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Global Robust Dissipativity of Neural Networks with Variable and Unbounded Delays

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Abstract: In this paper, the global robust dissipativity of a class of neural networks with variable and unbounded delays is investigated. Several criteria are obtained by constructing radically unbounded and positive definite Lyapunov functionals and using analytic techniques. Some numerical examples are given to compare our results with previous robust dissipativity results derived in the literature. It is shown that our results extend and improve earlier ones.

Keywords: *dissipativity; neural networks; attractive set; integro-differential models.*

Mathematics Subject Classification (2000): 92B20, 93D09, 34K35.

1 Introduction

In recent years, the stability of dynamical neural networks has received much attention and has been used in signal processing, pattern recognition, associative memory and optimization problems [1–10]. However, it is possible that there are no equilibrium points of dynamical systems in some situations. As pointed in [11–15], the global dissipativity is a more general concept and is of great importance to study in dynamical neural networks. It has found applications in the areas such as stability theory, chaos and synchronization theory and robust control [12]. The authors of [12] analyzed the global dissipation of neural networks with both variable and unbounded delays. In [11], some conditions for globally robust dissipativity of neural networks with time-varying delays are derived.

In this paper, motivated by the above discussions, we obtain several new sufficient conditions for the global robust dissipativity of integro-differential models of neural networks with variable and unbounded delays. The results compared with those presented in [11] can be checked easily. Some numerical examples illustrate the proposed conditions may provide useful and less conservative results for the problem.

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2 System Description

In this paper, we consider the model of neural network with variable and unbounded delays as follows [11]:

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds + u_i. \quad (1)$$

for $i = 1, 2, ..., n$, where n denotes the number of the neurons in the neural network, $x_i(t)$ is the state of the ith neuron at time t, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t))]^T$ is the activation function of the jth neuron at time t, $D = \mathrm{diag}(d_1, d_2, ..., d_n)$ is a positive definite diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ are the feedback matrix and the delayed feedback matrix, respectively, $u = (u_1, u_2, ..., u_n)^T$ is a constant external input vector. The assumption on the transmission delay $\tau(t)$ is proposed as $0 < \tau_{ij}(t) \le \sigma$, $\tau(t)$ is a differential function such that $\frac{d\tau_{ij}(t)}{dt} \le \tau^* \le 1$, for $i, j = 1, 2, ...n$.

The delay kernel function $k(\cdot) = (K_{ij}(\cdot))_{n\times n}, i, j = 1, 2, ..., n$ is assumed to satisfy the following conditions simultaneously:

(1) $K_{ij} : [0, \infty) \to [0, \infty)$;

(2) K_{ij} are bounded and continuous on $[0, \infty)$;

(3) $\int_0^\infty K_{ij}(s)ds = 1$;

(4) there exists a positive number ε such that $\int_0^\infty K_{ij}(s)e^{\varepsilon s}ds < \infty$,

(5) $\int_0^\infty e^{\beta s} K_{ij}(s)ds = p_{ij}(\beta)$, for $i, j = 1, 2, ..., n$, where $p_{ij}(\beta)$ is continuous function in $[0, \delta), \delta > 0$, and $p_{ij}(0) = 1$.

The initial conditions associated with the system (1) are given by $x_i(s) = \phi_i(s), -\sigma \le s \le 0, i = 1, 2, ..., n$, where $\phi_i(\cdot)$ is bounded and continuous on $[-\sigma, 0]$.

Throughout this paper, we will employ the following classes of activation functions :

(1) The set of bounded activation functions is defined as

$$\Gamma = \{f(x) | |f_i(x_i)| \le k_i, i = 1, 2, ..., n\}.$$

(2) The set of Lipschitz-continuous activation functions is defined as

$$\Psi = \{f(x)|0 \le \frac{f_i(x_i) - f_i(y_i)}{x_i - y_i} \le l_i, l_i > 0, \forall x_i, y_i \in R, x_i \ne y_i, i = 1, 2, ..., n\}.$$

(3) The general set of monotone non-decreasing activation functions is defined as

$$\Phi = \{f(x)|D^+ f_i(x_i) \ge 0, i = 1, 2, ..., n\}.$$

(4) There exist constants $\vartheta_i > 0$ such that $|f_i| \le \vartheta_i |x|, i = 1, 2, ..., n, \forall x \in R$. This class of functions will be denoted by $f(x) \in \Upsilon$.

The quantities d_i, a_{ij}, b_{ij} and c_{ij} may be considered as intervals as follows [15]:

$$\begin{aligned}
D_I : &= \{D = \mathrm{diag}(d_i) : \underline{D} \le D \le \overline{D}, i.e., \underline{d}_i \le d_i \le \overline{d}_i, i = 1, ..., n, \forall D \in D_I\},\\
A_I : &= \{A = (a_{ij})_{n\times n} : \underline{A} \le A \le \overline{A}, i.e., \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i, j = 1, ..., n, \forall A \in A_I\},\\
B_I : &= \{B = (b_{ij})_{n\times n} : \underline{B} \le B \le \overline{B}, i.e., \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, i, j = 1, ..., n, \forall B \in B_I\},\\
C_I : &= \{C = (c_{ij})_{n\times n} : \underline{C} \le C \le \overline{C}, i.e., \underline{c}_{ij} \le c_{ij} \le \overline{c}_{ij}, i, j = 1, ..., n, \forall C \in C_I\}. \quad (2)
\end{aligned}$$

Similar to [11], we give the following definitions.

Definition 2.1 The neural network definied by (1) is said to be a dissipative system, if there exists a compact set $S \subset R^n$, such that $\forall x_0 \in R^n, \exists T > 0$, when $t \geq t_0 + T, x(t, t_0, x_0) \subseteq S$, where $x(t, t_0, x_0)$ denotes the solution of Eq. (1) from initial state x_0 and initial time t_0. In this case, S is called a globally attractive set. A set S is called positive invariant if $\forall x_0 \in S$ implies $x(t, t_0, x_0) \subseteq S$ for $t \geq t_0$.

Definition 2.2 If $R \to R$ is a continuous function, then the upper right derivative $\frac{D^+ f(t)}{dt}$ of $f(t)$ is defined as

$$D^+ f(t) = \lim_{\theta \to 0^+} \frac{f(t+\theta) - f(t)}{\theta}. \tag{3}$$

Lemma 2.1 *[16] Let D, S and P be real matrices of appropriate dimensions with $P > 0$. Then for any vectors x, y with appropriate dimensions,*

$$2x^T DSy \leq x^T DPD^T x + y^T S^T P^{-1} Sy.$$

3 Main Results

Theorem 3.1 *Let $f(x) \in \Gamma$, then neural network system (1) is a robust dissipative system and the set S_1 is a positive invariant and globally attractive set, where*

$$S_1 = \{x||x_i| \leq d_i^{-1} \sum_{j=1}^n [(a_{ij}^* + b_{ij}^* + c_{ij}^*)k_j + |u_i|)], i = 1, 2, ..., n\}, \tag{4}$$

$a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij}), b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij})$ *and* $c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$.

Proof Let us use a radically unbounded and positive definite Lyapunov functional

$$V(x) = \sum_{i=1}^n \frac{1}{r}|x_i|^r.$$

Computing $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned} \frac{dV}{dt} &= \sum_{i=1}^n |x_i|^{r-1} sgn(x_i) \frac{dx_i}{dt} \\ &\leq \sum_{i=1}^n [-\underline{d}_i |x_i|^r + \sum_{j=1}^n (a_{ij}^* + b_{ij}^* + c_{ij}^*)k_j |x_i|^{r-1} + |u_i||x_i|^{r-1}] \\ &= -\sum_{i=1}^n |x_i|^{r-1} [\underline{d}_i |x_i| - \sum_{j=1}^n [(a_{ij}^* + b_{ij}^* + c_{ij}^*)k_j + |u_i|] < 0, \end{aligned} \tag{5}$$

where $x \in R^n \backslash S_1$, i.e., $x \overline{\in} S_1$. Eq. (5) implies that the neural network system (1) is a robust dissipative system and S_1 is a positive invariant and globally attractive set. □

Theorem 3.2 *Let $f(x) \in \Psi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If*

$$\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*} \overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1) I \leq 0,$$

where $C^* = (c_{ji}^*)_{n \times n}$, then the neural network system (1) is a robust dissipative system and the set $S_2 = \{x||f_i(x_i(t))| \leq \frac{l_i |u_i|}{\underline{d}_i}, i = 1, 2, ..., n$ is a positive invariant and globally attractive set.

Proof We use the following positive definite and unbounded Lyapunov functional:

$$V(x(t)) = 2\sum_{i=1}^{n}\int_{0}^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} f_i^2(x_i(s))ds$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}\int_{0}^{\infty} K_{ji}(s)(\int_{t-s}^{t} f_i^2(x_i(\xi)d\xi)ds.$$

Computing $\frac{dV}{dt}$ along the positive half trajectory of (1), we can conclude that

$$\begin{aligned}
\frac{dV}{dt} &= -2\sum_{i=1}^{n} d_i f_i(x_i(t))x_i(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(x_i(t))f_j(x_j(t)) + \sum_{i=1}^{n} f_i^2(x_i(t))\\
&+ 2\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} f_i(x_i(t))f_j(x_j(t-\tau_j(t))) - \sum_{i=1}^{n}(1-\frac{d\tau_i(t)}{dt})f_i^2(x_i(t-\tau_i(t)))\\
&+ 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_i(x_i(t))\int_{-\infty}^{t} K_{ij}(t-s)f_j(x_j(s))ds + 2\sum_{i=1}^{n} f_i(x_i(t))u_i\\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}\int_{0}^{\infty} K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds\\
&\le -2\sum_{i=1}^{n}\frac{\underline{d}_i}{l_i} f_i^2(x_i(t)) + f^T(x(t))(A+A^T)f(x(t)) + 2f^T(x(t))Bf(x(t-\tau(t)))\\
&+ f^T(x(t))f(x(t)) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{*}\int_{0}^{\infty} K_{ij} f_i(x_i(t))f_j(x_j(t-s))ds\\
&+ 2\sum_{i=1}^{n}|u_i||f_i(x_i(t))| + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*} f_j^2(x_j(t))\\
&- (1-\tau^*)f^T(x(t-\tau(t)))f(x(t-\tau(t))) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}\int_{0}^{\infty} K_{ij}(s)f_j^2(x_j(t-s))ds.
\end{aligned} \tag{6}$$

Using Lemma 1 and inequality technique, we have

$$\begin{aligned}
\frac{dV}{dt} &\le -2\sum_{i=1}^{n}\frac{\underline{d}_i}{l_i}|f_i(x_i(t))|[|f_i(x_i(t))| - \frac{l_i|u_i|}{\underline{d}_i}] + f^T(x(t))[\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*}\overline{B}\overline{B}^T\\
&+ (1+\|C^*\|_\infty + \|C^*\|_1)I]f(x(t)) \le -2\sum_{i=1}^{n}\frac{\underline{d}_i}{l_i}|f_i(x_i(t))|[|f_i(x_i(t))| - \frac{l_i|u_i|}{\underline{d}_i}] < 0
\end{aligned} \tag{7}$$

when $x \in R^n \backslash S_2$. Eq. (7) implies that the set S_2 is a positive invariant and globally attractive set. □

Theorem 3.3 *Let $f(x) \in \Psi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, ..., p_n)$ such that the matrix*

$$Q = P(\overline{A} - \underline{D}L^{-1}) + (\overline{A}^T - \underline{D}L^{-1})P + \frac{1}{1-\tau^*}P\overline{B}\overline{B}^T P + (1+\|PC^*\|_\infty + \|PC^*\|_1)I$$

is negative definite, then the neural network system (1) is a robust dissipative system and the set

$$S_3 = \left\{ x | \sum_{i=1}^{n} (f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)})^2 \le \sum_{i=1}^{n} (\frac{p_i u_i}{\lambda_M(Q)})^2, i = 1, 2, ..., n \right\}$$

is a positive invariant and globally attractive set, where $L = \mathrm{diag}(L_1, L_2, ..., L_n), P = \mathrm{diag}(p_1, p_2, ..., p_n)$ *and* $\lambda_M(Q)$ *is the maximum eigenvalue of the matrix* Q*.*

Proof We employ the following positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned} V(x(t)) = 2\sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s) ds + \sum_{i=1}^{n} \int_{t-\tau_i(t)}^{t} f_i^2(x_i(\xi)) d\xi \\ + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s) (\int_{t-s}^{t} f_i^2(x_i(\xi)) d\xi) ds. \end{aligned}$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we obtain that

$$\begin{aligned} \frac{dV}{dt} &= 2\sum_{i=1}^{n} p_i f_i(x_i(t))[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + u_i \\ &+ \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s)) ds] - \sum_{i=1}^{n} (1 - \frac{d\tau_i(t)}{dt}) f_i^2(x_i(t-\tau_i(t))) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s) [f_i^2(x_i(t)) - f_i^2(x_i(t-s))] ds + \sum_{i=1}^{n} f_i^2(x_i(t)) \\ &\le -2\sum_{i=1}^{n} \frac{p_i d_i}{l_i} f_i^2(x_i(t)) + f^T(x(t))(P\overline{A} + \overline{A}^T P) f(x(t)) + f^T(x(t)) f(x(t)) \\ &+ 2\sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ij}^* \int_0^{\infty} K_{ij}(s) f_i(x_i(t)) f_j(x_j(t-s)) ds + 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) \\ &- (1-\tau^*) f^T(x(t-\tau(t))) f(x(t-\tau(t))) + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* f_i^2(x_i(t)) \\ &- \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ji}^* \int_0^{\infty} K_{ji}(s) f_i^2(x_i(t-s)) ds + 2 f^T(x(t)) P B f(x(t-\tau(t))). \end{aligned} \tag{8}$$

From Lemma 1 and inequality technique, we can write the following inequalities:

$$\begin{aligned} \frac{dV}{dt} &\le -2\sum_{i=1}^{n} \frac{p_i d_i}{l_i} f_i^2(x(t)) + f^T(x(t))(P\overline{A} + \overline{A}^T P) f(x(t)) \\ &+ \frac{1}{1-\tau^*} f^T(x(t)) P \overline{B} \overline{B}^T P f(x(t)) + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i c_{ij}^* f_i^2(x_i(t)) \end{aligned}$$

$$\begin{aligned}
&+ \quad f^T(x(t))f(x(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i c_{ij}^* f_j^2(x_j(t)) + 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) \\
&= \quad 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) + f^T(x(t))Qf(x(t)) \\
&\leq \quad 2\sum_{i=1}^{n} p_i u_i f_i(x_i(t)) + \lambda_M(Q)\sum_{i=1}^{n} f_i^2(x_i(t)) \\
&= \quad \lambda_M(Q)\sum_{i=1}^{n}[(f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)})^2 - (\frac{p_i u_i}{\lambda_M(Q)})^2] < 0, \qquad (9)
\end{aligned}$$

when $x \in R^n \backslash S_3$. Eq. (9) implies that the set S_3 is a positive invariant and globally attractive set. □

Theorem 3.4 *Let $f(x) \in \Phi, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If the following condition holds:*

$$\overline{A} + \overline{A}^T + \overline{B} + \frac{1}{1-\tau^*}\overline{B}^T + (\|C^*\|_\infty + \|C^*\|_1)I \leq 0,$$

where $C^* = (c_{ji}^*)_{n\times n}$, then the neural network system (1) is a robust dissipative system and the set $S_4 = \{x||x_i(t) \leq \frac{|u_i|}{\underline{d}_i}, i = 1, 2, ..., n\}$ is a positive invariant and globally attractive set.

Proof Let us use the following positive definite and radially unbounded Lyapunov functional:

$$\begin{aligned}
V(x(t)) = 2\sum_{i=1}^{n}\int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{t-\tau_{ji}(t)}^{t} b_{ji}^* f_i^2(x_i(s))ds \\
+ \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^{\infty} K_{ji}(s)(\int_{t-s}^{t} f_i^2(x_i(\xi))d\xi)ds.
\end{aligned}$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned}
\frac{dV}{dt} \quad &= \quad 2\sum_{i=1}^{n} f_i(x_i(t))[-d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j(t))) + u_i \\
&+ \quad \sum_{j=1}^{n} c_{ij}\int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds] - \sum_{i=1}^{n}\sum_{j=1}^{n}(1 - \frac{d\tau_{ji}(t)}{dt}) f_i^2(x_i(t-\tau_{ji}(t))) \\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}^* f_i^2(x_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^{\infty} K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds \\
&\leq \quad -2\sum_{i=1}^{n} \underline{d}_i |f_i(x_i(t))||x_i(t)| + f^T(x(t))(\overline{A} + \overline{A}^T) f(x(t)) + 2\sum_{i=1}^{n} u_i f_i(x_i(t))
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^{*}\int_{0}^{\infty} K_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))ds + 2f^T(x(t))Bf(x(t-\tau(t))) \\
&- \quad (1-\tau^*)f^T(x(t-\tau(t)))Bf(x(t-\tau(t))) + f^T(x(t))Bf(x(t)) \\
&+ \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}f_i^2(x_i(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^{*}\int_{0}^{\infty} K_{ji}(s)f_i^2(x_i(t-s))ds. \qquad (10)
\end{aligned}
$$

From Lemma 1, it follows that

$$
\begin{aligned}
2f^T(x(t))Bf(x(t-\tau(t))) \quad &\leq \quad \frac{1}{1-\tau^*}f^T(x(t))B^Tf(x(t)) \\
&+ \quad (1-\tau^*)f^T(x(t-\tau(t)))B^TB^{-T}Bf(x(t-\tau(t))) \\
&= \quad \frac{1}{1-\tau^*}f^T(x(t))B^Tf(x(t)) \\
&+ \quad (1-\tau^*)f^T(x(t-\tau(t)))Bf(x(t-\tau(t))) \qquad (11)
\end{aligned}
$$

By using the inequality $2ab \leq a^2 + b^2$ for any $a, b \in R$, we have

$$
2\int_0^\infty K_{ij}(s)f_i(x_i(t))f_j(x_j(t-s))ds \leq \int_0^\infty K_{ij}(s)f_i^2(x_i(t))ds + \int_0^\infty K_{ij}(s)f_j^2(x_j(t-s))ds. \quad (12)
$$

From (10) to (12), we get

$$
\begin{aligned}
\frac{dV}{dt} \quad &\leq \quad -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| \\
&+ \quad f^T(x(t))(\overline{A} + \overline{A}^T + \overline{B} + \frac{1}{1-\tau^*}\overline{B}^T + (\|C^*\|_\infty + \|C^*\|_1)I)f(x(t)) \\
&\leq \quad -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| < 0, \qquad (13)
\end{aligned}
$$

when $x \in R^n \backslash S_4$. Eq. (13) implies that the set S_4 is a positive invariant and globally attractive set. □

Theorem 3.5 *Let $f(x) \in \Upsilon, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$. If the following condition holds:*

$$
\sum_{j=1}^{n}(\overline{a}_{ij} + \frac{1}{1-\tau^*}b_{ij}^* + c_{ij}^*) < 0,
$$

where $a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij}), b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij}), c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$, then the neural network system (1) is a robust dissipative system and the set $S_5 = \{x||x_i(t) \leq \frac{|u_i|}{\underline{d}_i}, i = 1, 2, ..., n\}$ is a positive invariant and globally attractive set.

Proof Let us use the following positive definite and radially unbounded Lyapunov functional:

$$
V(x(t)) = x_i(t) + \frac{1}{1-\tau^*}\sum_{j=1}^{n} b_{ij}^* \int_{t-\tau_j(t)}^{t} |f_j(x_j(s))|ds + \sum_{j=1}^{n} c_{ij}^*\vartheta_j \int_0^\infty K_{ij}(s)\int_{t-s}^{t} |x_j(\xi)|d\xi\, ds.
$$

Calculating $\frac{dV}{dt}$ along the positive half trajectory of (1), we have

$$\begin{aligned}\frac{dV}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau(t))) + u_i \\ &+ \frac{1}{1-\tau^*}\sum_{j=1}^{n} b_{ij}^* f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}\int_0^{\infty} K_{ij}(s) f_j(x_j(t-s))ds \\ &- \frac{1}{1-\tau^*}(1-\frac{d\tau(t)}{dt})\sum_{j=1}^{n} b_{ij}^* f_j(x_j(t-\tau(t))) \\ &+ \sum_{j=1}^{n} c_{ij}^*\vartheta \int_0^{\infty} K_{ij}(s)|x_j(t)|ds - \sum_{j=1}^{n} c_{ij}^*\vartheta_j \int_0^{\infty} K_{ij}(s)|x_j(t-s)|ds \\ &\le -\underline{d}_i|x_i(t)| + \sum_{j=1}^{n}\overline{a}_{ij}\vartheta_j|x_j| + \sum_{j=1}^{n} b_{ij}\vartheta_j|x_j(t-\tau(t))| + |u_i| + \frac{1}{1-\tau^*}\sum_{j=1}^{n} b_{ij}^*\vartheta_j x_j(t) \\ &- \sum_{j=1}^{n} b_{ij}\vartheta_j x_j(t-\tau(t))) + \sum_{j=1}^{n} c_{ij}^*\vartheta_j|x_j| \\ &= -\underline{d}_i|x_i(t)| + |u_i| + \sum_{j=1}^{n}\vartheta_j(\overline{a}_{ij} + \frac{1}{1-\tau^*}b_{ij}^* + c_{ij}^*)|x_j| < 0, \qquad (14)\end{aligned}$$

when $x \in R^n\backslash S_5$. Eq. (14) implies that the set S_5 is a positive invariant and globally attractive set. □

Remark 3.1 Our activation functions are more general than those in [11]. Hence, our results improve and generalizes the earlier results.

Remark 3.2 Our methods used in this paper, such as Lyapunov functional and matrix inequalities used in Theorem 4, are different to those in [11].

Remark 3.3 The neural network system in [14] can be seen as a special case for model (1). Therefore, the global robust dissipativity of that system can be studied similarly.

4 Comparison and Examples

To compare with [11], we restated Theorem 1 of [11].

Theorem 4.1 *Let $f(x) \in \Upsilon, f(0) = 0$ and $|f_i(x_i)| \to \infty$ as $|x_i| \to \infty$, the neural network defined by (1) is a robust dissipative system and the set $S_6 = \{x||x_i(t) \le \frac{|u_i|}{\underline{d}_i}, i = 1, 2, ..., n\}$ is a positive invariant and globally attractive set, if there exist positive constants $p_i > 0, i = 1, 2, ..., n$ such that*

$$p_i(-\overline{a}_{ii} - \frac{1}{1-\tau^*}b_{ii}^* - c_{ii}^*) - \sum_{j=1, j\ne i}^{n} p_j(a_{ji}^* + \frac{1}{1-\tau^*}b_{ji}^* + c_{ji}^*) \ge 0, \qquad (15)$$

where $i = 1, 2, ..., n$, $a_{ij}^* = \max(|\underline{a}_{ij}|, \overline{a}_{ij})$, $b_{ij}^* = \max(|\underline{b}_{ij}|, \overline{b}_{ij})$, $c_{ij}^* = \max(|\underline{c}_{ij}|, \overline{c}_{ij})$.

Example 4.1 Consider the system (1) with delays: $\tau_{ij}(t) = 1$ for $i, j = 1, 2$,

$$\underline{D} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1.2 \end{bmatrix}, \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}, \underline{A} = \begin{bmatrix} -2 & 0.7 \\ -0.9 & -3 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} -1.5 & 0.4 \\ 0.3 & -1.5 \end{bmatrix}, \underline{B} = \begin{bmatrix} 0.25 & -0.5 \\ -0.2 & -0.7 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0.5 & 0.25 \\ 0 & -0.5 \end{bmatrix},$$

$$\underline{C} = \overline{C} = 0, u_1 = 1.5, u_2 = -2, \sigma = 1, \tau^* = 0, p_i = 1.$$

The initial values of system (1) is assumed as $\phi(s) = 0.5, t \in [-1, 0)$. Since

$$\begin{cases} \overline{a}_{11} + \overline{a}_{12} + \frac{1}{1-\tau^*} b_{11}^* + \frac{1}{1-\tau^*} b_{12}^* + c_{11}^* + c_{12}^* = -0.1 < 0, \\ \overline{a}_{21} + \overline{a}_{22} + \frac{1}{1-\tau^*} b_{21}^* + \frac{1}{1-\tau^*} b_{22}^* + c_{21}^* + c_{22}^* = -0.3 < 0, \end{cases}$$

the condition of Theorem 5 in this paper is satisfied; the neural network system (1) is a globally robust dissipative system, and the set $S_5 = \{(x_1(t), x_2(t)) | |x_1(t)| \leq \frac{15}{4}, |x_2(t)| \leq \frac{5}{3}\}$ is positive invariant and globally attractive. Since

$$\begin{cases} -\overline{a}_{11} - \frac{1}{1-\tau^*} b_{11}^* - c_{11}^* - (a_{21}^* + \frac{1}{1-\tau^*} b_{21}^* + c_{21}^*) = -0.1 < 0, \\ -\overline{a}_{22} - \frac{1}{1-\tau^*} b_{22}^* - c_{22}^* - (a_{12}^* + \frac{1}{1-\tau^*} b_{12}^* + c_{12}^*) = -0.4 < 0, \end{cases}$$

the condition of Theorem 6 is not satisfied, one can not determine the dissipativity of the neural network (1). Therefore, our obtained criteria for the global robust dissipativity of neural networks with variable and unbounded delays are new.

Example 4.2 Consider the system (1) with delays: $\tau_{ij}(t) = 1$ for $i, j = 1, 2$,

$$\underline{D} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1.2 \end{bmatrix}, \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}, \underline{A} = \begin{bmatrix} -3.3 & -0.25 \\ \frac{1}{3} & -3 \end{bmatrix}, \overline{A} = \begin{bmatrix} -3 & 0.25 \\ 0.5 & -4 \end{bmatrix},$$

$$\underline{B} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \overline{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \underline{C} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix}, \overline{C} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix},$$

and $u_1 = 1.5, u_2 = -2, \sigma = 1, \tau^* = 0$. The initial values of system (1) are assumed as $\phi(s) = 0.5, t \in [-1, 0)$. Since that

$$\overline{A} + \overline{A}^T + \frac{1}{1-\tau^*} \overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1) I = \begin{bmatrix} -2 & \frac{7}{4} \\ \frac{7}{4} & -4 \end{bmatrix} \leq 0,$$

then the conditions of Theorem 2 are satisfied, and the neural networks system (1) is a globally robust dissipative system, and the set

$$S_2 = \{f_1(x_1(t)), f_2(x_2(t)) | |f_1(x_1(t))| \leq \frac{15}{4} l_1, |f_2(x_2(t))| \leq \frac{5}{3} l_2\}$$

is positive invariant and globally attractive.

5 Conclusion

This paper studies the global robust dissipativity of a class of neural networks with variable and unbounded delays. Several sufficient conditions are presented to characterize the global dissipation together with their sets of attraction. Our results would make good effects in studying the uniqueness of equilibria, global asymptotic stability, instability and the exsitence of periodic solutions. In addition, several examples are given to demonstrate the improvements and correctness of our results.

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