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Antagonistic Games with an Initial Phase †

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Abstract: We formalize and investigate an antagonistic game of two players (A and B), modeled by two independent marked Poisson processes forming casualties to the players. The game is observed by a third party point process. Unlike previous work on this topic, the initial observation moment is chosen not arbitrarily, but at some random moment of time following initial actions of the players. This caused an analytic complexity unresolved until recently. This, more realistic assumption, forms a new phase ("initial phase") of the game and it turns out to be a short game on its own. Following the initial phase, the main phase of the game lasts until one of the players' cumulative casualties exceed some specified threshold. We investigate the paths of the game in which player A loses the game.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time.*

Mathematics Subject Classification (2000): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

We model an antagonistic stochastic game by two marked Poisson processes $\mathcal{A}$ and $\mathcal{B}$, each representing casualties incurred to players A and B. The mutual attacks are rendered in accordance with associated Poisson point processes and their marks are distributed arbitrary and position independent. The game is observed by a third party process $\mathcal{T}$. Consequently, the information on the game is available upon $\mathcal{T}$, thereby forming the embedding $\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}$. (The latter is a more general bivariate marked point process with marks being mutually and position dependent.) The game lasts until one of the players

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gets “exhausted” or “ruined”. This happens whenever the total casualties to the players exceed some specified thresholds. The real exit from the game takes place with a delay in accordance with observations $\mathcal{T}$. This is one of the quite common scenarios of games, in which the co-authors [9] (and most recently, the first author [5–8, 12]) have been involved.

A realistic approach to the modeling was rendered through the embedded delayed process $\mathcal{A}_\mathcal{T} \otimes \mathcal{B}_\mathcal{T}$ distorting the real time information. However, in the previous models the position of the first observation epoch was placed arbitrarily on the positive time axis with no regard to the start of the conflict. As the result, the initial observation point could have been placed before the game began. In a recent article by Dshalalow and Huang, this deficiency was overcome by placing the first observation at some random time after the conflict has emerged. This alone formed a separate initial phase of the conflict with a joint functional, which included the time of the beginning of the conflict and the amount of casualties to the players, all the way to the first observation. To merge this initial phase with the rest of the game, required some past information (non-Markovian), all resulting in two separate phases, which we thereby have come to identify. From the modeling point of view, the present game is simpler than that of [7], which in contrast, also included a second phase following the initial and first phases.

The first phase of this game ends with player A losing to player B (while in [7] it was not specified who of the two exactly loses, as their casualties were then limited).

Even though our model is not entirely characterized as a sequential game, it comes close enough to this literature [1, 3, 5–7, 11, 12, 14, 15, 18, 21, 24]. The tools we are using in this paper are mainly self-contained and developed methods of fluctuation theory that originated from applications to random walk processes. We hold on classic random walk fluctuation analysis, only in a generalized forms. We mention just a few pieces of literature where applications of the fluctuation theory takes place in the areas such as economics [17] and physics [20]. More on this can be found in [5–9]. Topically, the paper falls into the category of antagonistic stochastic games widely applied to economics [2, 16, 19, 24] and warfare [9, 12, 22, 23]. As in all previous work by the authors and the first author, the results are directly applicable to economics and warfare, in particular, in light of a high volatility of the global economy in the recent months. The latter can be interpreted as an “antagonism” between the economic actions (such as bailout of credit institutions) against the panic of the market.

Another area of applied mathematics that relates to our work includes *hybrid systems* [4, 13], in particular hybrid stochastic games [5]. For more references on this topic see [5].

The layoff of the paper is as follows. Section 2 deals with the formalism of the game. Section 3 takes on the initial phase. Section 4 continues with the game beyond the initial phase until player A is ruined. The merge between the two phases is the main contribution to this section.

2 A Formal Description of the Model

The results of Sections 2 and 3 are based on Dshalalow and Huang [7]. To make it self-contained we follow the initial phase of [7].

Let $(\Omega, \mathcal{F}(\Omega), \mathfrak{F}_t, P)$ be a filtered probability space and let $\mathcal{F}_\mathcal{A}, \mathcal{F}_\mathcal{B}, \mathcal{F}_S \subseteq \mathcal{F}(\Omega)$ be independent sub-σ-algebras. We suppose that

$$\mathcal{A} := \sum_{j \geq 1} d_j \varepsilon_{r_j} \text{ and } \mathcal{B} := \sum_{k \geq 1} z_k \varepsilon_{w_k} \tag{2.1}$$

are $\mathcal{F}_{\mathcal{A}}$-measurable and $\mathcal{F}_{\mathcal{B}}$-measurable marked Poisson random measures (ε_a is a point mass at a) with respective intensities λ_A and λ_B and position independent marking. The random measures are specified by the transforms

$$Ee^{-u\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[h_A(u)-1]},\ h_A(u) = Ee^{-ud_1},\ Re(u) \geq 0, \tag{2.2}$$

$$Ee^{-v\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h_B(v)-1]},\ h_B(v) = Ee^{-vz_1},\ Re(v) \geq 0, \tag{2.3}$$

where $|\cdot|$ is the Borel–Lebesgue measure and d_j and z_k are nonnegative r.v.'s representing the successive strikes of players B and A against each other, respectively, while r_j and w_k are the times of the strikes.

The game starts with hostile actions initiated by one of the players A or B at r_1 or w_1. The players can exchange with several more strikes before the first information is noticed by an observer at time t_0. We therefore assume that

$$t_0 \geq \max\{r_1, w_1\}. \tag{2.4}$$

The initial observation time t_0 will be formalized below. All forthcoming observations will be rendered in accordance with a point process

$$T_0 = \sum_{i\geq 0} \varepsilon_{t_i} = \varepsilon_{t_0} + S,\ \text{with } S = \sum_{i\geq 1} \varepsilon_{t_i}, \\ 0 < t_0 < t_1 < \ldots < t_n < \ldots (t_n \to \infty,\ \text{with } n \to \infty). \tag{2.5}$$

We introduce the extension of $\mathcal{T}$:

$$\mathcal{T} := \varepsilon_{t_{-1}} + T_0,\ \text{with } t_{-1} := \min\{r_1, w_1\}, \tag{2.6}$$

such that the tail $S = \sum_{i\geq 1} \varepsilon_{t_i}$ of T_0 is $\mathcal{F}_S$-measurable. The increments $\Delta_1 := t_1 - t_0,\ \Delta_2 := t_2 - t_1,\ \Delta_3 := t_3 - t_2,\ \ldots$ are all independent and identically distributed, and all belong to the equivalence class $[\Delta]$ of r.v.'s with the common Laplace-Stieltjes transform

$$\delta(\theta) := Ee^{-\theta\Delta}. \tag{2.7}$$

Now we define the initial observation as

$$t_0 = \max\{r_1, w_1\} + \Delta_0, \tag{2.8}$$

where $\Delta_0 \in [\Delta]$ and Δ_0 is independent from the rest of the Δ's. t_0 is included in T_0 of equation (2.5) and because it contains some of the $\mathcal{A}$ and $\mathcal{B}$, T_0 is not $\mathcal{F}_S$-measurable. However, T_0 is a delayed renewal process, while $\mathcal{T}$ is not.

We assign to t_{-1} the genuine start of the game at time $\min\{r_1, w_1\}$ of (2.6). That is,

$$t_{-1} = \min\{r_1, w_1\}. \tag{2.9}$$

Now, since t_{-1} and $t_0 - t_{-1}$ are dependent (through r_1 and w_1), the extended process $\mathcal{T}$ of (2.6) is not a renewal process, and not even a delayed renewal, as it was in [5, 6, 8, 9, 12].

It should be clear that t_0 depends upon r_1 and w_1 and thus on $\mathcal{A}$ and $\mathcal{B}$, which makes T_0 $\mathcal{A} \otimes \mathcal{B}$-measurable. Define the continuous time parameter process

$$(\alpha(t), \beta(t)) := \mathcal{A} \otimes \mathcal{B}([0, t]),\ t \geq 0, \tag{2.10}$$

to be adapted to the filtration $(\mathfrak{F}_t)_{t\geq 0}$. Also introduce its embedding over T_0 :

$$(\alpha_j, \beta_j) := (\alpha(t_j), \beta(t_j)) = \mathcal{A}\otimes\mathcal{B}([0,t_j]),\ j = 0,1,\ldots, \tag{2.11}$$

which forms observations of $\mathcal{A}\otimes\mathcal{B}$ over T_0, with respective increments

$$(\xi_j, \eta_j) := \mathcal{A}\otimes\mathcal{B}((t_{j-1},t_j]),\ j = 1,\ldots. \tag{2.12}$$

In addition, let

$$(\xi_0, \eta_0) := \mathcal{A}\otimes\mathcal{B}((\max\{r_1,w_1\},t_0]) \tag{2.13}$$

to be used later on.

Introduce the embedded bivariate marked random measures

$$\mathcal{A}_{T_0}\otimes\mathcal{B}_{T_0} := (\alpha_0,\beta_0)\varepsilon_{t_0} + \sum_{j\geq 1}(\xi_j,\eta_j)\varepsilon_{t_j}, \tag{2.14}$$

where the marginal marked point processes

$$\mathcal{A}_{T_0} = \alpha_0\varepsilon_{t_0} + \sum_{i\geq 1}\xi_i\varepsilon_{t_i} \text{ and } \mathcal{B}_{T_0} = \beta_0\varepsilon_{t_0} + \sum_{i\geq 1}\eta_i\varepsilon_{t_i} \tag{2.15}$$

are with position dependent marking and with ξ_j and η_j being dependent. For the forthcoming sections we introduce the Laplace-Stieltjes transform

$$g(u,v,\theta) := Ee^{-u\xi_j - v\eta_j - \theta\Delta_j},\ Re(u)\geq 0,\ Re(v)\geq 0,\ Re(\theta)\geq 0,\ j\geq 1, \tag{2.16}$$

which will be evaluated as the follows:

$$\begin{aligned} E\big[e^{-u\xi_j - v\eta_j - \theta\Delta_j}\big] &= E\big[e^{-\theta\Delta_j}E\big[e^{-u\xi_j - v\eta_j}\big|\Delta_j\big]\big] \\ &= E\big[e^{-\theta\Delta_j}E\big[e^{-u\mathcal{A}((t_{j-1},t_j])}\big|\Delta_j\big]E\big[e^{-v\mathcal{B}((t_{j-1},t_j])}\big|\Delta_j\big]\big] \\ &= E\big[e^{-\theta\Delta_j}\cdot e^{\lambda_A\Delta_j(h_A(u)-1)}\cdot e^{\lambda_B\Delta_j(h_B(v)-1)}\big] \\ &= E\big[e^{-\{\theta+\lambda_A(1-h_A(u))+\lambda_B(1-h_B(v))\}\Delta_j}\big] \\ &= \delta(\theta^*),\ j = 1,2,\ldots, \end{aligned} \tag{2.17}$$

with

$$\theta^* := \theta + \lambda_A(1-h_A(u)) + \lambda_B(1-h_B(v)), \tag{2.18}$$

and δ defined in (2.7).

3 The Initial Phase of the Game

The entire *game* will include the recording of the conflict between players A and B known to an observer upon process $\mathcal{T}$ (informally, $\{t_{-1}, t_0, t_1, \ldots\}$) from its inception upon t_{-1} followed by the initial observation at time t_0. $\mathcal{T}$ is defined below. The actual start of the game at t_{-1} is unknown to the observer, as this moment takes place prior to t_0. From the construction of the extended game, the point process $\mathcal{T}$ is obviously "doubly delayed" (in light of its attachment t_{-1}). The information on t_{-1} will be used in section 4 during the merging process.

The initial phase of the game is specified as follows. Define the respective damages to the players at t_{-1} as

$$(\xi_{-1}, \eta_{-1}) := (\alpha_{-1}, \beta_{-1}) := (\alpha(t_{-1}), \beta(t_{-1})) = (d_1 \mathbf{1}_{\{r_1 \leq w_1\}}, z_1 \mathbf{1}_{\{r_1 \geq w_1\}}). \tag{3.1}$$

Therefore, the embedded process $\sum_{k\geq -1} \varepsilon_{t_k}(\alpha_k, \beta_k)$ satisfies the extended initial conditions

$$\mathcal{A}_{t_{-1}} \otimes \mathcal{B}_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (d_1, 0), \text{ on trace } \sigma\text{-algebra } \mathcal{F}(\Omega) \cap \{r_1 < w_1\}, \tag{3.2}$$

$$\mathcal{A}_{t_{-1}} \otimes \mathcal{B}_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (0, z_1), \text{ on } \mathcal{F}(\Omega) \cap \{r_1 > w_1\}, \tag{3.3}$$

$$\mathcal{A}_{t_{-1}} \otimes \mathcal{B}_{t_{-1}} = (\alpha_{-1}, \beta_{-1}) = (d_1, z_1), \text{ on } \mathcal{F}(\Omega) \cap \{r_1 = w_1\}. \tag{3.4}$$

The extended version of the game is defined as the bivariate marked point process

$$\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}} := (\xi_{-1}, \eta_{-1})\varepsilon_{t_{-1}} + (\alpha_0 - \xi_{-1}, \beta_0 - \eta_{-1})\varepsilon_{t_0} + \sum_{j\geq 1}(\xi_j, \eta_j)\varepsilon_{t_j} \tag{3.5}$$

(embedded over $\mathcal{T}$).

As we will see it in the next section, the game will require knowledge of $\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}$ at t_{-1} and t_0. Consequently, we begin to work on the functional

$$\phi_0 := \phi_0(a_0, b_0, \vartheta_0, u_0, v_0, \theta_0) = E[e^{-a_0\alpha_{-1} - u_0\alpha_0 - b_0\beta_{-1} - v_0\beta_0 - \vartheta_0 t_{-1} - \theta_0 t_0}] \tag{3.6}$$

that describes what we call, the *initial phase* of the game. The following theorem is due to Dshalalow and Huang [7].

Theorem 3.1 *The functional ϕ_0 of the initial phase of the game satisfies the following formula:*

$$\phi_0 = \frac{\lambda_A \lambda_B \delta(\theta_0^*)}{\vartheta_0 + \theta_0 + \lambda_A + \lambda_B} \left(\frac{1}{\theta_A + \lambda_B} h_A(a_0 + u_0) h_B(v_0) + \frac{1}{\theta_B + \lambda_A} h_A(u_0) h_B(b_0 + v_0) \right), \tag{3.7}$$

where

$$\theta_0^* := \theta_0 + \lambda_A(1 - h_A(u_0)) + \lambda_B(1 - h_B(v_0)), \tag{3.8}$$

$$\theta_A := \theta_0 - \lambda_A(h_A(u_0) - 1), \tag{3.9}$$

$$\theta_B := \theta_0 - \lambda_B(h_B(v_0) - 1), \tag{3.10}$$

$$\delta(\theta) := E\big[e^{-\theta\Delta_0}\big], \ \Delta_0 \in [\Delta]. \tag{3.11}$$

4 The Main Phase of the Game

After passing the initial phase, the game continues with its status registered at epochs $\mathcal{T}$ and it ends when at least one of the players sustains damages in excess of thresholds M or N. To further formalize the game past t_0 we introduce the following random *exit indices*

$$\mu := \inf\{j \geq 0 : \alpha_j = \alpha_0 + \xi_1 + \ldots + \xi_j > M\}, \tag{4.1}$$

$$\nu := \inf\{k \geq 0 : \beta_k = \beta_0 + \eta_1 + \ldots + \eta_k > N\}. \tag{4.2}$$

Related on μ and ν are the following r.v.'s:

t_μ is the nearest observation epoch when player A's damages exceed threshold M,
t_ν is the first observation of $\mathcal{T}$ when player B's damages exceed threshold N.
Apparently, α_μ and β_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. Accordingly, we will target the following functional

$$\phi_\mu := \phi_\mu(a, b, \vartheta, u, v, \theta) = E[e^{-a\alpha_{\mu-1} - u\alpha_\mu - b\beta_{\mu-1} - v\beta_\mu - \vartheta t_{\mu-1} - \theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}]. \tag{4.3}$$

To calculate a tractable form of ϕ_μ we will use the bivariate Laplace-Carson transform

$$\mathcal{LC}_{pq}(\bullet)(x, y) := xy \int_{p=0}^{\infty} \int_{q=0}^{\infty} e^{-xp-yq}(\bullet)d(p, q),\ Re(x) > 0,\ Re(y) > 0, \tag{4.4}$$

with the inverse

$$\mathcal{LC}_{xy}^{-1}(\bullet)(p, q) = \mathcal{L}_{xy}^{-1}(\bullet\,\frac{1}{xy}), \tag{4.5}$$

where $\mathcal{L}^{-1}$ is the inverse of the bivariate Laplace transform.

Theorem 4.1 *The functional ϕ_μ of the game on trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$ satisfies the following formula:*

$$\phi_\mu = \mathcal{LC}_{xy}^{-1}\left((\Phi_0^1 - \Phi_0) + \frac{\Phi_0^*}{1-g}(G^1 - G)\right)(M, N), \tag{4.6}$$

where

$$G := g(u + x, v + y, \theta), \tag{4.7}$$
$$G^1 := g(u, v + y, \theta), \tag{4.8}$$
$$\Phi_0^* := \phi_0(0, 0, 0, a + u + x, b + v + y, \vartheta + \theta), \tag{4.9}$$
$$\Phi_0 := \phi_0(a, b, \vartheta, u + x, v + y, \theta), \tag{4.10}$$
$$\Phi_0^1 := \phi_0(a + x, b, \vartheta, u, v + y, \theta), \tag{4.11}$$

with g and ϕ_0 of (2.16) and (3.7), respectively.

Proof: First we modify (4.1) and (4.2) for the random exit indices μ and ν which depend on parameters M and N, now to depend on p and q (being arbitrary nonnegative real numbers), respectively, and working with them as parametric families of r.v.'s:

$$\mu(p) := \inf\{j \geq 0 : \alpha_j = \alpha_0 + \xi_1 + \ldots + \xi_j > p\},\ p \geq 0, \tag{4.12}$$

$$\nu(q) := \inf\{k \geq 0 : \beta_k = \beta_0 + \eta_1 + \ldots + \eta_k > q\},\ q \geq 0. \tag{4.13}$$

The functional ϕ_μ will now change to

$$\Phi_{pq} = E[e^{-a\alpha_{\mu(p)-1} - u\alpha_{\mu(p)} - b\beta_{\mu(p)-1} - v\beta_{\mu(p)} - \vartheta t_{\mu(p)-1} - \theta t_{\mu(p)}}\mathbf{1}_{\{\mu(p)<\nu(q)\}}]. \tag{4.14}$$

This will follow the paths of the game on the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu(p) < \nu(q)\}$ and yield:

$$\Phi_{pq} = \sum_{j\geq 0}\sum_{k>j} E[e^{-a\alpha_{j-1} - u\alpha_j - b\beta_{j-1} - v\beta_j - \vartheta t_{j-1} - \theta t_j}\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}]. \tag{4.15}$$

By Fubini's theorem, and that

$$\mathcal{LC}_{pq}\left(\mathbf{1}_{\{\mu(p)=j,\nu(q)=k\}}\right)(x,y) = (e^{-x\alpha_{j-1}} - e^{-x\alpha_j})(e^{-y\beta_{k-1}} - e^{-y\beta_k}),$$

(which can be readily shown) we have

$$\begin{aligned}\mathcal{LC}_{pq}(\Phi_{pq})(x,y) = \sum_{j\geq 0}\sum_{k>j} & E[e^{-a\alpha_{j-1}-u\alpha_j-b\beta_{j-1}-v\beta_j-\vartheta t_{j-1}-\theta t_j} \\ & \times (e^{-x\alpha_{j-1}} - e^{-x\alpha_j})(e^{-y\beta_{k-1}} - e^{-y\beta_k})].\end{aligned} \tag{4.16}$$

We distinguish two cases.

***(i)* Case $j = 0$**. This case will include the entire information on the initial phase observed at t_0 and prior to t_0, including t_{-1}. In a few lines below, we are going to implement the result of Theorem 3.1 and utilize all necessary versions of the functional ϕ_0 :

$$\begin{aligned}
&\sum_{k>0} E[e^{-a\alpha_{-1}-u\alpha_0-b\beta_{-1}-v\beta_0-\vartheta t_{-1}-\theta t_0}(e^{-x\alpha_{-1}} - e^{-x\alpha_0})(e^{-y\beta_{k-1}} - e^{-y\beta_k})] \\
&\quad = \sum_{k>0} E[e^{-a\alpha_{-1}-u\alpha_0-b\beta_{-1}-v\beta_0-\vartheta t_{-1}-\theta t_0}(e^{-x\alpha_{-1}} - e^{-x\alpha_0}) \\
&\qquad \times e^{-y\beta_0} e^{-y(\eta_1+ldots+\eta_{k-1})}(1 - e^{-y\eta_k})] \\
&\quad = \Big\{E[e^{-(a+x)\alpha_{-1}-u\alpha_0-b\beta_{-1}-(v+y)\beta_0-\vartheta t_{-1}-\theta t_0}] \\
&\qquad - E[e^{-a\alpha_{-1}-(u+x)\alpha_0-b\beta_{-1}-(v+y)\beta_0-\vartheta t_{-1}-\theta t_0}]\Big\} \sum_{k>0} E[e^{-y(\eta_1+\ldots+\eta_{k-1})}(1 - e^{-y\eta_k})] \\
&\quad = \Big\{\phi_0(a+x, b, \vartheta, u, v+y, \theta) - \phi_0(a, b, \vartheta, u+x, v+y, \theta)\Big\} \\
&\qquad \times \sum_{k>0} [g(0,y,0)]^{k-1}\big(1 - g(0,y,0)\big) \\
&\quad = \Phi_0^1 - \Phi_0,
\end{aligned} \tag{4.17}$$

where the summation over $k > 0$ converges to 1 as per Lemma 1 of Dshalalow and Huang [5]: the associated convergence of $\sum_{k>0}[g(0,y,0)]^{k-1}$ is guaranteed provided that $Re(y) > 0$. The last line in (4.17) is due to notation (4.9-4.11).

***(ii)* Case $j > 0$**. This case also contains parts of functional ϕ_0 in the information related to the reference point t_0.

Transformation (4.16) for this case is

$$\begin{aligned}
&\sum_{j>0}\sum_{k>j} E[e^{-a\alpha_{j-1}-u\alpha_j-b\beta_{j-1}-v\beta_j-\vartheta t_{j-1}-\theta t_j}(e^{-x\alpha_{j-1}} - e^{-x\alpha_j})(e^{-y\beta_{k-1}} - e^{-y\beta_k})] \\
&\quad = \sum_{j>0}\sum_{k>j}\Big\{E[e^{-(a+u+x)\alpha_{j-1}-(b+v+y)\beta_{j-1}-(\vartheta+\theta)t_{j-1}}] \\
&\qquad \times E[e^{-u\xi_j}(1 - e^{-x\xi_j})e^{-(v+y)\eta_j-\theta\Delta_j}] E[e^{-y(\eta_{j+1}+\ldots+\eta_{k-1})}(1 - e^{-y\eta_k})]\Big\} \\
&\quad = \sum_{j>0}\Big\{E[e^{-(a+u+x)\alpha_0-(b+v+y)\beta_0-(\vartheta+\theta)t_0}] \\
&\qquad \times E[e^{-(a+u+x)(\xi_1+\ldots+\xi_{j-1})-(b+v+y)(\eta_1+\ldots+\eta_{j-1})-(\vartheta+\theta)\left(\Delta_1+\ldots+\Delta_{j-1}\right)}] \\
&\qquad \times E[e^{-u\xi_j}(1 - e^{-x\xi_j})e^{-(v+y)\eta_j-\theta\Delta_j}] \sum_{k>j} E[e^{-y(\eta_{j+1}+\ldots+\eta_{k-1})}(1 - e^{-y\eta_k})]\Big\},
\end{aligned} \tag{4.18}$$

where the third factor can be written as

$$E[e^{-u\xi_j-(v+y)\eta_j-\theta\Delta_j}]-E[e^{-(u+x)\xi_j-(v+y)\eta_j-\theta\Delta_j}]=G^1-G$$

(as per notation (4.7-4.8)) and the summation over $k>j$ converges to 1, for $Re(y)>0$, as per Lemma 1 of [5]. Then, after some algebra in (4.18) and the use of notation (4.7-4.8) and (4.18), we arrive at

$$\begin{aligned}&\phi_0(0,0,0,a+u+x,b+v+y,\vartheta+\theta)\cdot\sum_{j>0}g^{j-1}\cdot(G^1-G)\\&=\Phi_0^*\cdot\sum_{j>0}g^{j-1}\cdot(G^1-G)=\frac{\Phi_0^*}{1-g}(G^1-G),\end{aligned}\tag{4.19}$$

with the convergence of $\sum_{j>0}g^{j-1}$ under the condition that the parameters of g satisfy

$$Re(a+u+x)>0,\ Re(b+v+y)>0,\ Re(\vartheta+\theta)>0,\tag{4.20}$$

with any two of the three strict inequalities relaxed with $\geq$.

With the cases $j=0$ and $j>0$ combined together, we will arrive at

$$\mathcal{LC}_{pq}(\Phi_{pq})(x,y)=(\Phi_0^1-\Phi_0)+\frac{\Phi_0^*}{1-g}(G^1-G).\tag{4.21}$$

□

Remark 4.1 For the particular case

$$\varphi_\mu=\varphi_\mu(u,v,\vartheta)=E[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}]\tag{4.22}$$

of the functional ϕ_μ we get from (4.21)

$$\mathcal{LC}_{pq}(\varphi_{pq})(x,y)=\Phi_0^1-\Phi_0\frac{1-G^1}{1-G},\tag{4.23}$$

where φ_{pq} is the corresponding marginal reduction of Φ_{pq} while the rest of the marginal functionals G,G^1,Φ_0, and Φ_0^1 will shrink but for convenience carry the same characters:

$$G=g(u+x,v+y,\theta),\tag{4.24}$$

$$G^1=g(u,v+y,\theta),\tag{4.25}$$

$$\Phi_0^*=\Phi_0=\phi_0(0,0,0,u+x,v+y,\theta),\tag{4.26}$$

$$\Phi_0^1=\phi_0(x,0,0,u,v+y,\theta).\tag{4.27}$$

Explicitly,

$$\begin{aligned}\mathcal{LC}_{pq}(\varphi_{pq})(x,y)&=\phi_0(x,0,0,u,v+y,\theta)\\&\quad-\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)},\end{aligned}\tag{4.28}$$

where from (3.7-3.10) and (2.18), the marginal versions of ϕ_0 needed for (4.28) are

$$\begin{aligned}&\phi_0(x,0,0,u,v,\theta)=E[e^{-x\alpha_{-1}-u\alpha_0-v\beta_0-\theta t_0}]\\&=\frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\left(\frac{1}{\theta_A+\lambda_B}h_A(x+u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\right),\end{aligned}\tag{4.29}$$

$$\begin{aligned}&\phi_0(0,0,0,u,v,\theta)=E[e^{-u\alpha_0-v\beta_0-\theta t_0}]\\&=\frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\left(\frac{1}{\theta_A+\lambda_B}h_A(u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\right),\end{aligned}\tag{4.30}$$

and

$$\theta_0^* := \theta + \lambda_A(1 - h_A(u)) + \lambda_B(1 - h_B(v)), \tag{4.31}$$
$$\theta_A := \theta - \lambda_A(h_A(u) - 1), \tag{4.32}$$
$$\theta_B := \theta - \lambda_B(h_B(v) - 1). \tag{4.33}$$

□

Concluding Remarks. In this paper, we study fully antagonistic stochastic games of two players (A and B) (initiated in [5-7]), modeled by two independent marked Poisson processes recording times and quantities of casualties to the players. The game is observed by a third party renewal point process upon which the information is gathered (and a decision about upcoming steps can be made or modified). Unlike previous work in [5, 6, 8, 9], the initial observation moment is not arbitrarily chosen, but it is placed at random following some initial actions of the players. This caused an analytic complexity which was unresolved until recently. Due to this more realistic assumption a new phase in the game emerged, which we name the "initial phase". This initial phase turned out to be a short game on its own. Following the initial phase, the main phase of the game lasts until one of the players is ruined. This takes place when the cumulative casualties of a losing player exceed some specified threshold. We investigate the paths of the game in which player A loses the game. The general formulas are obtained in closed forms. In [10] we will render calculation for a variety of special cases.

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