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On Solutions of a Nonlinear Boundary Value Problem on Time Scales

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Abstract: We study a boundary value problem (BVP) for second order nonlinear dynamic equations on time scales. A condition is established that ensures existence and uniqueness of solutions to the BVP under consideration.

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Mathematics Subject Classification (2000): 34B15.

1 Introduction

Let $\mathbf{T}$ be a time scale and $a, b \in \mathbf{T}$ be fixed points with $a < b$ such that the time scale interval

$$(a, b) = \{t \in \mathbf{T} : a < t < b\}$$

is not empty. Throughout, all the intervals are time scale intervals. For standard notions and notations related to time scales calculus see [1, 2].

In this paper, we deal with the nonlinear boundary value problem (BVP)

$$y^{\Delta\nabla}(t) + f(t, y(t)) = 0, \quad t \in (a, b), \tag{1}$$

$$y(a) = y(b) = 0. \tag{2}$$

A function $y : [a, b] \to \mathbf{R}$ is called a *solution* of the BVP (1), (2) if the following conditions are satisfied:

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(a) y is continuous on $[a,b]$ and delta differentiable on (a,b) and such that there exist (finite) limits

$$y^{\Delta}(a) := \lim_{t\to a^+} y^{\Delta}(t) \quad \text{and} \quad y^{\Delta}(b) := \lim_{t\to b^-} y^{\Delta}(t).$$

(b) y^{Δ} is ∇-differentiable on $(a,b]$.

(c) y satisfies equation (1) and boundary conditions (2).

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose* $f : [a,b]\times\mathbf{R} \to \mathbf{R}$ *is continuous,* $f(b,0) = 0$ *in the case* b *is left-scattered, and suppose* f *satisfies the Lipschitz condition*

$$|f(t,\xi) - f(t,\eta)| \le L\,|\xi - \eta| \tag{3}$$

for all $t \in [a,b]$ *and* $\xi,\eta \in \mathbf{R}$*, where* $L > 0$ *is a constant (Lipschitz constant),* $\mathbf{R}$ *denotes the set of real numbers. Suppose further that*

$$L < \lambda_1, \tag{4}$$

where λ_1 *is the least positive eigenvalue of the problem*

$$y^{\Delta\nabla}(t) + \lambda y(t) = 0, \quad t \in (a,b), \tag{5}$$

$$y(a) = y(b) = 0. \tag{6}$$

Then the BVP (1), (2) has a unique solution.

Proof of Theorem 1.1 is presented in Section 2 and it uses a Hilbert space technique.

In Section 3, we compute the eigenvalues of (5), (6) explicitly in the cases $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$ (the set of integers) and show that

$$\lambda_1 = \frac{\pi^2}{(b-a)^2} \quad \text{if} \quad \mathbf{T} = \mathbf{R},$$

and

$$\lambda_1 = 4\sin^2 \frac{\pi}{2(b-a)} \ge \frac{8}{(b-a)^2} \quad \text{if} \quad \mathbf{T} = \mathbf{Z}.$$

Finally, in Section 4, we show that in the general case of arbitrary time scale $\mathbf{T}$ the estimation

$$\lambda_1 \ge \frac{4}{(b-a)^2}$$

holds and therefore the more explicit condition of the form

$$L < \frac{4}{(b-a)^2}$$

implies condition (4) of Theorem 1.1.

2 Proof of Theorem 1.1

Denote by $\mathcal{H}$ the Hilbert space of all real ∇-measurable functions $y : (a, b] \to \mathbf{R}$ such that $y(b) = 0$ in the case b is left-scattered, and that

$$\int_a^b y^2(t)\nabla t < \infty,$$

with the inner product

$$\langle y, z\rangle = \int_a^b y(t)z(t)\nabla t$$

and the norm

$$\|y\| = \sqrt{\langle y, y\rangle} = \left\{\int_a^b y^2(t)\nabla t\right\}^{\frac{1}{2}}.$$

Next denote by $\mathcal{D}$ the set of all functions $y \in \mathcal{H}$ satisfying the following three conditions:

(i) y is continuous on $(a, b]$, $y(b) = 0$, there exists $y(a) := \lim_{t\to a^+} y(t)$ and $y(a) = 0$.

(ii) y is continuously Δ-differentiable on (a, b), there exist (finite) limits

$$y^\Delta(a) := \lim_{t\to a^+} y^\Delta(t) \quad \text{and} \quad y^\Delta(b) := \lim_{t\to b^-} y^\Delta(t).$$

(iii) y^Δ is ∇-differentiable on $(a, b]$ and $y^{\Delta\nabla} \in \mathcal{H}$.

Define the operators $A : \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ by

$$(Ay)(t) = -y^{\Delta\nabla}(t) \quad \text{for} \quad y \in \mathcal{D},$$

$$(Fy)(t) = f(t, y(t)) \quad \text{for} \quad y \in \mathcal{H}.$$

Note that the operator A is linear, while F is nonlinear in general. The eigenvalues of problem (5), (6) coincide with the eigenvalues of the operator A.

As is shown in [3], the operator A is symmetric and positive:

$$\langle Ay, z\rangle = \langle y, Az\rangle \quad \text{for all} \quad y, z \in \mathcal{D},$$

$$\langle Ay, y\rangle > 0 \quad \text{for all} \quad y \in \mathcal{D},\ y \neq 0.$$

Further, A has $N = \dim\mathcal{H}$ (where $N \leq \infty$) orthonormal eigenfunctions φ_k which form a basis for $\mathcal{H}$ and the corresponding eigenvalues are simple and positive:

$$A\varphi_k = \lambda_k\varphi_k,$$

$$\langle\varphi_k, \varphi_l\rangle = 0 \text{ if } k \neq l \text{ and } \langle\varphi_k, \varphi_l\rangle = 1 \text{ if } k = l,$$

$$0 < \lambda_1 < \lambda_2 < \dots \quad .$$

For any function $u \in \mathcal{H}$ we have (expansion formula and Parseval's equality)

$$u = \sum_{k=1}^N c_k\varphi_k, \quad c_k = \langle u, \varphi_k\rangle\,, \tag{7}$$

$$\|u\|^2 = \langle u, u\rangle = \sum_{k=1}^{N} c_k^2.$$

In the case $N = \infty$ the sum in (7) becomes an infinite series and it converges to the function u in metric of the space $\mathcal{H}$. Since the operator A is positive, it is invertible. We have

$$Au = \sum_{k=1}^{N} c_k \lambda_k \varphi_k \quad \text{for all} \quad u \in \mathcal{D},$$

$$A^{-1}u = \sum_{k=1}^{N} \frac{c_k}{\lambda_k} \varphi_k \quad \text{for all} \quad u \in \mathcal{H},$$

where c_k are defined in (7). Hence

$$\left\|A^{-1}u\right\|^2 = \sum_{k=1}^{N} \frac{c_k^2}{\lambda_k^2} \le \frac{1}{\lambda_1^2} \sum_{k=1}^{N} c_k^2 = \frac{1}{\lambda_1^2} \|u\|^2 .$$

Thus we have established the following result: The operator A is invertible and

$$\left\|A^{-1}u\right\| \le \frac{1}{\lambda_1} \|u\| \quad \text{for all} \quad u \in \mathcal{H}. \tag{8}$$

The BVP (1), (2) is equivalent to the vector equation

$$Ay = Fy \quad \text{for} \quad y \in \mathcal{D},$$

which can be written in the form

$$y = A^{-1}Fy \quad \text{for} \quad y \in \mathcal{H}. \tag{9}$$

Note that the inverse operator A^{-1} maps $\mathcal{H}$ onto $\mathcal{D}$ and therefore if $y \in \mathcal{H}$ satisfies (9) then $y \in \mathcal{D}$. Let us set $S = A^{-1}F$. Then we get that the BVP (1), (2) is equivalent to the equation

$$y = Sy \quad (y \in \mathcal{H}).$$

The last equation is a fixed point problem.

We will use the following well-known contraction mapping theorem: *Let $\mathcal{H}$ be a Banach space and suppose that $S : \mathcal{H} \to \mathcal{H}$ is a contraction mapping, i.e., there is an α, $0 < \alpha < 1$, such that $\|Su - Sv\| \le \alpha \|u - v\|$ for all $u, v \in \mathcal{H}$. Then S has a unique fixed point in $\mathcal{H}$.*

It will be sufficient to show that the operator $S = A^{-1}F$ is a contraction mapping on the space $\mathcal{H}$. We have, using (8),

$$\begin{aligned} \|Su - Sv\| &= \left\|A^{-1}Fu - A^{-1}Fv\right\| \\ &= \left\|A^{-1}(Fu - Fv)\right\| \\ &\le \frac{1}{\lambda_1} \|Fu - Fv\| . \end{aligned} \tag{10}$$

Next, making use of the Lipschitz condition (3), we get

$$\begin{aligned}\|Fu - Fv\|^2 &= \int_a^b |f(t,u(t)) - f(t,v(t))|^2 \nabla t \\ &\le L^2 \int_a^b |u(t) - v(t)|^2 \nabla t \\ &= L^2 \|u - v\|^2\end{aligned}$$

so that

$$\|Fu - Fv\| \le L \|u - v\| \quad \text{for all} \quad u, v \in \mathcal{H}.$$

Thus, from (10) we obtain

$$\|Su - Sv\| \le \frac{L}{\lambda_1} \|u - v\| \quad \text{for all} \quad u, v \in \mathcal{H}.$$

Consequently, we see that under the condition (4), S is a contraction mapping and hence it has a unique fixed point in $\mathcal{H}$ by the contraction mapping theorem. Theorem 1.1 is proved.

Remark 2.1 The condition that functions $y \in \mathcal{H}$ satisfy $y(b) = 0$ in the case b is left-scattered guarantees the density of $\mathcal{D}$ in $\mathcal{H}$ (this is needed for the operator theory) and the condition that $f(b, 0) = 0$ in the case b is left-scattered guarantees $Fy \in \mathcal{H}$ for $y \in \mathcal{H}$.

3 Examples

In the case $\mathbf{T} = \mathbf{R}$, problem (1), (2) takes the form

$$y''(t) + f(t, y(t)) = 0, \quad t \in (a, b),$$

$$y(a) = y(b) = 0,$$

and eigenvalue problem (5), (6) takes the form

$$y''(t) + \lambda y(t) = 0, \quad t \in (a, b), \tag{11}$$

$$y(a) = y(b) = 0. \tag{12}$$

The eigenvalues of (11), (12) are

$$\lambda_k = \frac{\pi^2 k^2}{(b-a)^2} \qquad (k = 1.2, \ldots)$$

with the corresponding orthonormal eigenfunctions

$$\varphi_k(t) = \alpha_k \sin \frac{\pi k (t-a)}{b-a} \quad (k = 1, 2, \ldots),$$

where α_k are normirating constants. Therefore in this case

$$\lambda_1 = \frac{\pi^2}{(b-a)^2}$$

and condition (4) becomes

$$L < \frac{\pi^2}{(b-a)^2}.$$

In the case $\mathbf{T} = \mathbf{Z}$, problem (1), (2) takes the form

$$y(t-1) - 2y(t) + y(t+1) + f(t, y(t)) = 0, \quad t \in [a+1, b-1],$$

$$y(a) = y(b) = 0,$$

and eigenvalue problem (5), (6) takes the form

$$y(t-1) - 2y(t) + y(t+1) + \lambda y(t) = 0, \quad t \in [a+1, b-1], \tag{13}$$

$$y(a) = y(b) = 0. \tag{14}$$

The eigenvalues of (13), (14) are (cf. [4, Chap.7])

$$\lambda_k = 4\sin^2 \frac{\pi k}{2(b-a)} \quad (1 \le k \le b-a-1)$$

with the corresponding orthonormal eigenfunctions

$$\varphi_k(t) = \alpha_k \sin \frac{\pi k(t-a)}{b-a} \quad (1 \le k \le b-a-1),$$

where α_k are normirating constants. Therefore

$$\lambda_1 = 4\sin^2 \frac{\pi}{2(b-a)}$$

and condition (4) becomes

$$L < 4\sin^2 \frac{\pi}{2(b-a)}. \tag{15}$$

Since $b - a \ge 2$, using the inequality

$$\sin x \ge \frac{2\sqrt{2}}{\pi} x \quad \text{for} \quad 0 \le x \le \frac{\pi}{4},$$

we have that

$$\sin^2 \frac{\pi}{2(b-a)} \ge \frac{8}{\pi^2} . \frac{\pi^2}{4(b-a)^2} = \frac{2}{(b-a)^2}$$

and, therefore, the condition of the form

$$L < \frac{8}{(b-a)^2}$$

implies condition (15).

4 An Estimation for λ_1 in General Case

In the case of arbitrary time scale $\mathbf{T}$ we have (8). Besides, from $A\varphi_1 = \lambda_1\varphi_1$ we have

$$\left\|A^{-1}\varphi_1\right\| = \left\|\frac{1}{\lambda_1}\varphi_1\right\| = \frac{1}{\lambda_1}.$$

Consequently,

$$\left\|A^{-1}\right\| = \frac{1}{\lambda_1}. \tag{16}$$

On the other hand, the inverse operator A^{-1} has the form (see [3])

$$(A^{-1}u)(t) = \int_a^b G(t,s)u(s)\nabla s \quad \text{for any} \quad u \in \mathcal{H},$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases} (t-a)(b-s) & \text{if} \quad t \le s, \\ (s-a)(b-t) & \text{if} \quad t \ge s, \end{cases} \tag{17}$$

Hence

$$\begin{aligned} \left\|A^{-1}u\right\|^2 &= \int_a^b \left|\int_a^b G(t,s)u(s)\nabla s\right|^2 \nabla t \\ &\le \|u\|^2 \int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \end{aligned}$$

so that

$$\left\|A^{-1}\right\| \le \left\{\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t\right\}^{\frac{1}{2}}.$$

Therefore, taking into account (16), we get

$$\lambda_1 \ge \left\{\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t\right\}^{-\frac{1}{2}}. \tag{18}$$

Next, from (17) it follows that

$$0 \le G(t,s) \le \frac{1}{b-a}(s-a)(b-s)$$

for all t and s in $[a,b]$. Therefore

$$\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \le \frac{1}{(b-a)^2}\int_a^b \int_a^b (s-a)^2(b-s)^2 \nabla s \nabla t$$

and observing that

$$0 \le (s-a)(b-s) \le \frac{(b-a)^2}{4} \quad \text{for} \quad s \in [a,b],$$

we find

$$\int_a^b \int_a^b |G(t,s)|^2 \nabla s \nabla t \le \frac{(b-a)^4}{16}.$$

Comparing this with (18), we obtain

$$\lambda_1 \ge \frac{4}{(b-a)^2}.$$

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