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# Application of Generalized Hamiltonian Systems to Chaotic Synchronization◇

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**Abstract:** In this paper, a method of chaotic synchronization is introduced which is obtained from the perspective of passivity-based state observer design in the context of Generalized Hamiltonian systems including dissipative and destabilizing vector fields. Two cases of chaotic synchronization, namely, the synchronization of some famous chaotic systems with and without time delay are analyzed. The numerical results are obtained by the nonlinear dynamical software, WinPP in this paper. The numerical results are in very good agreement with the theoretical analysis.

**Keywords:** *chaotic synchronization; generalized Hamiltonian canonical form; observer approach; time-delay systems.*

**Mathematics Subject Classification (2000):** 37N35, 65P20, 68P25, 70K99, 93D20, 94A99.

## 1 Introduction

Because of the rapid development of chaos–based cryptography in secure communication, chaotic synchronization has become an active research area. Many results on all kinds of chaotic synchronization and its applications have been systematically summarized in [1]. Synchronization is ubiquitous in many natural and engineering systems. Synchronization literally means two identical, near-identical or even different chaotic systems tend to move at the same state, velocity, acceleration and phase, if one of them is coupled or both coupled with each other. The relevant research on synchronization can be dated back

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to Huygens who investigated frequency locking between two clocks, which is perhaps the first synchronization phenomenon observed. Chaotic synchronization has become an active research in nonlinear dynamics [2, 3] since the early 1990s when researchers realized that chaotic dynamical systems can be synchronized and recognized its potential applications to the secure communication. In addition to the classical complete synchronization [3, 4], there are some other new types of synchronization, such as Pecora–Carroll synchronization [5], phase synchronization [6], frequency synchronization [7], anticipating synchronization [8, 9], quasiperiodic synchronization [10], lag synchronization [11], inverse synchronization [12] and generalized synchronization [13].

Synchronization of chaotic dynamical systems has received a lot of attention in recent years. The potential application of chaotic synchronization to signal masking and private communication [14, 15, 16, 17, 18, 19] is very interesting. Chaos in nonlinear dynamical system is a well–established discipline in physics, chemistry, power, electronics, biology, ecology, economics, etc in the meantime. Chaotic behavior can be found in systems described by ordinary differential equations(ODE), discrete dynamical systems and delay differential equations(DDE), etc [20]. In other words, chaos is a multidisciplinary research field and ubiquitous phenomenon. The main property of chaotic dynamics is its critical sensitivity to initial conditions in the systems' evolution. For many years this property made chaos unpredictable, since the sensitivity to initial conditions reduces the long-terme predictability of such chaotic dynamical systems. But the recent investigations have shown, in fact, this property of chaotic dynamical systems could practically be very beneficial [21].

Time delay does also widely exist in the natural world and the human society. Finite signal transmission, switching speeds and memory effects make it ubiquitous in nature, technology and society [8]. Therefore the study of the effect of time delay on the systems' dynamics is of considerable practical importance. Time–delayed dynamical systems are also interesting since they have infinite-dimensional state spaces and the number of their positive Lyapunov exponents can be made arbitrarily large because of the existence of the time delay. From this point of view, such systems are especially appealing for secure communication scheme [1].

The objective of this paper is to apply the Generalized Hamiltonian forms and observer approach developed in [22] to the synchronization of some chaotic dynamical systems. Besides the observer perspective on synchronization, some works, such as the concept of synchronization is revisited in the light of the classical notion of observers from (non)linear control theory, are obtained in [23, 24]. As described in [25] this method has several advantages over the exiting synchronization methods: (1) it enables synchronization achieved in a systematic way; (2) it can be successfully applied to several well–known chaotic or hyperchaotic oscillators; (3) it does not require the computation of any Lyapunov exponent; (4) it does not require initial conditions belonging to the same basin of attraction. In Section 2, the Generalized Hamiltonian forms and observer approach [22, 25] are first introduced. Then the synchronization of some kinds of chaotic dynamical systems such as *Lü* system, Van der Pol-Duffing system, Genesio system and SMIB power system, which without time delay, employed by the Generalized Hamiltonian forms and observer approach is considered in Section 3. That of the delayed chaotic dynamical systems, i.e., SMIB power system and Van der Pol–Duffing system, is also investigated in Section 4. At last, the conclusion and discussion are presented.

## 2 The design in Generalized Hamiltonian system

Consider a smooth nonlinear system given in the following "Generalized Hamiltonian" canonical form,

$$\dot{x} = \mathcal{J}(x)\frac{\partial H}{\partial x} + \mathcal{S}(x)\frac{\partial H}{\partial x} + \mathcal{F}(x), \qquad x \in \mathbf{R}^n, \tag{2.1}$$

where $H(x)$ denotes a smooth energy function which is globally positive definite in $\mathbf{R}^n$. The column gradient vector of $H(x)$, denoted by $\partial H/\partial x$, is assumed to exist everywhere. We usually use quadratic energy function $H(x) = \frac{1}{2}x^T\mathcal{M}x$ with $\mathcal{M}$ being a, constant, symmetric positive definite matrix. So $\partial H/\partial x = \mathcal{M}x$. The square matrices $\mathcal{J}(x)$ and $S(x)$ satisfy for all $x \in \mathbf{R}^n$, the properties: $\mathcal{J}(x) + \mathcal{J}^T(x) = 0$, and $\mathcal{S}(x) = \mathcal{S}^T(x)$. The vector field $\mathcal{J}(x)\partial H/\partial x$ exhibits the conservative part of the system and it is also referred to as the workless part, or workless forces of the system; and $\mathcal{S}(x)$ depicts the working or nonconservative part of the system. For certain systems, $\mathcal{S}(x)$ is negative definite or negative semi–definite. If, on the other hand, $\mathcal{S}(x)$ is positive definite, positive semi–definite, or indefinite, it clearly represents, respectively, the global, semi–global, and local destabilizing part of the system. And where $\mathcal{F}(x)$ is a locally destabilizing vector field. Consider now the following dynamical system

$$\dot{x} = f(x,t). \tag{2.2}$$

It can be rewritten as

$$\dot{x} = A\frac{\partial H}{\partial x} + \mathcal{F}(x,t). \tag{2.3}$$

Since $A = \frac{A-A^T}{2} + \frac{A+A^T}{2}$, we have

$$\dot{x} = \frac{A-A^T}{2}\frac{\partial H}{\partial x} + \frac{A+A^T}{2}\frac{\partial H}{\partial x} + \mathcal{F}(x,t), \tag{2.4}$$

Let $\mathcal{J}(x) = \frac{A-A^T}{2}, \mathcal{S}(x) = \frac{A+A^T}{2}$. The equation (2.2) can be written in the Generalized Hamiltonian canonical form (2.1). This form is not only used for autonomous systems, but also for non-autonomous systems and delay differential equations.

In the context of observer design, we consider a special class of Generalized Hamiltonian systems with destabilizing vector field and liner output map, $y(t)$, given by

$$\begin{cases} \dot{x} = \mathcal{J}(y)\dfrac{\partial H}{\partial x} + (\mathcal{I} + \mathcal{S})\dfrac{\partial H}{\partial x} + \mathcal{F}(y), \qquad x \in \mathbf{R}^n, \\ y = \mathcal{C}\dfrac{\partial H}{\partial x}, \qquad y \in \mathbf{R}^m, \end{cases} \tag{2.5}$$

where $S$ is a constant symmetric matrix, the matrix $\mathcal{I}$ is a constant skew symmetric matrix. The vector variable $y(t)$ is referred to as the system output. The matrix $\mathcal{C}$ is a constant matrix.

We denote the estimate of the state vector $x$ by $\xi$, and consider the Hamiltonian energy function $H(\xi)$ to be the particularization of $H$ in terms of $\xi$. Similarly, we denote by $\eta$ the estimated output, computed in terms of the estimated state $\xi$. The gradient vector $\partial H/\partial \xi$ is, naturally, of the form$\mathcal{M}\xi$ with $\mathcal{M}$ being a constant symmetric positive definite matrix.

A dynamic nonlinear state observer for (2.5) is obtained as

$$\begin{cases} \dot{\xi} = \mathcal{J}(y)\dfrac{\partial H}{\partial \xi} + (\mathcal{I}+\mathcal{S})\dfrac{\partial H}{\partial \xi} + \mathcal{F}(y) + K(y-\eta), & \xi \in \mathbf{R}^n, \\ \eta = \mathcal{C}\dfrac{\partial H}{\partial \xi}, \qquad \eta \in \mathbf{R}^m, \end{cases} \tag{2.6}$$

where $K$ is a constant matrix, known as the observer gain. The state estimation error, defined as $e = x - \xi$ and the output estimation error, defined as $e_y = y - \eta$, are governed by

$$\begin{cases} \dot{e} = \mathcal{J}(y)\dfrac{\partial H}{\partial e} + (\mathcal{I}+\mathcal{S}-K\mathcal{C})\dfrac{\partial H}{\partial e}, & e \in \mathbf{R}^n, \\ e_y = \mathcal{C}\dfrac{\partial H}{\partial e}, \qquad e_y \in \mathbf{R}^m, \end{cases} \tag{2.7}$$

where the vector, $\partial H/\partial e$ actually stands, with some abuse of notation, for the gradient vector of energy function, $\partial H/\partial e = \partial H/\partial x - \partial H/\partial \xi = \mathcal{M}e$. When needed, set $\mathcal{I}+\mathcal{S} = \mathcal{W}$.

**Definition 2.1** (Synchronization) [1] We say that the receiver dynamics (2.6) synchronizes with the transmitter dynamics (2.5), if

$$\lim_{t\to\infty} \|x(t) - \xi(t)\| = 0, \tag{2.8}$$

no matter which initial conditions $x(0)$ and $\xi(0)$ have.

**Theorem 2.1** *(Stability of the estimation/synchronization error [22])The state $x$ of the nonlinear system* (2.5) *can be globally exponentially asymptotically estimated by the state $\xi$ of the nonlinear observer* (2.6) *if and only if there exists a constant matrix $K$ such that the symmetric matrix*

$$[\mathcal{W} - K\mathcal{C}] + [\mathcal{W} - K\mathcal{C}]^T = [\mathcal{S} - K\mathcal{C}] + [\mathcal{S} - K\mathcal{C}]^T = 2[\mathcal{S} - \frac{1}{2}(K\mathcal{C} + \mathcal{C}^T K^T)]$$

*is negative definite.*

In the latter synchronized programs, we mainly use Theorem 2.1. Most time we only consider the matrix $\mathcal{S}$, but not the matrix $\mathcal{I}+\mathcal{S}$.

And a sufficient but not necessary condition based on the observability condition for asymptotical stability of the synchronization was obtained.

**Theorem 2.2** *[22] The state x(t) of the nonlinear system* (2.5) *can be globally exponentially asymptotically estimated by the state $\xi$ of the nonlinear observer* (2.6)*, if the pair of matrices* $(\mathcal{C}, \mathcal{W})$ *or the pair* $(\mathcal{C}, \mathcal{S})$*, is either observable or, at least, detectable.*

## 3 Synchronization of some chaotic systems

### 3.1 *Lü* system

Consider *Lü* system [26]

$$\begin{cases} \dot{x_1} = a(x_2 - x_1), \\ \dot{x_2} = -x_1x_3 + cx_2, \\ \dot{x_3} = x_1x_2 - bx_3. \end{cases} \tag{3.1}$$

The system (3.1) can be easily written in the following Generalized Hamiltonian form,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix} \frac{\partial H}{\partial x}, \tag{3.2}$$

where $H(x)$ is the Hamiltonian energy scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.3}$$

we choose $y = [x_1, x_2]^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \frac{\partial H}{\partial x}$ as the output signal to be transmitted. The matrices $\mathcal{C}$, $\mathcal{S}$, $\mathcal{I}$, and $\mathcal{J}(y)$ are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{S} = \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix}, \mathcal{I} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{J}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix}.$$

Using Theorem **2.2** [22], the pair $(\mathcal{C}, \mathcal{W})$ or $(\mathcal{C}, \mathcal{S})$ already constitutes a detectable, but not observable pair for the chaotic parameters $a = 36, b = 3, c = 20$. Because the output contains two states, namely, $x_1$ and $x_2$, so the gain parameters should be chosen as $K_1$, ... ,$K_6$, this results in the receiver

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} & 0 \\ -\frac{a}{2} & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} -a & \frac{a}{2} & 0 \\ \frac{a}{2} & c & 0 \\ 0 & 0 & -b \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \tag{3.4}$$

where $\eta = \mathcal{C}\frac{\partial H}{\partial \xi}$. One may now choose the gain vector $K = \begin{pmatrix} K_1 & K_3 & K_5 \\ K_2 & K_4 & K_6 \end{pmatrix}^T$. The synchronization error, corresponding to this receiver, is

$$\begin{aligned} \begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{a}{2} - \frac{K_2}{2} + \frac{K_3}{2} & \frac{K_5}{2} \\ -\frac{a}{2} + \frac{K_2}{2} - \frac{K_3}{2} & 0 & -x_1 + \frac{K_6}{2} \\ -\frac{K_5}{2} & x_1 - \frac{K_6}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} \\ &+ \begin{pmatrix} -a - K_1 & \frac{a}{2} - \frac{K_2}{2} - \frac{K_3}{2} & -\frac{K_5}{2} \\ \frac{a}{2} - \frac{K_2}{2} - \frac{K_3}{2} & c - K_4 & -\frac{K_6}{2} \\ -\frac{K_5}{2} & -\frac{K_6}{2} & -b \end{pmatrix} \frac{\partial H}{\partial e}. \end{aligned} \tag{3.5}$$

From Theorem 2.1, the following expression is obtained

$$2[\mathcal{S} - \frac{1}{2}(K\mathcal{C} + \mathcal{C}^T K^T)] = \begin{pmatrix} -2a - 2K_1 & -K_2 - K_3 + a & -K_5 \\ -K_2 - K_3 + a & -2K_4 + 2c & -K_6 \\ -K_5 & -K_6 & -2b \end{pmatrix},$$

we may prescribe $K_1, K_2, K_3, K_4, K_5$ and $K_6$ in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester's Criterion–which provides a test for definite negativity of a matrix–thus, this is achieved by setting

$K_1 > -a,$
$(K_2 + K_3 - a)^2 < 4(K_1 + a)(K_4 - c),$
$(K_1 + a)[4b(K_4 - c) - K_6^2] - (K_2 + K_3 - a)[b(K_2 + K_3 - a) - K_5 K_6]$
$-K_5(K_4 - c) > 0.$

Figure 3.1 shows the performance of the designed receiver with the following parameter values and for the constant gains $a = 36,\ b = 3,\ c = 20,\ K_1 = 0,\ K_2 = 3,\ K_3 = 3,\ K_4 = 30,\ K_5 = 0,\ K_6 = 2$. From Figure 3.1, it can be easily known that after a very short time, $L\ddot{u}$ system is synchronized.

### 3.2 Van der Pol–Duffing system

The initial mathematical model is Van der Pol–Duffing system with external excitation given by

$$\ddot{x} + \omega_0^2 x - (\alpha - \gamma x^2)\dot{x} + \beta x^3 = k\cos(\Omega t). \tag{3.6}$$

By setting $x_1 = x, x_2 = \dot{x_1}$, we can write the system (14) in the following form

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = (\alpha - \gamma x_1^2)x_2 - \omega_0^2 x_1 - \beta x_1^3 + k\cos(\Omega t). \end{cases} \tag{3.7}$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2], \tag{3.8}$$

we write the system in Generalized Hamiltonian canonical form as

$$\begin{aligned} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix} \frac{\partial H}{\partial x} \\ &\quad + \begin{pmatrix} 0 \\ -\gamma x_1^2 x_2 - \beta x_1^3 + k\cos\Omega t \end{pmatrix}. \end{aligned} \tag{3.9}$$

The destabilizing vector requires two signals for complete cancellation at the receiver, namely, the variables, $x_1$ and $x_2$. The output is then chosen as the vector $y = [x_1, x_2]^T$. The matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix},$$

the pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. In order to achieve chaotic behavior, we should choose suitable parameters. The receiver would then be designed as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1+\omega_0^2}{2} \\ -\frac{1+\omega_0^2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1-\omega_0^2}{2} \\ \frac{1-\omega_0^2}{2} & \alpha \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &\quad + \begin{pmatrix} 0 \\ \\ -\gamma x_1^2 x_2 - \beta x_1^3 + k\cos\Omega t \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} (y - \eta), \end{aligned} \tag{3.10}$$

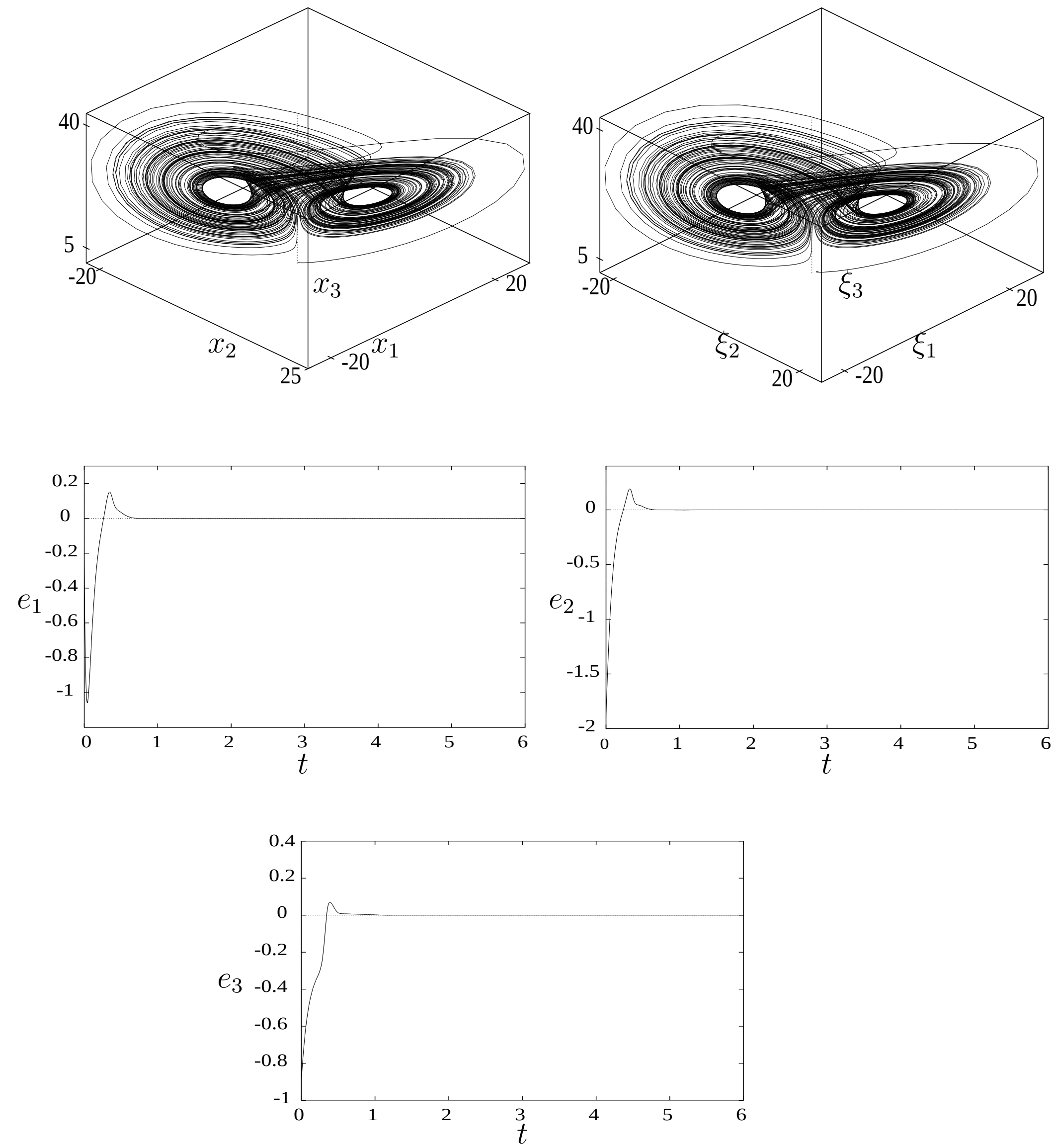


**Figure 3.1:** The synchronization of the $L\ddot{u}$ systems (3.2) and (3.4) with the following parameter values and for the constant gains $a = 36,\ b = 3,\ c = 20,\ K_1 = 0,\ K_2 = 3,\ K_3 = 3,\ K_4 = 30,\ K_5 = 0,\ K_6 = 2$ and the initial conditions $x(0) = (0.01,\ 0.1,\ 1)^T, \xi(0) = (1,\ 0.5,\ 2)^T$.

where $\eta = \mathcal{C}\frac{\partial H}{\partial \xi}$, the synchronization error dynamics is governed by

$$\begin{aligned}\begin{pmatrix} \dot{e_1} \\ \dot{e_1} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2}+\frac{1}{2}\omega_0^2+\frac{K_3}{2}-\frac{K_2}{2} \\ -\frac{1}{2}-\frac{1}{2}\omega_0^2+\frac{K_2}{2}-\frac{K_3}{2} & 0 \end{pmatrix}\frac{\partial H}{\partial e} \\ &\quad + \begin{pmatrix} -K_1 & \frac{1}{2}-\frac{1}{2}\omega_0^2-\frac{K_2}{2}-\frac{K_3}{2} \\ \frac{1}{2}-\frac{1}{2}\omega_0^2-\frac{K_2}{2}-\frac{K_3}{2} & -K_4+\alpha \end{pmatrix}\frac{\partial H}{\partial e}, \qquad (3.11)\end{aligned}$$

we could prescribe $K_1, K_2, K_3$, and $K_4$, in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester's Criterion , this is achieved by setting
$K_1 > 0$,
$K_4 > \alpha + \frac{1}{4K_1}(K_2 + K_3 + \omega_0^2 - 1)^2$.

Figure 3.2 shows the synchronization of the systems (3.9) and (3.10), the chosen parameters were set as following[27], $\alpha = 1, \gamma = 1, \omega_0^2 = 1, \beta = 0.01, k = 5, \Omega = 2.463$, with receiver parameter gains $K_1 = 1, K_2 = 0, K_3 = 0, K_4 = 9$.

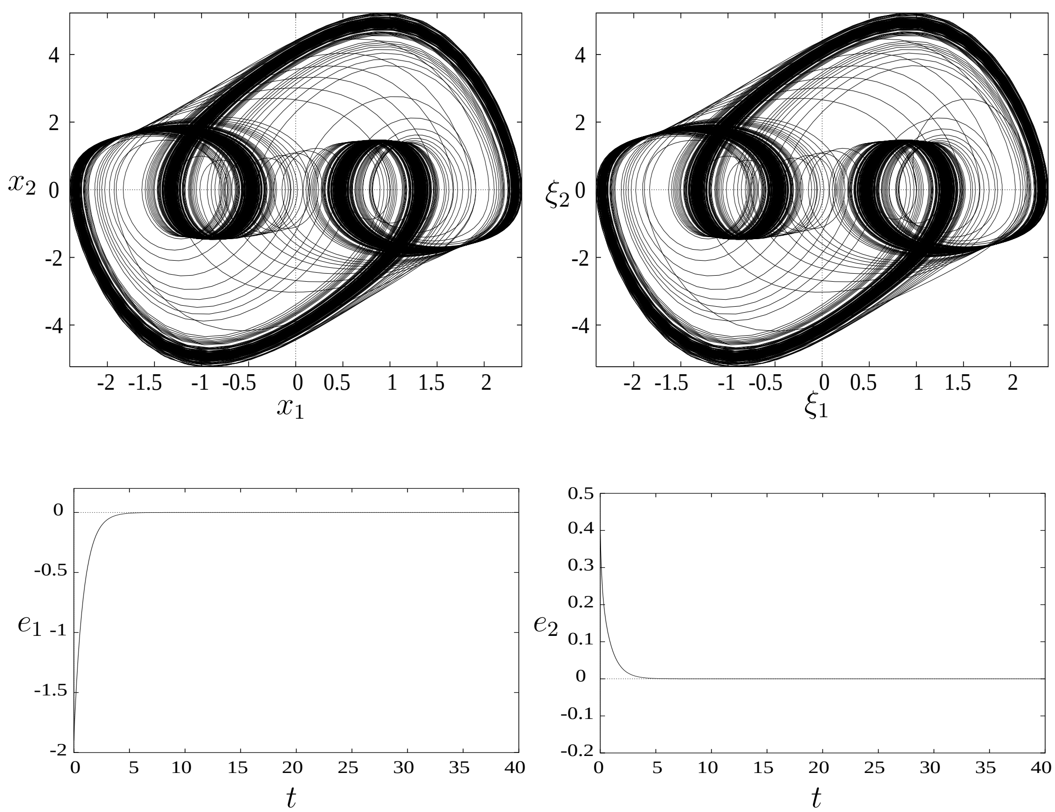


**Figure 3.2:** The synchronization of the van der Pol–Duffing systems (3.9) and (3.10) with the following parameter values and for the constant gains $\alpha = 1, \gamma = 1, \omega_0^2 = 1, \beta = 0.01, k = 5, \Omega = 2.463, K_1 = 1, K_2 = 0, K_3 = 0, K_4 = 9$ and the initial conditions $x(0) = (0.1,\ 0.5)^T, \xi(0) = (2,\ 0.1)^T$.

### 3.3 Genesio system

Genesio system, proposed by Genesio and Tesi [28], is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three negative parameters.

The dynamic equations of the system is given by

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = x_3, \\ \dot{x_3} = ax_1 + bx_2 + cx_3 + x_1^2, \end{cases} \tag{3.12}$$

where $x_1, x_2, x_3$ are state variables, indeed

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix}, \tag{3.13}$$

taking the Hamiltonian energy function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.14}$$

the destabilizing vector field and the lacking damping in $x_2$ variable, call for $y = [x_1, x_2]^T$ to be used as the output of the transmitter. The matrices $\mathcal{C}, \mathcal{S}$ are found to be

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix}, \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. The receiver would then be designed as

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{a}{2} \\ -\frac{1}{2} & 0 & \frac{1-b}{2} \\ \frac{a}{2} & -\frac{1-b}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & \frac{a}{2} \\ \frac{1}{2} & 0 & \frac{1+b}{2} \\ \frac{a}{2} & \frac{1+b}{2} & c \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &+ \begin{pmatrix} 0 \\ 0 \\ x_1^2 \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{3.15}$$

where $\eta = \mathcal{C}\frac{\partial H}{\partial \xi}$.

The synchronization error evolves according to

$$\begin{aligned} \begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} - \frac{K_2}{2} + \frac{K_3}{2} & -\frac{a}{2} + \frac{K_5}{2} \\ -\frac{1}{2} + \frac{K_2}{2} - \frac{K_3}{2} & 0 & \frac{1}{2} - \frac{b}{2} + \frac{K_6}{2} \\ \frac{a}{2} - \frac{K_5}{2} & -\frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} \\ &+ \begin{pmatrix} -K_1 & \frac{1}{2} - \frac{K_2}{2} - \frac{K_3}{2} & \frac{a}{2} - \frac{K_5}{2} \\ \frac{1}{2} - \frac{K_2}{2} - \frac{K_3}{2} & -K_4 & \frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} \\ \frac{a}{2} - \frac{K_5}{2} & \frac{1}{2} + \frac{b}{2} - \frac{K_6}{2} & c \end{pmatrix} \frac{\partial H}{\partial e}. \end{aligned} \tag{3.16}$$

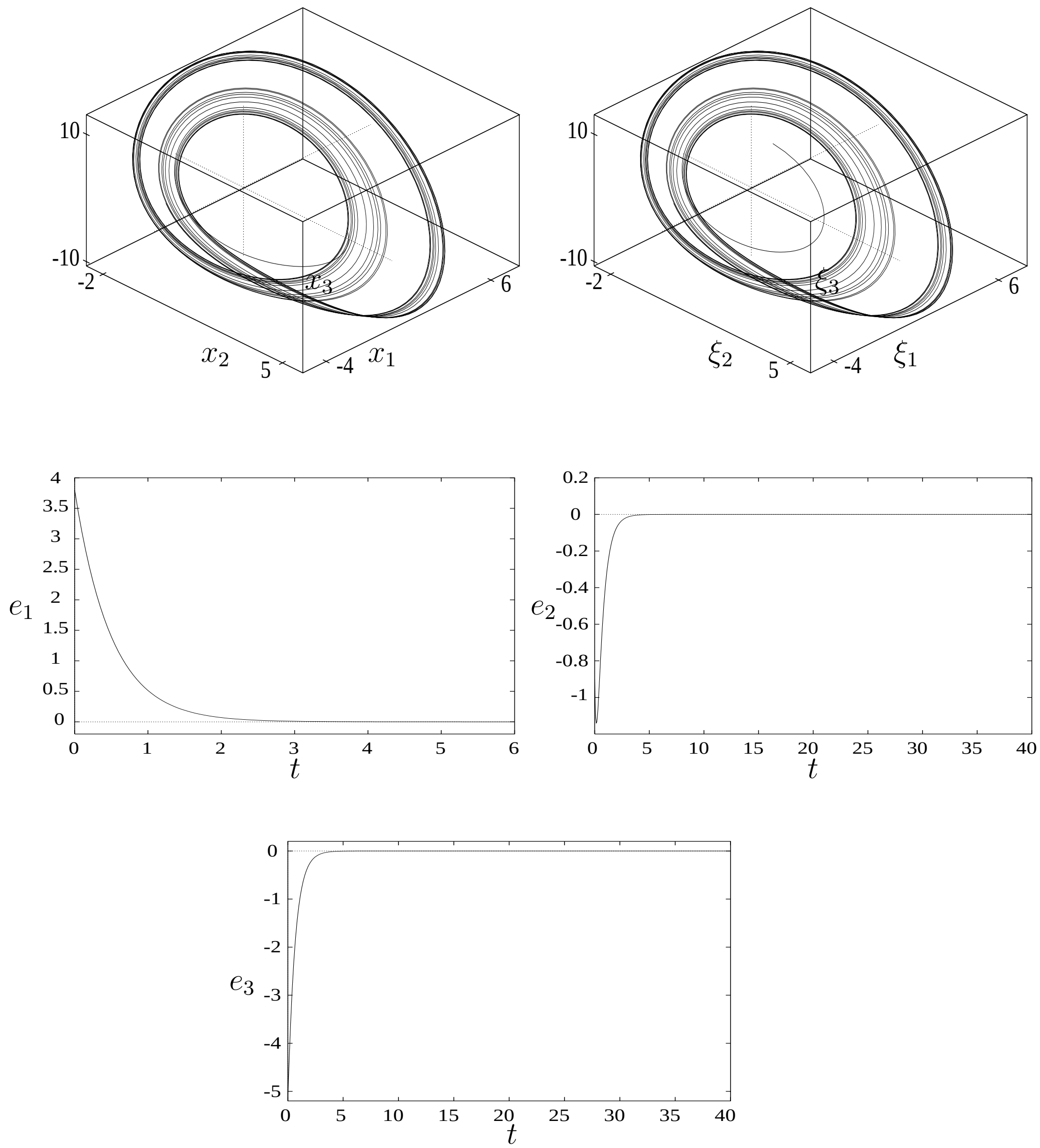


**Figure 3.3:** The synchronization of the Genesio systems (3.13) and (3.15) with the following parameter values and for the constant gains $a = -6, b = -2.92, c = -1.2, K_1 = 2, K_2 = 1, K_3 = 1, K_4 = 5, K_5 = -6, K_6 = -1.92$ and the initial conditions $x(0) = (4,\ 0.1,\ 0.8)^T, \xi(0) = (0.2,\ 1,\ 6)^T$.

To guarantee asymptotic stability of zero of the error dynamics, we should choose suitable $K_1, K_2, K_3, K_4, K_5, K_6$.By applying the Sylvester's Criterion , this is achieved by setting
$K_1 > 0$,
$(K_2 + K_3 - 1)^2 < 4K_1K_4$,
$2K_1[-4cK_4 - (K_6 - b - 1)^2] + 2c(K_2 + K_3 - 1)^2$
$+2(K_5 - a)(K_2 + K_3 - 1)(K_6 - b - 1) - 2K_4(K_5 - a)^2 > 0$.

Figure 3.3 shows the synchronization of the systems (3.13) and (3.15). The chosen parameters were set, following [21], as $a = -6, b = -2.92$, $c = -1.2$, with receiver parameter gains $K_1 = 2, K_2 = 1, K_3 = 1, K_4 = 5, K_5 = -6, K_6 = -1.92$.

### 3.4 SMIB power system

Consider SMIB power system [29], called swing equation

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = -cx_2 - \beta \sin x_1 + f \sin x_3, \\ \dot{x_3} = \omega. \end{cases} \tag{3.17}$$

Taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{3.18}$$

we write the system in Generalized Hamiltonian canonical form as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 \\ \omega \end{pmatrix} \tag{3.19}$$

The destabilizing vector field requires two signals for complete cancellation at the receiver. Namely, the variables, $x_1$ and $x_2$. The output is then chosen as the vector $y = [y_1, y_2]^T = [x_1, x_3]^T$, the matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable, $\mathcal{S}$ is therefore of indefinite sign. The receiver would then be designed as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi_1} \\ \dot{\xi_2} \\ \dot{\xi_3} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} \\ &+ \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 \\ \omega \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{3.20}$$

where $\eta = \mathcal{C}\frac{\partial H}{\partial \xi}$, the synchronization error, corresponding to this receiver, is found to be

$$\begin{pmatrix} \dot{e_1} \\ \dot{e_2} \\ \dot{e_3} \end{pmatrix} = \begin{pmatrix} 0 & \frac{K_3+1}{2} & \frac{K_5-K_2}{2} \\ -\frac{K_3+1}{2} & 0 & -\frac{K_4}{2} \\ \frac{K_2-K_5}{2} & \frac{K_4}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -K_1 & \frac{1-K_3}{2} & -\frac{K_2+K_5}{2} \\ \frac{1-K_2}{2} & -c & -\frac{K_4}{2} \\ -\frac{K_2+K_5}{2} & -\frac{K_4}{2} & -K_6 \end{pmatrix} \frac{\partial H}{\partial e}, \tag{3.21}$$

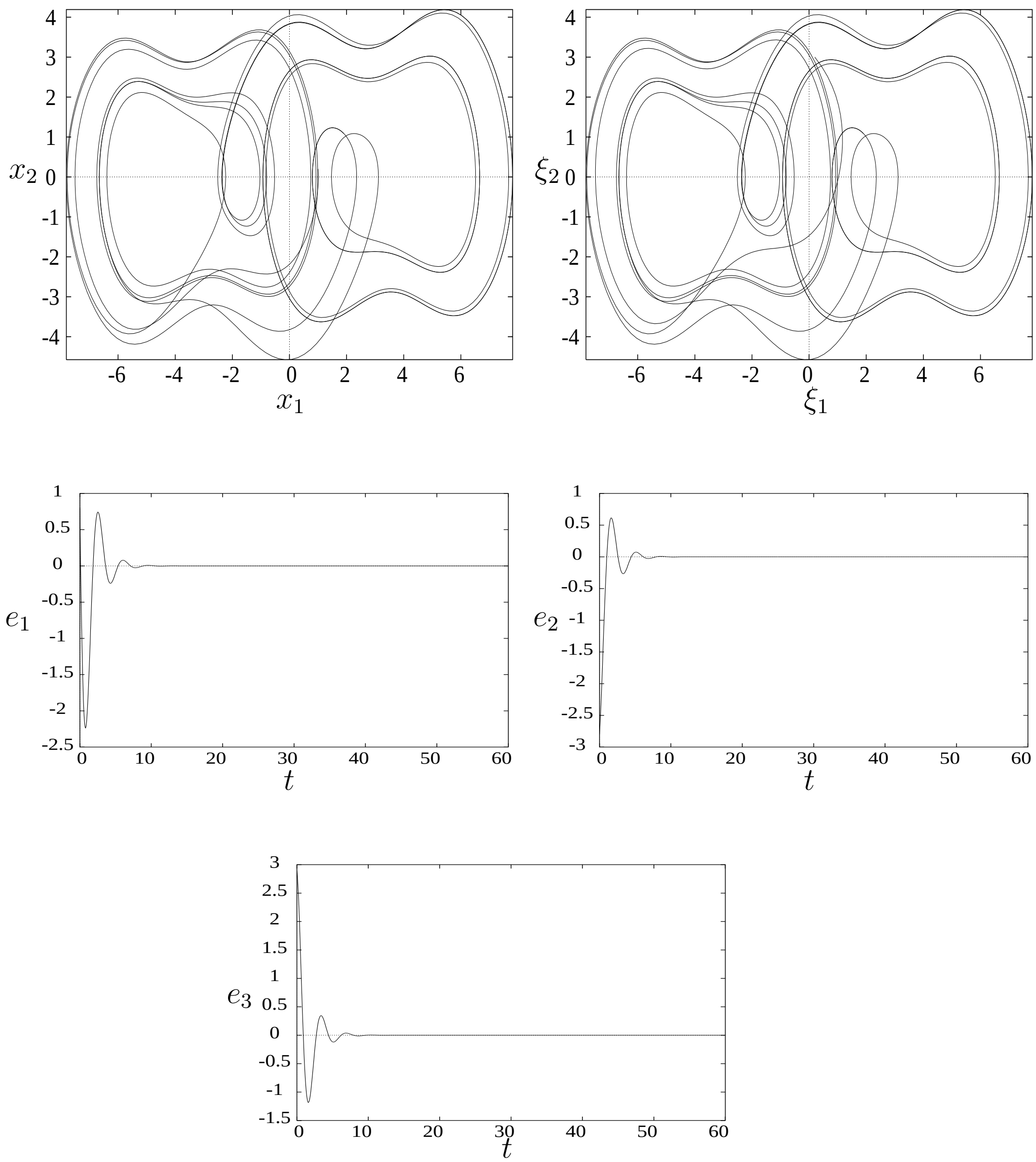


**Figure 3.4:** The synchronization of the SMIB power systems (3.19) and (3.20) with the following parameter values and for the constant gains $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$ and the initial conditions $x(0) = (1,\ 0.2,\ 3)^T, \xi(0) = (0.2,\ 3,\ 0.1)^T$.

we can easily know that $x_3$ and $\xi_3$ are synchronized with each other, so we only concern the synchronization of other variables,this was achieved by setting
$K_1 > 0,\quad (K_3 - 1)^2 < 4cK_1,$
$K_1(4cK_6 - K_4^2) + K_4(K_3 - 1)(K_2 + K_5) - K_6(K_3 - 1)^2 - c(K_2 + K_5)^2 > 0$

Figure 3.4 shows the performance of the designed receiver with the following parameter values for the system and for the constant gains: $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$.

## 4 Synchronization of Time–Delay Chaotic Systems

Following [22], that the time–delay system, a mathematic description by a delay differential equation (DDE), which in its simplest form of a single fixed time–delay $\tau$ is given by

$$\dot{x} = f(x, x(t-\tau)), \qquad x \in \mathbf{R}^n, \tag{4.1}$$

can be written in the Generalized Hamiltonian canonical form,

$$\dot{x} = \mathcal{J}\frac{\partial H}{\partial x} + \mathcal{S}(x)\frac{\partial H}{\partial x} + \mathcal{F}(x, x(t-\tau)), x \in \mathbf{R}^n. \tag{4.2}$$

The properties of $\mathcal{J}, \mathcal{S}$ and $\mathcal{F}(x, x(t-\tau))$ as before in Section 2. The observer design is also similar to (2.6). Now we use two examples to describe it.

### 4.1 Delayed Duffing–Van der Pol system

The system under consideration is a nonlinear oscillator governed by equation

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = -\alpha(1 - x_1^2)x_2 + F\cos(\omega t) - \beta x_1^3 + \gamma x_1(t-\tau), \end{cases} \tag{4.3}$$

taking $H(x) = \frac{1}{2}[x_1^2 + x_2^2]$ as the Hamiltonian energy function, we write the system in Generalized Hamiltonian form as

$$\begin{aligned} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix}\frac{\partial H}{\partial x} \\ &\quad + \begin{pmatrix} 0 \\ \alpha x_1^2 x_2 + F\cos(\omega t) - \beta x_1^3 + \gamma x_1(t-\tau) \end{pmatrix}. \end{aligned} \tag{4.4}$$

The destabilizing vector field requires for complete cancelation at the receiver, namely, the variables $x_1$ and $x_2$. Then the output is chosen as $y = [y_1, y_2]^T = [x_1, x_2]^T$, the matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

The pair $(\mathcal{C}, \mathcal{S})$ is observable, and hence detectable. But we can observe that the pair of matrices $(\mathcal{C}_1, \mathcal{S})$ is also a detectable pair. An injection of the synchronization error $e_1 = x_1 - \xi_1$ suffices to have an asymptotically stable trajectory convergence. The receiver would then be designed, exploiting this last observation, as follows

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\alpha \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 \\ \alpha x_1^2 x_2 + F\cos(\omega t) - \beta x_1^3 + \gamma x_1(t-\tau) \end{pmatrix} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} (y - \eta), \quad (4.5)$$

where $\eta = C_1 \frac{\partial H}{\partial \xi}$ corresponding to this receiver, we can obtain the synchronization error

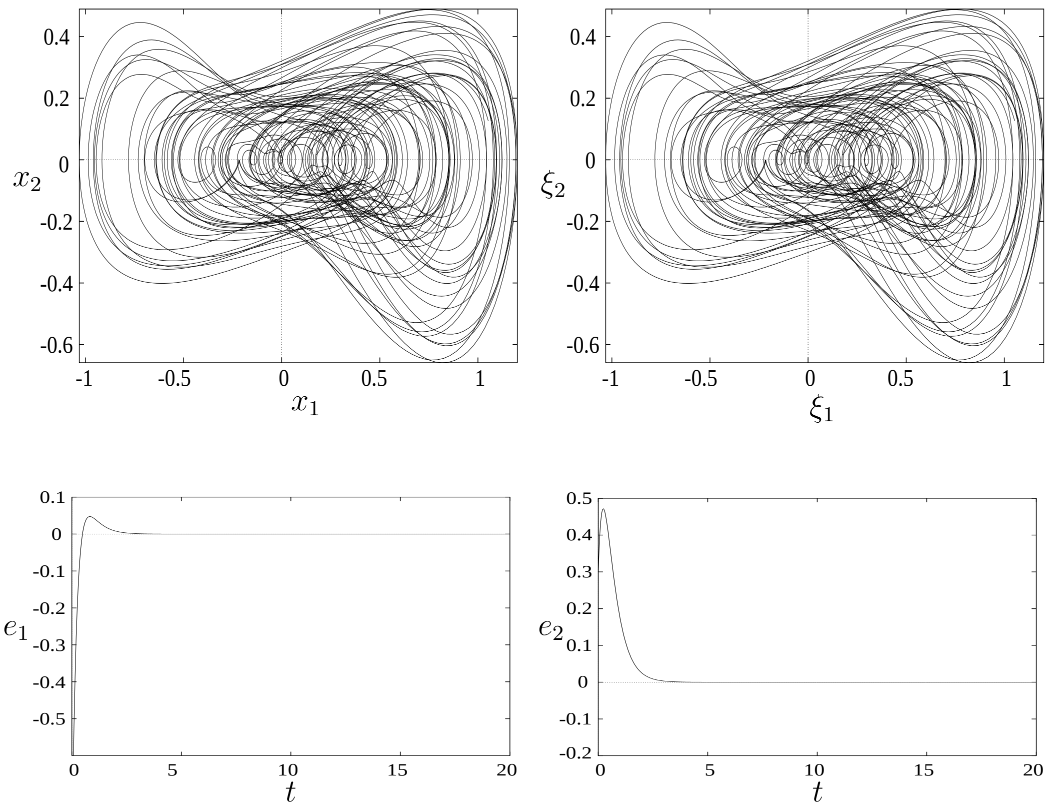


**Figure 4.1:** Delayed Duffing–Van der Pol system (4.4) and (4.5) with the following parameter values and for the constant gains $\alpha = 1.2,\ \beta = 0.75,\ F = 0.2,\ \gamma = 0.4,\ \omega = 0.5,\ \tau = 25,\ K_1 = 5,\ K_2 = 2.5.$ and the initial conditions $x(0) = (0.1,\ 0.5)^T,\ \xi(0) = (1,\ 0.2)^T$.

as

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} + \frac{K_2}{2} \\ -\frac{1}{2} - \frac{K_2}{2} & 0 \end{pmatrix} \frac{\partial H}{\partial e} + \begin{pmatrix} -K_1 & \frac{1}{2} - \frac{K_2}{2} \\ \frac{1}{2} - \frac{K_2}{2} & -\alpha \end{pmatrix} \frac{\partial H}{\partial e}. \quad (4.6)$$

We could prescribe $K_1$, and $K_2$, in order to ensure asymptotic stability of zero of the synchronization error. By applying the Sylvester's Criterion, this is achieved by setting $K_1 > 0$, $2K_1\alpha > (K_2 - 1)^2$.

Figure 4.1 shows the synchronization of the systems (4.4) and (4.5), the chosen parameters are set as the following $\alpha = 1.2,\ \beta = 0.75,\ F = 0.2,\ \gamma = 0.4,\ \omega = 0.5,\ \tau = 25$, with receiver parameter gains $K_1 = 5,\ K_2 = 2.5$.

### 4.2 Delayed SMIB power system

Let's also consider the classical SMIB power system (3.17) with a time–delay $\tau$,

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = -cx_2 - \beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t-\tau)), \\ \dot{x_3} = \omega, \end{cases} \tag{4.7}$$

taking as a Hamiltonian energy function the scalar function

$$H(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2], \tag{4.8}$$

we write the system in the Generalized Hamiltonian canonical form as

$$\begin{aligned} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial x} \\ &+ \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t-\tau)) \\ \omega \end{pmatrix}. \end{aligned} \tag{4.9}$$

Following the analysis in Section 3.4, taking the variable $y = [y_1, y_2]^T = [x_1, x_3]^T$ as the output, then the matrices $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{I}$ are given by

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the receiver would then be designed as follows

$$\begin{aligned} \begin{pmatrix} \dot{\xi_1} \\ \dot{\xi_2} \\ \dot{\xi_3} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial H}{\partial \xi} + \\ & \begin{pmatrix} 0 \\ -\beta \sin x_1 + f \sin x_3 + \epsilon \sin(Rx_1(t-\tau)) \\ \omega \end{pmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \\ K_5 & K_6 \end{pmatrix} (y - \eta), \end{aligned} \tag{4.10}$$

Figure 4.2 shows the performance of the designed receiver with the following parameter values of the system and the constant gains: $c = 2, \beta = 6, f = 9, R = 50, \omega = 1, \epsilon = 10, \tau = 0.6, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$.

## 5 Conclusion

In this paper, we have considered the problem of synchronization of several famous chaotic dynamical systems, including two types of synchronization which are respectively the dynamical systems with time–delay and that without any time–delay, from the perspective

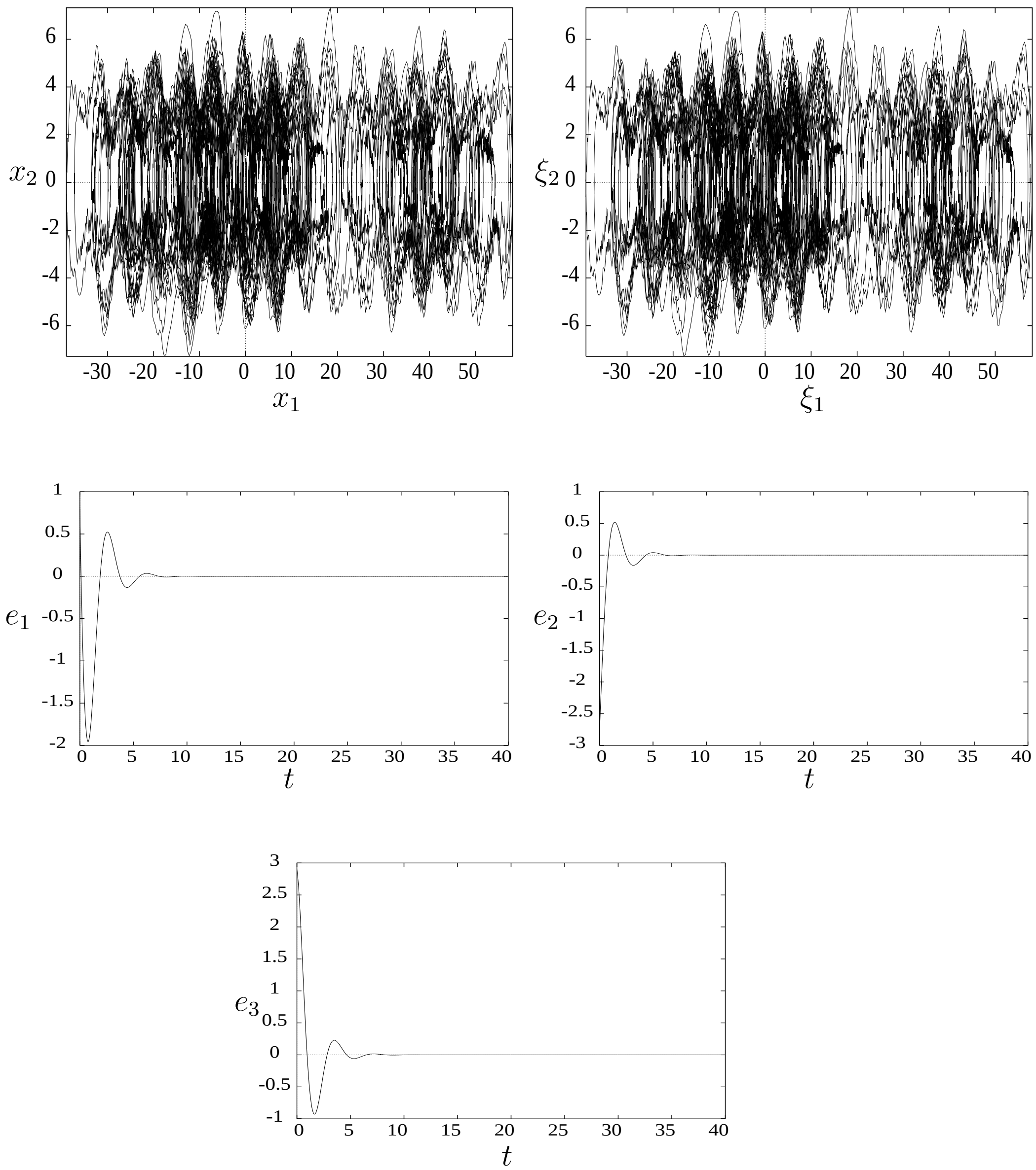


**Figure 4.2:** The synchronization of the SMIB power systems (4.9) and (4.10) with the following parameter values and for the constant gains $c = 1, \beta = 3, f = 5, \omega = 1, K_1 = 0.5, K_2 = 1.5, K_3 = 1, K_4 = 0.1, K_5 = -1.5, K_6 = 0.75$ and the initial conditions $x(0) = (1,\ 0.2\ ,3)^T, \xi(0) = (0.2,\ 3,\ 0.1)^T$.

of Generalized Hamiltonian systems (developed by Sira–Ramírez and Cruz[22]. Several chaotic dynamical systems, consisting of ones which are without any time–delay and that with time–delay, are analyzed from this perspective and their synchronization were both confirmed. The six figures show that the chaotic systems under consideration achieved synchronization with their receivers immediately, respectively.

In the course of applying this method to the synchronization of some dynamical systems, we confront one problem: for some systems as we choose one variable as the output signal, we can't find suitable values of $K_i,\ i = 1, 2, \cdots$, for the receiver to synchronize with the master. In order to overcome such problem we add one or more output signals, to increase the number of the constants $K_i,\ i = 1, 2, \cdots$, and extend the flexibility of the constants to be chosen, easily we obtain the synchronization. But a small problem is that the computation may be a little more complex.

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