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PERSONAGE IN SCIENCE

Professor G. Leitmann

on for 85th birthday

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This paper presents a biographical sketch and a review of scientific achievements of George Leitmann, who made outstanding contributions in contemporary nonlinear dynamics and systems theory.

1 Introduction

George Leitmann was born in Vienna, Austria on May 24, 1925. He emigrated, with his mother and two grandmothers, to the USA in 1940. Leitmann entered a technical high school in New York from which he graduated in 1943. Immediately after graduation he joined the US army and served in the reconnaissance unit of a Combat Engineer Battalion in France in 1944-45. For his role in the battle for Colmar, Leitmann was awarded the Croix de Guerre avec Palmes. At the end of the war, he was assigned to the Counterintelligence Corps as its youngest special agent and served as an interrogator at the Nuremburg war crimes trial in 1946. After his discharge from military service in 1946, Leitmann began his university education at Columbia University in New York. He received a BA degree in physics in 1949 and an MS degree in physics, with research on secondary electron emission, in 1950. From 1950-57 he worked as a physicist and then as head of aeroballistics at the rocket development center, China Lake, California. During that period he also enrolled in the University of California, Berkeley, from which he received the PhD in engineering science, with a dissertation on the exterior ballistics of high altitude rockets, in 1956. He joined the engineering faculty of the University of California, Berkeley as an Assistant Professor of engineering science in 1957; he was advanced to Associate Professor in 1959 and to Professor in 1963. During his employment at the rocket development center, Leitmann worked on the design and testing of a variety

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of rockets. From the outset in 1950 he was concerned with optimization of rocket trajectories by controlling the thrust magnitude and direction. He had already done research on this problem while preparing his PhD dissertation; he became especially interested in this topic meeting the famous Chinese scientist H.S. Tsien then at the California Institute of Technology. Of Leitmann's many published results in the 1950's and early 1960's, [1]–[6] and Chapter 5 of [A] constitute a sample. They are primarily based on the Calculus of Variations and extensions to allow for inequality constraints, already treated in the 1930's by G.A Bliss at the University of Chicago. To place Leitmann's contributions to rocket trajectory optimization within the extensive body of work in this area, see [7]. After exposure to the calculus of variations and somewhat later to the maximum principle as well as to Bellman's dynamic programming, Leitmann decided to compare various optimization techniques, particularly those employed to address aerospace — related problems; see [A]. In the 1960's, a meeting with Austin Blaquiere revealed that both Blaquiere and Leitmann were seeking an optimization method which is non-variational in contrast to the calculus of variations and the maximum principle, both of which are based on comparison of solutions. Fortuitously, both Blaquire and Leitmann were thinking about a geometric approach; this led to a long period of collaboration and a life-long friendship. In 1955 Leitmann married Nancy Lloyd. They have a son and a daughter as well as three grandchildren.

2 Basic Trends in Leitmann's Research

2.1 A geometric approach to optimal control and dynamic games

The geometric approach to optimal processes, initiated by A. Blaquiere and G. Leitmann [8], [B], Chapter 7 of [C], [9], [J], is not only an alternative avenue to the necessary optimality conditions embodied in the pioneering works of the Pontryagin school, but an investigation of the complex structure of optimal processes. This approach is based on global properties of optimal processes in cost-augmented state space and utilizes local aspects of these global properties to arrive at necessary conditions for optimality. This becomes even more important in dynamic games, initially treated for two-person zero-sum differential games by R. Isaacs. Various aspects of these zero-sum games were investigated at first for both qualitative and quantitative games [10], [11], [D], [E]. Subsequently, these results were extended to non-cooperative many-player games [F]. Many applied problems, especially in OR and economics, were treated via the geometric-approach-based theory, [12]–[15]. Since necessary optimality conditions yield only candidates for optimal solutions, there is considerable interest in sufficient conditions which assure optimality. Especially field type sufficient conditions for optimal control and even more so for dynamic games were inspired by the geometric approach [I], [16]–[18]. On the other hand, the direct sufficiency conditions in [I] can be applied to problems for which classical ones, requiring convexity, respectively concavity, do not apply[1].

2.2 Stabilization of dynamical systems with deterministic uncertainty

There is great interest in stabilizing, in some sense, dynamical system behavior in the presence of uncertainty. The subject has been considered and treated from two points

[1]A. Novak, Applying the Leitmann–Stalford Sufficiency Conditions to Maximization Control Problems with Non-Concave Hamiltonian, 11^{th} Workshop on Optimal Control, Dynamic Games and Nonlinear Systems, Amsterdam May–June, 2010.

of view: the uncertainty is statistical or it is deterministic. It is the latter model to which Leitmann has contributed. His approach appears to have been inspired by his exposure to Lyapunov stability theory on the one hand and to dynamic game theory on the other. Thus he has adopted a "worst case design" approach by allowing for a "game against nature", that is, a qualitative game with uncertainty modeled as a destabilizing opponent [19]–[21]. He soon dropped the game approach in favor of the simpler and more direct Lyapunov one, especially in the case of nonlinear systems. In order to assure continuous feedback control, he replaced the requirement for asymptotic stability by guaranteed ultimate boundedness as well as state feedback by output feedback [22]–[28]. Also of interest he considered the model following [29]. There exists a very large number of papers on applications in engineering, OR, economics, and related areas. A selected sample is the included of Section 3.

2.3 A coordinate transformation-based equivalent problem approach to optimization

Leitmann invoked invariance arguments to obtain the maximum propulsive efficiency of rockets, Chapter 13 of [A]. By the way, this topic had been the subject of many incorrect solutions up to that time, all of which violated required invariance. In 1967 Leitmann proposed a coordinate transformation which results in a problem which is "equivalent" to the originally posed problem and whose optimal solution is obtained directly, that is by simple inspection. He considered this approach for the simplest problem of the calculus of variations [30]. In the late 1990's, Leitmann returned to this idea and extended the method by obtaining results applicable to a wider class of optimization problems, [31]–[34]. Shortly thereafter, D.A. Carlson[2] pointed out a relation of Leitmann's approach to that of Caratheodory in his classic 1935 book on the calculus of variations and partial differential equations. As in the case with Blaquiere in the early 1960's, this led to a lasting collaboration and personal friendship which continues to this day and which has led to many contributions to the subject, [35]–[39], [41]. A relation to Noether's invariance transformations was noted by Torres [40]. And more recently, further aspects of the relation to the work of Caratheodory, in particular the substantial simplification due to the use of Leitmann's regularizing transformations, was noted by Wagener[3].

2.4 Avoidance control

There is ever-growing interest in collision avoidance, whether in the sense of evading a pursuier [42]–[44] or in view of such concerns due to increasing volume of highway and air traffic. To address this type of problem, Leitmann and Skowronski introduced the notion of avoidance control in the 1970's, not from a game–theoretic but rather from a Lyapunov theory point of view [45], [46]. This method has become the primary control scheme for collision avoidance currently in use[4].

[2] D.A. Carlson, An observation on two methods of obtaining solutions to variational problems, J. Optimiz. Theory and Appl., Vol. 115, No. 1, 2002.

[3] F.O.O. Wagener, On the Leitmann equivalent problem approach, J. Optimiz. Theory and Appl., Vol. 142, No. 1, 2009.

[4] D.M. Stipanović, A survey and some new results in avoidance control, 15$^{\text{th}}$ Int. Workshop on Dynamics and Control, eds. J. Rodellar and E. Reithmeier, p. 166f., CIMNE, Barcelona, 2009.

3 Applications-directed Research

While most of Leitmann's research has been and continues to be oriented toward applications, much has been applied to specific problems ranging over a wide array of subjects including some already mentioned in the preceding sections. Here then is a sample of his work dealing with specific problems: Economics [47, 48]; ecology [49]–[51]; earthquake engineering [52]–[54]; fisheries [55]–[57]; flight in wind shear [58]–[61]; robotics [62]–[64]; medical applications [65]–[67]; vibration suppression [68]–[69]; terrorism [70], [71].

4 University Activities

In addition to the professional appointment from 1957–1991, when he became an emeritus, he acted as a consultant to industrial companies and as member of governmental committees as well as University ones. He was the first University ombudsman, acting dean and associate in three areas. Afterwards, he served for four years as chairman of the engineering faculty and is at present associate dean for International Relations and professor in the graduate school.

5 Professional Activities

Leitmann has been on many professional society committees and was the founding president of the Alexander von Humboldt Association of America. He edited the Journal of Mathematical Analysis and Applications for 16 years, and he has served and continues to serve as Associate Editor of four journals and as member of eight editorial boards.

6 Awards

Leitmann is a member of the National Academy of Engineering of the USA and a foreign member of six academies of science or engineering. He received numerous medals and prizes including the Senior Scientist Award as well as the Humboldt and the Heisenberg medals of the Humboldt Foundation. He received the Levy Medal of the Franklin Institute and the Oldenburger Medal of the American Society of Mechanical Engineering. He was given the top awards of the professional societies in his field, the Isaacs Award of the International Society of Dynamic Games and the Bellman Control Heritage Award of the American Automatic Control Council. He is Commander of the Orders of Merit of Germany and of Italy, and he holds three honorary doctorates. Most recently, he was awarded the Medal of Honor of the Universität der Bundeswehr.

7 Public Activities

Leitmann is a keen supporter of the arts. Currently he is Chairman of the Board of the Artship Foundation of San Francisco. He has published his free translation of Bela Balasz' "Mantle of Dreams", Kodansha International, Tokyo. He performs occasionally in plays presented by the university drama club. G. Leitmann has published over 300 books and papers. The bibliography and other information can be seen on www.me.berkeley.edu/faculty/leitmann.

List of Monographs and Books by George Leitmann

[A] Leitmann, G. *Optimization Techniques.* New York: Academic Press, 1962. (Ed. and co-author)

[B] Leitmann, G. *An Introduction to Optimal Control.* New York: McGraw-Hill, 1966.

[C] Leitmann, G. *Topics in Optimization.* New York: Academic Press, 1967. (Ed. and co-author)

[D] Leitmann, G. and Blaquiere, A. *Jeux Quantitatifs.* Paris: Gauthier - Villars, 1968.

[E] Leitmann, G., Blaquiere, A. and Gerard, F. *Quantitative and Qualitative Games.* New York: Academic Press, 1969.

[F] Leitmann, G. *Cooperative and Non-Cooperative Many Player Differential Games.* Vienna: Springer-Verlag, 1974.

[G] Leitmann, G. *Multicriteria Decision Making.* Vienna: Springer-Verlag, 1975. (Ed. with A. Marzollo and co-author)

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[I] Leitmann, G. *Calculus of Variations and Optimal Control— An Introduction.* New York: Plenum Press, 1981.

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List of Selected Papers by George Leitmann

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A New Interconnected Observer Design in Power Converter: Theory and Experimentation

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Abstract: This paper deals with an observer design for a P-Cell Chopper. The goal is to reduce drastically the number of sensors in such system by using an observer in order to estimate all the capacitor voltages. Furthermore, considering an instantaneous model of a p-cell chopper, an interconnected observer is designed in order to estimate the capacitor voltages. This is realized by using only the load current measurement. Simulation results are given in order to illustrate the performance of such observer. To show the validity of our approach, experiments based an DSP results are presented.

Keywords: *p-cell chopper; observer design; interconnected observer.*

Mathematics Subject Classification (2000): 93C10, 93A15, 93C95.

1 Introduction

The power electronics knows important technological developments. This is carried out thanks to the developments of the power semiconductor but also of new energy conversion systems. Among these systems, Multi-Cell Chopper are based on the association in series of the elementary cells of commutation. This structure, appeared at the beginning of the 90's [20, 18], and makes it possible to share the constraints in tension and also to improve the harmonic contents of the waves forms [10].

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Form the practical point of view, the series multi-cell converter, designed by the LEEI (Toulouse-France), leads to a safe series association of components working in switching mode. This new structure combines additional benefits: attenuation of the voltage jump and modularity of the topologies. All these qualities make this new topology very attractive in many industrial applications. For instance, GEC/ACEC implements this proposal to realize the input chopper which supplies their "T13" locomotives in power. Three-phase inverters called "symphony" and developed by Alstom to drive electric motors are also based on the same principe.

To benefit as well as possible from the large potential of the multicellular structure, researcher went in various directions.

Furthermore, the normal operation of the series p-cell converter is obtained when the voltages are $v_{ci} = iE/p, i = 1, ..., p-1$ (see Figure 2). These voltages are generated when a suitable control of switches is applied in order to obtain a specific value. The control inserted of the switches allows cancelling the harmonics at the switching frequency F_{sw} and reducing the ripple of the chopped voltage. However, these properties are lost if the voltages of these capacitors drift. On the other hand, if a specific control is desired, it is advisable to measure these voltages in order to implement it. But, it is not easy because extra sensors are necessary to measure these voltages, then it increases the cost. For this reason, it should be avoided and the estimation of these voltages becomes an attractive and economical option. It is for such reason that, an original method to eliminate such sensors is the use of observers. From, control theory point of view, an observer is considered as a software sensor used to estimate the unmeasurable variables of a system.

On the other hand, several approaches have been considered to develop new methods of control and observation of the p-cell converter. Initially, models have been developed to describe their instantaneous behaviors [10], harmonic [11] or averaging [1]. These various models were used as the base for the development of control laws in open-loop [18] and in closed-loop [15, 21].

Until now, all these p-cell converters are driven successfully, by means of a fix frequency modulator based on pulse width modulation(PWM). Current control algorithms do not take into account the fact that any power converter is a discrete and discontinuous plant, or, at least a hybrid one. Nevertheless, the profitable skill of PWM technique is to ensure a well-known steady state behavior which is "optimal" for the electric load with respect to harmonic attenuation. Furthermore, some representations of the p-cell converter considered complex models and need to be discretized in order to design a discrete observer to be implemented.

Then, in all proposed methods a considerable number of feedback signals are required which are associated with extra cost of sensing devices. To reduce the cost of sensors, a methodology to estimate the voltages in the capacitors is necessary.

In [3, 4], the observer canonical form consisting of a linear output map and linear dynamics driven by a nonlinear output injection is used. The resulting observer has exactly linear error dynamics, i.e., nonlinearities are compensated exactly. The approaches suggested in [5, 6, 7] rely on the observability canonical form, which has significantly weaker existence conditions than the observer canonical form. In the observability canonical form, the observer is designed by a high-gain technique with a constant observer gain, i.e., the nonlinearities are not compensated but dominated by a linear part. For an implementation of the observer in the original coordinates one gets a Luenberger-like observer with a possibly nonlinear gain vector field [8, 9]. In the last decade, new approaches have been developed for nonlinear systems that are not uniformly observable. Several approaches use Kalman-like decompositions [8].

In this article, we develop an observer for p-cell chopper based on an instantaneous model describing the dynamical behavior of the p-cell converter. This model is constructed in order to design an observer estimating each flying capacitor voltage. The proposed observer design is based on the class of nonlinear systems which can be written in the form of affine state systems, for which the problem of state observer design has been studied. This class of observers is based on the excitation condition in order to guarantee its convergence.

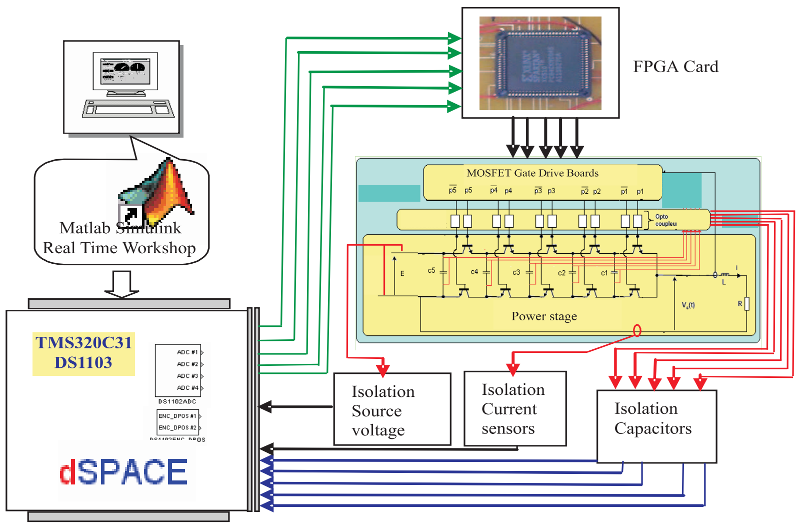

Figure 1: General structure of the dSPACE observer.

The objective of this work is to design an observer for a P-Cell Chopper converter in order to estimate the unmeasurable voltages of the capacitors using the load current i and the voltage of the source E, and give an experimental validation of it. The block diagram describing the proposed observation scheme is illustrated in Figure 1.

The paper is organized as follows: In Section 2, the instantaneous model of P-cell chopper is introduced. In Section 3, the observability properties of the P-Cell Chopper model are given. The observer design based on an new representation of the instantaneous model of the converter is presented in Section 4. In Section 5, using a model of 5 cells chopper, simulations results are shown in order to illustrate the performance of the proposed observer. The proposed observation scheme is validated and experimental results are given. Finally, some conclusions end the paper.

2 P-Cell Converter Model

Throughout the paper, the p-cell converter connects in series p elementary cells and a passive load R and L as illustrated in Figure 2. Each switching cell is controlled by a binary input signal $S_k(t)$ for $k = 1, ..., p$.

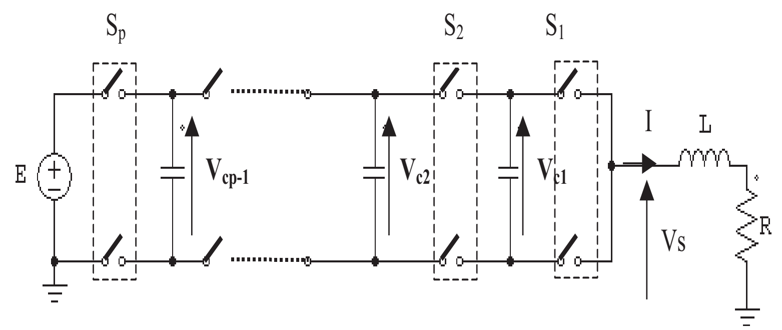

Figure 2: A p-cells converter.

This signal $S_k(t)$ is equal to 1 when the upper switch of the cell is conducting and to 0 when the lower complementary switch of the cell is conducting. The mathematical model describing the behavior of a p-cell converter is given by

$$\Sigma_{pcell}: \left\{ \begin{array}{rcl} \frac{dI}{dt} &=& -\frac{R}{L}I + \frac{E}{L}S_p - \frac{v_{c_{p-1}}}{L}\left(S_p - S_{p-1}\right)\ldots - \frac{v_{c_1}}{L}\left(S_2 - S_1\right), \\ \frac{dv_{c_1}}{dt} &=& \frac{1}{c_1}\left(S_2 - S_1\right)I, \\ \frac{dv_{c_2}}{dt} &=& \frac{1}{c_2}\left(S_3 - S_2\right)I, \\ &\vdots& \\ \frac{dv_{c_{p-1}}}{dt} &=& \frac{1}{c_p}\left(S_p - S_{p-1}\right)I, \\ y &=& I, \end{array} \right. \tag{1}$$

where v_{c_k} is the k^{th} flying capacitor voltage and I is the output load current, and is the only measurable output. c_k for $k = 1, ..., p$; are the capacitors, E is the voltage of the source, R is the resistance and L is the inductance.

Now, from the instantaneous state model of the p-cell converter given in (1), we will analyze the observability properties of such system in order to construct an observer. It is well known that the observability of nonlinear systems depends on the applied input, and a study of the different classes of inputs which render the system observable or unobservable is given in [16, 17].

Rewriting the model (1) in the state affine form, we have:

$$\Sigma: \left\{ \begin{array}{l} \dot{X} = \bar{A}(u)X + \bar{B}(u), \\ y = \bar{C}X, \end{array} \right. \tag{2}$$

where $X = (I, v_{c_1}, ..., v_{c_{p-1}})$ is the state vector, $u = \{S_1, ..., S_p\}$ is the input sequence applied to the converter,

$$\bar{A}(u) = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L}(S_2 - S_1) & \dots & -\frac{1}{L}(S_p - S_{p-1}) \\ -\frac{1}{c_1}(S_2 - S_1) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -\frac{1}{c_{p-1}}(S_p - S_{p-1}) & 0 & \dots & 0 \end{pmatrix},$$

$$\bar{B}(u) = \begin{pmatrix} \frac{E}{L} S_p \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \bar{C} = (1, 0, \dots, 0).$$

Regarding the instantaneous model of the multi-cell converter (2), we can see that there are several operating switching modes (S_k, S_{k+1}) which render the system unobservable, i.e. for the following operating switching modes

$$(S_k, S_{k+1}) = (1,1) \quad \text{and} \quad (S_k, S_{k+1}) = (0,0), \quad \text{for} \quad k = 1, ..., p-1,$$

the system becomes unobservable. These operating switching modes are not affected by the capacitor voltage. However, these cases occur only for a part of control sequence. If it occurs for all the control sequences this is not of physical interest because they represent particular situations in which the cell chopper is not operating.

Now, the sequence of corresponding input $u = \{u_1, ..., u_{p-1}\}$, where $u_k = S_{k+1} - S_k$, applied to the system (1), is sufficiently periodic. Furthermore, assuming that the current I is the only measurable variable of the system (2), from the observability rank condition, it follows that

$$\text{Rank}\left(\ \bar{C}, \bar{C}\bar{A}(u), ..., \bar{C}\bar{A}^{p-1}(u) \ \right)^T = 2. \tag{3}$$

It is clear that the system is not of full rank, i.e. the system is not observable. Then, in order to overcome this difficulty we consider a new representation of the multi-cell converted which is constituted of a set of subsystems of dimension 2. These subsystems are such that the whole system is represented as an interconnected structure. Furthermore, an analysis of the observability of each subsystem is required and is given in the next section.

3 Observer Design for a P-Cell Chopper

Now, in this section, the design of $p-1$ interconnected observers for p- cell chopper is given. For that, we will consider a different representation of system (1) such that the original system can be splitted into a suitable set of $p-1$ subsystems for which it will be possible to design an observer for estimating the capacitor voltages v_{c_j}, for $j = 1, ..., p-1$.

Next, considering that system (1) can be splitted into $p-1$ interconnected subsystems of the form

$$\Sigma_k : \begin{cases} \frac{dI}{dt} &= -\frac{R}{L} I + \frac{E}{L} S_P - \frac{1}{L} \sum\limits_{j=1}^{p-1} (S_{j+1} - S_j) v_{c_j}, \\ \frac{dv_{c_k}}{dt} &= \frac{1}{c_k} (S_{k+1} - S_k) i, \\ y &= I, \end{cases}$$

where the above system can be represented, for $k = 1, ..., p-1$, in a compact form as:

$$\Sigma_k : \begin{cases} \dot{X}_k = A_k(u_k) X_k + B_k(\bar{u}_k, \bar{X}_k), \\ y = \mathbf{C}_k X_k, \end{cases} \tag{4}$$

where $X_k = (I,\ v_{c_k})^T$ is the state vector of subsystem (4), $X = \left(I, v_{c_1}, ..., v_{c_{p-1}}\right)^T$ is the state of system (1), $\bar{X}_k = (v_{c_1}, ..., v_{c_{k-1}}, v_{c_{k+1}}, ..., v_{c_{p-1}})^T$, $u_k = S_{k+1} - S_k$, for $k = 1, ..., p-1$; and $\bar{u}_k = (u_1, ..., u_{k-1}, u_{k+1}, ..., u_p)^T$, are the inputs. Furthermore, $y = \mathbf{C}_k X_k = I$ is the output of subsystem (4) with $\mathbf{C}_k = (1\ 0)$ for $k = 1, ..., p-1$; and

$$A_k(u_k) = \begin{pmatrix} -\frac{R}{L} & -\frac{(u_k)}{L} \\ \frac{(u_k)}{c_k} & 0 \end{pmatrix}, \tag{5}$$

$$B_k(\bar{u}_k, \bar{X}_k) = \begin{pmatrix} -\frac{1}{L} \sum\limits_{j=1,\ j\neq k}^{p-1} (S_{j+1} - S_j) v_{c_j} + \frac{E}{L} S_p \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{L} \bar{u}_k^T \bar{X}_k + \frac{E}{L} S_p \\ 0 \end{pmatrix}. \tag{6}$$

It is clear that for $u_k = 0$, the system becomes unobservable. However, each subsystem k^{th}, which is of dimension 2, is observable for an appropriate input u_k and its rank is equal to 2. Furthermore, in order to estimate the unmeasurable variables, no feedback is applied to excite the converter as it has been proposed in other works. Instead of this, we consider an equivalent concept which is the well-known concept of regularly persistent input (see Appendix). More precisely, a regularly persistent input applied to the system allows to excite the system sufficiently to obtain the information necessary to be able to reconstruct the unmeasurable variables by means of an observer. If the input is not sufficiently persistent, then it is not possible to reconstruct the state of the system from the measured output and the applied input.

Furthermore, the function $B_k(\bar{u}_k, \bar{X}_k)$ is the interconnection term depending on inputs and states of each subsystem. Notice that the output is the current $I(t)$ and is the same for each subsystem. Then, the following system

$$O_k : \begin{cases} \dot{Z}_k = A_k(u_k) Z_k + B_k(\bar{u}_k, \bar{Z}_k) - P_k^{-1} \mathbf{C}_k^T (y_k - \hat{y}_k), \\ \dot{P}_k = -\theta_k P_k - A_k^T(u_k) P_k - P_k A_k(u_k) + \mathbf{C}_k^T \mathbf{C}_k, \end{cases} \tag{7}$$

is an observer for subsystem (4), for $k = 1, 2, ..., p-1$; where $\theta_k > 0$, $\hat{y}_k = \mathbf{C}_k X_k = \hat{I}$ and $P_k^{-1}\mathbf{C}_k^T$ is the gain of the observer which depends on the solution of the second equation of (7) for each subsystem with $Z_k = (\hat{I},\ \hat{v}_{c_k})^T$, $\bar{Z}_k = (\hat{v}_{c_1}, ..., \hat{v}_{c_{k-1}}, \hat{v}_{c_{k+1}}, ..., \hat{v}_{c_{p-1}})^T$ and $A_k(u_k)$ is given in (5) and for $k = 1, 2, ..., p-1$;

$$B_k(\bar{u}_k, \bar{Z}_k) = \begin{pmatrix} -\frac{1}{L} \sum\limits_{j=1,\ j\neq k}^{p-1} (S_{j+1} - S_j) \hat{v}_{c_j} + \frac{E}{L} S_p \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{L} \bar{u}_k^T \bar{Z}_k + \frac{E}{L} S_p \\ 0 \end{pmatrix}.$$

Now, consider that system (1) can be represented as follows:

$$\Sigma : \begin{cases} \dot{X}_1 = A_1(u_1) X_1 + B_1(\bar{u}_1, \bar{X}_1), \\ \dot{X}_2 = A_2(u_2) X_2 + B_2(\bar{u}_2, \bar{X}_2), \\ \vdots \\ \dot{X}_{p-1} = A_{p-1}(u_{p-1}) X_{p-1} + B_{p-1}(\bar{u}_{p-1}, \bar{X}_{p-1}). \end{cases} \tag{8}$$

Notice that the output is the current $I(t)$ and is the same for each subsystem. The main idea of the paper is to construct an observer for the whole system (1), from the separate observer design of each subsystem (4).

In general, if each (7) is an exponential observer for (4), for $k = 1, 2, ..., p-1$; then the following interconnected system

$$O : \begin{cases} \dot{Z}_1 = A_1 \left(u_1\right) Z_1 + B_1(\bar{u}_1, \bar{Z}_1) - P_1^{-1}\mathbf{C}_1^T \left(y - \hat{y}\right), \\ \dot{Z}_2 = A_2 \left(u_2\right) Z_2 + B_2(\bar{u}_2, \bar{Z}_2) - P_2^{-1}\mathbf{C}_2^T \left(y - \hat{y}\right), \\ \vdots \\ \dot{Z}_{p-1} = A_{p-1} \left(u_{p-1}\right) Z_{p-1} + B_{p-1}(\bar{u}_{p-1}, \bar{Z}_{p-1}) - P_{p-1}^{-1}\mathbf{C}_{p-1}^T \left(y - \hat{y}\right), \\ \dot{P}_1 = -\theta_1 P_1 - A_1^T(u_1) P_1 - P_1 A_1(u_1) + \mathbf{C}_1^T \mathbf{C}_1, \\ \dot{P}_2 = -\theta_2 P_2 - A_2^T(u_2) P_2 - P_2 A_2(u_2) + \mathbf{C}_2^T \mathbf{C}_2, \\ \vdots \\ \dot{P}_{p-1} = -\theta_{p-1} P_{p-1} - A_{p-1}^T(u_{p-1}) P_{p-1} - P_{p-1} A_{p-1}(u_{p-1}) + \mathbf{C}_{p-1}^T \mathbf{C}_{p-1}, \end{cases} \tag{9}$$

is an observer for the interconnected system (8).

Remark 3.1 The proposed observer 9 works for inputs satisfying the regularly persistent condition, which is equivalent to each subsystem (4) being observable, and hence, observer (7) works at the same time while the other subsystems become observable when their corresponding input satisfies the regularly persistent condition.

Now, we will give the sufficient conditions which ensure the convergence of the interconnected observer (9). For that, we introduce the following assumptions.

Assumption 3.1 Assume that the input $u_k = S_{k+1} - S_k$, for $k = 1, 2, ..., p-1$; is regularly persistent input for subsystem (4), and admits an exponential observer (7). The estimation error, defined as $\varepsilon_k = Z_k - X_k$, is bounded.

Assumption 3.2 The term $B_k(\bar{u}_k, \bar{X}_k)$ does not destroy the observability property of the subsystem (4), under the action of the regularly persistent input $u_k = (S_{k+1} - S_k)$, for $k = 1, 2, ..., p-1$. Moreover, $B_k(\bar{u}_k, \bar{X}_k)$ is Lipschitz with respect to $\bar{X}_k$ and uniform with respect to $\bar{u}_k$, for $k = 1, 2, ..., p-1$.

The observer convergence can be proved only if the inputs u_k are regularly persistent, i.e. it is a class of admissible inputs that allows to observe the system (for more details see [19, 20]). This assumption guarantees that the observer works and that its gain is well-defined, i.e. the matrices P_k, for $k = 1, 2, ..., p-1$, are nonsingular (see appendix).

The following result can be established.

Proposition 3.1 *Consider the system (1) can be represented in the form of system (8), where each subsystem (4) satisfies the assumptions 3.1 and 3.2, for $k = 1, 2, ..., p-1$. Then, system (9) is an exponential observer for system (8). Furthermore, the estimation error, defined as $\varepsilon = Z - X$, converges exponentially to zero.*

Proof In order to prove the convergence of the observer (9), first we consider the dynamics of subsystem (4), for which an observer of the form (7) can be designed. Then, defining the estimation error $\varepsilon_k = Z_k - X_k$ whose dynamics is given by

$$\dot{\varepsilon}_k = \{A(u_k) - P_k^{-1}\mathbf{C}_k^T\mathbf{C}_k\}\varepsilon_k + \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k) \tag{10}$$

for $k = 1, \dots, p-1$; where $\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k) = B_k(\bar{u}_k, \bar{Z}_k) - B_k(\bar{u}_k, \bar{X}_k)$.

From Assumption 3.1 and Lemma 7.1 (see Appendix), we can define $V = \sum_{l=1,}^{p-1} V_k$ as a Lyapunov function for the interconnected system (8), where $V(\varepsilon_k) = \varepsilon_k^T P_k \varepsilon_k$ is a Lyapunov function for subsystem (4). It is clear that these functions are well defined because the matrices P_k are nonsingular.

Taking the time derivative of $V(\varepsilon_k)$, it follows that

$$\dot{V}(\varepsilon_k) \leq -\theta_k V(\varepsilon_k) + \varepsilon_j^T P_k \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k) \quad \text{for} \quad k = 1, \dots, p-1. \tag{11}$$

Now, adding and subtracting the term $\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)^T P_k \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)$, we have

$$\dot{V}(\varepsilon_k) \leq -\theta_k V(\varepsilon_k) + 2\varepsilon_k^T P_k \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k) \pm \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)^T P_k \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k).$$

Next, regrouping the appropriate terms gives:

$$\begin{aligned} \dot{V}(\varepsilon_k) \quad \leq \quad & -(\theta_k - 1)\|\varepsilon_k\|_{P_k}^2 \\ & -\|\varepsilon_k\|_{P_k}^2 + 2\varepsilon_k^T P_k \Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k) - \|\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)\|_{P_k}^2 \\ & +\|\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)\|_{P_k}^2. \end{aligned} \tag{12}$$

It follows that

$$\dot{V}(\varepsilon_k) \leq -(\theta_k - 1)\|\varepsilon_k\|_{P_k}^2 + \|\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)\|_{P_k}^2. \tag{13}$$

Now, from assumption 3.2, $B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)$ is Lipschitz, it follows that

$$\|\Delta B_k(\bar{u}_k, \bar{X}_k, \bar{Z}_k)\|_{P_k}^2 < \sum_{l=1, l\neq k}^{p-1} \lambda_l \|\varepsilon_l\|_{P_k}^2. \tag{14}$$

we get:

$$\dot{V}(\varepsilon_k) \leq -(\theta_k - 1)\|\varepsilon_k\|_{P_k}^2 + \lambda_l \|\varepsilon_l\|_{P_k}^2, \tag{15}$$

the time derivative of V is given by

$$\dot{V}(\varepsilon) = \sum_{k=1,}^{p-1} \dot{V}(\varepsilon_k), \tag{16}$$

$$\dot{V}(\varepsilon) \leq \sum_{k=1,}^{p-1} \Big\{ -(\theta_k - 1)\|\varepsilon_k\|_{P_k}^2 + \sum_{l=1, l\neq k}^{p} \lambda_l \|\varepsilon_l\|_{P_k}^2 \Big\}. \tag{17}$$

Using the lemma on equivalence of norms, i.e. there exists a positive constant μ_l such that $\|\varepsilon_l\|_{P_k}^2 \leq \mu_l \|\varepsilon_l\|_{P_l}^2, \forall l = 1, ..., p-1$. Then, it follows that

$$\dot{V}(\varepsilon) \leq \sum_{k=1,}^{p-1} \Big\{ -(\theta_k - 1)\|\varepsilon_k\|_{P_k}^2 + \sum_{l=1, l\neq k}^{p-1} \lambda_l \mu_l \|\varepsilon_l\|_{P_k}^2 \Big\} \tag{18}$$

or

$$\dot{V}(\varepsilon) \leq \sum_{k=1}^{p-1} - \{(\theta_k - 1) - (p-1)\lambda_l \mu_l)\} \|\varepsilon_j\|_{P_k}^2. \tag{19}$$

Finally, we have $V(\varepsilon) \leq V(\varepsilon(t_0))e^{-\gamma(t-t_0)}$, for $\gamma = min(\gamma_1, ..., \gamma_{p-1})$ where $\gamma_k = (\theta_k - 1) - (p-1)\lambda_k \mu_k)$. Taking $\varepsilon = col(\varepsilon_1, ..., \varepsilon_{p-1})$, it is easy to see that

$$\|\varepsilon(t)\| \leq K\|\varepsilon(t_0)\|e^{-\gamma(t-t_0)}. \tag{20}$$

This ends the proof.

4 Observer for 5-Cell Chopper

Now, in this section we present the proposed methodology which is applied to a model of 5-Cell Chopper converter. For that, consider the following model of 5-cell chopper:

$$\Sigma_{5cell}: \begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{E}{L}S_5 - \frac{(S_2-S_1)}{L}v_{c_1} - \frac{(S_3-S_2)}{L}v_{c_2} - \frac{(S_4-S_3)}{L}v_{c_3} - \frac{(S_5-S_4)}{L}v_{c_4}, \\ \frac{dv_{c_1}}{dt} &= \frac{1}{c_1}(S_2 - S_1) I, \\ \frac{dv_{c_2}}{dt} &= \frac{1}{c_2}(S_3 - S_2) I, \\ \frac{dv_{c_3}}{dt} &= \frac{1}{c_3}(S_4 - S_3) I, \\ \frac{dv_{c_4}}{dt} &= \frac{1}{c_4}(S_5 - S_4) I. \end{cases} \tag{21}$$

Following the ideas of this original methodology, the model can be rewritten in the following form:

$$\Sigma_1: \begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{E}{L}S_5 - \frac{(S_2-S_1)}{L}v_{c_1} - \frac{(S_3-S_2)}{L}v_{c_2} - \frac{(S_4-S_3)}{L}v_{c_3} - \frac{(S_5-S_4)}{L}v_{c_4}, \\ \frac{dv_{c_1}}{dt} &= \frac{1}{c_1}(S_2 - S_1) I, \end{cases}$$

$$\Sigma_2: \begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{E}{L}S_5 - \frac{(S_2-S_1)}{L}v_{c_1} - \frac{(S_3-S_2)}{L}v_{c_2} - \frac{(S_4-S_3)}{L}v_{c_3} - \frac{(S_5-S_4)}{L}v_{c_4}, \\ \frac{dv_{c_2}}{dt} &= \frac{1}{c_2}(S_3 - S_2) I, \end{cases}$$

$$\Sigma_3: \begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{E}{L}S_5 - \frac{(S_2-S_1)}{L}v_{c_1} - \frac{(S_3-S_2)}{L}v_{c_2} - \frac{(S_4-S_3)}{L}v_{c_3} - \frac{(S_5-S_4)}{L}v_{c_4}, \\ \frac{dv_{c_3}}{dt} &= \frac{1}{c_3}(S_4 - S_3) I, \end{cases}$$

$$\Sigma_4: \begin{cases} \frac{dI}{dt} &= -\frac{R}{L}I + \frac{E}{L}S_5 - \frac{(S_2-S_1)}{L}v_{c_1} - \frac{(S_3-S_2)}{L}v_{c_2} - \frac{(S_4-S_3)}{L}v_{c_3} - \frac{(S_5-S_4)}{L}v_{c_4}, \\ \frac{dv_{c_4}}{dt} &= \frac{1}{c_4}(S_5 - S_4) I. \end{cases}$$

This set of subsystems can be represented in an interconnected compact form as follows

$$\Sigma_i: \begin{cases} \dot{X}_i = A_i(u_i)X_i + B_i(\bar{u}_i, \bar{X}_i), \\ y = \mathbf{C}_i X_i = I, \end{cases} \qquad \text{for} \quad i = 1, ..., 4.$$

It can be assumed that the control sequence of inputs provides the sufficient persistency to guarantee that the observer works correctly (see appendix and assumption 3.1). Using this assumption, an observer for the above interconnected subsystems is given by

$$O_i: \begin{cases} \dot{\hat{Z}}_i = A_i(u_i)\hat{X}_i + B_i(\bar{u}_i, \bar{Z}_i) + P_i^{-1}\mathbf{C}_i^T(y - \hat{y}), \\ \dot{P}_i = -\theta_i P_i - A_i^T(u_i)P_i + P_i A_i(u_i) + \mathbf{C}_i^T \mathbf{C}_i, \end{cases} \qquad \text{for} \quad i = 1, ..., 4.$$

5 Experimental Results

In this section, we show some experimental results obtained by using the proposed interconnected observer. In order to illustrate the performance of this observer, where the estimated states converge to the real states, the instantaneous converter model of 5 cells (21) is used for the observer design, where the capacitor voltages are estimated. The parameters of the model were chosen as follows:

$$f_d = 16kHz, \quad C = 40\mu F, \quad L = 1mH, \quad R = 100\Omega, \quad E = 120V.$$

Furthermore, to carry out the experimentation and show the efficiency of the proposed observer, we use a trajectory for the input voltage as given in Figure 3.

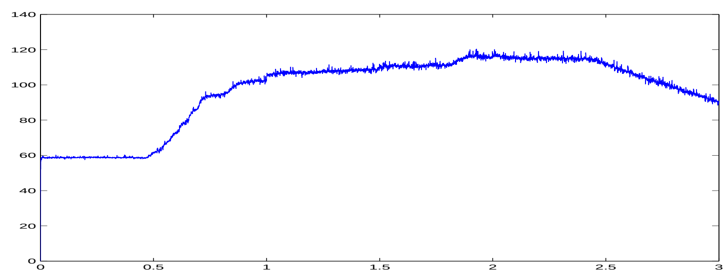

Figure 3: The input voltage E.

Finally, the following initial conditions of the system and the observer were selected as follows. For the system: $X_k = (i, v_{c_k})^T = (0,0)^T$ and for the observer: $Z_k = (\hat{\imath}, \hat{v}_{c_k})$ are given as $(1,20)$, $(1,30)$, $(1,35)$ and $(1,40)$, for $k = 1, ..., 4$. The parameters θ_k, for $k = 1, ..., 4$, which are the design parameters used to control the rate of convergence of each observer, were chosen as follows: $\theta_1 = 30$, $\theta_2 = 40$, $\theta_3 = 50$ and $\theta_4 = 60$.

5.1 Benchmark observation

The experimental setup realized based on the DS1103 dSPACE kit shown in Figure 1 gives the global scheme of the experimental setup. This kit allows real time implementation of converter, it includes several functions such as analog/digital converters and digital signal filtering. In order to run the application we must write our algorithm in C language. Then, we use the RTW and RTI packages to compile and load the algorithm on processor. To visualize and adjust the control parameters in real time we use the software controldesk which allows conducting the process by the computer.

The multi-cells chopper power stage is based on the use of MOSFET. The pulsewidth-modulator (PWM) blocks are generated by FPGA card. The observer is first designed in Simulink/Matlab, then, the Real-Time Workshop is used to automatically generate optimized C code for real time applications. Afterward, the interface between Simulink/Matlab and the digital signal processor (DSP) (DS1103 of dSpace) allows the control algorithm to be run on the hardware.

The master bit I/O is used to generate the required 5 gate signals, and six analog-to-digital converters (ADCs) are used for the sensed line-currents, capacitors voltage, and output voltages. An optical interface board is also designed in order to isolate the entire DSP master bit I/O and ADCS.

5.2 Experimental evaluation

The experimental results of Figures 4–8 are obtained under the following test conditions: The sample time was chosen equal to 50 micro-seconds, and the data acquisition is close to 1 sec in this experimental evaluation. We assume that all parameter are known.

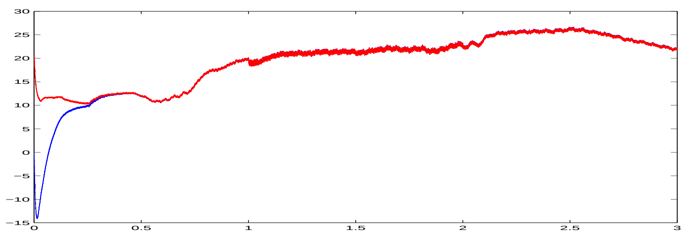

Figure 4: Capacitor voltage V_{c1} measured and estimated.

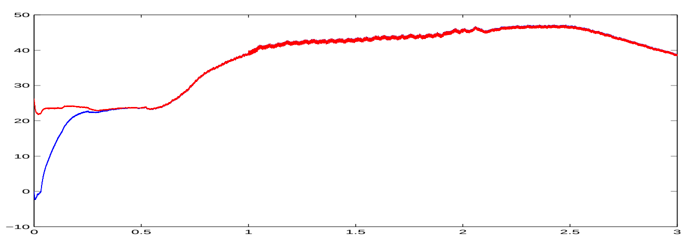

Figure 5: Capacitor voltage V_{c2} measured and estimated.

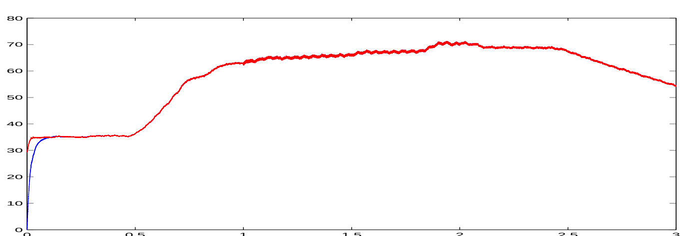

Figure 6: Capacitor voltage V_{c3} measured and estimated.

In order to compare the real and estimated voltages 4 sensors were used, an optical interface was used in this case. Furthermore, to reduce the noise in the signals, a low pass filter was required. In Figures 4–7, we can see the convergence of the estimated and real voltage given by the observer to the real variables, this highlights the well fader performance of the proposed observation scheme. From these plots, we can see that substantial transient of the voltages estimated, is due to the error in the initial conditions. However, these transients can be reduced choosing suitable initial conditions of the observer. In this experiment, the initial conditions were chosen far of them of

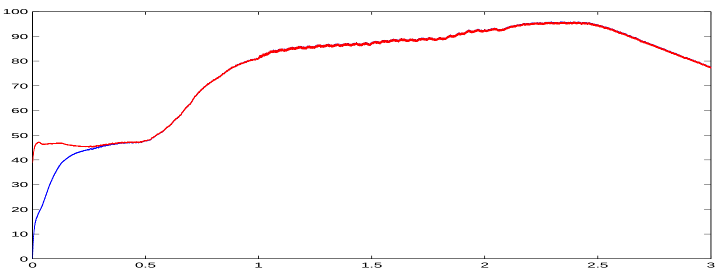

Figure 7: Capacitor voltage V_{c4} measured and estimated.

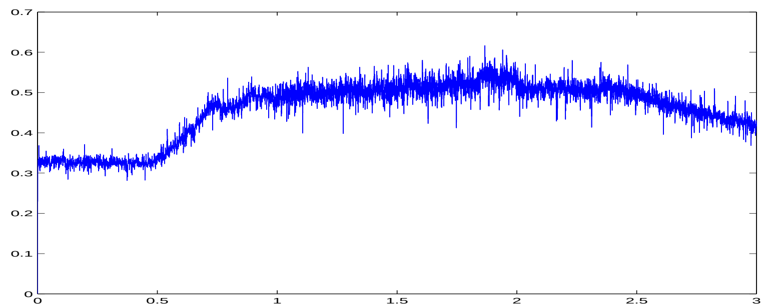

Figure 8: The output current load.

the converter to show the performance of the observer. The output current i is given in Figure 8.

Note that all experimental results are obtained by using a second order filter.

6 Conclusion

In this paper, using an instantaneous model of a Multi-Cell converter, an original methodology of observation has been presented. An observer design has been presented and validated experimentally, to estimate the capacitor voltages from the instantaneous measurement of the current. The practical interest of such observer has been illustrated by means of experimental results. Furthermore, sufficient conditions has been given in order to prove the exponential convergence to zero, with an arbitrary rate of convergence, of the proposed interconnected observer, which only depends on the persistence of the switching control sequence.

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7 Appendix: Some Mathematical Preliminaries

We introduce some definitions related to the inputs applied to the system. Consider a state-affine controlled system of the following form

$$\dot{x} = A(v)x + B(v), \quad y = Cx,$$

where $x \in \mathbf{R}^n; v \in \mathbf{R}^m; y \in \mathbf{R}^p$ with $A : R^m \to \mathcal{M}(n,m); B : R^m \to \mathcal{M}(n,1)$ continuous, and $\mathcal{C} \in \mathcal{M}(p,n)$, where $\mathcal{M}(k,l)$ denotes the space of $k \times l$ matrices with coefficients in R; k (resp.l) is the number of rows (resp. columns).

Notation. Let $\Phi_v(\tau,t)$ denote the transition matrix of:

$$\frac{d}{dt}\Phi_v(\tau,t) = A(v(\tau))\Phi_v(\tau,t), \quad \Phi_v(t,t) = \mathbf{I},$$

with the classical relation: $\Phi_v(t_1,t_2)\Phi_v(t_2,t_3) = \Phi_v(t_1,t_3)$. We then define:

- The *Observability Grammian*: $\Gamma(t,T,v) = \int\limits_t^{t+T} \Phi_v^T(\tau,t)C^TC\Phi_v(\tau,t)d\tau$.
- The *Universality index*: $\gamma(t,T,v) = \min\limits_i(\lambda_i(\Gamma(t,T,v)))$, where the $\lambda_i(M)$ stand for the eigenvalues of a given matrix M.

The input functions are assumed to be measurable and such that $A(v)$ is bounded on the set of admissible inputs of R^+. We recall below some required results of input functions ensuring the existence of an observer for (4).

Definition 7.1 (Regular Persistence). A measurable bounded input v is said to be regularly persistent for the state-affine system (4) if there exist $T > 0; \alpha > 0$ and $t_0 > 0$ such that $\gamma(t,T,v) > \alpha$ for every $t \geq t_0$.

Now, a further result based on regular persistence is introduced.

Lemma 7.1 *Assume that the input u_k is regularly persistent for system (2) and consider the following Lyapunov differential equation:*

$$\dot{P}_k = -\theta_k P_k - A^T(u_k)P_k - P_kA(u_k) + \mathcal{C}_k^T\mathcal{C}_k \tag{22}$$

with $P_k(0) > 0$. Then, $\exists\theta_{k0} > 0$ such that for any symmetric positive definite matrix $P_k(0)$, $\exists\theta_k \geq \theta_{k_0}$, $\exists\alpha_k, \beta_k > 0, t_0 > 0 : \forall t > t_0,\ \alpha_k I < P_k(t) < \beta_k I$, where I is the identity matrix.

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On the Absolute Stabilization of Dynamical-Delay Systems

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Abstract: In this paper we deal with the problem of absolute stabilization for Lur'e systems with time-varying delay in a range. An appropriate Lyapunov-Krasovskii functional is proposed to investigate the delay-range-dependant stabilization problem.The time-varying delay is assumed to belong to an interval and no restriction on its derivative is needed. Some relaxation matrices are introduced, which allow the delay to be a fast time-varying function. Furthermore, a numerical example is given to prove effectiveness of our main result.

Keywords: *time-varying delay system; absolute stability; Lur'e system; LMI; $\mathcal{S}$-procedure; Shur complement; Lyapunov–Krasovskii functional.*

Mathematics Subject Classification (2000): 93D05, 93D09, 93D15, 34D23.

1 Introduction

During the last two decades, considerable attention has been devoted to the problem of delay-dependent stability analysis and controller design for time-delay systems. For the recent progress, the reader is referred to [10, 11, 19, 27, 33, 37]. It is well known that the choice of an appropriate Lyapunov–Krasovskii functional (LKF) is crucial for deriving stability criteria and for obtaining a solution to various control problems.

We shall note that studies of stability of time-delay systems have grown steadily. Indeed, since 1940 all the results were delay independent see for examples [3, 9, 15, 20, 22, 29, 30]. But, the problem is that when the time-delay is small, these results are often overly conservative, especially, they are not applicable to closed-loop systems which are open-loop unstable and are stabilized using delayed inputs. That's why, many efforts were sacrificed to provide delay-dependant stability criteria.

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Since the introduction of absolute stability by Lur'e (1957), the absolute stability problem of nonlinear control systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraints has been the subject of many researches see [2, 18, 22, 25, 28, 34].

From the practical point of view and since in general the delay is not known, it is worth considering it as time-varying [5, 32, 35, 24]. For this object, one is interested in conditions that constrain the upper and lower bounds of the delay and the upper bound of the first derivative of the time-varying delay.

To the best of our knowledge, for the case where only the upper and lower bounds of the interval time-varying delay are precisely known and the lower bound of the delay is greater than zero, there is no result available for stability for such kinds of systems. It should also be mentioned that even for the case where the lower bound of the time-varying delay is zero and without considering the derivative of the time-varying delay, there are few works available in the existing literature [7, 13, 6] using Lyapunov–Krasovskii functional approach.

For this reason we are motivated to provide new stabilization criterion, in order to improve those in which some useful terms are ignored, when estimating the upper bound of the derivative of Lyapunov functional [8, 11].

Those resulting criteria are applicable to both fast and slow time-varying delay, in contrast with previous works in which the upper bound of the first derivative of the time-varying delay was either restricted to one or completely neglected, see [13, 31, 36]. It is important to mention that this became possible since the free matrices M_1 and M_2 of the proposition provide some extra freedom in their selection.

The stabilization criterion is formulated in the form of Linear Matrix Inequality (LMI). Moreover, we give an example to show the applicability of our main result.

Notation: Throughout this paper, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. I is the identity matrix. The set $\mathcal{C}_{n,\tau_M} := \mathcal{C}([-\tau_M, 0], \mathbb{R}^n)$ is the space of continuous functions mapping the interval $[-\tau_M, 0]$ to $\mathbb{R}^n$. The notation $A > 0$ is that the matrix A is positive definite.

2 Stabilization of Nonlinear Delay System

Consider the following time-varying-delay control system

$$\begin{array}{rcl} \dot{x}(t) & = & A_0x(t) + A_1x(t-\tau(t)) + B\omega(t) + Gu(t), \\ y(t) & = & C_0x(t) + C_1x(t-\tau(t)), \\ \omega(t) & = & -\psi(t, y(t)), \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ is the measured output, and the nonlinear function $\varphi(.,.) : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^p$ is assumed to be continuous and belongs to sector $[0, K]$, i.e $\varphi(.,.)$ satisfies

$$\varphi^\top(t, y)\,[\varphi(t, y) - Ky] \le 0,\ \forall (t, y) \in \mathbb{R}_+ \times \mathbb{R}^p, \tag{2}$$

where K is a positive definite matrix. The matrices $A_0,\ A_1,\ B,\ G,\ C_0$, and C_1 are real matrices with appropriate dimensions. The time delay $\tau(t)$ is a time-varying continuous function that satisfies

$$0 \le \tau_m \le \tau(t) < \tau_M \ \text{ and } \ \dot{\tau}(t) < \mu, \tag{3}$$

where τ_m, τ_M and μ are known constant reals.

Note that τ_m may not be equal to 0. The initial condition of 1 is given by

$$x(t) = \phi(t),\ t \in [-\tau_M, 0],\ \phi \in \mathcal{C}_{n,\tau_M}.$$

It is assumed that the right-hand side of (1) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition ϕ.

The closed-loop system with the state control feedback

$$u(t) = \tilde{K}x(t) \tag{4}$$

is given by

$$\dot{x}(t) = \left(A_0 + G\tilde{K}\right)x(t) + A_1 x(t-\tau(t)) + B\omega(t). \tag{5}$$

We first introduce the following definition.

Definition 2.1 The system (1) is said to be absolutely stabilizable in the sector $[0, K]$ if there exists a control $u(t) = Nx(t)$ such that the closed-loop system (5) is globally uniformly asymptotically stable for any nonlinear function $\varphi(t, y(t))$ satisfying (2).

The development of the work in this paper requires the following lemma which can be found in [36].

Lemma 2.1 *Let $x(t) \in \mathbb{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ and $X = X^\top > 0$, and a scalar function $\tau := \tau(t) \geq 0$:*

$$-\int_{t-\tau(t)}^{t} \dot{x}^\top(s)X\dot{x}(s)ds \quad \leq \quad \xi^\top(t)\Upsilon\xi(t) + \tau(t)\xi^\top(t)\Gamma^\top X^{-1}\Gamma\xi(t), \tag{6}$$

where

$$\Upsilon := \begin{bmatrix} M_1^\top + M_1 & -M_1^\top + M_2 \\ * & -M_2^\top - M_2 \end{bmatrix},\ \Gamma^\top := \begin{bmatrix} M_1^\top \\ M_2^\top \end{bmatrix},\ \xi(t) := \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}.$$

The following theorem gives a sufficient condition for stabilization of the system by means a state feedback when the nonlinearity $\psi(t, y)$ belongs to the sector $[0, K]$.

Theorem 2.1 *For given scalars $0 \leq \tau_m < \tau_M,\ \lambda_i,\ \alpha_i,\ \beta_i \in \mathbb{R},\ i = 1, 2$, if there exist a scalar $\epsilon > 0$, positive definite matrices $\overline{P} > 0,\ \overline{Q_1} > 0,\ \overline{Q_2} > 0,\ \overline{Q_3} > 0,\ \overline{R_1} > 0,\ \overline{R2} > 0,\ \overline{R_3} > 0$, and a matrix $Y \in \mathbb{R}^{r\times n}$ such that the LMI*

$$\Xi_2 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & \Xi_{15} & \Xi_{16} & \Xi_{17} & \Xi_{18} & 0 & \Xi_{110} & \Xi_{111} \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \Xi_{26} & 0 & \Xi_{28} & 0 & \Xi_{210} & \Xi_{211} \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 & \Xi_{39} & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & \Xi_{47} & 0 & \Xi_{49} & 0 & 0 \\ * & * & * & * & -2\varepsilon.I & \tau_M B^T & 0 & \Xi_{58} & 0 & \tau_M B^T & 0 \\ * & * & * & * & * & -\tau_M\overline{R_1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\tau_M\overline{R_1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\tau_M\overline{R_3} & 0 \\ * & * & * & * & * & * & * & * & * & * & -\tau_M\overline{R_3} \end{bmatrix} < 0, \tag{7}$$

where

$\Xi_{11} = \overline{P}(A_0 + (\lambda_1 + \alpha_1)I)^\top + (A_0 + (\lambda_1 + \alpha_1)I)\overline{P} + GY + Y^\top G^\top + \overline{Q_1} + \overline{Q_2} + \overline{Q_3},$
$\Xi_{12} = A_1\overline{P} + (\alpha_2 - \alpha_1)\overline{P},$
$\Xi_{14} = (\lambda_2 - \lambda_1)\overline{P},$
$\Xi_{15} = B - \epsilon\overline{P}C_0^\top K,$
$\Xi_{16} = \tau_M\overline{P}A_0^\top + \tau_M Y^\top G^\top,$
$\Xi_{17} = \lambda_1\tau_M\overline{R_1},$
$\Xi_{18} = (\tau_M - \tau_m)\overline{P}A_0^\top + (\tau_M - \tau_m)Y^\top G^\top,$
$\Xi_{110} = \tau_M\overline{P}A_0^\top + \tau_M Y^\top G^\top,$
$\Xi_{111} = \alpha_1\tau_M\overline{R_3},$
$\Xi_{22} = -(1 - \mu)\overline{Q_3} - 2\alpha_2\overline{P},$
$\Xi_{25} = -\epsilon\overline{P}C_1^\top K,$
$\Xi_{26} = \tau_M\overline{P}A_1^\top,$
$\Xi_{28} = (\tau_M - \tau_m)\overline{P}A_1^\top,$
$\Xi_{210} = \tau_M\overline{P}A_1^\top,$
$\Xi_{211} = \alpha_2\tau_M\overline{R_3},$
$\Xi_{33} = -\overline{Q_1} + 2\beta_1\overline{P},$
$\Xi_{34} = (\beta_2 - \beta_1)\overline{P},$
$\Xi_{39} = \beta_1(\tau_M - \tau_m)\overline{R_2},$
$\Xi_{44} = -\overline{Q_2} - 2(\lambda_2 + \beta_1)\overline{P},$
$\Xi_{47} = \lambda_2\tau_M\overline{R_1},$
$\Xi_{49} = \beta_2(\tau_M - \tau_m)\overline{R_2},$
$\Xi_{58} = (\tau_M - \tau_m)B^T,$
$\Xi_{88} = -(\tau_M - \tau_m)\overline{R_2},$
$\Xi_{99} = -(\tau_M - \tau_m)\overline{R_2},$

holds. Then the origin of the controlled system (1) is stabilized by the linear state feedback (4), where

$$\tilde{K} = Y\overline{P}^{-1}.$$

Proof Let $0 \le \tau_m < \tau_M$, $\lambda_1,\ \lambda_2,\ \alpha_1,\ \alpha_2,\ \beta_1$ and β_2 be fixed reals. Suppose that there exist a scalar $\epsilon > 0$, positive definite matrices $\overline{P} > 0,\ \overline{Q_1} > 0,\ \overline{Q_2} > 0,\ \overline{Q_3} > 0,\ \overline{R_1} > 0,\ \overline{R_2} > 0,\ \overline{R_3} > 0$, and a matrix $Y \in \mathbb{R}^{r\times n}$ such that the LMI (7) is satisfied. Let as denote by $\Xi_2^{'}$ the new matrix obtained after making these changes in the matrix Ξ_2 :
$\overline{P}^{-1} = P,\ \overline{P}^{-1}\ \overline{Q_1}\ \overline{P}^{-1} = Q_1,\ \overline{P}^{-1}\ \overline{Q_2}\ \overline{P}^{-1} = Q_2,\ \overline{P}^{-1}\ \overline{Q_3}\ \overline{P}^{-1} = Q_3,\ \overline{R_1}^{-1} = R_1,\ \overline{R_2}^{-1} = R_2,\ \overline{R_3}^{-1} = R_3,\ \tilde{K}P^{-1} = Y,\ M_i = \lambda_i P,\ N_i = \beta_i P,\ S_i = \alpha_i P,\ i = 1, 2.$
Then the LMI (7) is equivalent to the feasibility of the following LMI

$$T^\top\ \Xi_2^{'}\ T = \Xi_1 < 0, \tag{8}$$

where $T = \text{diag}\{P,\ P,\ P,\ P,\ I,\ R_1,\ R_1,\ R_2,\ R_2,\ R_3,\ R_3\}$,

$$\Xi_1 = \begin{bmatrix} \Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & \Xi_{15} & \Xi_{16} & \tau_M M_1^T & \Xi_{18} & 0 & \Xi_{110} & \tau_M S_1^T \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} & \tau_M A_1^T R_1 & 0 & \Xi_{28} & 0 & \tau_M A_1^T R_3 & \tau_M S_2^T \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & 0 & 0 & \Xi_{39} & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & \tau_M M_2^T & 0 & \Xi_{49} & 0 & 0 \\ * & * & * & * & \Xi_{55} & \tau_M B^T R_1 & 0 & \Xi_{58} & 0 & \tau_M B^T R_3 & 0 \\ * & * & * & * & * & -\tau_M R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\tau_M R_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\tau_M R_3 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\tau_M R_3 \end{bmatrix} < 0,$$

where

$$\begin{aligned}
\Xi_{11} &= (A_0 + G\tilde{K})^\top P + P(A_0 + G\tilde{K}) + Q_1 + Q_2 + Q_3 + M_1^\top + M_1 + S_1^\top + S_1, \\
\Xi_{12} &= PA_1 - S_1^\top + S_2, \\
\Xi_{14} &= -M_1^\top + M_2, \\
\Xi_{15} &= PB - \epsilon\, C_0^\top K, \\
\Xi_{16} &= \tau_M (A_0 + G\tilde{K})^T R_1, \\
\Xi_{110} &= \tau_M (A_0 + G\tilde{K})^T R_3, \\
\Xi_{22} &= -(1-\mu)Q_3 - S_2^\top - S_2, \\
\Xi_{25} &= -\epsilon\, C_1^\top K, \\
\Xi_{33} &= -Q_1 + N_1^\top + N_1, \\
\Xi_{34} &= -N_1^\top + N_2, \\
\Xi_{44} &= -Q_2 - M_2^\top - M_2 - N_2^\top - N_2, \\
\Xi_{55} &= -2\epsilon\, I, \\
\Xi_{18} &= (\tau_M - \tau_m)(A_0 + G\tilde{K})^T R_2, \\
\Xi_{28} &= (\tau_M - \tau_m) A_1^T R_2, \\
\Xi_{58} &= (\tau_M - \tau_m) B^T R_2, \\
\Xi_{88} &= -(\tau_M - \tau_m) R_2, \\
\Xi_{39} &= (\tau_M - \tau_m) N_1^T, \\
\Xi_{49} &= (\tau_M - \tau_m) N_2^T, \\
\Xi_{99} &= -(\tau_M - \tau_m) R_2.
\end{aligned}$$

Next let us consider the Lyapunov–Krasovskii functional candidate

$$\begin{aligned}
V(t, x_t) &= x^\top(t)Px(t) + \int_{t-\tau_m}^{t} x^\top(s)Q_1 x(s)ds + \int_{t-\tau_M}^{t} x^\top(s)Q_2 x(s)ds \\
&\quad + \int_{t-\tau(t)}^{t} x^\top(s)Q_3 x(s)ds + \int_{-\tau_M}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)R_1\dot{x}(s)dsd\theta \\
&\quad + \int_{-\tau_M}^{-\tau_m}\int_{t+\theta}^{t} \dot{x}^\top(s)R_2\dot{x}(s)dsd\theta \\
&\quad + \int_{-\tau(t)}^{0}\int_{t+\theta}^{t} \dot{x}^\top(s)R_3\dot{x}(s)dsd\theta.
\end{aligned}$$

Recall that matrices $P,\ Q_i,\ R_i,\ i = 1,2,3$ are positive definite as well as the matrices $\overline{P},\ \overline{Q_i},\ \overline{R_i},\ i = 1,2,3$. Then the derivative of V along the trajectories of system (1) is given by

$$\begin{aligned}\dot{V}(t,x_t) &= 2\dot{x}^\top(t)Px(t)+x^\top(t)Q_1x(t)-x^\top(t-\tau_m)Q_1x(t-\tau_m)\\ &+x^\top(t)Q_2x(t)-x^\top(t-\tau_M)Q_2x(t-\tau_M)\\ &+x^\top(t)Q_3x(t)-(1-\dot{\tau}(t))x^\top(t-\tau(t))Q_3x(t-\tau(t))\\ &+\tau_M\dot{x}^\top(t)R_1\dot{x}(t)-\int_{t-\tau_M}^{t}\dot{x}^\top(s)R_1\dot{x}(s)ds\\ &+(\tau_M-\tau_m)\dot{x}^\top(t)R_2\dot{x}(t)-\int_{t-\tau_M}^{t}-\tau_m\dot{x}^\top(s)R_2\dot{x}(s)ds\\ &+\tau(t)\dot{x}^\top(t)R_3\dot{x}(t)-\int_{t-\tau(t)}^{t}\dot{x}^\top(s)R_3\dot{x}(s)ds. \qquad (9)\end{aligned}$$

Using (3) and applying the integral inequality (4) to the right-hand side of (9), we obtain

$$\begin{aligned}\dot{V}(t,x_t) &\leq 2\dot{x}^\top(t)Px(t)+x^\top(t)[Q_1+Q_2+Q_3]x(t)-x^\top(t-\tau_m)Q_1x(t-\tau_m)\\ &-x^\top(t-\tau_M)Q_2x(t-\tau_M)-(1-\mu)x^\top(t-\tau(t))Q_3x(t-\tau(t))\\ &+\dot{x}^\top(t)[\tau_MR_1+(\tau_M-\tau_m)R_2+\tau_MR_3]\dot{x}(t)\\ &+\xi_1^\top(t)\Upsilon_1\xi_1(t)+\tau_M\xi_1^\top(t)\Gamma_1^\top R_1^{-1}\Gamma_1\xi_1(t)\\ &+\xi_2^\top(t)\Upsilon_2\xi_2(t)+(\tau_M-\tau_m)\xi_2^\top(t)\Gamma_2^\top R_2^{-1}\Gamma_2\xi_2(t)\\ &+\xi_3^\top(t)\Upsilon_3\xi_3(t)+\tau_M\xi_3^\top(t)\Gamma_3^\top R_3^{-1}\Gamma_3\xi_3(t)\end{aligned}$$

with

$$\xi_1(t)=\begin{bmatrix} x(t)\\ x(t-\tau_M)\end{bmatrix},\Gamma_1^\top=\begin{bmatrix} M_1^\top\\ M_2^\top\end{bmatrix},\Upsilon_1=\begin{bmatrix} M_1^\top+M_1 & -M_1^\top+M_2\\ * & -M_2^\top-M_2\end{bmatrix},$$

$$\xi_2(t)=\begin{bmatrix} x(t-\tau_m)\\ x(t-\tau_M)\end{bmatrix},\Gamma_2^\top=\begin{bmatrix} N_1^\top\\ N_2^\top\end{bmatrix},\Upsilon_2=\begin{bmatrix} N_1^\top+N_1 & -N_1^\top+N_2\\ * & -N_2^\top-N_2\end{bmatrix},$$

$$\xi_3(t)=\begin{bmatrix} x(t)\\ x(t-\tau(t))\end{bmatrix},\Gamma_3^\top=\begin{bmatrix} S_1^\top\\ S_2^\top\end{bmatrix},\Upsilon_3=\begin{bmatrix} S_1^\top+S_1 & -S_1^\top+S_2\\ * & -S_2^\top-S_2\end{bmatrix}.$$

Rearranging the terms of the right-hand side yields:

$$\dot{V}(t)\leq\eta^\top(t)\ \Pi\ \eta(t), \qquad (10)$$

where

$$\Pi:=\begin{bmatrix}\Pi_{11} & \Pi_{12} & 0 & \Pi_{14} & \Pi_{15}\\ * & \Pi_{22} & 0 & 0 & \Pi_{25}\\ * & * & \Pi_{33} & \Pi_{34} & 0\\ * & * & * & \Pi_{44} & 0\\ * & * & * & * & \Pi_{55}\end{bmatrix},\quad \eta(t):=\begin{bmatrix} x(t)\\ x(t-\tau(t))\\ x(t-\tau_m)\\ x(t-\tau_M)\\ \omega(t)\end{bmatrix}$$

with

$$
\begin{aligned}
\Pi_{11} &= (A_0+G\tilde{K})^\top P + P(A_0+G\tilde{K}) + Q_1 + Q_2 + Q_3 + \\
&\quad \tau_M (A_0+G\tilde{K})^\top R_1 (A_0+G\tilde{K}) + (\tau_M-\tau_m)(A_0+G\tilde{K})^\top R_2 (A_0+G\tilde{K}) + \\
&\quad \tau_M (A_0+G\tilde{K})^\top R_3 (A_0+G\tilde{K}) + M_1^\top + M_1 + \tau_M M_1^\top R_1^{-1} M_1 \\
&\quad +\tau_M S_1^\top R_3^{-1} S_1 + S_1^\top + S_1, \\
\Pi_{12} &= PA_1 + \tau_M (A_0+G\tilde{K})^\top R_1 A_1 + (\tau_M-\tau_m)(A_0+G\tilde{K})^\top R_2 A_1 + \\
&\quad \tau_M (A_0+G\tilde{K})^\top R_3 A_1 - S_1^\top + S_2 + \tau_M S_1 R_3^{-1} S_2, \\
\Pi_{14} &= -M_1^\top + M_2 + \tau_M M_1^\top R_1^{-1} M_2, \\
\Pi_{15} &= PB + \tau_M (A_0+G\tilde{K})^\top R_1 B + (\tau_M-\tau_m)(A_0+G\tilde{K})^\top R_2 B + \\
&\quad \tau_M (A_0+G\tilde{K})^\top R_3 B, \\
\Pi_{22} &= -(1-\mu)Q_3 - S_2^\top - S_2 + \tau_M A_1^\top R_1 A_1 + (\tau_M-\tau_m) A_1^\top R_2 A_1 + \tau_M A_1^\top R_3 A_1 \\
&\quad +\tau_M S_2^\top R_3^{-1} S_2, \\
\Pi_{25} &= \tau_M A_1^\top R_1 B + (\tau_M-\tau_m) A_1^\top R_2 B + \tau_M A_1^\top R_3 B, \\
\Pi_{33} &= -Q_1 + N_1^\top + N_1 + (\tau_M-\tau_m) N_1^\top R_2^{-1} N_1, \\
\Pi_{34} &= -N_1^\top + N_2 + (\tau_M-\tau_m) N_1^\top R_2^{-1} N_1, \\
\Pi_{44} &= -Q_2 - M_2^\top - M_2 + \tau_M M_2^\top R_1^{-1} M_2 + (\tau_M-\tau_m) N_2^\top R_2^{-1} N_2 - N_2^\top - N_2, \\
\Pi_{55} &= \tau_M B^\top R_1 B + (\tau_M-\tau_m) B^\top R_2 B + \tau_M B^\top R_3 B.
\end{aligned}
$$

A sufficient condition for asymptotic stability of the system (1) is to show that

$$\dot{V}(t) \le \eta^\top(t)\Pi\eta(t) < 0 \tag{11}$$

for all $\eta(t) \neq 0$. Then using (8) and Shur Complement we can see that the LMI (8) is equivalent to the following:

$$
\Sigma = \begin{bmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\
* & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} \\
* & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} \\
* & * & * & \Sigma_{44} & \Sigma_{45} \\
* & * & * & * & \Sigma_{55}
\end{bmatrix} < 0
$$

with $\Sigma_{ij} = \Pi_{ij}$, $(i,j = 1,2,3,4)$, $\Sigma_{15} = \Pi_{15} - \epsilon\, C_0^\top K$, $\Sigma_{25} = \Pi_{25} - \epsilon\, C_1^\top K$, $\Sigma_{35} = \Pi_{35}$, $\Sigma_{45} = \Pi_{45}$, $\Sigma_{55} = \Pi_{55} - 2\epsilon\, I$. On the other hand, by using the S-procedure and (2) we have

$$\eta^\top(t)\,\Sigma\,\eta(t) = \eta^\top(t)\,\Pi\,\eta(t) - 2\epsilon\,\omega^\top(t)\omega(t) - 2\epsilon\,\omega^\top(t)\,[KC_0x(t) + KC_1x(t-\tau)] < 0 \tag{12}$$

for all $\eta(t) \neq 0$. This completes the proof. □

Example 2.1 Consider the time delay system (1) with the nonlinear function satisfying (2) with

$$
A_0 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix},\ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ G = \begin{bmatrix} -1 & -1 \end{bmatrix},
$$

$$
C_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix},\ K = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}. \tag{13}
$$

Let the extra parameters be fixed to:

$$\tau_m = 10^{-4},\ \tau_M = 0.088,\ \mu = 0.01,\ \lambda_1 = -1,\ \lambda_2 = -1.2,$$
$$\alpha_1 = -0.2,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 0,$$

then by Theorem 1, we have $\epsilon = 0.6809$ and

$$\overline{Q_1} = \begin{bmatrix} 0.9955 & 0.1991 \\ 0.1991 & 0.9955 \end{bmatrix},\ \overline{Q_2} = \begin{bmatrix} 2.4678 & -0.5239 \\ -0.5239 & 2.0589 \end{bmatrix},\ \overline{Q_3} = \begin{bmatrix} 1.1643 & 0.0690 \\ 0.0690 & 1.1153 \end{bmatrix},$$

$$\overline{R_1} = \begin{bmatrix} 4.2714 & 0.4606 \\ 0.4606 & 4.1847 \end{bmatrix},\ \overline{R_2} = \begin{bmatrix} 13.6586 & -0.0274 \\ -0.0274 & 13.6260 \end{bmatrix},\ \overline{R_3} = \begin{bmatrix} 13.1236 & -0.0336 \\ -0.0336 & 13.0996 \end{bmatrix},$$

$$\overline{P} = \begin{bmatrix} 0.4781 & -0.3380 \\ -0.3380 & 0.3095 \end{bmatrix},\ Y = \begin{bmatrix} 1.1971 & 1.6605 \end{bmatrix},\ \tilde{K} = \begin{bmatrix} 27.6255 & 35.5345 \end{bmatrix}.$$

3 Conclusion

The problem of absolute stabilization of a class of time-varying delay systems with sector-bounded nonlinearity have been considered. New delay-dependant stabilization criterion with sector condition has been proposed. A new result is given and illustrated by numerical example, treated with Matlab, in order to show effectiveness of the main result. This criterion has been formulated in the form of linear matrix inequalities (LMI).

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Stabilization of Controllable Linear Systems

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Abstract: The work consists of two parts. The first part is devoted to linear continuous-time systems and the second one to linear discrete-time systems. In the first part stationary and nonstationary stabilization of linear time-invariant continuous-time systems is considered. A simple and direct proof of Zubov's and Wonham's Theorem on pole assignment in controllable linear systems by means of a suitable static time-invariant output feedback of the state is given. Brockett's problem of stabilization by means of a static time-varying output feedback of linear system is considered. To solve this problem two approaches are considered. Sufficient conditions of low- and high-frequency stabilization of controllable linear systems are given. Also examples of possibility of nonstationary low-frequency and high-frequency stabilization of two-dimensional and three-dimensional linear systems are given. In the second part the discrete-time version of Brockett's problem for linear control systems is considered. It is shown that under mild conditions stabilization for linear time-invariant discrete-time systems is possible by means of piecewise-constant periodic with a sufficiently large period static output feedback control. Sufficient conditions of low-frequency stabilization are given. For second-order systems a necessary and sufficient condition of stabilizability by periodic output feedback is given. Also pole assignment problem for linear time-invariant discrete-time systems by static periodic output feedback is considered.

Keywords: *stabilization, time-invariant system, Brockett problem, pole assignment, time-invariant/time-varying static output feedback.*

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1 Introduction

Within the last 130 years the methods of stabilization of control systems have been constructed, developed, and improved: from creating Vyshnegradsky's cataract to the analysis and synthesis of systems of rocket stabilization and distributed systems of clock-signal generators in multiprocessor clusters. At present the theory and practice of stabilization are the subject of many books and surveys. The various methods of stabilization have entered into textbooks on the control theory and became classical ones. But in the last thirty years there has been a rapid growth of publications, devoted to the methods of stabilization of linear control systems, and the above-mentioned books and surveys have already not reflected them completely.

The increasing interest to stabilization problems is motivated by the needs of the practice of control formulated in the open problems by many famous scholars such as V.I. Zubov, W.M. Wonham, D.S. Bernstein, R. Brockett, J. Rosenthal and J.C. Willems. For solving these problems the new methods of analysis and synthesis of linear control systems were developed.

In this survey the effort is made to describe new methods and results. The authors believe that the acquaintance with these methods and results will be useful for many specialists and will give an impetus to the further development of this interesting and substantial direction: the theory of stabilization of linear control systems.

A more detailed consideration of current methods of stabilization will be in our book [1].

2 Stabilizability and Pole Assignment in Linear Systems by Static Time-Invariant State Feedback

Here we consider the stabilization and pole assignment problems for linear time-invariant continuous-time systems.

Consider a linear time-invariant continuous-time system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is the state vector, $u = u(t) \in \mathbb{R}^m$ is the control input vector, and A, B are real constant matrices of dimension $n \times n$ and $n \times m$, respectively. (The point over the symbol x denotes the differentiation in t).

In the following all matrices have real-valued elements.

We consider for system (1) the classical feedback stabilization problem:

Under the assumption that the uncontrolled system is unstable, find an appropriate stabilizing feedback law.

It is well-known that this problem can be solved by means of a time-invariant static full state feedback $u = Sx$. This result follows from the following theorem on pole assignment.

Zubov's and Wonham's Theorem (on pole assignment) [2, 3]. *The system* (1) *is completely controllable if and only if for every choice of the self-conjugate set* $M = \{\mu_j\}_{j=1}^n$ *of complex numbers* μ_j *there exists* $(m \times n)$*-matrix* S *such that* $A + BS$ *has* M *for its set of eigenvalues.*

According to [4], this theorem was first obtained for the single-input case ($m = 1$) by Bertram in 1959 using locus method. In 1961, Bass independently formulated and proved the same result (but did not publish it) in the context of linear algebra. The single-input case was also considered by Rissanen [5] and Rosenbrock [6]. The above Theorem in the multi-input case for complex matrices A, B, S and arbitrary set M of complex numbers was proved by Popov [7] and by Langenhop [8]. Other contributions concerning pole assignment in multi-input systems by state feedback are due to Simon and Mitter [9], and Brunovsky [10]. In [9] the ability to relocate arbitrarily eigenvalues by state feedback was called *modal controllability*.

Zubov [2] and Wonham [3] were the first to prove the Theorem on pole assignment in the multi-input systems of the type (1) for *real* matrices A, B, S and self-conjugate set of complex numbers.

It should be noted that the proof of this Theorem in complex case (A, B, S and M are complex) is far simpler than real one.

Since then, when Zubov' and Wonham's works appeared, a great number (literally hundreds) of works, concerning pole assignment and its applications has been written. The primary impetus of most of the works mentioned concerns the stabilization problem for system (1).

The proof of Zubov's and Wonham's Theorem in multi-inpute case is rather tedious. Therefore after publication of works [2, 3] there were offered alternative proofs of this theorem (see, e.g., [11]-[13]; and also [14]-[19]). The goal of these papers was to give a simple proof of Zubov's and Wonham's Theorem.

Below we present another, different from the above-mentioned ones simple and direct new proof of Zubov's and Wonham's Theorem [20].

In the following instead of "complete controllability" of system (1) we will simply say about "controllability" of the pair (A, B).

2.1 Elementary Proof of Zubov's and Wonham's Theorem

A. ***Proof*** of Sufficiency.

Suppose that the pair (A, B) is not controllable. Then in system (1) we can separate from (1) a subsystem which contains no input variables. More precisely, there exists a nondegenerate linear transformation of coordinates $x \to Qx$ ($\det Q \neq 0$) such that the system (1) in new coordinates takes a form of the type (1) with the matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix},$$

$$A_{21} = 0, \quad B_2 = 0 \quad \text{or} \quad A_{12} = 0, \quad B_1 = 0.$$

Then it is clear that whatever $(m \times n)$-matrix

$$S = (\underbrace{S_1}_{n_1} \underbrace{S_2}_{n_2})\}m \quad (n_1 + n_2 = n)$$

we take the spectrum of the closed-loop system matrix $A + BS$ in the form

$$\sigma(A + BS) = \sigma(A_{11} + B_1 S_1) \cup \sigma(A_{22})$$

or

$$\sigma(A + BS) = \sigma(A_{11}) \cup \sigma(A_{22} + B_2 S_2).$$

We see that one of two parts of the spectrum of the matrix $A + BS$ is independent of the choice of matrix $A + BS$. Therefore, the matrix cannot have arbitrarily preassigned eigenvalues. The Sufficiency is proved. □

The proof of Necessity leans on a number of simple propositions.

Lemma 2.1 *Let*

$$\Gamma = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad (\beta \neq 0) \tag{2}$$

be a real (2×2)*-matrix. Let* B *be a real* (2×2)*- or* (2×1)*-matrix and* $B \neq 0$*. Then there exists a real matrix* R *such that the eigenvalues of the matrix* $\Gamma + BR$ *are real.*

The Proof of Lemma 2.1 is straightforward.

Using Lemma 2.1 we can easily prove the following proposition.

Lemma 2.2 *Let* Λ *and* B *be real* $(n \times n)$*- and* $(n \times m)$*-matrices, respectively. Suppose the pair* (Λ, B) *is controllable and all eigenvalues of the matrix* Λ *are nonreal. Then there exists a real* $(m \times n)$*-matrix* R *such that all eigenvalues of the matrix* $\Lambda + BR$ *are real.*

Proof Let $\lambda_1, \bar{\lambda}_1, \ldots, \lambda_\ell, \bar{\lambda}_\ell$ $(\lambda_j, \bar{\lambda}_j = \alpha_j \pm i\beta_j,\ \beta_j \neq 0,\ j = 1, \ldots, \ell;\ n = 2\ell)$ be the eigenvalues of the matrix Λ, listed according to their multiplicity.

By a similarity matrix Q $(\det Q \neq 0)$ transforms the matrix Λ to the real lower Jordan canonical form

$$\tilde{\Lambda} = Q^{-1}\Lambda Q = \operatorname{diag}\{J_1(\lambda_1), \ldots, J_q(\lambda_q)\},\ q \leq \ell.$$

Here $J_k(\lambda_k)$ $(k = 1, \ldots, q)$ is a lower λ_k – Jordan block of dimension $2\nu_k \times 2\nu_k$ $\left(\sum\limits_{k=1}^{q} \nu_k = \ell\right)$. That is, the block $J_k(\lambda_k)$ has (2×2)-matrices Γ_j $(j = 1, \ldots, \ell)$ of the type (2) on the diagonal, the identity (2×2)-matrices lower the diagonal, and zero – matrices elsewhere.

Let $\tilde{B} := Q^{-1}B$. Find $(m \times n)$-matrix such that the matrix $\tilde{\Lambda} + \tilde{B}\tilde{R}_1$ has two real (may be equal) and $n-2$ nonreal eigenvalues. Then it will be the same for the matrix $\Lambda_1 := \Lambda + BR_1$, where $R_1 = \tilde{R}Q^{-1}$. In this case the pair (Λ_1, B) will be controllable since (Λ, B) is controllable by assumption.

We seek $\tilde{R}_1$ in the form of a block matrix $\tilde{R}_1 = [\tilde{R}_{pq}]$ containing four blocks $\tilde{R}_{pq}$ $(p, q = 1, 2)$ such that $\tilde{R}_{12} = 0, \tilde{R}_{21} = 0, \tilde{R}_{22} = 0$ and (2×2)-block matrix $\tilde{R}_{11}$ is to be determined. (In the case $m = 1$ $\tilde{R}_{11}$ is a row matrix of size 1×2.)

Divide the matrices $\tilde{\Lambda}$ and $\tilde{B}$ into four blocks

$$\tilde{\Lambda} = [\tilde{\Lambda}_{pq}], \quad \tilde{B} = [\tilde{B}_{pq}] \quad (p, q = 1, 2)$$

in such a way that $\tilde{\Lambda}_{11}$ and $\tilde{B}_{11}$ are (2×2)-matrices. (In the case $m = 1$ $\tilde{B}_{11}$ is a column matrix of dimension 2×1.) It is clear that $\tilde{\Lambda}_{11} = \Gamma_1$, $\sigma(\tilde{\Lambda}_{22}) = \{\lambda_2, \bar{\lambda}_2, \ldots, \lambda_\ell, \bar{\lambda}_\ell\}$.

We have

$$\tilde{\Lambda}_1 := \tilde{\Lambda} + \tilde{B}\tilde{R}_1 \quad (p, q = 1, 2), \tag{3}$$

where $\tilde{M}_{12} = 0, \tilde{M}_{22} = \tilde{\Lambda}_{22}$ and

$$\tilde{M}_{11} = \Gamma_1 + \tilde{B}_{11}\tilde{R}_{11}. \tag{4}$$

The pair $(\tilde{\Lambda}, \tilde{B})$ is controllable, since the pair (Λ, B) is the same by assumption. Therefore it must be $(\tilde{B}_{11}, \tilde{B}_{12}) \neq 0$. Otherwise the pair $(\tilde{\Lambda}, \tilde{B})$ will not be controllable. Without loss of generality we assume that $\tilde{B}_{11} \neq 0$. Then by virtue of Lemma 2.1 there exists a matrix $\tilde{R}_{11}$ such that (2×2)-matrix (4) has real eigenvalues r_1 and r_2. Whence taking into account the structure of matrix (3) it follows that

$$\sigma(\tilde{\Lambda} + \tilde{B}\tilde{R}_1) = \{r_1, r_2\} \cup \{\lambda_2, \bar{\lambda}_2, \ldots, \lambda_\ell, \bar{\lambda}_\ell\}.$$

Rearrange the matrices $\tilde{M}_{11}, \Gamma_j (j = \overline{1, \ell})$ in the diagonal array of matrix (3) in such a way that Γ_2 appears in the top left-hand corner of matrix (3).

We apply to matrix Γ_2 the same procedure as above for Γ_1. Thus we change the matrix Γ_2 by matrix of the type (4) having real eigenvalues. Therefore we obtain a matrix Λ_2 having four (among them may be equal ones) real eigenvalues and $n-4$ remaining nonreal ones $\lambda_3, \bar{\lambda}_3, \ldots, \lambda_\ell, \bar{\lambda}_\ell$.

Repeating this process after ℓ steps as a result we obtain a matrix Λ_ℓ having only real eigenvalues. The Lemma 2.2 is proved. □

From Lemma 2.2 immediately it follows

Lemma 2.3 *Let A and B be arbitrary real $(n\times n)$- and $(n\times m)$-matrices, respectively. Let the pair (A, B) be controllable. Then there exists a real $(m \times n)$-matrix R such that all the eigenvalues of the matrix $A + BR$ are real.*

The following lemma solves the pole assignment problem in the field of real numbers $\mathbb{R}$.

Lemma 2.4 (Lemma on pole assignment in $\mathbb{R}$) *Let A and B be arbitrary real $(n \times n)$- and $(n \times m)$-matrices, respectively. Suppose the pair (A, B) is controllable. Let $\{\mu_1, \ldots, \mu_n\}$ be an arbitrary set of real numbers. Then there exists a real $(m \times n)$-matrix such that*

$$\sigma(A + BS) = \{\mu_1, \ldots, \mu_n\}. \tag{5}$$

Proof By virtue of Lemma 2.3 there exists a real $(m\times n)$-matrix R_0 such that all the eigenvalues of the matrix $A_0 := A + BR_0$ are real. We denote these ones by $\lambda_1, \ldots, \lambda_n$, listed according to multiplicity. That is,

$$\sigma(A_0) = \{\lambda_1, \ldots, \lambda_n\} \quad (\lambda_j \in \mathbb{R},\ j = 1, \ldots, n). \tag{6}$$

The pair (A_0, B) is controllable since the pair (A, B) is the same by assumption.

Let $\mu_1, \ldots, \mu_n$ be arbitrary n real numbers (among them may be repeating ones).

The proof of Lemma 2.4 is exactly analogous to that of Lemma 2.2 and consists of solution n intermediate tasks.

1) $\{A_0, B; \lambda_1|\mu_1\}$ – task: *Find real $(m \times n)$-matrix such that*

$$\sigma(A_0 + BS_1) = \{\mu_1; \lambda_2, \ldots, \lambda_n\}. \tag{7}$$

As above by a similarity matrix Q_0 we transform the matrix A_0 to the real lower Jordan form: $\tilde{A}_0 := Q_0^{-1}A_0Q_0$. Let $\tilde{B} = Q_0^{-1}B$.

We first solve $\{\tilde{A}_0, \tilde{B}; \lambda_1|\mu_1\}$ – task. For this purpose as above we seek a corresponding matrix $\tilde{S}_1$ in the form of a block matrix $\tilde{S}_1 = [\tilde{S}_{pq}]$ $(p, q = 1, 2)$ such that $\tilde{S}_{12} = 0$, $\tilde{S}_{21} = 0$, $\tilde{S}_{22} = 0$, and $\tilde{S}_{11} =: \tilde{s}_{11}$ is a real number to be determined.

As above divide the matrices $\tilde{A}_0$ and $\tilde{B}$ into four blocks

$$\tilde{A}_0 = [\tilde{A}_{pq}] \quad \tilde{B} = [\tilde{B}_{pq}]\ (p, q = 1, 2).$$

Here $\tilde{A}_{11} =: \tilde{a}_{11}$ and $\tilde{B}_{11} =: \tilde{b}_{11}$ are real numbers. Clearly, $\tilde{a}_{11} = \lambda_1$, $\sigma(\tilde{A}_{22}) = \{\lambda_2, \ldots, \lambda_n\}$. Then we have $\tilde{A}_1 := \tilde{A}_0 + \tilde{B}\tilde{S}_1 = [\tilde{M}_{pq}]$, where $\tilde{M}_{12} = 0$, $\tilde{M}_{22} = \tilde{A}_{22}$ and

$$\tilde{M}_{11} =: \tilde{m}_{11} = \lambda_1 + \tilde{B}_{11}\tilde{S}_{11}. \tag{8}$$

Since the pair (A_0, B) is controllable as above in the proof of Lemma 2.2 one must have $(\tilde{B}_{11}, \tilde{B}_{12}) \neq 0$. Without loss of generality we assume that $\tilde{B}_{11} = \tilde{b}_{11} \neq 0$.

We claim that in (8) $\tilde{m}_{11} = \mu_1$. From here and (8) we determine $\tilde{s}_{11} = (\mu_1 - \lambda_1)/\tilde{b}_{11}$. Therefore we have

$$\sigma(\tilde{A}_0 + \tilde{B}\tilde{S}_1) = \{\mu_1; \lambda_2, \dots, \lambda_n\}.$$

That is, the matrix $\tilde{S}_1$ is solution of the $\{\tilde{A}_0, \tilde{B}; \lambda_1|\mu_1\}$-task. Then the matrix $S_1 = \tilde{S}_1 Q_0^{-1}$ will be solution of the task (7), since the matrix $A_0 + BS_1$ is similar to the matrix $\tilde{A}_0 + \tilde{B}\tilde{S}_1$.

Denote $A_1 := A_0 + BS_1$. Then $A_1 = A + B(R_0 + S_1)$.

We solve the next

2) $\{A_1, B; \lambda_2|\mu_2\}$-task: *Find $(m \times n)$-matrix such that*

$$\sigma(A_1 + BS_2) = \{\mu_1, \mu_2; \lambda_3, \dots, \lambda_n\}. \tag{9}$$

We will exactly solve task (9) analogously to task (7). At first we rearrange the diagonal elements $\mu_1, \lambda_2, \dots, \lambda_n$ of the matrix $\tilde{A}_1 = \tilde{A}_0 + \tilde{B}\tilde{S}_1$ in such a way that λ_2 appears in the top left-hand corner of matrix $\tilde{A}_1$.

Apply to matrices A_1, B and numbers λ_2, μ_2 the same procedure of "the replacement λ_1 by μ_1" that we have made in the previous task. In the same way we determine a matrix S_2 and corresponding matrix $A_2 = A + B(R_0 + S_1 + S_2)$ such that

$$\sigma(A_2) = \{\mu_1, \mu_2; \lambda_3, \dots, \lambda_n\}.$$

In this case $S_2 = \tilde{S}_2 Q_0^{-1} Q_1^{-1}$, where Q_1 is a similarity matrix, and $\tilde{S}_2$ is determined analogously to $\tilde{S}_1$.

Repeating this process of solving corresponding $\{A_{j-1}, B; \lambda_j|\mu_j\}$-tasks we sequentially replace each eigenvalue λ_j $(j = 1, \dots, n)$ of the matrix A_0 from (6) by corresponding number μ_j from the given set $\{\mu_j\}_{j=1}^n$. As a result we obtain successively the matrices $S_1, \dots, S_n$ such that the matrix

$$A_n = A + B(R_0 + S_1 + \dots + S_n)$$

has $\{\mu_j\}_{j=1}^n$ for its desired set of eigenvalues. Hence, the matrix $S := R_0 + S_1 + \dots + S_n$ has the required property (5).

Lemma 2.4 is proved. $\Box$

Immediately from Lemma 2.4 it follows

Lemma 2.5 (Lemma on stabilization of the pair $(\mathbf{A}, \mathbf{B})$) *Let A and B be real $(n \times n)$- and $(n \times m)$-matrices, respectively. Let the pair (A, B) be controllable. Then there exists a real $(m \times n)$-matrix S such that the matrix $A + BS$ is stable, i.e. the pair (A, B) is stabilizable.*

Remark 2.1 A stabilization matrix S in Lemma 2.5 can be constructed by the algorithm described in the proof of Lemma 2.4.

Remark 2.2 In [21] another variant of elementary proof of the Lemma on Stabilization of the system (1) is proposed.

B. ***Proof*** of Necessity of Zubov's and Wonham's theorem.

Without loss of generality we may assume that $\operatorname{rank} B = m$.

If $m = n$, then the solution of pole assignment problem is given by the formula

$$S = B^{-1}(M - A),$$

where M is an arbitrary $(n\times n)$-matrix having the set $\{\mu_j\}_{j=1}^n$ as its spectrum. It remains to consider the case $1 \le m < n$.

Let $\{\mu_j\}_{j=1}^n$ be an arbitrary set of n complex numbers closed under complex conjugation. We will prove that there exists a $(m \times n)$-matrix S such that

$$\sigma(A + BS) = \{\mu_j\}_{j=1}^n.$$

Assume among the numbers μ_j $(j = 1, \dots, n)$ we have k real and ℓ complex-conjugate ones. Let $\mu_1, \dots, \mu_k$ be real numbers and the rest of 2ℓ numbers $\mu_{k+1}, \bar{\mu}_{k+1}, \dots, \mu_{k+1}, \bar{\mu}_{k+1}$ be complex conjugate ones. Let $\mu_{k+j}, \bar{\mu}_{k+j} = \sigma_{k+j} \pm i\omega_{k+j}$, $\omega_{k+j} \ne 0$ $(j = 1, \dots .\ell; k + 2\ell = n)$.

Let $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ be an arbitrary set of pairwise distinct real numbers λ_j $(j = 1, \dots, n)$. By virtue of Lemma 2.4 there exists a real $(m \times n)$-matrix S_0 such that

$$\sigma(A + BS_0) = \{\lambda_1, \dots, \lambda_n\} \quad (\lambda_p \ne \lambda_q,\ p \ne q,\ p, q = 1, \dots, n).$$

Denote $A_0 := A + BS_0$.

1. Applying sequentially the algorithm of solving of $\{A_{q-1}, B; \lambda_q|\mu_q\}$ – tasks described in the proof of Lemma 2.4 we construct matrices $S_1, \dots, S_k$ and the matrix

$$A_k = A_0 + B(S_1 + \dots + S_k)$$

such that $\sigma(A_k) = \{\mu_1, \dots, \mu_k; \lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n\}$.

2. We now solve the $\{A_k, B; \lambda_{k+1}, \lambda_{k+2}|\mu_{k+1}, \bar{\mu}_{k+1}\}$-task: *Find $(m \times n)$-matrix S_{k+1} such that*

$$\sigma(A_k + BS_{k+1}) = \{\mu_1, \dots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \lambda_{k+3}, \dots, \lambda_n\}. \tag{10}$$

Since $\lambda_{k+1} \ne \lambda_{k+2}$ by a similarity matrix P_0 one can reduce the matrix A_k to the form of the four block matrix

$$\tilde{A}_k := P_0^{-1} A_k P{+}_0 = [\tilde{\Lambda}_{pq}] \quad (p, q = 1, 2),$$

where $\tilde{\Lambda}_{11} = (\lambda_{k+1}, \lambda_{k+2})$, $\tilde{\Lambda}_{12} = 0$. It is clear that

$$\sigma(\tilde{\Lambda}_{22}) = \{\mu_1, \dots, \mu_k; \lambda_{k+3}, \lambda_{k+4}, \dots, \lambda_n\}.$$

Divide the matrix $\tilde{B} := P_0^{-1}B$ into four blocks such that the matrix $\tilde{B}_{11}$ has the dimension 2×2 (or 2×1 in the case $m = 1$): $\tilde{B} = [\tilde{B}_{pq}]$ $(p, q = 1, 2)$. Let

$$\tilde{B}_{11} = (\tilde{b}_{rt})_{r,t=1}^2.$$

(In the case $m = 1$ $\tilde{B}_{11} = \text{column}(\tilde{b}_{11}, \tilde{b}_{21}.)$

Since the pair (A, B) is controllable the pair (A_k, B), and therefore the pair $(\tilde{A}_k, \tilde{B})$ is controllable. Hence, as we have noted above in the proofs of Lemmas 2.2 and 2.4 one may assume that

$$\tilde{b}_{11} \ne 0 \quad \text{and} \quad \tilde{b}_{22} \ne 0. \tag{11}$$

To establish (10) we first solve $\{\tilde{A}_k, \tilde{B}; \lambda_{k+1}, \lambda_{k+2}|\mu_{k+1}, \bar{\mu}_{k+1}\}$-task. For this purpose as above we seek a matrix $\tilde{S}_{k+1}$ in the form of a block matrix $\tilde{S}_{k+1} = [\tilde{S}_{pq}]$ $(p, q = 1, 2)$, where $\tilde{S}_{12} = 0$, $\tilde{S}_{21} = 0$, $\tilde{S}_{22} = 0$ and (2×2)-matrix $\tilde{S}_{11}$ is to be determined (in the case $m = 1$ $\tilde{S}_{11}$ is a row (1×2)-matrix).

We have

$$\tilde{A}_{k+1} := \tilde{A}_k + \tilde{B}\tilde{S}_{k+1} = [\tilde{M}_{pq}], \tag{12}$$

where $\tilde{M}_{11} = \Lambda_{11} + \tilde{B}_{11}\tilde{S}_{11}$, $\tilde{M}_{12} = 0$, $\tilde{M}_{22} = \tilde{\Lambda}_{22}$. Two cases are possible: $a)$ $\det \tilde{B}_{11} \neq 0$ and $b)$ $\det \tilde{B}_{11} = 0$.

Case a). Here we claim that

$$\tilde{\Lambda}_{11} + \tilde{B}_{11}\tilde{S}_{11} = \Sigma_1, \tag{13}$$

where

$$\Sigma_1 = \begin{pmatrix} \sigma_{k+1} & -\omega_{k+1} \\ \omega_{k+1} & \sigma_{k+1} \end{pmatrix}.$$

From (13) we at once determine the matrix $\tilde{S}_{11} = (\tilde{B}_{11})^{-1}(\Sigma_1 - \tilde{\Lambda}_{11})$.

Case b). In this case we determine the matrix $\tilde{S}_{11}$ from the condition of equality of the characteristic polynomials of matrices in the right-hand and left-hand sides of (13):

$$\det(pI_2 - \tilde{\Lambda}_{11} - \tilde{B}_{11}\tilde{S}_{11}) = \det(pI_2 - \Sigma_1). \tag{14}$$

Here I_2 is the identity (2×2)-matrix.

Let $\tilde{S}_{11} = (c_{rt})^2_{r,t=j}$. (In the case $m = 1$ $\tilde{S}_{11} = (c_{11}, c_{12})$.) Taking into account inequalities (2.11) and equality $\tilde{b}_{11}\tilde{b}_{22} - \tilde{b}_{21}\tilde{b}_{12} = 0$, from (14) we can determine one of possible values of the entries c_{rt} of the matrix $\tilde{S}_{11}$:

$$\tilde{c}_{11} = d_1/\tilde{b}_{11} \quad c_{21} := 0; \quad c_{22} = d_2/\tilde{b}_{22} \quad (\tilde{b}_{22} \neq 0), \tag{15}$$

where

$$d_1 = (\sigma^2_{k+1} + \omega^2_{k+1} + \lambda^2_{k+1} - 2\lambda_{k+1}\sigma_{k+1})/(\lambda_{k+2} - \lambda_{k+1}),$$
$$d_2 = (\sigma^2_{k+1} + \omega^2_{k+1} + \lambda^2_{k+2} - 2\lambda_{k+2}\sigma_{k+1})/(\lambda_{k+1} - \lambda_{k+2}).$$

Since $\lambda_{k+1} \neq \lambda_{k+2}$ by choice of the set Λ the last expressions have a meaning. (In the case $m = 1$ $c_{11} = d_1/\tilde{b}_{11}$, $c_{12} = d_2/\tilde{b}_{21}$).

From (12) and (14) it follows that

$$\det(pI_n - \tilde{A}_{k+1}) = \det(pI_2 - \Sigma_1)\det(pI_{n-2} - M_{22}). \tag{16}$$

Here I_2, I_{n-2}, I_n are the identity matrices of respective dimensions. The equality (16) implies that for matrix (12) corresponding to the matrix $\tilde{S}_{k+1}$ with found above entries (15) the relation

$$\sigma(\tilde{A}_{k+1}) = \{\mu_1, \ldots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \lambda_{k+3}, \ldots, \lambda_n\} \tag{17}$$

holds.

Set $S_{k+1} := \tilde{S}_{k+1}P_0^{-1}$. Since the matrix $\tilde{A}_{k+1}$ is similar to the matrix $A_{k+1} := A_k + BS_{k+1}$, from (17) it follows that relation (10) is valid for the matrix S_{k+1}.

Further we solve the $\{A_{k+1}, B; \lambda_{k+3}, \lambda_{k+4}|\mu_{k+2}, \tilde{\mu}_{k+2}\}$ – task exactly analogously to the preceding one. As a result we find a matrix S_{k+2} and a corresponding matrix $A_{k+2} := A_{k+1} + BS_{k+2}$ such that

$$\sigma(A_{k+2}) = \{\mu_1, \ldots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \mu_{k+2}\bar{\mu}_{k+2}; \lambda_{k+5}, \ldots, \lambda_n\}.$$

Repeating this process as above after ℓ steps we find matrices $S_{k+1}, \dots, S_{k+\ell}$ and the matrix $A_{k+\ell} = A_k + B(S_{k+1} + \dots + S_{k+\ell})$ such that

$$\sigma(A_{k+\ell}) = \{\mu_1, \dots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}, \dots, \mu_{k+\ell}\bar{\mu}_{k+\ell}\}. \tag{18}$$

Since

$$A_{k+\ell} = A + B(S_0 + \sum_{q=1}^{k} S_q + \sum_{j=1}^{\ell} S_{k+j}),$$

from (18) it follows that the $(m \times n)$-matrix

$$S = S_0 + \sum_{q=1}^{k} S_q + \sum_{j=1}^{\ell} S_{k+j}$$

has the required property.

Zubov's and Wonham's Theorem is completely proved. $\square$

Remark 2.3 In just proposed proof of Zubov's and Wonham's theorem we only used the fact of possibility of matrices reduction to Jordan canonical form. But there is also an elementary proof of the theorem on reduction of a matrix to Jordan form proposed by A. F. Filippov [22]. Together with this Filippov's theorem our above proof of Zubov's and Wonham's theorem is completely elementary.

Remark 2.4 As is seen from the proofs of Lemmas 2.2, 2.4 and the proof of the sufficiency of Zubov's and Wonham's theorem there is no necessity to reduce matrices to Jordan form. It is sufficient only to reduce them to the following forms. In the proof of Lemma 2.2 in the top left-hand corner of the considered matrices we must have a (2×2)-matrix Γ of the type (2) and the elements of the first two rows except for the entries of matrix Γ must be equal to zero.

Also, in the proof of Lemma 2.4 in the top left-hand corner we must have a number λ_j and the elements of the first row except, may be, for λ_j must be equal to zero. This observation also applies to the proof of sufficiency of Zubov's and Wonham's Theorem. As a result the finding of the required matrix S becomes more "economical" for computations: much less number of operations must be done.

3 Pole Assignment in Linear Systems with Output Feedback

In the preceding section we have considered linear systems with full state feedback.

We now turn our attention to pole assignment for linear systems by output feedback.

Consider a linear time-invariant continuous-time system described by

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{19}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is an input vector, $y \in \mathbb{R}^\ell$ is an output vector, and A, B, C are real constant matrices of sizes $n \times n$, $n \times m$, $\ell \times n$, respectively ($\ell \leq n$).

Assume that the linear system (19) is controlled by a linear static output feedback

$$u = Sy \tag{20}$$

with a real constant $m \times \ell$-matrix S. Then the resulting closed-loop system (19), (20) is described by

$$\dot{x} = (A + BSC)x.$$

The poles of this system are the eigenvalues of the matrix $A + BSC$.

The problem of pole assignment arises in a natural way for closed-loop system using static output feedback.

Recall that this problem for system (19), (20) or simply triple matrices A, B, C is formulated as follows:

Given a triple real matrices (A, B, C) *and an arbitrary set* $\{\mu_j\}_{j=1}^n$ *of the complex numbers* μ_j *closed under complex conjugation, find a real matrix* S *such that the spectrum of the matrix* $A + BSC$ *coincides with the set* $\{\mu_j\}_{j=1}^n$*, i.e.*

$$\sigma(A + BSC) = \{\mu_j\}_{j=1}^n. \tag{21}$$

As we remarked above in the previous section this problem for two matrices A and B was first stated and solved by Zubov [2] and Wonham [3].

The problem of pole assignment by time-invariant static output feedback (20) has received much attention of researchers. Many works are devoted to solution of this problem and its various modifications (see surveys [23, 24]). Sufficient conditions have been obtained under which the pole assignment problem (21) can be resolved.

We note that for system (19) as for the system (1) property of controllability of the pair (A, B) is a necessary condition for the solvability of the pole assignment problem (21).

One of the pioneer works devoted to solving this problem was Davison's work [25]. In this work Davison proved the following assertion.

Theorem 3.1 (Davison [25]) *If the matrix* A *is cyclic (i.e. in its Jordan form to the distinct boxes correspond the distinct eigenvalues), the pair* (A, B) *is controllable and* $\operatorname{rank} B = m, \operatorname{rank} C = \ell$*, then there exists a matrix* S *such that the eigenvalues of the matrix* $A + BSC$ *of closed-loop system* (19), (20) *are arbitrary close to* ℓ *preassigned arbitrary numbers on the complex plane placed symmetrically with respect to the real axis.*

In the work [26] it was shown that if system (19) is controllable and observable, then there exists a matrix S such that the matrix $A + BSC$ is cyclic. Taking into account this result, in the paper [27] a theorem was proved which strengthen the Davison's Theorem. Namely, the following result is valid

Theorem 3.2 (Davison, Chatterjee [27]) *If* (A, B) *is controllable,* (A, C) *is observable, and* $\operatorname{rank} B = m, \operatorname{rank} C = \ell$*, then there exists a matrix* S *such that the* $\max\{\ell, m\}$ *eigenvalues of the matrix* $A + BSC$ *are arbitrary close to the* $\max\{\ell, m\}$ *preassigned arbitrary complex numbers closed under complex conjugation.*

In [28] an algorithm based on this theorem is given which allows pole assignment to be carried out on large linear systems (19) with output feedback (20).

In the case when A is a cyclic matrix an alternative proof of Davison's and Chatterjee's Theorem based on another approach was suggested in Sridhar's and Lindorff's work [29].

An analogous result under some other conditions is established in Jameson's work [30] for the systems with scalar input ($m = 1$). In this work for the case ($m = 1$) it is also proved that if the pair (A, B) is not controllable or the pair (A, C) is not observable and the eigenvalues λ_j $(j = 1, \dots, n)$ of the matrix A are distinct, then there is not any feedback matrix S such that the eigenvalues λ_{j_k} $(k = 1, \dots, r; r \leq n)$ which correspond to either the uncontrolled or unobserved variables, can be changed. Later, an alternative, more simple, proof of the second part of Jameson's assertion, extending his result to the systems with the vector input $(m > 1)$ was suggested in Nandi's and Herzog's note ([31]).

In later Davison's and Wang's [32] and Kimura's [33, 34] works it was established that under the same as above conditions on the matrices A, B and C for almost all A, B and C the $\min(n, m+\ell-1)$ eigenvalues of the matrix $A+BSC$ can be made arbitrarily close to the $\min(n, m+\ell-1)$ preassigned arbitrary complex numbers closed under complex conjugation.

This implies that if

$$m + \ell \geq n + 1,$$

then the pole assignment problem (21) is solvable for almost all matrices A, B and C.

Thus, the last inequality is a sufficient condition of solvability of the problem (21) in the typical case.

In Brockett's, Byrnes's [35] and Shumacher's [36] works there was given another sufficient condition of solvability of the problem (21) in the typical case. Namely, they show that

if $m\ell = n$ and the number

$$d(m, \ell) = \frac{1!2! \dots (\ell-1)!(m\ell)!}{m!(m+1)! \dots (m+\ell-1)!}$$

is odd, then the problem (21) *is solvable in the typical case.*

A sufficient condition in the case when the number $d(m, \ell)$ is even was obtained by Wang [37]:

if $m\ell > n$ and the number $d(m, \ell)$ is even, then the problem (21) *is solvable in the typical case.*

The distinct elementary proofs of this assertion were given in the works [38]-[41].

Another sufficient conditions of solvability of the problem (21) (and "near" problems) in the typical case were obtained in the works of many authors.

In Hermann's and Martin's [42] and Willems's and Hesselink's [43] papers it was established a general necessary condition

$$m\ell \geq n$$

of solvability of the problem (21) in the typical case. Later, this condition was strengthened in the work [44].

In [43] it is shown that, generally speaking, the inequality $m\ell \geq n$ is not a sufficient condition of solvability of the problem (21) in the typical case. Namely,

if $m = \ell = 2$ and $n = 4$, then the problem (21) *in the typical case is unsolvable.*

Note that in many works (see, for example, [45]-[51]) a more general than (21) eigenstructure assignment problem was considered. In this case the eigenvalues $\mu_1, \dots, \mu_r$ of the matrix of closed-loop system together with the corresponding to them eigenvectors $\xi_1, \dots, \xi_r$ are arbitrarily given or the elementary divisors, corresponding to these eigenvalues, are given. The problem is to find a matrix S such that either the spectrum of the matrix $A + BSC$ contains the set $\{\mu_j\}_{j=1}^r$ as a subset and the corresponding to the numbers μ_j eigenvectors of the matrix $A + BSC$ are equal to ξ_j (or are arbitrarily close to ξ_j) or the characteristic polynomial of the matrix $A+BSC$ has the preassigned polynomials $\psi_1,\dots,\psi_r(p)$ as its invariant factors (or elementary divisors).

One of the first works devoted to the eigenstructure assignment problem were the works of Rosenbrock [52], Kalman [53], Moore [54], and Srinathkumar [55]. The following result is valid.

Theorem 3.3 (Rosenbrock and Kalman [18, 52, 53]) *Suppose the pair* (A,B) *is controllable with the indices of controllability* $k_1 \geq k_2 \geq \cdots \geq k_m$. *Let* $\{\psi_i(p)\}_{i=1}^q$, $q \leq m$ *be a set of polynomials the leading coefficients of which are equal to* 1. *Assume that each polynomial* ψ_i $(i=1,\dots,q-1)$ *is divided by the successive one* ψ_{i+1} *without residue and* $\sum\limits_{i=1}^n deg\,\psi_i = n$.

Then for the existence of the matrix S *such that the given polynomials* ψ_i *are the nontrivial (not equal identically to the unity) invariant factors of the characteristic polynomial* $pI-A-BS$ *it is necessary and sufficient that the following inequalities hold*

$$\sum_{i=1}^{r} deg\psi_{q+1-i} \leq \sum_{i=1}^{r} k_{q+1-i},\ r=1,2,\dots,q.$$

In this case the equality occurs for $r=q=m$.

(Here "deg" denotes a "degree of polynomial".)

In the papers [56, 57] Rosenbrock's and Kalman's theorem (and results of other authors) are generalized.

In the work [54] it was described the class of all sets of the eigenvectors of the matrix $A+BS$ of closed-loop system with state feedback, which can correspond to the preassigned arbitrarily distinct eigenvalues of this matrix. In the same work in the case of distinct eigenvalues there was given the solution of the problem of simultaneous assignment of the eigenvalues and the corresponding eigenvectors of the matrix of closed-loop system.

In the paper [55] a tool developed in [33, 58] was used for study of the eigenstructure assignment problem for systems with state feedback. In [55] Srinathkumar has proved, in particular, the following assertion.

If the pair (A,B) is controllable, the pair (A,C) is observable and rank $(B)=m$, rank $(C)=\ell$, then there exists a matrix S such that the eigenvalues of the matrix $A+BSC$ are equal to the $\max(m,\ell)$ preassigned numbers with the corresponding $\max(m,\ell)$ eigenvectors with $\max(m,\ell)$ preassigned arbitrary components.

We also note Van der Woude's paper where a general theorem is proved giving a necessary and sufficient condition (in geometric terms) of solvability of pole assignment problem (21) by output feedback (20) for single-input system (19) $(m=1)$.

Theorem 3.4 (Van der Woude [59]) *Suppose the system* (19) *is controllable and* $f(p)$ *is an arbitrary real polynomial with leading coefficient 1 of degree* n.

Then for the existence of a real $(\ell\times 1)$*-matrix* S *such that*

$$\det(pI-(A+BSC)) = f(p)$$

it is necessary and sufficient that

$$f(A)Ker(C) \subset Lin(B,AB,\dots,A^{n-2}B).$$

Lately Van der Woude's theorem was essentially used by Aeyels and Willems [60, 61] for pole assignment in linear time-invariant discrete-time systems by periodic static output feedback. In the end we note that at the present time the pole assignment problem and the related with it adjoining questions are in the focus of attention of many scholars and the flow of literature in this direction does not weaken.

Remark 3.1 Some above-mentioned result can be regarded as results for output stabilization problem, since the latter is a special case of pole assignment problem. These results are formulated in terms of matrices whereas in the well-known Nyquist criterion the necessary and sufficient condition of stabilization of the system (19) is formulated in terms of behavior of hodograph of the frequency response of this system.

4 Nonstationary Stabilization. The Brockett Problem

In 1999, R. Brockett in the book [62] formulated the problem on stabilizability of a linear time-invariant system by means of a static time-varying output feedback.

To solve this problem two approaches are developed. The first of them is developed for constructing a low-frequency time-varying feedback, and the second approach for constructing a high-frequency one.

The Brockett problem is formulated as follows.

Problem 4.1 (Brockett Problem) Given a linear time-invariant continuous-time system (19), find a static time-varying output feedback

$$u = S(t)y, \tag{22}$$

such that the resulting closed-loop system

$$\dot{x} = (A + BS(t)C)x \tag{23}$$

is asymptotically stable.

In the previous section some aspects of the problem of stabilization of system (19) by output feedback (22) with a constant matrix $S(t) \equiv S=$ const are considered. In the Brockett problem it is required to find a variable stabilizing matrix $S = S(t)$ with the property mentioned above. In this case the Brockett problem can be reformulated in the following way.

Does the introduction of the time-dependent matrices $S(t)$ in feedback gain extend the possibility of stationary stabilization?

In the works [63]-[66] for some important cases the solution of the Brockett problem of nonstationary linear stabilization for system (19) in the class of piecewise-constant periodic with a sufficiently large period stabilizing functions $S(t)$ is given(a low-frequency stabilization).

In the works [67]-[70] for single-input single-output system (19) the Brockett problem is solved in the other class of the stabilizing functions. Namely, this is solved in the class of continuous periodic with a sufficiently small period functions $S(t)$ (a high-frequency stabilization). Below we consider these two types of nonstationary stabilization.

4.1 Nonstationary low-frequency stabilization

Basic hypotheses. Suppose that there exist real constant $(m \times \ell)$-matrices S_1 and S_2 such that the linear systems

$$\dot{x} = (A + BS_jC)x \quad (x \in \mathbb{R}^n) \quad (j = 1, 2) \tag{24}$$

possess stable invariant linear manifolds L_j and invariant linear manifolds M_j.

Suppose

$$M_j \cap L_j = \{0\}, \quad \dim M_j + \dim L_j = n.$$

We assume also that for solutions $x_j(t; x_0)$ $(x_j(0; x_0) = x_0)$ of systems (24) the following inequalities

$$|x_j(t; x_0)| \le \alpha_j |x_0| e^{-\lambda_j t} \quad \forall\, x_0 \in L_j, \tag{25}$$

$$|x_j(t; x_0)| \le \beta_j |x_0| e^{-\kappa_j t} \quad \forall\, x_0 \in M_j, \tag{26}$$

are satisfied for positive numbers $\lambda_j, \kappa_j, \alpha_j, \beta_j$.

Suppose that there exist a continuous $(m \times \ell)$-matrix $\Sigma(t)$ and a number $r > 0$ such that during the time from $t = 0$ to $t = r$ the phase flow $\{\theta^r_{t_0}\}$ of the system

$$\dot{x} = (A + B\Sigma(t)C)x \quad (x \in \mathbb{R}^n) \tag{27}$$

takes the manifold M_1 to a manifold lying in L_2:

$$\theta^r_0 M_1 \subset L_2. \tag{28}$$

Under these assumptions the following theorem holds.

Theorem 4.1 (The fundamental theorem) *Suppose the following inequality holds*

$$\lambda_1 \lambda_2 > \kappa_1 \kappa_2.$$

Then there exists a periodic $(m \times \ell)$-matrix $S(t)$ such that the system (23) is asymptotically stable. In this case stabilizing matrix $S(t)$ in (22) has the form

$$S(t) = \begin{cases} S_1 & \textit{for} \quad t \in [0, t_1), \\ \Sigma(t - t_1) & \textit{for} \quad t \in [t_1, t_1 + \tau), \quad S(t + T) = S(t), \\ S_2 & \textit{for} \quad t \in [t_1 + \tau, t_1 + t_2 + \tau), \end{cases} \tag{29}$$

where $T := t_1 + t_2 + \tau$ and positive numbers t_1 and t_2 are determined from conditions

$$\begin{cases} -\lambda_1 t_1 + \kappa_2 t_2 < -\tilde{T}, \\ -\lambda_2 t_2 + \kappa_1 t_1 < -\tilde{T}. \end{cases}$$

Here $\tilde{T}$ is a sufficiently large number.

Consider separately an important case of single-input single-output system (19). Let in (24)-(28)

$$S_1 = S_2 = S_0, \quad \Sigma(t) \equiv \Sigma_0, \quad S_0, \Sigma_0 \in \mathbb{R}, \tag{30}$$

$$S_0 \Sigma_0 < 0, \quad \lambda_1 = \lambda_2 = \lambda, \quad \kappa_1 = \kappa_2 = \kappa. \tag{31}$$

Suppose that all the eigenvalues λ_k of the matrix $A + B\Sigma_0 C$ have nonpositive real parts, in this case the eigenvalues with zero real parts have the prime divisors only.

Suppose there exists a sequence $\{\tau_j\} \to +\infty$ such that

$$\theta^{\tau_j} M_1 \subset L_2. \tag{32}$$

Here $\theta^t = e^{(A+\Sigma_0 BC)t}$ is the phase flow of system (27), where $\Sigma(t) \equiv \Sigma_0$. Then the following result is valid.

Theorem 4.2 *Suppose for system* (19) *the hypotheses* (30)-(32) *are satisfied. Suppose the inequality*

$$\lambda > \kappa$$

is valid. Then there exists T-periodic function with zero mean on the period such that the system (23) *is asymptotically stable. In this case the stabilizing function has the from*

$$S(t) = \begin{cases} S_0 & for \quad t \in [0, t^0), \\ \Sigma_0 & for \quad t \in [t^0, t^0 + \tau), \\ S_0 & for \quad t \in [t^0 + \tau_j, 2t^0 + \tau_j), \end{cases} \qquad S(t+T) = S(t),$$

Here $T = \tau_j(1 - \Sigma_0/S_0)$ is a period of the function $S(t)$, $t^0 = |\tau_j \Sigma_0 / 2S_0|$ and τ_j is a sufficiently large number satisfying condition (32).

We remark that there are propositions which provide effective test of the "condition of manifolds embedding" (28) [63]-[66].

Applying Theorem 4.1 to two-dimensional case of system (19) ($n = 2$) one can prove the following assertion.

Theorem 4.3 *Suppose there exist $(m \times \ell)$-matrices S_0 and Σ_0 satisfying the following hypotheses:*

1) $\det(BS_0C) \neq 0$, $Tr(BS_0C) \neq 0$*; if* $\det(BS_0C) = 0$*, then, at least one of inequalities* $\det A \neq 0$ *or* $\det(a_1, r_2) + \det(r_1, a_2) \neq 0$*, is valid. Here a_1, a_2 and r_1, r_2 are the first and the second columns of the matrices A and BS_0C, respectively.*

2) *The matrix $A + B\Sigma_0 C$ has complex-conjugate eigenvalues.*

Then there exists a periodic matrix S(t) such that the system (23) *is asymptotically stable.*

4.2 Stabilization of linear system in the scalar case

Consider the system (19) with scalar input u and scalar output y ($m = \ell = 1$).

In the sequel we shall assume that the transfer function $W(p) = C(A - pI)^{-1}B$ of system (19) is nondegenerate. This is equivalent to the fact that the pair (A, B) is controllable and the pair (A, C) is observable.

By applying Theorem 4.1 one can prove a number of assertions.

A. The case of codimension 1 of the stable manifold.

In this case the following theorems hold.

Theorem 4.4 *Suppose the systems* (24) *have a stable invariant manifold L_2 of dimension $n-1$ and an one-dimensional invariant manifold M_1, satisfying basic conditions* (25)-(28).

Suppose also that S_1, S_2 *and* Σ_0 *are numbers such that* $\Sigma_0 \neq S_j\,(j = 1, 2)$ *and the matrix* $Q = A + \Sigma_0 BC$ *has the complex-conjugate eigenvalues* $\alpha \pm i\beta$ *of multiplicity 1 and the rest of its eigenvalues* λ_k *satisfy the condition* $Re\lambda_k < \alpha$ $(k = 1, \dots, n-2)$.

Then there exists a periodic function $S(t)$ *of the type* (29) *with* $\Sigma(t) = \Sigma_0$ *such that the system* (23) *is asymptotically stable.*

Theorem 4.5 *Let the system* (24) $(j = 1, 2)$ *have a stable invariant manifold* L_2 *of dimension* $n-1$ *and an one-dimensional invariant manifold* M_1 *satisfying basic conditions* (25)-(28). *Suppose* $CB = 0$. *Then there exists a feedback* (22), *where* $S(t)$ *is a piecewise-constant periodic function of the type* (29), *such that the system* (23) *is asymptotically stable.*

Theorem 4.6 *Let in system* (19) $CB \neq 0$. *Suppose the matrix* A *has the eigenvalue* $\kappa > 0$ *and* $n-1$ *eigenvalues with the real part smaller than* $-\lambda$, *where* $\lambda > \kappa$. *Suppose that the inequality*

$$\frac{CB}{\lim\limits_{p \to \kappa} (\kappa - p) W(p)} < 1$$

is satisfied. Here $W(p)$ *is the transfer function of system* (19). *Then there exists a periodic function* $S(t)$ *of the type* (29) *such that the system* (23) *is asymptotically stable.*

Theorem 4.7 *Let* $CB \neq 0$. *Suppose that there exist numbers* $S_1 \neq S_2$ *such that:*

1) *the matrix* $A + S_1 BC$ *has the positive eigenvalue* κ_1.

2) *the matrix* $A + S_2 BC$ *has the one positive eigenvalue* κ_2 *and* $n-1$ *eigenvalues with negative real parts;*

3) *the inequality*

$$(CB)\frac{S_1 - S_2}{\kappa_2 - \kappa_1} < 1$$

holds. Suppose the condition $\lambda_1 \lambda_2 > \kappa_1 \kappa_2$ *of Fundamental Theorem is satisfied.*

Then there exists a periodic function $S(t)$ *of the type* (29) *such that the system* (23) *is asymptotically stable.*

B. The case of codimension 2 of the stable manifold

In this case the following result is valid.

Theorem 4.8 *Suppose the systems* (24) *have a* $n-2$*-dimensional stable invariant manifold* L_2 *and an one-dimensional invariant manifold* M_1 *satisfying basic conditions* (25)-(28). *Suppose that for a certain number* $\Sigma_0 \neq S_j$ $(j = 1, 2)$ *the matrix* $A + \Sigma_0 BC$ *has two complex-conjugate eigenvalues* $\alpha \pm i\beta$ *of multiplicity 1 and the rest of its eigenvalues* λ_j *satisfy the condition* $Re\lambda_j < \alpha$. *Then there exists a periodic function* $S(t)$ *of the type* (29), *where* $\Sigma(t) \equiv \Sigma_0$, $S_1, S_2, \Sigma_0 \in \mathbb{R}$, *such that the system* (23) *is asymptotically stable.*

4.3 Necessary conditions of stabilization

Above we derived some sufficient conditions of stabilizability of the system (19). Here we give necessary conditions of stabilizability of the system (19) with a scalar input u and a scalar output y.

A simple and general necessary condition of the impossibility of stabilization of system (19) is given by the following

Proposition 4.1 *If the inequality* $Tr(A + S(t)BC) \geq \alpha > 0$ *is satisfied for all* $t \in \mathbb{R}$ *and some positive number* α*, then the system* (23) *is not asymptotically stable.*

Here Tr denotes the trace of a matrix.

The statement of this proposition follows from the well-known Liouville formula.

Suppose now that the transfer function of system (19) is nondegenerate. Then it can be represented as the quotient

$$W(p) = \frac{\nu(p)}{\Delta(p)}$$

of the two polynomials

$$\nu(p) = c_n p^{n-1} + c_{n-1} p^{n-2} + \cdots + c_1, \;\; c_k \in \mathbb{R},$$
$$\Delta(p) = p^n + a_n p^{n-1} + \cdots + a_1, \;\; a_k \in \mathbb{R} \;\; (k = 1, \ldots, n),$$

with no common zeros. Here $\Delta(p)$ is the characteristic polynomial of the matrix A.

Assume that $c_n \neq 0$. In this case, without loss of generality, we set $c_n = 1$.

The following theorem gives sufficient conditions of the impossibility of stabilization of system (19).

Theorem 4.9 *Suppose for system* (19) *the following conditions are valid:*
1) *for* $n > 2$ $c_1 \leq 0, \ldots, c_{n-1} \leq 0$ *(for* $n = 2$ $c_1 \leq 0$*),*

$$\begin{aligned} 2) \quad & c_1(a_n - c_{n-1}) > a_1, \\ & c_1 + c_2(a_n - c_{n-1}) > a_2 \\ & \ldots\ldots\ldots\ldots\ldots \\ & c_{n-2} + c_{n-1}(a_n - c_{n-1}) > a_{n-1}. \end{aligned}$$

Then there does not exist a function $S(t)$ *such that the system* (23) *is asymptotically stable.*

Thus, a necessary condition of stabilization of the system (19) is the violation of at least one of hypotheses either 1) or 2) of Theorem 4.9 or the violation of inequality in the above Proposition.

4.4 Low-frequency stabilization of two-dimensional and three-dimensional systems

Now we apply the above results to the two-dimensional and three-dimensional systems.

A. Two-dimensional systems. Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$, the transfer function of which is equal to the following

$$W(p) = \frac{c_2 p + c_1}{p^2 + a_2 p + a_1}. \tag{33}$$

Here $a_1, a_2; c_1, c_2$ are real numbers.

Let $c_2 \neq 0$. Then without loss of generality we can assume that $c_2 = 1$. Suppose also that the function $W(p)$ is nondegenerate, i.e.

$$c_1^2 - a_2 c_1 + a_1 \neq 0. \tag{34}$$

Then the system with transfer function (33) can be realized in the phase space as a system of the type (19)

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -a_1 x_1 - a_2 x_2 - u. \quad y = c_1 x_1 + x_2. \end{cases} \tag{35}$$

From the Routh-Hurwitz conditions it follows that by the feedback $u = S_0 y$, $S_0 =$ const $\neq 0$ the stationary stabilization of system (35) is possible if and only if either the inequality $c_1 > 0$ or the relations $c_1 \leq 0, a_2 c_1 < a_1$, are valid.

Consider the case when the stationary stabilization is impossible: $c_1 \leq 0, \quad a_2 c_1 \geq a_1$.

Applying Theorem 4.3 or Theorem 4.6 we can obtain the following sufficient condition of nonstationary stabilization of system (35) $c_1^2 - a_2 c_1 + a_1 > 0$. If the inequality $c_1^2 - a_2 c_1 + a_1 < 0$. holds, then the hypotheses of Theorem 4.9 are satisfied. Therefore, system (35) cannot be stabilizable by any feedback $u = S(t) y$.

Thus, we have the following

Theorem 4.10 *Suppose that the transfer function $W(p)$ of system* (35) *is non-degenerate, i.e. inequality* (34) *is valid. Then a necessary and sufficient condition of stabilizability of system* (35) *is that at least one of the conditions holds:*

$$1)\, c_1 > 0 \quad \textit{or} \quad 2)\, c_1 \leq 0,\, c_1^2 - a_2 c_1 + a_1 > 0, \tag{36}$$

In this case for the stabilizing control $u = S(t)y$ the function $S(t)$ can be chosen as the piecewise-constant periodic one with sufficiently large period (a low-frequency stabilization).

Remark 4.1 Theorem 4.10 very well illustrates the fact that the introduction of a function $S(t) \neq S_0,\ S_0 =$ const, in the feedback $u = S(t)y$ (a nonstationary stabilization) extends the possibility of stationary stabilization ($S(t) \equiv S_0$).. Namely, in the space of parameters $\{(a_1, a_2; c_1)\}$ of system (35) conditions (36) select a more wide domain than the domain $\{c_1 > 0\} \cup \{c_1 < 0, a_2 c_1 < a_1\}$, defined by the Routh-Hurwitz conditions for stationary stabilization.

B. Three-Dimensional Systems

1) Suppose that the transfer function of system with a scalar input $u(t)$ and the scalar output $y(t)$ has the form

$$W(p) = \frac{1}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{37}$$

where α, β, γ are real numbers. Then such a system can be realized in the phase space $\mathbb{R}^3$ as a system of the type (19)

$$\begin{cases} \dot{x} = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u, \ y = x_1. \end{cases} \tag{38}$$

By the Routh-Hurwitz conditions the stationary stabilization $u = S_0 y$ of system (38) is possible if and only if

$$\alpha > 0 \quad \text{and} \quad \beta > 0.$$

Let $\alpha > 0$, $\beta \le 0$. In this case the stationary stabilization is impossible. Now we make use of Theorem 4.8.

By applying Theorem 4.8 to system (38) one can show that if $\alpha > 0$, $\beta \le 0$, there exists a control $u = S(t)y$, where $S(t)$ is a piecewise-constant periodic function with sufficiently large period, such that the system (38) with $u = S(t)y$ is asymptotically stable.

For system (38) with any feedback $u = S(t)y$ we have

$$\mathrm{Tr}(A + BS(t)C) = -\alpha \quad \forall t \in \mathbb{R}. \tag{39}$$

Then by Proposition from section 4.3 system (38) ($u = S(t)y$) is not asymptotically stable for $\alpha \le 0$.

Thus, we have the following

Theorem 4.11 *The system* (38) *with transfer function* (37) *is stabilized by feedback* (22) *if and only if $\alpha > 0$. In this case the function $S(t)$ for the stabilizing control can be chosen as the piecewise-constant periodic one with sufficiently large period (a low-frequency stabilization).*

2) Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$ and the transfer function of the form

$$W(p) = \frac{p}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{40}$$

where α, β and γ are real numbers.

Let $\gamma \neq 0$. This condition is a condition of nondegenaracy of the function (40). Then this system can be realized in the phase space $\mathbb{R}^3$ as a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u, \ y = x_2. \end{cases} \tag{41}$$

By the Routh-Hurwitz conditions the stationary stabilization of system (41) is possible if and only if $\alpha > 0, \gamma > 0$. Consider the case $\alpha > 0, \gamma < 0$. Then the stationary stabilization is impossible. We apply Theorem 4.5 with $S_1 = S_2$; $\lambda_1 = \lambda_2 = \lambda$, $\kappa_1 = \kappa_2 = \kappa$. Then we obtain that the conditions $\alpha > 0, \gamma < 0$, are sufficient for nonstationary stabilization of system (41).

Since for system (41) with any feedback $u = S(t)y$ the equality (39) holds, asymptotic stability of the system (41) is impossible for $\alpha \le 0$ by Proposition from section 4.3.

Thus, we have the following

Theorem 4.12 *Let $\alpha \neq 0, \gamma \neq 0$. Then for system* (41) *to be stabilized by feedback* (22) *it is necessary and sufficient that $\alpha > 0$.*

3) Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$ and the transfer function of the form

$$W(p) = \frac{p^2}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{42}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Suppose that the function (42) is nongenerate, i.e. $\gamma \neq 0$. Then this system can be realized in the phase space $\mathbb{R}^3$ as a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u, \ y = x_3. \end{cases} \tag{43}$$

The stationary stabilization $u = S_0 y$ of system (43) is possible if and only if $\beta > 0, \gamma > 0$. In the case $\beta < 0, \gamma < 0$ by Theorem 4.9 the stabilization (a stationary or nonstationary) is impossible.

Consider the case $\beta > 0, \gamma < 0$, when the stationary stabilization is impossible. By applying the fundamental theorem (Theorem 4.1) from section 4.1 and as above, letting $S_1 = S_2; \ \lambda_1 = \lambda_2 = \lambda, \ \kappa_1 = \kappa_2 = \kappa$, one can prove the following assertion.

Theorem 4.13 *Let $\beta \neq 0, \gamma < 0$. Then for system* (43) *to be stabilized by feedback* (22) *it is necessary and sufficient that $\beta > 0$.*

Remark 4.2 As Theorem 4.10 Theorems 4.11–4.13 very well illustrate advantages of nonstationary stabilization in comparison with the stationary one.

4.5 Nonstationary high-frequency stabilization

In the previous section for some important cases the solution of the Brockett problem of nonstationary linear stabilization of system (19) in the class of piecewise-constant periodic stabilizing functions $S(t)$ is given.

In the works [67]-[70] another approach is proposed for solving the Brockett problem. This approach differs from the technique considered in the previous section and is based on the averaging method and uses some ideas and methods from vibrational control theory [71]-[74].

Also in this approach some research methods are used developed for the investigation well-known phenomenon of stabilization of the upper pendulum equilibrium position when the suspension point performs sufficiently fast oscillations in the vertical direction.

In [67]-[70] the Brockett problem is solved in the class of continuous periodic functions with a sufficiently small period (a high-frequency stabilization). There there are considered the functions of the form $S(t) = \alpha + \beta\omega^k \cos(\omega t)$, where $k \in \mathbb{N}$ and ω is a sufficiently large parameter.

We present corresponding results. Consider two cases:

1) $CB \neq 0$ and 2) $CB = CAB = 0$.

A. Stabilization in the case $CB \neq 0$.

In this case the following theorem holds.

Theorem 4.14 ([70]). *Let in system* (19) $CB \neq 0$. *Suppose that there exist real numbers α and $\kappa \geq 0$ such that the matrix*

$$A + \kappa(CB)BCA + (\alpha - \kappa CAB)BC \tag{44}$$

is stable. Then there exists a periodic function

$$S(t) = \alpha + \beta\omega \cos\omega t, \tag{45}$$

where ω is a sufficiently large number and $\beta \in \mathbb{R}$ satisfies the relation

$$\frac{\left(\frac{1}{2\pi}\int\limits_0^{2\pi}\exp(\beta CB\sin t)dt\right)^2 - 1}{(CB)^2} = \kappa, \tag{46}$$

such that the closed-loop system (23) *is exponentially stable uniformly with respect to ω for all sufficiently large ω.*

B. Stabilization in the case $CB = CAB = 0$.

In this case the following result is valid.

Theorem 4.15 ([70]). *Let in system* (19) *$CB = CAB = 0$. Suppose that there exist real numbers α and $\kappa \geq 0$ such that the matrix*

$$A - 3\kappa(CA^2B)BCA + (\alpha + \kappa CA^3B)BC \tag{47}$$

is stable. Then there exists a periodic function

$$S(t) = \alpha + \gamma\omega^2\cos\omega t, \tag{48}$$

where ω is a sufficiently large number and $\gamma \in \mathbb{R}$ satisfies the relation

$$\gamma^2 = 2\kappa. \tag{49}$$

such that the closed-loop system (23) *is exponentially stable uniformly with respect to ω for sufficiently large ω.*

Remark 4.3 In the work [70] the case when in system (19) $CB = CAB = \cdots = CA^{2k-1}B = 0$ is also considered. In the case $k > 1$ ($k \in \mathbb{N}$) the corresponding stabilization theorem is formulated similarly to Theorem 4.15: instead of the stability property of matrix (47) the stability property of the matrix

$$A + (-1)^k(2k+1)\kappa(CA^{2k}B)BCA + [\alpha + (-1)^{k+1}(2k+1)\kappa(CA^{2k+1}B)BC]$$

is required. In this case the stabilizing function has the form $S(t) = \alpha + \beta\omega^{k+1}\cos\omega t$.

4.6 High-frequency stabilization of two-dimensional and three-dimensional systems

Here we consider examples of application of Theorems 4.14 and 4.15 to two-dimensional and three-dimensional systems.

A. Two-dimensional systems. Consider the system (35). Suppose that inequality (34) holds. For system (35) the condition $CB \neq 0$ is valid. Therefore, we can apply Theorem 4.14. In this case the matrix (44) takes the form

$$\begin{pmatrix} 0 & 1 \\ -a_1 - \alpha c_1 - \kappa(c_1^2 - a_2c_1 + a_1) & -a_2 - \alpha \end{pmatrix}. \tag{50}$$

The matrix (50) is stable if and only if there exist the values of parameters $\alpha \in \mathbb{R}$ and $\kappa \in [0, +\infty)$ such that the inequalities

$$a_2 + \alpha > 0, \quad a_1 + \alpha c_1 + \kappa(c_1^2 - a_2c_1 + a_1) > 0 \tag{51}$$

hold. Relations (51) are satisfied if at least one of the inequalities

$$c_1 > 0 \quad \text{or} \quad c_1^2 - a_2c_1 + a_1 > 0,\ c_1 \le 0 \tag{52}$$

is satisfied.

Thus, by Theorem 4.14 the condition (52) is sufficient for the existence of a control $u = S(t)y$, which stabilizes the system (35). Here $S(t)$ is a function of the type (45). As α, one may take an arbitrary number satisfying inequalities (51) for some $\kappa \ge 0$. As β, one should take a number satisfying the equation

$$\int\limits_0^{2\pi} e^{-\beta \sin t} dt = 2\pi\sqrt{1+\kappa}. \tag{53}$$

It is easy to show that the equation (53) has a solution with respect to β.

By Theorem 4.9 if the inequality $c_1^2 - a_2c_1 + a_1 < 0$ ($c_1 \le 0$) is satisfied, then system (35) cannot be stabilized by any feedback of the type $u = S(t)y$.

Thus, we have the following

Theorem 4.16 ([67, 70]) *Suppose the inequality* (34) *holds. Then*

1) *if at least one of inequalities* (52) *is satisfied, then there exists a feedback*

$$u = S(t)y, \quad S(t) = \alpha + \beta\omega\cos\omega t, \tag{54}$$

where α and β are determined from (51) *and* (53)*, respectively, such that the closed-loop system* (35),(54) *is exponentially stable uniformly with respect to ω for sufficiently large ω;*

2) *if condition* (52) *is not satisfied, then for any choice of function $S(t)$ the system* (35)*, where $u = S(t)y$, is not exponentially stable.*

Thus, condition (52) is necessary and sufficient one for the existence of feedback (54) such that it stabilizes uniformly exponentially system (35) in the class of continuous and periodic functions $S(t)$.

The same condition (52), as was shown in section 4.4, is also necessary and sufficient one for stabilization of system (35) in the other class of the piecewise-constant periodic functions $S(t)$.

B. Three-dimensional systems. Consider a system (38), where $\alpha := a_3, \beta := a_2, \gamma := a_1 (a_1, a_2, a_3$ are real numbers).

The stationary stabilization ($S(t) \equiv \text{const}$) is possible if and only if $a_2 > 0, a_3 > 0$.

For system (38) the relations $CB = CAB = 0$ are valid. We apply Theorem 4.15. Here the matrix (47) takes the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 + \alpha + \kappa a_3 & -a_2 - 3\kappa & -a_3 \end{pmatrix}. \tag{55}$$

The matrix (55) is stable if and only if there exist values of the parameters $\alpha \in \mathbb{R}$ and $\kappa \in [0, +\infty)$ such that the following inequalities

$$\left.\begin{aligned} &a_3 > 0,\ a_1 - \alpha - \kappa a_3 > 0. \\ &a_3(a_2 + 3\kappa) - a_1 + \alpha + \kappa a_3 > 0 \end{aligned}\right\} \tag{56}$$

hold. The relations (56) are equivalent to the inequality $a_3 > 0$.

Thus, by Theorem 4.15 the condition $a_3 > 0$ is sufficient for the existence of the periodic function $S(t)$ of the type (48) such that the feedback $u = S(t)y$ exponentially stabilizes the system (38). Here one may take as α an arbitrary number satisfying relation (56) for some $\kappa \geq 0$. As γ a number satisfying the equation (49) should be taken.

The relation (39) holds. Therefore, if $a_3 \leq 0$ by Proposition from section 4.3 the system (38) is not asymptotically (and exponentially) stable for any feedback $u = S(t)y$.

Thus, the following result is valid.

Theorem 4.17 ([69, 70]) 1) *If in system* (38) $a_3 > 0$*, then there exists a feedback of the type* (48) *such that the system* (38),(48) *is uniformly with respect to* ω *exponentially stable for sufficiently large values of* ω.

2) *If* $a_3 \leq 0$*, then for no function* $S(t)$ *the exponential stabilization of system* (38) *is possible by means of the feedback* $u = S(t)y$.

Thus, the condition $a_3 > 0$ is necessary and sufficient one for the existence of the feedback of the type (48), which stabilizes uniformly exponentially the system (38). As was shown above in section 4.4 (Theorem 4.11) the same condition $a_3 > 0$ is also necessary and sufficient one for stabilization of system (38) in the class of the piecewise-constant periodic functions $S(t)$ with sufficiently large period (a low-frequency stabilization).

5 Discrete-time systems. Problem statement

In this part the discrete-time version of Brockett stabilization problem and pole assignment in discrete-time systems by periodic output feedback will be considered.

Consider a linear time-invariant discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k \quad (k = 0, 1, 2, \ldots), \tag{57}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, $y_k \in \mathbb{R}^\ell$ is the output vector, A, B and C are real constant matrices of dimension $n \times n, n \times m$ and $\ell \times n$, respectively.

It is well known that if $C = I_n$ (I_n is the identity matrix) and the pair (A, B) is controllable then the poles of the system (57) can be assigned arbitrarily by time-invariant static state feedback [2, 3]. Hence the system (57) under the mentioned assumptions can be stabilizable. When only the output but not the state is available the problem of stabilizability and pole assignability by time-invariant static output feedback has also received much attention.

Necessary and/or sufficient conditions have been obtained under which stabilizability and pole assignability by time-invariant static output feedback are guaranteed. The basic results are available in the literature (see, for example, [18],[19] and surveys [23],[24]).

The question arises as to what extent the stabilization or pole assignment problem can be resolved by introducing time-varying static output feedback. The problem can be formulated as follows.

Problem 5.1 The Stabilization Problem.

Given a triple real constant matrices A, B and C, find a sequence of real $(m \times \ell)$-matrices $\{S_k\}$ $(k = 0, 1, 2, \ldots)$ such that the system (57) with the feedback

$$u_k = S_k y_k \quad (k = 0, 1, 2, \ldots), \tag{58}$$

i.e. the closed-loop system

$$x_{k+1} = (A + BS_kC)x_k \quad (k = 0, 1, 2, \ldots) \tag{59}$$

is asymptotically stable.

The Problem 5.1 is the discrete analog of the Brockett problem of stabilization of a linear continuous-time system by means of time-varying static output feedback.

It is important to notice that the discrete-time and continuous-time versions of Brockett problem are essentially different. This becomes clear, for example, from the fact that several difficulties and obstructions, which arise in solving of the Brockett problem in the continuous-time case, are lacking in the discrete-time case. For the statement the the next problem we assume that the time-dependent feedback (58) is periodic. i.e.

$$S_{k+p} = S_k \quad \forall k \in \{0, 1, 2, \ldots\}, \tag{60}$$

where p is a positive integer.

Then the system (59) is a periodic linear system of period p. This system can be considered as a time-invariant system with time interval equal to the period p:

$$\xi_{r+1} = M_s \xi_r \quad (r = 0, 1, 2, \ldots), \tag{61}$$

where

$$M_s = (A + BS_{p-1}C)(A + BS_{p-2}C) \ldots (A + BS_0C). \tag{62}$$

The eigenvalues of the composite matrix M_s determine the dynamics of the system (61), which in turn determines the dynamics of the periodic system (59),(60). These eigenvalues called multiplicators will be referred to as the poles of the periodic system (59). The matrix M_s is called the monodromy matrix for system (59),(60).

Now the pole assignment problem for the system (57) can be formulated in the following way:

Problem 5.2 The pole assignment problem.

Given a triple real constant matrices A, B and C, find real $(m \times \ell)$-matrices $S_0, S_1, \ldots, S_{p-1}$ such that the eigenvalues of the closed-loop system matrix M_s from (62) are the roots of a polynomial

$$f(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \tag{63}$$

with real coefficients $\alpha_i \, (i = 0, 1, \ldots, n-1)$.

Clearly, Problem 5.2 is more general than Problem 5.1. Problem 5.1 has been studied in [75], and Problem 5.2 in [60],[61]. In these works two different approaches were offered in solving these problems. Here we present the corresponding results. We begin with Problem 5.2.

5.1 Pole assignability

Consider the system (57) with scalar input and scalar output ($m = l = 1$). Then B is a column matrix, C is a row matrix, the feedback gains S_k are numbers.

A. Two-Dimensional Case

In this case the following theorem yields a complete solution of Problem 5.2 in the sense of giving conditions for the realizability of pole assignment by static periodic output feedback.

Theorem 5.1 (On pole assignment: $n = 2$ [60]). *Let in system* (57) $n = 2$. *Suppose that the pair* (A, B) *is controllable and the pair* (A, C) *is observable. Then for the problem of pole assignment in system* (57) *by means of periodic feedback* (58),(60) *with period 3 to be solvable it is necessary and sufficient that* 1) $CA^{-1}B \neq 0$ *and* 2) $|CB| + |TrA| \neq 0$.

Remarks to Theorem 5.1

1. In Theorem 5.1 it is assumed that the system (57) is controllable and observable. This assumption is necessary for pole assignability. This follows from the well-known fact that uncontrollable and unobservable modes cannot be moved by static output feedback: neither by time-invariant nor by time-varying feedback [76].

2. In Theorem 5.1 it is assumed implicitly that the system matrix A is non-singular. This assumption entails no restriction as soon as the system (57) is controllable and observable. The nonsingularity of matrix A in controllable and observable system can be realized by a preliminary output feedback. Really, this follows from the well-known formula

$$\det[zI - (A + SBC)] = \Delta(z) + S\nu(z),$$

where the characteristic polynomial $\Delta(z) = \det(zI - A)$ of the matrix A and the polynomial $\nu(z)$ of degree not greater than $n - 1$ have no common roots.

3. From the result stated in [77] it follows that the pole assignability of system (57) is not possible in general by means of a periodic static output feedback of period 2. Therefore, at least periodic static output feedback of period 3, as considered in Theorem 5.1, is necessary to realize pole assignment.

B. Multidimensional Case

Let $W(z)$ denote the transfer function of system (57). Consider its representation in the form of rational function

$$W(z) = C(Iz - A)^{-1}B = \frac{q_{n-1}z^{n-1} + \cdots + q_1 z + q_0}{z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0}, \tag{64}$$

where $p, q \in \mathbb{R}$ $(i = 0, 1, \ldots, n - 1)$. Here in the denominator of (64) we have the characteristic polynomial of the matrix A.

The following theorem gives sufficient conditions under which the poles of system (57) of arbitrary order can be assigned by means of periodic output feedback (58),(60).

Theorem 5.2 (On pole assignment: $n > 2$ [61]) *Suppose that the pair* (A, B) *is controllable and the pair* (A, C) *is observable. Suppose that the coefficients* q_i, p_i $(i = 0, 1, \ldots, n - 1)$ *of polynomials in the numerator and denominator of fraction* (64) *are non-zero and all quotients* p_i/q_i $(i = 0, 1, \ldots, n - 1)$ *are mutually different. Let* $\alpha_0 \neq 0$ *in* (63). *Then for the problem of pole assignment in system* (57) *by means of periodic feedback* (58),(60) *with period* $p = n + 1$ *to be solvable it is sufficient that*

$$rank\,[B, A\Pi_{s^0}B, \ldots, (A\Pi_{s^0})^{n-1}B] = n,$$

where

$$\Pi_{s^0} = (A + S^0_{n-1}BC)\ldots(A + S^0_0 BC),\;\; s^0 := (S^0_0, S^0_1, \ldots, S^0_{n-1}) = \left(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \cdots, \frac{p_{n-1}}{q_{n-1}}\right).$$

Remarks to Theorem 5.2

1. As in Theorem 5.1 (see Remark 1 to it) the conditions that (A, B) is controllable and (A, C) is observable are necessary for pole assignability. Therefore, without loss of

generality, the system matrix A is assumed to be nonsingular. By analogy with this we may also assume that the coefficients $p_i\,(i = 0, 1, \ldots, n-1)$ of characteristic polynomial of the matrix A are non-zero.

2. The assumption that the coefficients $q_i\,(i = 0, 1, \ldots, n-1)$ of the numerator in (64) are all non-zero is not necessary in general. This assumption is only a consequence of approach offered in [61]. The condition $q_0 \neq 0$ is equivalent to condition $CA^{-1}B \neq 0$, since $q_0 = -(CA^{-1}B)p_0$. This is necessary. Really, otherwise the determinant of the monodromy matrix M_s from (62) would be independent of the numbers $S_0, S_1, \ldots, S_{p-1}$ for any values of p, since

$$\det M_s = (\det A)^p \cdot (1 + S_0 CA^{-1}B) \ldots (1 + S_{p-1} CA^{-1}B).$$

On the other hand the condition $q_{n-1} \neq 0$ or equivalent (since $q_{n-1} = CB$) to it condition $CB \neq 0$ is not necessary in general. Indeed, in two-dimensional case (see Theorem 5.1) the zero value of CB may be allowed, but then the trace $\operatorname{Tr} A$ of the matrix A must be different from zero.

3. The condition that all quotients $p_i/q_i\,(i = 0, 1, \ldots, n-1)$ are mutually different is not necessary in general. It is a consequence of approach offered in [61]. For the second-order systems this condition is not necessary [60].

4. The condition that the period of the output feedback gain S_k is equal to $n+1$ is sufficient only. As remarked above for second order systems periodic feedback of period $p = 2$ cannot solve the pole assignment problem.

5. The condition $\alpha_0 \neq 0$ is equivalent to the condition that the poles of closed-loop system (59) must differ from the origin. This condition is not necessary. In [61] an example of third order system, where $\alpha_0 = 0$, is given but nevertheless the pole assignment is possible.

6. The result of Theorem 5.2 can be generalized for multi-input multi-output systems (see [61]).

5.2 Examples

A. A second order system ([60]).

Consider the system

$$x_{k+1} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k, \quad y_k = (-1\ 1)x_k. \tag{65}$$

The system (65) is controllable and observable. Here $CB = 1, CA^{-1}B = -1/2$. Therefore by Theorem 5.1 the pole assignment for system (65) is solvable by means of periodic feedback (58),(60) with period $p = 3$.

This result can be obtained also by Theorem 5.2. Really, the transfer function of system (65) is

$$W(z) = \frac{z-1}{z^2 - z - 2}.$$

We have $p_0 = -2, p_1 = -1, q_0 = -1, q_1 = 1, S_0^0 = 2, S_1^0 = -1$. Also,

$$\Pi_{s^0} = (A + S_0^0 BC)(A + S_1^0 BC) = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{rank}\,(B, A\Pi_{s^0}) = \text{rank} \begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} = 2.$$

Therefore, all conditions of Theorem 5.2 are satisfied.

It should be noted that the open-loop system (65) $x_{k+1} = Ax_k \, (u_k := 0)$ is unstable and cannot be stabilized by time-invariant output feedback $u_k = Sy_k \, (S = \text{const})$.

B. A third order system ([61]).

Consider the discrete-time third-order system of the type (57) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (q_0 \; q_1 \; q_1).$$

The transfer function is

$$W(z) = \frac{q_2 z^2 + q_1 z + q_0}{z^3 + p_2 z^2 + p_1 z + p_0}.$$

For the values of parameters $p_0 = 1.875, p_1 = 5.75, p_2 = 4.5, q_0 = 2, q_1 = 3, q_2 = 1$, the transfer function $W(z)$ has three poles $z_1 = -0.5, z_2 = -1.5, z_3 = -2.5$ and two real zeros $z_1^0 = -1, z_2^0 = -2$. By the root locus method it can be established that this system cannot be stabilized by constant output feedback.

Let the characteristic polynomial to be realized be $f(z) = z^3$, i.e. the closed-loop system is required to have all poles at the origin by introducing the periodic output feedback of period $p = 4$. A numerical analysis [61] yields the following result for feedback gains $S_0 = 0.9375, \; S_1 = 2.528322, \; S_2 = -8.928145, \; S_3 = 10$.

Note that the condition $\alpha_0 \neq 0$ of Theorem 5.2 is not satisfied. But nevertheless the pole assignment problem for considered system is solvable for the polynomial $f(z) = z^3$. Therefore, as is remarked above (Remark 5 to Theorem 5.2) the condition $\alpha_0 \neq 0$ in Theorem 5.2 is indeed not necessary.

Remark 5.1 Above for the pole assignment in system (57) the periodic memoryless output feedback has been used, i.e. the value of the input at a particular time $t = k$ depends on the output value at the same moment of time k. Contrary to this approach in the works [78]-[80] a memory in the periodic output feedback law is introduced. That is, value of the input at a moment $t = k$ depends on an output value at a time prior to this moment, namely at the beginning of the period. We adduce a result on pole assignment for single-input single-output system (57) by such kind (with memory) of periodic output feedback. In [79] the following theorem is established:

Let the pair (A, B) be controllable. Then the pole assignment problem has a solution if and only if the pair (A^n, C) is observable.

5.3 Stabilizability

We now turn to Problem 5.1 stated above.

5.3.1 Low-frequency stabilization of multi-input multi-output systems

Basic hypotheses. Suppose that there exist real constant $(m \times \ell)$-matrices $S_{(j)} (j = 1, 2)$ such that the systems

$$x_{k+1} = (A + BS_{(j)}C)x_k, \;\; x_k \in R^n \;\; (j = 1, 2)(k = 0, 1, 2, \ldots) \tag{66}$$

have stable invariant linear manifolds L_j and invariant linear manifolds M_j. Assume that $\dim M_j + \dim L_j = n$ and $M_j \cap L_j = \{0\}$. Suppose that for solutions $x_k^{(j)}(x_0^{(j)} = x_0)$ of systems (66) the following inequalities

$$\|x_k^{(j)}\| \leq \alpha_j \|x_0\| e^{-\lambda_j k} \quad \forall x_0 \in L_j, \tag{67}$$

$$\|x_k^{(j)}\| \leq \beta_j \|x_0\| e^{\mu_j k} \quad \forall x_0 \in M_j, \tag{68}$$

are satisfied for positive numbers $\lambda_j, \mu_j, \alpha_j, \beta_j$.

Assume that there exist a sequence of matrices $\{\Sigma_k\}_{k=0}^{\infty}$ and an integer $r \geq 1$ such that for the system

$$x_{k+1} = (A + B\Sigma_k C)x_k$$

the inclusion $\theta_0^r M_1 \subset L_2$ holds, where $\theta_0^r = \prod_{j=0}^{r-1}(A + B\Sigma_j C)$.

Under these assumptions we have the following

Theorem 5.3 (Fundamental Theorem on Stabilization [75]). *Suppose the inequality $\lambda_1\lambda_2 > \mu_1\mu_2$ holds. Then there exists a K-periodic matrix sequence $\{S_k\}(S_{k+K} = S_k, k = 0, 1, 2, \ldots; K \in \mathbb{N})$ such that the system* (59) *is asymptotically stable.*

In this case the stabilizing feedback gain matrix S_k has the form

$$S_k = \begin{cases} S_1 \quad \mathit{for} \quad k \in [0, k_1), \\ \Sigma_{k-k_1} \quad \mathit{for} \quad k \in [k_1, k_1 + r), \\ S_2 \quad \mathit{for} \quad k \in [k_1 + r, k_1 + k_2 + r), \end{cases}$$

where $K := k_1 + k_2 + r$, and positive integers k_1 and k_2 are determined from conditions

$$-\lambda_1 k_1 + \mu_2 k_2 < -T, \quad -\lambda_2 k_2 + \mu_1 k_1 < -T.$$

Here T is a sufficiently large positive number. (The notation $k \in [\alpha, \beta)$ means that k takes only integer values from the interval $[\alpha, \beta)$.)

5.3.2 Stabilization of single-input single-output systems

Consider the system (57) with scalar input u_k and scalar output y_k.

Theorem 5.3 implies the following assertion.

Theorem 5.4 (On Stabilization: $m = l = 1$ [75]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable. Let*

$$M_1 = M_2, \quad \dim M_1 = 1, \quad \mu_1 = \mu_2 = \mu, \qquad L_1 = L_2, \quad \dim L_1 = n - 1, \quad \lambda_1 = \lambda_2 = \lambda,$$

where $L_j, M_j (j = 1, 2)$ are linear manifolds introduced for systems (66)*, and λ_j, μ_j are numbers from* (67)*,* (68)*. Then if the inequality $\lambda > \mu$ is satisfied, there exists a K-periodic number sequence $\{S_k\}_{k=0}^{\infty}$ such that the system* (59) *is asymptotically stable.*

Using Theorem 5.4 one can prove the following

Theorem 5.5 (On Stabilization: $m = l = 1$ [75]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable. Suppose that there exists a number S_0 such that the matrix $A + S_0BC$ has $n-1$ eigenvalues $\rho_j (j = 1, \ldots, n-1)$ located inside the unit circle and for the eigenvalue ρ_n the inequality*

$$\max_j |\rho_n \cdot \rho_j| < 1 \tag{69}$$

holds. Then there exists a K-periodic number sequence $\{S_k\}_{k=0}^{\infty}$ such that the system (59) *is asymptotically stable.*

Remark 5.2 It is well known that for time-invariant system $x_{k+1} = Dx_k$ to be asymptotically stable it is necessary and sufficient that all eigenvalues of the matrix D should be located inside the unit circle $|z| < 1$. The matrix $A + S_0BC$ is closed-loop system obtained after introducing in system (57) a time-invariant output feedback $u_k = S_0y_k(S_0 \in \mathbb{R})$. As is seen the condition (69) of Theorem 5.5 relaxes the requirement of locating all eigenvalues of the matrix $A+S_0BC$ inside the unit circle. Hence Theorem 5.5 extends the possibility of stationary stabilization (by time-invariant output feedback).

5.3.3 Stabilization of linear second order systems

Consider a linear single-input single-output system with the transfer function

$$W(z) = \frac{c_2z + c_1}{z^2 + a_2z + a_1}, \tag{70}$$

Here $a_1, a_2.c_1, c_2$ are real numbers.

Suppose the function $W(z)$ is nondegenerate, i.e.

$$c_1^2 - a_2c_1c_2 + a_1c_2^2 \neq 0. \tag{71}$$

A state-space realization of the system considered is a system of the type (57) with

$$A = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (c_1 \;\; c_2). \tag{72}$$

Relation (71) is a necessary and sufficient condition for controllability of the pair (A, B) and observability of the pair (A, C).

Apply Theorem 5.5. The condition (69) is equivalent to the inequality $|a_1 - S_0c_1| < 1$ which have to be satisfied for some number S_0. This yields the conditions

$$c_1 \neq 0 \quad \text{or} \quad |a_1| < 1, \tag{73}$$

which by Theorem 5.5 are sufficient conditions for stabilizability of the system (57), (72).

One can show that conditions (73) are also necessary for stabilizability of the system considered.

Thus, since the similarity transformation $(A, B, C) \to (T^{-1}AT, T^{-1}B, CT)$ does not change the transfer function $W(z)$ and $\det A$, we have the following

Theorem 5.6 (On Stabilization: $n = 2, m = l = 1$ [75]). *Suppose that inequality* (71) *is satisfied. Then for the system* (57) *with transfer function* (70) *to be stabilizable it is necessary and sufficient that at least one of conditions*

$$W(0) \neq 0 \quad \text{or} \quad |\det A| < 1 \tag{74}$$

is valid.

We note that the conditions 1) and 2) of Theorem 5.1 are also sufficient but, in general, not necessary conditions for stabilization of system (57) in the two-dimensional case. Since $W(0) = CA^{-1}B$, it is clear that for the stabilization of system (57) conditions (74) are milder than the conditions 1) and 2) of Theorems 5.1.

Remark 5.3 Comparing the conditions (73) of nonstationary stabilization of system defined by matrices (72) for special case $c_2 := 0 (c_1 \neq 0)$ with necessary and sufficient condition $|a_2| < 2$ of stationary stabilization we see the additional possibilities opened up by introducing time-variance in the feedback gain.

6 Conclusion

It is well known that although some interesting results are obtained for arbitrary pole assignment in linear time-invariant systems by means of time-invariant static output feedback, the possibility of this approach is limited. Another approach to the pole assignment stabilization problems is to consider the potential of time-varying static output feedback. This approach was developed for stabilization of continuous-time systems by Brockett [62]. For pole assignment in discrete-time systems this approach was considered by Aeyels and Willems [60, 61].

It is shown that the stabilization by means of periodic output feedback is possible under weak conditions. Necessary and sufficient conditions for nonstationary low- and high-frequency stabilization of two- and three-dimensional systems are derived. It turns out that time-varying feedback control strategy can achieve results that cannot be obtained by time-invariant feedback.

Analogous problems are considered for pole assignment and stabilization of time-invariant discrete-time control systems.

It is shown that under mild conditions stabilization of time-invariant control systems is possible by means of piecewise-constant periodic with a sufficiently large period output feedback (low-frequency stabilization). For second order systems necessary and sufficient conditions of stabilizability are obtained. Also, it is shown that introducing time-variance in the feedback gain opens up additional possibilities of stabilization of time-invariant discrete-time control systems.

Further, the results of works [60, 61] on pole assignment in discrete-time systems by time-varying static output feedback are presented.

Finally, we remark that the problems of stabilization of linear controllable systems are the high-capacity impetus for the development of new mathematical methods, which are presented in the present paper. Here an attempt is made to represent a constantly increasing number of publications, concerning this subject. In these publications not only the classical problems of stabilization are solved but the new notions are introduced and the new problems, arising in different applications, are considered. For the solution of these problems the methods, suggested in the present paper, can be useful.

Some of the methods described here are useful for investigations of nonlinear systems [81].

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A Hierarchical Genetic Algorithm Coding for Constructing and Learning an Optimal Neural Network

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Abstract: Neural Networks (NN) proved to be a powerful problem solving mechanism with great ability to learn. The success and speed of training is based on the initial parameter settings such as architecture, initial weights, learning rates and others. The most used method of training Neuron Networks is the back propagation of the gradient. Although this method provides a global optimal solution in a reasonable time, it can converge towards local minimum, in addition to large number of parameters that should be fixed previously. Within this framework of study, we propose a new coding for a hierarchical genetic algorithm for the determination of the structure and the training of the Neuron Networks. These algorithms are known for structures' and parameters' optimization. We will prove that Hierarchical genetic algorithm can improve the result of backpropagation of gradient.

Keywords: *hierarchical genetic algorithms, neural networks, backpropagation algorithm, learning, multilayer perceptron, optimization.*

Mathematics Subject Classification (2000): 93A13, 92B20.

1 Introduction

The solutions to the problem of learning neural networks by back propagation of gradient are characterized by their incapacity to escape from local optima. The evolutionary algorithms bring a great number of solutions in certain fields: drive networks of variable architecture, automatic generation of Boolean Neurons Networks for the resolution of a certain class of problems of optimization [15]. But the research effort is especially related to the generation and the training of discrete networks.

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In this paper, we propose a solution of research for the learning and the determination of Neural Network structure. This solution is based on the hierarchical genetic algorithm which can escape local optima. The evolutionary algorithms are based on the study of the process of natural evolution. The important principles of this type of algorithms are the following:

- The algorithm works on a population of individuals. Each individual corresponds to a point of research on a space of solutions.
- The population is initialized by chance. It evolves/moves thanks to operations such as the change of an individual or the recombination of individuals.
- The adaptation of an individual to his environment is measured thanks to a function called Cost which associates a positive reality with each individual.

Genetic Algorithms (GA) form a subset of the evolutionary algorithms. An individual is entirely determined by a genotype (chromosome), a phenotype and a reality [12]. By analogy with the process of natural evolution, a chromosome is often a chain of bits or integer. The operations of change and recombination are carried out by specific operators. Empirical studies showed that GA converged towards a total optimum for a big class of problems of optimization defined for a discrete unit [9]. Theoretical studies demonstrated that under certain hypothesis of regularity for the force function, these algorithms converge asymptotically into total optimal solutions [9].

We will present in this work a Hierarchical Genetic Algorithm (HGA), which is an alternative to the Genetic Algorithms. HGA has as role to optimize parameters and the structure (number of neurons in hidden layer, input and output weights), and at the same time, to train the network for modelling a non-linear process. Algorithms will do a research in a very varied environment, in order to explore all the possible solutions and to avoid the local minima or to be able to leave them.

To summarize, our work consists in minimizing the function cost (the quadratic error) into two steps: an algorithm of back propagation of gradient starting the minimization, then a HGA continues the work by carrying out research for another architecture in order to leave the local minima and to find the global one.

Two examples will be employed to proof the improvement of the HGA solution compared by methods of gradient.

2 Learning Process

Once the architecture of a NN is selected, it is necessary to do a learning to determine the values of the weights making the NN output to be as close as possible to the laid down objective. This learning is carried out thanks to the minimization of a function, called function cost, calculated from the examples of the training and the output of the NN; this function determines the objective to reach.

2.1 Principle of the minimization algorithms

The principle of these methods is to use an initial point, to find a direction of descent of the cost in the space of the parameters W, then to move a step in this direction. Once a new point is reached, we reiterate the procedure until satisfaction of a criterion of stop. Thus, in the kth iteration, we calculate $\omega_k = \omega_{k-1} + \alpha_{k-1}.d_{k-1}$, where α_k is the step of the descent and d_k is the direction of descent. Various algorithms are characterized by the choice of these two parameters.

2.2 Problem of the local minima

The minimum found by the back propagation algorithms is local minima. The minimum found depends on the starting point of the research i.e. of the initialization of the weights. In practice, it is necessary to carry out several minimizations with different initializations, to find several minima and to keep the "best". It is nevertheless impossible and generally useless to make sure that the selected minimum is the global one.

3 Presentation of the Hierarchical Genetic Algorithms (HGA)

3.1 Principle

Contrary to the conventional genetic algorithms, the Hierarchical Genetic Algorithms (HGA) use variable chromosomes dimensions. These chromosomes have the specific name of Hierarchical Chromosomes (HC). They are used mainly for the joint optimization of structure and parameters. The principle of HC is the following: the activation of a parametric gene is guided by the value of a gene of control of the first level, which is activated by a gene of control of the second level, etc [7]. Figure 1 illustrates this mechanism.

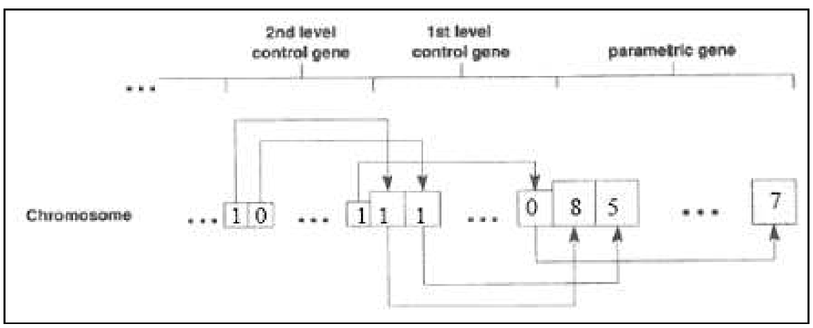

Figure 1: Structure of a Hierarchical Chromosome.

The HGA have a flexible hierarchical structure. Indeed, several hierarchical levels can be used if needed. Therefore it is a question of an ideal approach to model topologies or structures at the same time as the corresponding parameters. The HGA will seek a solution considering all the possible lengths (different structures) and all the values of parameters to meet the criteria of the objective function.

3.2 Genetic operations on the Hierarchical Chromosomes

Since the structure of the Hierarchical Chromosomes is fixed, the methods of crossing and mutation can be used independently for each level or on the entire chromosome if it is homogeneous. The genetic operations which affect genes of the higher levels can affect the number of active genes of the lower levels what makes it possible to jointly optimize the parameters and the topology of the system to be optimized [14].

3.3 Multi-criterion approach

Let us define the objective functions F associated with each chromosome X of a HGA: $F = [f_1 f_2 ... f_i]'$. The objective functions F are used by the HGA for the optimization of the parameters of the system. The task of a HGA is the joint optimization of the parameters and the topology of a system, so an additional objective function f_{i+1} must be considered by the HGA for the optimization of the topology of the system (Figure 2). Based on the specific structure of the chromosome of the HGA, information concerning topology to be optimized is directly acquired by genes of control.

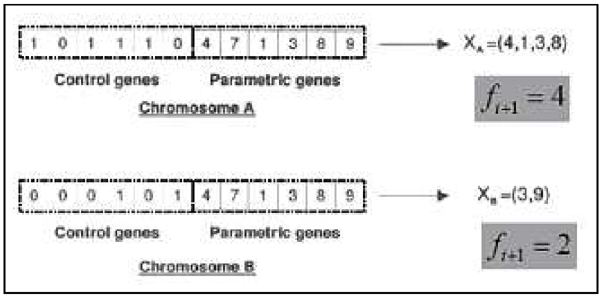

Figure 2: Example of optimization of the structure.

By regarding topology as an objective to be optimized, the problem is now formulated like a problem of multi-criterion optimization $F = \begin{bmatrix} F_i \\ f_{i+1} \end{bmatrix}$.

The genetic algorithm is used to minimize the vector of objective functions F.

Let us consider a problem of a system optimization, where:

- x_j represents a solution (chromosome),
- $F_i(x_j) = M_j$ represents the objective function to minimize.
- $f_{i+1}(x_j) = n_j$ represents the number of parameters of the system.

The whole of the acceptable solutions to this problem is represented by: $\{x : F_i(x) = 0 \text{ at } N1 \leq f_{i+1} \leq N2\}$. In short, the best solution to this problem is

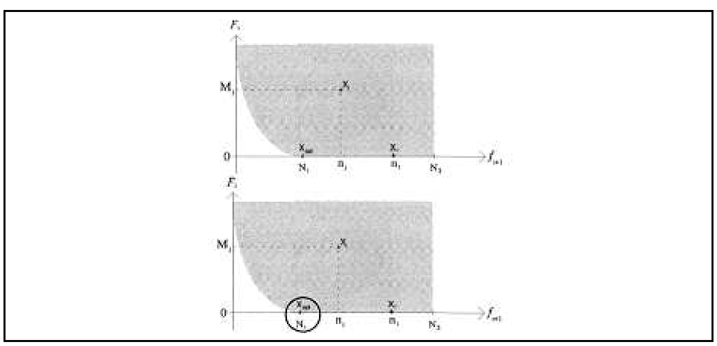

Figure 3: Whole of the acceptable solutions for a problem of joint optimization of the parameters and topology of a system.

represented by x_{opt}, where $f_{i+1}(x_{opt}) = N1$ and $F_i(x_{opt}) = 0$.

Other solutions where the value of the objective function $f_{i+1}(x_j) = n_j > N1$ with $F_i(x_j) = 0$ are acceptable solutions as regards the parameters of the system however the topology of the system is not optimal (Figure 3) [14].

4 HGA for NN learning

HGAs are used for the optimization of the parameters and the topology of NNs. The advantage of this approach is that the genes of the chromosome are classified in two categories (hierarchy). This approach is ideal to represent the relations between:

- layers of the network,
- neurons in hidden layer,
- weights.

The following Figure 4 illustrates the principle operation of the new strategy.

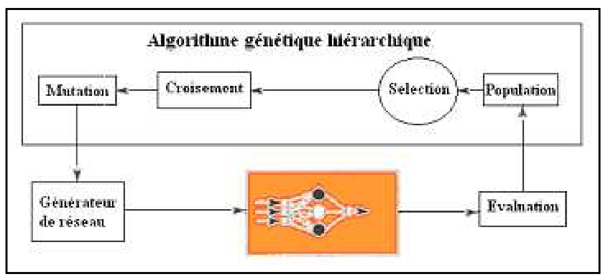

Figure 4: Principle of the new strategy.

Each chromosome is composed by 2 types of genes:

- Genes of control (bits) for activations of the layers and the neurons of the NN.
- Genes of connections (real) for the determination of the weights of connexion.

The following Figure 5 illustrates the Structure of a chromosome.

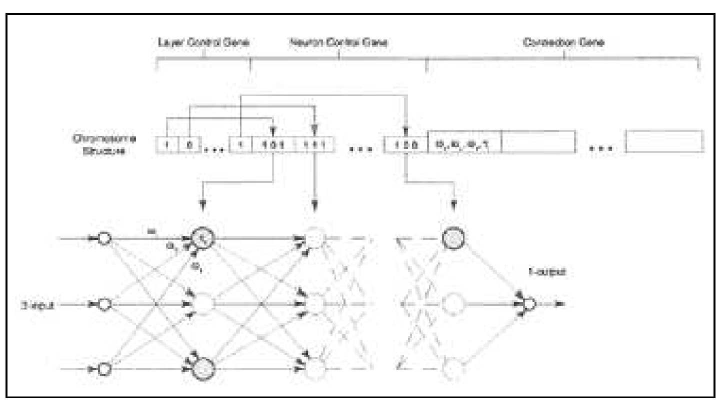

Figure 5: Structure of a chromosome for the optimization of a NN.

5 Presentation of New Approach for NN Learning

5.1 Objective

We build a HGA which selects a structure of NN (number of neurons in the hidden layer). This number is given after the evaluation of a cost function. The function to be minimized is the following quadratic criterion:

$$J = \frac{1}{N}\sum_{k=1}^{N}[y_{di} - y_i]^2, \tag{1}$$

where N is a number of example, y_{di} is an output of the process, y_i is an NN output.

5.2 Coding

We have N_{pop} chromosomes which are in the form of multi-dimensional table. The first line is coded with 0 and 1 which indicates the consideration or not of neuron in hidden layer. The lines which remain contain real which represent the whole of connexions of input and output of hidden layer.

Example of chromosome coding:
We consider a NN with 2 input layer, 1 output layer and maximum 5 neurons in hidden layer (this number can change).

Thus, our chromosome will have 5 columns and 4 lines:

- The columns represent neurons constituting the hidden layer,
- The 1st line contains 0 and 1. If 1 then the neuron exists (active) if 0 then it's not consider (inactive). In this example a hidden layer contains 3 neurons (There are 3 box with 1),
- 2nd and 3rd line represents input weights (2neurons of entry thus 2 lines)
- 4th line represents output weights (1 neuron of exit thus a line)

Figure 6 illustrates the description bellow.

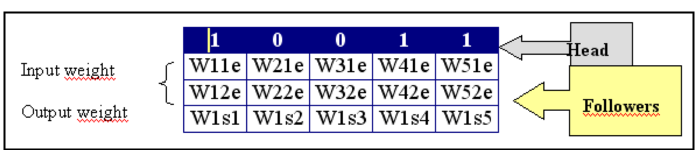

Figure 6: Chromosome code.

The corresponding NN is:

5.3 Evaluation

Feval represents the quadratic criterion to minimize. Each individual of the population is evaluated independently of the others

$$F_{eval}(x) = \frac{1}{N}\sum_{k=1}^{N}[y_{di} - y_i]^2. \tag{2}$$

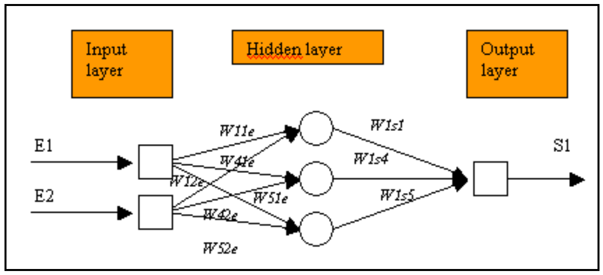

Figure 7: translate chromosome to RNA.

The function of evaluation is a function which depends on 2 independent variables : the first parameter is the desired output y_{di}. This output, we obtained it after having applied to our process an entry rich in frequency (learning sequence). The second variables y_i are the output which are calculated as follows:

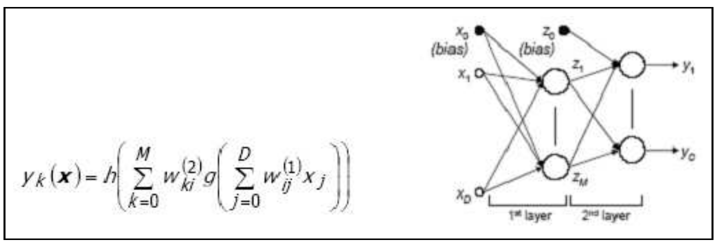

Figure 8: Calculation of the exit of the network.

5.4 Initial population

It is necessary to create and maintain a sufficient genetic diversity in the population. For this reason the initial population must be most heterogeneous as possible in order to prevent premature convergence. That is why we chose a random generation of N_{pop} individuals of the initial population. But for not having, a research space far from the required minimum, and to minimize the search time, we chose to inject into this initial population a certain number of individuals resulting from a training by back propagation.

5.5 Selection

We select a group of reproducers according to their evaluation by the cost function Feval. The individuals who have the minimal function will be selected. It is about an elitist selection. It was noted that this method induced a perfect convergence of the algorithm. Indeed, any Good individual which once combined with others also good, can appear to be interesting. For each individual:

- Calculate the function cost relative to each individual,
- Sort individuals in the ascending order of their function cost,
- Copy in the following generation the Pselect better individuals.

5.6 Criterion of stop

The iterations of the algorithm end when the maximum number of generation, defined in advance, is reached.

6 Application

6.1 System 1: Simple pendulum

We consider the non-linear system describing a simple pendulum and represented by the following model:

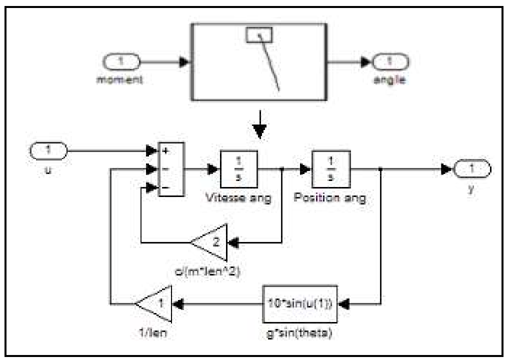

Figure 9: Model of the simple pendulum.

We propose to apply for this system the neuronal predictive control in order to control the position.

We will proceed by creating examples of learning. Then we will identify direct neuronal model of the system by two methods of training: backpropagation and HGA [4, 5].

Development of the learning examples.

These examples of learning constitute a database which will be used for the identification of the direct neuronal model by training. By considering a field of study, we excite the system with input vector (which is a control signal) of amplitude between [-10, 10] and we recover the corresponding vector which is in the interval [-1, 1]. We divide this database into two parts: One will be used for the learning and the other for validation. Figure 10 shows the sequence of test used.

Development of direct neuronal model: DNM :

Establish the system model to be used later in system control. We will follow the choice below:

- Choice of the representation: input/output,
- Choice of the model assumption: model NARX (model not looped),
- Choice of the order of the model: 2.

By considering these choices, the network predictor is governed by the following equation: $yr(k) = \varphi(y(k-1), y(k-2), u(k), u(k-2); C)$, where yr is NN output, y is a system output, φ is a function of network, C is network parameters (weight).

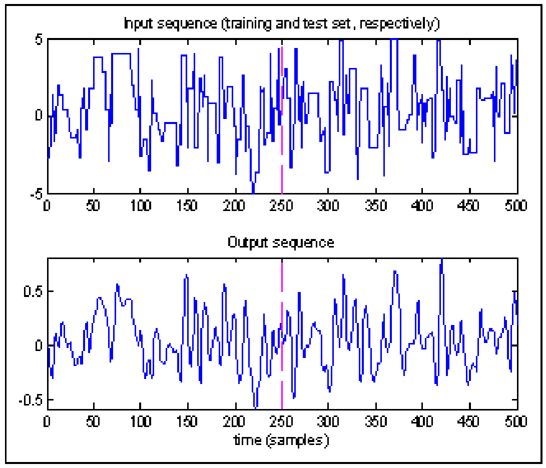

Figure 10: Input/output vectors (sequences of training and of test).

Architecture of the network.

We choose network not looped with 1 hidden layer: 4 inputs, 10 neurons hidden which have a sigmoid function (tansig) and 1 neuron of linear output. The input of the network is the vector: $Ed = [y(k-1); y(k-2); u(k-1); u(k-2)]$.

Consider the matrix of the input weights W_d and the matrix of the output weights W_{ds}. W_d is a matrix (10,4), W_{ds} is a matrix (1,10), ym is a network output, such as $ym(k) = W_{ds}$*tansig$(W_d * Ed)$;

Learning.

Used algorithm of training is the back propagation of the simple gradient to minimize the following criterion:

$$J = \frac{1}{2}\sum_{k=1}^{N}[y(k) - y_m(k)]^2, \tag{3}$$

where J is a quadratic error, N is a number of learning examples.

We use the following parameters: the iteration $N = 200$; a step of training $= 0.2$; values of initial input $y(1) = 0, u(1) = 0$; and a random initialization of the matrices of the weights.

Wd = Rand (10,4) = [0,52952 0,081759 -0,4405 -0,55428 -0,97313 0,63588 -0,020734 -0,028146 0,47375 0,40174 0,024 0,017082 -0,0060763 0,23539 -0,51866 0,32817 0,75842 -0,50884 0,0193 0,037063 0,25786 -0,08087 -0,25978 -0,25763 0,043251 0,44786 -0,42815 0,34426 -1,4512 0,69197 -0,021571 0,020547 0,21106 -0,073689 -0,28491 -0,38401 -0,13845 0,58937 -0,088476 -0,09446],

Wds = Rand (1,10) = [0,086723 -0,32983 0,17984 -0,068595 0,2788 0,015448 0,018001 -0,344 -0,072587].

Then we use the algorithm of back propagation for a given iteration.

Test of the model.

To estimate the error of generalization by calculating the average quadratic error on the sequence of test and of training according to the following formula:

$$EQMT = \sqrt{\frac{1}{N_T}\sum_{k=1}^{N_T}[y(k) - y_m(k)]}, \tag{4}$$

$$EQMA = \sqrt{\frac{1}{N_A}\sum_{k=1}^{N_A}[y(k) - y_m(k)]}. \tag{5}$$

The validation of the model found is done by applying to the NN input, after learning, test sequence and by comparing the output with the output wished. We evaluate the performance of the model found by the error analysis of prediction which represents the error of generalization of the network (Figure 11) [4, 5].

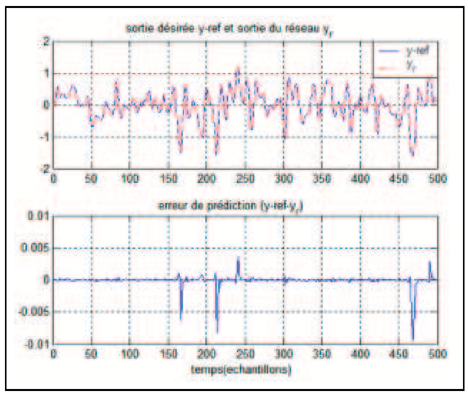

Figure 11: NN output and error of prediction of the DNM.

First, we made the learning of NN with the back propagation algorithm of the gradient. Now, we apply our HGA for determining the NN architecture and make learning. The parameters of simulation which we used are the following:

- The number of maximum neuron that our network can contain is 20 neurons.
- The number of neuron in the input layer is 4 (Ed=[y(k-1); y(k-2);u(k-1) ; u(k-2)]).
- The number of neuron in the output layer is 1.
- The number of individuals in a generation is 200.
- The number of generation (iteration) is 100.

After simulation, the algorithm gives us, a NN of 9 neurons in the hidden layer and the following matrices of weight:

Input Weight Wd: [0,52879 0,081639 -0,55371 -0,44001 -0,96342 0,62766 -0,031226 -0,023709 0,46174 0,40518 0,027177 0,020853 0,74751 -0,5037 0,021788 0,039943 0,25955 -0,079655 -0,25648 -0,24315 -1,4033 0,65602 -0,022435 0,014063 -0,10963 0,55455 -0,098256 -0,099649 -0,56835 1,0888 0,56752 0,020133 -0,074531 -0,87235 0,63455 0,3784].

Output Weight Wds:[0,081874 -0,33508 0,17908 0,28084 0,0172 -0,39829 -0,22687 -0,016373 0,008739].

Now we will evaluate the performance of the model found by the error analysis of prediction which represents the error of generalization of the network (see Figure 12).

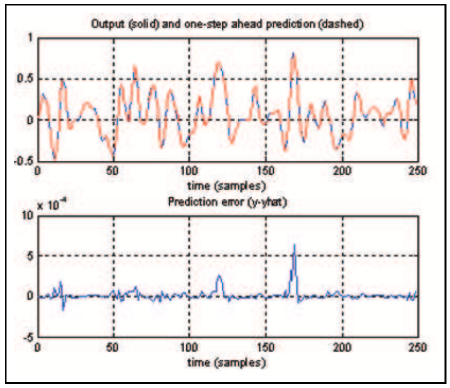

Figure 12: NN output and prediction error of the DNM after use of the HGA.

✓**Comments.**

According to Figure 12, we notice that the output of the network reproduces the evolutions of the output of the process. This results in an error of prediction very close to zero which reaches the maximum of 0.0005. So, the model established by the HGA gives a better result than that found previously by back propagation of the gradient. We can say that the solution given by back propagation of the gradient was local minima. HGA find another best minimum by using another structure and parameters of NN.

6.2 System 2: Rigid spring shock absorber

We consider the non-linear system describing a rigid spring shock absorber. It is represented by the following differential equation :

$$\ddot{y}(t) + \dot{y}(t) + y(t) + y^3(t) = u(t). \tag{6}$$

Figure 13 represents block of our system where y is a system output representing the linear position, u is a system input representing the force applied: y is a system output

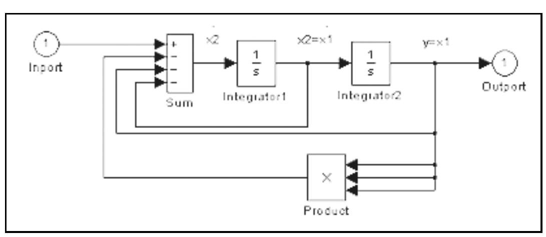

Figure 13: Model of spring shock absorber rigid.

representing the linear position, u is a system input representing the force applied.

Development of the examples of learning.

We recover the examples of learning by exciting the system with an input vector of frequency and amplitude variable between [-12, 12]. We recover the corresponding output vector (Figure 14). This database will be used for the identification of direct model MND and inverse model MNI of the process. We divide the vectors of input and output into two parts of which one will be used for the learning and the other for validation.

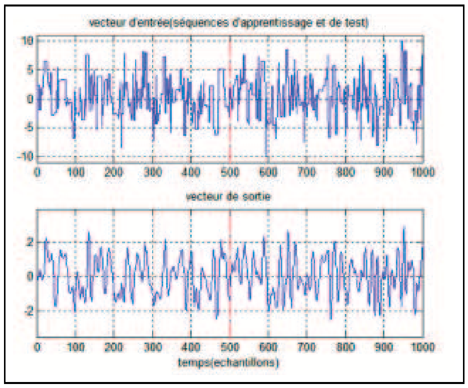

Figure 14: Input/Output Vectors (sequences of training and of test).

Inverse Model Control (IMC).

We propose to apply inverse model. We made learning of inverse model of the process by two methods which will be compared. The first one by direct learning and the other by HGA learning. Then, we will simulate the two inverse model control found in order to compare the performances of the two types of learning.

Learning direct inverse model.

Learning consists in minimizing the following criterion:

$$J = \frac{1}{2} \sum_{k=1}^{N} [u(k) - u_m(k)]^2, \tag{7}$$

where J is a quadratic error, N is a number of training examples.

The characteristics of the adopted network are the following ones:

- NN input : process order is 2 so the regressor is $(y(k+1), y(k-1), u(k-1), u(k-2))$,
- Network architecture: 5 neurons in hidden layer with sigmode activation functions, one linear output neuron.

Validation of the IMC.

We validate the model found after its learning by applying to the network the sequence of test and while comparing with the desired output. Figure 15 shows results found in order to evaluate the performance of the NN model [4, 5].

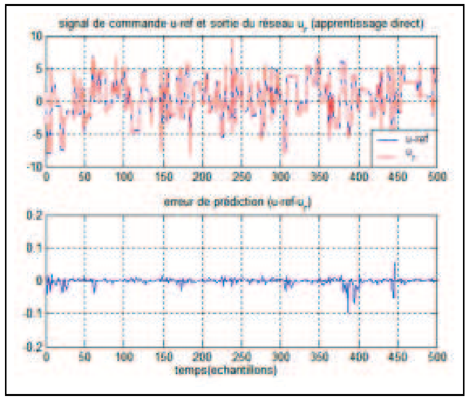

Figure 15: NN output and error of prediction of the IMC (direct learning).

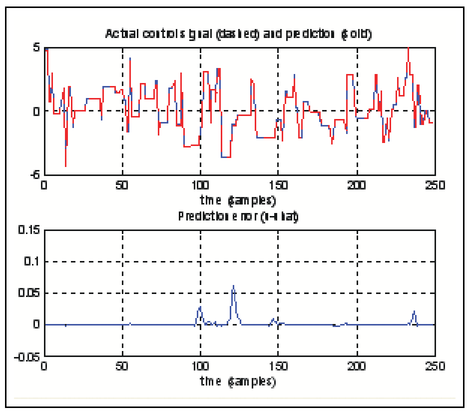

Figure 16: NN output and error of prediction of the IMC (direct learning) after use of AGH.

Now, we apply our HGA for learning inverse model. For the validation, we apply to the network found the sequence of test and we will compare it with the desired output. Figure 16 shows the results found and error of prediction in order to evaluate the performance of the network model.

✓**Comments.**

We note that the IMC found after the use of our hierarchical genetic algorithm is very satisfactory compared to the traditional method used at the beginning. Indeed we have an error of prediction much nearer to 0.

7 Conclusion

The algorithms performing local searches based on gradient information are sensitive to initialization and can provide a solution corresponding to a local optimum. This sensitivity to local optima is very limiting for multilayer perceptron learning.

Our main aim is the limitation of the defects of back propagation algorithm. The HGA gives us a very interesting result. Indeed this algorithm considered very great number of multi-layer architecture of perceptron and made a learning by applying its various operators (crossover and mutation).

In order to test our algorithm, precisely effectiveness of coding, we simulated two non-linear systems, by the Simulink tool of MatLab. We have initially made learning of direct and inverse model by using backpropagation and then our HGA. The results show that the variance of the error for the second method is better than for the first one and can escape local optima.

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On Nonlinear Abstract Neutral Differential Equations with Deviated Argument

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Abstract: In this paper we are concerned with a neutral differential equation with a deviated argument in an arbitrary Banach space X. To study the existence and uniqueness of a solution of the problem considered, we use the theory of the analytic semigroups and the fixed point arguments. Finally, we give an example to demonstrate an application of the abstract results.

Keywords: *neutral differential equation with a deviated argument; Banach fixed point theorem; analytic semigroup.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this study we are concerned with the following neutral differential equation with a deviated argument considered in a Banach space X:

$$\left\{ \begin{array}{c} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + Au(t) = f(t, u(t), u[h(u(t), t)]), \quad 0 < t \leq T < \infty, \\ u(0) = u_0, \end{array} \right. \tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup. f, g, h and a are suitably defined functions satisfying certain conditions to be stated later.

Initial results related to the differential equations with the deviated arguments can be found in some research papers of the last decade but still a complete theory seems to be missing. For the initial works on the existence, uniqueness and stability of various types of solutions of different kinds of differential equations, we refer to [1]-[14] and the references cited in these papers.

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Hernandez and Henriquez [9, 10] established some results concerning the existence, uniqueness and qualitative properties of a solution operator of the following general partial neutral functional differential equation with the infinite delay:

$$\begin{aligned} \frac{d}{dt}(u(t) - g(t, u_t)) &= Au(t) + f(t, u_t), \quad t \geq 0, \\ u_0 &= \varphi \in C_0, \end{aligned}$$

where A generates an analytic semigroup on a Banach space B, g and f are continuous functions from $[0, \infty) \times C_0$ into B and for each $u : (-\infty, b] \to B,\ b > 0$ and $t \in [0, b]$, u_t represents, as usual, the mapping defined from $(-\infty, 0]$ into B by

$$u_t(\theta) = u(t + \theta) \quad \text{for} \quad \theta \in (-\infty, 0].$$

Adimy *et al* [1] have studies the existence and stability of a solution of the following general class of nonlinear partial neutral functional differential equations:

$$\begin{aligned} \frac{d}{dt}(u(t) - g(t, u_t)) &= A(u(t) - g(t, u_t)) + f(t, u_t), \quad t \geq 0, \\ u_0 &= \varphi \in C_0, \end{aligned} \tag{1.2}$$

where the operator A is the Hille-Yosida operator not necessarily densely defined on the Banach space B. The functions g and f are continuous from $[0, \infty) \times C_0$ into B.

In this paper, we use the Banach fixed point theorem and the analytic semigroup theory to prove the existence and uniqueness of different kinds of solutions to the problem (1.1). The plan of the paper is as follows. In Section 3, we prove the existence and uniqueness of a local solution and in Section 4, the existence of a global solution for the problem (1.1) is given. In the last section, we give an example.

The results presented in this paper can be applied easily to the problem (1.1) with a nonlocal condition under some modified assumptions on the function f and the operator A.

2 Preliminaries and Assumptions

As pointed out earlier, we note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence, without loss of generality, we suppose that

$$\|S(t)\| \leq M \quad \text{for} \quad t \geq 0 \quad \text{and} \quad 0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ and being dense in E. We have $E_\kappa \hookrightarrow E_\alpha$, for $0 < \alpha < \kappa$ and the embedding is continuous. For more details on the fractional powers of the closed linear operators, we refer to Pazy [15].

It can be proved easily that $E_\alpha := D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha = \|A^\alpha x\|$ and it is equivalent to the graph norm of A^α. Also, for each $\alpha > 0$, we define $E_{-\alpha} = (E_\alpha)^*$, the dual space of E_α is a Banach space endowed with the norm $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$.

It can be seen easily that $\mathcal{C}_t^\alpha = C([0,t];E_\alpha)$, for all $t \in [0,T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} := \sup_{0\le\eta\le t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

We set $\mathcal{C}_T^{\alpha-1} = C([0,T];E_{\alpha-1}) = \{y \in \mathcal{C}_T^\alpha : \|y(t)-y(s)\|_{\alpha-1} \le L|t-s|, \forall\ t,s \in [0,T]\}$, where L is a suitable positive constant to be specified later and $0 \le \alpha < 1$.

To proceed further, we need to assume the following assumptions on operator A and function f, g, h, a:

(**A1**): $0 \in \rho(-A)$ and $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \ge 0\}$.

(**A2**): Let $U_1 \subset \mathrm{Dom(f)}$ be an open subset of $\mathbb{R}_+ \times E_\alpha \times E_{\alpha-1}$ and for each $(t,u,v) \in U_1$ there is a neighborhood $V_1 \subset U_1$ of (t,u,v). The nonlinear map $f : \mathbb{R}_+ \times E_\alpha \times E_{\alpha-1} \to E$ satisfies the following condition,

$$\|f(t,x_1,y_1) - f(s,x_2,y_2)\| \le L_f[|t-s|^{\theta_1} + \|x_1-x_2\|_\alpha + \|y_1-y_2\|_{\alpha-1}],$$

where $0 < \theta_1 \le 1$, $0 \le \alpha < 1$, $L_f > 0$ is a constant, $(t,x_1,y_1) \in V_1$, and $(s,x_1,y_2) \in V_2$.

(**A3**): Let $U_2 \subset \mathrm{Dom(h)}$ be an open subset of $E_\alpha \times \mathbb{R}_+$ and for each $(x,t) \in U_2$ there is a neighborhood $V_2 \subset U_2$ of (x,t). The map $h : E_\alpha \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition

$$|h(x,t) - h(y,s)| \le L_h[\|x-y\|_\alpha + |t-s|^{\theta_2}],$$

where $0 < \theta_2 \le 1$, $0 \le \alpha < 1$, $L_h > 0$ is a constant, $(x,t),\ (y,s) \in V_2$ and $h(.,0) = 0$.

(**A4**): Let $U_3 \subset \mathrm{Dom(g)}$ be an open subset of $[0,T] \times E_{\alpha-1}$ and for each $(t,x) \in U_3$, there is a neighborhood $V_3 \subset U_3$ of (x,t). The function $g : [0,T] \times E_{\alpha-1} \to E_\beta$ is continuous for $(t,u) \in [0,T_0] \times E_{\alpha-1}$ such that

$$\begin{aligned} \|A^\beta g(t,x) - A^\beta g(s,y)\| &\le L_g\{|t-s| + \|x-y\|_{\alpha-1}\}, \text{ and} \\ L_g\|A^{\alpha-\beta-1}\| &< 1, \end{aligned}$$

where $0 \le \alpha < 1$, $\ L_g > 0$ is a positive constant $(x,t),(y,s) \in V_3$.

(**A5**): The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:

(i) a satisfies the delay property $a(t) \le t$, for all $t \in [0,T]$;

(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$|a(t) - a(s)| \le L_a|t-s|, \text{ for all } t,s \in [0,T] \quad \text{and} \quad 1 > \|A^{-1}\|L_a.$$

Definition 2.1 A continuous function $u \in C_T^{\alpha-1} \cap C_T^\alpha$ is said to be a mild solution of equation (1.1) if u is the solution of the following integral equation

$$\begin{aligned} u(t) &= S(t)[u(0) + g(0,u_0)] - g(t,u(a(t))) + \int_0^t AS(t-s)g(s,u(a(s)))ds \\ &+ \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])ds,\ t \in [0,T] \end{aligned} \tag{2.3}$$

and satisfies the initial condition $u(0) = u_0$.

Definition 2.2 A function $u : [0,T] \to E$ is called a solution of 1.1 if u satisfies the following conditions,

(i) $u(.) + g(., u(a(.))) \in C_T^{\alpha-1} \cap C^1((0,T), E) \cap C([0,T], E)$,

(ii) $u(t) \in D(A)$, and $(t, u(t), u[h(u(t),t)]) \in U_1$,

(iii) $\frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t)] = f(t, u(t), u[h(u(t),t)])$ for all $t \in (0,T]$,

(iv) $u(0) = u_0$.

3 Existence of Local Solutions

In this section, we provide an existence and uniqueness theorem for a mild solution of (1.1). We set

$$\mathcal{W} = \{u \in C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1} : u(0) = u_0, \ \ \|u - u_0\|_{T_0,\alpha} \le \delta\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of $C_T^{\alpha-1}$.

Under the assumptions **(A2)-(A3)**, $0 \le \alpha < 1$ and $u \in \mathcal{C}_{T_0}^{\alpha}$ imply that $f(s, u(s), u[h(u(s),s)])$ is continuous on $[0, T_0]$. Therefore, we can show that there exists a positive constant N such that

$$\|f(s, u(s), u[h(u(s),s)])\| \le N = L_f[{T_0}^{\theta_1} + \delta(1 + LL_h) + LL_h T_0^{\theta_2}] + N_0,$$

where $N_0 = \|f(0, u_0, u_0)\|$. Similarly, with the help of the assumptions **(A4)-(A5)**, we can easily show that $\|A^{\beta} g(t, u(a(t)))\| \le L_g[T_0 + \delta] + \|g(0, u_0)\|_{\alpha} = N_1$. Also, we denote $\|A^{-1}\| = M_2$ and $\|A^{-\alpha}\| = M_3$.

Theorem 3.1 *Let us assume that the assumptions (A1)-(A5) are satisfied and $u_0 \in D(A^{\alpha})$, for $0 \le \alpha < 1$. Then, the differential equation (1.1) has a unique local mild solution $u(t)$, for $t \in (0, T_0)$, where $T_0 = T_0(\alpha, \beta, u_0) > 0$ is sufficiently small.*

Proof For a fixed $\delta > 0$, we choose $0 < T_0 = T_0(\alpha, \beta, u_0) \le T$ such that

$$C_{\alpha+1-\beta} L_g \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + C_{\alpha} L_f [2 + LL_h] \frac{T_0^{1-\alpha}}{(1-\alpha)} \le 1 - \eta, \tag{3.4}$$

where $\eta = L_g \|A^{\alpha-\beta-1}\| < 1$ and satisfying the following

$$\|(S(t) - I)A^{\alpha}[u_0 + g(0, u_0)]\| + \|A^{\alpha-\beta}\| L_g [T_0 + \delta] \le \frac{\delta}{2} \tag{3.5}$$

for all $t \in [0, T_0]$ and

$$C_{\alpha+1-\beta} N_1 \frac{T_0^{\beta-\alpha}}{\beta-\alpha} + C_{\alpha} N \frac{T_0^{1-\alpha}}{1-\alpha} \le \frac{\delta}{2}. \tag{3.6}$$

For more details of choosing such a T_0, we refer to Theorem 2.2 of [8].

We define a map $\mathcal{F} : C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1} \to C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1}$ as

$$\begin{aligned}(\mathcal{F}u)(t) \quad &= \quad S(t)[u_0 + g(0, u_0)] - g(t, u(a(t))) + \int_0^t AS(t-s) g(s, u(a(s))) ds \\ &+ \quad \int_0^t S(t-s) f(s, u(s), u[h(u(s),s)]) ds, \quad t \in [0,T]. \end{aligned} \tag{3.7}$$

In order to prove this theorem, we need to show that $\mathcal{F}u \in \mathcal{C}_{T_0}^{\alpha-1}$, for any $u \in \mathcal{C}_{T_0}^{\alpha-1}$. Clearly, $\mathcal{F} : \mathcal{C}_T^{\alpha} \to \mathcal{C}_T^{\alpha}$.

If $u \in \mathcal{C}_{T_0}^{\alpha-1}, \quad T > t_2 > t_1 > 0,$ and $0 \le \alpha < 1$, then we get

$$
\begin{aligned}
\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\|_{\alpha-1} &\le \|(S(t_2) - S(t_1))(u_0 + g(0,u_0))\|_{\alpha-1} \\
&+\|A^{\alpha-\beta-1}\|\|A^{\beta}g(t_2, u(a(t_2))) - A^{\beta}g(t_1, u(a(t_1)))\| \\
&+ \int_0^{t_1} \|(S(t_2-s) - S(t_1-s))A^{\alpha-\beta}\|\|A^{\beta}g(s, u(a(s)))\|ds \\
&+ \int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-\beta}\|\|A^{\beta}g(s, u(a(s)))\|ds. \\
&+ \int_0^{t_1} \|(S(t_2-s) - S(t_1-s))A^{\alpha-1}\| \\
&\qquad \times \|f(s, u(s), u[h(u(s), s)])\|ds \\
&+ \int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-1}\|\|f(s, u(s), u[h(u(s), s)])\|ds. \qquad (3.8)
\end{aligned}
$$

For the first part of the right hand side of (3.8), we have

$$
\begin{aligned}
\|(S(t_2) - S(t_1))(u_0 + g(0,u_0))\|_{\alpha-1} &\le \int_{t_1}^{t_2} \|A^{\alpha-1}S'(s)(u_0 + g(0,u_0))\|ds \\
&= \int_{t_1}^{t_2} \|A^{\alpha}S(s)(u_0 + g(0,u_0))\|ds \\
&\le \int_{t_1}^{t_2} \|S(s)\|[\|u_0\|_{\alpha} + \|A^{\alpha-\beta}\|\|g(0,u_0)\|_{\beta}]ds \\
&\le C_1(t_2 - t_1), \qquad (3.9)
\end{aligned}
$$

where $C_1 = [\|u_0\|_{\alpha} + \|A^{\alpha-\beta}\|\|g(0,u_0)\|_{\beta}]M$.

For the second part of the right hand side of (3.8), we can see that

$$
\begin{aligned}
&\|A^{\alpha-\beta-1}\|\|A^{\beta}g(t_2, u(a(t_2))) - A^{\beta}g(t_1, u(a(t_1)))\| \\
&\qquad \le \|A^{\alpha-\beta-1}\|L_g[|(t_2 - t_1)| + \|u(a(t_2)) - u(a(t_1))\|_{\alpha-1}] \\
&\qquad \le \|A^{\alpha-\beta-1}\|[L_g + LL_a]|(t_2 - t_1)| \\
&\qquad \le C_2|(t_2 - t_1)|. \qquad (3.10)
\end{aligned}
$$

where $C_2 = \|A^{\alpha-\beta-1}\|[L_g + LL_a]$.

To handle the third and fifth parts of the right hand side of (3.8), we observe that

$$
\begin{aligned}
\|(S(t_2-s) - S(t_1-s))\|_{\alpha-1} &\le \int_0^{t_2-t_1} \|A^{\alpha-1}S'(l)S(t_1-s)\|dl \\
&\le \int_0^{t_2-t_1} \|S(l)A^{\alpha}S(t_1-s)\|dl \\
&\le MC_{\alpha}(t_2-t_1)(t_1-s)^{-\alpha}. \qquad (3.11)
\end{aligned}
$$

Now we use the inequality (3.11) to get the bound for the third part we have

$$
\int_0^{t_1} \|(S(t_2-s) - S(t_1-s))A^{\alpha-\beta}\| \times \|A^{\beta}g(s, u(a(s))])\|ds \le C_4(t_2 - t_1), \qquad (3.12)
$$

where $C_4 = N_1 M C_{\alpha-\beta+1} \frac{T_0^{1-(\alpha-\beta)}}{1-(\alpha-\beta)}$. Similarly, bound for the fifth part is given as

$$\int_0^{t_1} \|(S(t_2-s) - S(t_1-s))A^{\alpha-1}\| \times \|f(s, u(s), u[h(u(s), s)])\| ds \le C_3(t_2 - t_1), \quad (3.13)$$

where $C_3 = NMC_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}$. For the bound for the sixth part, we have

$$\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-1}\| \|f(s, u(s), u[h(u(s), s)])\| ds \le C_5(t_2 - t_1), \quad (3.14)$$

where $C_5 = \|A^{\alpha-1}\| MN$. Finally, for the fourth part we have the following

$$\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-\beta}\| \|A^\beta g(s, u(a(s))\| ds \le C_6(t_2 - t_1), \quad (3.15)$$

where $C_6 = \|A^{\alpha-\beta}\| M N_1$.

We use the inequalities (3.9), (3.10), (3.13)-(3.15) in inequality (3.8) to get the following inequality

$$\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\|_{\alpha-1} \le \tilde{L}|t_2 - t_1|, \quad (3.16)$$

where, $\tilde{L} = \max\{C_i,\ i = 1, 2, \cdots 6\}$. Hence, $\mathcal{F} : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ follows.

Our next task is to show that $\mathcal{F} : \mathcal{W} \to \mathcal{W}$. Now, for $t \in (0, T_0]$ and $u \in \mathcal{W}$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - u_0\|_\alpha &\le \|(S(t) - I)A^\alpha[u_0 + g(0, u_0)]\| \\
&\quad + \|A^{\alpha-\beta}\| \|A^\beta g(s, u(a(s))) - A^\beta g(0, u(a(0)))\| \\
&\quad + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\| \|A^\beta g(s, u(a(s)))\| ds \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \|f(s, u(s), u[h(u(s), s)])\| ds \\
&\le \|(S(t) - I)A^\alpha[u_0 + g(0, u_0)]\| + \|A^{\alpha-\beta}\| L_g[T_0 + \delta] \\
&\qquad + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha} + C_{1+\alpha-\beta} N_1 \frac{T_0^{\beta-\alpha}}{\beta-\alpha}.
\end{aligned}$$

Hence, from inequalities (3.5) and (3.6), we get $\|\mathcal{F}u - u_0\|_{T_0,\alpha} \le \delta$. Therefore, $\mathcal{F} : \mathcal{W} \to \mathcal{W}$.

Now, if $t \in (0, T_0]$ and $u, v \in \mathcal{W}$, then

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|_\alpha &\le \|A^{\alpha-\beta}\| \|A^\beta g(t, u(a(s))) - A^\beta g(t, v(a(s)))\| \\
&\quad + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\| \|A^\beta g(s, u(a(s))) - A^\beta g(s, v(a(s)))\| ds. \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \\
&\quad \times \|f(s, u(s), u[h(u(s), s)]) - f(s, v(s), v[h(u(s), s)])\| ds. \qquad (3.17)
\end{aligned}$$

We have the following inequalities

$$\|A^\beta g(t, u(a(s))) - A^\beta g(t, v(a(t)))\| \quad \le \quad L_g \|A^{-1}\| \|u - v\|_{T_0,\alpha}, \quad (3.18)$$

$$\begin{aligned}&\|f(s,u(s),u[h(u(s),s)])-f(s,v(s),v[h(v(s),s)])\|\\ &\qquad\le L_f[2+LL_h]\|u-v\|_{T_0,\alpha}.\end{aligned}\tag{3.19}$$

We use the inequalities (3.18) and (3.19) in the inequality (3.17) and get

$$\begin{aligned}\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\alpha \quad\le\quad & [L_g(\|A^{\alpha-\beta-1}\|+C_{1+\alpha-\beta}\frac{T_0^{\beta-\alpha}}{\beta-\alpha})\\ &+C_\alpha L_f[2+LL_h]\frac{T_0^{1-\alpha}}{(1-\alpha)}]\|u-v\|_{T_0,\alpha}.\end{aligned}\tag{3.20}$$

Hence, from inequality (3.4), we get the following inequality given below

$$\|\mathcal{F}u-\mathcal{F}v\|_{T_0,\alpha}<\|u-v\|_{T_0,\alpha}.$$

Therefore, the map $\mathcal{F}$ has a unique fixed point $u\in\mathcal{W}$ which is given by

$$\begin{aligned}u(t)\quad=\quad & S(t)[u_0+g(0,u_0)]-g(t,u(a(t)))+\int_0^t AS(t-s)g(s,u(a(s)))ds\\ +\quad & \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])ds \;\; t\in[0,T_0].\end{aligned}\tag{3.21}$$

Hence, the mild solution u of equation (1.1) is given by the equation (3.21) and belongs to $C^{\alpha}_{T_0}\cap C^{\alpha-1}_{T_0}$. Also, on the similar lines of the proof of Theorem 6.3.1, we can easily check that

$$\|u(t+h)-u(t)\|\le L'|h|^\gamma$$

for some $0<\gamma<1-\alpha$. Furthermore, the inequality of (A2), implies the local Hölder continuity of the function f for $t,s\in[t_0,T], 0<t_0<T$. Precisely for $u\in C^{\alpha-1}_{T_0}$ and moreover, $u\in C^\gamma((0,T],E_\alpha)$ for $0<\gamma<1-\alpha$:

$$\begin{aligned}&\|f(t,u(t),u[h(u(t),t)])-f(s,u(s),u[h(u(s),s)])\|\\ &\le L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_\alpha+L|h(u(t),t)-h(u(s),s)|\}\\ &\le L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_\alpha+LL_h[|t-s|^{\theta_2}+\|u(t)-u(s)\|_\alpha]\}\\ &\le L_f\{|t-s|^{\theta_1}+L'|t-s|^\gamma+LL_h[|t-s|^{\theta_2}+L'|t-s|^\gamma]\}.\end{aligned}\tag{3.22}$$

Hence, the map $t\mapsto f(t,u(t),u[h(u(t),t)])$ is locally Hölder continuous. Therefore,

$$f(t,u(t),u[h(u(t),t)])\in C([0,T],E)\cap C^{\beta'}((0,T],E),$$

where $0<\beta'<\min\{\theta_1,\ \gamma,\ \theta_2\}$. Similarly, we can prove that $u(.)+g(.,u(a(.)))$ is also Hölder continuous on $(0,T_0]$. Therefore, from Theorem 3.1 pp. 110 and Corollary 3.3, pp. 113, Pazy [15], the function $u(.)+g(.,u(a(.)))\in C^{\alpha-1}_{T_0}\cap C^1((0,T_0),E)\cap C([0,T_0],E)$ and $u(.)$ is the unique solution of the problem (1.1) in the sense of definition (3.2) of Pazy [15]. This completes the proof of the Theorem. □

4 Existence of Global Solutions

In order to establish the global existence of a mild solution to (1.1), we need the following lemma.

Lemma 4.1 *Let $u_0(t,s) \geq 0$ be continuous on $0 \leq s \leq t \leq T < \infty$. If there are positive constants A, E and α such that*

$$u_0(t,s) \leq A + B \int_s^t (t-\sigma)^{\alpha-1} u_0(\sigma, s) d\sigma, \tag{4.1}$$

for $0 \leq s < t \leq T$, then there is a constant C such that $u_0(t,s) \leq C$.

Proof For $0 \leq s < t \leq T$, we have

$$\int_s^t (t-\tau)^{(\alpha-1)} (\tau - s)^{(\beta-1)} d\tau = (t-s)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{4.2}$$

which holds for every $\alpha, \beta > 0$. Integrating (4.1) $n-1$ times and using (4.2) and replacing $t-s$ by T, we get

$$u_0(t,s) \leq A \sum_{j=0}^{n-1} \left(\frac{BT^\alpha}{\alpha} \right)^j + \frac{(B\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_s^t (t-\sigma)^{n\alpha-1} u_0(\sigma, s) d\sigma. \tag{4.3}$$

Let n be large enough so that $n\alpha > 1$. We majorize $(t-\sigma)^{n\alpha-1}$ by $T^{n\alpha-1}$ to obtain

$$u_0(t,s) \leq c_1 + c_2 \int_s^t u_0(\sigma, s) d\sigma. \tag{4.4}$$

Application of Gronwall's inequality leads to

$$u_0(t,s) \leq c_1 e^{c_2(t-s)} \leq c_1 e^{c_2 T} \leq C. \tag{4.5}$$

This completes the proof of the lemma.

Theorem 4.1 *Suppose that $0 \in \rho(-A)$ and the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, the conditions **(A1)–(A5)** are satisfied and $u_0 \in D(A^\alpha)$. If there are continuous nondecreasing real valued functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\begin{aligned}
\|f(t,x,y)\| &\leq k_1(t)(1 + \|x\|_\alpha + \|y\|_{\alpha-1}), && (4.6)\\
|h(x,t)| &\leq k_2(t)(1 + \|z\|_\alpha), && (4.7)\\
\|g(t,y)\|_\beta &\leq k_3(t)(1 + \|v\|_{\alpha-1}), && (4.8)
\end{aligned}$$

for $t \geq 0$, $x \in E_\alpha$ and $y \in E_{\alpha-1}$, then the initial value problem (1.1) has a unique solution which exists for all $t \in [0,T]$.

Proof Let T_0 be sufficiently small as defined in the proof of Theorem 3.1 and let $u(t), t \in (0, T_0)$, be the local mild solution of (1.1). To prove the global existence of $u(t)$, we need to show that we can continue the solution of equation (1.1) as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if u exists on $[0,T)$, then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

We have the following inequality

$$\begin{aligned}
\|u[h(u(s),s)]\|_{\alpha-1} &\leq \|u[h(u(s),s)] - u(0)\|_{\alpha-1} + \|u_0\|_{\alpha-1}\\
&\leq L|h(u(s),s)| + \|u_0\|_{\alpha-1}\\
&\leq Lk_2(T) + Lk_2(T)\|u\|_{s,\alpha} + \|u_0\|_{\alpha-1}. && (4.9)
\end{aligned}$$

For $t \in [0,T)$, we have

$$
\begin{aligned}
\|u(t)\|_\alpha \quad &\leq \quad \|S(t)A^\alpha[u_0+g(0,u_0)]\| + \|A^{\alpha-\beta}\|\|g(t,u(a(t)))\|_\beta \\
&+ \quad \int_0^t \|A^{\alpha+1-\beta}S(t-s)\|\|A^\beta g(s,u(s)))\|ds \\
&+ \quad \int_0^t \|A^\alpha S(t-s)\|\|f(s,u(s),u[h(u(s),s)])\|ds \\
&\leq \quad M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1+\|A^{-1}\|\|u_0\|_\alpha\}] \\
&+ \quad k_3(T)\|A^{\alpha-\beta}\|[1+\|A^{-1}\|\|u\|_{t,\alpha}] \\
&+ \quad C_{\alpha+1-\beta}\int_0^t (t-s)^{-1+\beta-\alpha}k_3(T)[1+\|A^{-1}\|\|u\|_{s,\alpha}]ds, \\
&+ \quad C_\alpha \int_0^t (t-s)^{-\alpha}k_1[1+\|u\|_{s,\alpha}+\|u[h(u(s),s)]\|_{\alpha-1}]ds \\
&\leq \quad M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1+\|A^{-1}\|\|u_0\|_\alpha\}] \\
&+ \quad k_3(T)\|A^{\alpha-\beta}\|[1+\|A^{-1}\|\|u\|_{t,\alpha}] \\
&+ \quad k_3(T)C_{\alpha+1-\beta}\int_0^t (t-s)^{-(1+\alpha-\beta)}ds \\
&+ \quad \|A^{-1}\|k_3(T)C_{\alpha+1-\beta}\int_0^t (t-s)^{-(1+\alpha-\beta)}\|u\|_{s,\alpha}ds \\
&+ \quad k_1(T)C_\alpha\int_0^t (t-s)^{-\alpha}ds + k_1(T)C_\alpha\int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha}ds \\
&+ \quad (Lk_2(T)+\|u_0\|_{\alpha-1})k_1(T)C_\alpha\int_0^t (t-s)^{-\alpha}ds \\
&+ \quad Lk_2(T)k_1(T)C_\alpha\int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha}ds.
\end{aligned}
$$

Hence,

$$
\|u\|_{t,\alpha} \quad \leq \quad C_1 + \int_0^t (C_2(t-s)^{-\alpha} + C_3(t-s)^{\beta-\alpha-1})\|u\|_{s,\alpha}ds, \tag{4.10}
$$

where

$$
\begin{aligned}
C_1 \quad &= \quad \frac{M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1+\|A^{-1}\|\|u_0\|_\alpha\}] + k_3(T)\|A^{\alpha-\beta}\|}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)} \\
&+ \frac{k_1(T)C_\alpha T^{1-\alpha}}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)(1-\alpha)} \\
&+ \frac{(Lk_2(T)+\|u_0\|_{\alpha-1})k_1(T)C_\alpha T^{1-\alpha}}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)(1-\alpha)} \\
&+ \frac{k_3(T)C_{\alpha+1-\beta}T^{\alpha-\beta}}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)(\alpha-\beta)}, \\
C_2 \quad &= \quad \frac{k_1(T)C_\alpha[1+Lk_2(T)]}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)}, \\
C_3 \quad &= \quad \frac{\|A^{-1}\|k_3(T)C_{\alpha+1-\beta}}{(1-k_3(T)\|A^{\alpha-\beta-1}\|)}.
\end{aligned}
$$

Now, we rewrite (4.10) as follows

$$\|u\|_{t,\alpha} \quad \leq \quad C_1 + \int_0^t \widetilde{C}_{2,3}(t-s)^{-\widetilde{\gamma}} \|u\|_{s,\alpha} ds, \tag{4.11}$$

where

$$\widetilde{C}_{2,3}(t-s)^{-\widetilde{\gamma}} = 2 \times \max[C_2(t-s)^{-\alpha}, C_3(t-s)^{\beta-\alpha-1}]. \tag{4.12}$$

Hence, by applying Lemma 4.1 to the above inequality (4.11), we get the required results. This completes the proof of the theorem. □

5 Example

Let $E = L^2(0,1)$. We consider the following partial differential equations with a deviated argument,

$$\begin{cases} \partial_t[w(t,x) + \partial_x f_1(t, w(a(t),x))] - \partial_x^2[w(t,x)] \\ \quad = f_2(x, w(t,x)), + f_3(t,x,w(t,x)), \quad x \in (0,1),\ t > 0, \\ \quad w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \\ \quad w(0,x) = u_0,\ x \in (0,1), \end{cases} \tag{5.1}$$

where

$$f_2(x, w(t,x)) = \int_0^x K(x,s) w(s, h(t)(a_1|w(s,t)| + b_1|w_s(s,t)|))ds.$$

The function $f_3 : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u and uniformly continuous in x. Further, we assume that $a_1, b_1 \geq 0, (a_1, b_1) \neq (0,0)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $K : [0,1] \times [0,1] \to \mathbb{R}$.

We define an operator A, as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1(0,1) \cap H^2(0,1) :\ u'' \in E\}. \tag{5.2}$$

Here, clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2}) = H_0^1(0,1)$ is the Banach space endowed with the norm,

$$\|x\|_{1/2} := \|A^{1/2}x\|, \quad x \in D(A^{1/2})$$

and we denote this space by $E_{1/2}$. Also, for $t \in [0,T]$, we denote

$$C_t^{1/2} = C([0,t]; D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} := \sup_{0 \leq \eta \leq t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^{1/2}.$$

We observe some properties of the operators A and $A^{1/2}$ defined by (5.2). For $u \in D(A)$ and $\lambda \in \mathbb{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u\rangle = \langle \lambda u, u\rangle$; that is,

$$\langle -u'', u\rangle = |u'|_{L^2}^2 = \lambda |u|_{L^2}^2,$$

so $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle u_n, u_m\rangle = 0$ for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\,\alpha_n\,u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$. $E_{-\frac{1}{2}} = H^1(0,1)$ is a Sobolev space of negative index with the equivalent norm $\|.\|_{-\frac{1}{2}} = \sum_{n=1}^{\infty} |\langle ., u_n\rangle|^2$. For more details on the Sobolev space of negative index, we refer to Gal [8].

The equation (5.1) can be reformulated as the following abstract equation in $E = L^2(0,1)$:

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t)] &= f(t, u(t), u[h(u(t),t)]) \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{5.3}$$

where $u(t) = w(t, .)$ that is, $u(t)(x) = w(t,x)$, $x \in (0,1)$. The function $g : \mathbb{R}_+ \times E_{1/2} \to E$, such that $g(t, u(a(t)))(x) = \partial_x f_1(t, w(a(t), x))$ and the operator A is same as in equation (5.2).

The function $f : \mathbb{R}_+ \times E_{1/2} \times E_{-1/2} \to E$, is given by

$$f(t,\psi,\xi)(x) = f_2(x,\xi) + f_3(t,x,\psi), \tag{5.4}$$

where $f_2 : [0,1] \times E \to H_0^1(0,1)$ is given by

$$f_2(t,\xi) = \int_0^x K(x,y)\xi(y)dy, \tag{5.5}$$

and $f_3 : \mathbb{R} \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$, satisfies the following

$$\|f_3(t,x,\psi)\| \le Q(x,t)(1 + \|\psi\|_{H^2(0,1)}) \tag{5.6}$$

with $Q(.,t) \in E$ and Q is continuous in its second argument. We can easily verify that the function f satisfies the assumptions (A1)-(A4). For more details see [8].

For the function a we can take

(i) $a(t) = kt$, where $t \in [0,T]$ and $0 < k \le 1$;

(ii) $a(t) = kt^n$ for $t \in I = [0,1]$ $k \in (0,1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k\sin t$ for $t \in I = [0, \frac{\pi}{2}]$, and $k \in (0,1]$.

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Trajectory Planning and Tracking of Bilinear Systems Using Orthogonal Functions

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Abstract: This paper proposes a trajectory planning and tracking approach for bilinear systems that approximate weakly nonlinear systems, based on orthogonal functions and especially the use of operational integration and product matrices. These operational tools allow the conversion of a bilinear differential state equation into an algebraic one depending on initial and final conditions. Arranging and solving the obtained algebraic equation lead to an open loop control law that allows the planning of a system trajectory. The parameters setting of the tracking state feedback closed loop control is yielded by considering a reference model characterizing the desired performances.

Keywords: *bilinear systems; trajectory planning; orthogonal functions; trajectory tracking.*

Mathematics Subject Classification (2000): 42C05, 05E35, 93B52, 93C10.

1 Introduction

Trajectory planning and tracking are linked subjects. Indeed, trajectory planning is finding an open loop control that permits to reach a final fixed state from a known initial state, and tracking is designing a closed loop control that ensures stability of system round its planned trajectory. These subjects have been considered by different approaches for stationary linear systems and particular classes of nonlinear systems [1]–[4]. Orthogonal functions were used as a powerful tool for systems study, identification [5, 6] and control

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[7, 8]. For this purpose, different orthogonal functions were used as Walsh [9] and Block-pulse [10] functions as well as Laguerre [11], Chebychev [12], Hermite [13] and Legendre polynomials [14]. The projection of the system differential equation on an orthogonal basis leads to an algebraic system representation that turns out to be more convenient for equation resolution especially for bilinear systems. In this work, we start by pointing out that a weakly nonlinear systems can be approximated by a bilinear system [15], and then we propose to use orthogonal functions properties with the aim to turn away the integration difficulty caused by trajectory planning and tracking for bilinear systems. We will point out that the algebraic form of system obtained by the orthogonal basis approximation and the use of tools offered by orthogonal functions such as operational matrix of integration and of product makes possible the characterization of a planned system trajectory and the synthesis of tracking state feedback control.

2 Bilinear Approximation of Weakly Nonlinear Systems

Consider a nonlinear system described by the following state equation

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $f \in \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function with initial condition $x(0) = x_0$,$u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $B, C \in \mathbb{R}^n$ are constant vectors.

The system (1) can be linearized around an operating point (u_{op}, x_{op}, y_{op}) as

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t), \\ \tilde{y}(t) &= C\tilde{x}(t), \end{aligned} \tag{2}$$

where $\tilde{x} = x - x_{op}$, $\tilde{u} = u - u_{op}$, $\tilde{y} = y - y_{op}$ and $A = \frac{\partial f}{\partial u}|_{x=\tilde{x}}$. The matrix A can be also approached by means of an identification method [16].

The main inconvenience of the obtained linear model that describes the original nonlinear plant is its availability in a limited domain around the operating point. In order to simplify the nonlinear model in a large region, one may look for a bilinear model. In fact, the bilinear structure of dynamical system constitutes a medium structure between the complex nonlinear model and the simple linear one. It represents a good compromise between the simplicity and complexity of dynamical models. It is complex enough to preserve the nonlinear properties of the original system and it is simple enough to recall the linear representation. The bilinear model can be written in the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Nx(t)u(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{3}$$

The bilinearization of a nonlinear plant can be led by different techniques as the determination of A, B, N and C matrices by identification method [16]. Another known technique is the Carlemen bilinearization [15]. This technique is based on the development of the analytic function $f(.)$ in a polynomial form:

$$f(x) = A_1x^{[1]} + A_2x^{[2]} + A_3x^{[3]} + \cdots + A_rx^{[r]}, \tag{4}$$

where $x^{[i]}$ is the i-th Kronecker power of the vector x. Then the nonlinear system (1) with the polynomial approximation (4) can be bilinearized as

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{N}\hat{x}(t)u(t) + \hat{B}u(t), \\ y(t) &= \hat{C}\hat{x}(t), \end{aligned} \tag{5}$$

where $\hat{x}(t) = \begin{bmatrix} x^{(1)T} & x^{(2)T} & \cdots & x^{(r)T} \end{bmatrix}^T$ and $\hat{A}$, $\hat{B}$, $\hat{N}$, $\hat{C}$ are constant matrices, which can be expressed by A_n, B and C. $\hat{A}$ and $\hat{N}$ are square matrices of dimension $n + n^2 + \cdots + n^k$. $\hat{x}$, $\hat{B}$ and $\hat{C}$ are vectors with $n + n^2 + \cdots + n^k$ components.

As example, in particular case where $r = 3$ and $n = 1$ one has

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ 0 & \hat{A}_{22} & \hat{A}_{23} \\ 0 & 0 & \hat{A}_{33} \end{bmatrix}, \tag{6}$$

where $\hat{A}_{11} = A_1$, $\hat{A}_{12} = A_2$, $\hat{A}_{13} = A_3$, $\hat{A}_{22} = A_1 \otimes I_n + I_n \otimes A_1$, $\hat{A}_{23} = A_2 \otimes I_n + I_n \otimes A_2$, $\hat{A}_{33} = A_1 \otimes I_{n^2} + I_n \otimes A_1 \otimes I_n + I_{n^2} \otimes A_1$,

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \\ 0 \end{bmatrix} \text{ with } \hat{B}_1 = B, \tag{7}$$

$$\hat{N} = \begin{bmatrix} 0 & 0 & 0 \\ \hat{N}_{21} & 0 & 0 \\ 0 & \hat{N}_{32} & 0 \end{bmatrix}, \tag{8}$$

where $\hat{N}_{21} = B \otimes I_n + I_n \otimes B$, $\hat{N}_{22} = B \otimes I_{n^2} + I_n \otimes B \otimes I_n + I_{n^2} \otimes B$.

In the next section we will consider the class of bilinear system having the same representation as (5) for trajectory planning.

3 Proposed Approach for Trajectory Planning

3.1 Orthogonal functions

Consider a set of orthogonal functions $\Phi = \{\varphi_i(t), i \in \mathbb{N}\}$ defined on $[a, b] \subset \mathbb{R}$. The key idea is that all analytical function $f(t)$ absolutely integrable can be developed as follows

$$f(t) = \sum_{i=0}^{\infty} f_i \varphi_i(t), \ \forall t \in [a, b], \tag{9}$$

where the coefficients f_i are constant and given by

$$f_i = \frac{1}{r_i} \int_a^b w(x)\varphi_i(x)f(x)dx, \ \forall i \in \mathbb{N}. \tag{10}$$

To obtain practice function approximation, the projection (9) is shorten to an order N, such that:

$$f(t) \cong \sum_{i=0}^{N-1} f_i \varphi_i(t) = F_N^T \Phi_N(t), \tag{11}$$

where $F_N = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-1}^T \end{bmatrix}$ is a constant coefficient vector and $\Phi_N(t) = \begin{bmatrix} \varphi_0(t) & \varphi_1(t) & \cdots & \varphi_{n-1}(t)^T \end{bmatrix}$ is the vector composed by N orthogonal functions. The projection of a matrix $A(t) = [a_{ij}(t)]$ on the basis of the orthogonal functions is given by

$$A(t) \cong \sum_{i=0}^{N-1} A_{Ni}\varphi_i(t), \tag{12}$$

where $A_{Ni} \in \mathbb{R}^{n\times m}$ are constant matrices. More than approximation (12), orthogonal functions offers different operational tools like the operational matrix of integration and the operational matrix of product which are used for solving differential equations. The operational matrix of integration is the constant matrix $P_N \in \mathbb{R}^{N\times N}$ verifying:

$$\int_0^t \Phi_N(t)dt \cong P_N\Phi_N(t), \tag{13}$$

and the operational matrix of product M_{iN} is defined such that one has

$$\varphi_i(t)\Phi_N(t) \cong M_{iN}(V)\Phi_N(t) \tag{14}$$

with

$$M_{iN} = \begin{bmatrix} K_{0i} & K_{1i} & \cdots & f_{n-1,i} \end{bmatrix}, \tag{15}$$

where $\forall i, j \in \{0, 1, \cdots, N-1\}$, $\varphi_i(t)\varphi_j(t) \cong K_{ij}^T\Phi_N(t)$ Thus the following operational relation holds for any constant vector $V \in \mathbb{R}^n$ [16]:

$$\Phi_N(t)\Phi_N^T(t) \cong M_N(V)\Phi_N(t), \tag{16}$$

where $M_N(V) = \begin{bmatrix} M_{0N}V & \vdots & M_{1N}V & \vdots & \cdots & \vdots & M_{(N-1)N}V \end{bmatrix}$.

3.2 Proposed trajectory planning approach

Consider a bilinear system having the following state representation

$$\begin{aligned} \dot{x} &= Ax + Bu + \sum_{i=0}^{m} A_i x u_i, \\ y &= Cx, \end{aligned} \tag{17}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$. We intend to determine, using orthogonal functions as approximation tool, an open loop control that permits system (17) going to fixed final state $x(T)$ starting from known initial state $x(0)$. The projection of state variables of system (17) on a set of orthogonal functions $\{\varphi_i(t), i = 0, \cdots, N-1\}$ with a truncation of order N allows to write:

$$x(t) = x_N\Phi_N(t), \tag{18}$$

$$u_i(t) = u_{iN}\Phi_N(t), \tag{19}$$

$$u(t) = u_N\Phi_N(t), \tag{20}$$

and the state representation (17) can be put in the following approximated form

$$\dot{x} = Ax_N\Phi_N(t) + Bu_N\Phi_N(t) + \sum_{i=0}^{m} A_i x_N u_{iN}\Phi_N(t). \tag{21}$$

On the other hand, substituting the initial state $x(0)$ by its projection on the orthogonal basis $\Phi_N(t)$:

$$x(0) = x_{N,0}\Phi_N(t), \tag{22}$$

where $x_{N,0} = \begin{bmatrix} x(0) & \vdots & 0 & \vdots & \dots & \vdots & 0 \end{bmatrix}$ and integrating equation (21) between an initial time ($t_0 = 0$) and a time t and making use of the operational integration and product proprieties(13) and (14) one obtains

$$(x_N - x_{N,0}) = Ax_N P_N + Bu_N P_N + \sum_{i=0}^{m} A_i x_N M_N(u_{iN})P_N. \tag{23}$$

By using the Vec operator and its main following property [17]

$$Vec(ABC) = (C^T \otimes A)Vec(B), \tag{24}$$

the equation (23) yields the following relation

$$\begin{aligned} Vec(x_N) = \Big[I_{nN} - (P_N^T \otimes A) - \sum_{i=0}^{m} P_N^T M_N^T(u_{iN}) \otimes A_i\Big]^{-1} \\ \left[(P_N^T \otimes B)Vec(u_N) + Vec(x_{N,0})\right]. \end{aligned} \tag{25}$$

Integrating again relation (21) between instant t and final time T and replacing $x(T)$ by its projection on the orthogonal basis:

$$x(T) = x_{N,T}\Phi_N(t) \tag{26}$$

with $x_{N,T} = \begin{bmatrix} x(T) & \vdots & 0 & \vdots & \dots & \vdots & 0 \end{bmatrix}$ and using the fact that the orthogonal basis vector at final instant T verifies: $\Phi_N(t) = K_N\Phi_N(t)$ one obtains

$$(x_{N,T} - x_N) = Ax_N P_N(K_N - I_N) + Bu_N P_N(K_N - I_N) + \sum_{i=0}^{m} A_i x_N M_N(u_{iN})P_N(K_N - I_N), \tag{27}$$

putting $\Pi_N = P_N(K_N - I_N)$ and applying Vec operator yield:

$$\begin{aligned} Vec(x_N) = \left[I_{nN} + (\Pi_N^T \otimes A) + \textstyle\sum_{i=0}^{m} \Pi_N^T M_N^T(u_{iN}) \otimes A_i\right]^{-1} \\ \left[Vec(x_{N,T}) - (\Pi_N^T \otimes B)Vec(u_N)\right]. \end{aligned} \tag{28}$$

By equalizing (25) and (28) one obtains the following relation

$$H_N^{-1}\left[(P_N^T \otimes B)Vec(u_N) + Vec(x_{N,0}\right] = G_N^{-1}\left[Vec(x_{N,T} - (\Pi_N^T \otimes B)\right], \tag{29}$$

where

$$H_N = H_N(u_N) = I_{nN} - R_u, \tag{30}$$

$$R_u = (P_N^T \otimes A) + \sum_{i=0}^{m} P_N^T M_N^T(u_{iN}) \otimes A_i, \tag{31}$$

$$G_N = I_{nN} + (\Pi_N^T \otimes A) + \sum_{i=0}^{m} \Pi_N^T M_N^T(u_{iN}) \otimes A_i, \tag{32}$$

By substitution of Π_N by its expression $\Pi_N = P_N(K_N - I_N)$, one has

$$G_N = H_N + (K_N^T \otimes I_N)R_u, \tag{33}$$

and the relation (29) becomes:

$$\begin{aligned}&\left[I_{nN} - (K_N^T \otimes I_N)\right]\left[(P_N^T \otimes B)Vec(u_N) + Vec(x_{N,0})\right] + \\ &(K_N^T \otimes I_N)H_N^{-1}\left[(P_N^T \otimes B)Vec(u_N) + Vec(x_{N,0}\right] \\ &= (I_{nN} - K_N^T \otimes I_N)\left[(P_N^T \otimes B)Vec(u_N)\right] + Vec(x_{N,T})\end{aligned} \tag{34}$$

Let's put

$$Z(u_N) = H_N^{-1}(u_N)\left[(P_N^T \otimes B)Vec(u_N) + Vec(x_{N,0})\right], \tag{35}$$

$$\Gamma(x_{N,0}, x_{N,T}) = (K_N^T \otimes I_{nN})Vec(x_{N,0}) + Vec(x_{N,T}), \tag{36}$$

the relation (34) yields

$$(K_N^T \otimes I_N)Z(u_N) = \Gamma(x_{N,0}, x_{N,T}). \tag{37}$$

The planning open loop control is then derived by minimizing with respect to u_N the norm of the difference between the two parts of equality (37):

$$\zeta = \|(K_N^T \otimes I_N)Z(u_N) - \Gamma(x_{N,0}, x_{N,T})\|. \tag{38}$$

This minimization can be led using the Matlab optimization toolbox.

4 Trajectory Tracking: Closed Loop Control

Consider the difference variables

$$\delta x = x(t) - x_p(t), \quad \delta u = u(t) - u_p(t) \tag{39}$$

between the trajectory of system (17) and a planned trajectory $(x_p(t), u_p(t))$ verifying the system equation

$$\dot{x}_p = Ax_p + Bu_p + \sum_{i=0}^{m} A_i x_p u_{ip}, \tag{40}$$

the state equation of difference system can be written as

$$\delta\dot{x} = (A + \sum_{i=0}^{m} A_i u_{ip})\delta x(t) + (B + \sum_{i=0}^{m} A_i x_p)\delta u(t) + \sum_{i=0}^{m} A_i \delta x \delta u_i, \tag{41}$$

by neglecting the product term $\delta x \delta u_i$ compared with δx and δu_i, the state equation (41) can be simplified into a linear time variant state equation

$$\delta\dot{x} = \mathcal{A}(t)\delta x(t) + \mathcal{B}(t)\delta u(t), \tag{42}$$

where

$$\mathcal{A}(t) = A + \sum_{i=0}^{m} A_i u_{ip}(t), \quad \mathcal{B}(t) = B + \sum_{i=0}^{m} A_i x_p(t). \tag{43}$$

Our purpose is then to characterize a state feedback control law $\delta u(t) = -K\delta x(t)$ which confers to a controlled LTV (Linear Time Variant) system

$$\delta\dot{x} = (\mathcal{A}(t) - \mathcal{B}(t)K)\delta x(t), \tag{44}$$

desired performances. Such performances can be defined in a convenient linear reference model [18]

$$\delta\dot{x} = E\delta x(t). \tag{45}$$

The expansion of the time variant matrices $\mathcal{A}(t)$, $\mathcal{B}(t)$ and the state vector $\delta x(t)$ into a basis of orthogonal functions as follows

$$\mathcal{A}(t) = \sum_{i=0}^{N-1} \mathcal{A}_{Ni}\varphi_i(t), \tag{46}$$

$$\mathcal{B}(t) = \sum_{i=0}^{N-1} \mathcal{B}_{Ni}\varphi_i(t), \tag{47}$$

$$\delta x(t) = \delta x_N \Phi_N(t), \tag{48}$$

yields the following differential relation

$$\delta\dot{x} = \left[\textstyle\sum_{i=0}^{N-1} \mathcal{A}_{Ni}\varphi(t) - K \sum_{i=0}^{N-1} \mathcal{B}_{Ni}\varphi(t)\right] \delta x_N \Phi_N(t). \tag{49}$$

Integrating equation (49) and making use of operational matrices of integration and product and the Vec operator one obtains:

$$Vec(\delta x_N) - Vec(\delta x_{Np}) = (\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \mathcal{A}_{Ni} - k(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \mathcal{B}_{Ni})Vec(\delta x_N). \tag{50}$$

A similar development for the reference model (45) yields:

$$Vec(\delta x_{N,r}) - Vec(\delta x_{N,0}) = (P_N^T \otimes E)Vec(\delta x_N). \tag{51}$$

The equalization of $Vec(\delta x_N)$ coming from (49) and $Vec(\delta x_{N,r})$ derived from (50) allows to have the following linear algebraic equation where unknown is the feedback control gain K:

$$\phi K = \psi \tag{52}$$

with

$$\phi = \sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \mathcal{B}_{Ni}, \quad \psi = \sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \mathcal{A}_{Ni} - (P_N^T \otimes E). \tag{53}$$

Solving equation (52) by using least squares method leads to a closed loop control feedback law $\delta u(t) = -K\delta x(t)$ that ensures trajectory tracking for bilinear system (17). Note that the development (44) until (51) can be easily extended to look for a time variant feedback control law $\delta u(t) = -K(t)\delta x(t)$ where the time variant gain $K(t)$ can be determined as an expansion of orthogonal functions: $K(t) = \sum_{i=0}^{N-1} K_{Ni}\varphi_i(t)$.

5 Illustrating Example

In this section we present the implementation of the proposed approach for trajectory planning and tracking of the bilinear system described by the following equations

$$\begin{aligned} \dot{x}_1 = x_1 - x_2 u + u, \quad \dot{x}_2 = -x_2 - x_1 u - u, \\ y = x_2. \end{aligned} \tag{54}$$

A state representation of this system is the following

$$\begin{aligned} \dot{x} = Ax + Nxu + Bu, \\ y = Cx, \end{aligned} \tag{55}$$

with $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.
The application of the proposed planning approach based on modified Legendre orthogonal functions with a truncation order $N = 16$, for system (55) starting from an initial state $x_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ at initial time $t_0 = 0s$ to the final state $x_T = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ at final time $T = 10s$, yields the planned trajectories $x_{1p}(t)$ and $x_{2p}(t)$ and planning input $u_p(t)$ presented in Figure 1.

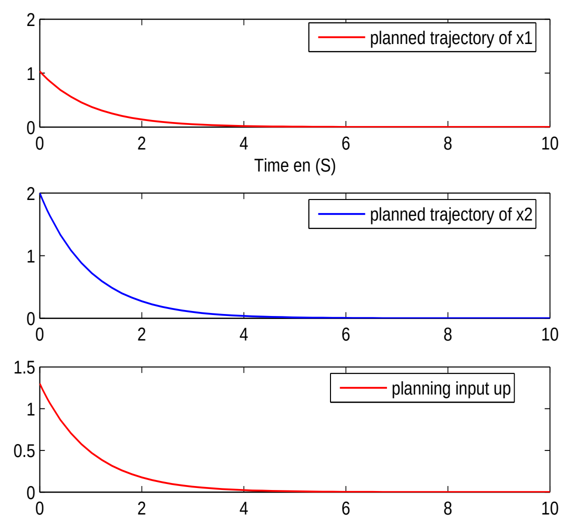

Figure 1: Trajectory planning and planning input.

These simulations show that the use of orthogonal approximation method yields an open loop control that allows to the bilinear system (55) to reach the fixed final state x_T starting from a chosen initial state x_0. Note that initial and final conditions can be modified and one obtains then an open loop control that yields another trajectory planning. The tracking of the obtained trajectory $(x_p(t), u_p(t))$ is given by the application of the orthogonal approximation to the following LTV system

$$\delta\dot{x} = \mathcal{A}(t)\delta x(t) + \mathcal{B}(t)\delta u(t) \tag{56}$$

with $\mathcal{A}(t) = A + Nu_p(t)$ and $\mathcal{B}(t) = B + Nx_p(t)$ derived from the linearization of the bilinear system round the planned trajectory.

The synthesis of the closed loop control $\delta u(t) = -K\delta x(t)$ that ensures the tracking of the planned trajectory is based on the following linear reference system that confers to the state feedback controlled LTV system the desired performances corresponding to the linear reference model:

$$\delta \dot{x} = E\delta x(t), \tag{57}$$

where $E = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. Figure 2 shows the effect of the obtained control law on the tracking of the planned trajectory affected by two instantenous disturbances at $t_1 = 3s$ and $t_2 = 15s$.

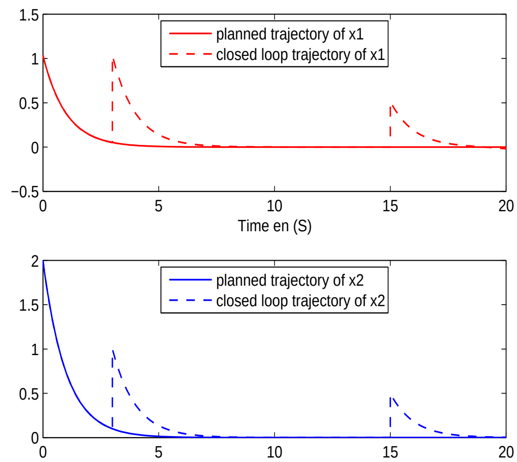

Figure 2: Trajectory tracking.

It appears that the designed control law ensures stability of the system around its planned trajectory. Note that the performances of the closed loop controlled system can be modified by choosing another linear reference model.

6 Conclusion

In this paper, a new approach has been introduced for trajectory planning and tracking of bilinear systems, which approximate weakly nonlinear systems, by using orthogonal functions as a tool of approximation. The presented method was applied to a class of bilinear invariant systems. The use of operational matrices of integration and product in planning problem has allowed the transformation of the system differential equation into an algebraic one depending on the control variable and the initial and final states. For trajectory tracking, this technique has allowed the synthesis of a closed feedback control which ensures for the considered system the performances of a prespecified reference model. Note that the proposed approach can be extended to other classes of systems such as time variant bilinear systems and affine control nonlinear systems.

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Multi-Point Boundary Value Problems on Time Scales

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Abstract: In this paper, we are interested in the existence of at least one,two and three positive solutions of a nonlinear second-order m-point boundary value problem on time scales by using fixed point theorems in cones. As an application, some examples are included to demonstrate the main results.

Keywords: *boundary value problems; cone; fixed point theorems; positive solutions; time scales.*

Mathematics Subject Classification (2000): 34B18, 39A10.

1 Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [7, 8]. Motivated by the study of Il'in and Moiseev [7, 8], Gupta [5] studied certain three-point boundary value problems for nonlinear ordinary differential equations. For the existence problems of positive solutions of multi-point boundary value problems on time scales, some authors have obtained many results in recent years, see [6, 9, 10, 12, 13, 14, 15, 16, 18] and the references therein.

Motivated by [17], in this paper, we are interested in the existence of multiple positive solutions of the following m-point boundary value problem (BVP)

$$\begin{cases} u^{\Delta\nabla}(t) + h(t)f(t, u(t)) = 0, \;\; t \in [t_1, t_m] \subset \mathbb{T}, \\ u^{\Delta}(t_m) = 0, \quad \alpha u(t_1) - \beta u^{\Delta}(t_1) = \sum\limits_{i=2}^{m-1} u^{\Delta}(t_i), \quad m \geq 3, \end{cases} \tag{1}$$

where $\mathbb{T}$ is a time scale, $0 \leq t_1 < \ldots < t_{m-1} < t_m$, $\alpha > 0$ and $\beta \geq 0$ are given constants. Some basic definitions and theorems on time scales can be found in the books [2, 3].

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The rest of paper is arranged as follows. In Section 2, we give several lemmas to prove the main results in this paper. In Section 3, we first establish the existence results of solutions of the BVP (1) as a result of Schauder fixed-point theorem. Second, we use Krasnosel'skii fixed-point theorem to show the existence of a positive solution for the BVP (1). Third, we apply the Avery-Henderson fixed-point theorem to prove the existence of at least two positive solutions to the BVP (1). Finally, we establish criteria for the existence of at least three positive solutions of the BVP (1) by using Legget-Williams fixed-point theorem. In Section 4, we give two examples to illustrate our results.

2 Preliminaries

We now state and prove several lemmas which are needed later. These lemmas are based on the linear BVP

$$\begin{cases} u^{\Delta\nabla}(t) + y(t) = 0, \;\; t \in [t_1, t_m] \subset \mathbb{T}, \\ u^{\Delta}(t_m) = 0, \quad \alpha u(t_1) - \beta u^{\Delta}(t_1) = \sum\limits_{i=2}^{m-1} u^{\Delta}(t_i), \quad m \geq 3. \end{cases} \tag{2}$$

Lemma 2.1 *Let $\alpha \neq 0$ and $y \in C_{ld}[t_1, t_m]$. Then the BVP (2) has the unique solution*

$$u(t) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s + \int_{t}^{t_m} (t - s) y(s) \nabla s. \tag{3}$$

Proof From $u^{\Delta\nabla}(t) + y(t) = 0$, we have

$$u(t) = u(t_m) + u^{\Delta}(t_m)(t_m - t) + \int_{t}^{t_m} (t - s) y(s) \nabla s.$$

By using the boundary conditions, we get

$$\alpha u(t_m) + \alpha \int_{t_1}^{t_m} (t_1 - s) y(s) \nabla s - \beta \int_{t_1}^{t_m} y(s) \nabla s = \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s.$$

Since

$$u(t_m) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s,$$

we obtain

$$u(t) = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s + \int_{t}^{t_m} (t - s) y(s) \nabla s. \;\square$$

Lemma 2.2 *If $\alpha > 0$, $\beta \geq 0$ and $y \in C_{ld}([t_1, t_m], [0, \infty))$, then the unique solution u of the BVP (2) given in (3) satisfies*

$$u(t) \geq 0, \;\; t \in [t_1, t_m] \subset \mathbb{T}.$$

Proof Since $u(t)$ is increasing on $[t_1, t_m]$, we know that if $u(t_1) \geq 0$, then $u(t) \geq 0$ for $t \in [t_1, t_m]$.

$$\begin{aligned} u(t_1) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s + \int_{t_1}^{t_m} (t_1 - s) y(s) \nabla s \\ &= \frac{\beta}{\alpha} \int_{t_1}^{t_m} y(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} y(s) \nabla s \\ &\geq 0. \end{aligned}$$

Hence the result holds. □

Lemma 2.3 *If $\alpha > 0$, $\beta \geq 0$ and $y \in C_{ld}([t_1, t_m], [0, \infty))$, then the unique solution to the BVP* (2) *satisfies*

$$u(t) \geq \frac{t - t_1}{t_m - t_1} \|u\|, \quad t \in [t_1, t_m] \subset \mathbb{T}, \tag{4}$$

where $\|u\| = \sup\limits_{t \in [t_1, t_m]} |u(t)|$.

Proof From the fact that $u(t)$ is increasing on $[t_1, t_m]$, we have $\|u\| = \sup\limits_{t \in [t_1, t_m]} |u(t)| = u(t_m)$. Let

$$g(t) = u(t) - \frac{t - t_1}{t_m - t_1} \|u\|, \quad t \in [t_1, t_m] \subset \mathbb{T}. \tag{5}$$

Since $g^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) = -y(t) \leq 0$, we know that the graph of g is concave on $[t_1, t_m] \subset \mathbb{T}$. We get

$$g(t_1) = u(t_1) \geq 0$$

and

$$g(t_m) = 0.$$

From the concavity of g,

$$g(t) \geq 0 \text{ for } t \in [t_1, t_m] \subset \mathbb{T}. \tag{6}$$

From (5) and (6), we obtain

$$u(t) \geq \frac{t - t_1}{t_m - t_1} \|u\| \text{ for } t \in [t_1, t_m] \subset \mathbb{T}. \quad \square$$

We assume the following hypotheses:

(H1) $h \in C_{ld}([t_1, t_m], [0, \infty))$ and there exists $t_0 \in [t_1, t_m]$ such that $h(t_0) > 0$.

(H2) $f : [t_1, t_m] \times [0, \infty) \to [0, \infty)$ is continuous such that $f(t, .) > 0$ on any subset of $\mathbb{T}$ containing t_0.

The solutions of the BVP (1) are the fixed points of the operator A defined by

$$\begin{aligned} Au(t) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\ &+ \int_{t}^{t_m} (t - s) h(s) f(s, u(s)) \nabla s. \end{aligned}$$

3 Existence of Solutions

To prove the existence of at least one solution for the BVP (1), we will apply the following Schauder Fixed Point Theorem: *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ be a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator A leaves the set $\mathcal{S}$ invariant, i.e. if $A(\mathcal{S}) \subset \mathcal{S}$, then A has at least one fixed point in $\mathcal{S}$.*

Let $\mathcal{B}$ denote the Banach space $C_{ld}[t_1, t_m]$ with the norm $\|u\| = \sup\limits_{t\in[t_1,t_m]} |u(t)|$.

Theorem 3.1 *Assume* (H1) and (H2) *are satisfied, $\alpha > 0$ and $\beta \geq 0$. Let there exists a number $r > 0$ such that*

$$\max_{\|u\|\leq r} |f(t, u)| \leq \frac{1}{k_1} u$$

for $t \in [t_1, t_m]$, where

$$k_1 = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)\nabla s.$$

Then the m-point BVP (1) *has at least one solution $u(t)$.*

Proof Let $\mathcal{S} = \{u \in \mathcal{B} : \|u\| \leq r\}$. Obviously, $\mathcal{S}$ is closed, bounded and convex subset of $\mathcal{B}$. Define $A : \mathcal{S} \to \mathcal{B}$ by

$$\begin{aligned} Au(t) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s, u(s))\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)f(s, u(s))\nabla s \\ &+ \int_{t}^{t_m} (t - s)h(s)f(s, u(s))\nabla s. \end{aligned}$$

for $t \in [t_1, t_m]$. Now, we will show that $A : \mathcal{S} \to \mathcal{S}$. If $u \in \mathcal{S}$,

$$\begin{aligned} \|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s, u(s))\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)f(s, u(s))\nabla s \\ &\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)\frac{1}{k_1}u(s)\nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s)\frac{1}{k_1}u(s)\nabla s \\ &\leq \|u\| \leq r. \end{aligned}$$

for every $t \in [t_1, t_m]$. Since $\|Au\| \leq r$, we have $A(\mathcal{S}) \subset \mathcal{S}$. Further, the operator A is completely continuous. Hence, A has at least one fixed point in $\mathcal{S}$ by Schauder fixed point theorem. Since the solutions of problem (1) are fixed points of operator A, the BVP (1) has at least one solution $u(t)$. □

We will need also the following (Krasnosel'skii) fixed point theorem [15] to prove the existence of at least one positive solution for the BVP (1).

Theorem 3.2 [4] *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 3.3 *Assume* (H1), (H2) *hold, and $\alpha > 0$, $\beta \geq 0$. In addition, let there exist numbers $0 < r < R < \infty$ such that*

$$f(s,u) \leq \frac{1}{k_1} u, \text{ if } 0 \leq u \leq r, \ \ s \in [t_1, t_m]$$

and

$$f(s,u) \geq \frac{t_m - t_1}{k_2(t_{m-1} - t_1)} u, \text{ if } R \leq u < \infty, \ \ s \in [t_{m-1}, t_m],$$

where

$$k_1 = \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) \nabla s$$

and

$$k_2 = \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \nabla s.$$

Then the BVP (1) has at least one positive solution.

Proof Define the cone $P \subset \mathcal{B}$ by

$$P = \{u \in \mathcal{B} : \ u \text{ is concave}, u(t) \geq 0 \text{ and } u^{\Delta}(t_m) = 0\}. \tag{7}$$

From (H1), (H2), Lemma 2.2 and Lemma 2.3, we have $AP \subset P$. Also it is easy to obtain that $A : P \to P$ is completely continuous. If $u \in P$ with $\|u\| = r$, then we get

$$\begin{aligned} \|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\ &\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) \frac{1}{k_1} u(s) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) \frac{1}{k_1} u(s) \nabla s \\ &\leq \|u\|. \end{aligned}$$

Thus, we have $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$, where $\Omega_1 := \{u \in C_{ld}([t_1, t_m], \mathbb{R}) : \|u\| < r\}$.

Let us now define

$$\Omega_2 := \{u \in C_{ld}([t_1, t_m], \mathbb{R}) : \|u\| < \frac{t_m - t_1}{t_{m-1} - t_1} R\}.$$

If $u \in P \cap \partial\Omega_2$, from (4)

$$u(t) \geq u(t_{m-1}) \geq \frac{t_{m-1} - t_1}{t_m - t_1}\|u\| = R, \quad t \in [t_{m-1}, t_m]$$

and so

$$\begin{aligned}
\|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s \\
&+ \frac{1}{\alpha}[(\int_{t_2}^{t_3} h(s) f(s, u(s)) \nabla s + \ldots + \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s) \\
&+ (\int_{t_3}^{t_4} h(s) f(s, u(s)) \nabla s + \ldots + \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s) + \ldots \\
&+ \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s] \\
&\geq \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s \\
&\geq \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \frac{t_m - t_1}{k_2(t_{m-1} - t_1)} u(s) \nabla s \\
&\geq \|u\|.
\end{aligned}$$

Hence, $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. By the first part of Theorem 3.2, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \leq \|u\| \leq \frac{t_m - t_1}{t_{m-1} - t_1} R$. Therefore, the BVP (1) has at least one positive solution. □

Now, we apply the following (Avery-Henderson) fixed point theorem [1] to prove the existence of at least two positive solutions to the nonlinear m-point BVP (1).

Theorem 3.4 [1] *Let P be a cone in a real Banach space E. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}.$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants r and M,

$$\phi(u) \leq \theta(u) \leq \eta(u) \text{ and } \|u\| \leq M\phi(u)$$

for all $u \in \overline{P(\phi, r)}$. *Suppose that there exist positive numbers* $p < q < r$ *such that*

$$\theta(\lambda u) \leq \lambda\theta(u), \text{ for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, q).$$

If $A : \overline{P(\phi, r)} \to P$ *is a completely continuous operator satisfying*

(*i*) $\phi(Au) > r$ *for all* $u \in \partial P(\phi, r)$,

(*ii*) $\theta(Au) < q$ *for all* $u \in \partial P(\theta, q)$,

(*iii*) $P(\eta, p) \neq \emptyset$ *and* $\eta(Au) > p$ *for all* $u \in \partial P(\eta, p)$,

then A *has at least two fixed points* u_1 *and* u_2 *such that*

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \text{ and } q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Define the constants

$$M := \Big(\int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1) h(s) \nabla s \Big)^{-1} \tag{8}$$

and

$$N := \Big(\int_{t_1}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) \nabla s \Big)^{-1}. \tag{9}$$

Theorem 3.5 *Assume* (H1), (H2) *hold and* $\alpha > 0$, $\beta \geq 0$. *Suppose there exist numbers* $0 < p < q < r$ *such that the function* f *satisfies the following conditions:*

(*i*) $f(s, u) > rM$ *for* $s \in [t_{m-1}, t_m]$ *and* $u \in [r, \frac{r(t_m - t_1)}{t_{m-1} - t_1}]$,

(*ii*) $f(s, u) < qN$ *for* $s \in [t_1, t_m]$ *and* $u \in [0, \frac{q(t_m - t_1)}{t_{m-1} - t_1}]$,

(*iii*) $f(s, u) > pM$ *for* $s \in [t_{m-1}, t_m]$ *and* $u \in [\frac{p(t_{m-1} - t_1)}{t_m - t_1}, p]$

where N *and* M *are defined in* (8) *and* (9), *respectively. Then the BVP* (1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$u_1(t_m) > p \text{ with } u_1(t_{m-1}) < q \;\text{ and }\; u_2(t_{m-1}) > q \text{ with } u_2(t_{m-1}) < r.$$

Proof Define the cone P as in (7). From (H1), (H2), Lemma 2.2 and Lemma 2.3, $AP \subset P$ and it is easy to obtain A is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(u) := u(t_{m-1}), \; \theta(u) := u(t_{m-1}), \; \eta(u) := u(t_m).$$

For each $u \in P$, we have

$$\phi(u) = \theta(u) \leq \eta(u)$$

and from (4) we have

$$\|u\| \leq \frac{t_m - t_1}{t_{m-1} - t_1} \phi(u). \tag{10}$$

Moreover, $\theta(0) = 0$ and for all $u \in P$, $\lambda \in [0, 1]$ we get $\theta(\lambda u) = \lambda\theta(u)$. In the following claims, we verify the remaining conditions of Theorem 3.5.

If $u \in \partial P(\phi, r)$, from (10) we have $r = u(t_{m-1}) \leq u(s) \leq \|u\| \leq \frac{r(t_m - t_1)}{t_{m-1} - t_1}$ for $s \in [t_{m-1}, t_m]$. Then using hypothesis (i) and (8), we obtain

$$\begin{aligned}
\phi(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s) h(s) f(s, u(s)) \nabla s \\
&= \int_{t_1}^{t_{m-1}} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} [\int_{t_2}^{t_{m-1}} h(s) f(s, u(s)) \nabla s + \dots \\
&+ \int_{t_{m-2}}^{t_{m-1}} h(s) f(s, u(s)) \nabla s + (m-2) \int_{t_{m-1}}^{t_m} h(s) f(s, u(s)) \nabla s] \\
&+ \int_{t_{m-1}}^{t_m} (\frac{\beta}{\alpha} + t_{m-1} - t_1) h(s) f(s, u(s)) \nabla s \\
&> \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1) h(s) r M \nabla s \\
&= r.
\end{aligned}$$

Thus the condition (i) of Theorem 3.4 holds. Next, we will show that the condition (ii) of Theorem 3.4 is satisfied. If $u \in \partial P(\theta, q)$, then from (10) we have $0 \leq u(s) \leq \|u\| \leq \frac{q(t_m - t_1)}{t_{m-1} - t_1}$ for $s \in [t_1, t_m]$. Thus, from hypothesis (ii) and (9) we get

$$\begin{aligned}
\theta(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\
&+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s) h(s) f(s, u(s)) \nabla s \\
&\leq \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} (m-2) \int_{t_1}^{t_m} h(s) f(s, u(s)) \nabla s \\
&< \int_{t_1}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1) h(s) q N \nabla s \\
&= q.
\end{aligned}$$

So condition (ii) of Theorem 3.4 holds. Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $u \in \partial P(\eta, p)$, from (4) we have $\frac{p(t_{m-1} - t_1)}{t_m - t_1} \leq u(t_{m-1}) \leq u(s) \leq \|u\| = p$ for $s \in [t_{m-1}, t_m]$.

Hence, we obtain

$$\begin{aligned}\eta(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1) h(s) f(s, u(s)) \nabla s + \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_i}^{t_m} h(s) f(s, u(s)) \nabla s \\ &> \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1) h(s) p M \nabla s \\ &= p\end{aligned}$$

using hypothesis (iii) and (8). Since all the conditions of Theorem 3.4 are satisfied, the m-point BVP (1) has at least two positive solutions u_1 and u_2 such that

$$u_1(t_m) > p \text{ with } u_1(t_{m-1}) < q \;\text{ and }\; u_2(t_{m-1}) > q \text{ with } u_2(t_{m-1}) < r. \;\square$$

Now, we will use the following (Legget-Williams) fixed point theorem [11] to prove the existence of at least three positive solutions to the nonlinear BVP (1).

Theorem 3.6 [11] *Let P be a cone in the real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi, a, b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \le \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \le r$ such that the following conditions hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \ne \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi, q, l)$;

(ii) $\|Au\| < p$ for $\|u\| \le p$;

(iii) $\psi(Au) > q$ for $u \in P(\psi, q, r)$ with $\|Au\| > l$,

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

Theorem 3.7 *Assume* (H1), (H2) *hold and $\alpha > 0$, $\beta \ge 0$. Suppose that there exist constants $0 < p < q < \frac{q(t_m - t_1)}{t_{m-1} - t_1} \le r$ such that the function f satisfies the following conditions:*

(i) $f(s, u) \le rN$ for $s \in [t_1, t_m]$ and $u \in [0, r]$;

(ii) $f(s, u) > qM$ for $s \in [t_{m-1}, t_m]$ and $u \in [q, \frac{q(t_m - t_1)}{t_{m-1} - t_1}]$;

(iii) $f(s, u) < pN$ for $s \in [t_1, t_m]$ and $u \in [0, p]$.

Then the BVP (1) *has at least three positive solutions u_1, u_2 and u_3 satisfying*

$$u_1(t_m) < p,\ u_2(t_{m-1}) > q,\ u_3(t_m) > p \text{ with } u_3(t_{m-1}) < q.$$

Proof We will show that the conditions of Theorem 3.6 are satisfied. For this purpose we first define the nonnegative continuous concave functional $\psi : P \to [0, \infty)$ to be

$\psi(u) := u(t_{m-1})$, the cone P as in (7), M as in (8) and N as in (9). We have $\psi(u) \le \|u\|$ for all $u \in P$. If $u \in \overline{P_r}$, then $0 \le u \le r$ and from the hypothesis (i), we get

$$\begin{aligned}
\|Au\| &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s,u(s))\nabla s + \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_i}^{t_m} h(s)f(s,u(s))\nabla s \\
&\le \int_{t_1}^{t_m} (\frac{\beta + m - 2}{\alpha} + s - t_1)h(s)rN\nabla s \\
&= r.
\end{aligned}$$

Thus, we obtain $A : \overline{P_r} \to \overline{P_r}$. Similarly, if $u \in \overline{P_p}$, then the hypothesis (iii) yields $f(s,u(s)) < pN$ for $s \in [t_1,t_m]$. Just as above, we have $A : \overline{P_p} \to P_p$. It follows that condition (ii) of Theorem 3.6 is satisfied.

Since $\frac{q(t_m-t_1)}{t_{m-1}-t_1} \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1})$ and $\psi(\frac{q(t_m-t_1)}{t_{m-1}-t_1}) = \frac{q(t_m-t_1)}{t_{m-1}-t_1} > q$, $\{u \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1}) : \psi(u) > q\} \neq \emptyset$. For all $u \in P(\psi, q, \frac{q(t_m-t_1)}{t_{m-1}-t_1})$, we get $q \le u(t_{m-1}) \le u(s) \le \|u\|$ for $s \in [t_{m-1}, t_m]$. Using the assumption (ii), we obtain

$$\begin{aligned}
\psi(Au) &= \int_{t_1}^{t_m} (\frac{\beta}{\alpha} + s - t_1)h(s)f(s,u(s))\nabla s + \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_i}^{t_m} h(s)f(s,u(s))\nabla s \\
&+ \int_{t_{m-1}}^{t_m} (t_{m-1} - s)h(s)f(s,u(s))\nabla s \\
&> \int_{t_{m-1}}^{t_m} (\frac{\beta + m - 2}{\alpha} + t_{m-1} - t_1)h(s)qM\nabla s \\
&= q.
\end{aligned}$$

Hence, the condition (i) of Theorem 3.6 holds.

For the condition (iii) of Theorem 3.6, we suppose that $u \in P(\psi, q, r)$ with $\|Au\| > \frac{q(t_m-t_1)}{t_{m-1}-t_1}$. Then, from (4) we obtain

$$\psi(Au) = Au(t_{m-1}) \ge \frac{t_{m-1} - t_1}{t_m - t_1}\|Au\| > q.$$

Since all conditions of the Legget-Williams fixed point theorem are satisfied, the nonlinear BVP (1) has at least three positive solutions u_1, u_2 and u_3 such that

$$u_1(t_m) < p,\ u_2(t_{m-1}) > q,\ u_3(t_m) > p \text{ with } u_3(t_{m-1}) < q. \ \Box$$

Using the ideas in the proof of the above problem, we can establish the existence of an arbitrary odd number of positive solutions of (1).

Theorem 3.8 *Assume that* (H1) *and* (H2) *are satisfied and* $\alpha > 0$, $\beta \ge 0$. *Suppose that there exist numbers*

$$0 < p_1 < q_1 < \frac{q_1(t_m - t_1)}{t_{m-1} - t_1} \le p_2 < q_2 < \frac{q_2(t_m - t_1)}{t_{m-1} - t_1} \le p_3 < \ldots \le p_n, \quad n \in \mathbb{N}$$

such that the function f satisfies the following conditions:

(i) $f(s,u) < p_i N$ *for* $s \in [t_1, t_m]$ *and* $u \in [0, p_i]$,

(ii) $f(s,u) > q_i M$ *for* $s \in [t_{m-1}, t_m]$ *and* $u \in [q_i, \frac{q_i(t_m - t_1)}{t_{m-1} - t_1}]$,

where N and M are defined in (8) *and* (9), *respectively. Then the m-point BVP* (1) *has at least $2n-1$ positive solutions.*

The proof of the theorem comes directly from induction. When $n = 1$, we obtain $A : \overline{P_{p_1}} \to P_{p_1} \subset \overline{P_{p_1}}$ from condition (i), which implies that A has at least one fixed point $u_1 \in \overline{P_{p_1}}$ by the Schauder fixed point theorem. When $n = 2$, by Theorem 3.7 we can obtain at least three positive solutions u_2, u_3 and u_4. Following this way, we can obtain that the operator A has $2n-1$ different fixed points by induction.

Remark 3.1 When $m = 3$, our results, i.e. Theorem 3.3, Theorem 3.5 and Theorem 3.7 reduce to Theorem 4, Theorem 5 and Theorem 6 in [12], respectively.

4 Examples

Example 4.1 Let $\mathbb{T} = \{(\frac{1}{5})^n : n \in \mathbb{N}_0\} \cup \{0\}$. We consider the following boundary value problem:

$$\begin{cases} u^{\Delta\nabla}(t) + \frac{29(u+4)^{\frac{1}{20}}}{(u+4)^{\frac{1}{2}}+1} = 0, \ \ t \in [0,1] \subset \mathbb{T}, \\ u^{\Delta}(1) = 0, \quad u(0) - 2u^{\Delta}(0) = u^{\Delta}(\frac{1}{25}) + u^{\Delta}(\frac{1}{5}). \end{cases} \tag{11}$$

Taking $t_1 = 0, t_2 = \frac{1}{25}, t_3 = \frac{1}{5}, t_4 = 1 = \alpha, \beta = 2, m = 4, h(t) = 1$ and $f(t,u) = \frac{29(u+4)^{\frac{1}{20}}}{(u+4)^{\frac{1}{2}}+1}$, we investigate the solvability of this problem by means of Theorem 3.5. By (8) and (9), we obtain $M = \frac{25}{84}$ and $N = \frac{6}{29}$.

If we take $p = 10, q = 17$ and $r = 19$, then $0 < p < q < r$ and the conditions $(i) - (iii)$ of Theorem 3.5 are satisfied. Thus, the BVP (11) has at least two positive solutions u_1 and u_2 satisfying

$$u_1(1) > 10 \text{ with } u_1(\frac{1}{5}) < 17 \text{ and } u_2(\frac{1}{5}) \text{ with } u_2(\frac{1}{5}) < 19.$$

Example 4.2 In problem (11), let $f(t,u) = \frac{2u^2}{(u+1)^2+1}$. If we take $p = 0.27, q = 1$ and $r = 8$ then $0 < p < q < \frac{q(t_m - t_1)}{t_{m-1} - t_1} \le r$ and the conditions $(i) - (iii)$ of Theorem 3.7 are satisfied. According to Theorem 3.7, the BVP

$$\begin{cases} u^{\Delta\nabla}(t) + \frac{2u^2}{(u+1)^2+1} = 0, \ \ t \in [0,1] \subset \mathbb{T}, \\ u^{\Delta}(1) = 0, \quad u(0) - 2u^{\Delta}(0) = u^{\Delta}(\frac{1}{25}) + u^{\Delta}(\frac{1}{5}), \end{cases}$$

has at least three positive solutions u_1, u_2 and u_3 satisfying

$$u_1(1) < 0.27, \quad u_2(\frac{1}{5}) > 1, \ \ u_3(1) > 0.27 \text{ with } u_3(\frac{1}{5}) < 1.$$

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