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Positive Solutions of a Second Order m-point BVP on Time Scales

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Abstract: In this study, we are concerned with proving the existence of multiple positive solutions of a general second order nonlinear m-point boundary value problem (m-PBVP)

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda h(t)f(t,u) = 0, \;\; t \in [0,1],$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

on time scales. The proofs are based on the fixed point theorems in a Banach space. We present an example to illustrate how our results work.

Keywords: *m-point boundary value problems, positive solutions, fixed point theorems, time scales.*

Mathematics Subject Classification (2000): 39A10, 34B18, 34B40, 45G10.

1 Introduction

The theory of dynamic equations on time scales unifies the well-known analogies in the concept of difference equations and differential equations. Some basic definitions and theorems on time scales can be found in the books [3, 4]. In the past few years starting with Il'in and Mossiev [8] and Gupta [6], the existence of positive solutions for nonlinear high-order and second order boundary value problems have been studied by many authors by using the coincidence degree theory and fixed point theorems in cones (see [1, 2, 7, 9, 11, 12, 15] and references therein).

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The m-point boundary value problems for dynamic equations on time scales arise in a variety of different areas of applied mathematics, physics and engineering. Recently Yaslan [14], Sun and Lee [13] obtained some existence results for three point and multi-point boundary value problems on time scales.

In 2003, Ma and Wang [12] studied the nonlinear boundary value problem

$$u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, \quad t \in (0,1), \quad u(0) = 0, \quad \alpha u(\eta) = u(1)$$

and obtained some existence results if f satisfies either superlinear and sublinear conditions by applying fixed point theorems in cones. We generalized the results of Ma and Wang in three aspects: (a) we generalized the three point BVP to m-point BVP with a dynamic equation; (b) we study the eigenvalue problem; (c) we obtain the existence of at least three positive solutions.

In this paper we deal with the determining the value of λ for which the following m-point BVP has a positive solution:

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + \lambda h(t)f(t,u) = 0, \ t \in [0,1], \tag{1}$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{2}$$

where $0 < \eta_i < 1, \ \forall i = 1,2,\ldots,m-2$, h, f, a and b satisfy:

(H1) $f \in \mathcal{C}([\rho(0),\sigma(1)] \times [0,\infty),[0,\infty))$;

(H2) $h \in \mathcal{C}([0,1],[0,\infty))$ and there exists $t_0 \in [0,1]$ such that $h(t_0) > 0$;

(H3) $a \in \mathcal{C}([0,1],[0,\infty)), \ b \in \mathcal{C}([0,1],(-\infty,0])$.

This paper is organized as follows. In Section 2, starting with some preliminary lemmas we state the Krasnosel'skii and Legget-Williams fixed point theorems. In Section 3, we give the main results which state the sufficient conditions for the m-point BVP (1)-(2) to have at least one or at least three solutions.

2 Preliminaries and Fixed Point Theorems

In this section we state the preliminary information that we need the prove the main results.

Lemma 2.1 *Assume that* (H3) *holds. Let* ϕ_1 *and* ϕ_2 *be the solutions of*

$$\phi_1^{\Delta\nabla}(t) + a(t)\phi_1^{\Delta}(t) + b(t)\phi_1(t) = 0, \tag{3}$$

$$\phi_1(\rho(0)) = 0, \ \phi_1(\sigma(1)) = 1, \tag{4}$$

$$\phi_2^{\Delta\nabla}(t) + a(t)\phi_2^{\Delta}(t) + b(t)\phi_2(t) = 0, \tag{5}$$

$$\phi_2(\rho(0)) = 1, \ \phi_2(\sigma(1)) = 0 \tag{6}$$

respectively. Then
(i) ϕ_1 *is strictly increasing on* $[\rho(0),1]$, **(ii)** ϕ_2 *is strictly decreasing on* $[\rho(0),1]$.

Lemma 2.2 *Assume that* (H3) *holds. Then* (3)-(4) *and* (5)-(6) *have unique solutions respectively.*

The proofs of the Lemma 2.1 and Lemma 2.2 can be obtained easily by generalizing the proofs of Lemma 2.1 and Lemma 2.2 in [12] to time scales.

For the rest of the paper we need the following assumption

(H4) $0 < \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i) < 1$.

In the following lemma we express the Green's function and the form of the solution of the linear m-point BVP corresponding to (1)-(2).

Lemma 2.3 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}[\rho(0), \sigma(1)]$. *Then the problem*

$$u^{\Delta\nabla}(t) + a(t)u^{\Delta}(t) + b(t)u(t) + y(t) = 0, \ t \in [0,1], \tag{7}$$

$$u(\rho(0)) = 0, \quad u(\sigma(1)) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \tag{8}$$

is equivalent to the integral equation

$$u(t) = \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)y(s)\nabla s + A\phi_1(t), \tag{9}$$

where

$$A = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \sum_{i=1}^{m-2} \alpha_i \left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i, s)p(s)y(s)\nabla s \right), \tag{10}$$

$$p(t) = e_a(\rho(t), \rho(0)), \tag{11}$$

$$G(t,s) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \begin{cases} \phi_1(t)\phi_2(s), & s \geq t, \\ \phi_1(s)\phi_2(t), & t \geq s. \end{cases} \tag{12}$$

Proof First we show that the unique solution of (7)-(8) can be represented by (9). From Lemma 2.1, we know that the homogenous part of (7) has two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$ since

$$\begin{vmatrix} \phi_1(\rho(0)) & \phi_1^{\Delta}(\rho(0)) \\ \phi_2(\rho(0)) & \phi_2^{\Delta}(\rho(0)) \end{vmatrix} = -\phi_1^{\Delta}(\rho(0)) \neq 0.$$

Now by the method of variations of constants, we can obtain the unique solution of (7)-(8) which can be represented by (9) where A and G are as in (10) and (12) respectively. Next we check the function defined in (9) is the solution of the BVP (7)-(8). For this purpose we first show that (9) satisfies (7). From the definition of the Green's function (12), we get

$$u(t) = \frac{1}{\phi_1^{\Delta}(\rho(0))} \left(\int_{\rho(0)}^{t} \phi_1(s)\phi_2(t)p(s)y(s)\nabla s + \int_{t}^{\sigma(1)} \phi_1(t)\phi_2(s)p(s)y(s)\nabla s \right) + A\phi_1(t).$$

Hence the derivatives u^{Δ} and $u^{\Delta\nabla}$ are as follows:

$$u^{\Delta}(t)=\frac{1}{\phi_1^{\Delta}(\rho(0))}\left(\phi_2^{\Delta}(t)\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s+\phi_1^{\Delta}(t)\int_{t}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s\right)+A\phi_1^{\Delta}(t)$$

and

$$\begin{aligned}u^{\Delta\nabla}(t)=\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(&\phi_2^{\Delta\nabla}(t)\int_{\rho(0)}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s+\phi_2^{\Delta}(t)\phi_1(t)p(t)y(t)\\&+\phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{\sigma(1)}\phi_2(s)p(s)y(s)\nabla s-\phi_1^{\Delta}(t)\phi_2(t)p(t)y(t)\Big)+A\phi_1^{\Delta\nabla}(t).\end{aligned}$$

Replacing the derivatives in (7), we deduce

$$\begin{aligned}u^{\Delta\nabla}(t)+a(t)u^{\Delta}(t)+b(t)u(t)&=A\big(\phi_1^{\Delta\nabla}(t)+a(t)\phi_1^{\Delta}(t)+b(t)\phi_1(t)\big)\\&+\Big(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{\rho(0)}^{t}\phi_1(s)p(s)y(s)\nabla s\Big)\big(\phi_2^{\Delta\nabla}(t)+a(t)\phi_2^{\Delta}(t)+b(t)\phi_2(t)\big)\\&+\Big(\frac{1}{\phi_1^{\Delta}(\rho(0))}\int_{t}^{\sigma(1)}\phi_s(s)p(s)y(s)\nabla s\Big)\big(\phi_1^{\Delta\nabla}(t)+a(t)\phi_1^{\Delta}(t)+b(t)\phi_1(t)\big)\\&+\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta\nabla}(t)\int_{t}^{\rho(t)}\phi_1(s)p(s)y(s)\nabla s+\phi_1^{\Delta\nabla}(t)\int_{\rho(t)}^{t}\phi_2(s)p(s)y(s)\nabla s\Big)\\&+\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta}(t)\phi_1(t)-\phi_1^{\Delta}(t)\phi_2(t)\Big)p(t)y(t)\\&=\frac{1}{\phi_1^{\Delta}(\rho(0))}\Big(\phi_2^{\Delta\nabla}(t)(\rho(t)-t)\phi_1(t)p(t)y(t)-\phi_1^{\Delta\nabla}(t)(\rho(t)-t)\phi_2(t)p(t)y(t)\\&\qquad+\phi_2^{\Delta}(t)\phi_1(t)y(t)-\phi_1^{\Delta}(t)\phi_2(t)y(t)\Big)\\&=\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big(\phi_2^{\Delta}(t)\phi_1(t)-\phi_1^{\Delta}(t)\phi_2(t)\big)\\&\qquad-\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)(\rho(t)-t)\big(\phi_1^{\Delta\nabla}(t)\phi_2(t)-\phi_2^{\Delta\nabla}(t)\phi_1(t)\big)\\&=\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big\{\big(\phi_2^{\Delta}(t)\phi_1(t)-\phi_1^{\Delta}(t)\phi_2(t)\big)\\&\qquad+(\rho(t)-t)\big(\phi_2^{\Delta}(t)\phi_1(t)-\phi_1^{\Delta}(t)\phi_2(t)\big)^{\nabla}\big\}\\&=\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)\big(\phi_2^{\Delta}(\rho(t))\phi_1(\rho(t))-\phi_1^{\Delta}(\rho(t))\phi_2(\rho(t))\big)\\&=\frac{1}{\phi_1^{\Delta}(\rho(0))}p(t)y(t)e_{\ominus a}(\rho(t),\rho(0))\big(-\phi_1^{\Delta}(\rho(0))\big)\\&=-y(t).\end{aligned}$$

Therefore the function defined in (9) satisfies (7). Further we obtain that (8) is satisfied by (9). The first boundary condition of (8) follows from (9), (10) and (12). Now we verify the second boundary condition. Since

$$G(\sigma(1),s)=\frac{1}{\phi_1^{\Delta}(\rho(0))}\phi_1(s)\phi_2(\sigma(1))=0,$$

we obtain

$$u(\sigma(1)) = \int_{\rho(0)}^{\sigma(1)} G(\sigma(1),s)p(s)y(s)\nabla s + A\phi_1(\sigma(1)) = A. \tag{13}$$

On the other hand, by using equation (10) we find

$$\begin{aligned}
\sum_{i=2}^{m-2}\alpha_i u(\eta_i) &= \sum_{i=2}^{m-2}\alpha_i\left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i,s)p(s)y(s)\nabla s + A\phi_1(\eta_i))\right)\\
&= \sum_{i=2}^{m-2}\alpha_i\left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i,s)p(s)y(s)\nabla s + \frac{\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)\int_{\rho(0)}^{\sigma(1)} G(\eta_i,s)p(s)y(s)\nabla s}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\right)\\
&= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\sum_{i=1}^{m-2}\alpha_i\left(\int_{\rho(0)}^{\sigma(1)} G(\eta_i,s)p(s)y(s)\nabla s\right) = A. \qquad (14)
\end{aligned}$$

Combining the equations (13) and (14) finishes the proof. □

In this study we consider the Banach space $\mathcal{B}$ of continuous functions defined on $[\rho(0),\sigma(1)]$ with the supremum norm. Now we set

$$q(t) = \min\left\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\right\}. \tag{15}$$

Lemma 2.4 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}([\rho(0),\sigma(1)],[0,\infty))$. *Then the unique solution of* (7)-(8) *satisfies* $u(t) \geq ||u||q(t)$.

Proof Let t_0 be the point in $(\rho(0),\sigma(1))$ such that $||u|| = u(t_0)$. Next we verify

$$G(t,s) \geq G(t_0,s)q(t). \tag{16}$$

For this purpose, we consider the following four cases:

(i) $t, t_0 \leq s$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(t)}{\phi_1(t_0)} \geq \frac{\phi_1(t)}{||\phi_1||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(ii) $t, t_0 \geq s$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_2(t)}{\phi_2(t_0)} \geq \frac{\phi_2(t)}{||\phi_2||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(iii) $t_0 \leq s \leq t$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(s)\phi_2(t)}{\phi_1(t_0)\phi_2(s)} \geq \frac{\phi_2(t)}{\phi_2(s)} \geq \frac{\phi_2(t)}{||\phi_2||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

(iv) $t \leq s \leq t_0$: In this case,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\phi_1(t)\phi_2(s)}{\phi_1(s)\phi_2(t_0)} \geq \frac{\phi_1(t)}{\phi_1(s)} \geq \frac{\phi_1(t)}{||\phi_1||} \geq \min\{\frac{\phi_1(t)}{||\phi_1(t)||}, \frac{\phi_2(t)}{||\phi_2(t)||}\} = q(t).$$

In the third and the fourth cases we make use of Lemma 2.1. It follows from the fact $1 \geq \phi_1(t) \geq q(t),\ \forall t \in [\rho(0), \sigma(1)]$ and the inequality (16) that

$$\begin{aligned}
u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)y(s)\nabla s + A\phi_1(t)\Big\} \geq \lambda\Big\{q(t)\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\phi_1(t)\Big\} \\
&\geq \lambda q(t)\Big(\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\Big) \geq \lambda q(t)\Big(\int_{\rho(0)}^{\sigma(1)} G(t_0,s)p(s)y(s)\nabla s + A\phi_1(t_0)\Big) \\
&= q(t)u(t_0) = q(t)||u||. \ \Box
\end{aligned}$$

Assume that $\xi := \inf\{t \in \mathbb{T} : t > \rho(0)\},\ w := \sup\{t \in \mathbb{T} : t < \sigma(1)\}$ both exist and are included in $[\rho(0), \sigma(1)]$, and also satisfy $\rho(0) < \xi < w < \sigma(1)$. Also assume that $\sigma(w) < \sigma(1)$ and $\rho(\xi) > \rho(0)$ hold.

Lemma 2.5 *Assume that* (H3) *and* (H4) *hold. Let* $y \in \mathcal{C}([\rho(0), \sigma(1)], [0, \infty))$. *Then there exists* $\gamma > 0$ *such that unique solution of* (7)–(8) *satisfies* $u(t) > \gamma||u||$.

Proof Choose

$$\gamma = \min\{q(t) : t \in [\xi, w]\}. \tag{17}$$

It is clear that $\gamma > 0$ and $u(t) \geq q(t)||u|| > \gamma||u||,\ \forall t \in [\xi, w]$. $\Box$

To make use of the fixed point theorems we consider the cone

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) > 0, t \in [\rho(0), \sigma(1)], \min_{t\in[\xi,w]} u(t) \geq \gamma||u||\} \tag{18}$$

on the Banach space $\mathcal{B}$, and set $\mathcal{P}_r = \{x \in \mathcal{P} : ||x|| < r\}$.

Theorem 2.1 [5] *(Krasnosel'skii Fixed Point Theorem) Let* E *be a Banach space, and let* $K \subset E$ *be a cone. Assume* Ω_1 *and* Ω_2 *are open, bounded subsets of* E *with* $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, *and let* $A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ *be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1, \|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1, \|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$

hold. Then A *has a fixed point in* $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.2 [11] *(Legget–Williams Fixed Point Theorem) Let* $\mathcal{P}$ *be a cone in a real Banach space* E. *Set*

$$\mathcal{P}(\psi, a, b) := \{x \in \mathcal{P} : a \leq \psi(x), \|x\| \leq b\}.$$

Suppose $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$ *be a completely continuous operator and* ψ *be a nonnegative, continuous, concave functional on* $\mathcal{P}$ *with* $\psi(u) \leq \|u\|$ *for all* $u \in \overline{\mathcal{P}_r}$. *If there exist* $0 < p < q < l \leq r$ *such that the following conditions hold:*

(i) $\{u \in \mathcal{P}(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ *and* $\psi(Au) > q$ *for all* $u \in \mathcal{P}(\psi, q, l)$,

(ii) $\|Au\| < p$ *for all* $\|u\| \leq p$,

(iii) $\psi(Au) > q$ *for* $u \in \mathcal{P}(\psi, q, r)$ *with* $\|Au\| > l$.

Then A *has at least three positive solutions* u_1, u_2 *and* u_3 *in* $\overline{\mathcal{P}_r}$ *satisfying*

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \ \text{ with } \psi(u_3) < q.$$

3 Main Results

We are concerned with determining values of λ, for which there exist positive solutions of m-point boundary value problem (1)-(2). We use Krasnosel'skii fixed point theorem and Legget-Williams fixed point theorem to prove the main results. From Lemma 2.3, it is clear that the solutions of (1)-(2) are the fixed points of the operator

$$\Phi_\lambda u(t) = \lambda\Big\{ \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s + A\phi_1(t)\Big\}. \tag{19}$$

To state the main results we need to define the following extended real numbers:

$$f_0 = \lim_{u\to 0^+} \inf \min_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{20}$$

$$f^0 = \lim_{u\to 0^+} \sup \max_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{21}$$

$$f_\infty = \lim_{u\to \infty} \inf \min_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}, \tag{22}$$

$$f^\infty = \lim_{u\to \infty} \sup \max_{t\in[\rho(0),\sigma(1)]} \frac{f(t,u)}{u}. \tag{23}$$

Let K and L be defined by

$$K = \min_{t\in[\xi,w]} \int_\xi^w G(t,s)p(s)h(s)\nabla s, \tag{24}$$

$$L = \max_{t\in[\rho(0),\sigma(1)]} \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)\nabla s \le \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s. \tag{25}$$

In the following three main results, we state the criteria on λ to make sure the existence of positive solutions of (1)-(2).

Theorem 3.1 *Assume that* (H1)-(H4) *are satisfied. Then for each λ satisfying either one of the following conditions*

$$\textbf{\textit{(a)}}\ \frac{1}{\gamma \mathrm{Kf}_\infty} < \lambda < \frac{1}{\mathrm{Lf}^0}\Big(\frac{1-\sum_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}\phi_1(\eta_{\mathrm{i}})}{1+\sum_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}}\Big); \quad \textbf{\textit{(b)}}\ \frac{1}{\gamma \mathrm{Kf}_0} < \lambda < \frac{1}{\mathrm{Lf}^\infty}\Big(\frac{1-\sum_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}\phi_1(\eta_{\mathrm{i}})}{1+\sum_{\mathrm{i}=1}^{\mathrm{m}-2}\alpha_{\mathrm{i}}}\Big),$$

there exists at least one positive solution of (1)-(2).

Proof We claim that $\Phi_\lambda : \mathcal{P} \to \mathcal{P}$ Let $u \in \mathcal{P}$. First from the nonnegativity of G and from the assumptions (H2) and (H3), it is clear that $\Phi_\lambda u(t) \ge 0$ for $t \in [\rho(0), \sigma(1)]$.

Next by using (16) and (15), we get

$$\begin{aligned}\min_{t\in[\xi,w]}\Phi_\lambda u(t) &= \min_{t\in[\xi,w]}\lambda\Big\{\int_{\rho(0)}^{\sigma(1)}G(t,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\geq \lambda\Big\{\int_{\rho(0)}^{\sigma(1)}q(t)G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\geq q(t)\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\Big\}\\ &\geq \gamma\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\Big\}\\ &\geq \gamma\Big\{\lambda\int_{\rho(0)}^{\sigma(1)}G(t_0,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t_0)\Big\}=\gamma||\Phi_\lambda u||.\end{aligned}$$

Thus $\Phi_\lambda u\in\mathcal{P}$. Also complete continuity of $\Phi_\lambda u(t)$ can be obtained easily by the analysis methods. Now we seek for the fixed points of $\Phi_\lambda u(t)$ which belongs to $\mathcal{P}$.

Assume **(a)** holds. Since $\lambda<\frac{1}{Lf^0}\Big(\frac{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}{1+\sum_{i=1}^{m-2}\alpha_i}\Big)$ there exists $\epsilon>0$ such that

$$\lambda L(f^0+\epsilon)\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\leq 1.$$

The use of the definition of f^0 guarantees that there exists $r_1>0$, sufficiently small such that

$$\frac{f(t,u)}{u}<f^0+\epsilon,\quad \forall u\in[0,r_1].$$

It follows that $f(t,u)<(f^0+\epsilon)u$ for $0\leq u\leq r_1$ and $t\in[\rho(0),\sigma(1)]$. If $u\in\partial\mathcal{P}_{r_1}$ then using the fact $G(t,s)\leq G(s,s)$ we obtain

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)}G(t,s)p(s)h(s)f(s,u(s))\nabla s+A\phi_1(t)\Big\}\\ &\leq \lambda\Big(1+\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\int_{\rho(0)}^{\sigma(1)}G(s,s)p(s)h(s)f(s,u(s))\nabla s\\ &\leq \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)\int_{\rho(0)}^{\sigma(1)}G(s,s)p(s)h(s)f(s,u(s))\nabla s\end{aligned}$$

$$\leq \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)(f^0+\epsilon)||u||\int_{\rho(0)}^{\sigma(1)}G(s,s)h(s)\nabla s$$
$$= \lambda\Big(\frac{1+\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i\phi_1(\eta_i)}\Big)(f^0+\epsilon)||u||L \leq ||u||.$$

Hence if we define the open bounded set

$$\Omega_1 = \{u \in \mathcal{P} : ||u|| < r_1\}, \tag{26}$$

then

$$||\Phi_\lambda u|| \leq ||u||, \ \forall u \in \partial\mathcal{P}_{r_1} = \mathcal{P} \cap \partial\Omega_1. \tag{27}$$

Now we use the other part of the inequality in part **(a)**, $\frac{1}{\gamma K f_\infty} < \lambda$. We distinguish this part of the proof into two parts and first consider the case $f_\infty < \infty$. In this case, we pick ϵ_1 such that $\gamma K(f_\infty - \epsilon_1) \geq 1$. The use of the definition of f_∞ guarantees that there exists $r > r_1$, sufficiently large so that

$$\frac{f(t,u)}{u} > f_\infty - \epsilon_1, \quad \forall u \geq r.$$

Therefore, $f(t,u) > (f_\infty - \epsilon_1)u$ for $(t,u) \in [\rho(0), \sigma(1)] \times [0, r_1]$. We pick r_2 such that $r_2 \geq \frac{r}{\gamma} > r_1$ and define

$$\Omega_2 = \{u \in \mathcal{P} : ||u|| < r_2\}. \tag{28}$$

If $u \in \partial\mathcal{P}_{r_2}$, then Lemma 2.5 leads us to have

$$\begin{aligned}\Phi_\lambda u(t) &= \lambda\Big\{\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s + A\phi_1(t)\Big\}\\ &\geq \lambda\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s\\ &\geq \lambda(f_\infty - \epsilon_1)\gamma||u||\int_{\xi}^{w} G(t,s)p(s)h(s)\nabla s\\ &\geq \lambda(f_\infty - \epsilon_1)\gamma||u||K\\ &\geq ||u||.\end{aligned} \tag{29}$$

Consequently, we consider the case $f_\infty = \infty$ for which the second part of the inequality in part **(a)** becomes $\lambda > 0$. If we choose M sufficiently large so that

$$\lambda M\gamma\int_{\xi}^{w} G(t,s)p(s)h(s)\nabla s \geq 1 \quad (or \ \ \lambda M\gamma K \geq 1)$$

for any $t \in [\rho(0), \sigma(1)]$, then there exists $r > r_1$ so that $f(t,u) > Mu$ for $u \geq r_1$. Let r_2 be defined as above and let $u \in \partial\mathcal{P}_{r_2}$. Then for all $t \in [\rho(0), \sigma(s)]$, we have

$$\begin{aligned}\Phi_\lambda u(t) &\geq \lambda M \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)u(s)\nabla s \\ &\geq \lambda M\gamma ||u|| \int_{\xi}^{w} G(t,s)p(s)h(s)\nabla = \lambda M\gamma K ||u|| \leq ||u|| \end{aligned} \tag{30}$$

From the inequalities (29) and (30)

$$||\Phi_\lambda u|| \geq ||u||, \ \forall u \in \partial\mathcal{P}_{r_2} = \mathcal{P} \cap \partial\Omega_2. \tag{31}$$

Inequalities (27) and (31) show that the conditions of Krasnosel'skii fixed point theorem (Theorem 2.1) are fulfilled. Thus from Theorem 2.1, we conclude that $\Phi_\lambda u$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$. □

The following result states the existence of at least one positive solution of problem (1)-(2) in a different manner and also bounds the positive solution.

Theorem 3.2 *Let $f(t,u)$ satisfy* (H1). *Assume that there exist two positive constants $r_2 > r_1 > 0$ such that the following conditions are satisfied:*

(H5) $f(t,u) \leq \dfrac{Mr_2}{\lambda}$ *for* $(t,u) \in [\rho(0), \sigma(1)] \times [0, r_2]$,

(H6) $f(t,u) \geq \dfrac{Nr_1}{\lambda}$ *for* $(t,u) \in [\rho(0), \sigma(1)] \times [0, r_1]$,

where

$$M = \frac{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)}{1 + \sum_{i=1}^{m-2} \alpha_i} \Big(\int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s \Big)^{-1}, \tag{32}$$

$$N = \Big(\gamma \int_{\xi}^{w} G(t_0,s)p(s)h(s)\nabla s \Big)^{-1} \tag{33}$$

and $t_0 \in (\rho(0), \sigma(1))$ such that $||u|| = u(t_0)$. Then the problem (1)-(2) *has at least one positive solution u satisfying $r_1 \leq ||u|| \leq r_2$.*

Proof Let Ω_2 be defined as in (28). If $u \in \partial\Omega_2$ then $||u|| = r_2$.

$$\begin{aligned}\Phi_\lambda u(t) &\leq \lambda \Big(\frac{1 + \sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \Big) \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)f(s,u(s))\nabla s \\ &\leq \lambda \Big(\frac{1 + \sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i \phi_1(\eta_i)} \Big) \frac{Mr_2}{\lambda} \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s = r_2. \end{aligned}$$

Therefore,

$$||\Phi_\lambda u|| \leq ||u||, \quad \forall u \in \partial\Omega_2. \tag{34}$$

Let Ω_1 be defined as in (26). Using (16) and (17) we obtain

$$\begin{aligned}\Phi_\lambda u \;&\geq\; \lambda \int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)h(s)f(s,u(s))\nabla s \geq \lambda q(t) \int_{\xi}^{w} G(t_0,s)p(s)h(s)f(s,u(s))\nabla s \\ &\geq\; \lambda\gamma \int_{\xi}^{w} G(t_0,s)p(s)h(s)f(s,u(s))\nabla s \geq \lambda\gamma \frac{Nr_1}{\lambda} \int_{\xi}^{w} G(t_0,s)p(s)h(s)\nabla s = r_1.\end{aligned}$$

Therefore,

$$||\Phi_\lambda u|| \geq ||u||, \quad \forall u \in \partial\Omega_1. \tag{35}$$

Inequalities (35) and (34) imply that the conditions of Theorem 2.1 hold. Hence $\Phi_\lambda u$ has at least one fixed point i.e., (1)-(2) has at least one positive solution in $\mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying $r_1 \leq ||u|| \leq r_2$. □

Theorem 3.3 *Let* $f(t,u)$ *satisfy* (H1) *and there exist constants* $0 < r_1 < r_2 < r_3$ *such that the following assumptions hold:*

(H7) $f(t,u) < \lambda^{-1}Mr_1$ *for all* $(t,u) \in [\rho(0),\sigma(1)] \times [0,r_1]$,

(H8) $f(t,u) \geq \lambda^{-1}Nr_2$ *for all* $(t,u) \in [\xi,w] \times [r_2,r_3]$,

(H9) $f(t,u) \leq \lambda^{-1}Mr_3$ *for all* $(t,u) \in [\rho(0),\sigma(1)] \times [\rho(0),r_3]$.

Then (1)-(2) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$||u_1|| < r_1, \quad r_2 < \min_{t\in[\xi,w]} |u_2(t)| \leq r_3, \quad r_1 < ||u_3|| \leq r_3 \quad \text{and} \quad \min_{t\in[\xi,w]} |u_3(t)| < r_2.$$

Proof We verify that the conditions of Legget-Williams fixed point theorem (Theorem 2.2) are satisfied. For this purpose we first define the nonnegative, continuous, concave functional $\psi : \mathcal{P} \to [0,\ \infty)$ to be $\psi(u) := \min_{t\in[\xi,w]} |u(t)|$, the cone $\mathcal{P}$ is as in (18), M as in (32) and N as in (33). Then $\psi(u) \leq \|u\|$ for all $u \in \mathcal{P}$.
If $u \in \overline{\mathcal{P}}_{r_3}$, then $\|u\| \leq r_3$. So by using assumption (H9) and the similar calculations as in Theorem (3.2), we get

$$\begin{aligned}\Phi_\lambda u(t) \;&\leq\; \lambda\Big(\frac{1+\sum\limits_{i=1}^{m-2}\alpha_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i\phi_1(\mu_i)}\Big) \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)f(s,u(s))\nabla s \\ &\leq\; \lambda\Big(\frac{1+\sum\limits_{i=1}^{m-2}\alpha_i}{1-\sum\limits_{i=1}^{m-2}\alpha_i\phi_1(\mu_i)}\Big)\lambda^{-1}M^{-1}r_3 \int_{\rho(0)}^{\sigma(1)} G(s,s)p(s)h(s)\nabla s = r_3.\end{aligned}$$

Hence $\Phi_\lambda : \overline{\mathcal{P}}_{r_3} \to \overline{\mathcal{P}}_{r_3}$.

In the same way, if $u \in \overline{\mathcal{P}}_{r_1}$, i.e. $||u|| \leq r_1$ assumption (H7) yields $||\Phi_\lambda u|| < r_1$. Therefore **(ii)** of Theorem 2.2 is satisfied.

To check the condition **(i)** of Theorem 2.2 we choose $u(t) = r_3, \ \ \forall t \in [\rho(0),\sigma(1)]$. It is clear that $u(t) = r_3 \in \mathcal{P}(\phi, r_2, r_3)$. Consequently, since $\phi(u) = \phi(r_3) = r_3 > r_2$ then

$\{u \in \mathcal{P}(\phi, r_2, r_3) : \phi(u) > r_2\} \neq \emptyset$. Moreover by taking assumption (H8) and Lemma 2.5 into account, we obtain

$$\begin{aligned}\phi(\Phi_\lambda u) &= \min_{t\in[\xi,w]} |\Phi_\lambda u(t)| \geq \lambda\gamma \int_\xi^w G(t_0,s)p(s)h(s)f(s,u(s))\nabla s \\ &\geq \lambda\gamma\lambda^{-1}Nr_2 \int_\xi^w G(t_0,s)p(s)h(s)\nabla s = r_2.\end{aligned}$$

Therefore **(i)** of Theorem 2.2 holds.

Similarly **(iii)** of Theorem 2.2 is satisfied. Hence $\Phi_\lambda u$ has at least three fixed points $u_1,\ u_2$ and u_3 satisfying

$$||u_1|| < r_1, \quad r_2 < \min_{t\in[\xi,w]} |u_2(t)| \leq r_3, \quad r_1 < ||u_3|| \leq r_3 \text{ and } \min_{t\in[\xi,w]} |u_3(t)| < r_2. \ \square$$

To illustrate how our results can be used in practice we present an example.

Example 3.1 Let $\mathbb{T} = \Big\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{5}{4}, ...\Big\}$. We consider the following four point boundary value problem:

$$u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) + 10^{-3}(35+u)e_1(t,0) = 0, \quad t \in [0,1],$$
$$u(0) = 0, \quad u(\frac{5}{4}) = \frac{1}{2}u(\frac{1}{4}) + \frac{1}{4}u(\frac{1}{2}).$$

This problem can be regarded as a BVP of the form (1)-(2), where $a(t) = 12/5, b(t) = -16/5, \lambda = 10^{-3}, h(t) = 1$ and $f(t,u) = (35+u)e_1(t,0)$. Clearly (H1)-(H3) are satisfied. Let $\phi_1(t)$ and $\phi_2(t)$ be the solutions of the following linear BVP's respectively.

$$u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) = 0 \ \ t \in [0,1], \quad u(0) = 0, \quad u(\frac{5}{4}) = 1,$$
$$u^{\Delta\nabla}(t) + \frac{12}{5}u^\Delta(t) - \frac{16}{5}u(t) = 0 \ \ t \in [0,1], \quad u(0) = 1, \quad u(\frac{5}{4}) = 0.$$

It is evident (from the the Corollaries 4.24 and 4.25 and Theorem 4.28 of [4]) that

$$\phi_1(t) = \frac{(\frac{5}{4})^{4t} - (\frac{1}{2})^{4t}}{(\frac{5}{4})^5 - (\frac{1}{2})^5} \quad \text{and} \quad \phi_2(t) = \frac{(\frac{5}{4})^5(\frac{1}{2})^{4t} - (\frac{1}{2})^5(\frac{5}{4})^{4t}}{(\frac{5}{4})^5 - (\frac{1}{2})^5}.$$

Also $\phi_1(t)$ satisfies (H4). The Green's function is of the following form:

$$G(t,s) = \frac{1024}{9279}\begin{cases} \{(\frac{5}{4})^{4t} - (\frac{1}{2})^{4t}\}\{(\frac{5}{4})^5(\frac{1}{2})^{4s} - (\frac{1}{2})^5(\frac{5}{4})^{4s}\}, & s \geq t, \\ \{(\frac{5}{4})^{4s} - (\frac{1}{2})^{4s}\}\{(\frac{5}{4})^5(\frac{1}{2})^{4t} - (\frac{1}{2})^5(\frac{5}{4})^{4t}\}, & t \geq s. \end{cases}$$

$p(t) = (\frac{2}{5})^{4t-1}$ follows from $e_\alpha(t,t_0) = \Big(1+\alpha h\Big)^{\frac{t-t_0}{h}}$ on $\mathbb{T} = h\mathbb{N}$. Furthermore we obtain $\gamma \approx 0,106$,

$$\int_0^{\frac{5}{4}} G(s,s)\Big(\frac{2}{5}\Big)^{4s-1}\nabla s \approx 0,44 \quad \text{and} \quad \int_{\frac{1}{2}}^{\frac{3}{4}} G(s,s)\Big(\frac{2}{5}\Big)^{4s-1}\nabla s \approx 0,0025.$$

and thus $M \approx 19,84$ and $N \approx 3650$.
If we choose $r_2 > r_1 > 0$ such that $r_1 = 5\cdot 10^{-6}$ and $r_2 = 0,1$, then it is straightforward from Theorem 3.2 that the four point BVP has at least one positive solution satisfying $5\cdot 10^{-6} \leq \|u\| \leq 0,1$.

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