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Dynamic Inequalities, Bounds, and Stability of Systems with Linear and Nonlinear Perturbations

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Abstract: Generalized dynamic inequalities are introduced to the time scales scene, mainly as generalizations of Gronwall's inequality. Linear systems with linear and nonlinear perturbations and their stability characteristics versus the unperturbed system are investigated. Bounds for solutions to linear dynamic systems are stated using the system matrix.

Keywords: *stability; perturbed linear system; dynamic inequality; system bounds; time scales.*

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1 Introduction

It is useful to consider state equations that are close (in an appropriate sense) to another linear state equation that is uniformly stable or uniformly exponentially stable. Prompted by Lyapunov [6], DaCunha [4] showed that if the stability of the uniformly regressive time varying linear dynamic system

$$x^{\Delta}(t) = A(t)x(t), \qquad x(t_0) = x_0, \tag{1.1}$$

has already been determined by an appropriate *generalized Lyapunov function*, then certain conditions on the perturbation matrix $F(t)$ guarantee specific stability characteristics of the perturbed linear system

$$z^{\Delta}(t) = [A(t) + F(t)]z(t), \qquad z(t_0) = z_0. \tag{1.2}$$

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In Brogan [2], Chen [3], and Rugh [8], the stability of linear systems and perturbed linear systems is investigated on the lackluster time scales of $\mathbb{R}$ and $\mathbb{Z}$. As is known in the time scales community, analysis on either of these two domains rarely offers the complexity and challenge of the same study on an arbitrary closed set of the reals. One of the main reasons for this is that the uniform graininess of each makes for a run of the mill investigation. Despite these shortcomings of $\mathbb{R}$ and $\mathbb{Z}$, this paper is motivated by these works to unify and extend to the more general area of time scales, as were Gard and Hoffacker [5] in the scalar dynamic equation case and Pötzsche, Siegmund, and Wirth [7] in the constant and Jordan reducible linear systems case. Our aim in this exposition is to prove analogous results for the universal time scales setting.

This paper is organized as follows. Section 2 introduces two dynamic inequalities which are generalizations of Gronwall's inequality. In addition to bounds for solutions to linear dynamic systems using the system matrix coefficients, linear systems with perturbations and their stability characteristics versus the unperturbed system are investigated in Section 3. Section 4 gives slightly more general stability results for linear systems with nonlinear perturbations. The author's conclusions end the paper.

2 Generalizations of Gronwall's Inequality

To begin with, we state two theorems from the introductory time scales text [1]. One important result that is supplied from the following is a way to show uniqueness of solutions for initial value problems of linear dynamic systems.

Theorem 2.1 *[1, Thm. 6.1] Let $f, x \in \mathrm{C_{rd}}$ and $p \in \mathcal{R}^+$. Then*

$$x^{\Delta}(t) \leq p(t)x(t) + f(t), \quad \textit{for all } t \in \mathbb{T}$$

implies

$$x(t) \leq e_p(t, t_0)x_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(t)\Delta s, \quad \textit{for all } t \in \mathbb{T}.$$

Theorem 2.2 (Gronwall's inequality) *[1, Thm. 6.4] Let $f, x \in \mathrm{C_{rd}}$, $p \in \mathcal{R}^+$, and $p \geq 0$ for all $t \geq t_0$. Then*

$$x(t) \leq f(t) + \int_{t_0}^{t} p(s)x(s)\Delta s, \quad \textit{for all } t \in \mathbb{T}$$

implies

$$x(t) \leq f(t) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)p(s)\Delta s, \quad \textit{for all } t \in \mathbb{T}. \tag{2.1}$$

By employing these previous two theorems, in particular, the generalized Gronwall inequality, we have the following two new generalized dynamic inequalities.

Theorem 2.3 *Let $x \in \mathrm{C_{rd}}$, $f \in \mathrm{C^1_{rd}}$, $p \in \mathcal{R}^+$, and $p \geq 0$ for all $t \geq t_0$. Then*

$$x(t) \leq f(t) + \int_{t_0}^{t} p(s)x(s)\Delta s, \quad \textit{for all } t \in \mathbb{T} \tag{2.2}$$

implies

$$x(t) \leq e_p(t, t_0)f(t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))f^{\Delta}(s)\Delta s, \quad \textit{for all } t \in \mathbb{T}. \tag{2.3}$$

Proof Applying Gronwall's inequality from Theorem 2.2 to the inequality (2.2), we obtain the inequality (2.1).

Defining the function $r(t)$ as the right hand side of the inequality (2.1), using the fact that $p \geq 0$, and then delta differentiating $r(t)$ we obtain

$$r^{\Delta}(t) = f^{\Delta}(t) + f(t)p(t) + \int_{t_0}^{t} p(t)e_p(t,\sigma(s))f(s)p(s)\Delta s = f^{\Delta}(t) + p(t)r(t).$$

Multiplying both sides by the positive function $e_{\ominus p}(\sigma(t), t_0)$ we have

$$e_{\ominus p}(\sigma(t), t_0)(r^{\Delta}(t) - p(t)r(t)) = e_{\ominus p}(\sigma(t), t_0)f^{\Delta}(t)$$

which is equivalent to

$$[e_{\ominus p}(t, t_0)r(t)]^{\Delta} = e_{\ominus p}(\sigma(t), t_0)f^{\Delta}(t).$$

On both sides, integrate from t_0 to t, then multiply by $e_p(t, t_0)$ and obtain

$$r(t) = e_p(t, t_0)r(t_0) + \int_{t_0}^{t} e_{\ominus p}(\sigma(s), t)f^{\Delta}(s)\Delta s.$$

Thus, the desired inequality (2.3) is obtained. □

Theorem 2.4 *Let $f, w, x \in C_{rd}$, where f is a constant, $p \in \mathcal{R}^+$, and $p \geq 0$ for all $t \geq t_0$. Then*

$$x(t) \leq f + \int_{t_0}^{t} w(s) + p(s)x(s)\Delta s, \qquad \text{for all } t \in \mathbb{T} \tag{2.4}$$

implies

$$x(t) \leq e_p(t, t_0)f + \int_{t_0}^{t} e_p(t,\sigma(s))w(s)\Delta s, \qquad \text{for all } t \in \mathbb{T}. \tag{2.5}$$

Proof We define the function $r(t)$ by writing the right hand side of the inequality (2.4). Observe that with (2.4) and the fact that $p \geq 0$,

$$r^{\Delta}(t) = w(t) + p(t)x(t) \leq w(t) + p(t)r(t).$$

Multiplying both sides by the positive function $e_{\ominus p}(\sigma(t), t_0)$ we have

$$e_{\ominus p}(\sigma(t), t_0)(r^{\Delta}(t) - p(t)r(t)) = e_{\ominus p}(\sigma(t), t_0)w(t)$$

which is equivalent to

$$[e_{\ominus p}(t, t_0)r(t)]^{\Delta} = e_{\ominus p}(\sigma(t), t_0)w(t).$$

On both sides, integrate from t_0 to t, then multiply by $e_p(t, t_0)$ and obtain

$$r(t) = e_p(t, t_0)r(t_0) + \int_{t_0}^{t} e_{\ominus p}(\sigma(s), t)w(s)\Delta s.$$

Thus, we obtain the desired inequality (2.5). □

Example 2.1 Given the time varying system (1.1), we can use Theorem 2.1 (with $f(t) \equiv 0$) or Theorem 2.4 (with $w \equiv 0$) to derive a bound on the solution using the system matrix. Observe that

$$||x(t)|| \leq ||x_0|| + \int_{t_0}^{t} ||A(s)||\,||x(s)||\Delta s \implies ||x(t)|| \leq e_{||A||}(t, t_0)||x_0||, \quad \text{for all } t \in \mathbb{T}.$$

3 Linear Perturbations

We begin this section with a few useful definitions.

Definition 3.1 [7, Lem. 4.5] A regressive mapping $\lambda \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{C})$ is *uniformly regressive* on the time scale $\mathbb{T}$ if there exists a constant $\delta > 0$ such that

$$0 < \delta^{-1} \leq |1 + \mu(t)\lambda(t)|, \tag{3.1}$$

for all $t \in \mathbb{T}$.

Further, the $n \times n$ linear dynamic system (1.1) is *uniformly regressive* if all eigenvalues $\{\lambda_i\}_{i=1}^{k}$, $k \leq n$, of A satisfy (3.1) for all $t \in \mathbb{T}$.

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the uniformly regressive time varying linear dynamic equation (1.1).

Definition 3.2 The time varying linear dynamic equation (1.1) is *uniformly stable* if there exists a finite constant $\gamma > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq \gamma ||x(t_0)||, \qquad t \geq t_0.$$

For the next definition, we define a stability property that not only concerns the boundedness of a solutions to (1.1), but also the asymptotic characteristics of the solutions as well. If the solutions to (1.1) possess the following stability property, then the solutions approach zero exponentially as $t \to \infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 3.3 The time varying linear dynamic equation (1.1) is called *uniformly exponentially stable* if there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$||x(t)|| \leq ||x(t_0)|| \gamma e_{-\lambda}(t, t_0), \qquad t \geq t_0.$$

It is obvious by inspection of the previous definitions that we must have $\gamma \geq 1$. By using the word uniform, it is implied that the choice of γ does not depend on the initial time t_0.

Definition 3.4 [7] The *regressive stability region* for the scalar IVP is defined to be the set

$$\mathcal{S}(\mathbb{T}) = \left\{ \gamma(t) \in \mathbb{C} : \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(\tau)} \frac{\log |1 + s\gamma(\tau)|}{s} \Delta\tau < 0 \right\}.$$

It is easy to see that the regressive stability region is always contained in $\{\gamma \in \mathbb{C} : \mathrm{Re}(\gamma) < 0\}$. The reader is referred to [7] for more explanation.

Theorem 3.1 *Suppose the linear system* (1.1) *is uniformly stable. Then there exists some* $\beta > 0$ *such that if*

$$\int_{\tau}^{\infty} ||F(s)|| \Delta s \leq \beta$$

for all $\tau \in \mathbb{T}$*, the perturbed system* (1.2) *is uniformly stable.*

Proof See [4] for proof. □

Theorem 3.2 *Suppose the linear system* (1.1) *is uniformly exponentially stable. Then there exists some* $\beta > 0$ *such that if*

$$\int_{\tau}^{\infty} ||F(s)||\Delta s \leq \beta$$

for all $\tau \in \mathbb{T}$*, the perturbed system* (1.2) *is uniformly exponentially stable.*

Proof For any initial conditions, the solution of (1.2) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t, t_0)$ is the transition matrix for the system (1.1). By the uniform exponential stability of (1.1), there exist constants λ, $\gamma > 0$ with $-\lambda \in \mathcal{R}^+$ uniformly such that $||\Phi_A(t, \tau)|| \leq \gamma e_{-\lambda}(t, \tau)$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. Taking the norms of both sides and utilizing the uniform regressivity, we see

$$||z(t)|| \leq \gamma e_{-\lambda}(t, t_0)||z_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t, s)\delta||F(s)||\, ||z(s)||\, \Delta s, \quad t \geq t_0.$$

Defining $\psi(t) := e_{-\lambda}(t_0, t)||z(t)||$, this implies

$$\psi(t) \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\delta||F(s)||\, \psi(s)\Delta s.$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}
||z(t)|| &\leq \gamma||z_0||e_{-\lambda\oplus\gamma\delta||F||}(t, t_0) \\
&= \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\int_{t_0}^{t} \frac{\mathrm{Log}(1 + \mu(s)\gamma\delta||F(s)||)}{\mu(s)}\Delta s\right) \\
&\leq \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\int_{t_0}^{\infty} \frac{\mathrm{Log}(1 + \mu(s)\gamma\delta||F(s)||)}{\mu(s)}\Delta s\right) \\
&\leq \gamma||z_0||e_{-\lambda}(t, t_0) \exp\left(\gamma\delta \int_{t_0}^{\infty} ||F(s)||\Delta s\right) \\
&\leq \gamma||z_0||e^{\gamma\delta\beta}e_{-\lambda}(t, t_0), \quad t \geq t_0.
\end{aligned}$$

Since γ and $-\lambda$ can be used for any initial conditions, the system (1.2) is uniformly exponentially stable. □

Theorem 3.3 *Suppose the linear system* (1.1) *is uniformly exponentially stable. Then there exists some* $\beta > 0$ *such that if*

$$||F(t)|| \leq \beta \tag{3.2}$$

for all $t \geq t_0$ *with* $t, t_0 \in \mathbb{T}$*, the perturbed system* (1.2) *is uniformly exponentially stable.*

Proof For any initial conditions, the solution of (1.2) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))F(s)z(s)\Delta s,$$

where $\Phi_A(t, t_0)$ is the transition matrix for the system (1.1). By the uniform exponential stability of (1.1), there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, \tau)|| \leq \gamma e_{-\lambda}(t, \tau)$, for all t, $\tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides, we have

$$||z(t)|| \leq \gamma e_{-\lambda}(t, t_0)||z_0|| + \int_{t_0}^{t} \gamma e_{-\lambda}(t, \sigma(s))||F(s)|| \, ||z(s)|| \, \Delta s, \quad t \geq t_0.$$

Rearranging and applying the uniform regressivity bound and the inequality (3.2), we obtain

$$e_{-\lambda}(t_0, t)||z(t)|| \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta e_{-\lambda}(t_0, s)||z(s)|| \, \Delta s, \quad t \geq t_0.$$

Defining $\psi(t) := e_{-\lambda}(t_0, t)||z(t)||$, we now have

$$\psi(t) \leq \gamma||z_0|| + \int_{t_0}^{t} \gamma\beta\delta\psi(s) \, \Delta s, \quad t \geq t_0.$$

By Gronwall's inequality, we obtain

$$\psi(t) \leq \gamma||z_0||e_{\gamma\beta\delta}(t, t_0), \quad t \geq t_0.$$

Thus, substituting back in for $\psi(t)$, we conclude

$$||z(t)|| \leq \gamma||z_0||e_{-\lambda \oplus \gamma\beta\delta}(t, t_0), \quad t \geq t_0.$$

We need $-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$ and negative for all $t \in \mathbb{T}$. Observe, since $\gamma\beta\delta > 0$, it is positively regressive, and so $\gamma\beta\delta \in \mathcal{R}^+$. Since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we see that $-\lambda \oplus \gamma\beta\delta \in \mathcal{R}^+$. So we must have

$$-\lambda \oplus \gamma\beta\delta < 0 \implies \beta < \frac{\lambda}{\gamma\delta(1 - \mu(t)\lambda)},$$

for all $t \in \mathbb{T}$. Thus, by choosing β accordingly and since γ is independent of the initial conditions, the system (1.2) is uniformly exponentially stable. $\square$

Theorem 3.4 *Consider the uniformly regressive linear dynamic system* (1.2)*, with the matrices $A(t)$ and $F(t)$ constant. Let the uniformly regressive constants $\lambda \in \mathcal{R}^+$ and $\gamma > 0$ such that*

$$||e_A(t, t_0)|| \leq \gamma e_\lambda(t, t_0), \qquad t \geq t_0.$$

Then the bound

$$||e_{A+F}(t, t_0)|| \leq \gamma e_{(\lambda \oplus \gamma\delta||F||)}(t, t_0), \qquad t \geq t_0,$$

is valid.

Proof We begin by noting that the solution X to (1.2) with constant system matrices is given by

$$e_{A+F}(t,t_0) = X(t) = e_A(t,t_0) + \int_{t_0}^{t} e_A(t,\sigma(s))FX(s)\Delta s. \tag{3.3}$$

The solution (3.3) can be bounded by the following

$$||X(t)|| \le \gamma e_\lambda(t,t_0) + \int_{t_0}^{t} \gamma e_\lambda(t,\sigma(s))||F||\ ||X(s)||\Delta s. \tag{3.4}$$

We now employ Gronwall's inequality on (3.4) by defining $\psi(t) := e_\lambda(t_0,t)||X(t)||$. Thus,

$$\psi(t) \le \gamma + \int_{t_0}^{t} \gamma e_\lambda(s,\sigma(s))||F||\ \psi(s)\Delta s \le \gamma + \int_{t_0}^{t} \gamma\delta||F||\ \psi(s)\Delta s$$

which implies

$$||e_{A+F}(t,t_0)|| \le \gamma e_{(\lambda\oplus\delta\gamma||F||)}(t,t_0). \ \square$$

Theorem 3.5 *Given the uniformly regressive system* (1.2) *with* $A(t)\equiv A$ *a constant matrix, suppose all eigenvalues of* A *belong to* $\mathcal{S}(\mathbb{T})$, *the matrix* $F(t)\in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ *satisfies*

$$\lim_{t\to\infty}||F(t)|| = 0, \tag{3.5}$$

and the solution $x(t)\in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$ *is defined for all* $t\ge t_0$. *Then given any initial conditions* $x(t_0)=x_0$, *the solution to* (1.2) *satisfies*

$$\lim_{t\to\infty} x(t) = 0. \tag{3.6}$$

Proof Since $\mathrm{spec}(A)\in\mathcal{S}(\mathbb{T})$ for all $t\in\mathbb{T}$ and the system is uniformly regressive, we have

$$||e_A(t,t_0)|| \le \gamma e_{-\lambda}(t,t_0), \tag{3.7}$$

for some $\gamma,\lambda>0$ with $-\lambda\in\mathcal{R}^+$, and all $t\ge t_0$. Using (3.7), we can bound the solution by

$$||x(t)|| \le \gamma e_{-\lambda}(t,t_0) + \int_{t_0}^{t} \gamma e_{-\lambda}(t,\sigma(s))||F(s)||\ ||x(s)||\Delta s.$$

Choose an $\varepsilon>0$ such that $-\lambda\oplus\varepsilon<0$ and $-\lambda\oplus\varepsilon\in\mathcal{R}^+$ for all $t\in\mathbb{T}$. By Gronwall's inequality, we have

$$||x(t)||e_{-\lambda}(t_0,t) \le \gamma||x_0||\exp\left[\int_{t_0}^{t}\lim_{s\searrow\mu(\tau)}\frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(\tau)||]\Delta\tau\right]. \tag{3.8}$$

Denoting the upper bound of the graininess of $\mathbb{T}$ by μ^* and employing the generalized version of L'Hôpital's rule [1] and (3.5), we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\int_{t_0}^{t}\lim_{s\searrow\mu(\tau)}\frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(\tau)||]\Delta\tau}{\int_{t_0}^{t}\lim_{s\searrow\mu(\tau)}\frac{1}{s}\mathrm{Log}[1+s\varepsilon]\Delta\tau} &= \lim_{t\to\infty}\frac{\lim_{s\searrow\mu(t)}\frac{1}{s}\mathrm{Log}[1+s\gamma\delta||F(t)||]}{\lim_{s\searrow\mu(t)}\frac{1}{s}\mathrm{Log}[1+s\varepsilon]}\\ &\le \frac{\gamma\delta\lim_{t\to\infty}||F(t)||}{\frac{1}{\mu^*}\mathrm{Log}[1+\mu^*\varepsilon]}\\ &= 0,\end{aligned}$$

thus implying that there exists a $T \in \mathbb{T}$ such that for $t \geq T$ we have

$$\int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \mathrm{Log}[1 + s\gamma\delta||F(\tau)||]\Delta\tau \leq \int_{t_0}^{t} \lim_{s \searrow \mu(\tau)} \frac{1}{s} \mathrm{Log}[1 + s\varepsilon]\Delta\tau.$$

From (3.8), for $t \geq T$ we obtain

$$||x(t)||e_{-\lambda}(t_0, t) \leq \gamma||x_0||e_{\varepsilon}(t, t_0).$$

With a correct choice of ε above, it easily follows that

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda \oplus \varepsilon}(t, t_0)$$

which implies the claim (3.6). □

4 Nonlinear Perturbations

In the following theorem, we show that under certain conditions on the linear and nonlinear perturbations, the resulting perturbed nonlinear initial value problem will still yield uniformly exponentially stable solutions.

Theorem 4.1 *Given the nonlinear uniformly regressive initial value problem*

$$x^{\Delta}(t) = [A(t) + F(t)]\, x(t) + g(t, x(t)), \qquad x(t_0) = x_0, \tag{4.1}$$

and an arbitrary time scale $\mathbb{T}$*, suppose* (1.1) *is uniformly exponentially stable, the matrix* $F(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n \times n})$ *satisfies* $||F(t)|| \leq \beta$ *for all* $t \in \mathbb{T}$*, the vector-valued function* $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *satisfies* $||g(t, x(t))|| \leq \epsilon||x(t)||$ *for all* $t \in \mathbb{T}$ *and* $x(t)$*, and the solution* $x(t) \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}^n)$ *is defined for all* $t \geq t_0$*. Then if* β *and* ϵ *are sufficiently small, there exist constants* γ*,* $\lambda^* > 0$ *with* $-\lambda^* \in \mathcal{R}^+$ *such that*

$$||x(t)|| \leq \gamma||x_0||e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$*.*

Proof Observe that the solution to (4.1) is given by

$$x(t) = \Phi_A(t, t_0)x_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))[F(s)x(s) + g(s, x(s))]\Delta s, \tag{4.2}$$

for all $t \geq t_0$. Since (1.1) is uniformly exponentially stable, there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$. Recall $||F(t)|| \leq \beta$, $||g(t, x(t))|| \leq \epsilon||x(t)||$ for all $t \in \mathbb{T}$, and since the decay factor $-\lambda$ is uniformly regressive on $\mathbb{T}$, there exists a $\delta > 0$ such that $0 < \delta^{-1} \leq (1 - \mu(t)\lambda)$ for all $t \in \mathbb{T}$ which implies that $0 < (1 - \mu(t)\lambda)^{-1} \leq \delta$. Taking the norms of both sides of (4.2), we obtain

$$\begin{aligned}
||x(t)|| &\leq ||\Phi_A(t, t_0)||\, ||x_0|| + \int_{t_0}^{t} ||\Phi_A(t, \sigma(s))||(||F(s)||\, ||x(s)|| + ||g(s, x(s))||)\Delta s \\
&= e_{-\lambda}(t, t_0)\left[\gamma||x_0|| + \int_{t_0}^{t} \gamma\delta(\beta + \epsilon)e_{-\lambda}(t_0, s)||x(s)||\Delta s\right],
\end{aligned}$$

for all $t \geq t_0$.

By Gronwall's inequality,

$$||x(t)|| \leq \gamma ||x_0|| e_{-\lambda \oplus \gamma\delta(\beta+\epsilon)}(t, t_0).$$

To conclude, we need $-\lambda \oplus \gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$ as well as $-\lambda \oplus \gamma\delta(\beta + \epsilon) < 0$. Observe that $\gamma\delta(\beta + \epsilon) > 0$ implies $\gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$ and since $\mathcal{R}^+$ is a subgroup of $\mathcal{R}$, we have $-\lambda \oplus \gamma\delta(\beta + \epsilon) \in \mathcal{R}^+$. So we need

$$-\lambda \oplus \gamma\delta(\beta + \epsilon) < 0 \implies \beta < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta} - \epsilon.$$

From this result, we must have $\frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta} - \epsilon > 0$ for all $t \in \mathbb{T}$, i.e. $\epsilon < \frac{\lambda}{(1-\mu(t)\lambda)\gamma\delta}$ for all $t \in \mathbb{T}$.

Thus, to fulfill the requirements of the theorem, we must satisfy the following:

$$0 < \epsilon < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta}, \quad 0 < \beta < \frac{\lambda}{(1 - \mu(t)\lambda)\gamma\delta} - \epsilon, \quad \text{and} \quad -\lambda^* := -\lambda \oplus \gamma\delta(\beta + \epsilon)$$

for all $t \in \mathbb{T}$. □

Corollary 4.1 *Given the nonlinear uniformly regressive initial value problem* (4.1) *with $A(t) \equiv A$ a constant matrix, suppose* $\operatorname{spec}(A) \in \mathcal{S}(\mathbb{T})$ *for all $t \in \mathbb{T}$, the matrix $F(t) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ satisfies $||F(t)|| \leq \beta$ for all $t \in \mathbb{T}$, the vector-valued function $g(t, x(t)) \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfies $||g(t, x(t))|| \leq \epsilon ||x(t)||$ for all $t \in \mathbb{T}$ and $x(t)$, and the solution $x(t) \in \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}^n)$ is defined for all $t \geq t_0$. Then if β and ϵ are sufficiently small, there exist constants γ, $\lambda^* > 0$ with $-\lambda^* \in \mathcal{R}^+$ such that*

$$||x(t)|| \leq \gamma ||x_0|| e_{-\lambda^*}(t, t_0)$$

for all $t \geq t_0$.

Proof The proof follows exactly as in Theorem 4.1, with the observation that $\Phi_A(t, t_0) \equiv e_A(t, t_0)$. Since $\operatorname{spec}(A) \in \mathcal{S}(\mathbb{T})$, there exist constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ such that $||e_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$ for all $t \geq t_0$, we now have the bound $||\Phi_A(t, t_0)|| \leq \gamma e_{-\lambda}(t, t_0)$, for some constants γ, $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$. □

Conclusions

The intent of this paper was to add to the completeness of bounds on solutions to linear systems on time scales. In particular, in Section 2 this was done via introduction of two generalizations of Gronwall's inequality, thereby creating addition possibilities for bounding solutions to systems of the form (1.1) and (1.2).

In Section 3 and Section 4, certain bounds were given on the linear and nonlinear perturbations which maintained stability of the system (1.2) were investigated. This included integral bounds and asymptotic bounds on the perturbation matrix F.

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