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State Dependent Generalized Inversion-Based Liapunov Equation for Spacecraft Attitude Control

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Abstract: Parametrization of nonunique linear equations solution via generalized inversion is utilized in nonlinear spacecraft control system design. A stable linear time-invariant ordinary differential equation in an attitude deviation norm measure is formed and is evaluated along the trajectories defined by the spacecraft mathematical model, yielding a linear relation in the control variables. Generalized inversion of the relation results in a control law that consists of auxiliary and particular parts. The *null-control vector* in the auxiliary part is designed by solving a state dependent Liapunov equation involving a *perturbed nullprojector* and by utilizing a *damped controls coefficient generalized inverse*, yielding globally uniformly ultimately bounded attitude trajectory tracking errors.

Keywords: *spacecraft attitude control; Moore–Penrose controls coefficient generalized inverse; null-control vector; damped controls coefficient generalized inverse; state dependent Liapunov equation; perturbed controls coefficient nullprojection.*

Mathematics Subject Classification (2000): 93B52, 93C10, 93C15, 93C35, 93C73, 93D05, 93D15, 93D30.

1 Introduction

Throughout the second half of the twentieth century, numerous control methodologies have been employed for spacecraft control, benefiting from the rapid development in nonlinear system theory. Among the methodologies applied to the attitude control problem of rigid spacecraft with known inertia parameters were those based on geometrical concepts, energy principles, optimal control, and feedback linearizing transformations.

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The present article introduces an algebraic control methodology that aims to utilize the simplicity of linear system theory by casting the nonlinear spacecraft control problem in a pointwise-linear form and utilizing a simple linear algebra relation to tackle the control problem. The primary tool used is the Moore–Penrose generalized matrix inverse (MPGI).

The procedure begins by defining a norm measure function of the spacecraft's attitude variables deviations from their desired values, and prespecifying a stable second-order linear differential equation in the measure function, resembling the desired attitude deviation dynamics. The differential equation is then transformed to a relation that is linear in the control vector by differentiating the norm measure function along the trajectories defined by the solution of the spacecraft's state space mathematical model. The MPGI is utilized thereafter to invert this relation for the control law required to realize the desired stable linear attitude deviation norm measure dynamics.

In addition to its algebraic simplicity, the derived control law has a special geometrical structure. It consists of auxiliary and particular parts, residing in the nullspace of the *controls coefficient* row vector and the range space of its generalized inverse, respectively. The auxiliary part contains a free nullvector, named the *null-control vector*, and is being projected onto the controls coefficient nullspace by means of a nullprojection matrix. Therefore, the choice of the null-control vector does not affect the dynamics of the attitude deviation norm measure function, and it parameterizes *all* control laws that are capable of realizing that dynamics.

The control problem is a problem of nonuniqueness; that is, if a dynamical system is controllable then there exists no unique strategy to control it. The MPGI was reintroduced in [1] to parameterize this *redundancy in control authority* in the context of program, or servo-constraints. The procedure is generalized in this work to the gas jet-actuated spacecraft control problem by considering nulling the deviation from desired spacecraft kinematics to be the servo-constraint that is to be realized.

Generalized inversion of the controls coefficient implies outer kinematics tracking exponential stability. However, not all choices from the infinite set of null-control vectors guarantee stability of the spacecraft internal dynamics. An observation is made in [1] that the null-control vector choice substantially affects the inner system states. Therefore, the primary objective in utilizing the null-control vector design freedom is to subdue internal instability of the closed loop control system.

To fulfill the internal stability objective, and inspired by the control law's affinity in the null-control vector, the later is chosen in this work to be proportional to the spacecraft angular velocity vector. The state dependent proportionality matrix is constructed by solving a state dependent Liapunov equation that is produced by a quadratic Liapunov function in the spacecraft angular velocity vector.

A fundamental property of the resulting Liapunov equation is its dependency on the controls coefficient generalized inverse (CCGI) and the corresponding nullprojector. This dependency is a source of two difficulties in the way of solving the equation. The first difficulty is due to rank deficiency of the controls coefficient nullprojector, and it is overcomed by perturbing the nullprojector to disencumber its rank deficiency.

The second difficulty is due to an inherent characteristic of the MPGI. Although well-defined for any matrix, regardless of its size or rank, the MPGI mapping of a matrix that is continuous in its elements suffers from a discontinuity, whenever the matrix changes rank. This appears as a divergence of the generalized inverse matrix elements to infinite values as the mapped matrix changes rank. Robustness against this generalized inversion instability is achieved by modifying the structure of the controls coefficient MPGI by means of a damping factor that limits its growth as steady state response is approached. Depending on the amount of modification, this *damped* CCGI results in a tradeoff between trajectory tracking accuracy and generalized inversion stability.

Modifying the definition of the controls coefficient MPGI results in an approximate realization of the desired spacecraft attitude deviation norm measure dynamics. It is shown that the closed loop attitude trajectories tracking errors resulting from applying the proposed generalized inversion-based control law are globally uniformly ultimately bounded, and that the ultimate bound is inversely proportional to the damping factor by which the generalized inverse is modified.

The present article introduces a nonlinear spacecraft attitude tracking control law, derived in the generalized inversion framework via a novel state dependent Liapunov equation, in a continuous development of Liapunov thoughts and results that remain after a century and a half from his birth anniversary to be the most famous criteria for nonlinear motion stability [2].

2 Spacecraft Mathematical Model

The spacecraft mathematical model is given by the following system of kinematical and dynamical differential equations

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0, \tag{1}$$

$$\dot{\omega} = J^{-1}\omega^{\times} J\omega + \tau, \quad \omega(0) = \omega_0, \tag{2}$$

where $\rho \in \mathbb{R}^{3\times 1}$ is the spacecraft vector of modified Rodrigues attitude parameters (MRPs) [3], $\omega \in \mathbb{R}^{3\times 1}$ is the vector of spacecraft angular velocity components in its body reference frame, $J \in \mathbb{R}^{3\times 3}$ is a diagonal matrix containing spacecraft's body principal moments of inertia, and $\tau := J^{-1}u \in \mathbb{R}^{3\times 1}$ is the vector of scaled control torques, where $u \in \mathbb{R}^{3\times 1}$ contains the applied gas jet actuator torque components about the spacecraft's principal axes. The cross product matrix $x^{\times}$ which corresponds to a vector $x \in \mathbb{R}^{3\times 1}$ is skew symmetric of the form

$$x^{\times} = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix}$$

and the matrix valued function $G(\rho) : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 3}$ is given by

$$G(\rho) = \frac{1}{2}\left(\frac{1 - \rho^T\rho}{2}\, I_{3\times 3} - \rho^{\times} + \rho\rho^T\right).$$

The MRPs are used as the attitude state variables, because of their validity in describing any angular displacement about the spacecraft's body axes up to 2π rad, such that $G(\rho)$ remains finite and invertible for any value of ρ that corresponds to such spacecraft angular displacement.

3 Attitude Deviation Norm Measure Dynamics

Let $\rho_d(t) \in \mathbb{R}^{3\times 1}$ be a prescribed desired spacecraft attitude vector such that $\rho_d(t)$ is at least twice continuously differentiable in t. The spacecraft attitude deviation vector from

$\rho_d(t)$ is defined as

$$z(\rho, t) := \rho - \rho_d(t). \tag{3}$$

Define the scalar attitude deviation norm measure function $\phi : \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}$ to be half the squared Euclidean norm of $z(\rho, t)$

$$\phi = \frac{1}{2} \parallel z(\rho, t) \parallel^2 = \frac{1}{2} \parallel \rho - \rho_d(t) \parallel^2 . \tag{4}$$

The first two time derivatives of ϕ along the spacecraft trajectories given by the solution of Eqs. (1) and (2) are

$$\dot{\phi} = \frac{\partial \phi}{\partial \rho} G(\rho)\omega + \frac{\partial \phi}{\partial t} = z^T(\rho, t) \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \tag{5}$$

and

$$\begin{aligned}\ddot{\phi} = {} & \left[G(\rho)\omega - \dot{\rho}_d(t)\right]^T \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \\ & + z^T(\rho, t) \left[\dot{G}(\rho, \omega)\omega + G(\rho)\left[J^{-1}\omega^{\times} J\omega + \tau\right] - \ddot{\rho}_d(t)\right], \end{aligned} \tag{6}$$

where $\dot{G}(\rho, \omega)$ is the time derivative of $G(\rho)$ obtained by differentiating the individual elements of $G(\rho)$ along the kinematical subsystem given by Eqs. (1). The procedure is to prespecify a desired stable linear second-order dynamics of ϕ in the form

$$\ddot{\phi} + c_1\dot{\phi} + c_2\phi = 0, \quad c_1, c_2 > 0. \tag{7}$$

With ϕ, $\dot{\phi}$, and $\ddot{\phi}$ given by Eqs. (4), (5), and (6), it is possible to write Eq. (7) in the quasi-linear form

$$\mathcal{A}(\rho, t)\tau = \mathcal{B}(\rho, \omega, t), \tag{8}$$

where the vector valued function $\mathcal{A}(\rho, t) : \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}^{1\times 3}$ is given by

$$\mathcal{A}(\rho, t) = z^T(\rho, t)G(\rho) \tag{9}$$

and the scalar valued function $\mathcal{B}(\rho, \omega, t) : \mathbb{R}^{3\times 1} \times \mathbb{R}^{3\times 1} \times \mathbb{R} \to \mathbb{R}$ is

$$\begin{aligned}\mathcal{B}(\rho, \omega, t) = {} & -\left[G(\rho)\omega - \dot{\rho}_d(t)\right]^T \left[G(\rho)\omega - \dot{\rho}_d(t)\right] \\ & - z^T(\rho, t)\left[\dot{G}(\rho, \omega)\omega + G(\rho)J^{-1}\omega^{\times} J\omega - \ddot{\rho}_d(t)\right] \\ & - c_1 z^T(\rho, t)\left[G(\rho)\omega - \dot{\rho}_d(t)\right] - \frac{c_2}{2} \parallel z(\rho, t) \parallel^2 . \end{aligned}$$

The row vector function $\mathcal{A}(\rho, t)$ is named the *controls coefficient* of the attitude deviation norm measure dynamics given by Eq. (7) along the spacecraft trajectories, and the scalar function $\mathcal{B}(\rho, \omega, t)$ is the corresponding *controls load*.

4 Linearly Parameterized Attitude Control Laws

The quasi-linear form given by Eq. (8) makes it feasible to assess realizability of the linear attitude deviation norm measure dynamics given by Eq. (7) in a pointwise manner.

Definition 4.1 For a given desired spacecraft attitude vector $\rho_d(t)$, the linear attitude deviation norm measure dynamics given by Eq. (7) is said to be realizable by the spacecraft equations of motion (1) and (2) at specific values of ρ and t if there exists a control vector τ that solves Eq. (8) for these values of ρ and t. If this is true for all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, then the linear attitude deviation norm measure dynamics is said to be globally realizable by the spacecraft equations of motion.

Proposition 4.1 *For any desired spacecraft attitude vector $\rho_d(t)$, the linear attitude deviation norm measure dynamics given by Eq. (7) is globally realizable by the spacecraft equations of motion (1) and (2). Furthermore, the infinite set of all control laws realizing that dynamics by the spacecraft equations of motion is parameterized by an arbitrarily chosen nullvector $y \in \mathbb{R}^{3\times 1}$ as*

$$\tau = \mathcal{A}^+(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y, \tag{10}$$

where $\mathcal{A}^+$ stands for the MPGI *of the controls coefficient given by*

$$\mathcal{A}^+(\rho, t) = \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\mathcal{A}(\rho,t)\mathcal{A}^T(\rho,t)}, & \mathcal{A}(\rho, t) \neq \mathbf{0}_{1\times 3}, \\ \mathbf{0}_{3\times 1}, & \mathcal{A}(\rho, t) = \mathbf{0}_{1\times 3}, \end{cases} \tag{11}$$

and $\mathcal{P}(\rho, t) \in \mathbb{R}^{3\times 3}$ is the corresponding nullprojector given by

$$\mathcal{P}(\rho, t) = I_{3\times 3} - \mathcal{A}^+(\rho, t)\mathcal{A}(\rho, t). \tag{12}$$

Proof A necessary and sufficient condition for the existence of a control vector τ that solves Eq. (8) at specific values of ρ and t is consistency of the equation at these values, i.e., $\mathcal{B}(\rho, \omega, t)$ is in the range space of $\mathcal{A}(\rho, t)$. This is guaranteed for all values of $\omega \in \mathbb{R}^{3\times 1}$, provided that $\mathcal{A}(\rho, t)$ does not vanish at the specified values of ρ and t, at which the linear attitude deviation norm measure dynamics given by Eq. (7) is realizable by the spacecraft equations of motion (1) and (2) according to definition 4.1. Since the matrix $G(\rho)$ is invertible for all values of ρ, it has a trivial nullspace, which implies from Eq. (9) that $\mathcal{A}(\rho, t)$ vanishes if and only if $z(\rho, t)$ does. Therefore, Eq. (8) is consistent at all ρ and t such that $z(\rho, t) \neq \mathbf{0}_{3\times 1}$, and the linear attitude deviation norm measure dynamics is globally realizable by the spacecraft equations of motion according to definition 4.1. Consequently, the infinite set of *all* control laws that realize the linear attitude deviation norm measure dynamics by the spacecraft equations of motion at all ρ and t such that $\mathcal{A}(\rho, t) \neq \mathbf{0}_{1\times 3}$ is given by Eq. (10) [4].

Since any choice of the nullvector y in the control law expression given by Eq. (10) yields a solution to Eq. (8), the choice of y does not affect realizability of the linear attitude deviation norm measure dynamics given by Eq. (7). Nevertheless, the choice of y substantially affects the spacecraft transient state response [1]. In particular, an inadequate choice of y can destabilize the spacecraft internal dynamics given by Eq. (2) or causes unsatisfactory closed loop performance. Due to the importance of the nullvector y in the present control system design development as a control vector by itself, we name it the *null-control vector*.

Corollary 4.1 *The infinite set of spacecraft closed loop systems equations realizing the linear attitude deviation norm measure dynamics given by Eq. (7) is parameterized by the null-control vector y as*

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0, \tag{13}$$

$$\dot{\omega} = J^{-1}\omega^{\times}J\omega + \mathcal{A}^+(\rho, t)\mathcal{B}(\rho, \omega, t) + \mathcal{P}(\rho, t)y, \quad \omega(0) = \omega_0. \tag{14}$$

Proof Equations (13) and (14) are obtained by substituting the control laws expressions given by Eqs. (10) in the spacecraft's mathematical model given by Eqs. (1) and (2).

5 Perturbed Controls Coefficient Nullprojector

The concept of perturbed controls coefficient nullprojector (PCCN) is crucial in the present development of the generalized inversion-based spacecraft control law.

Definition 5.1 The PCCN $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is defined as

$$\widetilde{\mathcal{P}}(\rho,\delta,t) := I_{3\times 3} - h(\delta)\mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t), \tag{15}$$

where $h(\delta) : \mathbb{R}^{1\times 1} \to \mathbb{R}^{1\times 1}$ is any continuous function such that

$$h(\delta) = 1 \quad \text{if and only if} \quad \delta = 0.$$

Proposition 5.1 *The* PCCN $\widetilde{\mathcal{P}}(\rho,\delta,t)$ *is of full rank for all* $\delta \neq 0$.

Proof The singular value decomposition of $\mathcal{A}(\rho,t)$ is given by

$$\mathcal{A}(\rho,t) = \mathbf{\Sigma}(\rho,t)\mathcal{V}^T(\rho,t),$$

where

$$\mathbf{\Sigma}(\rho,t) = \begin{bmatrix} \parallel \mathcal{A}(\rho,t) \parallel & 0 & 0 \end{bmatrix}$$

and $\mathcal{V}(\rho,t) \in \mathbb{R}^{3\times 3}$ is orthonormal, i.e.,

$$\mathcal{V}^{-1}(\rho,t) = \mathcal{V}^T(\rho,t), \quad \text{and} \quad \det\mathcal{V}(\rho,t) = 1.$$

By inspecting the four conditions defining the MPGI [4], it can be easily verified that it is given for $\mathcal{A}(\rho,t)$ by

$$\mathcal{A}^+(\rho,t) = \mathcal{V}(\rho,t)\mathbf{\Sigma}^+(\rho,t),$$

where $\mathbf{\Sigma}^+(\rho,t)$ is the MPGI of $\mathbf{\Sigma}(\rho,t)$ given by

$$\mathbf{\Sigma}^+(\rho,t) = \begin{bmatrix} \dfrac{1}{\parallel \mathcal{A}(\rho,t) \parallel} & 0 & 0 \end{bmatrix}^T.$$

Therefore,

$$\mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t) = \mathcal{V}(\rho,t)\mathbf{\Sigma}^+(\rho,t)\mathbf{\Sigma}(\rho,t)\mathcal{V}^T(\rho,t). \tag{16}$$

The right hand side of Eq. (16) is a singular value decomposition of $\mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t)$, where the diagonal matrix $\mathbf{\Sigma}^+(\rho,t)\mathbf{\Sigma}(\rho,t)$ contains the singular values of $\mathcal{A}^+(\rho,t)\mathcal{A}(\rho,t)$ as its diagonal elements

$$\mathbf{\Sigma}^+(\rho,t)\mathbf{\Sigma}(\rho,t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently, the PCCN $\widetilde{\mathcal{P}}(\rho,\delta,t)$ is

$$\begin{aligned}\widetilde{\mathcal{P}}(\rho,\delta,t) &= I_{3\times3} - h(\delta)\mathcal{A}^{+}(\rho,t)\mathcal{A}(\rho,t)\\ &= I_{3\times3} - h(\delta)\mathcal{V}(\rho,t)\mathbf{\Sigma}^{+}(\rho,t)\mathbf{\Sigma}(\rho,t)\mathcal{V}^{T}(\rho,t)\\ &= \mathcal{V}(\rho,t)[I_{3\times3} - h(\delta)\mathbf{\Sigma}^{+}(\rho,t)\mathbf{\Sigma}(\rho,t)]\mathcal{V}^{T}(\rho,t)\\ &= \mathcal{V}(\rho,t)\begin{bmatrix}1-h(\delta) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\mathcal{V}^{T}(\rho,t),\end{aligned}$$

which is of full rank for all $\delta \neq 0$.

Lemma 5.1 *The controls coefficient nullprojector $\mathcal{P}(\rho,t)$ commutes with its perturbation $\widetilde{\mathcal{P}}(\rho,\delta,t)$ for all $\delta \in \mathbb{R}$. Furthermore, their matrix multiplication is the controls coefficient nullprojector itself, i.e.,*

$$\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t) = \widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t) = \mathcal{P}(\rho,t). \tag{17}$$

Proof Equations (17) are verified by direct evaluation of the $\mathcal{P}(\rho,t)$ and $\widetilde{\mathcal{P}}(\rho,\delta,t)$ expressions given by Eqs. (12) and (15).

6 Null-Control Vector Design

The choice of the null-control vector y affects neither realizability of the attitude deviation norm measure dynamics given by Eq. (7) nor steady state spacecraft response. However, the choice of the null-control vector y affects both of spacecraft internal dynamics and spacecraft transient response. Hence, it provides a freedom that can be utilized to stabilize internal states of the spacecraft. Internal dynamics stability and stability robustness against controls coefficient singularity are the most important factors to be considered in designing the null-control vector y.

The structure of the control law τ given by Eqs. (10) has a special feature, namely the affinity of its auxiliary part in y, which provides a pointwise-linear parametrization to the nonlinear control law. Hence, let y be chosen as

$$y = K\omega,$$

where $K \in \mathbb{R}^{3\times3}$ is to be determined. With this choice of y, a class of control laws that globally realize the attitude deviation norm measure dynamics given by Eq. (7) is given by

$$\begin{aligned}\tau &= \mathcal{A}^{+}(\rho,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K\omega\\ &= [\mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K]\,\omega + \mathcal{H}_2(\rho,t),\end{aligned} \tag{18}$$

where

$$\begin{aligned}\mathcal{H}_1(\rho,\omega,t) = -\mathcal{A}^{+}(\rho,t)z^{T}(\rho,t)\Big[\dot{G}(\rho,\omega) + G(\rho)J^{-1}\omega^{\times}J + c_1 G(\rho)\Big]&\\ - \mathcal{A}^{+}(\rho,t)\Big[G(\rho)\omega - \dot{\rho}_d(t)\Big]^{T} G(\rho)&\end{aligned} \tag{19}$$

and

$$\mathcal{H}_2(\rho,t) = -\frac{c_2}{2}\mathcal{A}^{+}(\rho,t)z^T(\rho,t)z(\rho,t) + \mathcal{A}^{+}(\rho,t)z^T(\rho,t)\Big[\ddot{\rho}_d(t) + c_1\dot{\rho}_d(t)\Big] - \mathcal{A}^{+}(\rho,t) \parallel \dot{\rho}_d(t) \parallel_2^2 . \quad (20)$$

Hence, a class of closed loop dynamical subsystems realizing the dynamics given by Eq. (7) is obtained by substituting the control law given by Eqs. (18) in Eqs. (2), and it takes the form

$$\dot{\omega} = \left[J^{-1}\omega^{\times}J + \mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K\right]\omega + \mathcal{H}_2(\rho,t). \quad (21)$$

The term $\mathcal{H}_2(\rho,t)$ in the above closed loop dynamical subsystem can be viewed as a forcing term that drives the internal dynamics of the spacecraft to realize the desired attitude deviation norm measure dynamics.

7 Spacecraft Internal Stability

The cascaded nature of the spacecraft mathematical model given by Eqs. (1) and (2) implies that coupling between the spacecraft kinematics and dynamics is unidirectional, i.e., the open loop spacecraft dynamical subsystem is independent of the attitude parameters. This allows to independently analyze dynamical subsystem stability by using the following squared Euclidean norm of the spacecraft angular velocity vector as a control Liapunov function

$$V = \parallel \omega \parallel^2 .$$

Differentiating V along the trajectories of the unforced part of the closed loop dynamical subsystem Eqs. (21) obtained by setting $\mathcal{H}_2(\rho,t) = \mathbf{0}_{3\times1}$ and noticing skew-symmetry of $\omega^{\times}$ yields

$$\begin{aligned}\dot{V} &= 2\omega^T\left[J^{-1}\omega^{\times}J + \mathcal{H}_1(\rho,\omega,t) + \mathcal{P}(\rho,t)K\right]\omega \\ &= \omega^T\big[\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t)\big]\omega,\end{aligned}$$

where the matrix gain K is chosen to be symmetric. Global exponential stability of the unforced part of the closed loop dynamical subsystem given by Eqs. (21) at $\omega = \mathbf{0}_{3\times1}$ is guaranteed if $\dot{V}$ remains negative-definite as the spacecraft dynamics evolves in time, which implies the existence of a positive-definite constant matrix $Q \in \mathbb{R}^{3\times3}$ such that the Liapunov equation

$$\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t) + Q = 0 \quad (22)$$

is satisfied for all $t \geq 0$. Lemma 5.1 implies that Eq. (22) can be written as

$$\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + \widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t)K + K\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t) + Q = 0. \quad (23)$$

To solve the above matrix equation for the matrix gain K, the individual terms in the equation are vectorized by stacking their columns above each others such that [5]

$$\mathrm{vec}\Big[\widetilde{\mathcal{P}}(\rho,\delta,t)\mathcal{P}(\rho,t)K\Big] + \mathrm{vec}\Big[K\mathcal{P}(\rho,t)\widetilde{\mathcal{P}}(\rho,\delta,t)\Big] = -\mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t) + \mathcal{H}_1^T(\rho,\omega,t) + Q\right].$$

Employing the relation between the matrix vectorizing operation and the Kronecker product of matrices yields [5]

$$\left\{I_{3\times 3}\otimes\widetilde{\mathcal{P}}(\rho,\delta,t)\right\}\mathrm{vec}\left[\mathcal{P}(\rho,t)K\right]+\left\{\widetilde{\mathcal{P}}(\rho,\delta,t)\otimes I_{3\times 3}\right\}\mathrm{vec}\left[K\mathcal{P}(\rho,t)\right]= \\ -\,\mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right].$$

Therefore, the unique matrix gain solution of Liapunov equation (22) for $\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)$ is obtained as

$$\begin{aligned}\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t) &= -\mathrm{vec}^{-1}\left\{\left[I_{3\times 3}\otimes\widetilde{\mathcal{P}}(\rho,\delta,t)+\widetilde{\mathcal{P}}(\rho,\delta,t)\otimes I_{3\times 3}\right]^{-1}\right.\\ &\qquad \left.\mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right]\right\}\\ &= -\mathrm{vec}^{-1}\left\{\left[\widetilde{\mathcal{P}}(\rho,\delta,t)\oplus\widetilde{\mathcal{P}}(\rho,\delta,t)\right]^{-1}\right.\\ &\qquad \left.\mathrm{vec}\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{H}_1^T(\rho,\omega,t)+Q\right]\right\}\end{aligned} \tag{24}$$

and the control law

$$\tau=\left[\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\right]\omega$$

renders the equilibrium point $\omega=\mathbf{0}_{3\times 1}$ for the unforced part of the closed loop spacecraft dynamical subsystem equations (21) given by

$$\dot{\omega}=\left[J^{-1}\omega^{\times}J+\mathcal{H}_1(\rho,\omega,t)+\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\right]\omega \tag{25}$$

globally exponentially stable, where $\mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)$ is given by Eqs. (24).

8 Controls Coefficient Singularity Analysis

If the controls coefficient $\mathcal{A}(\rho,t)$ is singular at specific values of ρ and t, i.e., has zero elements, then its MPGI $\mathcal{A}^{+}(\rho,t)$ given by Eqs. (11) is infinite. The following proposition relates global realizability of the linear attitude deviation norm measure dynamics to controls coefficient singularity.

Proposition 8.1 *Given a desired spacecraft attitude vector $\rho_d(t)$ satisfying the smoothness assumption, a control law τ globally realizes the linear attitude deviation norm measure dynamics given by Eq. (7) by the spacecraft equations of motion (1) and (2) only if*

$$\lim_{t\to\infty}\mathcal{A}(\rho,t)=\mathbf{0}_{1\times 3}.$$

Proof Because of the equivalency of linear attitude deviation norm measure dynamics given by Eq. (7) and its quasi-linear form given by Eq. (8), global realizability of of the first implies the existence of a control law that drives ϕ according to the dynamics given by Eq. (7) at all ρ and t such that $z(\rho,t)\neq\mathbf{0}_{3\times 1}$. The norm property of ϕ implies that $z(\rho,t)=\mathbf{0}_{3\times 1}$ if and only if $\phi=0$. Therefore, global realizability of the stable dynamics given by Eq. (7) implies that

$$\lim_{t\to\infty}\phi=0 \quad\text{and}\quad \lim_{t\to\infty}z(\rho,t)=\mathbf{0}_{3\times 1}.$$

Since the matrix $G(\rho)$ is nonsingular for all finite values of ρ, Eq. (9) implies that

$$\lim_{t\to\infty} z(\rho,t) = \mathbf{0}_{3\times 1} \quad \text{if and only if} \quad \lim_{t\to\infty} \mathcal{A}(\rho,t) = \mathbf{0}_{1\times 3}.$$

With the expression of $\mathcal{A}(\rho,t)$ given by Eq. (9), the MPGI controls coefficient given by Eq. (11) can be written as

$$\mathcal{A}^{+}(\rho,t) = \frac{G^T(\rho)z(\rho,t)}{\| G^T(\rho)z(\rho,t) \|^2}.$$

Therefore,

$$\| \mathcal{A}^{+}(\rho,t) \| = \frac{\| G^T(\rho)z(\rho,t) \|}{\| G^T(\rho)z(\rho,t) \|^2} = \frac{1}{\| G^T(\rho)z(\rho,t) \|}. \tag{26}$$

Since $G(\rho)$ is finite for all finite values of ρ, Eq. (26) implies that

$$\lim_{z(\rho,t)\to\mathbf{0}_{3\times 1}} \| \mathcal{A}^{+}(\rho,t) \| = \infty.$$

In other words, unbounded CCGI $\mathcal{A}^{+}(\rho,t)$ in a control law given by Eqs. (10) is indispensable to globally realize the associated attitude deviation norm measure dynamics. For the purpose of controlling the growth of $\mathcal{A}^{+}(\rho,t)$, a limited-growth modified CCGI is introduced next.

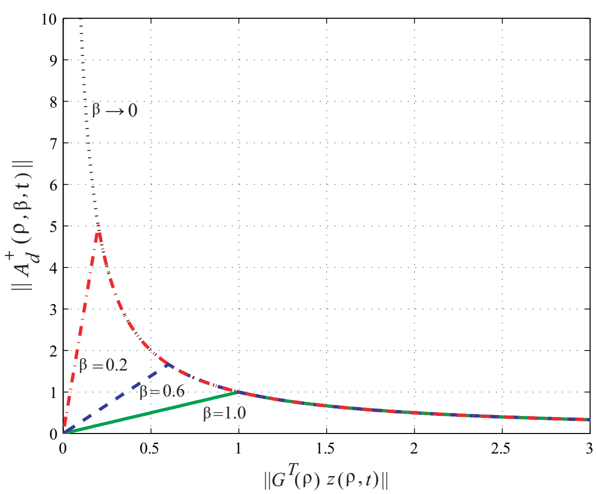

Figure 8.1: Damped CCGI.

Definition 8.1 The damped CCGI $\mathcal{A}_d^{+}(\rho,\beta,t)$ is defined as

$$\mathcal{A}_d^{+}(\rho,\beta,t) := \begin{cases} \frac{\mathcal{A}^T(\rho,t)}{\|\mathcal{A}(\rho,t)\|^2} & : \quad \| \mathcal{A}(\rho,t) \| \geqslant \beta, \\ \frac{\mathcal{A}^T(\rho,t)}{\beta^2} & : \quad \| \mathcal{A}(\rho,t) \| < \beta, \end{cases}$$

where the scalar β is a positive *generalized inverse damping factor*.

The above definition implies that

$$\|\mathcal{A}_d^+(\rho,\beta,t)\| \leqslant \frac{1}{\beta}$$

and that

$$\lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \| \mathcal{A}_d^+(\rho,\beta,t) \| = \lim_{z(\rho,t)\to \mathbf{0}_{3\times 1}} \frac{1}{\beta^2}\|G^T(\rho)z(\rho,t)\| = 0$$

and that $\mathcal{A}_d^+(\rho,\beta,t)$ pointwise converges to $\mathcal{A}^+(\rho,t)$ as β vanishes (see Figure 8.1). Accordingly, we define $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$ and $\mathcal{H}_{2d}(\rho,\beta,t)$ by replacing the CCGI $\mathcal{A}^+(\rho,t)$ in the $\mathcal{H}_1(\rho,\omega,t)$ and $\mathcal{H}_2(\rho,t)$ expressions given by Eqs. (19) and (20) with the damped CCGI $\mathcal{A}_d^+(\rho,\beta,t)$. Consequently, $K_d(\rho,\omega,\beta,\delta,t)$ is defined by replacing $\mathcal{H}_1(\rho,\omega,t)$ in the expression of $K(\rho,\omega,\delta,t)$ given by Eqs. (24) with $\mathcal{H}_{1d}(\rho,\omega,\beta,t)$.

9 Generalized Inversion-Based Attitude Tracking Control Law

Theorem 9.1 *The control law*

$$\tau_d = \mathcal{A}_d^+(\rho,\beta,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K_d(\rho,\omega,\beta,\delta,t)\omega \tag{27}$$

renders the trajectory tracking errors of the closed loop system given by Eqs. (1) and (2) globally uniformly ultimately bounded. Furthermore, any closed loop spacecraft attitude control trajectory with initial condition $\rho(0) \in \mathbb{R}^3$ enters the domain defined by

$$\|z(\rho,t)\| < \frac{\beta}{\sigma(G(\rho))} \tag{28}$$

in finite time and remains in it for all future time, where $\sigma(G(\rho))$ is the three times-repeated singular value of $G(\rho)$.

Proof Let ϕ_d be a norm measure function of the attitude deviation obtained by applying the control law given by Eqs. (27) to the spacecraft equations of motion (1) and (2), and let $\dot{\phi}_d$, $\ddot{\phi}_d$ be its first two time derivatives. Hence,

$$\begin{aligned} \phi_d &:= \phi_d(\rho,t) = \phi(\rho,t), \\ \dot{\phi}_d &:= \dot{\phi}_d(\rho,\omega,t) = \dot{\phi}(\rho,\omega,t), \\ \ddot{\phi}_d &:= \ddot{\phi}_d(\rho,\omega,\tau_d,t) = \ddot{\phi}(\rho,\omega,\tau,t) + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau, \end{aligned} \tag{29}$$

where τ is given by

$$\tau = \mathcal{A}^+(\rho,t)\mathcal{B}(\rho,\omega,t) + \mathcal{P}(\rho,t)K(\rho,\omega,\delta,t)\omega.$$

Adding $c_1\dot{\phi}_d + c_2\phi_d$ to both sides of Eq. (29) yields

$$\ddot{\phi}_d + c_1\dot{\phi}_d + c_2\phi_d = \ddot{\phi} + c_1\dot{\phi} + c_2\phi + \mathcal{A}(\rho,t)\tau_d - \mathcal{A}(\rho,t)\tau = \mathcal{A}(\rho,t)[\tau_d - \tau].$$

Therefore, let the state vector $\Phi_d \in \mathbb{R}^{2\times 1}$ be defined as

$$\Phi_d := \begin{bmatrix} \phi_d & \dot{\phi}_d \end{bmatrix}^T.$$

The attitude deviation norm measure closed loop dynamics becomes

$$\dot{\Phi}_d = \Lambda_1\Phi_d + \Delta_1(\rho,\beta,t), \tag{30}$$

where the asymptotically stable system matrix $\Lambda_1 \in \mathbb{R}^{2\times 2}$ is

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -c_2 & -c_1 \end{bmatrix}$$

and the input matrix valued function $\Delta_1 : \mathbb{R}^{5\times 1} \to \mathbb{R}^{2\times 1}$ is

$$\Delta_1(\rho,\omega,\beta,t) = \begin{cases} \mathbf{0}_{2\times 1} & : \quad \| \mathcal{A}(\rho,t) \| \geqslant \beta, \\ \begin{bmatrix} 0 \\ \frac{1}{\beta^2}\mathcal{B}(\rho,\omega,t) - \mathcal{B}(\rho,\omega,t) \end{bmatrix} & : \quad \| \mathcal{A}(\rho,t) \| < \beta. \end{cases}$$

On the other hand, the control law given by Eqs. (27) can be written as

$$\tau_d = [\mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K_d]\,\omega + \mathcal{H}_{2d}(\rho,\beta,t).$$

Using τ_d with the dynamical subsystem given by Eqs. (2) results in the closed loop dynamical subsystem

$$\dot{\omega} = \Lambda_2(\rho,\omega,\beta,\delta,t)\omega + \Delta_2(\rho,\beta,t), \tag{31}$$

where

$$\Lambda_2(\rho,\omega,\beta,\delta,t) = \left[J^{-1}\omega^{\times}J + \mathcal{H}_{1d}(\rho,\omega,\beta,t) + \mathcal{P}(\rho,t)K_d\right]$$

and

$$\Delta_2(\rho,\beta,t) = \mathcal{H}_{2d}(\rho,\beta,t).$$

Let the augmented state space vector ξ be defined as

$$\xi := \begin{bmatrix} \Phi_d^T & \omega^T \end{bmatrix}^T,$$

then Eqs. (30) and (31) form the augmented state space model

$$\dot{\xi} = \Lambda(\rho,\omega,\beta,\delta,t)\xi + \Delta(\rho,\omega,\beta,t), \tag{32}$$

where

$$\Lambda(\rho,\omega,\beta,\delta,t) = \begin{bmatrix} \Lambda_1 & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{3\times 2} & \Lambda_2(\rho,\omega,\beta,\delta,t) \end{bmatrix}, \qquad \Delta(\rho,\omega,\beta,t) = \begin{bmatrix} \Delta_1(\rho,\omega,\beta,t) \\ \Delta_2(\rho,\beta,t) \end{bmatrix}.$$

Now consider the unforced system

$$\dot{\xi}_p = \Lambda(\rho,\omega,\beta,\delta,t)\xi_p$$

and consider the positive definite $V_a(\xi) = \|\Phi_d\|^2 + \|\omega/\omega_0\|^2$, where ω_0 is a nondimensionalizing scalar. It can easily be verified that $\dot{V}_a$ is negative definite along the trajectories of ξ_p satisfying $\|\mathcal{A}(\rho,t)\| \geqslant \beta$, and that $\Delta(\rho,\omega,\beta,t)$ is a norm bounded nonvanishing perturbation vector. Therefore, the trajectories of the augmented dynamical system given by Eqs. (32) are globally uniformly ultimately bounded ([6], pp. 347). Furthermore, since $\Delta_1 = \mathbf{0}_{2\times 1}$ in the domain defined by $\|\mathcal{A}(\rho,t)\| \geqslant \beta$, it follows from Liapunov theory that the closed loop attitude trajectories move in the direction of decreasing $V_a(\xi)$ and must cross in finite time the boundary of the domain to its open complement domain defines by $\| \mathcal{A}(\rho,t) \| < \beta$, which becomes an invariant set. Moreover, $G(\rho)$ satisfies

$$\sigma_{min}(G(\rho)) = \sigma_{max}(G(\rho)) = \sigma(G(\rho)).$$

Therefore,

$$\|\mathcal{A}(\rho,t)\| = \|z^T(\rho,t)G(\rho)\| = \sigma(G(\rho))\|z(\rho,t)\|,$$

and the bound estimate of the attitude deviation vector norm given by Eq. (28) follows.

10 Numerical Simulations

The spacecraft model selected has inertia parameters $I_1 = 200$ Kg-m^2, $I_2 = 150$ Kg-m^2, $I_3 = 175$ Kg-m^2. Values of $c_1 = 0.9$ and $c_2 = 0.3$ are chosen, and the desired MRPs

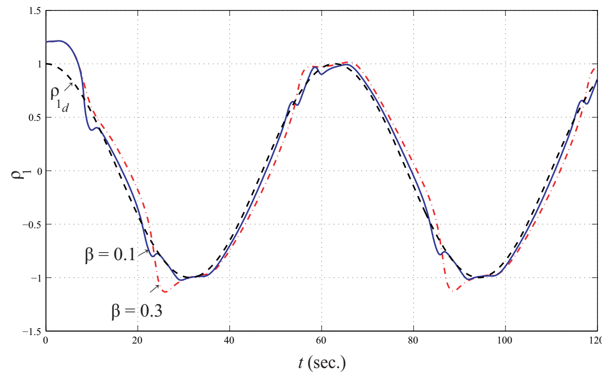

Figure 10.1: MRP ρ_1 vs. t: $\beta = 0.1, 0.3$.

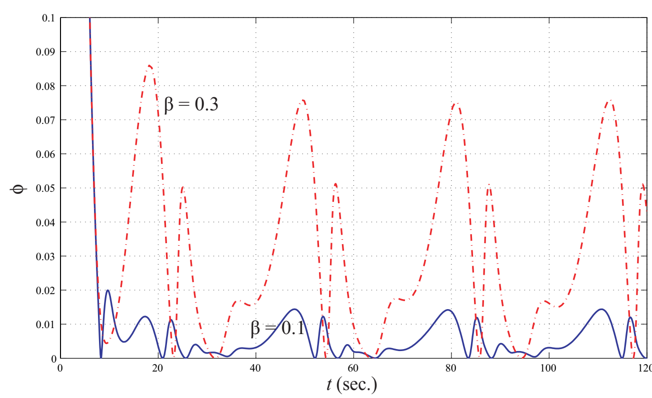

Figure 10.2: Attitude deviation norm measure ϕ vs. t: $\beta = 0.1, 0.3$.

trajectories are chosen to be $\rho_{d_i}(t) = \cos 0.1t$, $i = 1, 2, 3$, and $Q = I_{3\times3}$. All figures correspond to $\delta = 0.01$ and two values of $\beta = 0.1, 0.3$. Figure 10.1 shows the response of $\rho_1(t)$. Similar figures are obtained for $\rho_2(t)$ and $\rho_2(t)$, but are not shown. Figures 10.2 and 10.3 reveal the tradeoff between generalized inversion stability robustness against singularity and closed loop system tracking performance. The effect of changing δ on the closed loop response is minor.

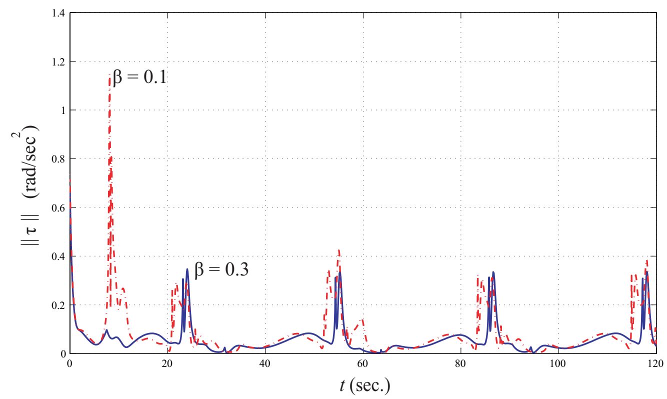

Figure 10.3: Scaled controls moments norm $\| \tau(\rho, \omega, t) \|_2$ vs. t: $\beta = 0.1, 0.3$.

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