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A New Approach for Construction of the Matrix-Valued Liapunov Functionals

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Abstract: In this paper we analyze the stability of large-scale functional differential equations with constant delays via matrix-valued functionals of Liapunov-Krasovskii. We establish a new approach for construction of the Liapunov-Krasovskii functional and present conditions which guarantee the uniform asymptotic stability of the trivial solution of linear and quasi-linear functional differential equations.

Keywords: *Functional differential equations; Liapunov's functional; uniform asymptotic stability; oscillator with delay.*

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1 Introduction

As is known, the direct Liapunov method [12] proves to be one of universal techniques of qualitative analysis of dynamical systems. Though the results achieved for the last decades in the development of this method (see [1, 4, 9, 15]) a series of general problems of motion stability theory shill remain in the focus of attention of many mathematicians and mechanical scientists. One of such problems is the problem of constructing suitable Liapunov functions (functionals) for certain classes of systems of equations.

For linear equations with constant coefficients and constant delay the problem on functional construction in [7, 6] is associated with solution of transcendent equations. Note that practical solution of the transcendent equation (see [7], p. 441)

$$\det(\lambda I - A - Be^{-\lambda\tau}) = 0, \tag{1.1}$$

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may prove to be a problem difficult to be solved, especially in the case when the system under investigation is of large dimensions.

In the case when equation (1.1) is solved, stability of zero solution follows automatically and construction of the functional has the meaning of converse theorem.

In the context of our approach the methods of construction of functionals from [6, 7] etc. may be useful in the construction of diagonal elements of matrix-valued functional provided all necessary constants and comparison functions in their above and below estimates will satisfy appropriate conditions.

General conclusion of the carried out analysis is as follows. The approaches involving solution of transcendent equation are hardly applicable for the construction of functionals for large-scale delay systems. In the case of systems of small dimensions the proposed methods of functional constructions can be applied to construct diagonal elements of matrix-valued functional, however the methods in themselves can not solve the problem on stability of the initial large scale system.

The aim of this paper is to present a new method of constructing the Liapunov functionals for the class of linear delay systems. The method is based on the idea of approximation of functional differential equation by the system of difference equations (see [4]) in combination with the idea of application of matrix-valued Liapunov functional (see [16]). This allows one to extend the class of admissible functionals suitable for construction of the Liapunov functionals for the system of equations under consideration.

An auxiliary result in this paper is a method of constructing the matrix-valued function for a system of difference equations of larger dimensions (see [18]).

2 Notation and Assumptions

In this section we introduce main designations used in the paper and assumptions on the systems under consideration.

Let $r > 0$ be given and $\mathcal{C} = C([-r,0],R^n)$ be the space of continuous functions mapping $[-r,0]$ into R^n. For $\varphi \in \mathcal{C}$ we define the norm

$$\|\varphi\| = \sup_{-r\le\theta\le 0} |\varphi(\theta)|, \tag{2.1}$$

where $|\cdot|$ means the Euclidean norm in R^n. Let C_H be an open subset of C for the elements of which $\|\varphi\| < H$ and $0 \in C_H$. If $x: \ [-r,a) \to R^n$ and is continuous, $0 < a \le +\infty$, then for each $t \in [0,a)$ in C_H $x_t(s) = x(t+s)$, $-r \le s \le 0$. In addition to norm (2.1) we apply the norm

$$\|\varphi\|_{L_2} = \left\{ \int\limits_{-r}^{0} |\varphi(\theta)|^2 d\,\theta \right\}^{1/2} \tag{2.2}$$

in the space $L_2([-r,0],R^n)$ of Lebesque functions integrated with square.

We study below the system with finite delay

$$\frac{dx}{dt} = F(x,x_t), \quad x_{t_0} = \varphi_0 \in \mathcal{C}, \quad t_0 \ge 0, \tag{2.3}$$

where $x \in R^n$, $F \in C(R^n \times \mathcal{C}, R^n)$, which has a linear approximation

$$\frac{dx}{dt} = Ax(t) + Bx(t-r) + f(x,x_t). \tag{2.4}$$

Here A and B are constant $n \times n$ matrices, $x(t)$ is n-dimensional vector, $r \geq 0$, $f \in C(R^n \times \mathcal{C}, R^n)$. The linear approximation of system (2.4)

$$\frac{dx}{dt} = Ax(t) + Bx(t-r) \tag{2.5}$$

is decomposed into two subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r), \\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r), \end{aligned} \tag{2.6}$$

where $x_i \in R^{n_i}$, $i = 1, 2$, $(x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}} = x$, A_{ij} and B_{ij} are constant matrices of the appropriate dimensions for which the independent sybsystems are

$$\begin{aligned} \frac{dx_1}{dt} &= A_{11}x_1(t) + B_{11}x_1(t-r), \\ \frac{dx_2}{dt} &= A_{22}x_2(t) + B_{22}x_2(t-r). \end{aligned} \tag{2.7}$$

For system (2.6) the matrix-valued functional

$$U(x, \varphi(\cdot)) : \quad R^n \times \mathcal{C}^n \to R^{2\times 2} \tag{2.8}$$

is constructed of some scalar elements $v_{ij}(\varphi_1, \varphi_2)$, $i, j = 1, 2$, under additional assumptions on matrices A_{ii} and $B_{ii}, i = 1, 2$ of system (2.7).

The scalar functional (cf. [3])

$$v(x, \varphi, \eta) = \eta^{\mathrm{T}} U(x, \varphi(\cdot))\eta, \quad \eta \in R^2_+, \quad \eta > 0, \tag{2.9}$$

together with upper right derivative number [9]

$$D^+v(x, \varphi, \eta)\big|_{(2.4)} : \quad R^n \times \mathcal{C} \to R$$

is the Liapunov–Krasovskii functional, if it solves the problem on stability of the state $x = 0$ of the system (2.4).

We recall that $D^+v(x, \varphi, \eta)\big|_{(2.4)}$ is calculated by the formula

$$D^+v(x, \varphi, \eta)\big|_{(2.4)} = \eta^{\mathrm{T}} D^+U(x, \varphi)\big|_{(2.4)}\eta,$$

where

$$D^+U(x, \varphi)\big|_{(2.4)} = \limsup_{\theta \to 0^+}\{[U(x + \theta F(x, \varphi), \varphi(0) + \theta F(x, \varphi)) - U(x, \varphi(0))]\theta^{-1}\}$$

is calculated element-wise. Before we start the solution of the problem on construction of functional (2.8) we need some results for the system of difference equations.

3 An Approach to Construction of Liapunov-Krasovskii Functionals

Now we consider autonomous linear system (2.5). Assume that for subsystems (2.7) the functionals

$$\begin{aligned}
v_{11}(\varphi_1) &= \varphi_1^{\mathrm T}(0)P_{11}\varphi_1(0) + 2\varphi_1^{\mathrm T}(0)\int\limits_{-r}^{0} K_1(\theta)\varphi_1(\theta)\,d\theta \\
&\quad + \int\limits_{-r}^{0} \varphi_1^{\mathrm T}(\theta)\Gamma_1(\theta)\varphi_1(\theta)\,d\theta + \int\limits_{-r}^{0}\int\limits_{-r}^{0} \varphi_1^{\mathrm T}(\xi)\gamma_1(\xi,\eta)\varphi_1(\eta)\,d\xi\,d\eta, \\
v_{22}(\varphi_2) &= \varphi_2^{\mathrm T}(0)P_{22}\varphi_2(0) + 2\varphi_2^{\mathrm T}(0)\int\limits_{-r}^{0} K_2(\theta)\varphi_2(\theta)\,d\theta \\
&\quad + \int\limits_{-r}^{0} \varphi_2^{\mathrm T}(\theta)\Gamma_2(\theta)\varphi_2(\theta)\,d\theta + \int\limits_{-r}^{0}\int\limits_{-r}^{0} \varphi_2^{\mathrm T}(\xi)\gamma_2(\xi,\eta)\varphi_2(\eta)\,d\xi\,d\eta,
\end{aligned} \tag{3.1}$$

are constructed somehow, where P_{11}, P_{22} are constant symmetric positive definite matrices,

$$\begin{aligned}
&K_1,\Gamma_1 \in C([-r,0],\, R^{n_1\times n_1}), && K_2,\Gamma_2 \in C([-r,0],\, R^{n_2\times n_2}),\\
&\gamma_1 \in C([-r,0]\times[-r,0],\, R^{n_1\times n_1}), && \gamma_2 \in C([-r,0]\times[-r,0],\, R^{n_2\times n_2}).
\end{aligned}$$

Further we employ the idea of approximation of system (2.6) by system of difference equations. With this in mind we divide the segment $[-r,0]$ into N equal parts of length h, i.e. $Nh = r$; the derivatives $\dfrac{dx_i}{dt}$, $i = 1,2$, are approximated by the differences $(x_i(t+h) - x_i(t))h^{-1}$. The system of difference equations corresponding to system (2.7) is (cf. Hale [4])*

$$\begin{aligned}
&\widetilde x_{11}(\tau+1) = (I_{n_1} + hA_{11})\widetilde x_{11}(\tau) + hB_{11}\widetilde x_{1N}(t) + hA_{12}\widetilde x_{21}(t) + hB_{12}\widetilde x_{2N}(t),\\
&\widetilde x_{12}(\tau+1) = \widetilde x_{11}(\tau),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\widetilde x_{1N}(\tau+1) = \widetilde x_{1N-1}(\tau),\\
&\widetilde x_{21}(\tau+1) = (I_{n_2} + hA_{22})\widetilde x_{21} + hB_{22}\widetilde x_{2N}(\tau) + hA_{12}\widetilde x_{21}(\tau) + hB_{12}\widetilde x_{2N}(\tau),\\
&\widetilde x_{22}(\tau+1) = \widetilde x_{21}(\tau),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\widetilde x_{2N}(\tau+1) = \widetilde x_{2N-1}(\tau),
\end{aligned} \tag{3.2}$$

where I_{n_1}, I_{n_2} are identity matrices of the corresponding dimensions.

The point $\widetilde x_i(0) = (\varphi_i(0),\, \varphi_i(-h),\, \dots,\, \varphi_i(-Nh))^{\mathrm T}$ corresponds to the initial function specifying solution $\widetilde x = (\widetilde x_1^{\mathrm T}, \widetilde x_2^{\mathrm T})^{\mathrm T}$ of system of difference equations (3.2).

Further we present system (3.2) in matrix form

$$\begin{aligned}
\widetilde x_1(\tau+1) &= A_{11}\widetilde x_1(\tau) + A_{12}\widetilde x_2(\tau),\\
\widetilde x_2(\tau+1) &= A_{21}\widetilde x_1(\tau) + A_{22}\widetilde x_2(\tau),
\end{aligned} \tag{3.3}$$

* It should be noted here that for stability analysis of the zero solution of system (2.3) with decomposition (2.6) a formal approach presented by Hale [4], p. 138 – 141, is employed.

where $\widetilde{x}_1 \in R^{n_1(N+1)}$, $\widetilde{x}_2 \in R^{n_2(N+1)}$ and

$$\widetilde{A}_{11} = \begin{pmatrix} I_{n_1} + hA_{11} & O_{n_1} & \dots & O_{n_1} & hB_{11} \\ I_{n_1} & O_{n_1} & \dots & O_{n_1} & O_{n_1} \\ \dots & \dots & \dots & \dots & \dots \\ O_{n_1} & \dots & \dots & I_{n_1} & O_{n_1} \end{pmatrix},$$

$$\widetilde{A}_{22} = \begin{pmatrix} I_{n_2} + hA_{22} & O_{n_2} & \dots & O_{n_2} & hB_{22} \\ I_{n_2} & O_{n_2} & \dots & O_{n_2} & O_{n_2} \\ \dots & \dots & \dots & \dots & \dots \\ O_{n_2} & \dots & \dots & I_{n_2} & O_{n_2} \end{pmatrix},$$

$$\widetilde{A}_{12} = \begin{pmatrix} hA_{12} & O_{n_1\times n_2} & \dots & hB_{12} \\ O_{n_1\times n_2} & O_{n_1\times n_2} & \dots & O_{n_1\times n_2} \\ \dots & \dots & \dots & \dots \\ O_{n_1\times n_2} & \dots & \dots & O_{n_1\times n_2} \end{pmatrix},$$

$$\widetilde{A}_{21} = \begin{pmatrix} hA_{21} & O_{n_2\times n_1} & \dots & hB_{21} \\ O_{n_2\times n_1} & O_{n_2\times n_1} & \dots & O_{n_2\times n_1} \\ \dots & \dots & \dots & \dots \\ O_{n_2\times n_1} & \dots & \dots & O_{n_2\times n_1} \end{pmatrix}.$$

Let k be arbitrary number, then vector $(\widetilde{x}_1^{\mathrm{T}}(kh),\, x_2^{\mathrm{T}}(kh))^{\mathrm{T}}$ is a phase vector for system (3.3) for any $t = kh$. For sufficiently small h vector $\widetilde{x}_i(kh)$, $i = 1, 2$, is an exact enough approximation of solutions of system (2.7) at points kh, $k = 0, -1, \dots, -N$.

Functionals $v_{11}(\varphi_1)$ and $v_{22}(\varphi_2)$ are approximated by the quadratic forms

$$\widetilde{v}_{11}(\widetilde{x}_1) = \widetilde{x}_1^{\mathrm{T}} \widetilde{P}_{11} \widetilde{x}_1, \quad \widetilde{v}_{22}(\widetilde{x}_2) = \widetilde{x}_2^{\mathrm{T}} \widetilde{P}_{22} \widetilde{x}_2, \tag{3.4}$$

where

$$\widetilde{P}_{11} = \begin{pmatrix} NP_{11} & k_{11}^{\mathrm{T}} & k_{12}^{\mathrm{T}} & \dots & k_{1N}^{\mathrm{T}} \\ k_{11} & \alpha_{11}^1 & \alpha_{12}^1 & \dots & \alpha_{1N}^1 \\ k_{12} & \alpha_{12}^1 & \alpha_{22}^1 & \dots & \alpha_{2N}^1 \\ \dots & \dots & \dots & \dots & \dots \\ k_{1N} & \alpha_{1N}^1 & \alpha_{2N}^1 & \dots & \alpha_{NN}^1 \end{pmatrix},$$

$$\widetilde{P}_{22} = \begin{pmatrix} NP_{22} & k_{21}^{\mathrm{T}} & k_{22}^{\mathrm{T}} & \dots & k_{2N}^{\mathrm{T}} \\ k_{21} & \alpha_{11}^2 & \alpha_{12}^2 & \dots & \alpha_{1N}^2 \\ k_{22} & \alpha_{12}^2 & \alpha_{22}^2 & \dots & \alpha_{2N}^2 \\ \dots & \dots & \dots & \dots & \dots \\ k_{2N}^2 & \alpha_{1N}^2 & \alpha_{2N}^2 & \dots & \alpha_{NN}^2 \end{pmatrix}.$$

Here the constant matrices $P_{11}, P_{22}, k_{ji}, \alpha_{ij}^j$, $i = 1, 2, \dots, N$, $j = 1, 2$, of the corresponding dimensions are determined as

$$k_{ji} = K_j(-hi), \quad \alpha_{ii}^j = \Gamma_j(-ih), \quad i = 1, 2, \dots, N, \quad j = 1, 2,$$
$$\alpha_{ij}^1 = \gamma_1(-hi, -hj), \quad \alpha_{ij}^2 = \gamma_2(-hi, -hj), \quad i, j = 1, 2, \dots, N, \quad i \neq j.$$

We construct the non-diagonal element $v_{12}(\widetilde{x}_1, \widetilde{x}_2)$ of the matrix-valued functional $U(\widetilde{x}_1, \widetilde{x}_2)$ in the bilinear form

$$v_{12}(\widetilde{x}_1, \widetilde{x}_2) = \widetilde{x}_1^{\mathrm{T}} \widetilde{P}_{12} \widetilde{x}_2, \tag{3.5}$$

where matrix $\widetilde{P}_{12}$ satisfies the equation

$$\widetilde{A}_{11}^{\mathrm{T}}\widetilde{P}_{12}\widetilde{A}_{22}-\widetilde{P}_{12}=-\frac{\eta_1}{\eta_2}\widetilde{A}_{11}^{\mathrm{T}}\widetilde{P}_{11}\widetilde{A}_{12}-\frac{\eta_2}{\eta_1}\widetilde{A}_{21}^{\mathrm{T}}\widetilde{P}_{22}\widetilde{A}_{22} \tag{3.6}$$

and has the form

$$\widetilde{P}_{12}=\begin{pmatrix} NP_{12} & s_1^2 & s_2^2 & \dots & s_N^2\\ s_1^1 & q_{11} & q_{12} & \dots & q_{1N}\\ s_2^1 & q_{21} & q_{22} & \dots & q_{2N}\\ \dots & \dots & \dots & \dots & \dots\\ s_N^1 & q_{N1} & q_{N2} & \dots & q_{NN}\end{pmatrix}.$$

Here P_{12}, s_i^j, q_{ij} are matrices and η_1, η_2 are positive constants.

In terms of equation (3.6) we get

$$\begin{aligned}(NP_{12}+rA_{11}^{\mathrm{T}}P_{12}+s_1^1)(I_{n_2}+hA_{22})+s_1^2+hA_{11}^{\mathrm{T}}s_1^2+q_{11}-NP_{12}\\ =-\frac{r\eta_1}{\eta_2}P_{11}A_{12}-hr\frac{\eta_1}{\eta_2}A_{11}P_{11}A_{12}-h\frac{\eta_1}{\eta_2}k_{11}A_{11}k_{1N}^{\mathrm{T}}A_{12}\\ -r\frac{\eta_2}{\eta_1}A_{21}^{\mathrm{T}}P_{22}-r\frac{\eta_1}{\eta_2}hA_{21}^{\mathrm{T}}P_{22}A_{22}-\frac{\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}k_{2N}A_{21}^{\mathrm{T}}k_{21}^{\mathrm{T}},\end{aligned} \tag{3.7}$$

$$s_i^2-s_{i-1}^2+hA_{11}^{\mathrm{T}}s_i^2+q_{1i}=-\frac{\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}k_{2i},\quad i=2,\dots,N, \tag{3.8}$$

$$hs_i^1B_{22}-q_{i-1,N}=-\frac{\eta_1}{\eta_2}hk_{1i}^{\mathrm{T}}B_{12},\quad i=2,\dots,N, \tag{3.9}$$

$$hB_{11}^{\mathrm{T}}s_i^2-q_{N,i-1}=-\frac{\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}k_{2i},\quad i=2,\dots,N, \tag{3.10}$$

$$\begin{aligned}rP_{12}B_{22}+hrA_{11}^{\mathrm{T}}P_{12}B_{22}+hs_1^1B_{22}-s_N^2=-\frac{\eta_1}{\eta_2}rP_{11}B_{12}\\ -\frac{r\eta_1}{\eta_2}hA_{11}^{\mathrm{T}}P_{11}B_{12}-\frac{\eta_1}{\eta_2}hk_{11}B_{12}-\frac{r\eta_2}{\eta_1}hA_{21}^{\mathrm{T}}P_{22}B_{22},\end{aligned} \tag{3.11}$$

$$\begin{aligned}rB_{11}^{\mathrm{T}}P_{12}+hrB_{11}^{\mathrm{T}}P_{12}A_{22}+hB_{11}^{\mathrm{T}}s_1^2-s_N^1=-\frac{r\eta_1}{\eta_2}hB_{11}^{\mathrm{T}}P_{11}A_{12}\\ -\frac{r\eta_2}{\eta_1}B_{21}^{\mathrm{T}}P_{22}-\frac{r\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}P_{22}A_{22}-\frac{\eta_2}{\eta_1}hB_{21}^{\mathrm{T}}k_{12}^{\mathrm{T}},\end{aligned} \tag{3.12}$$

$$s_i^1+hs_i^1A_{22}-s_{i-1}^1+q_{i1}=-hk_{1i}A_{12},\quad i=2,\dots,N, \tag{3.13}$$

$$q_{NN}=h\left(B_{11}^{\mathrm{T}}B_{22}+\frac{\eta_1}{\eta_2}B_{11}B_{12}+\frac{\eta_2}{\eta_1}B_{21}B_{22}\right), \tag{3.14}$$

$$q_{ii}=\mathrm{const},\quad q_{ij}=q_{i-1,j-1},\quad i,j=2,\dots,N. \tag{3.15}$$

From equation (3.7), in view of (3.4) and $h\to 0$ we get*

$$A_{11}^{\mathrm{T}}P_{12}+P_{12}A_{22}=-\frac{\eta_1}{\eta_2}P_{11}A_{12}-\frac{\eta_2}{\eta_1}A_{21}^{\mathrm{T}}P_{22}-\frac{1}{r}\,(S_1(0)+S_2(0)). \tag{3.16}$$

Similarly in view of (3.15) we get from (3.9)

$$q_{1i}=q_{N+1-i,N}=h(s_{N+2-i}^1B_{22}+k_{1,N+2-i}^{\mathrm{T}}B_{12}). \tag{3.17}$$

* From here on in formulas (3.7) – (3.14) and (3.18) passage to the limit as $h\to 0$ is formal.

Then, in view of (3.17) equations (3.9) imply

$$(s_i^2 - s_{i-1}^2)h^{-1} + A_{11}^{\mathrm{T}} s_i^2 + s_{N+2-i}^1 B_{22} + \frac{\eta_1}{\eta_2} k_{1,N+2-i} B_{12} = -\frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} k_{2i}^{\mathrm{T}}, \quad i = 2,\dots,N, \tag{3.18}$$

and passing to the limit as $h \to 0$ we obtain

$$-\frac{dS_2}{d\theta} + A_{11}^{\mathrm{T}} S_2(\theta) + S_1(-r-\theta)B_{22} + \frac{\eta_1}{\eta_2} K_1(-r-\theta)B_{12} = -\frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} K_2^{\mathrm{T}}(\theta). \tag{3.19}$$

Similarly to the above, in view of (3.14) we get from (3.19)

$$-\frac{dS_1}{d\theta} + S_1(\theta)A_{22} + B_{11}^{\mathrm{T}} S_2(-r-\theta) + \frac{\eta_2}{\eta_1} B_{21}^{\mathrm{T}} K_2^{\mathrm{T}}(-r-\theta) = -\frac{\eta_1}{\eta_2} K_1(\theta)A_{12}. \tag{3.20}$$

Taking into account (3.14) we find from (3.11) and (3.12) as $h \to 0$ the initial conditions

$$\begin{aligned} S_2(-1) &= r\left(P_{12}B_{22} + \frac{\eta_1}{\eta_2} P_{11} B_{12}\right), \\ S_1(-1) &= r\left(B_{11}^{\mathrm{T}} P_{12} + \frac{\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22}\right). \end{aligned} \tag{3.21}$$

In the expression of the bilinear form $\frac{1}{N} v_{12}(\widetilde{x}_1, \widetilde{x}_2)$ the formal limiting passage $(h \to 0)$ yields the expression for the functional

$$\begin{aligned} v_{12}(\varphi_1,\varphi_2) &= \varphi_1^{\mathrm{T}}(0) P_{12} \varphi_2(0) + \frac{1}{r}\varphi_1^{\mathrm{T}}(0) \int_{-r}^{0} S_2(\theta)\varphi_2(\theta)\,d\theta \\ &+ \frac{1}{r}\varphi_2^{\mathrm{T}}(0)\int_{-r}^{0} S_1^{\mathrm{T}}(\theta)\varphi_1(\theta)\,d\theta + \frac{1}{r}\int_{-r}^{0} d\xi \varphi_1^{\mathrm{T}}(\xi) \int_{-r}^{\xi} \Big\{ S_1(\xi-\eta-r)B_{22} \\ &+ \frac{\eta_1}{\eta_2} K_1^{\mathrm{T}}(\xi-\eta-r)B_{12} \Big\} \varphi_2(\eta)\,d\eta + \frac{1}{r}\int_{-r}^{0} d\xi \varphi_1^{\mathrm{T}}(\xi) \int_{\xi}^{0} \Big\{ B_{11}^{\mathrm{T}} S_2(\eta-\xi-r) \\ &+ \frac{\eta_1}{\eta_2} B_{21}^{\mathrm{T}} K_2(\eta-\xi-r) \Big\} \varphi_2(\eta) d\eta. \end{aligned} \tag{3.22}$$

In order to formulate stability conditions for system (2.6) in terms of the matrix-valued functional $U(\varphi_1,\varphi_2)$ with components (3.1) and (3.22) it is necessary to estimate its and their upper right derivative numbers along solutions of system (2.6). To this end we define concretely the choice of functionals (3.1) as

$$v_{11}(\varphi_1) = \varphi_1^{\mathrm{T}}(0) P_{11}\varphi_1(0) + \int_{-r}^{0} k(\theta)\varphi_1^{\mathrm{T}}(\theta) D_1 \varphi_1(\theta)\,d\theta, \tag{23}$$

$$v_{22}(\varphi_2) = \varphi_2^{\mathrm{T}}(0) P_{22}\varphi_2(0) + \int_{-r}^{0} k(\theta)\varphi_2^{\mathrm{T}}(\theta) D_2 \varphi_2(\theta)\,d\theta, \tag{24}$$

where P_{11}, P_{22}, D_1 and D_2 are positive definite matrices of the corresponding dimensions and $k(\theta) = 1 + \dfrac{1}{2r}\theta$.

Basing on the system of equations

$$\frac{dS_2}{d\theta} = A_{11}^{\mathrm{T}} S_2(\theta) + S_1(-r-\theta)B_{22}, \tag{25}$$

$$\frac{dS_1}{d\theta} = S_1(\theta)A_{22} + B_{11}^{\mathrm{T}} S_2(-r-\theta), \tag{26}$$

$$A_{11}^{\mathrm{T}} P_{12} + P_{12}A_{22} = -\frac{\eta_1}{\eta_2}P_{11}A_{12} - \frac{\eta_2}{\eta_1}A_{21}^{\mathrm{T}}P_{22} - \frac{1}{r}(S_1(0) + S_2(0)) \tag{27}$$

under initial conditions

$$\begin{aligned} S_2(-r) &= r\Big(P_{12}B_{22} + \frac{\eta_1}{\eta_2}P_{11}B_{12}\Big),\\ S_1(-r) &= r\Big(B_{11}^{\mathrm{T}}P_{12} + \frac{\eta_2}{\eta_1}B_{21}^{\mathrm{T}}P_{22}\Big), \end{aligned} \tag{3.28}$$

where $P_{12} \in R^{n_1\times n_2}$, $S_1, S_2 \in C^1([-r,0], R^{n_1\times n_2})$, η_1, η_2 are positive constants we construct functional $v_{12}(\varphi_1,\varphi_2)$ in the form

$$\begin{aligned} v_{12}(\varphi_1,\varphi_2) &= \varphi_1^{\mathrm{T}}(0)P_{12}\varphi_2(0) + \frac{1}{r}\varphi_1^{\mathrm{T}}(0)\int_{-r}^{0} S_2(\theta)\varphi_2(\theta)\,d\theta\\ &+ \frac{1}{r}\varphi_2^{\mathrm{T}}(0)\int_{-r}^{0} S_1^{\mathrm{T}}(\theta)\varphi_1(\theta)\,d\theta + \frac{1}{r}\int_{-r}^{0} d\xi\varphi_1^{\mathrm{T}}(\xi)\int_{-r}^{\xi} S_1(\xi-\eta-r)B_{22}\varphi_2(\eta)\,d\eta\\ &+ \frac{1}{r}\int_{-r}^{0} d\xi\varphi_1^{\mathrm{T}}(\xi)\int_{\xi}^{0} B_{11}^{\mathrm{T}}S_2(\eta-\xi-r)\varphi_2(\eta)\,d\eta. \end{aligned} \tag{3.29}$$

Since for the functionals $v_{ij}(\cdot)$, $i,j = 1,2$, the lower estimates

$$\begin{aligned} v_{11}(\varphi_1) &\ge \lambda_m(P_{11})|\varphi_1(0)|^2 + \frac{1}{2}\lambda_m(D_1)\|\varphi_1\|_{L_2}^2\\ v_{22}(\varphi_2) &\ge \lambda_m(P_{22})|\varphi_2(0)|^2 + \frac{1}{2}\lambda_m(D_2)\|\varphi_2\|_{L_2}^2\\ v_{12}(\varphi_1,\varphi_2) &\ge -\|P_{12}\||\varphi_1(0)||\varphi_2(0)| - \varkappa_2|\varphi_1(0)|\|\varphi_2\|_{L_2}\\ &\quad - \varkappa_1|\varphi_2(0)|\|\varphi_1\|_{L_2} - (\varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|)\|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2}, \end{aligned} \tag{3.30}$$

and the upper estimates

$$\begin{aligned} v_{11}(\varphi_1) &\le \lambda_M(P_{11})|\varphi_1(0)|^2 + \frac{1}{2}\lambda_M(D_1)\|\varphi_1\|_{L_2}^2\\ v_{22}(\varphi_2) &\le \lambda_M(P_{22})|\varphi_2(0)|^2 + \frac{1}{2}\lambda_M(D_2)\|\varphi_1\|_{L_2}^2\\ v_{12}(\varphi_1,\varphi_2) &\le \|P_{12}\||\varphi_1(0)||\varphi_2(0)| + \varkappa_2|\varphi_1(0)|\|\varphi_2\|_{L_2}\\ &\quad + \varkappa_{12}|\varphi_2(0)|\|\varphi_1\|_{L_2} + (\varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|)\|\varphi_1\|_{L_2}\|\varphi_2\|_{L_2}, \end{aligned} \tag{3.31}$$

are satisfied, where

$$\varkappa_1 = \frac{1}{r}\left\{\int_{-r}^{0} \|S_1(\theta)\|^2 d\theta\right\}^{1/2}, \quad \varkappa_2 = \frac{1}{r}\left\{\int_{-r}^{0} \|S_2(\theta)\|^2 d\theta\right\}^{1/2},$$

$$\varkappa_{12} = \frac{1}{r}\left\{\int_{-r}^{0}\int_{-r}^{0} \|S_1(\xi-\eta-r)\|^2 d\xi d\eta\right\}^{1/2},$$

$$\varkappa_{21} = \frac{1}{r}\left\{\int_{-r}^{0}\int_{-r}^{0} \|S_2(\xi-\eta-r)\|^2 d\xi d\eta\right\}^{1/2},$$

for the functional

$$v(\varphi_1,\varphi_2,\eta) = \eta^{\mathrm{T}} U(\varphi_1,\varphi_2)\eta = \eta_1^2 v_{11}(\varphi_1) + 2\eta_1\eta_2 v_{12}(\varphi_1,\varphi_2) + \eta_2^2 v_{22}(\varphi_2)$$

the bilateral estimate

$$u^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H u \le v(\varphi_1,\varphi_2,\eta) \le u^{\mathrm{T}} H^{\mathrm{T}} \overline{C} H u, \tag{3.32}$$

is valid, where

$$u = (|\varphi_1(0)|,\ |\varphi_2(0)|,\ \|\varphi_1\|_{L_2},\ \|\varphi_2\|_{L_2}),$$
$$H = \operatorname{diag}(\eta_1,\eta_2,\eta_1,\eta_2), \quad \zeta = \varkappa_{21}\|B_{11}\| + \varkappa_{12}\|B_{22}\|,$$
$$\overline{C} = \begin{pmatrix} \lambda_M(P_{11}) & \|P_{12}\| & 0 & \varkappa_2 \\ \|P_{12}\| & \lambda_M(P_{22}) & \varkappa_1 & 0 \\ 0 & \varkappa_1 & \lambda_M(D_1) & \zeta \\ \varkappa_2 & 0 & \zeta & \lambda_M(D_2) \end{pmatrix},$$
$$\underline{C} = \begin{pmatrix} \lambda_m(P_{11}) & -\|P_{12}\| & 0 & -\varkappa_2 \\ -\|P_{12}\| & \lambda_m(P_{22}) & -\varkappa_1 & 0 \\ 0 & -\varkappa_1 & \frac{1}{2}\lambda_m(D_1) & -\zeta \\ -\varkappa_2 & 0 & -\zeta & \frac{1}{2}\lambda_m(D_2) \end{pmatrix}.$$

Further together with functionals (3.23), (3.24) and (3.29) we use the upper right derivative numbers $D^+ v_{ij}(\cdot)\big|_{(2.6)}$, $i,j = 1,2$:

$$\begin{aligned} D^+ v_{11}(\varphi_1)\big|_{(2.6)} &= \varphi_1^{\mathrm{T}}(0)(A_{11}^{\mathrm{T}} P_{11} + P_{11}A_{11} + D_1)\varphi_1(0) \\ &- \frac{1}{2}\varphi_1^{\mathrm{T}}(-r)D_1\varphi_1(-r) + \varphi_1^{\mathrm{T}}(0)P_{11}B_{11}\varphi_1(-r) + \varphi_1^{\mathrm{T}}(0)P_{11}A_{12}\varphi_2(0) \qquad (33) \\ &+ \varphi_1^{\mathrm{T}}(0)P_{11}B_{12}\varphi_2(-r) - \frac{1}{2r}\int_{-r}^{0} \varphi_1^{\mathrm{T}}(\theta)D_1\varphi_1(\theta)\, d\theta, \end{aligned}$$

$$\begin{aligned} D^+ v_{22}(\varphi_2)\big|_{(2.6)} &= \varphi_2^{\mathrm{T}}(0)(A_{22}^{\mathrm{T}} P_{22} + P_{22}A_{22} + D_2)\varphi_2(0) \\ &- \frac{1}{2}\varphi_2^{\mathrm{T}}(-r)D_1\varphi_2(-r) + \varphi_2^{\mathrm{T}}(0)P_{22}B_{22}\varphi_2(-r) + \varphi_2^{\mathrm{T}}(0)P_{22}A_{21}\varphi_1(0) \qquad (34) \end{aligned}$$

$$
+\varphi_2^{\mathrm T}(0)P_{22}B_{21}\varphi_1(-r)-\frac{1}{2r}\int\limits_{-r}^{0}\varphi_2^{\mathrm T}(\theta)D_2\varphi_2(\theta)\,d\theta,
$$

$$
\begin{aligned}
D^+v_{12}(\varphi_1,\varphi_2)\big|_{(2.6)} &= \varphi_1^{\mathrm T}(0)\Big(A_{11}^{\mathrm T}P_{12}+P_{12}A_{22}+\frac{1}{r}(S_1(0)+S_2(0))\Big)\varphi_2(0)\\
&+\frac12\varphi_1^{\mathrm T}(0)(P_{12}A_{21}+A_{21}^{\mathrm T}P_{12}^{\mathrm T})\varphi_1(0)+\frac12\varphi_2^{\mathrm T}(0)(A_{12}^{\mathrm T}P_{12}+P_{12}^{\mathrm T}A_{12})\varphi_2(0)\\
&+\varphi_1^{\mathrm T}(0)P_{12}B_{21}\varphi_1(-r)+\varphi_2^{\mathrm T}(-r)B_{12}^{\mathrm T}P_{12}\varphi_2(0)\\
&+\frac1r\varphi_2^{\mathrm T}(-r)B_{12}^{\mathrm T}\int\limits_{-r}^{0}S_2(\theta)\varphi_2(\theta)\,d\theta+\frac1r\varphi_2^{\mathrm T}(0)A_{12}^{\mathrm T}\int\limits_{-r}^{0}S_2(\theta)\varphi_2(\theta)\,d\theta \qquad (35)\\
&+\frac1r\varphi_1^{\mathrm T}(0)A_{21}^{\mathrm T}\int\limits_{-r}^{0}S_1^{\mathrm T}(\theta)\varphi_1(\theta)\,d\theta+\frac1r\varphi_1^{\mathrm T}(-r)B_{21}^{\mathrm T}\int\limits_{-r}^{0}S_1^{\mathrm T}(\theta)\varphi_1(\theta)\,d\theta\\
&-\frac{\eta_1}{\eta_2}\varphi_1^{\mathrm T}(0)P_{11}B_{12}\varphi_2(-r)-\frac{\eta_2}{\eta_1}\varphi_1^{\mathrm T}(-r)B_{21}^{\mathrm T}P_{22}\varphi_2(0).
\end{aligned}
$$

In view of expressions (3.33) – (3.35) for the upper right derivative number of functional $v(\varphi_1,\varphi_2,\eta)$ in the domain of values $R^n\times\mathcal{C}^n$ we have the estimate

$$
D^+v(\varphi_1,\varphi_2,\eta)\big|_{(2.6)}\le u_1^{\mathrm T}\Sigma_1u_1+u_2^{\mathrm T}\Sigma_2u_2, \tag{3.36}
$$

where

$$
u_1=(|\varphi_1(0)|,\,|\varphi_1(-r)|,\,\|\varphi_1\|_{L_2})^{\mathrm T},
$$
$$
u_2=(|\varphi_2(0)|,\,|\varphi_2(-r)|,\,\|\varphi_2\|_{L_2})^{\mathrm T}
$$

and $\Sigma_1=[\sigma_{ij}^1]_{i,j=1}^3$, $\Sigma_2=[\sigma_{ij}^2]_{i,j=1}^3$ are constant matrices with the elements

$$
\begin{gathered}
\sigma_{11}^1=\lambda_M(A_{11}^{\mathrm T}P_{11}+P_{11}A_{11}+D_1)\eta_1^2+\eta_1\eta_2\lambda_M(P_{12}A_{21}+A_{21}^{\mathrm T}P_{12}^{\mathrm T}),\\
\sigma_{22}^1=-\frac12\lambda_m(D_1)\eta_1^2,\quad \sigma_{33}^1=-\frac{1}{2r}\lambda_m(D_1)\eta_1^2,\\
\sigma_{12}^1=\|P_{11}\|\,\|B_{11}\|\eta_1^2+\|P_{12}\|\,\|B_{21}\|\eta_1\eta_2,\\
\sigma_{23}^1=\varkappa_1\|B_{21}\|\eta_1\eta_2,\quad \sigma_{13}^1=\varkappa_1\|A_{21}\|\eta_1\eta_2,\\
\sigma_{11}^2=\lambda_M(A_{22}^{\mathrm T}P_{22}+P_{22}A_{22}+D_2)\eta_2^2+\eta_1\eta_2\lambda_M(P_{21}A_{12}+A_{12}^{\mathrm T}P_{21}^{\mathrm T}),\\
\sigma_{22}^2=-\frac12\lambda_m(D_2)\eta_2^2,\quad \sigma_{33}^2=-\frac{1}{2r}\lambda_m(D_2)\eta_2^2,\\
\sigma_{12}^2=\|P_{22}\|\,\|B_{22}\|\eta_2^2+\|P_{12}\|\,\|B_{12}\|\eta_1\eta_2,\\
\sigma_{23}^2=\varkappa_2\|B_{12}\|\eta_1\eta_2,\quad \sigma_{13}^2=\varkappa_2\|A_{12}\|\eta_1\eta_2,\\
\sigma_{ij}^1=\sigma_{ji}^1,\quad \sigma_{ij}^2=\sigma_{ji}^2,\quad i,j=1,2,3,\quad i\ne j.
\end{gathered}
$$

In the partial case when $B_{11}=0$ and $B_{22}=0$ system (2.6) becomes

$$
\begin{aligned}
\frac{dx_1}{dt}&=A_{11}x_1(t)+A_{12}x_2(t)+B_{12}x_2(t-r),\\
\frac{dx_2}{dt}&=A_{21}x_1(t)+A_{22}x_2(t)+B_{21}x_1(t-r).
\end{aligned} \tag{3.37}
$$

Besides, system of equations (3.25) – (3.27) becomes

$$\begin{aligned} &\frac{dS_2}{d\theta} = A_{11}^{\mathrm{T}} S_2(\theta), \qquad \frac{dS_1}{d\theta} = S_1(\theta) A_{22} \\ &A_{11}^{\mathrm{T}} P_{12} + P_{12} A_{22} = -\frac{\eta_1}{\eta_2} P_{11} A_{12} - \frac{\eta_2}{\eta_1} A_{21}^{\mathrm{T}} P_{22} - \frac{1}{r}(S_1(0) + S_2(0)) \end{aligned} \tag{3.38}$$

under the initial conditions

$$S_2(-r) = \frac{r\eta_1}{\eta_2} P_{11} B_{12}, \quad S_1(-r) = \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22}. \tag{3.39}$$

The first group of equations (3.38) can be integrated in the explicit form

$$\begin{aligned} S_1(\theta) &= \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22}(\theta + r)\}, \\ S_2(\theta) &= \frac{r\eta_1}{\eta_2} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\} P_{11} B_{12}. \end{aligned} \tag{3.40}$$

Letting $\theta = 0$ we find

$$\begin{aligned} S_1(0) &= \frac{r\eta_2}{\eta_1} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22} r\}, \\ S_2(0) &= \frac{r\eta_1}{\eta_2} \exp\{A_{11}^{\mathrm{T}} r\} P_{11} B_{12}. \end{aligned}$$

Therefore equation (3.27) becomes

$$\begin{aligned} A_{11}^{\mathrm{T}} P_{12} + P_{12} A_{22} = &-\frac{\eta_1}{\eta_2}(P_{11} A_{12} + \exp\{A_{11}^{\mathrm{T}} r\} P_{11} B_{12}) \\ &- \frac{\eta_2}{\eta_1}(A_{21}^{\mathrm{T}} P_{22} + B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22} r\}). \end{aligned} \tag{3.41}$$

Necessary and sufficient existence conditions for unique solution of equation (3.41) follow from Lankaster [11].

Diagonal elements of the matrix-valued functional $U(\varphi_1, \varphi_2)$ are taken in the form of (3.23), (3.24) for $w_1(\varphi_1) = v_{11}(\varphi_1)$ and $w_2(\varphi_2) = v_{22}(\varphi_2)$, and non-diagonal element $w_{12}(\varphi_1, \varphi_2)$ is represented as

$$\begin{aligned} w_{12}(\varphi_1, \varphi_2) = \varphi_1^{\mathrm{T}}(0) P_{12} \varphi_2(0) &+ \frac{\eta_1}{\eta_2} \varphi_1^{\mathrm{T}}(0) \int_{-r}^{0} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\} P_{11} B_{12} \varphi_2(\theta)\, d\theta \\ &+ \frac{\eta_2}{\eta_1} \varphi_2^{\mathrm{T}}(0) \int_{-r}^{0} B_{21}^{\mathrm{T}} P_{22} \exp\{A_{22}(\theta + r)\} \varphi_1(\theta)\, d\theta. \end{aligned} \tag{3.42}$$

For estimation of functional (3.42) we shall formulate one auxiliary result (see [2]).

Lemma 3.1 *Let A be a constant $n \times n$–matrix, then estimate*

$$\| \exp At \| \le e^{\Delta t} \sum_{k=0}^{n-1} \frac{1}{k!} (2t\|A\|)^k, \quad t \ge 0,$$

is valid, where $\Delta = \max\{\operatorname{Re}\lambda \ \lambda \in \sigma(A)\}$, $\sigma(A)$ is a spectrum of matrix A.

Using this result one can estimate $\|\exp At\|$ as follows. Let $\varepsilon > 0$ be a sufficiently small positive number. Consider function [17]

$$f(t) = e^{-\varepsilon t} \sum_{k=0}^{n-1} \frac{1}{k!}(2t\|A\|)^k, \quad t \geq 0.$$

In view of the fact that $f(t) \to 0$ as $t \to \infty$ we conclude that there exists $M_\varepsilon = \max\limits_{t \geq 0} f(t)$ and find the estimate

$$\|\exp\{At\}\| \leq M_\varepsilon e^{(\Delta+\varepsilon)t} \quad \text{for} \quad t \geq 0. \tag{3.43}$$

Applying estimate (3.43) it is easy to find

$$\varkappa_1 = \frac{1}{r}\left\{\int_{-r}^{0} \lambda_M(S_1(\theta)S_1^{\mathrm{T}}(\theta))\right\}^{1/2} \leq \frac{\eta_2}{\eta_1}\|B_{21}\|\,\|P_{22}\|M_{\varepsilon 1}\left[\frac{e^{2(\Delta_1+\varepsilon)r}-1}{2(\Delta_1+\varepsilon)}\right]^{1/2} = \xi_1,$$

$$\varkappa_2 = \frac{1}{r}\left\{\int_{-r}^{0} \lambda_M(S_2(\theta)S_2^{\mathrm{T}}(\theta))\right\}^{1/2} \leq \frac{\eta_1}{\eta_2}\|B_{12}\|\,\|P_{11}\|M_{\varepsilon 2}\left[\frac{e^{2(\Delta_2+\varepsilon)r}-1}{2(\Delta_2+\varepsilon)}\right]^{1/2} = \xi_2,$$

where $M_{\varepsilon 1}$ and $M_{\varepsilon 1}$ are the corresponding constants and Δ_1 and Δ_2 are maximal real values of spectra of matrices A_{11} and A_{22} respectively.

Thus, for the scalar functional

$$w(\varphi_1, \varphi_2, \eta) = \eta^{\mathrm{T}}\, U(\varphi_1, \varphi_2)\eta \tag{3.44}$$

the estimate

$$u^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H u \leq w(\varphi_1, \varphi_2, \eta) \leq u^{\mathrm{T}} H^{\mathrm{T}}\, \overline{C} H u, \tag{3.45}$$

is valid, where

$$u = (|\varphi_1(0)|,\, |\varphi_2(0)|,\, \|\varphi_1\|_{L_2},\, \|\varphi_2\|_{L_2})^{\mathrm{T}}, \quad H = \mathrm{diag}(\eta_1, \eta_2, \eta_1, \eta_2),$$

$$\overline{C} = \begin{pmatrix} \lambda_M(P_{11}) & \|P_{12}\| & 0 & \xi_2 \\ \|P_{12}\| & \lambda_M(P_{22}) & \xi_1 & 0 \\ 0 & \xi_1 & \lambda_M(D_1) & 0 \\ \xi_2 & 0 & 0 & \lambda_M(D_2) \end{pmatrix},$$

$$\underline{C} = \begin{pmatrix} \lambda_m(P_{11}) & -\|P_{12}\| & 0 & -\xi_2 \\ -\|P_{12}\| & \lambda_m(P_{22}) & -\xi_1 & 0 \\ 0 & -\xi_1 & \frac{1}{2}\lambda_m(D_1) & 0 \\ -\xi_2 & 0 & 0 & \frac{1}{2}\lambda_m(D_2) \end{pmatrix}.$$

In view of estimates (3.36) and (3.43) in the region of values $R^n \times \mathcal{C}^n$ it is easy to find the estimate of the upper right derivative number of functional (3.44) along solutions of system (3.37)

$$D^+ v(\varphi_1, \varphi_2, \eta)\big|_{(3.37)} \leq u_1^{\mathrm{T}}\Omega_1 w_1 + u_2^{\mathrm{T}}\Omega_2 w_2, \tag{3.46}$$

and $\Omega_1 = [\omega^1_{ij}]^3_{i,j=1}$, $\Omega_2 = [\omega^2_{ij}]^3_{i,j=1}$ are constant matrices with the elements

$$\omega^1_{11} = \lambda_M(A^{\mathrm T}_{11}P_{11} + P_{11}A_{11} + D_1)\eta_1^2 + \eta_1\eta_2\lambda_M(P_{12}A_{21} + A^{\mathrm T}_{21}P^{\mathrm T}_{12}),$$
$$\omega^1_{22} = -\frac{1}{2}\lambda_m(D_1)\eta_1^2, \quad \omega^1_{33} = -\frac{1}{2r}\lambda_m(D_1)\eta_1^2,$$
$$\omega^1_{12} = \|P_{12}\|\,\|B_{21}\|\eta_1\eta_2, \quad \omega^1_{23} = \xi_1\|B_{21}\|\eta_1\eta_2, \quad \omega^1_{13} = \xi_1\|A_{21}\|\eta_1\eta_1,$$
$$\omega^2_{11} = \lambda_M(A^{\mathrm T}_{22}P_{22} + P_{22}A_{22} + D_2)\eta_2^2 + \eta_1\eta_2\lambda_M(P_{21}A_{12} + A^{\mathrm T}_{12}P^{\mathrm T}_{21}),$$
$$\omega^2_{22} = -\frac{1}{2}\lambda_m(D_2)\eta_2^2, \quad \omega^2_{33} = -\frac{1}{2r}\lambda_m(D_2)\eta_2^2,$$
$$\omega^2_{12} = \|P_{12}\|\,\|B_{12}\|\eta_1\eta_2, \quad \omega^2_{23} = \xi_2\|B_{12}\|\eta_1\eta_2, \quad \omega^2_{13} = \xi_2\|A_{12}\|\eta_1\eta_2,$$
$$\omega^1_{ij} = \omega^1_{ji}, \quad \omega^2_{ij} = \omega^2_{ji}, \quad i,j = 1,2,3, \quad i \neq j.$$

Under some restrictions on the sign-definiteness of matrices $\overline{C}$, $\underline{C}$, and Σ_1, Σ_2 the constructed functional is the Liapunov-Krasovskii functional and applying this functional in Section 4 we shall establish new sufficient conditions for asymptotic stability of the equilibrium state $x = 0$ of quasilinear system. For system (3.37) the proposed method of constructing matrix-valued functional is more efficient, since system of equations (3.38) – (3.39) is integrable in the explicit form. By means of functional $v(\varphi_1, \varphi_2)$ in Section 4 we shall establish sufficient stability conditions for the equilibrium state of system (3.37).

4 Stability Analysis of Quasilinear Delay Systems

We consider an autonomous quasilinear delay system (2.4) with decomposition

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r) + f_1(x, x_t),\\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r) + f_2(x, x_t),\end{aligned} \tag{4.1}$$

where $x_i \in R^{n_i}$, $i = 1,2$, $x = (x_1^{\mathrm T}, x_2^{\mathrm T})^{\mathrm T}$, A_{ij} and B_{ij} are constant matrices of appropriate dimensions.

We make the following assumptions on the functions $f_i(x, x_t)$, $i = 1,2$.

Assumption 1 *The functions $f_i(x, x_t)$, $i = 1,2$, satisfy the following conditions*

(1) *the functions $f_i \in C(R^n \times \mathcal{C}^n, R^n)$ for $i = 1,2$;*

(2) *the functions $f_i(0,0) = 0$ iff $x = x_t = 0$;*

(3) *there exist constants $c_{ij}, l_{ij} > 0$, $i,j = 1,2$, such that*

$$|f_i(x, x_t)| \le c_{i1}|x_1(0)| + c_{i2}|x_2(0)| + l_{i1}\|x_1\|_{L_2} + l_{i2}\|x_2\|_{L_2},$$

where $|\cdot|$ is an Euclidean norm in R^{n_i}, $\|\cdot\|_{L_2}$ is the L_2-norm.

For the linear system

$$\begin{aligned}\frac{dx_1}{dt} &= A_{11}x_1(t) + A_{12}x_2(t) + B_{11}x_1(t-r) + B_{12}x_2(t-r),\\ \frac{dx_2}{dt} &= A_{21}x_1(t) + A_{22}x_2(t) + B_{21}x_1(t-r) + B_{22}x_2(t-r),\end{aligned} \tag{4.2}$$

using the results of Section 3 we construct the matrix-valued functional

$$U:\ R^n\times\mathcal{C}^n\to R^{2\times 2}$$

with the elements (3.23), (3.24) and (3.29).

Applying functional $U(\varphi_1,\varphi_2)$ one can establish sufficient stability conditions for solution $x=0$ of system (4.1). First, we introduce the designations

$$\Delta\Sigma_1=\begin{pmatrix}2c_{11}\|P_{11}\|+c_{21}\|P_{12}\| & 0 & \|P_{11}\|l_{11}+\dfrac{1}{2}\|P_{12}\|l_{21}\\ 0 & 0 & 0\\ \|P_{11}\|l_{11}+\dfrac{1}{2}\|P_{12}\|l_{21} & 0 & 0\end{pmatrix},$$

$$\Delta\Sigma_2=\begin{pmatrix}2c_{22}\|P_{22}\|+c_{12}\|P_{12}\| & 0 & \|P_{22}\|l_{22}+\dfrac{1}{2}\|P_{12}\|l_{12}\\ 0 & 0 & 0\\ \|P_{22}\|l_{22}+\dfrac{1}{2}\|P_{12}\|l_{12} & 0 & 0\end{pmatrix},$$

$$\Delta\Sigma_{12}=\begin{pmatrix}2c_{12}\|P_{11}\|+2c_{21}\|P_{22}\|+c_{22}\|P_{12}\|+c_{11}\|P_{12}\| & 0 & 2\|P_{22}\|l_{21}+\|P_{12}\|l_{12}\\ 0 & 0 & 0\\ 2\|P_{11}\|l_{12}+\|P_{12}\|l_{22} & 0 & 0\end{pmatrix}.$$

Theorem 4.1 *Let system of equations (4.1) be such that*

(1) *there exist solutions of equations (3.25) – (3.27) under initial conditions (3.28) for some* $\eta\in R^2_+,\ \eta>0$*;*

(2) *matrices* $\underline{C}$ *and* $\overline{C}$ *in estimate (3.32) are positive definite;*

(3) *matrices* $\Sigma_1+\Delta\Sigma_1$ *and* $\Sigma_2+\Delta\Sigma_2$ *are negative definite;*

(4) *inequality*

$$\|\Sigma_{12}\|<\lambda_M(\Sigma_1+\Delta\Sigma_1)\lambda_M(\Sigma_2+\Delta\Sigma_2)$$

holds true.

Then the solution $x=0$ *of system (4.1) is uniformly asymptotically stable.*

Proof Condition (2) of Theorem 4.1 ensures the possibility of constructing the "scalar" functional $v\ R^n\times\mathcal{C}^n\times R^2_+\to R_+,\ v(\varphi_1,\varphi_2,\eta)=\eta^{\mathrm T}U(\varphi_1,\varphi_2)\eta,$ satisfying the conditions of definite positiveness and decrease. The upper right derivative number of the functional $v(\varphi_1,\varphi_2,\eta)$ admits the estimate

$$D^+v(\varphi_1,\varphi_2,\eta)\big|_{(4.1)}\le u_1^{\mathrm T}(\Sigma_1+\Delta\Sigma_1)u_1+2u_1^{\mathrm T}\Sigma_{12}u_2+u_2^{\mathrm T}(\Sigma_2+\Delta\Sigma_2)u_2,$$

where

$$u_1=(|\varphi_1(0)|,\,|\varphi_1(-r)|,\,\|\varphi_1\|_{L_2})^{\mathrm T},\quad u_2=(|\varphi_2(0)|,\,|\varphi_2(-r)|,\,\|\varphi_2\|_{L_2})^{\mathrm T}.$$

Conditions (3) and (4) ensure definite negativeness of $D^+v(\varphi_1,\varphi_2,\eta)|_{(4.1)}$. Thus, the solution $x=0$ of system (4.1) is uniformly asymptotically stable and the constructed functional $v(\varphi_1,\varphi_2,\eta)$ is the matrix-valued Liapunov-Krasovskii functional. □

Corollary 4.1 Let system of equations (4.2) be such that

(i) there exist solutions of equations (3.25) – (3.27) under initial conditions (3.28) for some $\eta \in R^2_+,\ \eta > 0$;

(ii) matrices $\underline{C}$ and $\overline{C}$ in estimate (3.32) are positive definite;

(iii) matrices Σ_1 and Σ_2 are negative definite.

Then solution $x = 0$ of system (4.2) is uniformly asymptotically stable.

In the partial case, when $B_{11} = 0$ and $B_{22} = 0$ sufficient conditions of uniform asymptotic stability of solution $x = 0$ are formulated in terms of estimates (3.45) and (3.46) for matrix-valued functional $U(\varphi_1, \varphi_2)$ with the elements

$$
\begin{aligned}
w_{11}(\varphi_1) &= \varphi_1^{\mathrm{T}}(0)P_{11}\varphi_1(0) + \int_{-r}^{0} k(\theta)\varphi_1^{\mathrm{T}}(\theta)D_1\varphi_1(\theta)\,d\,\theta,\\
w_{22}(\varphi_2) &= \varphi_2^{\mathrm{T}}(0)P_{22}\varphi_2(0) + \int_{-r}^{0} k(\theta)\varphi_2^{\mathrm{T}}(\theta)D_2\varphi_2(\theta)\,d\,\theta,\\
w_{12}(\varphi_1, \varphi_2) &= \varphi_1^{\mathrm{T}}(0)P_{12}\varphi_2(0)\\
&\quad + \frac{\eta_1}{\eta_2}\varphi_1^{\mathrm{T}}(0)\int_{-r}^{0} \exp\{A_{11}^{\mathrm{T}}(\theta + r)\}P_{11}B_{12}\varphi_2(\theta)\,d\theta\\
&\quad + \frac{\eta_2}{\eta_1}\varphi_2^{\mathrm{T}}(0)\int_{-r}^{0} B_{21}^{\mathrm{T}}P_{22}\exp\{A_{22}(\theta + r)\}\varphi_2(\theta)\,d\theta.
\end{aligned}
$$

Theorem 4.2 *Let system of equations (4.1) be such that*

(1) $B_{11} = 0,\ B_{22} = 0$*;*

(2) *matrices $\underline{C}$ and $\overline{C}$ in estimates (3.45) are positive definite;*

(3) *matrices $\Omega_1 + \Delta\Sigma_1$ and $\Omega_2 + \Delta\Sigma_2$ from estimate (3.46) are negative definite;*

(4) *inequality*

$$\|\Sigma_{12}\| < \lambda_M(\Omega_1 + \Delta\Sigma_1)\lambda_M(\Omega_2 + \Delta\Sigma_2)$$

holds true.

Then solution $x = 0$ of system (4.1) is uniformly asymptotically stable.

The proof is similar to the proof of Theorem 4.1.

Corollary 4.2 Let system of equations (4.2) be such that

(i) matrices $B_{11} = 0$ and $B_{22} = 0$;

(ii) matrices $\underline{C}$ and $\overline{C}$ in estimate (3.45) are positive definite;

(iii) matrices Ω_1 and Ω_2 in estimate (3.46) are negative definite.

Then solution $x = 0$ of system (4.2) is uniformly asymptotically stable.

5 Application

As applications of results of Section 3.3 we consider oscillations of a harmonic oscillator. The perturbed motion equation of the oscillator is

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \omega^2 x(t) + cx(t-r) = 0, \tag{5.1}$$

where x is a state variable, $\omega,\ c,\ \mu > 0$ are constants. Introduce an auxiliary variable $y = \dfrac{dx}{dt}$ and present equation (5.1) as a system

$$\begin{aligned} \frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -\omega^2 x(t) - \mu y(t) - cx(t-r). \end{aligned} \tag{5.2}$$

Applying the proposed technique of construction of the Liapunov functionals for system (5.2) we construct a scalar functional $w(\varphi_1, \varphi_2)$ as

$$w(\varphi_1, \varphi_2) = v_{11}(\varphi_1) + 2v_{12}(\varphi_1, \varphi_2) + v_{22}(\varphi_2), \tag{5.3}$$

where

$$\begin{aligned} v_{11}(\varphi_1) &= \gamma^2\varphi_1^2(0) + \gamma^2 d_1 \int_{-r}^{0} \Big(1 + \frac{\theta}{2r}\Big)\varphi_1^2(\theta)\, d\theta,\\ v_{22}(\varphi_2) &= \varphi_2^2(0) d_2 \int_{-r}^{0} \Big(1 + \frac{\theta}{2r}\Big)\varphi_2^2(\theta)\, d\theta,\\ v_{12}(\varphi_1,\varphi_2) &= 2\frac{\gamma^2 - \omega^2 - ce^{-\mu r}}{\mu}\,\varphi_1(0)\varphi_2(0) - 2ce^{-\mu r}\varphi_2(0)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\, d\theta, \end{aligned}$$

and γ, d_1 and d_2 are indefinite positive constants.

Functional (5.3) can be estimated from below by means of the Cauchy–Bunyakovsky inequality

$$\begin{aligned} w(\varphi_1,\varphi_2) &\ge \gamma^2\varphi_1^2(0) + \varphi_2^2(0) - 2\frac{\gamma^2 - \omega^2 - ce^{-\mu r}}{\mu}\,|\varphi_1(0)|\,|\varphi_2(0)|\\ &+ \frac{\gamma^2 d_1}{2}\|\varphi_1(\theta)\|_{L_2}^2 + \frac{d_2}{2}\|\varphi_2(\theta)\|_{L_2}^2 - 2|c|e^{-\mu r}\sqrt{\frac{e^{2\mu r}-1}{2\mu}}|\varphi_2(0)|\,\|\varphi_1(\theta)\|_{L_2}. \end{aligned}$$

The derivative of functional (5.3) along solutions of system (5.2) is

$$
\begin{aligned}
D^+w(\varphi_1,\varphi_2)\big|_{(5.2)} &= \Big(d_1\gamma^2 - \frac{2\omega^2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_1^2(0)\\
&- \frac{d_1\gamma^2}{2}\,\varphi_1^2(-r) - \frac{d_1\gamma^2}{2r}\,\|\varphi_1(\theta)\|_{L_2}^2 - \frac{2c(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\,\varphi_1(0)\varphi_1(-r)\\
&+ 2ce^{-\mu r}\omega^2\varphi_1(0)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\,d\theta + 2c^2e^{-\mu r}\varphi_1(-r)\int_{-r}^{0} e^{-\mu\theta}\varphi_1(\theta)\,d\theta\\
&+ \Big(-2\mu + d_2 + \frac{2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_2^2(0) - \frac{d_2}{2}\,\varphi_2^2(-r) - \frac{d_2}{2r}\,\|\varphi_2(\theta)\|_{L_2}^2.
\end{aligned}
\tag{5.4}
$$

The analysis of (5.4) shows that it is reasonable to take constants $\gamma^2 = \omega^2 + ce^{-\mu r} + \dfrac{\mu^2}{2}$, $d_1 = \dfrac{\mu\omega^2}{2\gamma}$ and $d_2 = \dfrac{\mu}{2}$. Applying the Cauchy-Bunyakovsky inequality once again we estimate derivative (5.4) of functional (5.3)

$$
\begin{aligned}
D^+w(\varphi_1,\varphi_2)\big|_{(5.1)} &\le \Big(d_1\gamma^2 - \frac{2\omega^2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_1^2(0)\\
&- \frac{d_1\gamma^2}{2}\,\varphi_1^2(-r) - \frac{d_1\gamma^2}{2r}\,\|\varphi_1(\theta)\|_{L_2}^2 + \frac{2|c|(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\,|\varphi_1(0)||\varphi_1(-r)|\\
&+ 2|c|e^{-\mu r}\omega^2\sqrt{\frac{e^{2\mu r}-1}{2\mu}}|\varphi_1(0)|\,\|\varphi_1(\theta)\|_{L_2}\\
&+ 2c^2e^{-\mu r}\sqrt{\frac{e^{2\mu r}-1}{2\mu}}|\varphi_1(-r)|\,\|\varphi_1(\theta)\|_{L_2}^2\\
&+ \Big(-2\mu + d_2 + \frac{2(\gamma^2-\omega^2-ce^{-\mu r})}{\mu}\Big)\varphi_2^2(0) - \frac{d_2}{2}\,\varphi_2^2(-r) - \frac{d_2}{2r}\,\|\varphi_2(\theta)\|_{L_2}^2.
\end{aligned}
$$

Conditions of positive definiteness of functional $w(\varphi_1,\varphi_2)$ and negative definiteness of functional $D^+w(\varphi_1,\varphi_2)|_{(5.2)}$ yield new conditions of asymptotic stability of zero solution of equation (5.2) in the form of the system of inequalities

$$
|c| < \frac{\mu}{2}\sqrt{\frac{\mu}{r(1-e^{-2\mu r})}}, \quad \omega^2 > |c|\sqrt{\frac{24r(1-e^{-2\mu r})+2\mu^3}{\mu^3-4c^2r(1-e^{-2\mu r})}},
$$
$$
\big(2\omega^2 + 2ce^{-\mu r} + \mu^2\big)\big(\mu^2\omega^2 - 2c^2(1-e^{-2\mu r})\big) \ge \frac{\mu^4\omega^2}{2}.
$$

6 Concluding Remarks

It is of interest to apply the proposed approach for a class of neutral functional differential equations with time-varying delay. In [14] the Liapunov functional $V(x(t))$ is used and a condition for asymptotic stability of zero solution of the system under consideration is established.

Another class of equations being of interest for the application of the method are the logic-dynamical hybrid systems given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay (see [10]).

An urgent direction of applications is the analysis of the robust stability of nonlinear uncertain neural networks with constant or time-varying delay (see [13]) as well as the problem of robust dynamic parameter-dependent output feedback stabilization under H_∞ performance index for a class of linear time-invariant parameter-dependent systems with multi-time delays in the state vector and in the presence of norm-bounded nonlinear uncertainties (see [8]).

References

[1] Burton, T.A. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations.* Orlando, San Diego, etc.: Academic Press, 1985.

[2] Bylov, V.F., Vinograd, R.E. et al. *Lyapunov Characteristic Exponents Theory and Its Applications to Stability Issues.* Moscow: Nauka, 1966. [Russian]

[3] Djordjevic, M.Z. Stability analysis of nonlinear systems by the matrix Liapunov method. *Large Scale Systems* **5** (1983) 252–262.

[4] Hale, J.K. *Theory of Functional Differential Equations.* New York: Wiley, 1977.

[5] Hara, T. and Sugie, J. Stability region for systems of differential difference equations. *Funkcialaj Ekvacioj* **39** (1996) 69–86.

[6] Huang Wenzhang. Generalization of Liapunov's theorem in a linear delay system. *J. Math. Anal. Appl.* **142** (1989) 83–94.

[7] Infante, E.F. and Castelan, W.B. A Liapunov functional for a matrix difference-differential equation. *J. Differ. Eqns* **29** (1978) 439–451.

[8] Karimi, H.R. Robust dynamic parameter-dependent output feedback control of uncertain parameter-dependent state-delayed systems. *Nonlinear Dynamics ans Syst. Theory* **6**(2) (2006) 143–158.

[9] Krasovskii, N.N. *Stability of Motion.* Stanford, California: Stanford Univ. Press, 1963.

[10] Khusainov, D., Langerak, R. and Kuzmych, O. Estimations of solutions convergence of hybrid systems consisting of linear equations with delay. *Nonlinear Dynamics ans Syst. Theory* **7**(2) (2007) 169–186.

[11] Lankaster, P. *Matrix Theory.* Moscow: Nauka, 1978. [Russian]

[12] Liapunov, A.M. *General Problem of Stability of Motion.* Math. Soc., Harkov, 1892. [Russian]

[13] Lou, X.Y. and Cui, B.T. Robust stability for nonlinear uncertain neural networks with delay. *Nonlinear Dynamics ans Syst. Theory* **7**(2) (2007) 369–378.

[14] Lou, X.Y. and Cui, B.T. Output feedback passive control of neutral systems with time-varying delays in state and control input. *Nonlinear Dynamics ans Syst. Theory* **8**(2) (2008) 195–204.

[15] Martynyuk, A.A. *Stability by Liapunov's Matrix Functions Method with Aplications.* New York: Marcel Dekker, 1998.

[16] Martynyuk, A.A. Matrix-valued functional approach for stability analysis of functional differential equations. *Nonlin. Analysis* **56** (2004) 793–802.

[17] Martynyuk, A.A., Lakshmikantham, V. and Leela, S. *Stability of Motion: The Method of Integral Inequalities.* Kiev: Naukova Dumka, 1989. [Russian]

[18] Martynyuk, A.A. and Slyn'ko, V.I. Stability results for large-scale difference systems via matrix-valued liapunov functions. *Nonlinear Dynamics ans Syst. Theory* **7**(2) (2007) 217–224.

[19] Sakata, S. Asymptotic stability for a linear system of differential difference equations. *Funkcialaj Ekvacioj* **41** (1998) 435–449.