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The Fell Topology for Dynamic Equations on Time Scales

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Abstract: In order to study the changing dynamics of solutions of a dynamic equation on time scales as the time scales change, we must determine appropriate topologies on the set of time scales and the set of solutions of dynamic equations. As a first step, we prove a natural characterization of the Fell topology on the space of time scales.

Keywords: *time scales; dynamic equations; Fell topology.*

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1 Introduction

Dynamic equations on times scales were introduced by S. Hilger in [10] in 1988. A thorough introduction is contained in [2]. A time scale is a nonempty closed subset of $\mathbb{R}$. Hilger's Δ-derivative is defined for a real-valued function f whose domain is a time scale $\mathbb{T}$ and is denoted by $f^{\Delta}(t)$ at any $t \in \mathbb{T}$, where $t < \sup T$.

By design, $f^{\Delta}(t)$ mimics the standard right-hand derivative $f'(t)$ when there exists a strictly decreasing sequence convergent to t in $\mathbb{T}$ and a scaled difference operator otherwise. In particular, $f^{\Delta}(t) = f'(t)$ on $\mathbb{R}$ and $f^{\Delta}(t) = \Delta f(t)$ on $\mathbb{Z}$. While the Δ-derivative is a "forwards" operator, an analogous "backwards" operator exists called the ∇-derivative.

Generalizing differential and difference equations are dynamic equations, which involve Δ-derivatives (or ∇-derivatives, etc.). Given a dynamic equation, say the initial value problem

$$x^{\Delta} = f(t,x), \quad x(t_0) = x_0, \tag{1.1}$$

the solution inherently depends on the time scale. Broadly, we would like to examine how the solution of (1.1) depends on the time scale that is its domain.

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1.1 An example

The following illustrative example has been considered in [4], [8], [12], and [14]. Consider the initial value problem:

$$x^{\Delta} = 4x\left(\tfrac{3}{4} - x\right), \quad x(0) = x_0.$$

Over the eulerian time scales $\mu\mathbb{Z}_+$ for $0 < \mu \leq 1$, the solution is found by iterating

$$L_\mu(x) = 4\mu x\left(\tfrac{3\mu+1}{4\mu} - x\right).$$

starting from $x(0) = x_0$. When $\mu = 1$, the difference equation is solved by iteration of

$$L_1(x) = 4x(1 - x)$$

over $\mathbb{Z}_+$. On the other hand, as $\mu \to 0$, the solutions appear to tend towards the solution of the logistic differential equation over $\mathbb{R}_+$.

The dynamics of the quadratic polynomial L_μ is easily understood: L_μ is topologically conjugate to

$$Q_c(x) = x^2 + c, \text{where } c = \frac{1}{4}(1 - 9\mu^2).$$

Every value of $\mu \in (0, 1]$ corresponds exactly to one value of $c \in \big[-2, 1/4\big)$, with $\mu = 1$ corresponding to $c = -2$ and $c \to 1/4$ as $\mu \to 0$.

Note that the real interval $\big[-2, 1/4\big]$ is the real part of the Mandelbrot set for the family Q_c. Hence, passing through the time scales $\mu\mathbb{Z}_+$—from a difference equation when $\mu = 1$ towards a differential equation as $\mu \to 0$—all of the interesting dynamics of real quadratic polynomials, including all of their bifurcations, are displayed! (Of course, the issue of $\mu\mathbb{Z}_+$ converging to $\mathbb{R}_+$ must be dealt with also.)

1.2 The goal

In the example of subsection 1.1, we have realized the domain of the solutions on eulerian time scales as a parameter of a family of dynamical systems. This is a simple case. We do not know what happens when non-eulerian time scales are used in this example. Also, we have not dealt with an equation that has non-unique solutions.

As indicated in [14], we propose the following project. For any given initial value problem, treat the time scales as a parameter. Let A denote the set of all time scales and let B denote the set of all solutions of the initial value problem on all possible time scales. Consider the canonical projection:

$$\begin{array}{c} B \\ \Big\downarrow \pi \\ A \end{array} \tag{1.2}$$

That is, an element of B, a solution $f : \mathbb{T} \to \mathbb{R}$, projects to its domain, $\mathbb{T}$. What can be said about this projection? Hopefully, this approach will help explain the changes in dynamics of solutions caused by changes in time scales and make for better modeling of applications.

In Section 2, we examine the Fell topology on the space of time scales. We prove a recent conjecture in [14] giving a natural characterization of convergence in the Fell topology.

Section 3 considers the compatible topology on the space of partial mappings, *i.e.*, continuous function on time scales.

The first natural example, that of equations with unique solutions, is treated in Section 3.3. Of course, the projection is a homeomorphism onto its image in this case.

2 Convergence of Sets in Terms of Convergence of Their Elements

A *hyperspace* is a set of closed subsets of a topological space X. The set of all closed subsets of X is denoted $\mathrm{CL}(X)$. See [11] for an introduction. For example, $\mathrm{CL}(\mathbb{R})$ is the set of all time scales.

Hausdorff (for metrizable X), Vietoris, and Fell defined topologies on hyperspaces in [9], [15], and [6], respectively. These are all equivalent on a compact metrizable space. However, the Vietoris and Fell topologies are not metrizable on $\mathrm{CL}(\mathbb{R})$.

2.1 The Fell topology on CL(X)

We set the following notation that will assist in defining the Fell topology on $\mathrm{CL}(X)$. For any $E \subset X$, let

$$E^- = \{\mathbb{A} \in \mathrm{CL}(X) | \mathbb{A} \cap E \neq \emptyset\}$$

and

$$\begin{aligned} E^+ &= \{\mathbb{A} \in \mathrm{CL}(X) | \mathbb{A} \subset E\} \\ &= \{\mathbb{A} \in \mathrm{CL}(X) | \mathbb{A} \cap (X - E) = \emptyset\}. \end{aligned}$$

We say that every $\mathbb{A} \in E^-$ *hits* E and every $\mathbb{A} \in E^+$ *misses* $X - E$; E^- and E^+ are called *hit* and *miss* sets, respectively. Note that $E^+ \subset E^-$ for every E. Also, we call a subset of X *cocompact* if its complement is compact.

The Fell, as well as the Vietoris, topologies are defined by hit and miss sets; these topologies are called *hit-and-miss* topologies. (In fact, the Hausdorf metric topology is also a hit-and-miss topology. See [13].) The Fell topology, denoted by $\tau(F)$, is generated by the hit sets U^- for all open subsets U of X and the miss sets V^+ for all cocompact subsets V of X. The Vietoris topology, denoted by $\tau(V)$, is similarly generated except that the V's need only be open. (If X is Hausdorff, then the Vietoris topology is finer than the Fell topology.)

Remark 2.1 By convergence in $\mathrm{CL}(X)$, we will mean convergence with respect to the Fell topology on $\mathrm{CL}(X)$ unless otherwise indicated.

2.2 Convergence through a sequence in CL(X)

In [14], we defined another kind of convergence. (This was also discussed in [12] and it inspired [3] and [4].)

Let $\{\mathbb{T}_n\}$ be a sequence in $\mathrm{CL}(X)$ and let $t \in X$. t is called a *sequential limit point* of the sequence $\{\mathbb{T}_n\}$ if there exists a sequence $\{t_n\}$ such that $t_n \in \mathbb{T}_n$ for all $n \in \mathbb{N}$ and t_n converges to t in X. Analogously, t is called a *subsequential limit point* of the sequence $\{\mathbb{T}_n\}$ if t is a sequential limit point of a subsequence $\{\mathbb{T}_{n_i}\}$. We denote the set of all sequential limit points of $\{\mathbb{T}_n\}$ by $\mathbb{T}$ and the set of all subsequential limit points of

$\{\mathbb{T}_n\}$ by $\mathbb{T}'$. We say that $\{t_n\}$ converges to a sequential limit point t *through the* $\mathbb{T}_n$*'s.* Similarly, $\{t_{n_i}\}$ converges to a subsequential limit point t *through the* $\mathbb{T}_{n_i}$*'s.*

It is always the case that $\mathbb{T} \subset \mathbb{T}'$. Two obvious questions are whether $\mathbb{T}$ is in $\mathrm{CL}(X)$ and whether $\mathbb{T}$ is the limit of the sequence $\mathbb{T}_n$.

Lemma 2.1 *If X is metrizable, then $\mathbb{T}$ is closed in X.*

Proof Choose a metric d on X. Suppose that a sequence $\{s_i\}$ in $\mathbb{T}$ converges to t in X. We wish to show that $t \in \mathbb{T}$.

Since the sequence $\{s_i\}$ converges to t, for every $n \in \mathbb{N}$, there exists a natural number N_n such that

$$d(s_i, t) < \frac{1}{2n} \tag{2.1}$$

whenever $i \geq N_n$.

Since, for each $i \in \mathbb{N}$, $s_i \in \mathbb{T}$, there exist sequences $\{t_{i,j}\}$ converging to s_i through the $\mathbb{T}_j$'s. Set $M_0 = 1$. For all $i \in \mathbb{N}$, there exists a natural number M_i such that $M_i > M_{i-1}$ and

$$d(t_{i,j}, s_i) < \frac{1}{2i} \tag{2.2}$$

whenever $j \geq M_i$.

We wish to construct a sequence $\{t_j\}$ converging to t through the $\mathbb{T}_j$'s. For each i, for $M_{i-1} \leq j < M_i$, set $t_j = t_{i,j}$.

Take an arbitrary $\varepsilon > 0$. When $i > 1/\varepsilon$ and $i \geq N_n$, by (2.2) and (2.1),

$$\begin{aligned} d(t_j, t) &\leq d(t_j, s_i) + d(s_i, t) \\ &< \frac{1}{2i} + \frac{1}{2n} < \varepsilon. \end{aligned}$$

Therefore, the sequence $\{t_j\}$ converges to t through the $\mathbb{T}_j$'s and $t \in \mathbb{T}$. □

Remark 2.2 Therefore, in the setting of a metric space X, the sequential limit set of a sequence in $\mathrm{CL}(X)$ is either empty or in $\mathrm{CL}(X)$. For example, the sequence of singleton sets $\{\{n\}\}$ in $\mathrm{CL}(\mathbb{R})$ has empty sequential limit set and $\emptyset \notin \mathrm{CL}(\mathbb{R})$ by definition.

2.3 A characterization of the Fell topology on CL(X)

In [14], it was conjectured that a sequence is convergent in $\mathrm{CL}(\mathbb{R})$ if and only if the sequential and subsequential limit sets of the sequence are equal. We prove this in the more general setting of a metric space X.

Theorem 2.1 *Let X be metrizable. Let $\{\mathbb{T}_n\}$ be a sequence in CL(X). $\{\mathbb{T}_n\}$ converges in CL(X) if and only if $\mathbb{T} = \mathbb{T}' \neq \emptyset$. Moreover, in this situation, $\{\mathbb{T}_n\}$ converges to $\mathbb{T}$.*

Proof Choose a metric d on X. First, let us suppose that $\mathbb{T} = \mathbb{T}' \neq \emptyset$. We consider two cases of subbasic open sets containing $\mathbb{T}$ in order to prove that $\{\mathbb{T}_n\}$ converges to $\mathbb{T}$.

Case 1: Let $U \subset X$ be open such that $\mathbb{T} \in U^-$. Choose $t \in \mathbb{T} \cap U$ and $\varepsilon > 0$ sufficiently small such that

$$B_\varepsilon(t) = \{\, b \in X \mid d(b, t) < \varepsilon \,\} \subset U.$$

Since $t \in \mathbb{T}$ is a sequential limit point, there exists a sequence $\{t_n\}$ that converges to t through the T_n's. Therefore, there exists N such that $t_n \in B_\varepsilon(t) \subset U$ for all $n \geq N$. Hence, $\mathbb{T}_n \cap U \neq \emptyset$ and $\mathbb{T}_n \in U^-$ for all $n \geq N$.

Case 2: Let $K \subset X$ be compact such that $V = X - K$ and $\mathbb{T} \in V^+$. Assume that there is no N such that $\mathbb{T}_n \in V^+$ for all $n \geq N$. So there exists a subsequence $\{\mathbb{T}_{n_i}\}$ such that $\mathbb{T}_{n_i} \notin V^+$. Therefore, for each i, there exists $t_i \in \mathbb{T}_{n_i} \cap K$. If the set $\{t_i\}$ is finite, then $\{t_i\}$ has a constant subsequence $\{t\}$ for some $t \in K$. Alternatively, the infinite set $\{t_i\}$ has a limit point t in the compact set K. In either case, $t \in \mathbb{T}'$, but $t \notin \mathbb{T}$. This contradicts the fact that $\mathbb{T} = \mathbb{T}'$.

Therefore, $\mathbb{T} = \mathbb{T}' \neq \emptyset$ implies that $\{\mathbb{T}_n\}$ converges to $\mathbb{T}$, which is in $\mathrm{CL}(X)$ by Lemma 2.1.

Conversely, let us suppose that $\{\mathbb{T}_n\}$ converges to $\mathbb{S}$ in $\mathrm{CL}(X)$. We know that $S \neq \emptyset$ since $\mathbb{S} \in \mathrm{CL}(X)$. We wish to show that $\mathbb{S} \subset \mathbb{T} \subset \mathbb{T}' \subset \mathbb{S}$. We know that $\mathbb{T} \subset \mathbb{T}'$. It remains to show that $\mathbb{S} \subset \mathbb{T}$ and $\mathbb{T}' \subset \mathbb{S}$.

First, we choose $s \in \mathbb{S}$. For every $m \in \mathbb{N}$, let

$$U_m = B_{1/m}(s) = \left\{ u \in X \;\middle|\; d(u,s) < \frac{1}{m} \right\}.$$

For every m, $\mathbb{S} \in U_m^-$ since $s \in \mathbb{S} \cap U_m$. Since $\{\mathbb{T}_n\}$ converges to $\mathbb{S}$, for every m, there exists an integer N_m such that $\mathbb{T}_n \in U_m^-$ whenever $n \geq N_m$. If necessary, adjust the sequence $\{N_m\}$ to be increasing. For every m and every integer n such that $N_m \leq n < N_{m+1}$, choose $t_n \in \mathbb{T}_n \cap U_m$. This yields a sequence $\{t_n\}$ that converges to s through the $\mathbb{T}_n$'s. Therefore, $\mathbb{S} \subset \mathbb{T}$.

Next, we choose $t \in \mathbb{T}'$. Thus, there exists a sequence t_{n_i} that converges to t through the $\mathbb{T}_{n_i}$'s. Assume that $t \notin \mathbb{S}$. Choose a cocompact V such that $\mathbb{S} \subset V$ and choose $\varepsilon > 0$ such that

$$B_\varepsilon(t) \cap V = \{\, u \in X \mid d(u,t) < \varepsilon \,\} \cap V = \emptyset.$$

Since $\{\mathbb{T}_n\}$ converges to $\mathbb{S}$, there exists N such that $\mathbb{T}_n \in V^+$ whenever $n \geq N$. For every $n \geq N$ and for every $t' \in \mathbb{T}_n$,

$$d(t', t) \geq \varepsilon > 0.$$

This contradicts that $\{t_{n_i}\}$ converges to t through the $\mathbb{T}_{n_i}$'s. Therefore, $\mathbb{T}' \subset \mathbb{S}$. □

Remark 2.3 In particular, Theorem 2.1 characterizes convergence in $\mathrm{CL}(\mathbb{R})$, the space of all time scales.

2.4 Examples

Example 2.1 The sequence of singleton sets $\{\{n\}\}$ does not converge since its sequential limit set is empty. While we could say the sequence converges to the empty set, we do not include the empty set in $\mathrm{CL}(X)$. Similarly, the sequence of intervals $\{[n, n+1]\}$ fails to converge.

Example 2.2 The sequence of intervals $\{[-n, n]\}$ converges to its sequential limit set $\mathbb{R}$. This fails to converge in the Hausdorff topology since the distance between $\{[-n, n]\}$ and $\mathbb{R}$ is bounded away from 0. (See [14].) How about in the Vietoris topology?

Let us see if the proof of Theorem 2.1 holds for $\{[-n, n]\}$ in the Vietoris topology rather than the Fell topology. That is, we allow V to just be open rather than cocompact.

The sequential limit set is $\mathbb{R}$. If $\mathbb{R} \in V^+$, then $V = \mathbb{R}$. But then $\{[-n, n]\} \in V^+$ for every n and the convergence holds in the Vietoris topology.

Example 2.3 The sequence $\{\mathbb{Z} + \frac{1}{n}\}$ converges to its sequential limit set $\mathbb{Z}$. It also converges in the Hausdorff topology, but not in the Vietoris topology. Here the proof would break down for

$$V = \bigcup_{k=1}^{\infty} \left(k - \frac{1}{k}, k + \frac{1}{k} \right),$$

which is not cocompact. (See [14].)

Example 2.4 The sequence $\frac{1}{n}\mathbb{Z}_+$ converges to its sequential limit set $\mathbb{R}_+$.

2.5 Properties of the Fell topology on $\mathbf{CL}(X)$

Many properties of the Fell topology on $\mathrm{CL}(X)$ for a metrizable space X may be found in [1], wherein references to primary sources can be found.

The Fell topology on the one-point compactification of $\mathrm{CL}(X)$—extended to include the empty set—is compact Hausdorff; we denote this by $\overline{\mathrm{CL}(X)}$. The Fell topology on $\mathrm{CL}(X)$ is locally compact Hausdorff. For example, this implies that the Fell topology on $\mathrm{CL}(X)$ is completely regular.

Since $\overline{\mathrm{CL}(X)}$ is compact, every sequence $\{\mathbb{T}_n\}$ in $\mathrm{CL}(X)$ or $\overline{\mathrm{CL}(X)}$ must have a convergent subsequence. So the subsequential limit set in $\overline{\mathrm{CL}(X)}$ is never empty, but may be $\{\emptyset\}$.

Giving a subset $\mathcal{S} \subset \mathrm{CL}(X)$, the induced topology, the Hausdorff, Vietoris, and Fell topologies always agree if X is a compact metric space. So, when considering uniformly bounded time scales, we can revert to Hausdorff metric.

3 The Topology on The Solution Spaces

Recall that for Hausdorff spaces X and Y, a subbasis for the *compact-open topology* on the set, $C(X, Y)$, of continuous functions from X to Y is given by

$$S(K, U) = \{f \in C(X, Y) \mid K \subset X \text{ is compact, } U \text{ is open in } Y \text{, and } f(K) \subset U\}.$$

If Y is a metric space, this is the *topology of compact convergence*, *i.e.*, sequences converge if and only if they converge uniformly on compact subsets. If X is compact and Y is a metric space, this is the *topology of uniform convergence*.

3.1 The space of continuous functions on time scales

Since we are interested in function spaces over variable domains, we must unite the standard function spaces.

For a closed subset K of X, a function $f : K \to Y$ can be thought of as a *partial function* from X to Y—the domain of definition is K rather than X. By a *partial mapping*, we will mean a continuous partial function. (See [7].) The set of all partial mappings from X to Y is

$$C_F(X, Y) = \cup \{\, C(K, Y) \mid K \in \mathrm{CL}(X) \,\}\,.$$

The subscript "F" is a reminder that we will be using the Fell topology to build a compatible topology on this set. *E.g.*, $C_F(\mathbb{R},\mathbb{R})$ is the set of all continuous real-valued functions on time scales.

Suppose that X and Y are metric spaces. So $X \times Y$ is metrizable. We wish to give a topology on $C_F(X,Y)$ that is consistent with the compact-open topology on $C(X,Y)$.

Consider the function $\mathrm{Gr} : C_F(X,Y) \to \mathrm{CL}(X \times Y)$ that sends each partial mapping to its graph. Since Gr is injective, we can pull back the Fell topology on $\mathrm{CL}(X \times Y)$ to give a topology on $C_F(X,Y)$: $\mathcal{S}$ is open in $C_F(X,Y)$ if and only if $\mathrm{Gr}(\mathcal{S})$ is open in $\mathrm{Gr}(C_F(X,Y))$ as a subspace of $\mathrm{CL}(X \times Y)$.

Following [7], Theorem 3.1 follows from the facts that projection from $X \times Y$ to X is continuous and induces a continuous mapping from $\mathrm{CL}(X \times Y)$ to $\mathrm{CL}(X)$.

Theorem 3.1 *The canonical projection $\pi : C_F(X,Y) \to CL(X)$ is continuous.*

3.2 The case of unique solutions

Recall the goal proposed in subsection 1.2. We examine the case of a dynamic equation whose solutions are always unique (for example, $x^{\Delta} = 0$).

Let $\mathcal{S}$ denote the set of all solutions of a given initial value problem over all possible time scales. Consider the restriction of the projection π:

$$\pi_{\mathcal{S}} : \mathcal{S} \to \mathrm{CL}(\mathbb{R}).$$

That is, an element of $\mathcal{S}$, a solution $f : \mathbb{T} \to \mathbb{R}$ of the initial value problem, projects to its domain, $\mathbb{T}$. Since all solutions are unique on their domains, $\pi_{\mathcal{S}}$ is a bijection onto its range. The construction of the topology on $C_F(X,Y)$ now shows the following:

Corollary 3.1 *$\pi_{\mathcal{S}}$ is a homeomorphism onto its range.*

3.3 Open problem: the case of non-unique solutions

In the non-unique case, the projection $\pi_{\mathcal{S}}$ may be far more interesting. Hopefully, the topology will tell us something about the dynamics. A question to whet one's appetite: can there be monodromy ? Can we lift a loop with a base point in the space of time scales so that we start and end at different solutions?

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Independent of this paper, Esty and Hilger have concluded, in [5], that the Fell topology is best suited for the space of time scales. They give an interesting characterization of the Fell topology that extends the topologies induced by the Hausdorff metric on compact sets. The present paper seeks to extend the same topologies from the viewpoint of the Vietoris topology as a hit-and-miss topology. This seems to be somewhat more natural and dynamic. Probably that is because of the similarity of the hit-and-miss constructions of the Vietoris and Fell topologies; it is far less natural to think of the Hausdorff topology as a hit-and-miss topology.

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