Nonlinear Dynamics and Systems Theory, **9** (1) (2009) 11–22

Eigenvalues for Iterative Systems of Nonlinear Boundary Value Problems on Time Scales

M. Benchohra [1], F. Berhoun [1], S. Hamani [1], J. Henderson [2*], S.K. Ntouyas [3], A. Ouahab [1] and I.K. Purnaras [3]

[1] *Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000, Sidi Bel Abbès, Algérie*
[2] *Department of Mathematics, Baylor University, Waco, Texas 76798-7328 USA*
[3] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*

Received: February 6, 2008; Revised: December 19, 2008

Abstract: Values of $\lambda_1, \ldots, \lambda_n$ are determined for which there exist positive solutions of the iterative system of dynamic equations, $u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0$, $1 \le i \le n$, $u_{n+1}(t) = u_1(t)$, for $t \in [0,1]_{\mathbb{T}}$, and satisfying the boundary conditions, $u_i(0) = 0 = u_i(\sigma^2(1))$, $1 \le i \le n$, where $\mathbb{T}$ is a time scale. A Guo-Krasnosel'skii fixed point theorem is applied.

Keywords: *time scales; boundary value problem; iterative system of dynamic equations; nonlinear; eigenvalue.*

Mathematics Subject Classification (2000): 39A10, 34B18.

1 Introduction

Let $\mathbb{T}$ be a time scale with $0,\ \sigma^2(1) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation,

$$J_{\mathbb{T}} := J \cap \mathbb{T}.$$

We are concerned with determining values of λ_i, $1 \le i \le n$, for which there exist positive solutions for the iterative system of dynamic equations,

$$\begin{gathered} u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0,\ 1 \le i \le n,\ t \in [0,1]_{\mathbb{T}}, \\ u_{n+1}(t) = u_1(t),\ t \in [0,1]_{\mathbb{T}}, \end{gathered} \tag{1}$$

satisfying the boundary conditions,

$$u_i(0) = 0 = u_i(\sigma^2(1)),\ 1 \le i \le n, \tag{2}$$

where

* Corresponding author: Johnny_Henderson@baylor.edu

© 2009 InforMath Publishing Group/1562-8353 (print)/1813-7385 (online)/www.e-ndst.kiev.ua

(A) $f_i \in C([0,\infty),[0,\infty)),\ 1 \leq i \leq n$;

(B) $a_i \in C([0,\sigma(1)]_{\mathbb{T}},[0,\infty)),\ 1 \leq i \leq n$, and a_i does not vanish identically on any closed subinterval of $[0,\sigma(1)]_{\mathbb{T}}$;

(C) Each of $f_{i0} := \lim\limits_{x\to 0^+} \frac{f_i(x)}{x}$ and $f_{i\infty} := \lim\limits_{x\to\infty} \frac{f_i(x)}{x}$, $1 \leq i \leq n$, exists as a positive real number.

There is a great deal of research activity devoted to positive solutions of dynamic equations on time scales; see, for example [1, 3, 4, 5, 8, 10, 14]. This work entails an extension of the paper by Chyan and Henderson [9] to eigenvalue problems for systems of nonlinear boundary value problems on time scales, and also, in a very real sense, an extension of the recent paper by Benchohra, Henderson and Ntouyas [7]. Also, in that light, this paper is closely related to the works by Li and Sun [27, 29].

On a larger scale, there has been a great deal of study focused on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from both a theoretical sense [11, 13, 21, 24, 31] and as applications for which only positive solutions are meaningful [2, 12, 25, 26]. These considerations are formulated primarily for scalar problems, but good attention also has been given to boundary value problems for systems of differential equations [6, 15, 16, 17, 18, 19, 20, 22, 23, 28, 30, 32].

The main tool in this paper is an application of the Guo-Krasnosel'skii fixed point theorem for operators leaving a Banach space cone invariant [13]. A Green's function plays a fundamental role in defining an appropriate operator on a suitable cone.

2 Some preliminaries

In this section, we state the well-known Guo-Krasnosel'skii fixed point theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta\Delta} = 0,$$
$$y(0) = 0 = y(\sigma^2(1)).$$

Erbe and Peterson [10] have found,

$$G(t,s) = \frac{1}{\sigma^2(1)} \begin{cases} t(\sigma^2(1)-\sigma(s)), & \text{if } t \leq s, \\ \sigma(s)(\sigma^2(1)-t), & \text{if } \sigma(s) \leq t, \end{cases}$$

from which

$$G(t,s) > 0, (t,s) \in (0,\sigma^2(1))_{\mathbb{T}} \times (0,\sigma(1))_{\mathbb{T}}, \tag{3}$$

$$G(t,s) \leq G(\sigma(s),s) = \frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}, t \in [0,\sigma^2(1)]_{\mathbb{T}}, s \in [0,\sigma(1)]_{\mathbb{T}}, \tag{4}$$

and it is also shown in [10] that

$$G(t,s) \geq kG(\sigma(s),s) = k\frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)},\ t \in \left[\frac{\sigma^2(1)}{4}, \frac{3\sigma^2(1)}{4}\right]_{\mathbb{T}}, s \in [0,\sigma(1)]_{\mathbb{T}}, \tag{5}$$

where

$$k = \min\left\{\frac{1}{4}, \frac{\sigma^2(1)}{4(\sigma^2(1)-\sigma(0))}\right\}.$$

We note that an n-tuple $(u_1(t), \dots, u_n(t))$ is a solution of the eigenvalue problem (1), (2) if, and only if

$$u_i(t) = \lambda_i \int_0^{\sigma(1)} G(t,s)a_i(s)f_i(u_{i+1}(\sigma(s)))\Delta s,\ 0 \le t \le \sigma^2(1),\ 1 \le i \le n$$

and

$$u_{n+1}(t) = u_1(t),\ 0 \le t \le \sigma^2(1),$$

so that, in particular,

$$\begin{aligned} u_1(t) &= \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2) \times \\ &\quad \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2),s_3)a_3(s_3) \cdots \times \\ &\quad \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u_1(\sigma(s_n)))\Delta s_n\Big) \cdots \Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

Values of $\lambda_1, \dots, \lambda_n$, for which there are positive solutions (positive with respect to a cone) of (1), (2), will be determined via applications of the following fixed point theorem [13].

Theorem 2.1 *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_2,$ *or*

(ii) $\|Tu\| \ge \|u\|, u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \le \|u\|, u \in \mathcal{P} \cap \partial\Omega_2.$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Positive solutions in a cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (that is, positive solutions) of (1), (2). Assume throughout that $[0, \sigma^2(1)]_{\mathbb{T}}$ is such that

$$\xi = \min\left\{t \in \mathbb{T} |\, t \ge \frac{\sigma^2(1)}{4}\right\}$$

and

$$\omega = \max\left\{t \in \mathbb{T} |\, t \le \frac{3\sigma^2(1)}{4}\right\}$$

both exist and satisfy

$$\frac{\sigma^2(1)}{4} \le \xi < \omega \le \frac{3\sigma^2(1)}{4}.$$

Next, let $\tau_i \in [\xi, \omega]_{\mathbb{T}}$ be defined by

$$\int_\xi^\omega G(\tau_i, s)a(s)\Delta s = \min_{t \in [\xi,\omega]_{\mathbb{T}}} \int_\xi^\omega G(t,s)a_i(s)\Delta s.$$

Finally, we define

$$l = \min_{s\in[0,\sigma^2(1)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}$$

and let

$$m = \min\{k, l\}. \tag{6}$$

For our construction, let $\mathcal{B} = \{x \mid x : [0, \sigma^2(1)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm, $\|x\| = \sup\{|x(t)| : t \in [0, \sigma^2(1)]_{\mathbb{T}}\}$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ x \in \mathcal{B} \mid x(t) \geq 0 \text{ on } [0, \sigma^2(1)]_{\mathbb{T}} \text{ and } \min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} x(t) \geq m\|x\| \right\}.$$

We next define an integral operator $T : \mathcal{P} \to \mathcal{B}$, for $u \in \mathcal{P}$, by

$$\begin{aligned} Tu(t) \;=\; & \lambda_1 \int_0^{\sigma(1)} G(t, s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big) \cdots \Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned} \tag{7}$$

Notice from (A), (B) and (3) that, for $u \in \mathcal{P}$, $Tu(t) \geq 0$ on $[0, \sigma^2(1)]_{\mathbb{T}}$. Also, for $u \in \mathcal{P}$, we have from (4) that

$$\begin{aligned} Tu(t) \;\leq\; & \lambda_1 \int_0^{\sigma(1)} G(\sigma(s_1), s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big) \cdots \Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

so that

$$\begin{aligned} \|Tu\| \;\leq\; & \lambda_1 \int_0^{\sigma(1)} G(\sigma(s_1), s_1) a_1(s_1) f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) \times \\ & \times f_2\Big(\lambda_3 \int_0^{\sigma(1)} G(\sigma(s_2), s_3) a_3(s_3) \cdots \times \\ & \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big) \cdots \Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned} \tag{8}$$

Next, if $u \in \mathcal{P}$, we have from (5), (6) and (8),

$$\begin{aligned}
& \min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} Tu(t) \\
= & \min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2)\, \times \\
& \times f_2\Big(\cdots f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\cdots\Big)\Delta s_2\Big)\Delta s_1 \\
\geq & \lambda_1 m \int_0^{\sigma(1)} G(\sigma(s_1),s_1)a_1(s_1)f_1\Big(\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1),s_2)a_2(s_2)\, \times \\
& \times f_2\Big(\cdots f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\cdots\Big)\Delta s_2\Big)\Delta s_1 \\
\geq & m\|Tu\|.
\end{aligned}$$

Consequently, $T : \mathcal{P} \to \mathcal{P}$. In addition, the standard arguments can be used to verify that T is completely continuous.

By the remarks in Section 2, we seek suitable fixed points of T belonging to the cone $\mathcal{P}$.

For our first result, define positive numbers L_1 and L_2 by

$$L_1 := \max_{1\leq i\leq n}\left\{\left[m\int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s f_{i\infty}\right]^{-1}\right\},$$

and

$$L_2 := \min_{1\leq i\leq n}\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a_i(s)\Delta s f_{i0}\right]^{-1}\right\},$$

where we recall that $G(\sigma(s),s) = \frac{\sigma(s)(\sigma^2(1)-\sigma(s))}{\sigma^2(1)}$.

Theorem 3.1 *Assume conditions* (A), (B) *and* (C) *are satisfied. Then, for* $\lambda_1,\ldots,\lambda_n$ *satisfying*

$$L_1 < \lambda_i < L_2,\ 1 \leq i \leq n, \tag{9}$$

there exists an n-tuple $(u_1,\ldots,u_n)$ *satisfying* (1), (2) *such that* $u_i(t) > 0$ *on* $(0,\sigma^2(1))_{\mathbb{T}}$, $1 \leq i \leq n$.

Proof. Let λ_j, $1 \leq j \leq n$, be as in (9). And let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n}\left\{\left[m\int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s(f_{i\infty}-\epsilon)\right]^{-1}\right\} \leq \min_{1\leq j\leq n}\lambda_j$$

and

$$\max_{1\leq j\leq n}\lambda_j \leq \min_{1\leq i\leq n}\left\{\left[\int_0^{\sigma(1)} G(\sigma(s),s)a_i(s)\Delta s(f_{i0}+\epsilon)\right]^{-1}\right\}.$$

We seek fixed points of the completely continuous operator $T : \mathcal{P} \to \mathcal{P}$ defined by (7).

Now, from the definitions of f_{i0}, $1 \leq i \leq n$, there exists an $H_1 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \leq (f_{i0} + \epsilon)x, \; 0 < x \leq H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (4) and the choice of ϵ, for $0 \leq s_{n-1} \leq \sigma(1)$,

$$\begin{aligned}
& \lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
\leq \; & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
\leq \; & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) (f_{n0} + \epsilon)(u(\sigma(s_n)))\Delta s_n \\
\leq \; & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_n), s_n) a_n(s_n) \Delta s_n (f_{n0} + \epsilon)\|u\| \\
\leq \; & \|u\| \\
= \; & H_1.
\end{aligned}$$

It follows in a similar manner from (4) and the choice of ϵ that, for $0 \leq s_{n-2} \leq \sigma(1)$,

$$\begin{aligned}
& \lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-2}), s_{n-1}) a_{n-1}(s_{n-1}) \times \\
& \qquad \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
\leq \; & \lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_{n-1}) a_{n-1}(s_{n-1}) \Delta s_{n-1} (f_{n-1,0} + \epsilon)\|u\| \\
\leq \; & \|u\| \\
= \; & H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we reach, for $0 \leq t \leq \sigma^2(1)$,

$$\lambda_1 \int_0^{\sigma(1)} G(t, s_1) a_1(s_1) f_1(\cdots f_n(u(\sigma(s_n)))\Delta s_n \cdots)\Delta s_1 \leq H_1,$$

so that, for $0 \leq t \leq \sigma^2(1)$,

$$Tu(t) \leq H_1,$$

or

$$\|Tu\| \; \leq H_1 = \|u\|.$$

If we set

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < H_1\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{10}$$

Next, from the definition of $f_{i\infty}$, $1 \leq i \leq n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \geq (f_{i\infty} - \epsilon)x, \; x \geq \overline{H}_2.$$

Let

$$H_2 := \max\left\{2H_1, \frac{\overline{H}_2}{m}\right\}.$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then

$$\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} u(t) \geq m\|u\| \geq \overline{H}_2.$$

Consequently, from (5) and the choice of ϵ, for $0 \leq s_{n-1} \leq \sigma(1)$,

$$\begin{aligned}
&\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
\geq\ &\lambda_n \int_\xi^{\omega} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
\geq\ &\lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)(f_{n\infty} - \epsilon)(u(\sigma(s_n)))\Delta s_n \\
\geq\ &m\lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)\Delta s_n(f_{n\infty} - \epsilon)\|u\| \\
\geq\ &\|u\| \\
=\ &H_2.
\end{aligned}$$

It follows similarly from (5) and the choice of ϵ that, for $0 \leq s_{n-2} \leq \sigma(1)$,

$$\begin{aligned}
&\lambda_{n-1} \int_0^{\sigma(1)} G(\sigma(s_{n-2}), s_{n-1})a_{n-1}(s_{n-1}) \times \\
&\qquad \times f_{n-1}\Big(\lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
\geq\ &m\lambda_{n-1} \int_\xi^{\omega} G(\tau_{n-1}, s_{n-1})a_{n-1}(s_{n-1})\Delta s_{n-1}(f_{n-1,\infty} - \epsilon)\|u\| \\
\geq\ &\|u\| \\
=\ &H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we reach

$$Tu(\tau_1) = \lambda_1 \int_0^{\sigma(1)} G(\tau_1, s_1)a_1(s_1)f_1(\cdots f_n(u(\sigma(s_n)))\Delta s_n \cdots)\Delta s_1 \geq \|u\| = H_2,$$

so that $\|Tu\| \geq \|u\|$. So, if we set

$$\Omega_2 = \{x \in \mathcal{B} \ |\|x\| < H_2\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{11}$$

Applying Theorem 2.1 to (10) and (11), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. As such, setting $u_1 = u_{n+1} = u$, we obtain a positive solution $(u_1, \ldots, u_n)$ of (1), (2) given iteratively by

$$u_j(t) = \lambda_j \int_0^{\sigma(1)} G(t, s)a_j(s)f_j(u_{j+1}(\sigma(s)))\Delta s, \quad j = n, n-1, \ldots, 1.$$

The proof is complete. $\square$

Prior to our next result, let ξ_i, $1 \le i \le n$, be defined by

$$\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s = \max_{t\in[1,\sigma^2(1)]_{\mathbb{T}}} \int_0^{\sigma(1)} G(t,s)a_i(s)\Delta s.$$

Then, we define positive numbers L_3 and L_4 by

$$L_3 := \max_{1\le i\le n} \left\{ \left[m \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s f_{i0} \right]^{-1} \right\}$$

and

$$L_4 := \min_{1\le i\le n} \left\{ \left[\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s f_{i\infty} \right]^{-1} \right\}.$$

Theorem 3.2 *Assume conditions* (A)–(C) *are satisfied. Then, for each* $\lambda_1, \ldots, \lambda_n$ *satisfying*

$$L_3 < \lambda_i < L_4, \quad 1 \le i \le n, \tag{12}$$

there exists an n*-tuple* $(u_1, \ldots, u_n)$ *satisfying* (1), (2) *such that* $u_i(t) > 0$ *on* $(0, \sigma^2(1))_{\mathbb{T}}$, $1 \le i \le n$.

Proof Let $\lambda_j, 1 \le j \le n$, be as in (12). And let $\epsilon > 0$ be chosen such that

$$\max_{1\le i\le n} \left\{ \left[m \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s(f_{i0} - \epsilon) \right]^{-1} \right\} \le \min_{1\le j\le n} \lambda_j$$

and

$$\max_{1\le j\le n} \lambda_j \le \min_{1\le i\le n} \left\{ \left[\int_0^{\sigma(1)} G(\sigma(s), s)a_i(s)\Delta s(f_{i\infty} + \epsilon) \right]^{-1} \right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (7). From the definition of f_{i0}, $1 \le i \le n$, there exists $\overline{H_3} > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge (f_{i0} - \epsilon)x, \ 0 < x \le \overline{H_3}.$$

Also, from the definition of f_{i0}, it follows that $f_{i0}(0) = 0$, $1 \le i \le n$, and so there exist $0 < K_n < K_{n-1} < \cdots < K_2 < \overline{H_3}$ such that

$$\lambda_i f_i(t) \le \frac{K_{i-1}}{\int_0^{\sigma(1)} G(\xi_i, s)a_i(s)\Delta s}, \ t \in [0, K_i]_{\mathbb{T}}, \ 3 \le i \le n,$$

and

$$\lambda_2 f_2(t) \le \frac{\overline{H_3}}{\int_0^{\sigma(1)} G(\xi_2, s)a_2(s)\Delta s}, \ t \in [0, K_2]_{\mathbb{T}}.$$

Choose $u \in \mathcal{P}$ with $\|u\| = K_n$. Then, we have

$$\begin{aligned} & \lambda_n \int_0^{\sigma(1)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\ \leq \; & \lambda_n \int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\ \leq \; & \frac{\int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) K_{n-1} \Delta s_n}{\int_0^{\sigma(1)} G(\xi_n, s_n) a_n(s_n) \Delta s_n} \\ \leq \; & K_{n-1}. \end{aligned}$$

Bootstrapping yields the standard iterative pattern, and it follows that

$$\lambda_2 \int_0^{\sigma(1)} G(\sigma(s_1), s_2) a_2(s_2) f_2(\cdots)\Delta s_2 \leq \overline{H_3}.$$

Then

$$\begin{aligned} Tu(\tau_1) &= \lambda_1 \int_0^{\sigma(1)} G(\tau_1, s_1) a_1(s_1) f_1\left(\lambda_2 \ \cdots \right) \Delta s_1 \\ &\geq \lambda_1 m \int_\xi^\omega G(\tau_1, s_1) a_1(s_1)(f_{1,0} - \epsilon)\|u\| \Delta s_1 \\ &\geq \|u\|. \end{aligned}$$

So, $\|Tu\| \geq \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} \mid \|x\| < K_n\},$$

then

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1.$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leq i \leq n$, is unbounded at ∞.

For each $1 \leq i \leq n$, set

$$f_i^*(x) = \sup_{0 \leq s \leq x} f_i(s).$$

Then, it is straightforward that, for each $1 \leq i \leq n$, f_i^* is a nondecreasing real-valued function, $f_i \leq f_i^*$, and

$$\lim_{x \to \infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leq i \leq n$, there exists $\overline{H_4}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq (f_{i\infty} + \epsilon)x, \ x \geq \overline{H_4}.$$

It follows that there exists $H_4 > \max\{2\overline{H_3}, \overline{H_4}\}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq f_i^*(H_4), \quad 0 < x \leq H_4.$$

Choose $u \in \mathcal{P}$ with $||u|| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Tu(t) &= \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1(\lambda_2 \cdots)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(t,s_1)a_1(s_1)f_1^*(\lambda_2 \cdots)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(\xi_1,s_1)a_1(s_1)f_1^*(H_4)\Delta s_1 \\ &\leq \lambda_1 \int_0^{\sigma(1)} G(\xi_1,s_1)a_1(s_1)\Delta s_1(f_{1\infty}+\epsilon)H_4 \\ &\leq H_4 \\ &= \|u\|, \end{aligned}$$

and so $\|Tu\| \leq \|u\|$. So, if we let

$$\Omega_2 = \{x \in \mathcal{B} \mid \|x\| < H_4\},$$

then

$$\|Tu\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2.$$

Application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which in turn, with $u_1 = u_{n+1} = u$, yields an n-tuple $(u_1, \ldots, u_n)$ satisfying (1), (2) for the chosen values of λ_i, $1 \leq i \leq n$. The proof is complete. $\square$

References

[1] Agarwal, R.P. and O'Regan, D. Triple solutions to boundary value problems on time scales. *Applied Math. Letters* **13** (2000) 7–11.

[2] Agarwal, R.P., O'Regan, D. and Wong, P.J.Y. *Positive solutions of differential, difference and integral equations.* Kluwer, Dordrecht, 1999.

[3] Anderson, D.R. Eigenvalue intervals for a second-order mixed-conditions problem on time scale. *Int. J. Nonlin. Differential Equations* **7** (2002) 97–104.

[4] Anderson, D. R. Eigenvalue intervals for a two-point boundary value problem on a measure chain. *J. Comput. Appl. Math.* **141** (2002) 57–64.

[5] Bohner, M. and Peterson, A. *Dynamic equations on time scales: an introduction with applications.* Birkhauser, New York, 2001.

[6] Benchohra, M., Hamani, S., Henderson, J., Ntouyas, S. K. and Ouahab, A. Positive solutions for systems of nonlinear eigenvalue problems. *Global J. Math. Anal.* **1** (2007) 19–28.

[7] Benchohra, M., Henderson, J. and Ntouyas, S. K. Eigenvalue problems for systems of nonlinear boundary problems on time scales. *Adv. Difference Equ.* **2007** (2007), Article ID 31640, 10 pp.

[8] Chyan, C. J., Davis, J. M., Henderson, J. and Yin, W. K.C. Eigenvalue comparisons for differential equations on a measure chain. *Electron. J. Differential Equations* **1998** (1998), No. 35, 7 pp.

[9] Chyan, C. and Henderson, J. Eigenvalue problems for nonlinear differential equations on a measure chain. *J. Math. Anal. Appl.* **245** (2000) 547–559.

[10] Erbe, L. H. and Peterson, A. Positive solutions for a nonlinear differential equation on a measure chain. *Math. Comput. Modelling* **32** (2000) 571–585.

[11] Erbe, L. H. and Wang, H. On the existence of positive solutions of ordinary differential equations. *Proc. Amer. Math. Soc.* **120** (1994) 743–748.

[12] Graef, J. R. and Yang, B. Boundary value problems for second order nonlinear ordinary differential equations. *Comm. Appl. Anal.* **6** (2002) 273–288.

[13] Guo, D. and Lakshmikantham, V. *Nonlinear problems in abstract cones.* Academic Press, Orlando, 1988.

[14] He, Z. Double positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales. *Appl. Anal.* **84** (2005) 377–390.

[15] Henderson, J. and Ntouyas, S. K. Positive solutions for systems of nonlinear boundary value problems. *Nonlin. Studies* **15** (2008) 51–60.

[16] Henderson, J. and Ntouyas, S. K. Positive solutions for systems of three-point nonlinear boundary value problems. *Australian J. Math. Anal. Appl.* **5** (1), Article 11 (2008) 1–9.

[17] Henderson, J. and Ntouyas, S. K. Positive solutions for systems of nonlinear eigenvalue problems for functional differential equations. *Appl. Anal.* **86** (2007) 1365–1374.

[18] Henderson, J. and Ntouyas, S. K. Positive solutions for systems of nth order three-point nonlocal boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **18** (2007) 1–12.

[19] Henderson, J., Ntouyas, S.K. and Purnaras, I.K. Positive solutions for systems of generalized three-point nonlinear boundary value problems. *Comment. Math. Univ. Carolinae* **49** (2008) 79–91.

[20] Henderson, J., Ntouyas, S.K. and Purnaras, I.K. Positive solutions for systems of second order four-point nonlinear boundary boundary value problems. *Commun. Appl. Anal.* **12** (1) (2008) 29–40.

[21] Henderson, J. and Wang, H. Positive solutions for nonlinear eigenvalue problems. *J. Math. Anal. Appl.* **208** (1997) 1051–1060.

[22] Henderson, J. and Wang, H. Nonlinear eigenvalue problems for quasilinear systems. *Comput. Math. Appl.* **49** (2005) 1941–1949.

[23] Henderson, J. and Wang, H. An eigenvalue problem for quasilinear systems. *Rocky Mtn. J. Math.* **37** (2007) 215–228.

[24] Hu, L. and Wang, L.L. Multiple positive solutions of boundary value problems for systems of nonlinear second order differential equations. *J. Math. Anal. Appl.* **335** (2007) 1052–1060.

[25] Infante, G. Eigenvalues of some nonlocal boundary value problems. *Proc. Edinburgh Math. Soc.* **46** (2003) 75–86.

[26] Infante, G. and Webb, J.R.L. Loss of positivity in a nonlinear scalar heat equation. *Nonlin. Differ. Equ. Appl.* **13** (2006) 249–261.

[27] Li, W.T. and Sun, H.R. Multiple positive solutions for nonlinear dynamical systems on a measure chain. *J. Comput. Appl. Math.* **162** (2004) 421–430.

[28] Ma, R. Y. Multiple nonnegative solutions of second order systems of boundary value problems. *Nonlin. Anal.* **42** (2000) 1003–1010.

[29] Sun, H.R. and Li, W.T. Existence of positive solutions to second-order time scale systems. *Comput. Math. Appl.* **49** (2005) 131–145.

[30] Wang, H. On the number of positive solutions of nonlinear systems. *J. Math. Anal. Appl.* **281** (2003) 287–306.

[31] Webb, J. R. L. Positive solutions of some three point boundary value problems via fixed point index theory. *Nonlin. Anal.* **47** (2001) 4319–4332.

[32] Zhou, Y. and Xu, Y. Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations. *J. Math. Anal. Appl.* **320** (2006) 578–590.