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Stability Properties for Some Non-autonomous Dissipative Phenomena Proved by Families of Liapunov Functionals

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Abstract: We prove some new results regarding the boundedness, stability and attractivity of the solutions of a class of initial-boundary-value problems characterized by a quasi-linear third order equation which may contain time-dependent coefficients. The class includes equations arising in superconductor theory, and in the theory of viscoelastic materials. In the proof we use a family of Liapunov functionals W depending on two parameters, which we adapt to the 'error', i.e. to the size σ of the chosen neighbourhood of the null solution.

Keywords: *nonlinear higher order PDE-stability, boundedness-boundary value problems.*

Mathematics Subject Classification (2000): 35B35, 35G30.

1 Introduction

In this paper we study the boundedness and stability properties of a large class of initial-boundary-value problems of the form

$$\begin{cases} -\varepsilon(t)u_{xxt} + u_{tt} - C(t)u_{xx} + a'u_t = F(u) - au_t, & x \in]0,\pi[, \quad t > t_0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \end{cases} \tag{1.1}$$

$$u(x,t_0) = u_0(x), \qquad u_t(x,t_0) = u_1(x). \tag{1.2}$$

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Here $t_0 \geq 0$, $\varepsilon \in C^2(I,I)$, $C \in C^1(I,\mathbb{R}^+)$ (with $I := [0,\infty[$) are functions of t, with $C(t) \geq \overline{C} = \mathrm{const} > 0$, the conservative force fulfills $F(0) = 0$, so that the equation admits the trivial solution $u(x,t) \equiv 0$; $a' = \mathrm{const} \geq 0$, $a = a(x,t,u,u_x,u_t,u_{xx}) \geq 0$, $\varepsilon(t) \geq 0$, so that the corresponding terms are dissipative [1].

Solutions u of such problems describe a number of physically remarkable continuous phenomena occurring on a finite space interval.

For instance, when $F(u) = b\sin u$, $a = 0$ we deal with a perturbed Sine–Gordon equation which is used to describe the classical Josephson effect [8] in the theory of superconductors, which is at the base (see e.g. [12, 1] and references therein) of a large number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3–6 in [2]): $u(x,t)$ is the phase difference of the macroscopic quantum wave functions describing the Bose–Einstein condensates of Cooper pairs in two superconductors separated by a very thin and narrow dielectric strip (a socalled "Josephson junction"), the dissipative term $(a'+a)u_t$ is due to Joule effect of the residual current across the junction due to single electrons, whereas the third order dissipative term is due to the surface impedence of the two superconductors of the strip. Usually the model is considered with constant (dimensionless) coefficients $\varepsilon, C, (a'+a)$, but in fact the latter depend on other physical parameters like the temperature or the voltage difference applied to the junction (see e.g. [12]), which can be controlled and varied with time; in a more accurate description of the model one should take a non-constant $a = \beta\cos u$, where β also depends on temperature and voltage difference applied and therefore can be varied with time.

Other applications of problem (1.1)–(1.2) include heat conduction at low temperature [13, 7], sound propagation in viscous gases [10], propagation of plane waves in perfect incompressible and electrically conducting fluids [15], motions of viscoelastic fluids or solids [9, 14, 16]. For instance, problem (1.1)–(1.2) with $a = 0 = a'$ describes [14] the evolution of the displacement $u(x,t)$ of the section of a rod from its rest position x in a Voigt material when an external force F is applied; in this case $c^2 = E/\rho$, $\varepsilon = 1/(\rho\mu)$, where ρ is the (constant) linear density of the rod at rest, and E, μ are respectively the elastic and viscous constants of the rod, which enter the stress-strain relation $\sigma = E\nu + \partial_t\nu/\mu$, where σ is the stress, ν is the strain. Again, some of these parameters, like the viscous constant of the rod, may depend on the temperature of the rod, which can be controlled and varied with time.

The problem (1.1)–(1.2) considered here generalizes those considered in [3, 4, 5, 6], in that the square velocity C and the dissipative coefficient ε can depend on t. The physical phenomena just described provide the motivations for such a generalization. While we require C to have a positive lower bound, in order not to completely destroy the wave propagation effects due to the operator $\partial_t^2 - C\partial_x^2$, we wish to include the cases that ε goes to zero as $t \to \infty$, vanishes at some point t, or even vanishes identically. To that

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[1] This follows from the non-positivity of the corresponding terms in the time derivative of the Hamiltonian:

$$H = \int_0^\pi dx \left[\frac{u_t^2 + Cu_x^2}{2} - \int_0^{u(x)} F(z)dz\right] \quad \Rightarrow \quad \dot{H} = -\int_0^\pi dx\,\left[(a+a')u_t^2 + \varepsilon u_{xt}^2\right] + \int_0^\pi dx\,\dot{C}\frac{u_x^2}{2}.$$

We also see that the last term is respectively dissipative, forcing if $\dot{C}$ is negative, positive. H can play the role of Liapunov functional w.r.t. the reduced norm $d_{\varepsilon=0}(u,u_t)$.

end, we consider the t-dependent norm

$$d^2(\varphi,\psi) \equiv d^2_\varepsilon(\varphi,\psi) = \int_0^\pi dx\,[\varepsilon^2(t)\varphi^2_{xx}+\varphi^2_x+\varphi^2+\psi^2]. \tag{1.3}$$

ε^2 plays the role of a weight for the second order derivative term φ^2_{xx} so that for $\varepsilon = 0$ this automatically reduces to the proper norm needed for treating the corresponding second order problem. Imposing the condition that φ,ψ vanish in $0,\pi$ one easily derives that $|\varphi(x)|, \varepsilon|\varphi_x(x)| \le d(\varphi,\psi)$ for any x; therefore a convergence in the norm d implies also a uniform (in x) pointwise convergence of φ and a uniform (in x) pointwise convergence of φ_x for $\varepsilon(t)\neq 0$. To evaluate the distance of u from the trivial solution we shall use the t-dependent norm $d(t) \equiv d_{\varepsilon(t)}\big[u(x,t),u_t(x,t)\big]$; we use the abbreviation $d(t)$ whenever this is not ambiguous.

In Section 2 we state the hypotheses necessary to prove our results, give the relevant definitions of boundedness and (asymptotic) stability, introduce a 2-parameter family of Liapunov functionals W and tune these parameters in order to prove bounds for $W, \dot{W}$. In Sections 3, 4 we prove the main results: a theorem of stability and (exponential) asymptotic stability of the null solution (Section 3), under stronger assumptions theorem of eventual and/or uniform boundedness of the solutions and eventual and/or exponential asymptotic stability in the large of the null solution (Section 4). In Section 5, we mention some examples to which these results can be applied.

We note that for constant ε the existence and uniqueness of the solution of the problem (1.1)–(1.2) follows from the theorem in section 2 of [6], as we can replace at the left-hand side $C(t)$ by $\inf_t C$ and include in the right-hand side the difference $[\inf_t C - C(t)]u_{xx}$.

2 Main Assumptions, Definitions and Preliminary Estimates

For any function $f(t)$, we denote $\overline{f} = \inf_{t>0} f(t)$, $\overline{\overline{f}} = \sup_{t>0} f(t)$. We assume that there exist constants $A\ge 0$, $\tau>0$, $k\ge 0$, $\rho>0$, $\mu>0$ such that

$$F(0)=0 \quad \& \quad F_z(z)\le k \quad \text{if } |z|<\rho. \tag{2.1}$$

$$\overline{C}\ge k, \quad C-\dot{\varepsilon}\ge\mu(1+\varepsilon), \quad \mu+\frac{\overline{C}}{2}-2k>0, \quad \overline{\overline{\varepsilon}}>-\infty. \tag{2.2}$$

$$0 \le a\le Ad^\tau(u,u_t), \qquad a'+\frac{\overline{\varepsilon}}{2}>0 \tag{2.3}$$

We are not excluding the following cases: $\varepsilon(t) = 0$ for some t, $\varepsilon \xrightarrow{t\to\infty} 0$, $\varepsilon(t) \equiv 0$, $\varepsilon \xrightarrow{t\to\infty} \infty$ [in view of $(2.2)_2$ the latter condition requires also $C \xrightarrow{t\to\infty} \infty$]; but by condition $(2.3)_2$ at least one of the dissipative terms must be nonzero. Eq. (2.1) implies

$$\int_0^\varphi F(z)dz\le k\frac{\varphi^2}{2}, \qquad \varphi F(\varphi)\le k\varphi^2 \quad \text{if } |\varphi|<\rho. \tag{2.4}$$

We shall consider also the cases that, in addition to (2.1), either one of the following inequalities [which are stronger than (2.4)] holds:

$$\int_0^\varphi F(z)dz\le 0, \qquad \varphi F(\varphi)\le 0 \quad \text{if } |\varphi|<\rho. \tag{2.4$'$}$$

To formulate our results, we need the following definitions. Fix once and for all $\kappa \in \mathbb{R}$, $\xi > 0$ and let $I_\kappa := [\kappa, \infty[$, $d(t) := d_{\varepsilon(t)}\big[u(x,t), u_t(x,t)\big]$.

Definition 2.1 The solution $u(x,t) \equiv 0$ of (1.1) is stable if for any $\sigma \in]0, \xi]$ and $t_0 \in I_\kappa$ there exists a $\delta(\sigma, t_0) > 0$ such that

$$d(t_0) < \delta(\sigma, t_0) \quad \Rightarrow \quad d(t) < \sigma \quad \forall t \geq t_0.$$

If δ can be chosen independent of t_0, $\delta = \delta(\sigma)$, $u(x,t) \equiv 0$ is uniformly stable.

Definition 2.2 The solution $u(x,t) \equiv 0$ of (1.1) is asymptotically stable if it is stable and moreover for any $t_0 \in I_\kappa$ there exists a $\delta(t_0) > 0$ such that $d(t_0) < \delta(t_0)$ implies $d(t) \to 0$ as $t \to \infty$, namely for any $\nu > 0$ there exists a $T(\nu, t_0, u_0, u_1) > 0$ such that

$$d(t_0) < \delta(t_0) \quad \Rightarrow \quad d(t) < \nu \quad \forall t \geq t_0 + T.$$

The solution $u(x,t) \equiv 0$ is uniformly asymptotically stable if it is uniformly stable and moreover δ, T can be chosen independent of t_0, u_0, u_1, i.e. $d(t) \to 0$ as $t \to \infty$ uniformly in t_0, u_0, u_1.

Definition 2.3 The solutions of (1.1) are eventually uniformly bounded if for any $\delta > 0$ there exist a $s(\delta) \geq 0$ and a $\beta(\delta) > 0$ such that if $t_0 \geq s(\delta)$, $d(t_0) \leq \delta$, then $d(t) < \beta(\delta)$ for all $t \geq t_0$. If $s(\delta) = 0$ the solutions of (1.1) are uniformly bounded.

Definition 2.4 The solutions of (1.1) are bounded if for any $\delta > 0$ there exist a $\tilde{\beta}(\delta, t_0) > 0$ such that if $d(t_0) \leq \delta$, then $d(t) < \tilde{\beta}(\delta, t_0)$ for all $t \geq t_0$.

Definition 2.5 The solution $u(x,t) \equiv 0$ of (1.1) is eventually exponential-asymptotically stable in the large if for any $\delta > 0$ there are a nonnegative constant $s(\delta)$ and positive constants $D(\delta), E(\delta)$ such that if $t_0 \geq s(\delta)$, $d(t_0) \leq \delta$, then

$$d(t) \leq D(\delta) \exp\left[-E(\delta)(t - t_0)\right] d(t_0), \qquad \forall t \geq t_0. \tag{2.5}$$

If $s(\delta) = 0$ then $u(x,t) \equiv 0$ is exponential-asymptotically stable in the large.

Definition 2.6 The solution $u(x,t) \equiv 0$ of (1.1) is (uniformly) exponential-asymptotically stable if there exist positive constants δ, D, E such that

$$d(t_0) < \delta \quad \Rightarrow \quad d(t) \leq D \exp\left[-E(t - t_0)\right] d(t_0), \quad \forall t \geq t_0. \tag{2.6}$$

Definition 2.7 The solution $u(x,t) \equiv 0$ of (1.1) is asymptotically stable in the large if it is stable and moreover for any $t_0 \in I_\kappa$, $\nu, \alpha > 0$ there exists $T(\alpha, \nu, t_0, u_0, u_1) > 0$ such that

$$d(t_0) < \alpha \quad \Rightarrow \quad d(t) < \nu \quad \forall t \geq t_0 + T.$$

We recall Poincaré inequality, which easily follows from Fourier analysis:

$$\phi \in C^1(]0, \pi[), \quad \phi(0) = 0, \quad \phi(\pi) = 0 \quad \Rightarrow \quad \int_0^\pi dx\, \phi_x^2(x) \geq \int_0^\pi dx\, \phi^2(x). \tag{2.7}$$

We introduce the non-autonomous family of Liapunov functionals

$$\begin{aligned} W \equiv W(\varphi, \psi, t; \gamma, \theta) \quad := \quad & \int_0^\pi \frac{1}{2}\Big\{\gamma\psi^2 + (\varepsilon\varphi_{xx} - \psi)^2 + [C(1+\gamma) - \dot{\varepsilon} + \varepsilon(a' + \theta)]\varphi_x^2 \\ & + a'\theta\varphi^2 + 2\theta\varphi\psi - 2(1+\gamma)\int_0^{\varphi(x)} F(z)dz\Big\} dx \end{aligned} \tag{2.8}$$

where θ, γ are for the moment unspecified positive parameters. W coincides with the Liapunov functional of [3] for constant ε, C and $\gamma = 3$, $\theta = a'$. Let $W(t;\gamma,\theta) := W(u, u_t, t; \gamma, \theta)$. Using (1.1), from (2.8) one finds

$$\begin{aligned}
\dot W(t;\gamma,\theta) &= \int_0^\pi \Big\{ (\varepsilon u_{xx} - u_t)(\varepsilon u_{xxt} - u_{tt} + \dot\varepsilon u_{xx}) + [\dot C(1+\gamma) - \ddot\varepsilon + \dot\varepsilon(a'+\theta)]\frac{u_x^2}{2} \\
&\quad + [C(1+\gamma) - \dot\varepsilon + \varepsilon(a'+\theta)] u_x u_{xt} + a'\theta u u_t + \theta u_t^2 + (\gamma u_t + \theta u) u_{tt} - (1+\gamma)F(u)u_t \Big\}\, dx \\
&= \int_0^\pi \Big\{ (\varepsilon u_{xx} - u_t)[(a+a')u_t - C u_{xx} - F(u) + \dot\varepsilon u_{xx}] + [\dot C(1+\gamma) - \ddot\varepsilon + \dot\varepsilon(a'+\theta)]\frac{u_x^2}{2} \\
&\quad - [C(1+\gamma) - \dot\varepsilon + \varepsilon(a'+\theta)] u_{xx} u_t + a'\theta u u_t + \theta u_t^2 \\
&\quad + (\gamma u_t + \theta u)[C u_{xx} + \varepsilon u_{xxt} + F(u) - (a+a')u_t] - (1+\gamma)F(u)u_t \}\, dx \\
&= \int_0^\pi \Big\{ \varepsilon u_{xx}[(\dot\varepsilon - C) - F(u)]u_{xx} + [\varepsilon u_{xx}(a+a') - (a+a')u_t + C u_{xx} + F(u) - \dot\varepsilon u_{xx} - C(1+\gamma)u_{xx} \\
&\quad + \dot\varepsilon u_{xx} - \varepsilon(a'+\theta)u_{xx} + a'\theta u + \theta u_t + \gamma C u_{xx} + \gamma\varepsilon u_{xxt} + \gamma F(u) - (a+a')\gamma u_t - \theta(a+a')u \\
&\quad - (1+\gamma)F(u)]u_t + \theta u[C u_{xx} + \varepsilon u_{xxt} + F(u)] + [\dot C(1+\gamma) - \ddot\varepsilon + \dot\varepsilon(a'+\theta)]\frac{u_x^2}{2} \Big\}\, dx \\
&= \int_0^\pi \Big\{ \varepsilon[(\dot\varepsilon - C)u_{xx} - F(u)]u_{xx} + u_t[\varepsilon a u_{xx} - (a+a')(1+\gamma)u_t - \varepsilon\theta u_{xx} \\
&\quad + \theta u_t + \gamma\varepsilon u_{xxt} - a\theta u] + \theta u[C u_{xx} + \varepsilon u_{xxt} + F(u)] + [\dot C(1+\gamma) - \ddot\varepsilon + \dot\varepsilon(a'+\theta)]\frac{u_x^2}{2} \Big\}\, dx \\
&= -\int_0^\pi \Big\{ \varepsilon(C - \dot\varepsilon)u_{xx}^2 + [(a+a')(1+\gamma) - \theta]\, u_t^2 + \Big[2\theta C + \ddot\varepsilon - \dot\varepsilon(a'+\theta) - (1+\gamma)\dot C\Big]\frac{u_x^2}{2} + \varepsilon\gamma u_{xt}^2 \\
&\quad + \theta a u u_t - \theta u F(u) + \varepsilon[-a u_t + F(u)]u_{xx} \Big\} dx.
\end{aligned} \tag{2.9}$$

2.1 Upper bound for $\dot W$

After some rearrangement of terms and integration by parts of the last term, we obtain

$$\begin{aligned}
\dot W = -\int_0^\pi \Big\{ & \varepsilon\gamma u_{xt}^2 + \Big[(a+a')(1+\gamma) - \theta - \varepsilon\frac{a^2}{C - \dot\varepsilon} - \theta\frac{a^2}{C}\Big] u_t^2 + \varepsilon(C - \dot\varepsilon)\Big[\frac{a}{C - \dot\varepsilon}u_t - \frac{u_{xx}}{2}\Big]^2 \\
& + \frac{3}{4}\varepsilon(C - \dot\varepsilon)u_{xx}^2 + \Big[C\Big(\frac{\theta}{2} - a'\Big) + \ddot\varepsilon + (C - \dot\varepsilon)(a'+\theta) - (1+\gamma)\dot C - 2\varepsilon F_u\Big]\frac{u_x^2}{2} \\
& + \frac{\theta C}{4}(u_x^2 - u^2) + \frac{\theta C}{4}\Big[u + \frac{2a}{C}u_t\Big]^2 - \theta u F(u) \Big\}\, dx.
\end{aligned}$$

Using (2.7) with $\phi(x)=u_t(x,t), u(x,t)$ we thus find, provided $|u|<\rho$, $\theta>2a'$, $\mu(a'+\theta)>2k$

$$\begin{aligned}\dot{W}\leq &-\int_0^{\pi}\Bigg\{\left[\varepsilon\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{C}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\ &\left[C\left(\frac{\theta}{2}-a'\right)+\ddot{\varepsilon}+\mu(1+\varepsilon)(a'+\theta)-(1+\gamma)\dot{C}-2\varepsilon k\right]\frac{u_x^2}{2}-\theta k u^2\Bigg\}dx\\ \leq &-\int_0^{\pi}\Bigg\{\left[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\ &\left[\overline{C}\left(\frac{\theta}{2}-a'\right)+\overline{\ddot{\varepsilon}}+\mu(a'+\theta)+[\mu(a'+\theta)-2k]\varepsilon-(1+\gamma)\dot{C}-2k\theta\right]\frac{u_x^2}{2}\Bigg\}dx.\end{aligned}\tag{2.10}$$

We now assume that there exists $\bar{t}(\gamma)\in[0,\infty[$ such that

$$\dot{C}(1+\gamma)\leq 1 \quad \text{for } t\geq\bar{t}, \qquad \dot{C}(1+\gamma)>1 \quad \text{for } 0\leq t<\bar{t}.\tag{2.11}$$

This is clearly satisfied with $\bar{t}(\gamma)\equiv 0$ if $\dot{C}\leq 0$, whereas it is satisfied with some $\bar{t}(\gamma)\geq 0$ if $\dot{C}\stackrel{t\to\infty}{\longrightarrow}0$. We fix θ by choosing

$$\theta>\theta_1:=\max\left\{2a',\frac{2k}{\mu}-a',\frac{5-\overline{\ddot{\varepsilon}}-a'(\mu-\overline{C})}{\mu+\overline{C}/2-2k}\right\}.\tag{2.12}$$

Then for all $t>\bar{t}$

$$\theta\left(\mu+\frac{\overline{C}}{2}-2k\right)+[\mu(a'+\theta)-2k]\overline{\varepsilon}+\overline{\ddot{\varepsilon}}-(1+\gamma)\dot{C}+a'(\mu-\overline{C})>4.\tag{2.13}$$

Next, provided $d(u,u_t)\leq\sigma<\rho$, we choose

$$\gamma>\gamma_1(\sigma):=\frac{1+\theta}{a'+\overline{\varepsilon}}+\gamma_{32}\sigma^{2\tau}, \qquad \gamma_{32}:=\frac{A^2}{(a'+\overline{\varepsilon})}\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right),\tag{2.14}$$

what implies, for $d\leq\sigma$,

$$\begin{aligned}&\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)=a+a'+(a+a'+\overline{\varepsilon})\gamma-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\\ &\geq a'+\frac{a+a'+\overline{\varepsilon}}{a'+\overline{\varepsilon}}\left[(1+\theta)+A^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\sigma^{2\tau}\right]-\theta-A^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)d^{2\tau}\geq 1+a'.\end{aligned}\tag{2.15}$$

Equations (2.10), (2.13) and (2.15) imply for all $t\geq\bar{t}$

$$\begin{aligned}\dot{W}(u,u_t,t;\gamma,\theta)\leq &-\int_0^{\pi}\Bigg\{\left[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\left(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\right)\right]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\\ &\left[\theta\left(\mu+\frac{\overline{C}}{2}-2k\right)+[\mu(a'+\theta)-2k]\overline{\varepsilon}+\overline{\ddot{\varepsilon}}-(1+\gamma)\dot{C}+a'(\mu-\overline{C})\right]\frac{u_x^2+u^2}{4}\Bigg\}dx\\ &<-\eta\, d^2(t), \qquad \eta:=\min\{1,3\mu/4\}\end{aligned}\tag{2.16}$$

provided $0<d(t)<\sigma$. If, in addition to (2.3) with $k>0$, the inequality (2.4') [which is stronger than (2.4)] holds, then it is easy to check that we can avoid assuming $(2.2)_3$ and obtain again the previous inequality, provided we replace k by 0 in the definition (2.12) of θ_1.

Remark 2.1 One can check that if we had adopted the same Liapunov functional as in [5, 6] formulae (4.2), i.e. W of (2.8) with $\theta=0=a'$, we would have not been able to obtain (2.16) (which is essential to prove the asymptotic stability of the null solution) in a number of situations, e.g. if $\varepsilon\to 0$ sufficiently fast as $t\to\infty$.

2.2 Lower bound for W

From the definition (2.8) it immediately follows

$$W(\varphi,\psi,t;\gamma,\theta) = \int_0^\pi \frac{1}{2}\Bigg\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\frac{(\varepsilon\varphi_{xx}-2\psi)^2}{4}+\frac{(\varepsilon\varphi_{xx}-\psi)^2}{2}+\varepsilon^2\frac{\varphi_{xx}^2}{4}$$
$$+[C(1+\gamma)-\dot\varepsilon+\varepsilon(a'+\theta)]\varphi_x^2+(a'\theta-1)\varphi^2+[\theta\psi+\varphi]^2-2(1+\gamma)\int_0^{\varphi(x)}F(z)dz\Bigg\}dx. \quad (2.17)$$

Using $(2.2)_2$, (2.4) and (2.7) with $\phi(x)=\varphi(x)$ we find for $|\varphi|<\rho$

$$W\geq\int_0^\pi\frac{1}{2}\left\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\varepsilon^2\frac{\varphi_{xx}^2}{4}+[(C-k)\gamma+\mu+(\mu+a'+\theta)\varepsilon]\,\varphi_x^2+[a'\theta-1-k]\,\varphi^2\right\}dx$$
$$\geq\int_0^\pi\frac{1}{2}\Bigg\{\left(\gamma-\theta^2-\frac{1}{2}\right)\psi^2+\varepsilon^2\frac{\varphi_{xx}^2}{4}+\left[(\overline{C}-k)\gamma+\mu+\left(\mu+a'+\frac{\theta}{2}\right)\overline{\varepsilon}\right]\varphi_x^2$$
$$+\left[\left(a'+\frac{\overline{\varepsilon}}{2}\right)\theta-1-k\right]\varphi^2\Bigg\}dx. \quad (2.18)$$

Choosing

$$\theta>\theta_2:=\max\left\{\theta_1,\frac{k+5/4}{a'+\overline{\varepsilon}/2}\right\},\qquad \gamma\geq\gamma_2(\sigma):=\gamma_1(\sigma)+\theta^2+1, \quad (2.19)$$

we find that for $d\leq\sigma$

$$W(\varphi,\psi,t;\gamma,\theta)\geq\chi d^2(\varphi,\psi),\qquad \chi:=\frac{1}{2}\min\left\{\frac{1}{4},(\overline{C}-k)\gamma+\mu+\left(\mu+a'+\frac{\theta}{2}\right)\overline{\varepsilon}\right\}. \quad (2.20)$$

(Note that $0<\chi\leq 1/8$). If, in addition to (2.1) (with some $k>0$), the inequality $(2.4')_1$ holds, then it is easy to check that we obtain (2.20) [with the replacement $k\to 0$ in the definition of χ] by choosing θ,γ as in (2.19), but replacing $k\to 0$ there.

Finally, we note that if $\tau=0$ in (2.3), i.e. $a\leq A=$const, then $\gamma,\bar t(\gamma)$ are independent of σ.

2.3 Upper bound for W

As argued in [3],

$$\left|\int_0^\varphi F(z)dz\right|=\left|\int_0^\varphi dz\int_0^\zeta F_\zeta(\zeta)d\zeta\right|=\left|\int_0^\varphi F_\zeta(\zeta)(\varphi-\zeta)d\zeta\right|.$$

Consequently, introducing the non-decreasing funtion $m(r):=\max\{|F_\zeta(\zeta)|\ :\ |\zeta|\leq r\}$ and in view of the inequality $|\varphi|\leq d(\varphi,\psi)$ we obtain

$$\left|\int_0^\varphi F(z)dz\right|\leq m(|\varphi|)\frac{\varphi^2}{2}\leq m(d)\frac{d^2}{2}. \quad (2.21)$$

Thus, from definition (2.8) and the inequalities $-2\epsilon\varphi_{xx}\psi \leq \epsilon^2\varphi_{xx}^2+\psi^2$, $2\theta\varphi\psi\leq\theta(\varphi^2+\psi^2)$, $(2.2)_3$ we easily find

$$W(\varphi,\psi,t;\gamma,\theta)\leq\int_0^\pi\frac{1}{2}\{(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2+[C(1+\gamma)-\dot{\varepsilon}$$

$$+\varepsilon(a'+\theta)]\,\varphi_x^2+(a'+1)\theta\varphi^2\Big\}\,dx+(1+\gamma)m(d)\frac{d^2}{2}\leq\int_0^\pi\frac{1}{2}\Big\{(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2$$

$$+\left[C\gamma+(C-\dot{\varepsilon})\left(1+\frac{a'+\theta}{\mu}\right)\right]\varphi_x^2+(a'+1)\theta\varphi^2\Big\}\,dx+(1+\gamma)m(d)\frac{d^2}{2}.$$

Choosing

$$\gamma\geq\gamma_3(\sigma):=\gamma_2(\sigma)+1+\tfrac{a'+\theta}{\mu}+(a'+1)\theta=\gamma_{31}+\gamma_{32}\sigma^{2\tau},\tag{2.22}$$

where $\gamma_{31}:=\frac{1+\theta}{a'+\overline{\varepsilon}}+\theta^2+2+\frac{a'+\theta}{\mu}+(a'+1)\theta$ and setting

$$g(t):=C(t)-\dot{\varepsilon}(t)/2+1>1,\qquad B^2(d):=[1+m(d)]\,d^2,\tag{2.23}$$

we find that for $d\leq\sigma$

$$\begin{aligned}W(\varphi,\psi,t;\gamma,\theta)\ &\leq\ \int_0^\pi\frac{1}{2}\big[(\gamma+2+\theta)\psi^2+2\varepsilon^2\varphi_{xx}^2+\gamma\,(2C-\dot{\varepsilon})\,\varphi_x^2+\gamma\varphi^2\big]\,dx+(1+\gamma)m(d)\frac{d^2}{2}\\ &\leq\ [2\gamma g(t)+(1+\gamma)m(d)]\,\frac{d^2}{2}\leq(1+\gamma)\,[g(t)+m(d)]\,d^2\\ &\leq\ [1+\gamma(\sigma)]\,g(t)B^2(d).\end{aligned}\tag{2.24}$$

The map $d\in[0,\infty[\to B(d)\in[0,\infty[$ is continuous and increasing, therefore also invertible. Moreover, $B(d)\geq d$.

3 Asymptotic Stability of the Null Solution

Theorem 3.1 *Assume that conditions (2.1)-(2.3) are fulfilled. Then the null solution $u(x,t)$ of (1.1) is stable if one of the following conditions is fulfilled:*

$$\dot{C}\leq 0,\quad\forall t\in I,\tag{3.1}$$

$$\dot{C}\overset{t\to\infty}{\longrightarrow}0;\tag{3.2}$$

the stability is uniform if the function $g(t)$ defined by (2.23) fulfills $\overline{\overline{g}}<\infty$. The ξ appearing in Definition 2.1 is a suitable positive constant, more precisely $\xi\in]0,\rho]$ if $\rho<\infty$. The null solution is asymptotically stable if, in addition,

$$\int_0^\infty\frac{dt}{g(t)}=\infty,\tag{3.3}$$

and uniformly exponential-asymptotically stable if $\overline{\overline{g}}<\infty$.

Proof As a first step, we analyze the behaviour of

$$\frac{\sigma^2}{1+\gamma_3(\sigma)} = \frac{\sigma^2}{1+\gamma_{31}+\gamma_{32}\sigma^{2\tau}} =: r^2(\sigma).$$

The positive constants γ_{31}, γ_{32}, defined in (2.22), are independent of σ, t_0. The function $r(\sigma)$ is an increasing and therefore invertible map $r : [0, \sigma_M[\to [0, r_M[$, where:

$$\begin{aligned} &\sigma_M = \infty, && r_M = \infty, && \text{if } \tau \in [0,1[, \\ &\sigma_M = \infty && r_M = 1/\sqrt{\gamma_{32}}, && \text{if } \tau = 1, \\ &\sigma_M^{2\tau} := \tfrac{1+\gamma_{31}}{\gamma_{32}(\tau-1)}, && r_M = [\tfrac{\tau-1}{1+\gamma_{31}}]^{\frac{\tau-1}{2\tau}} / \sqrt{\tau}\gamma_{32}^{\frac{1}{2\tau}}, && \text{if } \tau > 1, \end{aligned} \tag{3.4}$$

(in the latter case $r(\sigma)$ is decreasing beyond σ_M).

Next, let $\xi := \min\{\sigma_M, \rho\}$ if the rhs is finite, otherwise choose $\xi \in \mathbb{R}^+$; we shall consider an "error" $\sigma \in]0, \xi[$. We define

$$\delta(\sigma, t_0) := B^{-1}\left[r(\sigma)\frac{\sqrt{\chi}}{\sqrt{g(t_0)}}\right], \qquad \kappa := \bar{t}[\gamma_3(\xi)]. \tag{3.5}$$

$\delta(\sigma, t_0)$ belongs to $]0, \sigma[$, because $B(d) \geq d$ implies $B^{-1}\left[r(\sigma)\sqrt{\chi}/\sqrt{g(t_0)}\right] \leq \sqrt{\chi}\sigma \leq \sigma/2$ and is an increasing function of σ. The function $\bar{t}(\gamma)$ was defined in (2.11); $\bar{t}[\gamma_3(\sigma)] \leq \kappa$ as the function $\bar{t}[\gamma_3(\sigma)]$ is non-decreasing. Mimicking an argument of [6], we can show that for any $t_0 \geq \kappa$

$$d(t_0) < \delta(\sigma, t_0) \quad \Rightarrow \quad d(t) < \sigma \quad \forall t \geq t_0. \tag{3.6}$$

Ad absurdum, assume that there exists a finite $t_1 > t_0$ such that (3.6) is fulfilled for all $t \in [t_0, t_1[$, whereas

$$d(t_1) = \sigma. \tag{3.7}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W[u, u_t, t; \gamma_3(\sigma), \theta]$ is a decreasing function of t in $[t_0, t_1]$. Using (2.20), (2.24) we find the following contradiction with (3.7):

$$\begin{aligned} \chi d^2(t_1) &\leq W(t_1) < W(t_0) \leq [1+\gamma_3(\sigma)]\, g(t_0) B^2\, [d(t_0)] < [1+\gamma_3(\sigma)]\, g(t_0) B^2(\delta) \\ &= [1+\gamma_3(\sigma)]\, g(t_0) \left\{ B\left[B^{-1}\left(\sigma\frac{\sqrt{\chi}}{\sqrt{[1+\gamma_3(\sigma)]g(t_0)}}\right)\right]\right\}^2 = \chi\sigma^2. \end{aligned}$$

Eq. (3.6) amounts to the stability of the null solution; if $\overline{\overline{g}} < \infty$ we obtain the uniform stability replacing $(3.5)_1$ by $\delta(\sigma) := B^{-1}\left[r(\sigma)\sqrt{\chi}/\sqrt{\overline{\overline{g}}}\right]$.

Let now $\delta(t_0) := \delta(\xi, t_0)$. By (3.6) and the monotonicity of $\delta(\cdot, t_0)$ we find that for any $t_0 \geq \kappa$

$$d(t_0) < \delta(t_0) \quad \Rightarrow \quad d(t) < \xi \qquad \forall t \geq t_0. \tag{3.8}$$

Choosing $W(t) \equiv W[u, u_t, t; \gamma_3(\xi), \theta]$, (2.24) becomes

$$W(t) \leq h(\xi) g(t) d^2(t), \qquad h(\xi) := [1+\gamma_3(\xi)]\, [1+m(\xi)]\,, \tag{3.9}$$

which together with (2.16), implies $\dot{W}(t) \leq -\eta W(t)/[hg(t)]$ and (by means of the comparison principle [17]) $W(t) < W(t_0)\exp\left[-\eta\int_{t_0}^{t} dz/[hg(z)]\right]$, whence

$$d^2(t) \leq \frac{W(t)}{\chi} < \frac{W(t_0)}{\chi}\exp\left[-\frac{\eta}{h}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right]$$
$$\leq \frac{hg(t_0)}{\chi}d^2(t_0)\exp\left[-\frac{\eta}{h}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right] < \frac{h(\xi)g(t_0)}{\chi}\xi^2\exp\left[-\frac{\eta}{h(\xi)}\int\limits_{t_0}^{t}\frac{dz}{g(z)}\right]$$

Condition (3.3) implies that the exponential goes to zero as $t \to \infty$, proving the asymptotic stability of the null solution; if $\overline{\overline{g}}<\infty$ we can replace $g(t_0), g(z)$ by $\overline{\overline{g}}$ in the last but one inequality and obtain

$$d^2(t) < \frac{h(\xi)\overline{\overline{g}}}{\chi}\exp\left[-\frac{\eta}{h(\xi)\overline{\overline{g}}}(t-t_0)\right]d^2(t_0),$$

which proves the uniform exponential-asymptotic stability of the null solution (just set $\delta = B^{-1}\left[r(\xi)\sqrt{\chi}/\sqrt{\overline{\overline{g}}}\right]$, $D=\sqrt{h(\xi)\overline{\overline{g}}/\chi}$, $E=\eta/\left[2h(\xi)\overline{\overline{g}}\right]$ in Def. 2.6). □

Remark 3.1 We stress that the theorem holds also if $\rho = \infty$. In the latter case ξ is σ_M, if the latter is finite, an arbitrary positive constant, if also $\sigma_M = \infty$.

Next, we are going to extend some of the previous results *in the large*.

4 Boundedness of the Solutions and Asymptotic Stability in the Large

Theorem 4.1 *Assume that: conditions (2.1)-(2.3), and possibly either one of (2.4'), are fulfilled with $\rho = \infty$ and $\tau < 1$; the function $g(t)$ defined by (2.23) fulfills $\overline{\overline{g}} < \infty$; (3.1) is fulfilled. Then:*
1. the solutions of (1.1) are uniformly bounded;
2. the null solution of (1.1) is exponential-asymptotically stable in the large.
If only (3.2), instead of (3.1), is satisfied, then:
3. the solutions of (1.1) are eventually uniformly bounded;
4. the null solution of (1.1) is eventually exponential-asymptotically stable in the large.

Proof As noted, $r(\sigma)$ can be inverted to an increasing map $r^{-1} : [0, r_M[\mapsto [0, \sigma_M[$, whence also

$$\beta(\delta) := r^{-1}\left[\frac{\sqrt{\overline{\overline{g}}}B(\delta)}{\sqrt{\chi}}\right] \tag{4.1}$$

defines an increasing map $\beta : [0, \delta_M[\mapsto [0, \sigma_M[$, where $\delta_M := B^{-1}(r_M\sqrt{\chi}/\sqrt{\overline{\overline{g}}})$. Note that $\beta(\delta) > \delta$. An immediate consequence of (4.1) is

$$\frac{\overline{\overline{g}}B^2(\delta)}{\chi} = r^2[\beta(\delta)] = \frac{\beta^2(\delta)}{1+\gamma_3[\beta(\delta)]}. \tag{4.2}$$

From (2.11) it immediately follows that

$$s(\delta) := \bar{t}\{\gamma_3[\beta(\delta)]\}\begin{cases} = 0, & \text{if (3.1) is fulfilled,} \\ < \infty, & \text{if (3.2) is fulfilled.}\end{cases} \tag{4.3}$$

We can now show that for any $\delta\in]0,\delta_M[$, $t_0\ge s(\delta)$

$$d(t_0) < \delta \quad \Rightarrow \quad d(t) < \beta(\delta), \quad \forall t \ge t_0. \tag{4.4}$$

Ad absurdum, assume that there exists a finite $t_2 > t_0$ such that (4.4) is fulfilled for all $t \in [t_0, t_2[$, whereas

$$d(t_2) = \beta(\delta). \tag{4.5}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W\{u, u_t, t; \gamma_3[\beta(\delta)], \theta\}$ is a decreasing function of t in $[t_0, t_2]$. Using (2.20), (2.24) and the (4.2) we find the following contradiction with (4.5):

$$\chi d^2(t_2) \le W(t_2) < W(t_0) \le \{1+\gamma_3[\beta(\delta)]\}g(t_0)B^2\,[d(t_0)] < \{1+\gamma_3[\beta(\delta)]\}\overline{\overline{g}}B^2(\delta) = \chi\beta^2(\delta).$$

Formula (4.4) together with (4.3) proves statements 1., 3. under the assumption $\tau\in[0,1[$, because then by (3.4) $\delta_M = \infty$, so that we can choose any $\delta > 0$ in Definition 2.3.

With the above choice of θ, by (4.4), (3.9) we find that for $t\ge t_0\ge s(\delta)$ the Liapunov functional $W_\delta(t) \equiv W\big\{u, u_t, t; \gamma_3\big[\beta(\delta)\big], \theta(\delta)\big\}$ fulfills

$$W_\delta(t)\le h(\delta)\overline{\overline{g}}d^2(t); \tag{4.6}$$

this, together with (2.16) implies $\dot{W}_\delta(t) \le -\eta W_\delta(t)/[h(\delta)\overline{\overline{g}}]$ and (by means of the comparison principle [17]) $W_\delta(t) < W_\delta(t_0)\exp\big[-\eta(t-t_0)/[h(\delta)\overline{\overline{g}}]\big]$. From the latter inequality, (2.20) and (4.6) with $t=t_0$ it follows

$$d^2(t) \le \frac{W_\delta(t)}{\chi} < \frac{W_\delta(t_0)}{\chi}\exp\left[-\frac{\eta}{h(\delta)\overline{\overline{g}}}(t-t_0)\right] \le \frac{h(\delta)\overline{\overline{g}}}{\chi}\exp\left[-\frac{\eta}{h(\delta)\overline{\overline{g}}}(t-t_0)\right] d^2(t_0)$$

for all $t \ge t_0 \ge s(\delta)$. Recalling again (4.3), we see that the latter formula proves statements 2., 4. □

In the case $\tau\ge 1$ we find, by (3.4),

$$\delta_M = B^{-1}\left(r_M\frac{\sqrt{\chi}}{\sqrt{\overline{\overline{g}}}}\right) = B^{-1}\left\{\left[\frac{\tau-1}{1+\gamma_{31}}\right]^{\frac{\tau-1}{2\tau}}\frac{\sqrt{\chi}}{\sqrt{\overline{\overline{g}}\tau\gamma_{32}^{1/\tau}}}\right\}.$$

The finiteness of δ_M prevents us from extending the results in the large of the previous theorem to the case $\tau\ge 1$. One might think to exploit the freedom in the choice of θ to make δ_M as large as we wish. From the θ-dependence of γ_{31}, γ_{32} [formulae (2.22), (2.14)] we see that δ_M decreases with θ, so this is impossible. However, we can prove boundedness and asymptotic stability in the large even for some unbounded $g(t)$, provided $\tau = 0$.

Theorem 4.2 *Assume that: conditions (2.3–2.1), and possibly either one of (2.4'), are fulfilled with $\rho = \infty$ and $\tau = 0$; the function $g(t)$ defined by (2.23) fulfills (3.3); either (3.1) or (3.2) is fulfilled. Then:*
1. the solutions of (1.1) are bounded;
2. the null solution of (1.1) is asymptotically stable in the large.

Proof The condition $\tau = 0$ means that γ does not depend on σ; then $r^{-1}(\beta) = \beta\sqrt{1+\gamma}$, which is an increasing map $r^{-1}: I \to I$. For any fixed t_0 setting

$$\tilde{\beta}(\alpha; t_0) := r^{-1}\left[\frac{\sqrt{g(t_0)}B(\alpha)}{\sqrt{\chi}}\right] = B(\alpha)\frac{\sqrt{g(t_0)(1+\gamma)}}{\sqrt{\chi}} \tag{4.7}$$

also defines an increasing map $\tilde{\beta}: I \to I$, with $\tilde{\beta}(\alpha;t_0) > \alpha$. We now prove statement 1, i.e. for any $\alpha > 0$, $t_0 \geq \kappa := \bar{t}(\gamma)$,

$$d(t_0) < \alpha \quad \Rightarrow \quad d(t) < \tilde{\beta}(\alpha;t_0) \quad \forall t \geq t_0. \tag{4.8}$$

Ad absurdum, assume that there exist a finite $t_2 \in [t_0, t]$ such that (4.8) is fulfilled for all $t \in [t_0, t_2[$, whereas

$$d(t_2) = \tilde{\beta}(\alpha;t_0). \tag{4.9}$$

The negativity of the rhs(2.16) implies that $W(t) \equiv W\{u(t), u_t(t), t; \gamma, \theta\}$ is a decreasing function of t in $[t_0, t_2]$. Using (2.20), (2.24) and (4.7) we find the following contradiction with (4.9):

$$\chi d^2(t_2) \leq W(t_2) < W(t_0) \leq (1+\gamma) g(t_0) B^2\left[d(t_0)\right] < (1+\gamma) g(t_0) B^2(\alpha) = \chi \tilde{\beta}^2(\alpha;t_0), \text{ Q.E.D.}$$

By Theorem 3.1 the null solution of (1.1) is stable. Moreover, by (4.8) relation (2.24) becomes

$$W(t) \leq \tilde{h}(\alpha, t_0) g(t) d^2(t), \qquad \tilde{h}(\alpha, t_0) := (1+\gamma)\left\{1+m\left[\tilde{\beta}(\alpha;t_0)\right]\right\},$$

which, together with (2.16), implies $\dot{W}(t) \leq -\eta W(t)/[\tilde{h} g(t)]$ and employing usual arguments, $W(t) < W(t_0) \exp\left[-\eta \int_{t_0}^{t} dz/[\tilde{h} g(z)]\right]$, whence, for all $t > t_0 \geq \kappa$,

$$\begin{aligned} d^2(t) \quad &\leq \quad \frac{W(t)}{\chi} < \frac{W(t_0)}{\chi} \exp\left[-\frac{\eta}{\tilde{h}} \int_{t_0}^{t} \frac{dz}{g(z)}\right] \leq \frac{\tilde{h} g(t_0)}{\chi} d^2(t_0) \exp\left[-\frac{\eta}{\tilde{h}} \int_{t_0}^{t} \frac{dz}{g(z)}\right] \\ &< \quad \frac{\tilde{h}(\alpha, t_0) g(t_0)}{\chi} \alpha^2 \exp\left[-\frac{\eta}{h(\alpha, t_0)} \int_{t_0}^{t} \frac{dz}{g(z)}\right]. \end{aligned}$$

The function $G_{t_0}(t) := \int_{t_0}^{t} dz/g(z)$ is increasing and by (3.3) diverges with t, what makes the rhs go to zero as $t \to \infty$; more precisely, we can fulfill Definition 2.7 defining the corresponding function $T(\alpha, \nu, t_0, u_0, u_1)$ by the condition that the rhs of the previous equation equals $\nu_0^2 := \min\{\nu^2, \alpha^2\}$ at $t = t_0 + T$, or equivalently

$$T = G_{t_0}^{-1}\left\{-\frac{\tilde{h}(\alpha, t_0)}{\eta} \log\left[\frac{\chi\, \nu_0^2}{\tilde{h}(\alpha, t_0)\, g(t_0)\, \alpha^2}\right]\right\} - t_0$$

(the rhs is positive as the argument of the logarithm is less than 1, by the definitions of $\chi, \tilde{h}$ and by the inequality $\nu_0/\alpha \leq 1$); this proves statement 2. □

5 Examples

Out of the many examples of forcing terms fulfilling (2.1) we just mention $F(z) = b\sin(\omega z)$ (this has $F_z(z) \leq b\omega =: k$), which makes (1.1) into a modification of the sine-Gordon equation, and the possibly non-analytic ones $F(z) = -b|z|^q z$ with $b > 0$, $q \geq 0$ (this has $F_z(z) \leq 0 =: k$), or $F(z) = b|z|^q z$ (this has $F_z(z) = b(q+1)|z|^q < b(q+1)|\rho|^q =: k$ if $|z| < \rho$). Out of the many examples of t-dependent coefficients that fulfill (2.2-2.3) and either (3.1) or (3.2), but not the hypotheses of the theorems of [4, 5, 6], we just mention the following ones:

Example 5.1 $\varepsilon(t) = \varepsilon_0(1+t)^{-p}$ with constant $\varepsilon_0, p \geq 0$ and $C \equiv C_0 \equiv$constant, with $C_0 > \frac{4(1+\varepsilon_0)k}{3+\varepsilon_0}$. As a consequence $\underline{\varepsilon} = 0 \leq \varepsilon \leq \varepsilon_0 = \overline{\overline{\varepsilon}}$, $\underline{\dot{\varepsilon}} = -p\varepsilon_0 \leq \dot{\varepsilon} = -p\varepsilon_0[1+t]^{-p-1} \leq 0 = \overline{\overline{\dot{\varepsilon}}}$, $\ddot{\varepsilon} = p(p+1)\varepsilon_0[1+t]^{-p-2} \geq 0 = \underline{\ddot{\varepsilon}}$ [condition $(2.2)_4$ is fulfilled], ($\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon} \to 0$ as $t \to \infty$). Conditions $(2.2)_1$-$(2.2)_3$ are fulfilled with $\mu = C/(1+\varepsilon_0)$. We find $g(t) = C_0 + p\varepsilon_0[1+t]^{-p-1} + 1$, whence $\overline{\overline{g}} = C_0 + p\varepsilon_0 + 1$. Finally we assume that $a' > 0$ and a fulfills $(2.3)_1$. Then Theorems 3.1, 4.1, apply: the null solution of (1.1) is uniformly stable and uniformly exponential-asymptotically stable; it is also uniformly bounded and exponential-asymptotically stable in the large if in addition $\rho = \infty$, $\tau < 1$.

One can check that if we had adopted the same Liapunov functional as in [5, 6] formulae (4.2), i.e. W of (2.8) with $\theta = 0 = a'$, for $p > 1$ (namely $\varepsilon \to 0$ sufficiently fast as $t \to \infty$) we would have not been able to prove the asymptotic stability.

Example 5.2 $\varepsilon(t) = \varepsilon_0(1+t)^p$, $C(t) = C_0(1+t)^q$, with $1 > q \geq p \geq 0$, $\varepsilon_0 \geq 0$ and C_0 fulfilling

$$C_0 > p\varepsilon_0, \qquad C_0 > \frac{4(1+\varepsilon_0)k + 2p\,\varepsilon_0}{3+\varepsilon_0}.$$

If $q, p > 0$ then $C(t), \varepsilon(t)$ diverge as $t \to \infty$. We immediately find $\varepsilon(t) \geq \varepsilon_0 = \underline{\varepsilon}$, $\dot{\varepsilon} = p\varepsilon_0(1+t)^{p-1} \geq 0$, $\ddot{\varepsilon} = p(p-1)\varepsilon_0(1+t)^{p-2} \leq 0$, $\underline{\ddot{\varepsilon}} = p(p-1)\varepsilon_0$ [condition $(2.2)_4$ is fulfilled], $C(t) \geq C_0$,

$$\frac{C - \dot{\varepsilon}}{1+\varepsilon} = \frac{C_0(1+t)^q - p\varepsilon_0(1+t)^{p-1}}{1+\varepsilon_0(1+t)^p} = \frac{C_0(1+t)^{q-p} - p\varepsilon_0(1+t)^{-1}}{(1+t)^{-p} + \varepsilon_0} \geq \frac{C_0 - p\varepsilon_0}{1+\varepsilon_0},$$

and conditions $(2.2)_1$-$(2.2)_3$ are fulfilled with $\mu = (C_0 - p\varepsilon_0)/(1+\varepsilon_0)$. Moreover, $\dot{C} = qC_0(1+t)^{q-1} \to 0$ as $t \to \infty$ [condition (3.2) is fulfilled]; $g(t)$ grows as t^q, implying that (3.3) is fulfilled. Finally we assume that a fulfills $(2.3)_1$ [condition $(2.3)_2$ is already satisfied] . Then Theorem 3.1 applies: the null solution of (1.1) is asymptotically stable. If in addition $\rho = \infty$, $\tau = 0$ then Theorem 4.2 applies, and the null solution is also bounded and asymptotically stable in the large.

Example 5.3 $\varepsilon(t)$ fulfilling $\overline{\overline{\varepsilon}} < \infty$, $\overline{\overline{\dot{\varepsilon}}} < \infty$, $\underline{\dot{\varepsilon}} > -\infty$, $\underline{\ddot{\varepsilon}} > -\infty$ [condition (3.2)]; we note that this includes regular, periodic $\varepsilon(t)$. $C(t) = C_0 + C_1(1+t)^{-q}$ with constant C_0, C_1, q fulfilling $C_1 > 0$, $q \geq 0$ and

$$C_0 > \max\left\{0, \overline{\overline{\dot{\varepsilon}}}, \frac{4(1+\overline{\overline{\varepsilon}})k + 2\overline{\overline{\dot{\varepsilon}}}}{3+\overline{\overline{\varepsilon}}}\right\}, \qquad C_0 \geq k.$$

Then conditions $(2.2)_1$–$(2.2)_3$ are fulfilled with $\mu = (C_0 - \overline{\overline{\dot{\varepsilon}}})/(1+\overline{\overline{\varepsilon}})$. Moreover, $\dot{C} \leq 0$ (condition (3.1) is fulfilled). We find $g(t) \leq C_0 + C_1 - \underline{\dot{\varepsilon}} + 1 =: \overline{\overline{g}} < \infty$. Finally we assume that $a' > 0$ and a fulfills $(2.3)_1$. Then Theorems 3.1, 4.1, apply: the null solution of (1.1) is uniformly stable and uniformly exponential-asymptotically stable. It is also uniformly bounded and exponential-asymptotically stable in the large if in addition $\rho = \infty$, $\tau < 1$.

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