proved to be very important for me. I also thank the colleagues in the Department of Processes Stability of the S.P. Timoshenko Institute of Mechanics for their help in preparing this paper.

References

[1] Barbashin, E.A. *Introduction to the Theory of Stability.* Nauka, Moscow, 1967.

[2] De Luca, F. and Manes, C. Hybrid force/position control for robots on contact with dynamic environments. *Proc. Robot Control, SYROCO'91*, 1988, 377–382.

[3] Denysenko, V.S., Martynyuk, A.A. and Slyn'ko, V.I. Stability analysis of impulsive Takagi-Sugeno systems. *International Journal of Innovative Computing, Information and Control* **5** (10A) (2009) 3141–3155.

[4] Egorov, V.A., Kallistov, Yu.N., Mitrofanov, V.B., and Piontkovski, A.A. *Mathematical Models of Sustainable Development.* Gidrometeoizdat, Leningrad, 1980. [Russian]

[5] Forrester, J.W. *World Dynamics.* Nauka, Moscow, 1978. [Russian]

[6] Hilger, S. Analysis on measure chains — a unified approach to continuous and discrete calculus. *Result. Math.* **18** (1/2) (1990) 18–56.

[7] Hopfield, J.J. Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. Nat. Acad. of Science USA* **81** (1984) 3088–3092.

[8] Lakshmikantham, V., Leela, S. and Martynyuk A.A. *Stability Analysis of Nonlinear systems.* New York, Marcel Dekker, 1989.

[9] Levashov, V.K. *Sustainable Development of Society.* Moscow, Academia, 2001. [Russian]

[10] Likhanski, V.V. and Napartovich, A.P. Radiation emitted by optically coupled lasers. *Usp. Phys. Nauk* **160** (3) (1990) 101–143. [Russian]

[11] Likhanski, V.V., Napartovich, A.P. and Sykharev, A.G. Phase locking of optically coupled and periodically pumped lasers. *Kvantovaya Electronika* **22** (1) (1995) 47–48. [Russian]

[12] Lila, D.M. and Martynyuk, A.A. Stability of periodic motions of quasilinear systems. *Int. Appl. Mech.* **44** (2008) 1161–1172.

[13] Lila, D.M. and Martynyuk, A.A. On stability of some solutions for equations of locked lasing of optically coupled lasers with periodical pumping. *Nonlinear oscillations* **12** (4) (2009) 464–473.

[14] Lila, D.M. and Martynyuk, A.A. Setting up Lyapunov functions for the class of systems with quasiperiodic coefficients. *Int. Appl. Mech.* **44** (12) (2008) 1421–1429.

[15] Liu, X. Progress in stability of impulsive systems with applications to populations growth models. In: *Advances in Stability Theory on the End of 20th Century* (Ed.: A.A. Martynyuk). Stability and Control: Theory, Methods and Applications, Taylor and Francis, London, Vol. 13, 2003, 321–338.

[16] Martynyuk, A.A. *Stability by Liapunov's Matrix Functions Method with Applications.* Marcel Dekker, New York, 1998.

[17] Martynyuk, A.A. *Stability of Motion: The Role of Multicomponent Liapunov Functions.* Cambridge Scientific Publishers, Cambridge, 2007.

[18] Martynyuk, A.A. On a generalization of Richardson's model of the arms race. *Dokl. Akad. Nauk* **339** (1) (1994) 15–17. [Russian]

[19] Martynyuk, A.A. The models of the world dynamics and sustainable development. *Dokl. Nats. Akad. Nauk Ukr.* (7) (2010) 16–21. [Russian]

[20] Martynyuk, A.A. On exponential stability on time scale. *Dokl. Akad. Nauk* **421** (3) (2008) 312–317. [Russian]

[21] Martynyuk, A.A. and Chernienko A.N. To the theory of motion stability of robot interacting with dynamic environment. *Electronnoje Modelirovanije* **21** (5) (1999) 3–15. [Russian]

[22] Martynyuk, A.A. and Lukyanova T.A. On stability of the neural networks on time scales. *Dokl. Nats. Akad. Nauk Ukr.* (1) (2010) 21–26. [Russian]

[23] Meadows, D.L. and Meadows, D.H. *Toward Global Equilibrium.* Wright-Allen Press, Cambridge, 1973.

[24] Michel, A.N., Jay, A., Farrell, A. and Hung-Fa Sun. Analysis and synthesis techniques for Hopfield type synchronous discrete time neural networks with application to associative memory. *IEEE Trans. Circ. and Syst.* **37** (11) (1990) 1356–1366.

[25] Samoilenko, A.M. and Ronto, V.I. *Numerical-Analitical Methods of Investigation of Periodic Solutions.* Vussh. Shkola, Kiev, 1976. [Russian]

[26] Strauss, A. and Yorke, J. Perturbation theorems for ordinary differential equations. *J. Diff. Eqns.* **3** (1967) 15–30.

[27] Wang, K. and Michel, A.N. Robustness and perturbation analysis of a class of artificial neural networks. *Neural Networks* **7** (2) (1994) 251–259.

[28] Zhang, J. Global stability analysis in Hopfield neural networks. *Appl. Math. Let.* **16** (2003) 925–931.

[29] Zhang, X., Li, D. and Dan, Y. Impulsive control of Takagi–Sugeno fuzzy systems. *Fourth Int. Conf. on Fuzzy Systems and Knowledge Discovery.* Vol. 1, 2007, 321–325.

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 53–64

Stability of Hybrid Mechanical Systems with Switching Linear Force Fields

A.Yu. Aleksandrov [1*], Y. Chen [2], A.A. Kosov [3] and L. Zhang [2]

[1] *St. Petersburg State University, 35 Universitetskij Pr., Petrodvorets, St. Petersburg, 198504, Russia*
[2] *Beijing University of Technology, 100 Pingleyuan, Chaoyang, Beijing 100124, China*
[3] *Institute of System Dynamics and Control Theory, SB of RAS, 134 Lermontova Str., Irkutsk, 664033, Russia*

Received: August 26, 2010; Revised: January 24, 2011

Abstract: Linear hybrid mechanical systems with switchings of force fields are studied. Some sufficient conditions are brought forward for the switched systems being asymptotically stable for any switched law. The results are obtained based on two approaches. The first one is called as the decomposition method, and the second one consists in an explicit construction of the common Lyapunov functions for the families of systems corresponding to the switched systems. Different cases of domination concerning one of the force field components (e.g., velocity, gyroscopic, dissipative, potential) are considered.

Keywords: *hybrid mechanical systems; switched systems; stability; decomposition; common Lyapunov functions.*

Mathematics Subject Classification (2000): 34A38, 34D20, 70H14.

1 Introduction

The stability analysis of hybrid systems which are described by differential equations with switching right-hand sides is one of the most important problems in modern automatic control theory [3–5, 9, 15]. In various cases, after the design of continuous controller has been finished, it is required to verify the stability of the closed system for any admissible switching law [7, 9, 17]. Such a situation naturally arises when the switching law is either unknown or is too complex to consider in the stability investigation.

* Corresponding author: mailto:alex43102006@yandex.ru

A well-known approach for the stability analysis is to construct a common Lyapunov function for the family of subsystems corresponding to the switched system. That is, the function is positive and monotonically decreases along the solutions to each subsystem from the family. However, the problem of the existence of a common Lyapunov function is not completely solved until to now even for families of linear stationary systems [7, 8]. Only in some special cases, e.g., for two-dimensional or three-dimensional linear systems [14, 15], necessary and sufficient conditions for the existence of common quadratic Lyapunov function are found. For the linear systems with higher dimension, the existence of common quadratic Lyapunov function is proved only under some additional conditions, for instance, under commutativity of systems matrices [13].

This paper deals with mechanical systems with switching force fields. The switchings can be caused by both intrinsic reasons, such as using computer or microprocessor in control loop, and external reasons, for instance, when movement of mechanical system occurs in environment with changeable resistance [4, 6, 10, 15]. Motion of mechanical systems is usually described by differential equations of the second order, that results in the occurrence of some special properties. In particular, in the presence of switchings in acting force field, conditions of commutativity will be obviously broken. Therefore, the corresponding results based on commutativity of systems matrices for the existence of common quadratic Lyapunov functions are nonapplicable to mechanical systems. This motives us to study extendedly the problem of the existence of common Lyapunov functions for mechanical systems.

In the paper, we present two approaches for constructing common Lyapunov functions for mechanical systems with switching force fields. The first one is to decompose the original system consisting of n differential equations of the second order into two first-order subsystems of the same dimension. The approach is also available for the systems without switchings in the force fields since it allows one to solve stability problem on the basis of analysis of matrices of twice smaller dimensions than that in the original system. In the presence of switchings, decomposition makes it possible to use the conditions of matrix coefficients commutativity, which guarantees the existence of a common quadratic Lyapunov function for the family of systems corresponding to the switched system. The second approach is to give out an explicit construction of the common Lyapunov functions for the mechanical systems. These Lyapunov functions are constructed on the base of elements possessing clear mechanical meaning. It should be noted that in certain situations both approaches stated above are practically close to each other and lead to similar stability conditions.

In sum, this paper provides some stability conditions on the basis of construction of the common Lyapunov functions with essential use of mechanical system specificity. This specificity impels us to investigate this special subclass of hybrid systems not following the ordinal line of thought. Thus, the results obtained possess certain theoretical features and are of undoubted practical interest.

2 Statement of the Problem

Consider a family of linear systems

$$A\ddot{q} + F_s\dot{q} + C_s q = 0, \quad s = 1, \dots, N, \tag{1}$$

where q and $\dot{q}$ are n-dimensional vectors of generalized coordinates and generalized velocities, respectively; A, F_s, C_s are constant matrices, and matrix A is nonsingular. A switching law is the piecewise constant function $\sigma : [0, +\infty) \to D = \{1, \dots, N\}$. Thus, the switched system generated by the family (1) and a switching signal σ is

$$A\ddot{q} + F_\sigma \dot{q} + C_\sigma q = 0. \tag{2}$$

In this paper, we assume that on every bounded time interval, the switching function has finite number of discontinuities, which are called switching instants of time, and takes a constant value on every interval between two consecutive switching instants. This kind of switching law is called admissible.

We will look for the conditions to guarantee the switched system (2) is asymptotically stable for any admissible switching law. As is well known, it is sufficient [7, 9] to construct a common Lyapunov function for the family (1) such that it satisfies the assumptions of Lyapunov asymptotic stability theorem.

Linear systems (1) can be represented in the form

$$\dot{x} = P_s x, \quad s = 1, \dots, N, \tag{3}$$

where $x = (q^T, \dot{q}^T)^T$,

$$P_s = \begin{pmatrix} 0 & I \\ -A^{-1}C_s & -A^{-1}F_s \end{pmatrix},$$

and I denotes the identity matrix. Thus, one can investigate the stability of (3) for the general switched linear systems, and some well known conditions of the existence of a common quadratic Lyapunov function [8, 9, 13] can be used.

However, we point out that such approach is not always effective, partly owing to the following difficulties:

1) The transformation of (1) to the form (3) lose the mechanical meaning of the conditions;

2) The dimension of (3) becomes higher;

3) Systems (1) possess a special structure, therefore known results obtained for the linear switched systems of general form may be nonapplicable for (3).

For instance, commutativity of matrices $P_1, \dots, P_N$ in the family (3) is a simple condition of the existence of common Lyapunov function [13]. But due to the special structure of matrices of $P_1, \dots, P_N$ in systems (3), the commutativity results in the equalities $F_s = F_r$, $C_s = C_r$, $s, r = 1, \dots, N$, which is a trivial case.

In the paper, we consider two approaches for the stability analysis of switched mechanical system (2). The first one is based on a decomposition procedure, and the second one consists in an explicit construction of the Lyapunov functions for the switched system.

3 Decomposition Approach for Stability Analysis of Linear Mechanical Systems

In the section, we consider the decomposition conditions for mechanical systems without switchings.

3.1 Systems with the domination of velocity forces

The systems with the domination of velocity forces are described by the following differential equations

$$A\ddot{q} + hF\dot{q} + Cq = 0. \tag{4}$$

This kind of equations are generally considered as the linearized ones of motions for gyroscopic systems [18], where q and $\dot{q}$ are n-dimensional vectors of generalized coordinates and generalized velocities respectively; A, F and C are constant matrices; h is a large positive parameter. We assume that all the matrices in (4) are nonsingular.

System (4) is linear and stationary one. Therefore, some well-known criteria, for instance, the Hurwitz criterion or equivalent ones [12], can be used to determine the stability conditions for this system. However, in the case of high dimension of (4) or under the uncertainties in the matrices A, F, C this approach may be inefficient or even nonapplicable in practice.

Another way to investigate the stability in such situations is to decompose the system into several simpler systems, to study each of them separately, and then to appropriately apply the obtained results to the original system [1, 16].

V. I. Zubov has proposed the following result, which allows one to decompose, for sufficiently large values of parameter h, the problem of stability analysis of system (4) consisting of n differential equations of the second order into two analogical problems for the first-order systems of the same dimension.

Theorem 3.1 [18] *Let the following isolated subsystems*

$$F\dot{y} + Cy = 0, \tag{5}$$

$$A\dot{z} + Fz = 0 \tag{6}$$

be asymptotically stable. Then there exists $h_0 > 0$ *such that for any* $h > h_0$ *system* (4) *is also asymptotically stable.*

In applications, it is important to get the estimation of the lower bound h_0 for admissible values of h. Theorem 3.1 in [18] was proved on the base of the first Lyapunov method and by means of the expansion of the roots of the characteristic equation for (4) in the series with respect to the negative powers of h. However, this process did not give constructive estimation of h_0 value.

In this paper, we suggest another approach to prove Theorem 3.1, which is based on using Lyapunov direct method. The new proof contains a constructive procedure for determining the set of admissible values of large parameter h.

Proof Making the substitution of variables

$$\dot{q} = z, \qquad A\dot{q} + hFq = hFy, \tag{7}$$

we transform (4) into

$$\begin{aligned} F\dot{y} &= -\tfrac{1}{h}Cy + \tfrac{1}{h^2}CF^{-1}Az, \\ A\dot{z} &= -hFz - Cy + \tfrac{1}{h}F^{-1}Az. \end{aligned} \tag{8}$$

From the asymptotic stability of isolated subsystems (5) and (6), it follows [1] the existence of quadratic forms $V_1(y)$ and $V_2(z)$ such that the inequalities

$$a_{11}\|y\|^2 \le V_1 \le a_{12}\|y\|^2, \qquad a_{21}\|z\|^2 \le V_2 \le a_{22}\|z\|^2,$$

$$\left\|\frac{\partial V_1}{\partial y}\right\| \le a_{13}\|y\|, \qquad \left\|\frac{\partial V_2}{\partial z}\right\| \le a_{23}\|z\|, \qquad \dot{V}_1\big|_{(5)} \le -a_{14}\|y\|^2, \qquad \dot{V}_2\big|_{(6)} \le -a_{24}\|z\|^2$$

are valid for any $y, z \in R^n$, where a_{ij} are positive constants, $i = 1, 2$, $j = 1, 2, 3, 4$. Construct the function $V(y, z) = \varepsilon h^2 V_1(y) + V_2(z)$, where ε is a positive parameter. Differentiating $V(y, z)$ along the solutions to (8), we get that the inequality

$$\dot{V}\Big|_{(8)} \leq -a_{14}\varepsilon h\|y\|^2 - \left(ha_{24} - \frac{b_1}{h}\right)\|z\|^2 + (b_2\varepsilon + b_3)\|y\|\|z\|$$

holds for any $y, z \in R^n$, where $b_1 = a_{23}\|A^{-1}CF^{-1}A\|$, $b_2 = a_{13}\|F^{-1}CF^{-1}A\|$, $b_3 = a_{23}\|A^{-1}C\|$. Hence, if the condition

$$h > \sqrt{\frac{(b_2\varepsilon + b_3)^2}{4\varepsilon a_{14}a_{24}} + \frac{b_1}{a_{24}}} \tag{9}$$

is satisfied, function $\dot{V}\Big|_{(8)}$ is negative definite.

To complete the proof, it remains to find a $\varepsilon_0 > 0$ such that for $\varepsilon = \varepsilon_0$ (9) gives us the largest region of admissible values of h. It is easy to show that $\varepsilon_0 = b_3/b_2$, and estimation (9) becomes $h > \sqrt{b_2 b_3/(a_{14}a_{24}) + b_1/a_{24}}$. □

3.2 Systems with the domination of gyroscopic forces

Along with (4), the following equations

$$A\ddot{q} + (B + hG)\dot{q} + Cq = 0 \tag{10}$$

are also used as a linear approximation for the equations of motions of gyroscopic systems [11], where $q, \dot{q} \in R^n$; A, B, G, C are constant matrices; h is a large positive parameter. It is assumed [11] that A is symmetric and positive definite matrix of inertial characteristics; B is symmetric matrix of dissipative and accelerating forces; G is skew-symmetric and nonsingular matrix of gyroscopic forces. Thus, the dominating forces in (4) are the velocity ones, while in (10) they are the gyroscopic ones.

The conditions of decomposition for (10) have been established in [11]. As mentioned above, in [11] as well as in [18], for justifying the possibility of decomposition, the first Lyapunov method was used, and the constructive estimation for the admissible values of large parameter was not obtained.

Next we propose the same approach based on Lyapunov direct method as in the proof of Theorem 3.1 to study the stability analysis of system (10).

Consider the isolated subsystems

$$G\dot{y} + Cy = 0, \tag{11}$$

$$A\dot{z} + (B + hG)z = 0. \tag{12}$$

Theorem 3.2 *Let the matrix B be positive definite and subsystem* (11) *be asymptotically stable. Then there exists $h_0 > 0$ such that for any $h > h_0$ system* (10) *is also asymptotically stable.*

Proof By using the substitution of variables $\dot{q} = z$, $A\dot{q} + (B + hG)q = (B + hG)y$, we transform (10) into the following system

$$\begin{aligned} \dot{y} &= -\tfrac{1}{h}G^{-1}Cy + \tfrac{1}{h}(B + hG)^{-1}BG^{-1}Cy + (B + hG)^{-1}C(B + hG)^{-1}Az, \\ A\dot{z} &= -(B + hG)z - Cy + C(B + hG)^{-1}Az. \end{aligned} \tag{13}$$

From the asymptotic stability of (11), it follows that for this subsystem there exists a quadratic Lyapunov function $V_1(y)$ satisfying all the assumptions of the Lyapunov asymptotic stability theorem. If the matrix B is positive definite, then subsystem (12) is asymptotically stable for any $h > 0$, and let $V_2(z) = z^T A z$ be its Lyapunov function.

Construct the function $V(y,z) = \varepsilon h^2 V_1(y) + V_2(z)$, $\varepsilon = const > 0$. Let $\bar{h}$ be a positive number. Differentiating $V(y,z)$ along the solutions to (13), one gets that the inequality

$$\dot{V}\Big|_{(13)} \leq -\varepsilon(a_1 h - a_2)\|y\|^2 - \left(a_3 - \frac{a_4}{h}\right)\|z\|^2 + (a_5\varepsilon + a_6)\|y\|\|z\|$$

holds for $h \geq \bar{h}$ and for all $y, z \in R^n$, where a_i are positive constants, $i = 1, \ldots, 6$. It should be noted that values of a_2, a_4 and a_5 depend on the chosen value of $\bar{h}$. Hence, if the conditions $h \geq \bar{h}$, $h > a_2/a_1$ and

$$(a_1 h - a_2)\left(a_3 - \frac{a_4}{h}\right) > \frac{(a_5\varepsilon + a_6)^2}{4\varepsilon} \tag{14}$$

are satisfied, function $\dot{V}\Big|_{(13)}$ is negative definite.

It is easy to verify that for $\varepsilon = a_6/a_5$ inequality (14) gives out the largest region of admissible values h: $(a_1 h - a_2)(a_3 - a_4/h) > a_5 a_6$. □

4 Decomposition of Switched Mechanical Systems

Now we turn to consider the linear mechanical system with switching positional forces

$$A\ddot{q} + hF\dot{q} + C_\sigma q = 0. \tag{15}$$

The corresponding family of systems are

$$A\ddot{q} + hF\dot{q} + C_s q = 0, \quad s = 1, \ldots, N, \tag{16}$$

where $q, \dot{q} \in R^n$; A, F, C_s are constant nonsingular matrices; h is a large positive parameter.

The decomposition method stated above is still used to obtain the asymptotic stability conditions for (15). We point out that in this case the approach suggested in [18] for justifying the possibility of decomposition can not anymore be used for switched system (15) since the negativeness of real parts of all roots of characteristic equations for systems (16) does not provide asymptotic stability of (15) [9].

Now we show that the approach proposed in the proof of Theorem 3.1 allows us to obtain decomposition conditions for the systems with switching positional forces.

Consider the isolated subsystem

$$A\dot{z} + Fz = 0 \tag{17}$$

and the family of isolated subsystems

$$F\dot{y} + C_s y = 0, \quad s = 1, \ldots, N. \tag{18}$$

Theorem 4.1 *Let the following conditions be fulfilled:*

(a) Subsystem (17) *is asymptotically stable;*

(b) Subsystems (18) *are asymptotically stable, and moreover the family* (18) *admits a common quadratic Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

Then, for sufficiently large values of h and for any switching law, system (15) *is asymptotically stable.*

Proof By using the substitution (7), we transform (16) into the systems

$$\begin{aligned} F\dot{y} &= -\tfrac{1}{h}C_s y + \tfrac{1}{h^2}C_s F^{-1}Az, \\ A\dot{z} &= -hFz - C_s y + \tfrac{1}{h}C_s F^{-1}Az, \quad s = 1,\dots,N. \end{aligned} \tag{19}$$

Let $V_1(y)$ be a common quadratic Lyapunov function of family (18), and $V_2(z)$ be a quadratic Lyapunov function of (17), respectively, both of which satisfy all the assumptions of the Lyapunov asymptotic stability theorem. Construct the function $V(y,z) = \varepsilon h^2 V_1(y) + V_2(z)$, where ε is a positive parameter. By the analogy with the proof of Theorem 3.1, it is easy to show that, for sufficiently small values of ε and for sufficiently large values of h, the derivative of $V(y,z)$ along the solution to each of the systems in (19) is negative definite. Thus, $V(y,z)$ is a common Lyapunov function for family (19). It implies that, for any switching law σ, the zero solution of the system

$$\begin{aligned} F\dot{y} &= -\tfrac{1}{h}C_\sigma y + \tfrac{1}{h^2}C_\sigma F^{-1}Az, \\ A\dot{z} &= -hFz - C_\sigma y + \tfrac{1}{h}C_\sigma F^{-1}Az \end{aligned}$$

is asymptotically stable. Hence, the zero solution of (15) possesses the same property. □

Now we turn to consider the linear mechanical system with the dominating gyroscopic forces and with the switching positional forces

$$A\ddot{q} + (B + hG)\dot{q} + C_\sigma q = 0. \tag{20}$$

The corresponding family of systems has the form

$$A\ddot{q} + (B + hG)\dot{q} + C_s q = 0, \quad s = 1,\dots,N,$$

where $q, \dot{q} \in R^n$; A, B, G, C_s are constant matrices; h is a large positive parameter. We assume that A is symmetric and positive definite matrix, B is symmetric matrix, G is skew-symmetric and nonsingular matrix.

Theorem 4.2 *Let the following conditions be fulfilled:*
(a) Matrix B is positive definite;
(b) Subsystems

$$G\dot{y} + C_s y = 0, \quad s = 1,\dots,N, \tag{21}$$

are asymptotically stable, and for family (21) *there exists a common quadratic Lyapunov function satisfying the assumptions of the Lyapunov asymptotic stability theorem.*
Then, for sufficiently large values of h and for any switching law, system (20) *is asymptotically stable.*

The proof of Theorem 4.2 is similar to that one of Theorem 3.2.

Remark 4.1 Just as in Section 3, the suggested approach permits one to develop a constructive procedure for the estimation of lower bounds of admissible values of parameter h in systems (15) and (20).

Remark 4.2 As mentioned in Section 2, for systems (15) and (20), the direct application of known results on the existence of a common Lyapunov functions may be ineffective or even impossible. Theorems 4.1 and 4.2 provide a possibility to reduce the problem of constructing a common Lyapunov function for system of dimension $2n$ with the special structure to the analogical problem for the subsystem of dimension n which, generally, does not possess the special structure.

For instance, in Section 2, it has been shown that well-known commutativity condition is nonapplicable to system (15). However, according to Theorem 4.1, under the sufficiently large values of parameter h, instead of (15), one can consider subsystem (17) and family of subsystems (18). In fact, for (18), the commutativity condition becomes $C_sF^{-1}C_r = C_rF^{-1}C_s$, $s, r = 1, \dots, N$.

5 Construction of the Common Lyapunov Functions

5.1 Domination of potential forces

Consider the linear switched mechanical system

$$A\ddot{q} + (B_\sigma + G_\sigma)\,\dot{q} + (hK + P_\sigma)\,q = 0. \tag{22}$$

The corresponding family of systems is described as follows

$$A\ddot{q} + (B_s + G_s)\,\dot{q} + (hK + P_s)\,q = 0, \qquad s = 1, \dots, N, \tag{23}$$

where A, B_s, G_s, K, P_s are constant matrices, h is positive parameter. We assume that matrices K and B_s are symmetric, while matrices G_s and P_s are skew-symmetric. Moreover, here and in what follows it is assumed that A is symmetric and positive definite matrix.

Theorem 5.1 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \dots, B_N$ are positive definite;
(b) Matrix K is positive definite;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of all components of force field, with the exception of potential component, system (22) *is asymptotically stable.*

Proof Construct the common Lyapunov function for the family (23) in the form

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^TA\dot{q} + \frac{h}{2}q^TKq + \varepsilon q^TA\dot{q}, \tag{24}$$

where $\varepsilon > 0$ is sufficiently small positive number.

For arbitrary symmetric matrix M, let $\lambda_{min}(M)$ and $\lambda_{max}(M)$ be the minimal and maximal eigenvalues of M, respectively. Introduce the following notations: $k_1 = \lambda_{min}(K)$, $k_2 = \lambda_{max}(K)$, $a_1 = \lambda_{min}(A)$, $a_2 = \lambda_{max}(A)$,

$$b_1 = \min_{s=1,\dots,N} \lambda_{min}(B_s), \quad b_2 = \max_{s=1,\dots,N} \lambda_{max}(B_s),$$

$$p = \max_{s=1,\dots,N} \sqrt{\lambda_{max}\left(P_s^TP_s\right)}, \qquad g = \max_{s=1,\dots,N} \sqrt{\lambda_{max}\left(G_s^TG_s\right)}.$$

For all $q, \dot{q} \in R^n$ the estimations

$$\frac{a_1}{2}\|\dot{q}\|^2 - \varepsilon a_2\|q\|\|\dot{q}\| + \frac{h}{2}k_1\|q\|^2 \le V(q,\dot{q}) \le \frac{a_2}{2}\|\dot{q}\|^2 + \varepsilon a_2\|q\|\|\dot{q}\| + \frac{h}{2}k_2\|q\|^2 \tag{25}$$

are valid. Differentiating the Lyapunov function (24) along the solution to sth system in family (23), we get

$$\begin{aligned} \dot{V} &= -\dot{q}^TB_s\dot{q} + \varepsilon\dot{q}^TA\dot{q} - \varepsilon hq^TKq - \dot{q}^TP_sq - \varepsilon q^T\left(B_s + G_s\right)\dot{q} \\ &\le (-b_1 + \varepsilon a_2)\|\dot{q}\|^2 - \varepsilon hk_1\|q\|^2 + (p + \varepsilon(b_2 + g))\|q\|\|\dot{q}\|. \end{aligned} \tag{26}$$

By using the estimations (25) and (26), it is easy to show that, for sufficiently large value h and sufficiently small value ε, function (24) is positive definite, while its derivative along the solutions to any system in (23) is negative definite.

For instance, if $\varepsilon = b_1/(2a_2)$, then we have the following condition for h:

$$h > \max\left\{ \frac{b_1^2}{4a_1k_1};\ \frac{(2pa_2 + b_1(b_2+g))^2}{4a_2b_1^2k_1} \right\}.$$

For these values of parameters, there exist positive numbers $\beta_1, \beta_2, \beta_3$ such that for all $q, \dot{q} \in R^n$ the inequalities

$$\beta_1 \left(\|\dot{q}\|^2 + \|q\|^2\right) \le V(q,\dot{q}) \le \beta_2 \left(\|\dot{q}\|^2 + \|q\|^2\right), \qquad \dot{V}\big|_{(22)} \le -\beta_3 \left(\|\dot{q}\|^2 + \|q\|^2\right)$$

hold. Hence, system (22) is asymptotically stable. □

5.2 Domination of dissipative forces

Now we consider the switched system

$$A\ddot{q} + (hB + G_\sigma)\,\dot{q} + (K_\sigma + P_\sigma)\,q = 0. \tag{27}$$

The corresponding family of systems has the form

$$A\ddot{q} + (hB + G_s)\,\dot{q} + (K_s + P_s)\,q = 0, \qquad s = 1,\dots,N, \tag{28}$$

where A, B, G_s, K_s, P_s are constant matrices, h is positive parameter. We assume that matrices B and K_s are symmetric, while matrices G_s and P_s are skew-symmetric.

Theorem 5.2 *Let the following conditions be fulfilled:*
(a) Matrices $K_1,\dots,K_N$ are positive definite;
(b) Matrix B is positive definite;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of all components of force field, with the exception of dissipative component, system (27) *is asymptotically stable.*

Proof After constructing the common Lyapunov function for the family of systems (28) in the form

$$V(q,\dot{q}) = \frac{1}{2}\dot{q}^TA\dot{q} + \frac{h}{2}q^TBq + q^TA\dot{q},$$

the subsequent proof is similar to that one of Theorem 5.1. □

5.3 System with small nonconservative forces

Next consider the switched system with the small parameter at the nonconservative forces

$$A\ddot{q} + (B_\sigma + G_\sigma)\,\dot{q} + (K + \varepsilon P_\sigma)\,q = 0, \tag{29}$$

and the corresponding family of systems

$$A\ddot{q} + (B_s + G_s)\,\dot{q} + (K + \varepsilon P_s)\,q = 0, \qquad s = 1,\dots,N, \tag{30}$$

where A, B_s, G_s, K, P_s are constant matrices, ε is small positive parameter. Assume that matrices K and B_s are symmetric, while matrices G_s and P_s are skew-symmetric.

Theorem 5.3 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \dots, B_N$ are positive definite;
(b) Matrix K is positive definite;
(c) The value of parameter ε is sufficiently small.
Then, for any switching law of all components of force field, with the exception of potential component, system (29) *is asymptotically stable.*

Proof Construct the Lyapunov function in the form

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{1}{2}q^T K q + \varepsilon q^T A\dot{q}. \tag{31}$$

It is easy to verify that, for sufficiently small values of ε, function (31) is positive definite, and its derivative along the solutions of each system from (30) is negative definite. □

5.4 Domination of gyroscopic forces

Next we consider the switched system in the form

$$A\ddot{q} + (B_\sigma + hG)\,\dot{q} + (K + P)\,q = 0. \tag{32}$$

The corresponding family of systems is

$$A\ddot{q} + (B_s + hG)\,\dot{q} + (K + P)\,q = 0, \qquad s = 1, \dots, N, \tag{33}$$

where A, B_s, G, K, P are constant matrices, h is positive parameter. We assume that matrices B_s and K are symmetric, while matrices G and P are skew-symmetric, and moreover matrix G is nonsingular.

Theorem 5.4 *Let the following conditions be fulfilled:*
(a) Matrices $B_1, \dots, B_N$ are positive definite;
(b) The subsystem

$$\dot{y} = -G^{-1}(K + P)y \tag{34}$$

is asymptotically stable;
(c) The value of parameter h is sufficiently large.
Then, for any switching law of dissipative forces, system (32) *is asymptotically stable.*

Proof The asymptotic stability of subsystem (34) implies that for any given symmetric positive definite matrix D, there exists a symmetric positive definite matrix L such that

$$LG^{-1}(K + P) + (K + P)^T \left(G^{-1}\right)^T L = D.$$

Construct the Lyapunov function

$$V(q, \dot{q}) = \frac{1}{2}\dot{q}^T A\dot{q} + \frac{1}{2}q^T L q - \frac{1}{h}q^T F A\dot{q}, \tag{35}$$

where $F = (K - P - L)G^{-1}$. Differentiating $V(q, \dot{q})$ along the solutions of sth system from the family (33), one gets

$$\dot{V} = -\dot{q}^T B_s \dot{q} - \frac{1}{2h}q^T D q - \frac{1}{h}\left(\dot{q}^T F A\dot{q} - q^T F B_s \dot{q}\right).$$

Hence, for sufficiently large values of h, (35) is a common Lyapunov function for family (33). □

Remark 5.1 The approach for the Lyapunov function construction, which was used in the proof of Theorem 5.4, is a generalization of that one suggested in [2].

Remark 5.2 Theorem 5.4 is similar to Theorem 4.2. However, switching forces in system (20) are positional ones, while in system (32) they are the dissipative ones.

Remark 5.3 The Lyapunov functions considered in the present section are constructed on the base of elements possessing clear mechanical meaning (kinetic energy, potential, matrices of acting forces).

Remark 5.4 By the use of the Lyapunov functions constructed the estimations for the admissible values of parameters h and ε in the systems investigated can be obtained.

6 Conclusion

Theorems 3.1 and 3.2 about decomposition of linear mechanical systems with a large parameter are very significant for the justification of precession theory of gyroscopic devices. These theorems were proved primarily in [11, 18] on the base of the first Lyapunov method by means of the expansion of the roots of the characteristic equations for systems considered in the series with respect to negative powers of parameter. However, it is not convenient in applying such approach, since in [11, 18] no constructive estimations for the lower bounds of the admissible values of large parameter were given. Furthermore, for mechanical system with switching force fields, negativity of real parts of all characteristic equation roots doesn't guarantee the stability of equilibrium position. This paper presents new proofs of the above theorems, which are based on Lyapunov direct method. The Lyapunov functions found for an auxiliary isolated subsystems are used for constructing the common Lyapunov function, which guarantees the asymptotic stability of the equilibrium position for the mechanical system with switching force fields.

For mechanical systems with two degrees of freedom and with switching linear force fields, Theorems 4.1 and 4.2 permit one to use the necessary and sufficient conditions of the existence of a common quadratic Lyapunov function for family of switched two-dimensional systems [14]. Direct application of the criterion in [14] to the system with two degrees of freedom is impossible since, in this case, dimension of full system is equal to four. Moreover, decomposition allows one to use for switched linear isolated subsystems the commutativity conditions guaranteeing the existence of a common Lyapunov function for them [13]. We note that these conditions can not be used directly for full original system.

In the present paper, linear systems are studied. However, the theorems proved guarantee exponential stability of equilibrium positions. Hence, these theorems determine the asymptotic stability conditions for nonlinear systems by the linear approximation. By the way, the decomposition method can also be utilized for the mechanical systems with essentially nonlinear forces. We will deal with them in our future work. Moreover, the results obtained can be used for the design of stabilizing controls for mechanical systems.

Acknowledgment

This work was supported by the Russian Foundation of Basic Researches under grant no. 08-08-92208-GFEN_a, the National Science Foundation of China under grants no. 60774037, 60911120067, 60904069, 10926048, and the Doctoral Fund of Ministry of Education of China under grant no. 20091103120008.

References

[1] Abdullin, R.Z., Anapolsky, L.Yu., et al. *Vector Lyapunov Functions in Stability Theory. Advanced series in mathematical science and engineering.* World Federation Publishers Company, Atlanta, 1996.

[2] Agafonov, S.A. The stability of non-conservative systems and an estimate of the domain of attraction. *Journal of Applied Mathematics and Mechanics* **67** (2) (2003) 209–213.

[3] Collins, P. Chaotic dynamics in hybrid systems. *Nonlinear Dynamics and Systems Theory* **8** (2) (2008) 169–194.

[4] DeCarlo, R., Branicky, M., Pettersson, S. and Lennartson, B. Perspectives and results on the stability and stabilisability of hybrid systems. *Proc. IEEE* **88** (2000) 1069–1082.

[5] Hui, Q. and Haddad, W.M. Semistability of switched dynamical systems, Part I: Linear system theory. *Nonlinear Analysis: Hybrid Systems* **3** (3) (2009) 343–353.

[6] Kovalev, A.M., Martynyuk, A.A., Boichuk, O.A., Mazko, A.G., Petryshyn, R.I., Slyusarchuk, V.Yu., Zuyev, A.L. and Slyn'ko, V.I. Novel qualitative methods of nonlinear mechanics and their application to the analysis of multifrequency oscillations, stability, and control problems. *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 117–145.

[7] Liberzon, D. Lie algebras and stability of switched nonlinear systems. In: *Unsolved problems in mathematical systems and control theory* (V.D. Blondel and A. Megretski, eds.). Princeton university press, Princeton, Oxford, 2004, pp. 90–92.

[8] Liberzon, D., Hespanha, J.P. and Morse, A. S. Stability of switched systems: a Lie-algebraic condition. *Syst. Control Lett.* **37** (3) (1999) 117–122.

[9] Liberzon, D. and Morse, A.S. Basic problems in stability and design of switched systems. *IEEE Control Syst. Magazin* **19** (15) (1999) 59–70.

[10] Marton, L. and Lantos, B. Control of mechanical systems with Stribeck friction and backlash. *Syst. Control Lett.* **58** (2) (2009) 141–147.

[11] Merkin, D.R. *Gyroscopic Systems.* Moscow: Nauka, 1974. [Russian]

[12] Merkin, D.R. *Introduction to the Theory of Stability.* Springer, New York, 1997.

[13] Narendra, K.S. and Balakrishnan, J. A common Lyapunov function for stable LTI systems with commuting A-matrices. *IEEE Trans. Automat. Control* **39** (12) (1994) 2469–2471.

[14] Pakshin, P.V. and Pozdyayev, V.V. Existence criterion of the common quadratic Lyapunov function for a set of linear second-order systems. *Journal of Computer and Systems Sciences International* **44** (4) (2005) 519–524.

[15] Shorten R., Wirth, F., Mason, O., Wulf, K., and King, C. Stability criteria for switched and hybrid systems. *SIAM Rev.* **49** (4) (2007) 545–592.

[16] Siliak, D.D. *Decentralized Control of Complex Systems.* Academic Press, New York, 1991.

[17] Zheng, G., Boutat, D. and Barbot, J.P. On uniform controller design for linear switched systems. *Nonlinear Analysis: Hybrid Systems* **4** (1) (2010) 189–198.

[18] Zubov, V.I. *Analytical Dynamics of Gyroscopic Systems.* Leningrad: Sudostroenie, 1970. [Russian]

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 65–82

Existence and Uniqueness of Solutions of Strongly Damped Wave Equations with Integral Boundary Conditions

J. Dabas [1*] and D. Bahuguna [2]

[1] *Department of Paper Technology, Indian Institute of Technology Roorkee, Saharanpur Campus, Saharanpur – 247001, UP India.*
[2] *Department of Mathematics & Statistics Indian Institute of Technology, Kanpur – 208 016, UP India.*

Received: November 25, 2009; Revised: January 24, 2011

Abstract: In this work, we consider a strongly damped wave equation with integral boundary conditions. We apply the method of semi-discretization in time, also known as the method of lines, to establish the existence and uniqueness of a weak solution.

Keywords: *method of lines; strongly damped wave equation equation; integral boundary conditions; weak solution.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this paper, we are concerned with the following strongly damped wave equation involving nonlocal boundary conditions

$$\frac{\partial^2 w}{\partial t^2}(x,t) - \frac{\partial^3 w}{\partial x^2 \partial t}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) = g(x,t),\ (x,t) \in (0,1) \times [0,T], \tag{1}$$

with the initial conditions

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = w_1(x), \quad x \in [0,1], \tag{2}$$

* Corresponding author: mailto:jay.dabas@gmail.com

and the integral boundary conditions

$$\int_0^1 w(x,t)dx = m(t), \quad \int_0^1 xw(x,t)dx = k(t), \quad (x,t) \in (0,1) \times [0,T], \tag{3}$$

where $0 < T < \infty$, the map g is defined from $(0,1) \times [0,T]$ into $\mathbb{R}$. Such type of the equations aries in the motion of mechanical systems. Our aim is to apply the method of semi-discretization in time, also known as the method of lines or Rothe's method, to establish the existence, uniqueness of a weak solution.

In [1] Bahuguna studied a strongly damped wave equation as an abstract differential equation in a Banach space and established the existence and uniqueness of a strong solution with the help of Rothe's method. Beilin [13] has considered the wave equation with an integral condition using the method of separation of variables and Fourier series. Pulkina [14] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of the generalized solutions using the fixed point arguments. Bouziani and Merazga [10] have considered the quasilinear wave equation with the two integral boundary conditions and proved the existence and uniqueness of a solution by Rothe's method. The initial work on the nonlocal boundary conditions (integral conditions) has been carried out by Cannon [12]. Subsequently, similar studies have been carried out by Kamynin [16], Ionkin [15] and others.

Recently, the study of an initial boundary value problem with the integral boundary conditions has received considerable attention of researchers. For relevant references with the consideration of the nonlocal boundary conditions we refer to the papers [2, 3, 5, 8, 9, 10, 11] and the references cited in these papers. In these papers authors have used the method of semi-discretization in time and have established the existence and uniqueness of a weak solution. Our analysis is motivated by the works of Bahuguna [1], Bahuguna and Dabas [2, 3, 5] and Bouziani and Merazga [9, 10]. For more references on Rothe's method we refer to the papers [4, 6, 7] and the references cited in these papers.

Using the transformation $u(x,t) = w(x,t) - r(x,t)$ we reduce the nonhomogeneous integral boundary conditions in the problem (1)–(3) into homogeneous boundary conditions. We look for $r(x,t) := \chi(t)x + \xi(t)$, where χ and ξ are to be chosen suitably, with

$$\int_0^1 r(x,t)dx = m(t) \quad \text{and} \quad \int_0^1 xr(x,t)dx = k(t). \tag{4}$$

From (4), we have

$$\begin{aligned} \frac{1}{2}\chi(t) + \xi(t) &= m(t), & (5)\\ \frac{1}{3}\chi(t) + \frac{1}{2}\xi(t) &= k(t). & (6) \end{aligned}$$

Hence the linear system (5)-(6) is uniquely solvable and $\chi(t)$ and $\xi(t)$ are given by

$$\begin{aligned} \chi(t) &= 12k(t) - 6m(t), & (7)\\ \xi(t) &= 4m(t) - 6k(t). & (8) \end{aligned}$$

By using this transformation problem (1)–(3) equivalently reduces to the problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^3 u}{\partial x^2 \partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t),\ (x,t) \in (0,1)\times[0,T], \tag{9}$$

$$u(x,0) = U_0(x), \quad \frac{\partial u}{\partial t}(x,0) = U_1(x), \quad x \in [0,1], \tag{10}$$

$$\int_0^1 u(x,t)dx = 0, \quad \int_0^1 xu(x,t)dx = 0, \quad (x,t) \in (0,1)\times[0,T], \tag{11}$$

where $f(x,t) = g(x,t)+\frac{\partial^2 r}{\partial t^2}$, $U_0(x) = w_0(x)-r(x,0)$ and $U_1(x) = w_1(x)-\frac{\partial r}{\partial t}(x,0)$. Hence the solution of the problem (1)–(3) will be directly obtained by $w(x,t) = u(x,t)+r(x,t)$.

In the next section we define some function spaces required to establish the existence and uniqueness of weak solution to (9)–(11). The definition of weak solution and assumptions are also stated in this section.

2 Preliminaries

The problem (9)–(11) may be treated as an abstract equation in the real Hilbert space $\mathbf{H} = L^2(0,1)$ of square-integrable functions defined from $(0,1)$ into $\mathbb{R}$ with the inner product and the norm respectively

$$(u,v) = \int_0^1 u(x)v(x)\, dx, \quad \|u\|^2 = \int_0^1 |u(x)|^2\, dx, \quad u,v \in \mathbf{H}.$$

For $k \in \mathbb{N}$, the Sobolev space $\mathbf{H}^k$ is the Hilbert space of all functions $u \in \mathbf{H}$ such that the distributional derivative $u^{(j)} \in \mathbf{H}$ with the inner product and the norm respectively

$$(u,v)_k = \sum_{j=0}^{k} (u^{(j)}, v^{(j)}), \quad \|u\|_k^2 = \sum_{j=0}^{k} \|u^{(j)}\|^2, \quad u,v \in \mathbf{H}^k.$$

We shall incorporate the integral condition (11) with the space itself under consideration by taking $\mathbf{V} \subset \mathbf{H}$ defined by

$$\mathbf{V} = \left\{ u \in \mathbf{H} : \int_0^1 u(x)\, dx = \int_0^1 xu(x)dx = 0 \right\}. \tag{12}$$

$\mathbf{V}$ is a closed subspace of $\mathbf{H}$ and hence is a Hilbert space itself with the inner product $(.,.)$. and the corresponding norm $\|.\|$.

For any Banach space X with the norm $\|.\|_X$ and an interval $I = [a,b]$, $-\infty < a < b < \infty$, we shall denote $C(I;X)$ the space of all continuous functions u from $[a,b]$ into X with the norm

$$\|u\|_{C(I;X)} = \max_{a \le t \le b} \|u(t)\|_X.$$

The space $L^2(I;X)$ consists of all square-Bochner integrable functions (equivalent classes) u such that with the norm

$$\|u\|^2_{L^2(I;X)} = \int_a^b \|u(t)\|_X^2\, dt.$$

Similarly $L^\infty(I;X)$ is the Banach space of all essentially bounded functions from I into X with the norm

$$\|u\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t\in I} \|u(t)\|_X,$$

and the Banach space $Lip(I;X)$ is the space of all Lipschitz continuous functions from I into X with the norm

$$\|u\|_{Lip(I;X)} = \|u\|_{C(I;X)} + \sup_{t,s\in I;\, t\neq s} \frac{\|u(t)-u(s)\|}{|t-s|}.$$

In addition, to the spaces mentioned above, we need the space $B_2^1(0,1)$ introduced by Merazga and A. Bouziani [9] being the completion of the space $C_0(0,1)$ of all real continuous functions having compact supports in (0,1) with the inner product

$$(u,v)_{B_2^1} = \int_0^1 \left(\int_0^x u(\xi)d\xi\right)\left(\int_0^x u(\xi)d\xi\right)dx.$$

It is clear that $v \in B_2^1(0,1)$ if and only if $\int_0^x v(\xi)d\xi \in L^2(0,1)$ and the corresponding norm $\|u\|_{B_2^1}^2 = \int_0^1 \left(\int_0^x u(\xi)d\xi\right)^2 dx$. It follows that the inequality $\|v\|_{B_2^1}^2 \leq \frac{1}{2}\|v\|^2$ holds for every $v \in L^2(0,1)$, and the embedding $L^2(0,1) \to B_2^1(0,1)$ is continuous.

Given a function $h : (0,1)\times[a,b] \to \mathbb{R}$ such that for each $t\in[a,b]$, $h(.,t) : [a,b] \to \mathbf{H}$, we may identify it with the function $h : [a,b] \to \mathbf{H}$ given by $h(t)(x) = h(x,t)$. We assume the following conditions:

(A1) The function $f : [0,T]\times H \to H$ satisfies the Lipschitz condition, i.e., there exists a positive constant L_f such that

$$\|f(t)-f(s)\|_{B_2^1} \leq L_f|t-s| \quad \text{for} \quad t,s\in[0,T] \quad \text{and} \quad u,v\in H.$$

(A2) $U_0(x), U_1(x) \in H^2(0,1)$ and $U_0(x), U_1(x) \in V$, i.e.

$$\int_0^1 U_0(x)dx = \int_0^1 xU_0(x)dx = 0, \text{ and } \int_0^1 U_1(x)dx = \int_0^1 xU_1(x)dx = 0.$$

Definition 2.1 By a weak solution of the problem (9)–(11) we mean a function $u : [0,T] \to \mathbf{H}$ such that

1. $u \in Lip([0,T],V)$,
2. u has $a.e.$ in $[0,T]$ a strong derivative $\frac{du}{dt} \in L^\infty([0,T];V)\cap Lip([0,T],B_2^1(0,1))$, and $\frac{d^2u}{dt^2} \in L^\infty([0,T],B_2^1(0,1))$,
3. u satisfying the initial boundary conditions (2) and the integral conditions (11),
4. also the following integral identity is satisfied

$$\left(\frac{d^2u(t)}{dt^2}, v\right)_{B_2^1} + \left(\frac{du(t)}{dt}, v\right) + (u(t),v) = (f(t),v)_{B_2^1}. \tag{13}$$

for all $v \in L^2([0,T],V)$ and a.e. $t\in[0,T]$.

We have the need of the following lemma due to Sloan and Thomme [17] for latter use.

Lemma 2.1 *Let $\{w_l\}$ be a sequence of nonnegative real numbers satisfying*

$$w_l \leq \alpha_l + \sum_{i=0}^{l-1} \beta_i w_i, \quad l > 0,$$

where $\{\alpha_l\}$ is a nondecreasing sequence of nonnegative real numbers and $\beta_l \geq 0$. Then

$$w_l \leq \alpha_l \exp\{\sum_{i=0}^{l-1} \beta_i\}, \quad l > 0.$$

3 Discretization Scheme and Priori Estimates

In this section we discretized the problem (9)–(11) and established the estimates. We shall prove Theorem 5.1 given in the last section with the help of Lemma 3.2 and 4.2 stated and proved in subsequent sections. For a positive integer n, we consider the discretization

$$[t_{j-1}^n, t_j^n], \quad t_j^n = jh_n, \quad h_n = \frac{T}{n}, \quad j = 0, 1, 2, \ldots, n;$$

of the interval $[0, T]$. We call u^n an approximate solution and set $u_0^n = U_0$,

$$u_{-1}^n(x) = U_0(x) - h_n U_1(x), \tag{14}$$

$$u_{-2}^n(x) = h_n^2 \left[f(0) + \frac{d^2 U_0}{dx^2} + \frac{d^2 U_1}{dx^2} \right] + U_0 - 2h_n U_1, \tag{15}$$

for all $n \in \mathbb{N}$. For $j = 1, 2, \ldots, n$, we define u_j^n the unique solutions of each of the equations

$$\delta^2 u_j^n - \frac{d^2 \delta u_j^n}{dx^2} - \frac{d^2 u_j^n}{dx^2} = f_j^n, \quad x \in (0, 1), \tag{16}$$

$$\int_0^1 u_j^n(x) dx = 0, \tag{17}$$

$$\int_0^1 x u_j^n(x) dx = 0, \tag{18}$$

where

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h_n}, \quad \delta^2 u_j^n = \frac{\delta u_j^n - \delta u_{j-1}^n}{h_n}, \quad f_j^n = f(t_j^n). \tag{19}$$

The existence of unique $u_j^n \in \mathbf{H^2}$ satisfying $(16) - (18)$ is ensured similarly as established in [8] Lemma 3.1.

Lemma 3.1 *For each $n \in \mathbb{N}$ and each $j = 1, \ldots, n$, the problem (16)–(18) admits a unique solution $u_j \in H^2(0, 1)$.*

Proof For this purpose, we introduce the following functions

$$q_j^n = u_j^n + \delta u_j^n, \quad j = 1, \ldots, n. \tag{20}$$

If we solve this for u_j^n we have

$$u_j^n = \frac{h_n}{1+h_n} q_j^n + \frac{1}{1+h_n} u_{j-1}^n, \quad j = 1, 2, \ldots, n.$$

And also

$$\delta u_j^n = \frac{1}{1+h_n}(q_j^n - u_{j-1}^n), \quad \delta^2 u_j^n = \frac{1}{1+h_n}(\delta q_j^n - \delta u_{j-1}^n),\ j = 1, \ldots, n. \tag{21}$$

Then the problem (16)–(18) is equivalent to the following problem

$$-\tfrac{d^2 q_j^n}{dx^2} + \tfrac{1}{h_n(1+h_n)} q_j^n = f_j^n + \tfrac{1}{1+h_n}[\delta u_{j-1}^n + \tfrac{1}{h_n} q_{j-1}^n], \quad x \in (0,1), \tag{22}$$

$$\textstyle\int_0^1 q_j^n(x)dx = 0, \quad \int_0^1 x q_j^n(x)dx = 0. \tag{23}$$

For solving the system (22)–(23) we use an idea from [8]. Details are as follows. We first solve the equation (22) with classical Dirichlet boundary conditions

$$q_j^n(0) = \lambda, \quad \text{and} \quad q_j^n(1) = \mu, \tag{24}$$

where (λ, μ) is for the moment an arbitrary fixed ordered pair of real numbers. For $j = 1$, we have

$$F_1 = f_1 + \frac{1}{1+h_n}[\delta u_0^n + \frac{1}{h_n} q_0^n] \in \mathbf{H},$$

the Lax-Milgram Lemma guarantees the existence and uniqueness of a strong solution $q_1^n \in H^2(0,1)$ of the problem (22)–(24). Step by step each q_j is then uniquely determined in terms of $U_0, U_1, q_1^n, \ldots, q_{j-1}^n$. Let us show that the parameters λ and μ can be chosen in a way such that the corresponding function $q_j^n(., \lambda, \mu)$ is also a solution of the problem (22)–(23) provided that n is large enough. The function $q_j^n(., \lambda, \mu)$ shall be a solution to problem (22)–(23) if and only if the pair (λ, μ) satisfies

$$\int_0^1 q_j^n(x, \lambda, \mu)dx = 0, \tag{25}$$

$$\int_0^1 x q_j^n(x, \lambda, \mu)dx = 0. \tag{26}$$

Solving (25)–(26) will provide all the solutions to the problem (22)-(23). Let us write $q_j^n(., \lambda, \mu)$ as the sum of two functions

$$q_j^n(x, \lambda, \mu) = q_j^n(x, 0, 0) + \overline{q}_j^n(x, \lambda, \mu),$$

where $q_j^n(x, 0, 0)$ and $\overline{q}_j^n(x, \lambda, \mu)$ are solutions to the following problems, respectively:

$$\begin{cases} -\frac{d^2 q_j^n}{dx^2} + \frac{1}{h_n(1+h_n)} q_j^n = F_j, \\ q_j^n(0) = 0 = q_j^n(1), \end{cases} \quad \begin{cases} -\frac{d^2 \overline{q}_j^n}{dx^2} + \frac{1}{h_n(1+h_n)} \overline{q}_j^n = 0, \quad x \in (0,1), \\ \overline{q}_j^n(0) = \lambda,\ \overline{q}_j^n(1) = \mu. \end{cases} \tag{27}$$

The solution of the second problem is given by

$$\overline{q}_j^n(x) = a_1 e^{px} + a_2 e^{-px}, \quad \text{where} \quad p = \frac{1}{\sqrt{h_n(1+h_n)}}, \tag{28}$$

using the boundary conditions, we find the constants as

$$a_1 = \frac{\mu - \lambda e^{-p}}{e^p - e^{-p}}, \quad a_2 = \frac{\mu - \lambda e^{p}}{e^{-p} - e^p}.$$

Now putting the function $q_j(x, \lambda, \mu)$ in (25)–(26), we have

$$\lambda + \mu = \frac{p \sinh p}{\cosh p - 1} \int_0^1 q_j(x, 0, 0)dx, \tag{29}$$

$$(p - \sinh p)\lambda + (\sinh p - p\cosh p)\mu = p^2 \sinh p \int_0^1 q_j(x, 0, 0)dx. \tag{30}$$

Determinant of the coefficient of the above system is

$$D(h_n) = 2\sinh p - p\cosh p - p.$$

It can be shown that the real function $D(h_n)$ admits a unique real root for all $h_n > 0$, hence the system (16)-(18) which is equivalent to (22)-(23) is uniquely solvable. This completes the proof of the lemma. $\square$

We first obtain the estimates for δu_j^n and difference quotients $\{\frac{\delta u_j^n - \delta u_{j-1}^n}{h_n}\}$ using $(A1)$ and $(A2)$ which in turn imply the uniform bounds of $\{u_j^n\}$. To derive the estimates first we reformulate the discretized problem in the variational form. Let v be any function from the space V and

$$\int_0^x (x - \xi)v(\xi)d\xi = \Im_x^2 v, \quad \forall x \in (0, 1), \tag{31}$$

where

$$\Im_x v = \int_0^x v(\xi)d\xi, \quad \text{and} \quad \Im_x^2 v = \Im_x(\Im_x v) = \int_0^x d\xi \int_0^\xi v(s)ds, \tag{32}$$

with $x = 1$ in (31), for any $v \in V$ we have $\Im_1^2 v = 0$. Now multiplying (16) by $\Im_x^2 v$, $j = 1, 2, \ldots, n$, and integrating over $(0, 1)$, we get

$$\begin{aligned} &\textstyle\int_0^1 \delta^2 u_j^n(x)\Im_x^2 v \, dx - \int_0^1 \frac{d^2\delta u_j^n}{dx^2}(x)\Im_x^2 v dx - \int_0^1 \frac{d^2 u_j^n}{dx^2}(x)\Im_x^2 v dx \\ &\qquad = \textstyle\int_0^1 f_j^n(x)\Im_x^2 v dx. \end{aligned} \tag{33}$$

Now integrating by parts for each term in (33) we have

$$\begin{aligned} \int_0^1 \delta^2 u_j^n(x)\Im_x^2 v \, dx &= \int_0^1 \frac{d}{dx}\left(\Im_x(\delta^2 u_j^n)\right)\Im_x^2 v dx \\ &= \Im_x(\delta^2 u_j^n)\Im_x^2 v|_{x=0}^{x=1} - \int_0^1 \Im_x(\delta^2 u_j^n)\Im_x v dx \\ &= -(\delta^2 u_j^n, v)_{B_2^1}. \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{d^2\delta u_j^n}{dx^2}(x)\Im_x^2 v dx &= \frac{d\delta u_j^n}{dx}(x)\Im_x^2 v|_{x=0}^{x=1} - \int_0^1 \frac{d\delta u_j^n}{dx}(x)\Im_x v dx \\ &= -\int_0^1 \frac{d\delta u_j^n}{dx}(x)\Im_x v dx \\ &= -\delta u_j^n(x)\Im_x v|_{x=0}^{x=1} + \int_0^1 \delta u_j^n(x) v \, dx \\ &= (\delta u_j^n, v), \end{aligned}$$

and

$$\int_0^1 \frac{d^2u_j^n}{dx^2}(x)\Im_x^2 v dx = (u_j, v), \quad \int_0^1 f_j^n \Im_x^2 v \, dx = -(f_j^n, v)_{B_2^1}.$$

Finally, we have the following variational identity

$$(\delta^2 u_j^n, v)_{B_2^1} + (\delta u_j^n, v) + (u_j^n, v) = (f_j^n, v)_{B_2^1}. \tag{34}$$

Throughout, C will represent a generic constant independent of j, h_n and n and CT, Ce^{CT} are again replaced by C.

Lemma 3.2 *Assume that the hypotheses (A1) and (A2) are satisfied. Then there exists a positive constant C, independent of j, h_n and n such that*

$$\begin{aligned} \|\delta u_j^n\| &\leq C, &(35)\\ \|\delta^2 u_j^n\|_{B_2^1} &\leq C, &(36) \end{aligned}$$

$n \geq 1$ *and* $j = 1, \ldots, n$.

Proof For $2 \leq j \leq n$, putting $v = \delta^2 u_j^n$, in (34) we have

$$\begin{aligned} (\delta^2 u_j^n - \delta^2 u_{j-1}^n, \delta^2 u_j^n)_{B_2^1} + h_n(\delta^2 u_j^n, \delta^2 u_j^n) + (\delta u_j^n, \delta u_j^n - \delta u_{j-1}^n) \\ = (f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1}. \end{aligned}$$

Using the identity

$$2(u, u-w) = \|u\|^2 - \|w\|^2 + \|u-w\|^2,$$

we obtain

$$\begin{aligned} &\|\delta^2 u_j^n\|_{B_2^1}^2 - \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta^2 u_j^n - \delta^2 u_{j-1}^n\|_{B_2^1}^2 + h_n\|\delta^2 u_j^n\|^2 \\ &+ \|\delta u_j^n\|^2 - \|\delta u_{j-1}^n\|^2 + \|\delta u_j^n - \delta u_{j-1}^n\|^2 = 2(f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1}. \end{aligned} \tag{37}$$

We neglect the third, forth and the last terms on the left hand side of the equation (37) to get

$$\begin{aligned} \|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 &\leq \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta u_{j-1}^n\|^2 + 2(f_j^n - f_{j-1}^n, \delta^2 u_j^n)_{B_2^1} \\ &\leq \|\delta^2 u_{j-1}^n\|_{B_2^1}^2 + \|\delta u_{j-1}^n\|^2 + 2\|f_j^n - f_{j-1}^n\|_{B_2^1}\|\delta^2 u_j^n\|_{B_2^1}. \end{aligned}$$

Repeating the above inequality, we obtain

$$\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 \leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + 2\sum_{i=2}^{j-1}\|f_j^n - f_{j-1}^n\|_{B_2^1}\|\delta^2 u_j^n\|_{B_2^1}.$$

Using the Cauchy inequality $2ab \leq \frac{1}{\epsilon}a^2 + \epsilon b^2, \ a, b \in \mathbb{R}, \ \epsilon > 0$, with $\epsilon = h_n$ and using assumption $(A1)$, we have the estimate

$$\begin{aligned} \|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 &\leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + \frac{1}{h_n}\sum_{i=2}^{j-1}\|f(t_j^n) - f(t_{j-1}^n)\|_{B_2^1}^2 \\ &\quad + h_n\sum_{i=2}^{j-1}\|\delta^2 u_j^n\|_{B_2^1}^2 \\ &\leq \|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 + CT + h_n\sum_{i=0}^{j-1}\|\delta^2 u_j^n\|_{B_2^1}^2. \end{aligned} \tag{38}$$

From Lemma 2.1 we get the estimate

$$\|\delta^2 u_j^n\|^2_{B_2^1} + \|\delta u_j^n\|^2 \quad \leq \quad \Big[\|\delta^2 u_1^n\|^2_{B_2^1} + \|\delta u_1^n\|^2 + CT\Big] \exp\{(j-1)h_n\}. \tag{39}$$

To estimate the right hand side in (39), we use the variational identity (34) for $j = 1$ and $v = \delta^2 u_1^n = \frac{\delta u_1^n - U_1}{h_n}$, to obtain

$$(\delta^2 u_1^n, \delta^2 u_1^n)_{B_2^1} + (\delta u_1^n, \delta^2 u_1^n) + \left(u_1, \frac{\delta u_1^n - U_1}{h_n}\right) = (f_1^n, \delta^2 u_1^n)_{B_2^1}. \tag{40}$$

Rearranging the terms, we get

$$\begin{aligned} \|\delta^2 u_1^n\|^2_{B_2^1} + h_n\|\delta^2 u_1^n\|^2 + (\delta u_1^n, \delta u_1^n - U_1) \quad &= \quad (f_1^n, \delta^2 u_1^n)_{B_2^1} - (\delta u_0, \delta^2 u_1^n) \\ &\quad -(U_0, \delta^2 u_1^n). \end{aligned} \tag{41}$$

Again by using the equality $2(u, u-w) = \|u\|^2 - \|w\|^2 + \|u-w\|^2$, we have

$$\begin{aligned} &\|\delta^2 u_1^n\|^2_{B_2^1} + h_n\|\delta^2 u_1^n\|^2 + \tfrac{1}{2}\{\|\delta u_1^n\|^2 + \|\delta u_1^n - U_1\|^2 - \|U_1\|^2\} \\ &\quad = (f_1^n, \delta^2 u_1^n)_{B_2^1} - (U_1, \delta^2 u_1^n) - (U_0, \delta^2 u_1^n). \end{aligned} \tag{42}$$

The second term on the right hand side of (42) gives us

$$\begin{aligned} (U_1, \delta^2 u_1^n) \quad &= \quad \int_0^1 U_1(x) \frac{d}{dx}(\Im_x \delta^2 u_1^n) dx \\ &= \quad U_1(x)\Im_x \delta^2 u_1|_{x=0}^{x=1} - \int_0^1 \frac{dU_1}{dx}(x)\Im_x \delta u_1^n dx \\ &= \quad -\int_0^1 \frac{dU_1}{dx}(x)\Im_x \delta u_1^n dx. \end{aligned} \tag{43}$$

Using $\Im_x(\frac{d^2U_1}{dx^2}) = \frac{dU_1}{dx}(x) - \frac{dU_1}{dx}(0)$, for all $x \in (0,1)$, in equation (43) we obtain

$$(U_1, \delta^2 u_1^n) = -\left(\frac{d^2U_1}{dx^2}, \delta^2 u_1^n\right)_{B_2^1}.$$

Similarly, we may write

$$(U_0, \delta^2 u_1^n) = -\left(\frac{d^2U_0}{dx^2}, \delta^2 u_1^n\right)_{B_2^1}.$$

From (42), we obtain

$$\begin{aligned} &2\|\delta^2 u_1^n\|^2_{B_2^1} + \|\delta u_1^n\|^2 \\ &\leq \|U_1\|^2 + 2\left[\left\|f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2}\right\|_{B_2^1}\right]\|\delta^2 u_1^n\|_{B_2^1} \\ &\leq \|U_1\|^2 + \left\|f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2}\right\|^2_{B_2^1} + \|\delta^2 u_1^n\|^2_{B_2^1}. \end{aligned} \tag{44}$$

Thus, from (44), we have

$$\|\delta^2 u_1^n\|_{B_2^1}^2 + \|\delta u_1^n\|^2 \quad \leq \quad \|U_1\|^2 + \left\| f_1^n + \frac{d^2U_1}{dx^2} + \frac{d^2U_0}{dx^2} \right\|_{B_2^1}^2 = C_1. \tag{45}$$

Finally we estimate (39) as

$$\|\delta^2 u_j^n\|_{B_2^1}^2 + \|\delta u_j^n\|^2 \leq C_1 \exp\{CT\}. \tag{46}$$

This completes the proof of the lemma. □

Remark 3.1 Estimates of Lemma 3.2 imply that for all n and $j = 1, 2, \ldots, n$, $\|u_j^n\| \leq C$.

Definition 3.1 We define Rothe's sequence $\{U^n\}$ and $\{V^n\}$ of functions from $[0, T]$ into $\mathbf{H}^2 \cap \mathbf{V}$, given by

$$\begin{aligned} U^n(t) &= u_{j-1}^n + (t - t_{j-1}^n)\delta u_j^n, \quad t \in [t_{j-1}^n, t_j^n], \quad j = 1, 2, \ldots, n, \\ V^n(t) &= \delta u_{j-1}^n + (t - t_{j-1}^n)\delta^2 u_j^n, \quad t \in [t_{j-1}^n, t_j^n], \quad j = 1, 2, \ldots, n. \end{aligned}$$

Furthermore, we define another set of sequences $\{X^n\}$, $\{Y^n\}$ and $\{\tilde{Y}^n\}$ of step functions given by

$$\begin{aligned} X^n(t) &= U_0, \quad t \in (-h_n, 0], \quad X^n(t) = u_j^n, \quad t \in (t_{j-1}^n, t_j^n], \\ Y^n(t) &= U_1, \quad t \in (-h_n, 0], \quad Y^n(t) = \delta u_j^n, \quad t \in (t_{j-1}^n, t_j^n], \\ \tilde{Y}^n(t) &= \delta^2 u_1^n, \quad t = 0, \qquad \tilde{Y}^n(t) = \delta^2 u_j^n, \quad t \in (t_{j-1}^n, t_j^n]. \end{aligned}$$

for $j = 1, 2, \ldots, n$.

Remark 3.2 From Lemma 3.2 it follows that

1. The functions $\{U^n(t)\}$ and $\{V^n(t)\}$ are Lipschitz continuous on $[0, T]$ with uniform Lipschitz constant C, i.e.

$$\|U^n(t) - U^n(s)\| \leq C|t - s|, \quad \|V^n(t) - V^n(s)\|_{B_2^1} \leq C|t - s|.$$

2. The sequences $\{U^n(t)\}, \{X^n(t)\}$ are bounded in the space $L^2([0, T]; V)$ and the sequences $\{V^n(t)\}, \{Y^n(t)\}$ are bounded in the space $L^2([0, T]; B_2^1(0, 1))$ uniformly for all $t \in [0, T]$ and $n \in \mathbb{N}$. Also we have

$$\left\| \frac{dU^n}{dt}(t) \right\| \leq C, \quad \left\| \frac{dV^n}{dt}(t) \right\|_{B_2^1} \leq C.$$

3. The sequence $X^n(t) - U^n(t)$, $U^n(t) - X^n(t - h_n)$ and $Y^n(t) - Y^n(t - h_n) \to 0$ in $L^2([0, T], V)$ as $n \to \infty$. Also the sequence $Y^n(t) - V^n(t) \to 0$ in $L^2([0, T], B_2^1(0, 1))$ as $n \to \infty$. These results follow due to the following inequalities

$$\begin{aligned} &\left\| V^n(t) - \frac{dU^n}{dt}(t) \right\|_{B_2^1} \leq Ch_n, \\ &\|X^n(t) - U^n(t)\| \leq \frac{C}{n}, \quad \text{and} \quad \|U^n(t) - X^n(t - h_n)\| \leq \frac{C}{n}, \\ &\|Y^n(t) - V^n(t)\|_{B_2^1} \leq Ch_n, \text{ and } \|Y^n(t) - Y^n(t - h_n)\| \leq Ch_n. \end{aligned}$$

4 Convergence and Existence Result

In this section we establish the existence and uniqueness of a weak solution to (9)–(11).

Lemma 4.1 *There exist two functions* $u \in L^2([0,T];V) \cap L^\infty([0,T];V)$ *with* $u' \in L^2([0,T];B_2^1(0,1)) \cap L^\infty([0,T];B_2^1(0,1))$ *and* $w \in L^2([0,T];B_2^1(0,1)) \cap L^\infty([0,T];B_2^1(0,1))$ *with* $w' \in L^2([0,T];B_2^1(0,1))$ *such that*

$$U^{n_p} \rightharpoonup u \quad in \quad L^2([0,T];V), \tag{47}$$

$$V^{n_p} \rightharpoonup w, \quad in \quad L^2([0,T];B_2^1(0,1)). \tag{48}$$

Furthermore, we have that

$$\frac{dU^{n_p}}{dt} \rightharpoonup \frac{du}{dt} \quad in \quad L^2([0,T];B_2^1(0,1)), \tag{49}$$

$$\frac{du}{dt} = w \quad on \quad [0,T] \quad and \quad \frac{d^2u}{dt^2} = \frac{dw}{dt} \quad a.e. \quad on \quad [0,T], \tag{50}$$

where " $\rightharpoonup$ *" stands for the weak convergence.*

Proof Remark 3.2 we know that the sequences $\{X^n\}$ and $\{Y^n\}$ are bounded in $L^2([0,T];V)$, while the sequence $\{\tilde{Y}^n\}$ is bounded in $L^2([0,T];B_2^1(0,1))$. It follows that subsequences $\{X^{n_p}\}$, $\{Y^{n_p}\}$ and $\{\tilde{Y}^{n_p}\}$ can be found such that

$$\begin{array}{lcll} X^{n_p} & \rightharpoonup & u & \text{in} \quad L^2([0,T];V), \\ Y^{n_p} & \rightharpoonup & w & \text{in} \quad L^2([0,T];B_2^1(0,1)), \\ \tilde{Y}^{n_p} & \rightharpoonup & \tilde{w} & \text{in} \quad L^2([0,T];B_2^1(0,1)). \end{array}$$

Similarly as in the preceding chapters, one finds that

$$\begin{array}{lcll} U^{n_p} & \rightharpoonup & u & \text{in} \quad L^2([0,T];V), \\ \dfrac{dU^{n_p}}{dt} & \rightharpoonup & \dfrac{du}{dt} & \text{in} \quad L^2([0,T];V), \\ V^{n_p} & \rightharpoonup & w & \text{in} \quad L^2([0,T];B_2^1(0,1)). \end{array}$$

Now, we show that $w = \frac{du}{dt}$. For all $v \in L^2([0,T],B_2^1(0,1))$, we have

$$\begin{aligned} &\left(V^{n_p} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} \\ &= \left(V^{n_p} - \frac{dU^{n_p}}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} + \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} \\ &\le \left\| V^{n_p} - \frac{dU^{n_p}}{dt} \right\|_{L^2([0,T];B_2^1(0,1))} \|v\|_{L^2([0,T];B_2^1(0,1))} \\ &+ \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))}. \end{aligned} \tag{51}$$

From Remark 3.2 and $\frac{dU^{n_p}}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2([0,T];V)$, we have

$$\begin{aligned} \left(V^{n_p} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))} \le\ & Ch_n \|v\|_{L^2([0,T];B_2^1(0,1))} \\ &+ \left(\frac{dU^{n_p}}{dt} - \frac{du}{dt}, v\right)_{L^2([0,T];B_2^1(0,1))}. \end{aligned} \tag{52}$$

Hence we conclude that as $p \to \infty$,

$$V^{n_p} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)).$$

Since $V^{n_p} \rightharpoonup w$ as $p \to \infty$, we have $w = \frac{du}{dt}$ and also $\tilde{w} = \frac{dw}{dt} = \frac{d^2u}{dt^2}$. This can also be achieved by another way by considering the following equalities

$$U^{n_p}(t) - U_0 = \int_0^t Y^{n_p}(s)ds \quad \text{in} \quad L^2([0,T]; V), \tag{53}$$

$$V^{n_p}(t) - U_1 = \int_0^t \tilde{Y}^{n_p}(s)ds \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)). \tag{54}$$

The above equalities can be ensured directly from the construction of U^n, V^n, Y^n and $\tilde{Y}^n$. It follows due to the above convergence result that

$$u(t) - U_0 = \int_0^t w(s)ds \quad \text{in} \quad L^2([0,T]; V), \tag{55}$$

$$w(t) - U_1 = \int_0^t \tilde{w}(s)ds \quad \text{in} \quad L^2([0,T]; B_2^1(0,1)), \tag{56}$$

which imply that $u \in C([0,T]; V)$ and strongly differentiable a.e. in $[0,T]$ with $w = \frac{du}{dt}$ and also $\tilde{w} = \frac{dw}{dt} = \frac{d^2u}{dt^2}$. Now we show that $u \in L^\infty([0,T]; V)$ and $u', w \in L^\infty([0,T]; B_2^1(0,1))$. The estimate $\|u_j^n\| \leq C$, implies that the Rothe' sequence $\{U^n\}$ is bounded in $L^\infty([0,T]; V)$. Hence a subsequence $\{U^{n_k}\}$ of $\{U^n\}$ can be found converging weakly to a function $z \in L^\infty([0,T], V)$, which is easily shown to be equal to the function u. The second assertion $u', w \in L^\infty([0,T]; B_2^1(0,1))$ is obtained similarly. This completes the proof of the lemma. □

Thus, from Lemma 4.1 we conclude the following:

$$\begin{aligned}
&u \in AC([0,T]; V),\\
&u' \in L^2([0,T]; V) \cap AC([0,T]; B_2^1(0,1)),\\
&u'' \in L^2([0,T]; B_2^1(0,1)),\\
&u(0) = U_0 \quad \text{and} \quad u'(0) = U_1 \quad \text{in} \quad C([0,T); B_2^1(0,1)),
\end{aligned}$$

where $AC([0,T]; V)$ denotes a space of all absolutely continuous functions from $[0,T]$ into V.

For the notational convenience, let

$$f^n(0) = f_0, \quad f^n(t) = f(t_j^n), \quad t \in (t_{j-1}^n, t_j^n], \quad 1 \leq j \leq n.$$

Then (34) may be rewritten as

$$\left(\frac{dV^n}{dt}(t), v\right)_{B_2^1} + (Y^n(t), v) + (X^n(t - h_n), v) = (f^n(t), v)_{B_2^1}, \tag{57}$$

for all $v \in \mathbf{V}$ and a.e. $t \in (0,T]$.

Lemma 4.2 *There exist $u \in C([0,T]; V)$ and $w \in C([0,T]; B_2^1(0,1))$ such that*

$$\|U^n - u\|_{C([0,T];V)} \to 0 \quad \text{and} \quad \|V^n - w\|_{C([0,T];B_2^1(0,1))} \to 0, \tag{58}$$

as $n \to \infty$. Moreover u and w are Lipschitz continuous on $[0,T]$.

Proof For $m > n > n_0$, we consider the Rothe functions U^n and U^m corresponding to the step lengths $h_n = \frac{T}{n}$ and $h_m = \frac{T}{m}$. From (57) taking $v = Y^n(t) - Y^m(t)$, we have

$$\begin{aligned}
&\left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - Y^m(t)\right)_{B_2^1} \\
&+ (Y^n(t) - Y^m(t), Y^n(t) - Y^m(t)) \\
&+ (X^n(t-h_n) - X^m(t-h_m), Y^n(t) - Y^m(t)) \\
&= \; (f^n(t) - f^m(t), Y^n(t) - Y^m(t))_{B_2^1}.
\end{aligned} \tag{59}$$

Now the first term of the left hand side in (59) can be written as

$$\begin{aligned}
&\left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - Y^m(t)\right)_{B_2^1} \\
&= \; \left(\frac{d}{dt}(V^n(t) - V^m(t)), Y^n(t) - V^n(t) + V^m(t) - Y^m(t)\right)_{B_2^1} \\
&+ \; \left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - V^m(t)\right)_{B_2^1}.
\end{aligned} \tag{60}$$

Similarly we may write the third term of (59) as

$$\begin{aligned}
&(X^n(t-h_n) - X^m(t-h_m), Y^n(t) - Y^m(t)) \\
&= \; (X^n(t-h_n) - U^n(t) + U^m(t) - X^m(t-h_m), Y^n(t) - Y^m(t)) \\
&+ (U^n(t) - U^m(t), Y^n(t) - Y^m(t)).
\end{aligned} \tag{61}$$

Combining the equations (60)–(61) and using the fact that $Y^n(t) = \frac{dU^n}{dt}(t)$, equation (59) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|V^n(t) - V^m(t)\|^2_{B_2^1} + \frac{1}{2}\frac{d}{dt}\|U^n(t) - U^m(t)\|^2 + \|Y^n(t) - Y^m(t)\|^2 \\
&= \left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - Y^n(t) + Y^m(t) - V^m(t)\right)_{B_2^1} \\
&+ (X^n(t-h_n) - U^n(t) + U^m(t) - X^m(t-h_m), Y^m(t) - Y^n(t)) \\
&+ (f^n(t) - f^m(t), Y^n(t) - Y^m(t))_{B_2^1}.
\end{aligned} \tag{62}$$

The first term on the right hand side of (62) is estimated as

$$\begin{aligned}
&\left(\frac{d}{dt}(V^n(t) - V^m(t)), V^n(t) - Y^n(t) + Y^m(t) - V^m(t)\right)_{B_2^1} \\
&\le \left[\left\|\frac{dV^n(t)}{dt}\right\|_{B_2^1} + \left\|\frac{dV^m(t)}{dt}\right\|_{B_2^1}\right]\left[\|V^n(t) - Y^n(t)\|_{B_2^1} + \|Y^m(t) - V^m(t)\|_{B_2^1}\right] \\
&\le C(h_n + h_m).
\end{aligned} \tag{63}$$

Similarly, we have

$$\begin{aligned}
&(X^n(t-h_n) - U^n(t) + U^m(t) - X^m(t-h_m), Y^m(t) - Y^n(t)) \\
&\le [\|X^n(t-h_n) - U^n(t)\| + \|U^m(t) - X^m(t-h_m)\|]\,[\|Y^m(t)\| + \|Y^n(t)\|] \\
&\le C(h_n + h_m).
\end{aligned} \tag{64}$$

The last term in (62) is estimated as

$$\begin{aligned}
&(f^n(t) - f^m(t), Y^n(t) - Y^m(t))_{B_2^1} \\
&\leq \|f^n(t) - f^m(t)\|_{B_2^1} \|Y^n(t) - Y^m(t)\|_{B_2^1} \\
&\leq \frac{1}{2}\|f^n(t) - f^m(t)\|^2_{B_2^1} + \frac{1}{2}\|Y^n(t) - Y^m(t)\|^2_{B_2^1} \\
&\leq \epsilon_{nm} + \frac{1}{2}\|V^n(t) - V^m(t)\|^2_{B_2^1}, \qquad (65)
\end{aligned}$$

where

$$\epsilon_{nm} = C(h_n + h_m) + C(h_n + h_m)^2 + C(h_n + h_m)\|V^n(t) - V^m(t)\|_{B_2^1},$$

is a sequence of real numbers tending to zero as $n, m \to \infty$. Now using (63)–(64) and (65), (62) becomes

$$\begin{aligned}
&\frac{d}{dt}\|V^n(t) - V^m(t)\|^2_{B_2^1} + \frac{d}{dt}\|U^n(t) - U^m(t)\|^2 \\
&= \epsilon^1_{nm} + \|V^n(t) - V^m(t)\|^2_{B_2^1} + \|U^n(t) - U^m(t)\|^2, \qquad (66)
\end{aligned}$$

where ϵ^1_{nm} is another sequence of numbers tending to zero as $n, m \to \infty$. Integrating the last inequality over $(0, t)$ and using $U^n(0) = U^m(0) = U_0,\ V^n(0) = V^m(0) = U_1$, we have

$$\begin{aligned}
&\|V^n(t) - V^m(t)\|^2_{B_2^1} + \|U^n(t) - U^m(t)\|^2 \\
&= \epsilon^1_{nm}T + \int_0^t \|V^n(s) - V^m(s)\|^2_{B_2^1} ds + \int_0^t \|U^n(s) - U^m(s)\|^2 ds. \qquad (67)
\end{aligned}$$

Application of Gronwall's inequality implies that

$$\|V^n(t) - V^m(t)\|^2_{B_2^1} + \|U^n(t) - U^m(t)\|^2 \leq (\epsilon^1_{nm}T)\exp\{T\}. \qquad (68)$$

Taking the supremum over $t \in [0, T]$ we conclude that there exist functions $u \in C([0, T]; V)$ and $w \in C([0, T]; B_2^1(0, 1))$ such that $U^n \to u$ and $V^n \to w$ as $n \to \infty$. By Remark 3.2 it follows that u, and w are Lipschitz continuous functions. This completes the proof of the lemma. □

5 Main Result

In this section we conclude our main result. We summarize the result so far obtained by previous Lemmas 4.1 and 4.2 in Remark 5.1 below.

Remark 5.1 By Remark 3.2 and Lemma 4.2, we conclude the following:

1. $u \in L^2([0, T]; V) \cap Lip([0, T]; V)$;
2. u is strongly differentiable $a.e.$ in $[0, T]$ and $\frac{du}{dt} \in L^\infty([0, T]; V)$;
3. $X^n(t) \rightharpoonup u(t)$ in V for all $t \in [0, T]$; and $\frac{dU^n}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2([0, T]; V)$;
4. $w \in Lip([0, T]; B_2^1(0, 1))$; w is strongly differentiable $a.e$ in $[0, T]$ and $\frac{dw}{dt} \in L^\infty([0, T]; B_2^1(0, 1))$;

5. $Y^n(t) \rightharpoonup w(t)$ in V for all $t \in [0,T]$; and $\frac{dV^n}{dt} \rightharpoonup \frac{dw}{dt}$ in $L^2([0,T]; B_2^1(0,1))$.

Thus, by the definition of weak solution stated in Definition 2.1 the function $u(t)$ possesses several characteristic properties. Since $u \in L^2([0,T_0]; V)$ we have for almost all $t \in [0,T]$, $u \in V$. Hence the integral boundary conditions (11) are satisfied. The initial condition is fulfilled in the sense of the equations (55) and (56). Now the question is in what sense the given differential equation equation (9) is satisfied. The answer to this question lies in the proof of the main theorem of this article.

Theorem 5.1 *Suppose that the conditions (A1) and (A2) are satisfied. Then problem (9)–(11) has a unique weak solution on* $[0,T]$. *For the sets of data* (U_0^i, U_1^i, f^i), *the corresponding solutions* u^i, $i = 1,2$, *satisfy the following estimate*

$$\left\| \frac{d}{dt}(u^1(t) - u^2(t)) \right\|_{B_2^1}^2 + \left\| u^1(t) - u^2(t) \right\|^2$$
$$\leq \left(\|U_1^1 - U_1^2\|_{B_2^1}^2 + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|_{B_2^1}^2 ds \right) \exp\{t\} \tag{69}$$

which shows the continuous dependence of the solutions on the data.

Proof Now we prove the existence on $[0,T]$. Integrating the identity (57) over $(0,t) \subset [0,T]$ and invoking the fact that $V^n(0) = U_1$, we have

$$(V^n(t) - U_1, v)_{B_2^1} + \int_0^t (Y^n(s), v)\, ds + \int_0^t (X^n(s), v)\, ds$$
$$= \int_0^t (f^n(s), v)_{B_2^1}\, ds. \tag{70}$$

Since $V^n(t) \rightharpoonup \frac{du(t)}{dt}$ in $\mathbf{V}$ for all $t \in [0,T]$, we have

$$(V^n(t) - U_1, v)_{B_2^1} \to \left(\frac{du(t)}{dt} - U_1, v \right)_{B_2^1}, \quad \text{as} \quad n \to \infty. \tag{71}$$

The linear functionals $(Y^n(s), v)$ and $(X^n(s), v)$ are bounded on $\mathbf{V}$, hence by the bounded convergence theorem as $n \to \infty$,

$$\int_0^t (Y^n(s), v) ds \to \int_0^t \left(\frac{du(s)}{dt}, v \right) ds, \quad \forall\, t \in [0,T], \tag{72}$$

$$\int_0^t (X^n(s), v) ds \to \int_0^t (u(s), v) ds, \quad \forall\, t \in [0,T]. \tag{73}$$

Assumption (A1) implies that $\|f^n(s) - f(s)\|_{B_2^1} \leq \frac{C}{n}$ a.e. in $[0,T]$. Hence

$$\|f^n(s) - f(s)\|_{L^2([0,T]; B_2^1(0,1))} \to 0 \quad \text{as} \quad n \to \infty. \tag{74}$$

This implies that $f^n(s) \to f(s)$ in $L^2([0,T]; B_2^1(0,1))$ as $n \to \infty$. Now, by taking into account the convergence result (71)–(74) and passing to the limit as $n \to \infty$, in (70) we have

$$\left(\frac{du}{dt}(t) - U_1, v \right)_{B_2^1} + \int_0^t \left(\frac{du}{dt}(s), v \right) ds + \int_0^t (u(s), v) ds = \int_0^t (f(s), v)_{B_2^1} ds,$$

for all $v \in \mathbf{V}$ and $t \in [0,T]$. Differentiating the above identity we get the desired result,

$$\left(\frac{d^2u}{dt^2}(t), v\right)_{B^1_2} + \left(\frac{du}{dt}(t), v\right) + (u(t), v) = (f(t), v)_{B^1_2}.$$

Uniqueness: Let u_1 and u_2 be two such solutions of (9)-(11). Let we denote the difference of these two solutions by $u(t) = u_1(t) - u_2(t)$, Then from (13), by taking $v = \frac{du(t)}{dt}$, we have

$$\left(\frac{d^2u(t)}{dt^2}, \frac{du(t)}{dt}\right)_{B^1_2} + \left\|\frac{du(t)}{dt}\right\|^2 + \left(u(t), \frac{du(t)}{dt}\right) = 0. \tag{75}$$

Since

$$\left(\frac{d^2u(t)}{dt^2}, \frac{du(t)}{dt}\right)_{B^1_2} = \frac{1}{2}\frac{d}{dt}\left\|\frac{du(t)}{dt}\right\|^2_{B^1_2} \quad \text{and} \quad \left(u(t), \frac{du(t)}{dt}\right) = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2.$$

Then, (75) is written as

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{du(t)}{dt}\right\|^2_{B^1_2} + \left\|\frac{du(t)}{dt}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = 0. \tag{76}$$

Integrating over $(0,s)$ for $0 \leq s \leq t \leq T$ and using the fact that $u(0) \equiv 0$ and $\frac{du(0)}{dt} = 0$, we get

$$\left\|\frac{du(t)}{dt}\right\|^2_{B^1_2} + \int_0^t \left\|\frac{du(s)}{ds}\right\|^2 ds + \|u(t)\|^2 = 0,$$

consequently

$$\left\|\frac{du(t)}{dt}\right\|^2_{B^1_2} + \|u(t)\|^2 \leq 0.$$

Application of the Gronwall's inequality implies that $u \equiv 0$ on $[0,T]$.

Continuous dependence: let u^1 and u^2 be two weak solutions of the problem (9)–(11), corresponding to (U^1_0, U^1_1, f^1) and (U^2_0, U^2_1, f^2), respectively and the initial data satisfy the assumptions (A1) and (A2), from (13), putting $v = \frac{d}{dt}(u^1(t) - u^2(t))$, we have

$$\begin{aligned}&\left(\frac{d^2}{dt^2}(u^1(t) - u^2(t)), \frac{d}{dt}(u^1(t) - u^2(t))\right)_{B^1_2} + \left\|\frac{d}{dt}(u^1(t) - u^2(t)\right\|^2 \\ &+ \left(u^1(t) - u^2(t), \frac{d}{dt}(u^1(t) - u^2(t)\right) = \left(f^1(t) - f^2(t), \frac{d}{dt}(u^1(t) - u^2(t)\right)_{B^1_2}.\end{aligned}$$

Similarly, as in the uniqueness we may drop the middle term, we get

$$\begin{aligned}&\frac{d}{dt}\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B^1_2} + \frac{d}{dt}\|u^1(t) - u^2(t)\|^2 \\ &\leq 2\|f^1(t) - f^2(t)\|_{B^1_2}\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|_{B^1_2} \\ &\leq \|f^1(t) - f^2(t)\|^2_{B^1_2} + \left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B^1_2}.\end{aligned}$$

Integrating over $(0, s)$ for $0 \le s \le t \le T$ and using the fact that $u^i(0) = U_0^i$ and $du^i(0)/dt = U_1^i$, for $i = 1, 2$, we get

$$\begin{aligned}
&\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1} + \|u^1(t) - u^2(t)\|^2 \\
&\le \|U_1^1 - U_1^2\|^2_{B_2^1} + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|^2_{B_2^1} ds \\
&+ \int_0^t \left\|\frac{d}{dt}(u^1(s) - u^2(s))\right\|^2_{B_2^1} ds + \int_0^t \|u^1(s) - u^2(s)\|^2 ds.
\end{aligned}$$

Application of the Gronwall inequality leads to the estimate

$$\begin{aligned}
&\left\|\frac{d}{dt}(u^1(t) - u^2(t))\right\|^2_{B_2^1} + \|u^1(t) - u^2(t)\|^2 \\
&\le \{\|U_1^1 - U_1^2\|^2_{B_2^1} + \|U_0^1 - U_0^2\|^2 + \int_0^t \|f^1(s) - f^2(s)\|^2_{B_2^1} ds\} \exp\{t\}.
\end{aligned}$$

This completes the proof of the theorem. □

6 Application

Example 6.1 In this example we consider the following problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^3 u}{\partial t \partial x^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \sin x \cos t,\ (x,t) \in (0,\pi) \times [0,T], \quad (77)$$

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = \sin x, \quad x \in (0,\pi), \quad (78)$$

$$\int_0^\pi u(x,t)dx = 2\sin t,\ \int_0^\pi x u(x,t)dx = \pi \sin t, \quad t \in [0,T]. \quad (79)$$

We notice that $u = \sin x \sin t$ is an exact solution of the above problem. The results of the earlier sections may be used to ensure the well-posedness of this model. We shall be dealing with the problem involving the Neumann condition together with nonlocal integral conditions of first kind in our subsequent study.

References

[1] Bahuguna, D. Rothe's method to strongly damped wave equations. *Acta Appl. Math.* **38** (1995) 185–196.

[2] Bahuguna, D. and Dabas, J. Existence and uniqueness of solution to a semilinear partial delay differential equation with an integral condition. *Nonlinear Dynamics and System Theory* **8** (1) (2008) 7–19.

[3] Dabas, J. and Bahuguna, D. An integro-differential equation with an integral boundary condition. *Mathematical and Computer Modelling* **50** (2009) 123–131.

[4] Bahuguna, D., Dabas, J. and Shukla, R.K. Method of lines to Hyperbolic integro-differential Equations. *Nonlinear Dynamics and System Theory* **8** (4) (2008) 317–328.

[5] Bahuguna, D., Abbas, S.and Dabas,J. Partial functional differential equation with an integral condition and applications to population dynamics. *Nonlinear Analysis, TMA* **69** (2008) 2623–2635.

[6] Bahuguna, D. and Raghavendra, V. Application of Rothe's method to nonlinear integro-differential equations in Hilbert spaces. *Nonlinear Analysis, TMA.* **23** (1) (1994) 75–81.

[7] Kacur, J. Method of Rothe in evolution equations. *Teubner-Texte zur Mathematik*, vol.80, BSBB. G. Teubner Verlagsgesellschaft, Leipzig, 1985.

[8] Merazga, N. and Bouziani, A. On a time-discretization method for a semilinear heat equation with purly integral conditions in a nonclassical function space. *Nonlinear Analysis, TMA* **66** (2007) 604–623.

[9] Merazga, D. and Bouziani, A. Rothe time-discretization method for a nonlocal problem arising in thermoelasticity. *J. Appl. Math. Stoch. Anal.* **1** (2005) 13–28.

[10] Bouziani, A. and Merazga, N. Rothe time-discretization method applied to a quasilinear wave equation subject to integral conditions. *Adv. Difference Equ.* **3** (2004) 211–235.

[11] Merazga, N. and Bouziani, A. Rothe method for a mixed problem with an integral condition for the two dimensional diffusion equation. *Abstract and applied analysis.* **16** (2003) 899–922.

[12] Cannon, J.R. The solution of the heat equation subject to the specification of energy. *Quart. Appl. Math.* **21** (1963) 155–160.

[13] Belin, S.A. Existence of solutions for one dimensional wave equations with nonlocal conditions. *Electronic Journal of Differential Equations* **76** (2001) 1–8.

[14] Pulkina, L.S. A non-local problem with integral conditions for hyperbolic equations. *Electronic Journal of Differential Equations* **45** (1999) 1–6.

[15] Ionkin, N.I. Solutions of boundary value problem in heat conduction theory with nonlocal boundary conditions. *Differents. Urav.* **13** (1977) 294–304.

[16] Kamynin, L.I. A boundary value problem in the theory of the heat conduction with non-classical boundary condition. *Z. Vychisl. Mat. Fiz.* **6** (1964) 1006–1024.

[17] Sloan, I.H. and Thomee, V. Time discretization of an integro-differential equation of parabolic type. *SIAM J. Numer. Anal.* **23** (1986) 1052–1061.

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 83–92

Mean Square Stability of Itô–Volterra Dynamic Equation

S. Sanyal *

Department of Mathematics, Marshall University, Huntington, WV 25755-2560, USA

Received: March 25, 2010; Revised: January 24, 2011

Abstract: This paper presents a sufficient condition for the mean square stability of the Itô–Volterra dynamic equation on isolated time scales.

Keywords: *Itô–Volterra dynamic equation; mean square stability; convolution; time scale; stochastic dynamic equations.*

Mathematics Subject Classification (2000): 37C75, 45D05, 60H20.

1 Introduction

Given a time scale $\mathbb{T}$, a collection of measurable real functions $X = \{X(t) : t \in \mathbb{T}\}$, defined on a measurable space $(\Omega, \mathcal{F})$, will be referred to as a stochastic process indexed by $\mathbb{T}$ [15, 19, 30]. We consider the Itô–Volterra dynamic equation of the form

$$\Delta X = (a * X)(t)\Delta t + (b * X)(t)\Delta V, \quad X(t_0) = X_0, \tag{1.1}$$

where $a, b : \mathbb{T} \to \mathbb{R}$, $a * X$ is the convolution of a and X defined in Definition 2.2, V is the solution of

$$\Delta V = \sqrt{\mu(t)}\Delta W, \quad V(t_0) = V_0, \tag{1.2}$$

and $X = \{X(t) : t \in \mathbb{T}\}$ is a stochastic process indexed by an isolated time scale $\mathbb{T}$, and $\mu(t) = \sigma(t) - t$ with $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. In (1.2), W is one-dimensional Brownian motion indexed by a time scale $\mathbb{T}$ which is defined as an adapted stochastic process $W = \{W(t), \mathcal{F}(t) : t \in \mathbb{T}\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the following properties: (a) $W(t_0) = 0$ a.s.; (b) if $t_0 \le s < t$ and $s, t \in \mathbb{T}$, then the increment $\Delta W(t) = W(\sigma(t)) - W(t)$ is independent of $\mathcal{F}(s)$ and is normally distributed with mean zero and variance $\mu(t)$.

* Corresponding author: mailto:sanyal@marshall.edu

Since $V^{\Delta}(t)=\Delta V(t)/\Delta t=\Delta W(t)/\sqrt{\mu(t)}$, we observe that $\{V^{\Delta}(t):t\in\mathbb{T}\}$ are i.i.d. random variables which generate a natural filtration $\{\mathcal{F}(t):t\in\mathbb{T}\}$ on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{E}[V^{\Delta}(t)]=0$ and $\mathbb{E}\left[\left(V^{\Delta}(t)\right)^2\right]=1$, where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. Throughout the paper we assume that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau\in[t_0,t)$.

For time scale calculus we refer to [14]; for integral equations of Volterra type we refer to [16,23,24]. Stability and convergence of solutions of Volterra equations, likewise, has been discussed in [2–6, 17, 18, 20–22, 25–29]. For improper integrals and multiple integration on time scales we refer to [1,7,8,10,11,13], and for partial differentiation on time scales we refer to [9].

The organization of the paper is as follows. Section 2 presents core definitions and concept of convolution on a time scale. In Section 3, we derive new conditions that guarantee the mean square stability of (1.1) on an isolated time scale. Our attempt is to make the mathematical discussion that follows as self contained as is practical.

2 Convolution

Convolution on time scales was introduced by Bohner and Guseinov in [12]. In this section we present a brief survey. Let $\sup\mathbb{T}=\infty$ and fix $-\infty<t_0\in\mathbb{T}$.

Definition 2.1 For $b:\mathbb{T}\to\mathbb{R}$, the *shift* (or delay) $\tilde{b}$ of b is the function $\tilde{b}:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ given by

$$\begin{aligned}\tilde{b}^{\Delta_t}(t,\sigma(s))&=-\tilde{b}^{\Delta_s}(t,s),\quad t,s\in\mathbb{T},\ t\ge s\ge t_0,\\ \tilde{b}(t,t_0)&=b(t),\quad t\in\mathbb{T},\ t\ge t_0,\end{aligned}\tag{2.1}$$

where Δ_t is the partial Δ-derivative with respect to t.

Example 2.1 For the forward difference operator, the problem (2.1) takes the form

$$\begin{aligned}\mu(s)\Delta_t\tilde{b}(t,\sigma(s))&=-\mu(t)\Delta_s\tilde{b}(t,s),\quad t,s\in\mathbb{T},\ t\ge s\ge t_0,\\ \tilde{b}(t,t_0)&=b(t),\quad t\in\mathbb{T},\ t\ge t_0.\end{aligned}\tag{2.2}$$

Example 2.2 For $\mathbb{T}=\mathbb{R}$, the problem (2.1) takes the form

$$\frac{\partial\tilde{b}(t,s)}{\partial t}=-\frac{\partial\tilde{b}(t,s)}{\partial s},\quad \tilde{b}(t,t_0)=b(t),\tag{2.3}$$

and its unique solution is $\tilde{b}(t,s)=b(t-s+t_0)$.

Example 2.3 For $\mathbb{T}=\mathbb{Z}$, the problem (2.1) takes the form

$$\tilde{b}(t+1,s+1)-\tilde{b}(t,s+1)=-\tilde{b}(t,s+1)+\tilde{b}(t,s),\quad \tilde{b}(t,t_0)=b(t),\tag{2.4}$$

and its unique solution is again $\tilde{b}(t,s)=b(t-s+t_0)$.

Lemma 2.1 *If $\tilde{b}$ is the shift of b, then $\tilde{b}(t,t)=b(t_0)$ for all $t\in\mathbb{T}$.*

Definition 2.2 The convolution of two functions $b, r : \mathbb{T} \to \mathbb{R}$, $b * r$ is defined as

$$(b * r)(t) = \int_{t_0}^{t} \tilde{b}(t, \sigma(s)) r(s) \Delta s, \quad t \in \mathbb{T}, \tag{2.5}$$

where $\tilde{b}$ is given by (2.1).

Example 2.4 For $\mathbb{T} = \mathbb{N}_0$ and $n \in \mathbb{N}_0$, (2.5) reduces to

$$(b * r)(n) = \sum_{i=0}^{n-1} b(n - i - 1) r(i). \tag{2.6}$$

Theorem 2.1 *The shift of a convolution is given by the formula*

$$(\widetilde{b * r})(t, s) = \int_{s}^{t} \tilde{b}(t, \sigma(l)) \tilde{r}(l, s) \Delta l. \tag{2.7}$$

Example 2.5 For $\mathbb{T} = \mathbb{N}_0$ and $m, n \in \mathbb{N}_0$, (2.7) reduces to

$$(\widetilde{b * r})(n, m) = \sum_{i=m}^{n-1} b(n - i - 1) r(i - m). \tag{2.8}$$

Theorem 2.2 *The convolution is associative, that is,*

$$(a * f) * r = a * (f * r). \tag{2.9}$$

Proof We use Theorem 2.1. Then

$$\begin{aligned}
((a * f) * r)(t) &= \int_{t_0}^{t} (\widetilde{a * f})(t, \sigma(s)) r(s) \Delta s \\
&= \int_{t_0}^{t} \int_{\sigma(s)}^{t} \tilde{a}(t, \sigma(u)) \tilde{f}(u, \sigma(s)) r(s) \Delta u \Delta s \\
&= \int_{t_0}^{t} \int_{t_0}^{u} \tilde{a}(t, \sigma(u)) \tilde{f}(u, \sigma(s)) r(s) \Delta s \Delta u \\
&= \int_{t_0}^{t} \tilde{a}(t, \sigma(u)) (f * r)(u) \Delta u \\
&= (a * (f * r))(t),
\end{aligned}$$

where on the second equality we have used (2.7). Hence, the associative property holds.

Theorem 2.3 *If r is delta differentiable, then*

$$(r * f)^{\Delta} = r^{\Delta} * f + r(t_0) f \tag{2.10}$$

and if f is delta differentiable, then

$$(r * f)^{\Delta} = r * f^{\Delta} + r f(t_0). \tag{2.11}$$

Proof First note that

$$(r*f)^{\Delta}(t) = \int_{t_0}^{t} r^{\Delta_t}(t,\sigma(t))f(s)\Delta s + \tilde{r}(\sigma(t),\sigma(t))f(t).$$

From here, since $\tilde{r}(\sigma(t),\sigma(t)) = r(t_0)$ by Lemma 2.1, and since

$$\widetilde{r^{\Delta}}(t,s) = \tilde{r}^{\Delta_t}(t,s),$$

the first equal sign of the statement follows. For the second equal sign, we use the definition of $\tilde{r}$ and integration by parts:

$$\begin{aligned}
(r*f)^{\Delta}(t) &= -\int_{t_0}^{t} \tilde{r}^{\Delta_s}(t,s)f(s)\Delta s + r(t_0)f(t) \\
&= -\int_{t_0}^{t} \left((\tilde{r}(t,\cdot)f)^{\Delta} - \tilde{r}(t,\sigma(s))f^{\Delta}(s)\right)\Delta s + r(t_0)f(t) \\
&= -\tilde{r}(t,t)f(t) + \tilde{r}(t,t_0)f(t_0) + \int_{t_0}^{t} \tilde{r}(t,\sigma(s))f^{\Delta}(s)\Delta s + r(t_0)f(t) \\
&= (r*f^{\Delta})(t) + r(t)f(t_0).
\end{aligned}$$

This completes the proof.

3 Mean-Square Stability

In this section we study the mean-square stability of (1.1).

Definition 3.1 A stochastic process indexed by a time scale $X = \{X(t) : t \in \mathbb{T}\}$ and defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is mean square stable if $\mathbb{E}\left[X^2\right] \in L^1_{\Delta}(\mathbb{T})$, i.e.,

$$\int_{\mathbb{T}} \mathbb{E}\left[X^2(\tau)\right]\Delta\tau < \infty,$$

where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$.

In the definition above and henceforth, $L^p_{\Delta}(\mathbb{T})$ for $p > 0$ would represent the space of all functions $f : \mathbb{T} \to \mathbb{R}$, such that $\int_{\mathbb{T}} |f|^p(\tau)\Delta\tau < \infty$.

Theorem 3.1 *If $X(t)$ is represented as*

$$X(t) = r(t)X_0 + (r*f)(t), \tag{3.1}$$

where

$$r^{\Delta}(t) = (a*r)(t), \quad r(t_0) = 1, \tag{3.2}$$

and

$$f(t) = (b*X)(t)V^{\Delta}(t). \tag{3.3}$$

then X is a solution of the Itô–Volterra dynamic equation

$$\Delta X = (a*X)(t)\Delta t + (b*X)(t)\Delta V, \quad X(t_0) = X_0. \tag{3.4}$$

Proof From (3.1) we have

$$\begin{aligned}
\Delta X(t) &= r^{\Delta}(t)X_0\Delta t + (r*f)^{\Delta}(t)\Delta t \\
&= (a*r)(t)X_0\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t \\
&= (a*(rX_0))(t)\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t \\
&= (a*(X-r*f))(t)\Delta t + (r^{\Delta}*f)(t)\Delta t + f(t)\Delta t \\
&= (a*X)(t)\Delta t - (a*(r*f))(t)\Delta t + ((a*r)*f)(t)\Delta t + f(t)\Delta t \\
&= (a*X)(t)\Delta t + f(t)\Delta t \\
&= (a*X)(t)\Delta t + (b*X)(t)\Delta V(t),
\end{aligned}$$

where on the second equality we have used (2.10) and on the sixth equality we have used Theorem 2.2.

Lemma 3.1 *If f is given by* (3.3)*, then* $\mathbb{E}[f(t)] = 0$ *and*

$$\mathbb{E}[f(t)f(s)] = \begin{cases} \displaystyle\int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(t_1))\tilde{b}(t,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\Delta t_1\Delta t_2 =: \phi(t) & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases}$$

Proof We first note that

$$\begin{aligned}
\mathbb{E}[f(t)] &= \mathbb{E}\left[\int_{t_0}^{t} \tilde{b}(t,\sigma(\tau))X(\tau)V^{\Delta}(t)\Delta\tau\right] \\
&= \int_{t_0}^{t} \tilde{b}(t,\sigma(\tau))\mathbb{E}[X(\tau)V^{\Delta}(t)]\Delta\tau \\
&= \int_{t_0}^{t} \tilde{b}(t,\sigma(\tau))\mathbb{E}\left[X(\tau)\right]\mathbb{E}[V^{\Delta}(t)]\Delta\tau \\
&= 0,
\end{aligned}$$

by the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau \in [t_0, t)$ and $\mathbb{E}\left[V^{\Delta}(t)\right] = 0$. Next, we consider

$$\begin{aligned}
&\mathbb{E}[f(t)f(s)] \\
&= \mathbb{E}\left[\int_{t_0}^{t} \tilde{b}(t,\sigma(t_1))X(t_1)V^{\Delta}(t)\Delta t_1 \int_{t_0}^{s} \tilde{b}(s,\sigma(t_2))X(t_2)V^{\Delta}(s)\Delta t_2\right] \\
&= \mathbb{E}\left[\int_{t_0}^{t}\int_{t_0}^{s} \tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\Delta t_1\Delta t_2\right] \\
&= \int_{t_0}^{t}\int_{t_0}^{s} \tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1\Delta t_2 \\
&= \int_{t_0}^{t}\int_{t_0}^{s} \tilde{b}(t,\sigma(t_1))\tilde{b}(s,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\mathbb{E}\left[V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1\Delta t_2 \\
&= \begin{cases} \displaystyle\int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(t_1))\tilde{b}(t,\sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\Delta t_1\Delta t_2 & \text{if } s = t \\ 0 & \text{if } s \neq t, \end{cases}
\end{aligned}$$

where on the fourth equation we have used the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau \in [t_0, t)$ and on fifth equation we have used $\mathbb{E}[V^{\Delta}(t)] = 0$ and $\mathbb{E}[(V^{\Delta}(t))^2] = 1 > 0$.

Lemma 3.2 *If $X(t) = r(t)X_0 + (r * f)(t)$, then*

$$\mathbb{E}[X(l)X(m)] = r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s,$$

where ϕ is as in Lemma 3.1 and $l \wedge m = \min(l,m)$.

Proof From (3.1) we have,

$$\begin{aligned}
\mathbb{E}[X(l)X(m)] &= \mathbb{E}\left[\{r(l)X_0 + (r*f)(l)\}\{r(m)X_0 + (r*f)(m)\}\right] \\
&= r(l)r(m)X_0^2 \\
&\quad + \int_{t_0}^{l}\int_{t_0}^{m} \tilde{r}(l,\sigma(s_1))\tilde{r}(m,\sigma(s_2))\mathbb{E}\left[f(s_1)f(s_2)\right]\Delta s_1\Delta s_2 \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\mathbb{E}\left[f^2(s)\right]\Delta s \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s,
\end{aligned}$$

where on the second equality we have used the fact that $\mathbb{E}\left[f(t)\right] = 0$ and on the third equality we have used Lemma 3.1.

Lemma 3.3 *The function ϕ defined in Lemma 3.1 is given by*

$$\phi(t) = (b*r)^2(t)X_0^2 + \int_{t_0}^{t} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s.$$

Proof Using Lemma 3.1, Lemma 3.2 and (2.5), we have

$$\begin{aligned}
\phi(t) &= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\mathbb{E}\left[X(l)X(m)\right]\Delta l\Delta m \\
&= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))r(l)r(m)X_0^2\Delta l\Delta m \\
&\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\int_{t_0}^{l\wedge m} \tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s\Delta l\Delta m \\
&= \int_{t_0}^{t}\int_{t_0}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))r(l)r(m)X_0^2\Delta l\Delta m \\
&\quad + \int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{l\wedge m} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta s\Delta l\Delta m \\
&= \left(\int_{t_0}^{t} \tilde{b}(t,\sigma(l))r(l)\Delta l\right)^2 X_0^2 \\
&\quad + \int_{t_0}^{t}\int_{\sigma(s)}^{t}\int_{\sigma(s)}^{t} \tilde{b}(t,\sigma(l))\tilde{b}(t,\sigma(m))\tilde{r}(l,\sigma(s))\tilde{r}(m,\sigma(s))\phi(s)\Delta m\Delta l\Delta s \\
&= (b*r)^2(t)X_0^2 + \int_{t_0}^{t}\left(\int_{\sigma(s)}^{t} \tilde{b}(t,\sigma(l))\tilde{r}(l,\sigma(s))\Delta l\right)^2 \phi(s)\Delta s \\
&= (b*r)^2(t)X_0^2 + \int_{t_0}^{t} (\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s,
\end{aligned}$$

where on the last equality we have used Theorem 2.1.

Theorem 3.2 *If X is a solution of* (3.4), *then*

$$\mathbb{E}\left[X^2(t)\right] = r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s.$$

Proof Squaring both sides of (3.1), we have

$$\begin{aligned} X^2(t) &= r^2(t)X_0^2 + 2r(t)X_0(r*f)(t) \\ &\quad + \int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))f(s_1)\Delta s_1 \int_{t_0}^{t} \tilde{r}(t,\sigma(s_2))f(s_2)\Delta s_2 \\ &= r^2(t)X_0^2 + 2r(t)X_0\ (r*f)(t) \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))\tilde{r}(t,\sigma(s_2))f(s_1)f(s_2)\Delta s_1\Delta s_2. \end{aligned}$$

Now taking the expectation on both sides of the above expression, we have

$$\begin{aligned} \mathbb{E}\left[X^2(t)\right] &= r^2(t)X_0^2 + 2r(t)X_0 \int_{t_0}^{t} \tilde{r}(t,\sigma(s))\mathbb{E}[f(s)]\Delta s \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{t} \tilde{r}(t,\sigma(s_1))\tilde{r}(t,\sigma(s_2))\mathbb{E}[f(s_1)f(s_2)]\Delta s_1\Delta s_2 \\ &= r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s, \end{aligned}$$

where on the second equality we have used Lemma 3.1.

Theorem 3.3 *Suppose X is a solution of* (3.4) *and r is a solution of* (3.2). *Then*

$$r,\ \tilde{r}(\cdot,s),\ \text{and } b*r \in L^2_\Delta(\mathbb{T}) \tag{3.5}$$

and

$$\int_{\sigma(s)}^{\infty} (\widetilde{b*r})^2(t,\sigma(s))\Delta t \le k < 1 \quad \text{for all} \quad s\in\mathbb{T}, \tag{3.6}$$

imply that $\mathbb{E}\left[X^2\right] \in L^1_\Delta(\mathbb{T})$.

Proof From Lemma 3.3, we have

$$\begin{aligned} \int_{t_0}^{\infty} \phi(t)\Delta t &= X_0^2\int_{t_0}^{\infty}(b*r)^2(t)\Delta t + \int_{t_0}^{\infty}\int_{t_0}^{t}(\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta s\Delta t \\ &= X_0^2\int_{t_0}^{\infty}(b*r)^2(t)\Delta t + \int_{t_0}^{\infty}\int_{\sigma(s)}^{\infty}(\widetilde{b*r})^2(t,\sigma(s))\phi(s)\Delta t\Delta s \\ &\le X_0^2\int_{t_0}^{\infty}(b*r)^2(t)\Delta t + k\int_{t_0}^{\infty}\phi(s)\Delta s. \end{aligned}$$

Simplifying and using the fact that $b*r\in L^2_\Delta(\mathbb{T})$, we have

$$\int_{t_0}^{\infty} \phi(t)\Delta t \le \frac{X_0^2}{1-k}\int_{t_0}^{\infty}(b*r)^2(t)\Delta t < \infty, \tag{3.7}$$

which implies that $\phi \in L^1_\Delta(\mathbb{T})$. Then from Theorem 3.2, we have

$$\begin{aligned}
\int_{t_0}^{\infty} \mathbb{E}\left[X^2(t)\right]\Delta t &= X_0^2\int_{t_0}^{\infty} r^2(t)\Delta t + \int_{t_0}^{\infty}\int_{t_0}^{t} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta s\Delta t \\
&\leq \alpha + \int_{t_0}^{\infty}\int_{\sigma(s)}^{\infty} \tilde{r}^2(t,\sigma(s))\phi(s)\Delta t\Delta s \\
&\leq \alpha + \beta\int_{t_0}^{\infty}\phi(s)\Delta s \\
&< \infty,
\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ such that

$$X_0^2\int_{t_0}^{\infty} r^2(t)\Delta t < \alpha$$

and

$$\int_{\sigma(s)}^{\infty} \tilde{r}^2(t,\sigma(s))\Delta t < \beta,$$

and this concludes the proof.

Example 3.1 For $\mathbb{T} = \mathbb{N}_0$, equation (3.5) reduces (with redundancy) to

$$\sum_{n=0}^{\infty} r^2(n) < \infty,$$

$$\sum_{n=m+1}^{\infty} r^2(n-m) < \infty, \quad \text{for all } m \in \mathbb{N}_0,$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} b(n-i-1)r(i)\right)^2 < \infty.$$

Similarly, equation (3.6) reduces to

$$\sum_{n=m+1}^{\infty}\left(\sum_{i=m+1}^{n-1} b(n-i-1)r(i-m-1)\right)^2 \leq k < 1 \quad \text{for all } m \in \mathbb{N}_0.$$

Remark 3.1 For $\mathbb{T} = \mathbb{R}$, $t_0 = 0$, and $\sup\mathbb{T} = \infty$, equations (3.5) and (3.6) reduce to

$$\int_0^{\infty} r^2(\tau)\mathrm{d}\tau < \infty,$$

$$\int_s^{\infty} r^2(\tau - s)\mathrm{d}\tau < \infty \quad \text{for all } s \in \mathbb{T},$$

$$\int_0^{\infty}\left(\int_0^{\tau} b(\tau-s)r(s)\mathrm{d}s\right)^2 d\tau < \infty,$$

and

$$\int_s^{\infty}\left(\int_s^{t} b(t-\tau)r(\tau-s)\mathrm{d}\tau\right)^2 \mathrm{d}t \leq k < 1 \quad \text{for all } s \in \mathbb{T}.$$

Acknowledgment

The author would like to thank Dr. Martin Bohner, Dr. Bonita Lawrence, Dr. A.A. Martynyuk, Dr. Alexey G. Mazko, and Dr. V. Sree Hari Rao for valuable suggestions that helped in improvement of this paper.

References

[1] Agarwal, R. P., Victoria, O., Perera, K., and Vivero, D. R. Basic properties of Sobolev's spaces on time scales. *Adv. Difference Equ.*, pages 1687-1839 Art. ID 38121, 14, 2006.

[2] Appleby, J. A. D., Devin, S., and Reynolds, D. W. Mean square convergence of solutions of linear stochastic Volterra equations to non-equilibrium limits. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 13B(suppl.): 515–534, 2006.

[3] Appleby, J. A. D. and Flynn, A. Stabilization of Volterra equations by noise. *J. Appl. Math. Stoch. Anal.*, pages Art. ID 89729, 29, 2006.

[4] Appleby, J. A. D., Győri, I., and Reynolds, D. W. On exact convergence rates for solutions of linear systems of Volterra difference equations. *J. Difference Equ. Appl.* **12** (12) (2006) 1257–1275.

[5] Appleby, J. A. D., Mao, X., and Riedle, M. Geometric Brownian motion with delay: mean square characterisation. *Proc. Amer. Math. Soc.* **137** (1) (2009) 339–348.

[6] Appleby, J. A. D. and Riedle, M. Almost sure asymptotic stability of stochastic Volterra integro-differential equations with fading perturbations. *Stoch. Anal. Appl.* **24** (4) (2006) 813–826.

[7] Bohner, M. and Guseinov, G. S. Improper integrals on time scales. *Dynam. Systems Appl.* **12** (1-2) (2003) 45–65.

[8] Bohner, M. and Guseinov, G. S. Riemann and Lebesgue integration. In: *Advances in dynamic equations on time scales*. Birkhäuser, Boston, 2003, 117–163.

[9] Bohner, M. and Guseinov, G. S. Partial differentiation on time scales. *Dynam. Systems Appl.* **13** (3-4) (2004) 351–379.

[10] Bohner, M. and Guseinov, G. S. Multiple integration on time scales. *Dynam. Systems Appl.* **14** (3-4) (2005) 579–606.

[11] Bohner, M. and Guseinov, G. S. Multiple Lebesgue integration on time scales. *Adv. Difference Equ.*, pages Art. ID 26391, 12, 2006.

[12] Bohner, M. and Guseinov, G. S. The convolution on time scales. *Abstr. Appl. Anal.*, pages Art. ID 58373, 24, 2007.

[13] Bohner, M. and Guseinov, G. S. Double integral calculus of variations on time scales. *Comput. Math. Appl.* **54** (1) (2007) 45–57.

[14] Bohner, M. and Peterson, A. *Dynamic equations on time scales. An introduction with applications.* Birkhäuser, Boston, 2001.

[15] Bohner, M. and Sanyal, S. The stochastic dynamic exponential and geometric Brownian motion on isolated time scales. *Commun. Math. Anal.* **8** (3) (2010) 120–135.

[16] Burton, T. A. *Volterra integral and differential equations*, volume 202 of *Mathematics in Science and Engineering*. Elsevier B. V., Amsterdam, Second edition, 2005.

[17] Burton, T. A. Integral equations, Volterra equations, and the remarkable resolvent: contractions. *Electron. J. Qual. Theory Differ. Equ.* (2006) 2–17. (electronic)

[18] Burton, T. A. and Mahfoud, W. E. Instability and stability in Volterra equations. In: *Trends in theory and practice of nonlinear differential equations*, *Lecture Notes in Pure and Appl. Math.* V. 90, Marcel Dekker, New York, 1984, 99–104.

[19] Grow, G. and Sanyal, S. Brownian motion indexed by a time scale. *Stoch. Anal. Appl.* To appear.

[20] Eloe, P. W., Islam, M. N., and Raffoul, Y. N. Uniform asymptotic stability in nonlinear Volterra discrete systems. *Comput. Math. Appl.* **45** (6-9) (2003) 1033–1039.

[21] Islam, M. N., and Raffoul, Y. N. Stability in linear Volterra integrodifferential equations with nonlinear perturbation. *J. Integral Equations Appl.* **17** (3) (2005) 259–276.

[22] Kaufmann, E. R. and Raffoul, Y. N. Discretization scheme in Volterra integro-differential equations that preserves stability and boundedness. *J. Difference Equ. Appl.* **12** (7) (2006) 731–740.

[23] Kulik, T., and Tisdell, C. C. Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains. *Int. J. Difference Equ.* **3** (1) (2008) 103–133.

[24] Lakshmikantham, V., Leela, S., and Martynyuk, A. A. *Stability analysis of nonlinear systems.* Marcel Dekker, Inc., New York, 1989.

[25] Mahdavi, M. Asymptotic behavior in some classes of functional differential equations. *Nonlinear Dyn. Syst. Theory.* **4** (1) (2004) 51–57.

[26] Rodkina, A. E. Stochastic Volterra integral equations. *Izv. Akad. Nauk Respub. Moldova Mat.* **93** (3) (1992) 9–15.

[27] Rodkina, A. E. and Basin, M. On delay-dependent stability for vector nonlinear stochastic delay-difference equations with Volterra diffusion term. *Systems Control Lett.* **56** (6) (2007) 423–430.

[28] Rodkina, A. E., Mao, X., and Kolmanovskii, V. On asymptotic behaviour of solutions of stochastic difference equations with Volterra type main term. *Stoch. Anal. Appl.* **18** (5) (2000) 837–857.

[29] Rodkina, A. E. and Schurz, H. A theorem on asymptotic stability of solutions of nonlinear stochastic difference equations with Volterra type noise. *Stab. Control Theory Appl.* **6** (1) (2004) 23–34.

[30] Sanyal, S. *Stochastic Dynamic Equations.* Ph.D. dissertation, Missouri University of Science and Technology, Rolla, Missouri, USA, 2008.

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 93–97

An Oscillation Criteria for Second-order Linear Differential Equations

J. Tyagi *

Department of Mathematics, Indian Institute of Technology Gandhinagar
Vishwakarma Government Engineering College Complex,
Chandkheda, Visat–Gandhinagar Highway, Ahmedabad, Gujarat, India – 382424

Received: April 15, 2010; Revised: January 24, 2011

Abstract: We establish an oscillation criteria for a class of second-order linear differential equations

$$(p(t)x'(t))' + q(t)x(t) = 0, \; t \in [0, \infty),$$

via Levin's comparison theorem. We employ an interval oscillation technique for oscillation of the above equation. This approach depends only on the behavior of q in certain interval. In this study, we allow the sign-changing nature of q. Using this approach, we also ascertain to answer the oscillatory behavior of a number of linear differential equations.

Keywords: *linear ordinary differential equations; oscillation.*

Mathematics Subject Classification (2000): 34Cxx, 34C10.

1 Introduction

We consider the second-order linear differential equations of the form

$$(p(t)x'(t))' + q(t)x(t) = 0, \tag{1}$$

where $p,\, q \in C([0,\infty), \mathbb{R})$, $p(t) > 0$ and $p\,x' \in C^1([0, \infty), \mathbb{R})$. When $p(t) \equiv 1$, (1) reduces to

$$x''(t) + q(t)x(t) = 0. \tag{2}$$

There is an extensive literature for the oscillation/non-oscillation of (1) and (2) (see [1–12]). Most of these results require the integral of the function q on the entire half interval

* Corresponding author: mailto:jtyagi1@gmail.com

$[0, \infty)$. Also, it is well-known that if $q(t)$ is of mean value zero and $q(t) \neq 0$, then (2) is oscillatory, (cf. [1]). We emphasize that the behavior of nonoscillatory solutions to certain second-order functional differential equations can be ascertained in terms of the oscillatory behavior of (2) (see [9]). Assuming the nonoscillation of (1), Tunc obtained some nonoscillation theorem for third-order nonlinear differential equations (see [7]). Let us recall the definition of interval oscillation.

If for each given solution of (1), we find a sequence of intervals $[\tau_n, \eta_n]$, $\tau_n \to \infty$, $\eta_n < \tau_{n+1}$ such that the given solution has at least one zero in (τ_n, η_n), for each $n \in \mathbb{N}$, then the solution is oscillatory.

By the above approach El–Sayed [2], gave some interval oscillation criteria for forced second-order linear differential equations. In the present study, the ideas of [2] are used to establish an interval oscillation criteria for (1). This approach depends only on the behavior of q in certain interval. Also, we do not restrict the sign of q. By this approach, we ascertain to answer the oscillatory behavior of a number of linear differential equations. Section 2 contains the preliminaries. Section 3 is devoted to the main result and its applications.

2 Preliminaries

We need the following lemmas for the proof of our main result. We consider

$$(p_1(t)x'(t))' + q(t)x(t) = 0, \tag{3}$$

$$(p_2(t)y'(t))' + r(t)y(t) = 0, \ \alpha \leq t \leq \beta, \tag{4}$$

where $p_1, p_2, q, r \in C([\alpha, \beta], \mathbb{R})$, $p_1(t) > 0$, $p_2(t) > 0$ and $p_1x', p_2x' \in C^1([\alpha, \beta], \mathbb{R})$.

Lemma 2.1 *Let $p_2(t) \geq p_1(t) > 0, \ \forall\, t \in [\alpha, \beta]$. Let x and y be nontrivial solutions of (3) and (4), respectively such that $x(t)$ does not vanish on $[\alpha, \beta]$, $y(\alpha) \neq 0$ and the inequality*

$$\frac{-p_1(\alpha)x'(\alpha)}{x(\alpha)} + \int_\alpha^t q(s)ds > \left| \frac{-p_2(\alpha)y'(\alpha)}{y(\alpha)} + \int_\alpha^t r(s)ds \right|, \tag{5}$$

holds for all $t \in [\alpha, \beta]$. Then $y(t)$ does not vanish on $[\alpha, \beta]$ and

$$-\frac{p_1(t)x'(t)}{x(t)} > \left| \frac{p_2(t)y'(t)}{y(t)} \right|, \ \alpha \leq t \leq \beta.$$

Proof Since $x(t)$ does not vanish on $[\alpha, \beta]$, so $w(t) = -\frac{p_1(t)x'(t)}{x(t)}$ on $[\alpha, \beta]$ transforms (3) to

$$w'(t) = q(t) + \frac{(w(t))^2}{p_1(t)},$$

which is equivalent to the integral equation

$$w(t) = w(\alpha) + \int_\alpha^t q(s)ds + \int_\alpha^t \frac{(w(s))^2}{p_1(s)} ds.$$

Since $y(\alpha) \neq 0$, so with the substitution $z(t) = -\frac{p_2(t)y'(t)}{y(t)}$ on some interval $[\alpha, \gamma]$, $\alpha < \gamma \leq \beta$ and using the hypothesis that $p_2(t) \geq p_1(t) > 0$, the proof of Lemma 2.1 is similar to the proof of Theorem 1.35 [6]. We omit the proof for the sake of brevity.

Lemma 2.2 Let $p_2(t) \geq p_1(t) > 0,\ \forall\, t \in [\alpha,\, \beta]$. Let x and y be nontrivial solutions of (3) and (4), respectively such that $x(t)$ does not vanish on $[\alpha,\, \beta]$, $y(\beta) \neq 0$ and the inequality

$$\frac{p_1(\beta)x'(\beta)}{x(\beta)} + \int_t^\beta q(s)ds > \left| \frac{p_2(\beta)y'(\beta)}{y(\beta)} + \int_t^\beta r(s)ds \right|, \tag{6}$$

holds for all $t \in [\alpha,\, \beta]$. Then $y(t)$ does not vanish on $[\alpha,\, \beta]$ and

$$\frac{p_1(t)x'(t)}{x(t)} > \left| \frac{p_2(t)y'(t)}{y(t)} \right|,\ \alpha \leq t \leq \beta.$$

Proof The proof of this lemma is similar to the proof of Theorem 1.36 [6]. For convenience, we give a brief sketch. We define new functions x_1, y_1, q_1, r_1, p_1^* and p_2^* on $[\alpha,\, \beta]$ by

$$\begin{aligned} &x_1(t) = x(\alpha+\beta-t), \quad y_1(t) = y(\alpha+\beta-t). \\ &q_1(t) = q(\alpha+\beta-t), \quad r_1(t) = r(\alpha+\beta-t). \\ &p_1^*(t) = p_1(\alpha+\beta-t), \quad p_2^*(t) = p_2(\alpha+\beta-t). \end{aligned}$$

Then $x_1(t)$ does not vanish on $[\alpha,\, \beta]$, $y_1(\alpha) = y(\beta) \neq 0$ and

$$\begin{aligned} -\frac{p_1^*(\alpha)x_1'(\alpha)}{x_1(\alpha)} + \int_\alpha^{\alpha+\beta-t} q_1(s)ds &= \frac{p_1(\beta)x'(\beta)}{x(\beta)} + \int_t^\beta q(s)ds, \\ -\frac{p_2^*(\alpha)y_1'(\alpha)}{y_1(\alpha)} + \int_\alpha^{\alpha+\beta-t} r_1(s)ds &= \frac{p_2(\beta)y'(\beta)}{y(\beta)} + \int_t^\beta r(s)ds. \end{aligned}$$

It is easy to observe that inequality (6) is equivalent to inequality (5) of Lemma 2.1 and using the fact that $t \in [\alpha,\, \beta] \Leftrightarrow \alpha+\beta-t \in [\alpha,\, \beta]$, the required conclusion follows from Lemma 2.1.

Lemma 2.3 *Let y be a nontrivial solution of* (4) *satisfying the conditions* $y(\alpha) = 0 = y(\beta) = y'(\gamma),\ \alpha < \gamma < \beta$. *Let* $p_2(t) \geq p_1(t) > 0,\ \forall\, t \in [\alpha,\, \beta]$. *If the inequalities*

$$\begin{aligned} \int_t^\gamma q(s)ds &\geq \left| \int_t^\gamma r(s)ds \right|, \\ \int_\gamma^t q(s)ds &\geq \left| \int_\gamma^t r(s)ds \right| \end{aligned}$$

hold for all $t \in [\alpha,\, \gamma]$ *and* $[\gamma,\, \beta]$ *respectively, then every solution of* (3) *has at least one zero on* $[\alpha,\, \beta]$.

Proof The proof of this lemma is similar to the proof of Theorem 1.37 [6] with the account of Lemmas 2.1 and 2.2. We omit the details.

3 Main Result

In this section, we prove the main result on oscillation for second-order linear differential equations.

Theorem 3.1 *Let there exist a monotonic sequence $\{\tau_n\} \subset \mathbb{R}^+$ such that $\tau_n \to \infty$, as $n \to \infty$ and a sequence $\{k_n\}$ of positive numbers such that*

$$\int_t^{\tau_n+\frac{\pi}{2\sqrt{k_n}}} q(s)ds \geq k_n \left(\tau_n + \frac{\pi}{2\sqrt{k_n}} - t\right), \forall\, t \in \left[\tau_n, \tau_n + \frac{\pi}{2\sqrt{k_n}}\right], \tag{7}$$

$$\int_{\tau_n+\frac{\pi}{2\sqrt{k_n}}}^t q(s)ds \geq k_n \left(t - \tau_n - \frac{\pi}{2\sqrt{k_n}}\right), \forall\, t \in \left[\tau_n + \frac{\pi}{2\sqrt{k_n}}, \tau_n + \frac{\pi}{\sqrt{k_n}}\right], \tag{8}$$

$\forall\, n \in \mathbb{N}$. *Also, let* $0 < p(t) \leq 1,\ \forall\, t \in [\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}]$. *Then* (1) *is oscillatory.*

Proof We prove this theorem by contradiction. Let x be a nontrivial solution of (1). Suppose x has finitely many zeros on $[0, \infty)$, so there exists a $\tau_0 > 0$ such that $x(t) \neq 0,\ \forall\, t \geq \tau_0$. We consider

$$y''(t) + k_n y(t) = 0,\ t \in [\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}],\ \tau_n \geq \tau_0 \text{ for some } n \in \mathbb{N}. \tag{9}$$

(9) has a solution $y(t) = \sin\sqrt{k_n}(t - \tau_n)$ which has two consecutive zeros at $t = \tau_n$ and at $t = \tau_n + \frac{\pi}{\sqrt{k_n}}$. Also, $y'(t) = 0$ at $t = \tau_n + \frac{\pi}{2\sqrt{k_n}}$. From (7) and (8), it is easy to observe that the hypotheses of Lemma 2.3 are fulfilled. An application of Lemma 2.3 yields that x has at least one zero on $[\tau_n,\, \tau_n + \frac{\pi}{\sqrt{k_n}}]$, which leads to a contradiction. Hence the proof is complete.

Remark 3.1 We introduce Liouville's transformation $x(t) = \sqrt{t}\, y(s),\ s = \log t$, which converts (2) to

$$y''(s) + Q(s)y(s) = 0, \tag{10}$$

where $Q(s) = q(e^s)e^{2s} - \frac{1}{4}$. Let $q \in C([0, \infty], \mathbb{R})$ and satisfies (7) and (8) $\forall\, n \in \mathbb{N}$, then (10) is oscillatory.

Remark 3.2 Let $P \in C^2([0, \infty), (0, \infty))$. The substitution $x(t) = y(t)P^{\frac{1}{2}}(t)$ converts (2) to

$$(P(t)y'(t))' + Q(t)y(t) = 0, \tag{11}$$

where $Q(t) = \frac{P''(t)}{2} + P(t)q(t) - \frac{(P'(t))^2}{4P(t)}$. An oscillation criteria for (2) gives an oscillation criteria for (11) and conversely.

Remark 3.3 Consider the equation

$$x''(t) + \frac{1}{t^2}x(t) = 0. \tag{12}$$

Let $\{\tau_n\} \subset \mathbb{R}^+$ be any monotonic, divergent sequence. We choose

$$k_n = \frac{1}{(\tau_n + \frac{\pi}{2\sqrt{k_n}})(\tau_n + \frac{\pi}{\sqrt{k_n}})},\ n \in \mathbb{N},$$

or after simplifying we have $k_n = \frac{8+5\pi^2+3\pi\sqrt{\pi^2+16}}{8\tau_n^2}$. With this choice of τ_n and k_n, it is easy to satisfy the hypotheses of Theorem 3.1. So, an application of Theorem 3.1 implies that (12) is oscillatory, while none of the known criteria (see [4, 5, 12]) can be applied to (12).

Example 3.1 Consider the differential equation

$$((1-\alpha\sin^2 t)x'(t))' + (1+2\cos t)x(t) = 0, \ 0 \le \alpha < 1. \tag{13}$$

(13) can be viewed as (1) with $p(t) = 1-\alpha\sin^2 t$, $q(t) = 1+2\cos t$. With the choice of $\tau_n = 2n\pi$, $k_n = \frac{1}{16}$, inequalities (7) and (8) are converted to

$$2\sin t + \frac{15t}{16} \le \frac{15}{16}(2n\pi + 2\pi), \ \forall\, t \in [2n\pi,\ (n+1)2\pi], \tag{14}$$

$$2\sin t + \frac{15t}{16} \ge \frac{15}{16}(2n\pi + 2\pi), \ \forall\, t \in [(n+1)2\pi,\ (n+2)2\pi]. \tag{15}$$

By simple calculus, it is easy to verify the inequalities (14) and (15). An application of Theorem 3.1 implies that (13) is oscillatory.

Remark 3.4 In (13), $q(t) = 1+2\cos t$, which mean value is non-zero and therefore the result given in [1] cannot apply to (13).

References

[1] Coppel, W. A. *Disconjugacy.* Lecture Notes in Math., Vol. 220, Springer-Varleg, Berlin, 1971.

[2] El-Sayed, M. A. An oscillation criterion for a forced second order linear differential equations. *Proc. Amer. Math. Soc.* **118** (1993) 813–817.

[3] Hartman, P. On nonoscillatory linear differential equations of second order. *Amer. J. Math.* **74** (1952) 389–400.

[4] Kamenev, I. V. An integral criterion for oscillation of linear differential equations of second order. *Math. Zametki* **23** (1978) 249–251.

[5] Leighton, W. The detection of oscillation of solutions of a second order linear differential equation. *Duke. Math. J.* **17** (1950) 57–62.

[6] Swanson, C. A. Comparison and oscillation theory of linear differential equations, Vol. 148, Academic Press, New York and London, 1968.

[7] Tunc, C. On the non-oscillation of solutions of some nonlinear differential equations of third order. *Nonlinear Dynamics and Systems Theory* **7** (4) 419–430.

[8] Tyagi, J. An oscillation theorem for a second order nonlinear differential equations with variable potential. *Electron. J. Diff. Equations* (19) (2009) 1–5.

[9] Tyagi, J. Oscillation of solutions and behavior of the nonoscillatory solutions of second-order nonlinear functional equations. *Nonlinear Dynamics and Systems Theory* **9** (3) (2009) 317–326.

[10] Tyagi, J. and Raghavendra, V. A note on a generalization of Sturm's comparison theorem. *Nonlinear Dynamics and Systems Theory.* **8** (2) (2008) 213–216.

[11] Tyagi, J. and Raghavendra, V. An oscillation criteria for second-order nonlinear differential equations with functional arguments. *Electron. J. Diff. Equations* (30) (2009) 1–7.

[12] Wintner, A. A criterion of oscillatory stability. *Quat. Appl. Math.* **7** (1949) 115–117.

Nonlinear Dynamics and Systems Theory, **11** (1) (2011) 99–112

Internal Multiple Models Control Based on Robust Clustering Algorithm

A. Zribi *, M. Chtourou and M. Djemel

Research Unit on Intelligent Control, Design and Optimisation of Complex Systems (ICOS)
Department of Electrical Engineering, ENIS, B.P. 1173, Sfax, Tunisia.

Received: June 6, 2010; Revised: February 1, 2011

Abstract: In this paper, Internal Multiple Model Control (IMMC) based on Robust Clustering Algorithm (RCA) is proposed. The IMMC requires, firstly, the definition of set a of local models each one valid in a given region. Different strategies exist in the literature dealing with the determination of the local models base. However, most of these strategies need a priori knowledge of the system. In order to overcome this difficulty, a RCA is proposed to find the optimum number of clusters. In the second step, the obtained data relative to each cluster will be used to build the local models base. Finally, the internal model control (IMC) structure will be developed using the models base where a linear controller will be constructed for every model. The efficiency of the IMMC based on RCA is demonstrated through an uncertain linear system and by the control of a neutralization of PH reaction in a Continuously Stirred Tank Reactor (CSTR).

Keywords: *IMMC; multiple models; RCA; PH neutralization system.*

Mathematics Subject Classification (2000): 93C42, 93B12, 92B20.

1 Introduction

In the case of linear plants, IMC have been extensively studied due to its robustness properties against disturbances and a model mismatch [21, 17, 7]. It uses the process model as the internal model to predict the process output. However, many industrial systems exhibit strong nonlinear behavior and they may be required to operate over a wide range of operating conditions. Additionally, there are situations where the nonlinear plants are extremely difficult to model and they exhibit high uncertainties [2, 3]. Under these

* Corresponding author: mailto:ali_zribi@yahoo.fr

conditions, the multi-model approach is an efficient and a powerful way to resolve problem of modeling and control of complex and non-linear processes [14, 15, 20, 10, 18, 16]. The past few years have shown an increase in the use of the multi-model representation combined with the IMC. The modeling concept includes a number of approaches such as: Takagi and Sugeno Fuzzy Inference Systems [14], local neural networks [1, 22]. However, these approaches remain so confronted with several difficulties such as the determination of the local models base. To resolve this problem, a RCA is proposed to determine the models base for complex systems. This approach is an unsupervised classification which does not require a priori knowledge about the system and uses a robust estimator to find the optimal number of clusters by repeatedly merging similar clusters [4]. The IMMC can be summarizing in three steps. The first step consists in dividing the systematic space in some subspaces using the RCA where a criterion is developed to find the optimal partition. In the second step, a local model is built for every subspace. Finally, the local models base will be combined with the IMC structure. We will show that IMMC has strong robustness under the conditions of modelling uncertainties. It can effectively compensate the modeling error of the plant by using this error as a feedback signal.

The remainder of this paper is organized as follows. In Section 2, we present the RCA. Section 3 presents a description of the IMMC and the validities computations. In Section 4, to check the ability of the proposed approaches, two examples have been considered. Finally, Section 5 provides the conclusion.

2 Robust Clustering Algorithm

The Robust Competitive Agglomeration (RCA) algorithm [5, 6] is a fuzzy partitional algorithm which does not require the number of clusters to be specified. Let $X = \{x_i/i \in \{1 \dots N\}\}$ be a set of N inputs vectors. Let $V = \{V_j/j \in \{1 \dots C\}\}$ represent prototypes of the clusters. The RCA algorithm minimizes the following objective function:

$$J_R(U,V) = \sum_{j=1}^{C}\sum_{i=1}^{N}(u_{ji})^2\rho_j\left(d_{ji}^2\right) - \alpha\sum_{j=1}^{C}\left[\sum_{i=1}^{N}w_{ji}u_{ji}\right]^2. \tag{1}$$

In (1), d_{ji} stands for the distance from the input vector x_i to the center v_j and u_{ji} is the membership of x_i to cluster j. $\rho_j()$ is a robust loss function associated with cluster j, and $w_{ji} = w_j\left(d_{ji}^2\right) = \frac{\partial\rho_j\left(d_{ji}^2\right)}{\partial d_{ji}^2}$ represents the typicality of point x_i with respect to cluster j. The function $\rho_j\left(\right)$ corresponds to the loss function used in M-estimators of robust statistics and $w_j\left(\right)$ represents the weight function of an equivalent W-estimator [4]. By minimizing both terms in (1) simultaneously, the data set will be partitioned into the optimal number of clusters while clusters will be arranged in order to minimize the sum of intracluster distances [4].

Membership of cluster s can be written as [5]:

$$u_{st} = \frac{\frac{1}{\rho_s\left(d_{st}^2\right)}}{\sum\limits_{j=1}^{C}\frac{1}{\rho_j\left(d_{jt}\right)^2}} + \frac{\alpha}{\rho_s\left(d_{st}^2\right)}(N_s - \overline{N_t}) = u_{st}^{RR} + u_{st}^{Bias}, \tag{2}$$

where N_s represents the robust cardinality of the cluster s. It is defined by [5]:

$$N_s = \sum_{i=1}^{N} w_{si} u_{si}, \tag{3}$$

where $\overline{N_t}$ is the weighted average of cardinalities and it is defined by the following equation [5]:

$$\overline{N_t} = \frac{\sum\limits_{j=1}^{C} \frac{1}{\rho_j (d_{jt})^2} N_j}{\sum\limits_{j=1}^{C} \frac{1}{\rho_j (d_{jt})^2}}. \tag{4}$$

The first term in equation (2) is the membership term in the FCM algorithm using a robust distance. The second term leads to a reduction of cardinality of spurious clusters, which are discarded if their cardinality drops below a threshold [6]. So only good clusters are conserved. For clusters with cardinality higher than the average, the bias term is positive, thus appreciating the membership value. On the other hand, for low cardinality clusters, the bias term is negative, thus depreciating the membership value. It should be noted that when a feature point x_t is close to only one cluster s, and far from other clusters, we have:

$$N_s \approx N_t \Leftrightarrow u_{st}^{Bias} = 0. \tag{5}$$

In this case the membership value, u_{st} is independent of the cluster cardinalities, and is reduced to u_{st}^{RR}. In other words, if a point is close to only one cluster, it will have high membership value in this cluster and no competition is involved. On the other hand, if a point is close to many clusters, these clusters will compete for this point based on cardinality.

The parameter α should provide a balance between the two terms of (1), so α at iteration k is defined by [6]:

$$\alpha(k) = \eta_0 \exp\left(-\frac{k}{\tau}\right) \frac{\sum\limits_{j=1}^{C} \sum\limits_{i=1}^{N} \left(u_{ji}^{(k\text{-}1)}\right)^2 \rho_j\left(d_{ji}^2\right)}{\sum\limits_{j=1}^{C} \left[\sum\limits_{i=1}^{N} w_{ji}^{(k-1)} u_{ji}^{(k-1)}\right]^2}, \tag{6}$$

where η_0 is the initial value, and τ is the time constant. The exponential factor makes the second term preponderant in a first time to reduce the number of cluster, and then the first term dominates to seek the best partition of the data.

2.1 Weight function

The RCA technique proposed by Frigui and Krishnapuram [4, 5] tried to make the data partitioning robust by using the weight functions of a robust statistical law. The argument of the weight function consists of the squares of the distances. The weight function is chosen a monotonically nonincreasing function as defined below [4]:

$$w_j\left(d_{ji}^2\right) = \begin{cases} 1 - \frac{d_{ji}^4}{2T_j^2}, & if\ d_{ji}^2 \in [0, T_j], \\ \frac{\left[d_{ji}^2 - (T_j + cS_j)\right]^2}{2cS_j^2}, & if\ d_{ji}^2 \in]T_j, T_j + cS_j], \\ 0, & if\ d_{ji}^2 \succ T_j + cS_j, \end{cases} \tag{7}$$

where $T_j = MED_j\left(d_{ji}^2\right)$ and $S_j = MAD_j\left(d_{ji}^2\right)$, MED_j is the median of the residuals of the *j-th* cluster and MAD_j is the median of absolute deviations of the *j-th* cluster.

The loss function associated with this weight function can be derived by integrating (7). This yields:

$$\rho_j\left(d_{ji}^2\right) = \begin{cases} d_{ji}^2 - \frac{d_{ji}^6}{6T_j^2}, & if\ d_{ji}^2 \in [0, T_j], \\ \frac{\left[d_{ji}^2 - (T_j + cS_j)\right]^3}{6cS_j^2} + \frac{5T_j + cS_j}{6}, & if\ d_{ji}^2 \in]T_j, T_j + cS_j], \\ \frac{5T_j + cS_j}{6} + K_j, & if\ d_{ji}^2 \succ T_j + cS_j. \end{cases} \tag{8}$$

In (8) K_j is a constant used to make all $\rho_j\,()$ functions reach the same maximum value.

$$K_j = \max_{1 \le j \le C}\left\{\frac{5T_j + cS_j}{6}\right\} - \frac{5T_j + cS_j}{6}. \tag{9}$$

The constants K_j are added to prevent assigning all noise points to the most compact cluster. By forcing all functions to have the same maximum value, all noise points will have the same membership value in all clusters.

2.2 Algorithm outline

The RCA can be summarized by the following steps:
Step 1: Fix the maximum number of clusters, initialise the vector center, and set $k = 1$, w_{ji}=1 and c_0=12.
Step 2: Compute d_{ji} , estimate T_j and S_j.
Step 3: Update the weight w_j, $\alpha(k)$ and the partition matrix $U^{(k)}$.
Step 4: Compute the robust cardinality N_j , if $N_j \prec \varepsilon$ discard cluster *j*.
Step 5: Update the number of cluster C and $k = k + 1$.
Step 6: Update the tuning factor $c_k = \max(4, c_{k-1} - 1)$ and the center parameters.
Step 7 : Test of the convergence: if the center parameters stabilize then stop otherwise go to step 2.

The Mahanobis distance given by (10) has been used in this investigation. The update equations for the centers v_j and the covariance matrices are given by (11) and (12):

$$d^2\left(x_i, v_j\right) = \left|A_j\right|^{\frac{1}{n}}\left(x_i - v_j\right)^T A_j^{-1}\left(x_i - v_j\right), \tag{10}$$

$$v_j = \frac{\sum_{i=1}^{N}\left(u_{ji}\right)^2 w_{ji} x_i}{\sum_{i=1}^{N}\left(u_{ji}\right)^2 w_{ji}}, \tag{11}$$

$$A_j = \frac{\sum_{i=1}^{N}\left(u_{ji}\right)^2\left(x_i - v_j\right)^T\left(x_i - v_j\right)}{\sum_{i=1}^{N}\left(u_{ji}\right)^2 w_{ji}}. \tag{12}$$

2.3 Validity criteria

To assure an accurate modeling step, a robust competitive criterion D_c has been introduced. It is based on the comparison between the global cardinality and the average cardinality. The global cardinality of a partition is defined by:

$$C_G = \frac{1}{N}\sum_{i=1}^{N}\frac{\sum_{j=1}^{C}\frac{1}{\rho_j\left(d_{ji}^2\right)}N_j}{\sum_{j=1}^{C}\frac{1}{\rho_j\left(d_{ji}^2\right)}}. \tag{13}$$

The cardinality for one class represents the average distribution of the points around the center. Then the average cardinality is computed from the cardinality of all the classes in order to obtain a value which translates the distribution of the points around the center of the classes. The average cardinality of the created clusters is defined by:

$$C_V = \frac{1}{C}\sum_{j=1}^{C}\sum_{i=1}^{N} w_{ji}u_{ji} \tag{14}$$

The average and the global cardinality are equivalent on condition that the partitioning is optimal. That is to say when the obtained clusters correspond to the real classes, the ratio between average and global cardinality tends towards one.

The robust criterion validity D_c which reflects the state of the partition is given by:

$$D_c = \left|1 - \frac{C_G}{C_V}\right|. \tag{15}$$

When the optimal partition is attained the ratio in (15) tends to *1* and so the criterion D_c is minimal. So, the optimal partition is obtained for the minimum value of D_c.

3 The Principle of IMMC Based on Classifier

In IMC, a plant model is placed in parallel with the real plant [17]. The difference between the plant and the model outputs is used for feedback purposes. The feedback signal is an estimate of the plant disturbances or the model mismatch. For linear plant, IMC have been shown good robustness properties against disturbances and model mismatches [19]. In this paper, the linear IMC strategy will be investigated to control uncertain systems using multiple models. According to the characteristics of the controlled plant, the design principle of IMMC based on RCA is as follows, firstly, divide the data into some subspaces using the RCA. The second step is a structural and parametric estimations step in order to determine the local base models. In fact, a partional linear model is built to every subspace. In the third step, the IMC structure will be combined with the local models base where a linear controller will be constructed for every local model.

The filter F in the IMC structure is employed to introduce robustness into the controller to deal with plant uncertainty [11].

3.1 Development of local linear models

The core idea is to represent the uncertain nonlinear dynamic system by a set of locally valid sub-linear models across the operating range. Each model is developed around

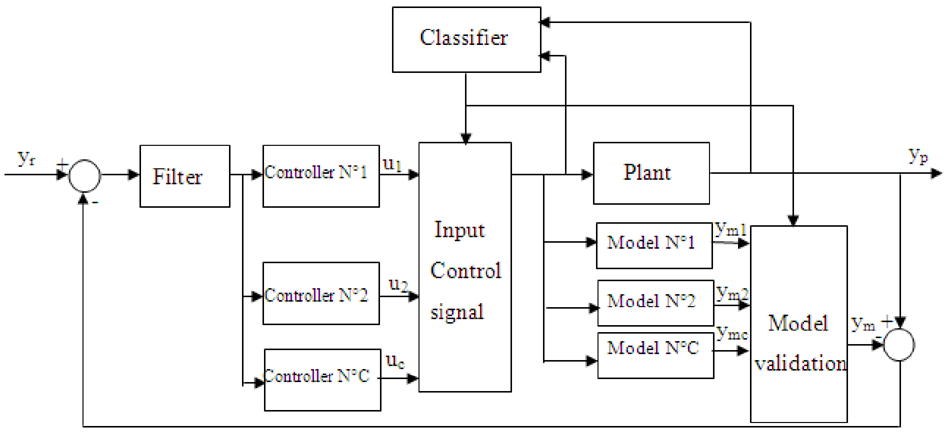

Figure 1: Structure of IMMC.

an operation range. A structural and a parametric identification must be carried out to elaborate the related local model. The established models are constructed using the ARX structure given by the following relation:

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{j=1}^{m} b_j u(k-j), \tag{16}$$

where u is the input to the unknown system, y is the output system, a_i and b_j are the parameters of the ARX model. The parametric identification uses the Recursive Least Square (RLS) method and exploits the observation-vectors related to every cluster.

3.2 Development of local controllers

For minimum-phase processes (stable with no time delay or zeros outside the unit circle), IMC can produce perfect control based on a controller designed as the inverse of the process. When dealing with a non minimum phase process, this procedure cannot be used directly since the transfer function of plant $\hat{G}$ is not invertible. One approach to handle a noninvertible process model is to apply the following factorization

$$\hat{G}(z) = \hat{G}^{-}\ \hat{G}^{+}, \tag{17}$$

where $\hat{G}^{+}$ contains all the zeros outside the unit circle and all the time delays and $\hat{G}^{-}$ is then inverted for controller design [9].

3.3 Control strategies

In IMMC, two control strategies are considered. The first one is based on the switching between different models. This method consists in choosing the nearest model to the process which leads to the least modelling error. The appropriate controller is then obtained from the validated inverse model. In the second strategy, the internal model

and the controller outputs are obtained by fusion of generated models and controllers pondered by a validity criterion. The global model and control outputs are presented by the following equations.
In the first strategy:

$$y_m(k) = y_{mj}(k), \quad where \quad d_{jk} = \min_{1\le t\le C}(d_{tk}), \tag{18}$$

$$u_c(k) = u_j(k), \qquad where \quad d_{jk} = \min_{1\le t\le C}(d_{tk}). \tag{19}$$

In the second strategy:

$$y_m(k) = \sum_{j=1}^{C} r_j(k)\; y_{mj}(k), \tag{20}$$

$$u_c(k) = \sum_{j=1}^{C} r_j(k)\; u_j(k), \tag{21}$$

where C is the number of models in the library. y_{mj} and u_j are respectively the outputs of model M_j and its corresponding controller. r_j is the validity criterion.

Several validities computation methods were proposed in the literature [8, 13, 12]. All these methods are based on measuring the distance between the current state of the process and the model M_j. The proposed method of validities computation is inspired from the fuzzy version where the cluster's parameters obtained from the RCA are exploited. It evaluates the contribution of the model to describe the system behaviors in its full range

$$r_j(k) = \frac{1 - \frac{d_{ji}}{\sum\limits_{j=1}^{C} d_{ji}}}{C-1}. \tag{22}$$

4 Simulation Results

In order to evaluate the performances of the presented algorithms, two examples will be considered. The first one concerns the control of an uncertain linear system. The second example treats the control of a chemical process which is a PH neutralization process.

4.1 Uncertain linear system

Let us consider a linear system with uncertain parameters described by [20]:

$$y^p(k) = -a_1(k)y^p(k-1) - a_2(k)y^p(k-2) + b_1(k)u(k-1) + b_2(k)u(k-2), \tag{23}$$

$$a_1(k) = -0.8 + 0.08 \sin\left(\frac{2\pi k}{200}\right), \tag{24}$$

$$a_2(k) = 0.1 + 0.01 \sin\left(\frac{2\pi k}{200}\right), \tag{25}$$

$$b_1(k) = 0.5 + 0.04 \sin\left(\frac{2\pi k}{200}\right), \tag{26}$$

$$b_2(k) = 0.2 + 0.02 \sin\left(\frac{2\pi k}{200}\right). \tag{27}$$

The system is excited by a random sequence $[0, 3]$ to generate the necessary data base. 2000 samples are used where $[y^p(k),\ y^p(k-1), u(k-1), u(k-2)]$ is the vector to be clustered. The minimization of the criterion D_c allows to find the optimal partition corresponding to the number of clusters. Table 1 illustrates the results obtained for this example where the minimum is detected for two clusters.

Number of cluster	2	3	4	5	6
D_c	0.08	0.401	0.46	0.586	0.667

Table 1: D_c criterion of the first example.

The clustered data based on D_c criterion are presented in Figure 2.

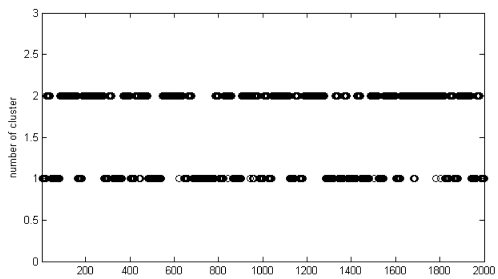

Figure 2: Clustered data based on RCA (example 1).

The following stage is to estimate the parameters of each local model. A second order ARX model has been used as the model structure. The parameters of each local model are given in Table 2.

Models	M_1	M_2
a_1	1.2973	1.6661
a_2	-0.4457	-0.7151
b_1	0.5203	0.5064
b_2	-0.1706	-0.396

Table 2: Model Parameters of the first example.

In order to compare performances of the control strategies, a criterion will be defined and given by the following equation:

$$E = \frac{1}{N} \sum_{k=1}^{N} [y^r(k) - y^p(k)]^2. \tag{28}$$

The responses of IMMC using the first and the second control strategies are given by the following figures.

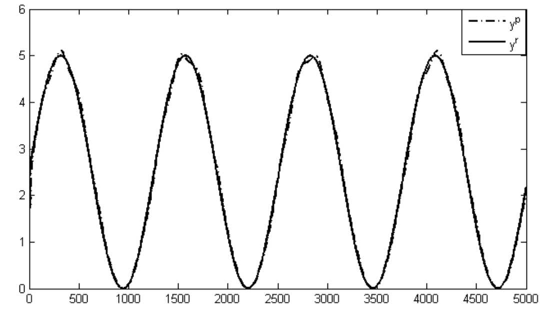

Figure 3: IMMC output evolution using the first strategy for set point tracking.

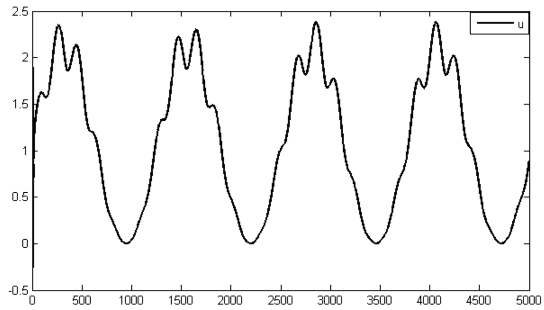

Figure 4: IMMC control input evolution using the first strategy.

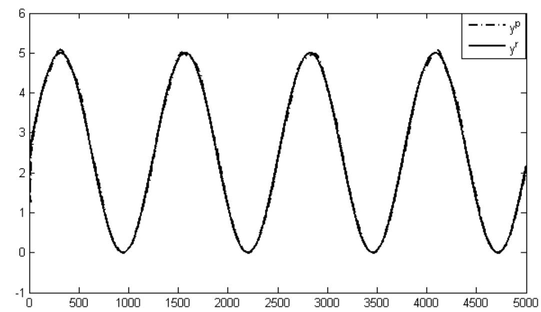

Figure 5: IMMC output evolution using the second strategy for set point tracking.

In the first strategy control $E = 0.0658$. In the second strategy control $E = 0.0522$. These figures show that the plant output y^p follows the desired output y^r of the uncertain process. The obtained performances show that second approach is slightly more accurate.

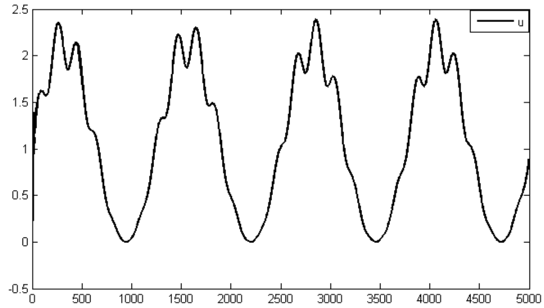

Figure 6: IMMC control input evolution using the second strategy.

4.2 Uncertain nonlinear system: PH neutralization system

In order to show the performance of the IMMC, it will be applied to the case of the PH neutralization system. It is a well-known benchmark problem and it has two input streams: sodium hydroxide and acetic acid. For collection of the data, a sampling time of 12 second has been used [1].

Parameters	Description	Nominal Value
v	Volume of the tank	1000 [l]
q_1	Flow rate of acetic acid	81 [l/min]
C2	Inlet concentration of NaOH	0.05 [mol/l]
C1	Inlet concentration of acetic acid	0.32 [mol/l]
C_A	Initial concentration of sodium in the CSTR	0.0432 [mol/l]
C_B	Initial concentration of acetate in the CSTR	0.0432 [mol/l]
K_a	Acid equilibrium constant	1.753 10-5
K_b	Inlet concentration of acetic acid	10-14

Table 3: Parameters description.

Number of cluster	2	3	4	5	6
D_c	0.5899	0.0303	0.2088	0.1307	0.156

Table 4: D_c criterion for the second example.

The process model consists of two nonlinear ordinary differential equations and a nonlinear output equation for the PH.

$$\begin{cases} v\frac{dC_A}{dt} = q_1C_1 - (q_1+q_2)C_A, \\ v\frac{dC_B}{dt} = q_2C_2 - (q_1+q_2)C_B, \\ 10^{3PH} + 10^{2PH}\,(K_a + C_B) + 10^{PH}(C_BK_a - C_AK_a - K_w) - K_a + K_w. \end{cases} \tag{29}$$

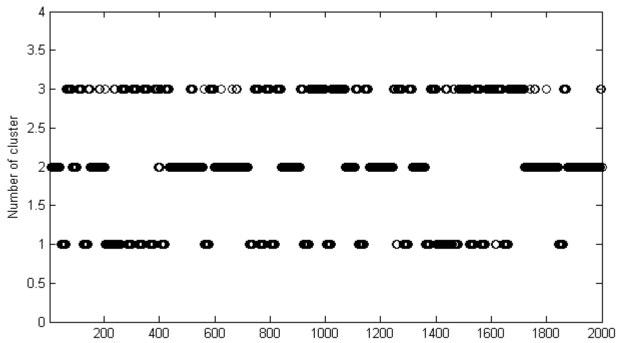

Figure 7: Clustered data based on RCA (example 2).

Models	M_1	M_2	M_3
a_1	1.15	1.16	1.131
a_2	-0.2379	-0.2148	-0.1797
b_1	0.01025	0.04474	0.009975
b_2	0.001234	- 0.01256	- 0.004141

Table 5: Model Parameters of the second example.

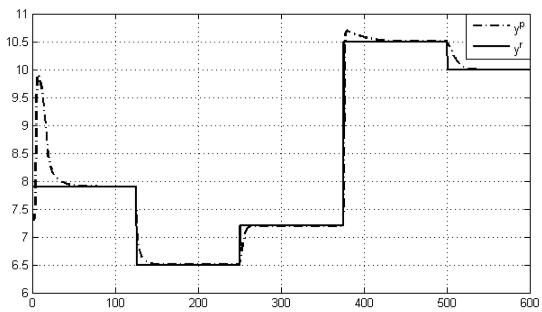

Figure 8: IMMC output evolution for PH tracking using the first strategy.

The process output is the PH and the input is the sodium flowrate q_2. The parameters used in the simulations are described and given in Table 3. The system is excited by a control input q_2 of random amplitude in the range $[512; 525]$ with duration of 20 sampling periods; the total length of the sequence is 2000. Three clusters are identified. The number of cluster was determined using the Table 4 where the minimum is detected for three clusters. The clustering results of the PH neutralization system are presented in the Figures 7.

The model basis considered consists of three local models. They are described by

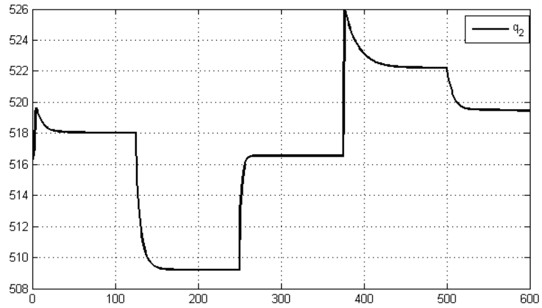

Figure 9: IMMC control input evolution using the first strategy.

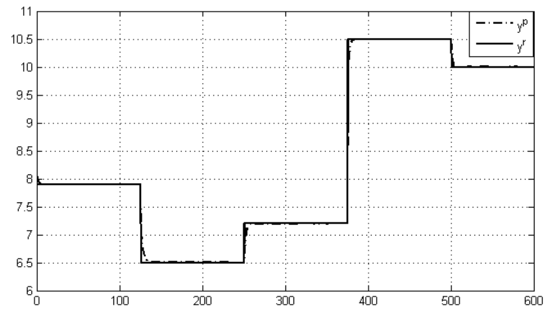

Figure 10: IMMC output evolution for PH tracking using the second strategy.

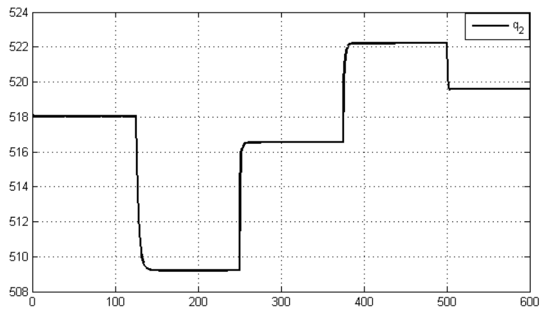

Figure 11: IMMC control input evolution using the second strategy.

discrete transfer functions having the same structure where the parameters of the process are given by the Table 5. The PH responses of the system using different strategies are shown in Figures 8 and 10. In the first strategy control $E = 0.0753$. In the second strategy control $E = 0.0079$.

Using the first strategy, Figure 8 shows that there is clearly a poor performance. However, when models are pondered more satisfactory tracking behavior (especially in control action) is detected. Indeed, the obtained relative errors confirm the robustness of the second strategy. The difference from the previous simulations can be attributed to the model plant mismatch.

5 Conclusion

Considering the complexity to control uncertain plants, IMMC based on RCA is proposed in view of all the advantage of multiple model and internal model control. The IMMC can be applied in three steps. The primary step consists in determining the suitable number of local base models using the RCA. The second step is parametric estimations step in order to determine the local base models. In fact, a partional linear model is created to every subspace. In the third step, the IMC structure will be combined with the local base models where a controller will be constructed for every model. The application of this approach is carried out on two simulation examples of uncertain systems. The simulation results show that IMMC is more robust when the models and controllers are weight on global model and controller.

References

[1] Bhat, N.V. and McAvoy, T.J. Determining model structure for neural models by network stripping. *Computers and Chemical Engineering* **16** (1992) 271–281.

[2] Boukezzoula, R., Galichet, S. and Foulloy L. Nonlinear Internal Model Control: Application of Inverse Model Based Fuzzy Control. *IEEE Transactions on Fuzzy Systems* **11** (6) (2003) 814–829.

[3] Brown, M.D., Lightbody, G. and Irwin G.W. Nonlinear internal model control using local model networks. *IEEE Proc. Control Theory Appl.* **144** (6) (1997) 505–514.

[4] Frigui,H. and Krishnapuram, R. A Robust Competitive Clustering Algorithm With Applications in Computer Vision. *IEEE Trans. Pattern Anal. Mach. Intell* **21** (5) (1999) 450–465.

[5] Frigui, H. and Krishnapuram, R. A robust algorithm for automatic extraction of an unknown number of clusters from noisy data. *Pattern Recognition Letters* **17** (1996) 1223–1232.

[6] Frigui, H. and Krishnapuram, R. Clustering by competitive agglomeration. *Pattern Recognition* **30** (7) (1997) 1109–1119.

[7] Gala, O., Romagnoli, J.A., Palazoglu, A. and Arkun Y. Gap Metric Concept and Implications for Multilinear Model-Based Controller Design. *Ind. Eng. Chem. Res.* **42** (2003) 2189–2197.

[8] Gasso, K. *Identification des systmes dynamiques non linaires: Approche multimodle*. Thesis, Institut National Polytechnique de Lorraine, France, 2000.

[9] Garcia, C.E. and Moran, M. Internal Model Control. I. A Unifying Review and Some Neu Result.*Ind. Eng. Chem. Process Des. Dev.* **21** (1982) 308–323.

[10] Gawthrop, P.J. Continuous-time local state local model networks. *Proceeding of IEEE Conference on Systems, Man and Cybernetics* (1995) 852–857.

[11] Hunt, K.J. and Sbarbaro, D. Adaptive filtering and neural networks for realisation of internal model control. *Intelligent Systems Engineering* **2** (2) (1993) 67–76.

[12] Johansen, T.A. and Foss, B.A. Constructing NARMAX models using ARMAX models. *Int. J. Control* **58** (1993) 1125–1153.

[13] Johansen, T.A. and Foss, B.A. Multiple model approaches to modelling and control. *Int. J. Control* **72** (1999) 575.

[14] Jun, L., Wan-li, W. and Xiu-hua, D. Multiple Model Internal Model Control Based on Fuzzy Membership Function. *Proceedings of the IEEE International Conference on Automation and Logistics Qingdao*, 2008.

[15] Lahmari, K. *Contribution la commande multi-modle des processus complexes.* Thesis, UST de Lille, 1999.

[16] Mezghani, S., Elkamel, A. and Borne, P. Multi-model control of discrete systems with uncertainties. *The Electronic International Journal of Advance Modilling and Optimization* **32** (2001) 7–17.

[17] Morari, M. and Zafiriou, E. *Robust Process Control.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[18] Narendra, K.S. and Balakrishnan. Adaptative Control Using Multiple Models. *IEEE Transactions on Automatic Control* **42** (1997) 171–187.

[19] Rivals, I. and Personnaz, L. Nonlinear internal model control using neural networks: Application to processes with delay and design issues. *IEEE Transactions on Neural Networks* **11** (2000) 80–90.

[20] Talmoudi, S., Abderrahim, k., Ben Abdennour, R; and Ksouri, M. Multi-model Approach using Neural Networks for Complex Systems Modeling and Identification. *Nonlinear Dynamics and Systems Theory* **8** (3) (2008) 299–316.

[21] Wang, Q.G., Bi, Q. and Zhang, Y. Partial Internal Model Control. *IEEE Transactions on Industrial Electronics* **48** (5) (2001).

[22] Zhao, Z., Liu, Z., Wen, X. and Zhang, J. *A new multi-model internal model control scheme based on neural network.* Proceedings of the 7th World Congress on Intelligent Control and Automation, Chongqing, China, 2008.