Nonlinear Dynamics and Systems Theory, **9** (4) (2009) 361–374

Positive Solutions of Semipositone Singular Dirichlet Dynamic Boundary Value Problems

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Received: June 25, 2008; Revised: September 3, 2009

Abstract: We obtain a sufficient condition for the existence of a positive solution for a second-order superlinear semipositone singular Dirichlet dynamic boundary value problem by constructing a special cone. As a special case when $\mathbb{T} = \mathbb{R}$, this result includes those of Zhang and Liu [9]. This result is new in a general time scale setting and can be applied to q-difference equations. Two examples are given at the end of this paper to demonstrate the result.

Keywords: *semipositone; cone; time scale; delta derivative; nabla derivative.*

Mathematics Subject Classification (2000): 39A10, 34B10.

1 Introduction

We consider the following Dirichlet boundary value problem (BVP)

$$Lx = f(t,x) + h(t), \quad t \in (\rho(a), \sigma(b))_{\mathbb{T}}, \tag{1.1}$$

$$x(\rho(a)) = 0, \tag{1.2}$$

$$x(\sigma(b)) = 0, \tag{1.3}$$

where the operator L is defined by $Lx := -\left(p(t)x^{\Delta}\right)^{\nabla}$, and $\mathbb{T}$ is a time scale containing a and b. We define the time scale interval $(a,b)_{\mathbb{T}}$ by $(a,b)_{\mathbb{T}} := (a,b) \cap \mathbb{T}$, and similarly for other types of intervals. If $\mathbb{T}$ has a right-scattered minimum m, we define $\mathbb{T}_{\kappa} := \mathbb{T} \setminus \{m\}$; otherwise, we set $\mathbb{T}_{\kappa} = \mathbb{T}$. The backward graininess ν is defined by $\nu := t - \rho(t)$. Then

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the nabla derivative of x at t, denoted by $x^{\nabla}(t)$, is defined to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|x(\rho(t)) - x(s) - x^{\nabla}(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \forall s \in U.$$

For $\mathbb{T} = \mathbb{R}$, we have $x^{\nabla} = x'$, and for $\mathbb{T} = \mathbb{Z}$, we have $x^{\nabla}(t) = \nabla x(t) = x(t) - x(t-1)$, which is the *backward difference operator*. An introduction of Time Scales Calculus can be found in Chapter 1 of [4], and in [5]. The domain D of L is the set of functions $x : \mathbb{T} \to \mathbb{R}$ such that x is continuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$, x^{Δ} is continuous on $[\rho(a), b]_{\mathbb{T}}$, and $\big(p(t)x^{\Delta}\big)^{\nabla}$ is continuous on $[a, b]_{\mathbb{T}}$. Since f may have singularities with respect to t at one or both end points, we shall assume, either f is continuous on $(a, b)_{\mathbb{T}} \times \mathbb{R}$ if f is singular at both a and b, or f is continous on $(a, b]_{\mathbb{T}} \times \mathbb{R}$ if f is not singular at b, or f is continuous on $[a, b)_{\mathbb{T}} \times \mathbb{R}$ if f is not singular at a. If either f or h has a singularity at a, we assume $\rho(a) = a = \sigma(a)$, and if f or h has a singularity at b, then we assume $\rho(b) = b = \sigma(b)$. Let a and b be such that $0 \leq \rho(a) \leq a < b < \infty$ with $(a, b)_{\mathbb{T}} \neq \phi$, and $h : (\rho(a), \sigma(b))_{\mathbb{T}} \to (-\infty, \infty)$ is Lebesgue ∇-integrable. Also $p > 0$ is continuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$, and there are constants m, M such that

$$0 < m \leq p(t) \leq M.$$

The BVP (1.1) $-$ (1.3) arises in chemical reactor theory [2] when we consider the domain to be the set of real numbers. Since the function $h(t)$ in the above BVP may change sign we say this type of problem is semipositone. Special cases are studied in [8], [1] and the references therein. In the applications one is interested in finding positive solutions.

We impose the following conditions:

($\mathbf{H_1}$) For any $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, $f(t, 1) > 0$, and there exist constants $\lambda_1 \geq \lambda_2 > 1$ such that for any $(t, u) \in (\rho(a), \sigma(b))_{\mathbb{T}} \times [0, \infty)$

$$c^{\lambda_1} f(t, u) \leq f(t, cu) \leq c^{\lambda_2} f(t, u), \qquad c \in [0, 1]. \tag{1.4}$$

($\mathbf{H_2}$) Let $r := \frac{M^3(\sigma(b)-\rho(a))}{m^4} \int_{\rho(a)}^{b} h^{-}(t)\nabla t > 0$, where m, and M are such that $0 < m \leq p(t) \leq M$, and $h^{\pm}(t) := \max\{\pm h(t), 0\}$, and assume

$$\int_{\rho(a)}^{b} (s - \rho(a))(\sigma(b) - s)[f(s, 1) + h^{+}(s)]\nabla s < \frac{m^2 r(\sigma(b) - \rho(a))}{M[(r+1)^{\lambda_1} + 1]}. \tag{1.5}$$

Remark 1.1 Note that it is easy to see for $c \geq 1$, from (1.4) that

$$c^{\lambda_2} f(t, u) \leq f(t, cu) \leq c^{\lambda_1} f(t, u) \tag{1.6}$$

for any $(t, u) \in (\rho(a), \sigma(b))_{\mathbb{T}} \times [0, \infty)$.

A solution u_0 of the BVP (1.1) $-$ (1.3) with $u_0(t) > 0$, $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, is called positive solution of the BVP (1.1) $-$ (1.3).

2 Preliminary Lemmas

We state the following lemmas which we will use later in this section.

Lemma 2.1 *[7] Let X be a real Banach space, Ω be a bounded open subset of X with $0 \in \Omega$, and $A : \bar{\Omega} \cap P \to P$ be a completely continuous operator, where P is a cone in X.*
(i) Suppose that $Au \neq \lambda u$, for all $u \in \partial\Omega \cap P$, $\lambda \geq 1$. Then $i(A, \Omega \cap P, P) = 1$.
(ii) Suppose that $Au \nleq u$, for all $u \in \partial\Omega \cap P$. Then $i(A, \Omega \cap P, P) = 0$.

Lemma 2.2 *If $f(t,u)$ satisfies (H_1), then for any $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, $f(t,u)$ is non-decreasing in $u \in [0,\infty)$, and for any nonempty $[\alpha,\beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$,*

$$\lim_{u\to\infty} \min_{t\in[\alpha,\beta]_{\mathbb{T}}} \frac{f(t,u)}{u} = \infty.$$

Proof Let $t \in (\rho(a), \sigma(b))_{\mathbb{T}}$, and $x, y \in [0,\infty)$ be arbitrary. Without loss of generality assume $0 \leq x \leq y$. Now, if $y = 0$, then $f(t,x) \leq f(t,y)$ is clear. If $y \neq 0$, let $c_0 = \frac{x}{y}$, then $0 \leq c_0 \leq 1$. Now by (1.4),

$$f(t,x) = f(t, c_0 y) \leq c_0^{\lambda_2} f(t,y) \leq f(t,y).$$

Thus $f(t,u)$ is non-decreasing in u on $[0,\infty)$.

Next choose $u > 1$. Then it follows from (1.6) that $f(t,u) \geq u^{\lambda_2} f(t,1)$. So we get

$$\frac{f(t,u)}{u} \geq u^{\lambda_2 - 1} f(t,1), \quad \forall t \in (\rho(a), \sigma(b))_{\mathbb{T}}.$$

So for any nonempty $[\alpha,\beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$\min_{t\in[\alpha,\beta]_{\mathbb{T}}} \frac{f(t,u)}{u} \geq u^{\lambda_2 - 1} \min_{t\in[\alpha,\beta]_{\mathbb{T}}} f(t,1).$$

Since $f(t,1) > 0$ (by (H_1)),

$$\lim_{u\to\infty} \min_{t\in[\alpha,\beta]_{\mathbb{T}}} \frac{f(t,u)}{u} = \infty. \quad \square$$

Let $X := \{x \in C([\rho(a), \sigma(b)]_{\mathbb{T}}, \mathbb{R})\}$ with $||x|| = \sup_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}} |x(t)|$, and define

$$P := \{x \in X : x(t) \geq 0,\ t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\},$$
$$Q := \{x \in P : x(t) \geq ||x|| \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2},\ t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\},$$

where $0 < m \leq p(t) \leq M$.

Then one can easily verify that X is a real Banach space, and P, Q are cones in X, and clearly $Q \subset P$.

Note that the Green's function for the BVP

$$\begin{aligned} -(p(t)x^{\Delta})^{\nabla} &= 0, \qquad t \in (\rho(a), \sigma(b))_{\mathbb{T}} \\ x(\rho(a)) &= 0 \\ x(\sigma(b)) &= 0 \end{aligned}$$

can be shown to be given by (see [3] for more information)

$$G(t,s) = \frac{1}{\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)}\Delta\tau} \begin{cases} \int_{\rho(a)}^{t} \frac{1}{p(\tau)}\Delta\tau \int_{s}^{\sigma(b)} \frac{1}{p(\tau)}\Delta\tau, & \text{for } t \le s; \\ \int_{\rho(a)}^{s} \frac{1}{p(\tau)}\Delta\tau \int_{t}^{\sigma(b)} \frac{1}{p(\tau)}\Delta\tau, & \text{for } s \le t. \end{cases} \tag{2.1}$$

Also note that

$$0 \le G(t,s) \le G(s,s) = \frac{\int_{\rho(a)}^{s} \frac{1}{p(\tau)}\Delta\tau \int_{s}^{\sigma(b)} \frac{1}{p(\tau)}\Delta\tau}{\int_{\rho(a)}^{\sigma(b)} \frac{1}{p(\tau)}\Delta\tau} \le \frac{M(s-\rho(a))(\sigma(b)-s)}{m^2(\sigma(b)-\rho(a))}. \tag{2.2}$$

Now set $w(t) := \int_{\rho(a)}^{b} G(t,s)h^{-}(s)\nabla s,$ where $G(t,s)$ is as defined above. Then $w(t)$ is the unique solution of the BVP

$$\left(p(t)x^{\Delta}\right)^{\nabla} + h^{-}(t) = 0, \quad t \in (\rho(a), \sigma(b))_{\mathbb{T}}, \quad x(\rho(a)) = 0 = x(\sigma(b)). \tag{2.3}$$

To see that $w(t)$ is well defined note that

$$\begin{aligned} w(t) &= \int_{\rho(a)}^{b} G(t,s)h^{-}(s)\nabla s \\ &\le \int_{\rho(a)}^{b} G(s,s)h^{-}(s)\nabla s \\ &\le \frac{M(\sigma(b)-\rho(a))}{m^2} \int_{\rho(a)}^{b} h^{-}(s)\nabla s \\ &< \infty, \qquad \text{for all } t \in [\rho(a), \sigma(b)]_{\mathbb{T}}. \end{aligned}$$

Also,

$$\begin{aligned} w(\rho(a)) &= \int_{\rho(a)}^{b} G(\rho(a),s)h^{-}(s)\nabla s = 0. \\ w(\sigma(b)) &= \int_{\rho(a)}^{b} G(\sigma(b),s)h^{-}(s)\nabla s = 0. \end{aligned}$$

It remains to show that

$$-\left(p(t)w^{\Delta}\right)^{\nabla} = h^{-}(t). \tag{2.4}$$

To verify this last statement we will use the formulas [5][Theorem 5.37]

$$\begin{aligned} \left(\int_a^t f(t,s)\nabla s\right)^{\Delta} &= \int_a^t f^{\Delta}(t,s)\nabla s + f(\sigma(t),\sigma(t)); \\ \left(\int_a^t f(t,s)\nabla s\right)^{\nabla} &= \int_a^t f^{\nabla}(t,s)\nabla s + f(\rho(t),t). \end{aligned}$$

Note that

$$w(t)=\frac{1}{\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau}\left[\int_{\rho(a)}^{t}\left(\int_{\rho(a)}^{s}\frac{1}{p(\tau)}\Delta\tau\int_{t}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right)h^{-}(s)\nabla s\right.$$
$$\left.+\int_{t}^{b}\left(\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau\int_{s}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right)h^{-}(s)\nabla s\right]$$

Then we get,

$$\begin{aligned}\left(\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right)w^{\Delta}(t)&=\left(\int_{\rho(a)}^{t}\int_{\rho(a)}^{s}\frac{1}{p(\tau)}\Delta\tau\int_{t}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\right)^{\Delta}\\&\quad+\left(\int_{t}^{b}\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau\int_{s}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\right)^{\Delta}\\&=-\int_{\rho(a)}^{t}\frac{1}{p(t)}\int_{\rho(a)}^{s}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\\&\quad+\int_{\rho(a)}^{\sigma(t)}\frac{1}{p(\tau)}\Delta\tau\int_{\sigma(t)}^{\sigma(b)}\frac{1}{p(\tau)}\nabla\tau\, h^{-}(\sigma(t))\\&\quad+\int_{t}^{b}\frac{1}{p(t)}\int_{s}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\\&\quad-\int_{\rho(a)}^{\sigma(t)}\frac{1}{p(\tau)}\Delta\tau\int_{\sigma(t)}^{\sigma(b)}\frac{1}{p(\tau)}\nabla\tau\, h^{-}(\sigma(t))\\&=-\int_{\rho(a)}^{t}\frac{1}{p(t)}\int_{\rho(a)}^{s}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\\&\quad+\int_{t}^{b}\frac{1}{p(t)}\int_{s}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s.\end{aligned}$$

So,

$$\begin{aligned}-\left(p(t)w^{\Delta}\right)^{\nabla}(t)&=\frac{1}{\left(\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right)}\left[\left(\int_{\rho(a)}^{t}\int_{\rho(a)}^{s}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\right)^{\nabla}\right.\\&\qquad\left.-\left(\int_{t}^{b}\int_{s}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(s)\nabla s\right)^{\nabla}\right]\\&=\frac{1}{\left(\int_{\rho(a)}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\right)}\left(\int_{\rho(a)}^{t}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(t)+\int_{t}^{\sigma(b)}\frac{1}{p(\tau)}\Delta\tau\, h^{-}(t)\right)\\&=h^{-}(t).\end{aligned}$$

Now we define an operator T on P by

$$(Tu)(t):=\int_{\rho(a)}^{b}G(t,s)\left[f(s,[u-w]^{+}(s))+h^{+}(s)\right]\nabla s,\quad t\in[\rho(a),\sigma(b)]_{\mathbb{T}}.$$

Claim: $T : P \to P$.
Proof of claim: Let $u \in P$ be fixed but arbitrary. Choose $0 < c < 1$ such that $c\,||u|| < 1$. Then

$$c[u-w]^+(s) \le cu(s) \le c\,||u|| < 1$$

Then by (1.4), (1.6), and Lemma 2.2, we get, for all $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$f(t,[u-w]^+(t)) \le \left(\frac{1}{c}\right)^{\lambda_1} f(t, c[u-w]^+(t)) \le c^{\lambda_2-\lambda_1}\,||u||^{\lambda_2} f(t,1). \tag{2.5}$$

So for any $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get using (2.2), (2.5), and (1.5) that

$$\begin{aligned}(Tu)(t) &= \int_{\rho(a)}^{b} G(t,s)[f(s,[u-w]^+(s)) + h^+(s)]\,\nabla s \\ &\le \int_{\rho(a)}^{b} G(s,s)\left[c^{\lambda_2-\lambda_1}\,||u||^{\lambda_2} f(s,1) + h^+(s)\right]\nabla s \\ &\le \frac{M\left(c^{\lambda_2-\lambda_1}\,||u||^{\lambda_2}+1\right)}{m^2(\sigma(b)-\rho(a))}\int_{\rho(a)}^{b}(s-\rho(a))(\sigma(b)-s)[f(s,1)+h^+(s)]\,\nabla s \\ &< \infty.\end{aligned}$$

Note that $Tu \in C[\rho(a), \sigma(b)]_{\mathbb{T}}$, and $Tu(t) \ge 0, \quad \forall t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ are clear.

Thus $T : P \to P$ is well defined.

So from the definition of the operator T, we can easily prove the following theorem:

Theorem 2.1 *Suppose that* (H_1)*, and* (H_2) *hold. Then the operator* T *has a fixed point in* $C[\rho(a), \sigma(b)]_{\mathbb{T}}$ *iff the BVP*

$$\begin{cases}\left(p(t)u^{\Delta}\right)^{\nabla} + f(t,[u-w]^+(t)) + h^+(t) = 0 & \rho(a) < t < \sigma(b)\\ u(\rho(a)) = 0 = u(\sigma(b))\end{cases} \tag{2.6}$$

has a positive solution where w *is given as in* (2.3).

Proof The operator T has a fixed point u,

$$\begin{aligned}&\Longrightarrow\ u(t) = (Tu)(t)' \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}} \\ &\Longrightarrow\ u(t) = \int_{\rho(a)}^{b} G(t,s)\left[f(s,[u-w]^+(s)) + h^+(s)\right]\nabla s, \quad u(\rho(a)) = 0 = u(\sigma(b))\end{aligned}$$

Now using properties of the Green's function (the same steps that are used above to verify (2.4)), we get

$$-\left(p(t)u^{\Delta}\right)^{\nabla} = f(t,[u-w]^+(t)) + h^+(t), \quad u(\rho(a)) = 0 = u(\sigma(b)).$$

The other direction is similar. □

Now we have the following lemma:

Lemma 2.3 *If the singular BVP* (2.6) *has a positive solution* $u(t) \geq w(t)$ *for all* $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, *then the BVP* (1.1) – (1.3) *has a* $C[a,b]_{\mathbb{T}} \cap C^2(a,b)_{\mathbb{T}}$ *positive solution* $y(t) = u(t) - w(t), \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$.

Proof Let $u(t) = y(t) + w(t), \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. Then by the first equation of (2.6), it follows that

$$\begin{aligned}
\left(p(t)y^{\Delta}\right)^{\nabla} + \left(p(t)w^{\Delta}\right)^{\nabla} + f(t, y(t)) + h^{+}(t) &= 0,\\
\text{i.e.,}\quad \left(p(t)y^{\Delta}\right)^{\nabla} - h^{-}(t) + f(t, y(t)) + h^{+}(t) &= 0,\\
\text{i.e.,}\quad \left(p(t)y^{\Delta}\right)^{\nabla} + f(t, y(t)) + h(t) &= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
y(\rho(a)) &= u(\rho(a)) - w(\rho(a)) = 0,\\
y(\sigma(b)) &= u(\sigma(b)) - w(\sigma(b)) = 0.
\end{aligned}$$

Thus $y(t) = u(t) - w(t)$ is a positive solution of the BVP (1.1) – (1.3). □

Lemma 2.4 *Assume* (H_1) *and* (H_2) *hold. Then* $T : Q \to Q$ *is a completely continuous operator.*

Proof For any $u \in Q$, let $y(t) = Tu(t)$. Then $y(\rho(a)) = 0 = y(\sigma(b))$. So there exists $t_0 \in (\rho(a), \sigma(b))$ such that $||y|| = y(t_0)$. Note that for any $t, s \in (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$\begin{aligned}
\frac{G(t,s)}{G(t_0,s)} &\geq \begin{cases}
\frac{m(t-\rho(a))}{M(t_0-\rho(a))}, & \text{for } t, t_0 \leq s;\\
\frac{m^2(t-\rho(a))(\sigma(b)-s)}{M^2(s-\rho(a))(\sigma(b)-t_0)}, & \text{for } t \leq s \leq t_0;\\
\frac{m(\sigma(b)-t)}{M(\sigma(b)-t_0)}, & \text{for } t, t_0 \geq s;\\
\frac{m^2(s-\rho(a))(\sigma(b)-t)}{M^2(t_0-\rho(a))(\sigma(b)-s)}, & \text{for } t \geq s \geq t_0;
\end{cases}\\
&\geq \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2}.
\end{aligned}$$

Then for all $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$\begin{aligned}
y(t) = (Tu)(t) &= \int_{\rho(a)}^{b} G(t,s)\left[f(s, [u-w]^{+}(s)) + h^{+}(s)\right]\nabla s\\
&= \int_{\rho(a)}^{b} \frac{G(t,s)}{G(t_0,s)} G(t_0,s)[f(s,[u-w]^{+}(s)) + h^{+}(s)]\nabla s\\
&\geq \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} y(t_0)\\
&= \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2}\, ||y||\,.
\end{aligned}$$

Thus, $Tu \in Q$, and hence $T : Q \to Q$.

Next we show that $T : Q \to Q$ is a completely continuous operator.

First we show $T : Q \to Q$ is continuous. Let $\{x_n\}_{n=0}^{\infty} \subset Q$ be such that $x_n \to x_0$ when $n \to \infty$. Then there is a constant $M_1 > 0$ such that $||x_n|| \le M_1$ for all $n = 0, 1, 2, \cdots$. Since for any $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$[x_n - w]^+(s) \le x_n(s) \le ||x_n|| \le M_1 < M_1 + 1,$$

by (1.6), and Lemma 2.2 (since (H_1) holds for f), we get

$$\begin{aligned} f(s, [x_n - w]^+(s)) + h^+(s) &\le f(s, M_1 + 1) + h^+(s) \\ &\le (M_1 + 1)^{\lambda_1} f(s, 1) + h^+(s) \\ &\le \left[(M_1 + 1)^{\lambda_1} + 1\right] \left[f(s, 1) + h^+(s)\right]. \end{aligned}$$

Then using (2.2) and (1.5), we get

$$\begin{aligned} &\int_{\rho(a)}^{b} G(t, s) \left[f(s, [x_n - w]^+(s)) + h^+(s)\right] \nabla s \\ &\quad \le \left[(M_1 + 1)^{\lambda_1} + 1\right] \int_{\rho(a)}^{b} G(s, s) \left[f(s, 1) + h^+(s)\right] \nabla s \\ &\quad \le \frac{M\left[(M_1 + 1)^{\lambda_1} + 1\right]}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^{b} (s - \rho(a))(\sigma(b) - s) \left[f(s, 1) + h^+(s)\right] \nabla s \\ &\quad < \infty. \end{aligned}$$

Note that by the continuity of f,

$$\lim_{n\to\infty} f(s, [x_n - w]^+(s)) = f(s, [x_0 - w]^+(s)).$$

Then by the Lebesgue Dominated Convergence Theorem [5, page 159], we get

$$\begin{aligned} &\lim_{n\to\infty} ||Tx_n - Tx_0|| \\ &\quad = \lim_{n\to\infty} \sup_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}} |Tx_n - Tx_0| \\ &\quad \le \lim_{n\to\infty} \sup_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}} \int_{\rho(a)}^{b} G(t, s) \left|f(s, [x_n - w]^+(s)) - f(s, [x_0 - w]^+(s))\right| \nabla s \\ &\quad \le \lim_{n\to\infty} \int_{\rho(a)}^{b} \frac{M(s - \rho(a))(\sigma(b) - s)}{m^2(\sigma(b) - \rho(a))} \left|f(s, [x_n - w]^+) - f(s, [x_0 - w]^+)\right| \nabla s \\ &\quad \le \frac{M}{m^2(\sigma(b) - \rho(a))} \int_{\rho(a)}^{b} (s - \rho(a))(\sigma(b) - s) \lim_{n\to\infty} \big|f(s, [x_n - w]^+(s)) \\ &\qquad\qquad - f(s, [x_0 - w]^+(s))\big| \nabla s \\ &\quad = 0. \end{aligned}$$

Thus $T : Q \to Q$ is continuous.

Finally, we show that $T : Q \to Q$ is relatively compact.
To see this let $D \subset Q$ be any bounded set. Then there exists $M_2 > 0$ such that $||x|| \le M_2$ for all $x \in D$. So for any $x \in D$ and $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get

$$[x - w]^+(s)) \le x(s) \le ||x|| \le M_2 < M_2 + 1.$$

So for all $s \in [\rho(a), \sigma(b)]_{\mathbb{T}}$,

$$f(s, [x-w]^+(s)) + h^+(s) \le f(s, M_2+1) + h^+(s) \le \left[(M_2+1)^{\lambda_1}+1\right]\left[f(s,1)+h^+(s)\right].$$

Then for all $x \in D$, and $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, we get using (2.2) and (1.5),

$$\begin{aligned} |Tx(t)| &= \left|\int_{\rho(a)}^{b} G(t,s)\left[f(s,[x-w]^+(s))+h^+(s)\right]\nabla s\right| \\ &\le \frac{M\left[(M_2+1)^{\lambda_1}+1\right]}{m^2(\sigma(b)-\rho(a))}\int_{\rho(a)}^{b}(s-\rho(a))(\sigma(b)-s)\left[f(s,1)+h^+(s)\right]\nabla s \\ &< \infty. \end{aligned}$$

Thus $T(D)$ is uniformly bounded.

Again by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} |Tx(t_1)-Tx(t_2)| &\le \int_{\rho(a)}^{b}|G(t_1,s)-G(t_2,s)|\left[f(s,[x-w]^+(s))+h^+(s)\right]\nabla s \\ &\le \left[(M_2+1)^{\lambda_1}+1\right]\int_{\rho(a)}^{b}|G(t_1,s)-G(t_2,s)|\left[f(s,1)+h^+(s)\right]\nabla s \\ &\longrightarrow 0 \quad \text{as } t_1 \to t_2. \end{aligned}$$

Since this is true for any $t_1, t_2 \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and the RHS is independent of x, $T(D)$ is equicontinuous on $[\rho(a), \sigma(b)]_{\mathbb{T}}$. Then by the Arzela-Ascoli Theorem, $T : Q \to Q$ is relatively compact.

Thus, $T : Q \to Q$ is a completely continuous operator. □

Lemma 2.5 *Assume* (H_1) *and* (H_2) *hold. Let* $Q_r = \{x \in Q : ||x|| < r\}$, *and* $\partial Q_r = \{x \in Q : ||x|| = r\}$, *where* $r := \frac{M^3(\sigma(b)-\rho(a))}{m^4}\int_{\rho(a)}^{b} h^-(t)\nabla t$ *as defined in* (H_2). *Then* $i(T, Q_r, Q) = 1$.

Proof Assume that there exist $z_0 \in \partial Q_r$, $\mu \ge 1$ such that $\mu z_0 = Tz_0$. Then $z_0 = \frac{1}{\mu}Tz_0$, and $0 < \frac{1}{\mu} \le 1$. Since $z_0 \in \partial Q_r$,

$$[z_0-w]^+(s) \le z_0(s) \le ||z_0|| = r < r+1,$$

then for $s \in (\rho(a), \sigma(b))_{\mathbb{T}}$, we get

$$f(s,[z_0-w]^+(s)) + h^+(s) \le \left[(r+1)^{\lambda_1}+1\right]\left[f(s,1)+h^+(s)\right].$$

Now

$$\begin{aligned} r = ||z_0|| &= \left|\left|\frac{1}{\mu}Tz_0\right|\right| \\ &\le ||Tz_0|| \\ &= \sup_{t\in[\rho(a),\sigma(b)]}\left|\int_{\rho(a)}^{b} G(t,s)\left[f(s,[z_0-w]^+(s))+h^+(s)\right]\nabla s\right| \\ &\le \int_{\rho(a)}^{b} G(s,s)\left[f(s,[z_0-w]^+(s))+h^+(s)\right]\nabla s \\ &\le \frac{M[(r+1)^{\lambda_1}+1]}{m^2[\sigma(b)-\rho(a)]}\int_{\rho(a)}^{b}(s-\rho(a))(\sigma(b)-s)[f(s,1)+h^+(s)]\nabla s. \end{aligned}$$

This implies,

$$\int_{\rho(a)}^{b}(s-\rho(a))(\sigma(b)-s)[f(s,1)+h^{+}(s)]\nabla s \geq \frac{m^2 r(\sigma(b)-\rho(a))}{M[(r+1)^{\lambda_1}+1]}$$

which is a contradiction to (1.5). So $Tz_0 \neq \mu z_0$ for all $z_0 \in \partial Q_r,\ \mu \geq 1$.

Then by Lemma 2.1, $i(T, Q_r, Q) = 1$. □

Lemma 2.6 *Assume* (H_1) *holds. Then there exists a constant* $R > r$ *such that* $i(T, Q_R, Q) = 0$ *where* $Q_R := \{x \in Q : ||x|| < R\}$*, and* $\partial Q_R := \{x \in Q : ||x|| = R\}$.

Proof Assume $x \not\geq Tx$ for all $x \in \partial Q_R$ is false. Then there exists $y_1 \in \partial Q_R$ such that $y_1 \geq Ty_1$.

Choose constants α, β so that $[\alpha, \beta]_{\mathbb{T}} \subset (\rho(a), \sigma(b))_{\mathbb{T}}$, and K such that

$$K > \frac{2M^2(\sigma(b)-\rho(a))^2}{m^2(\alpha-\rho(a))(\sigma(b)-\beta)\max_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}}\int_{\alpha}^{\beta}G(t,s)\nabla s}. \tag{2.7}$$

From Lemma 2.2 there exists $R_1 > 2r$ such that when $t \in [\alpha, \beta]_{\mathbb{T}}$, and $x \geq R_1$, we get

$$\frac{f(t,x)}{x} \geq K$$

That is,

$$f(t,x) \geq Kx, \quad t \in [\alpha,\beta]_{\mathbb{T}}, \quad x \geq R_1.$$

Let

$$R \geq \frac{2R_1 M^2(\sigma(b)-\rho(a))^2}{m^2(\alpha-\rho(a))(\sigma(b)-\beta)}. \tag{2.8}$$

Then clearly $R > R_1 > 2r$, and so $\frac{r}{R} < \frac{1}{2}$.

Now for the above mentioned y_1, we have for all $t \in [\alpha, \beta]_{\mathbb{T}}$,

$$\begin{aligned}
y_1(t) - w(t) &= y_1(t) - \int_{\rho(a)}^{b} G(t,s)h^{-}(s)\nabla s \\
&\geq y_1(t) - \frac{M(t-\rho(a))(\sigma(b)-t)}{m^2(\sigma(b)-\rho(a))}\int_{\rho(a)}^{b} h^{-}(s)\nabla s \\
&= y_1(t) - \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2}r \\
&\geq y_1(t) - \frac{y_1(t)}{||y_1||}r = y_1(t) - \frac{r}{R}y_1(t) \\
&\geq y_1(t) - \frac{1}{2}y_1(t) = \frac{1}{2}y_1(t) \\
&\geq \frac{1}{2}\,||y_1||\,\frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} \quad (\text{as } y_1 \in Q) \\
&\geq \frac{1}{2}R\frac{m^2(\alpha-\rho(a))(\sigma(b)-\beta)}{M^2(\sigma(b)-\rho(a))^2} \qquad (2.9) \\
&\geq R_1 > 0. \quad (\text{using (2.8)})
\end{aligned}$$

So,

$$\begin{aligned}
R &= \|y_1\| \geq y_1(t) \\
&\geq Ty_1(t) = \int_{\rho(a)}^{b} G(t,s)\left[f(s,[y_1-w]^+(s)) + h^+(s)\right]\nabla s \\
&\geq \int_{\alpha}^{\beta} G(t,s)\left[f(s,(y_1(s)-w(s)) + h^+(s)\right]\nabla s \\
&\geq \int_{\alpha}^{\beta} G(t,s)f(s,(y_1(s)-w(s)))\nabla s \\
&\geq \int_{\alpha}^{\beta} G(t,s)K(y_1(s)-w(s))\nabla s \\
&\geq \int_{\alpha}^{\beta} G(t,s)K\frac{1}{2}R\frac{m^2(\alpha-\rho(a))(\sigma(b)-\beta)}{M^2(\sigma(b)-\rho(a))^2}\nabla s \quad \text{(using (2.9))} \\
&= \frac{1}{2}KR\frac{m^2(\alpha-\rho(a))(\sigma(b)-\beta)}{M^2(\sigma(b)-\rho(a))^2}\int_{\alpha}^{\beta} G(t,s)\nabla s, \quad \forall t\in[\rho(a),\sigma(b)]_{\mathbb{T}} \\
&\geq \frac{1}{2}KR\frac{m^2(\alpha-\rho(a))(\sigma(b)-\beta)}{M^2(\sigma(b)-\rho(a))^2}\max_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}}\int_{\alpha}^{\beta} G(t,s)\nabla s \\
\Rightarrow K &\leq \frac{2M^2(\sigma(b)-\rho(a))^2}{m^2(\alpha-\rho(a))(\sigma(b)-\beta)\max_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}}\int_{\alpha}^{\beta} G(t,s)\nabla s}
\end{aligned}$$

which is a contradiction to our choice of K above.

Thus $x \ngeq Tx$ for all $x \in \partial Q_R$, so by Lemma 2.1, we get

$$i(T, Q_R, Q) = 0. \ \square$$

3 Main Result

Now we state and prove our main result.

Theorem 3.1 *Suppose that* (H_1)*, and* (H_2) *hold. Then the BVP* $(1.1)-(1.3)$ *has at least one* $C[a,b]_{\mathbb{T}} \cap C^2(a,b)_{\mathbb{T}}$ *positive solution* $u_0(t)$*, and there exists* $k > 0$ *such that* $u_0(t) \geq k(t-\rho(a))(\sigma(b)-t)$*,* $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$*.*

Proof By Lemmas 2.5, 2.6, and by a property of the fixed point index, we get

$$\begin{aligned}
i(T, Q_R \setminus \bar{Q}_r, Q) &= i(T, Q_R, Q) - i(T, Q_r, Q) \\
&= 0 - 1 \\
&= -1 \quad (\neq 0).
\end{aligned}$$

So T has a fixed point z_0 in $Q_R \setminus \bar{Q}_r$, with $r < \|z_0\| < R$.

Then for all $t \in [\rho(a), \sigma(b)]$,

$$\begin{aligned}
z_0(t) - w(t) &\geq ||z_0|| \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} - \int_{\rho(a)}^{b} G(t,s)h^-(s)\nabla s \\
&\geq ||z_0|| \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} - \frac{M(t-\rho(a))(\sigma(b)-t)}{m^2(\sigma(b)-\rho(a))} \int_{\rho(a)}^{b} h^-(s)\nabla s \\
&= ||z_0|| \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} - r\frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2} \\
&= \frac{m^2(t-\rho(a))(\sigma(b)-t)}{M^2(\sigma(b)-\rho(a))^2}[||z_0|| - r] \\
&= k(t-\rho(a))(\sigma(b)-t) \quad \text{where } k := \frac{m^2[||z_0|| - r]}{M^2(\sigma(b)-\rho(a))^2} > 0 \\
&\geq 0, \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}.
\end{aligned}$$

Now let $u_0(t) := z_0(t) - w(t),\ t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. Then from Lemma 2.3, it follows that $u_0(t)$ is a positive solution of the BVP (1.1) – (1.3), and there exists a $k > 0$ such that $u_0(t) \geq k(t-\rho(a))(\sigma(b)-t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. The proof is now completed. □

4 Examples

In this section we give two examples as applications of Theorem 3.1.

Example 4.1 Let $\mathbb{T} = \left\{\frac{1}{q^n}\right\}_{n=0}^{\infty} \cup \{0, 2\}$, $q > 1$. Then we claim the BVP

$$\begin{cases} u^{\Delta\nabla} + \frac{u^{3/2}}{5t} - \frac{1}{\sqrt{t}} = 0, \quad t \in (0,2)_{\mathbb{T}}, \\ u(0) = 0 = u(2) \end{cases} \tag{4.1}$$

has a positive solution.

First note that the BVP (4.1) is of the from (1.1) – (1.3) with $a = 0$, $b = 1$ and

$$p(t) \equiv 1, \quad f(t,u) = \frac{u^{3/2}}{5t}, \quad h^-(t) = \frac{1}{\sqrt{t}}, \quad h^+(t) = 0.$$

Also note that f and h have a singularity at $t = 0$, and $m = M = 1$. Then since $q > 1$,

$$\begin{aligned}
r &= \frac{M^3(\sigma(b)-\rho(a))}{m^4} \int_{\rho(a)}^{b} h^-(t)\nabla t \\
&= 2\int_0^1 \frac{1}{\sqrt{t}}\nabla t \\
&= 2\left[1\left(1-\frac{1}{q}\right) + \sqrt{q}\left(\frac{1}{q}-\frac{1}{q^2}\right) + \sqrt{q^2}\left(\frac{1}{q^2}-\frac{1}{q^3}\right) + \cdots\right] \\
&= 2\left[1 + \frac{1}{\sqrt{q}}\right].
\end{aligned}$$

Take $\lambda_1 = \lambda_2 = 3/2$, then (H_1) is satisfied.

For (H_2) note that,

$$\int_{\rho(a)}^{b} (s-\rho(a))(\sigma(b)-s)\left[f(s,1)+h^{+}(s)\right]\nabla s$$
$$= \frac{1}{5}\int_{0}^{1}(2-s)\nabla s = \frac{2+q}{5+5q}.$$

Also note that,

$$\frac{m^2 r(\sigma(b)-\rho(a))}{M((r+1)^{\lambda_1}+1)} \geq \frac{2r}{(r+1)^2+1} = \frac{2q+2\sqrt{q}}{5q+6\sqrt{q}+2}.$$

Now, it is easy to see that $\frac{2+q}{5+5q} < \frac{2q+2\sqrt{q}}{5q+6\sqrt{q}+2}$ for $q>1$.

Thus, (H_2) is also satisfied. Hence the existence of a positive solution is now guaranteed from Theorem 3.1.

Example 4.2 Let $\mathbb{T}=$ The Cantor Set. (See pages 18 − 19 of [4] for more information regarding this time scale.)

Consider the following BVP for $k > \frac{20}{7}$,

$$\begin{cases} u^{\Delta\nabla} + \frac{u^2}{k(1-t)} - \frac{1}{\sqrt{t}+\sqrt{\rho(t)}} = 0, & t \in (0,1)_{\mathbb{T}} \\ u(0) = 0 = u(1). \end{cases} \tag{4.2}$$

Again we apply Theorem 3.1. First note that

$$\begin{aligned} r &= \frac{M^3(\sigma(b)-\rho(a))}{m^4}\int_{\rho(a)}^{b} h^{-}(t)\,\nabla t \\ &= \int_0^1 \frac{1}{\sqrt{t}+\sqrt{\rho(t)}}\,\nabla t \\ &= \int_0^1 (\sqrt{t})^{\nabla}\,\nabla t = 1. \end{aligned}$$

Take $\lambda_1 = \lambda_2 = 2$, then (H_1) is satisfied.

In [6] the authors show that

$$\int_0^1 t\,\Delta t = \frac{3}{7},$$

where $t \in \mathbb{T}$, and $\mathbb{T}$ is the Cantor set. Using similar arguments we get that

$$\int_0^1 t\,\nabla t = \frac{4}{7}$$

which we use below.

To see that (H_2) holds, note that

$$\begin{aligned}
&\int_{\rho(a)}^{b} (s-\rho(a))(\sigma(b)-s)\left[f(s,1)+h^{+}(s)\right]\nabla s \\
&= \int_0^1 s(1-s)\frac{1}{k(1-s)}\,\nabla s \\
&= \frac{1}{k}\int_0^1 s\,\nabla s = \frac{4}{7k}.
\end{aligned}$$

Now it is clear that

$$\frac{4}{7k} < \frac{m^2 r(\sigma(b)-\rho(a))}{M[(r+1)^{\lambda_1}+1]} = \frac{r}{(r+1)^2+1} = \frac{1}{5} \text{ for } k > \tfrac{20}{7}.$$

Thus (H_2) is also satisfied. Hence the existence of a positive solution is now guaranteed from Theorem 3.1.

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