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Neutral Functional Equations with Causal Operators on a Semi-Axis

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Abstract: This paper is concerned with the global study of a certain class of functional differential equation involving causal (abstract Volterra) operators on a certain function space $\mathrm{E}(R_+, R^n)$. It is closely related to our previous joint papers, listed in the References, the difference being motivated by the fact that we consider new function spaces on the half-axis R_+. The approach in this paper is also somewhat different than in preceding papers, by C. Corduneanu, the results being also different. A dynamical interpretation is also indicated.

Keywords: *Neutral functional equations; causal operators; global existence.*

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1 Statement of the Problem

Let us consider the functional differential equation

$$\frac{d}{dt}\left[\frac{dx(t)}{dt} - (Lx)(t)\right] = (Vx)(t), \quad t \in R_+, \tag{1}$$

where $x \in R^n$, $n \geq 1$ is an integer, and L, V are causal operators acting on the function space $C(R_+, R^n)$, consisting of all continuous maps from R_+ into R^n, the topology/convergence being defined by the family of semi-norms $\{|x|_k : \ k \geq 1\}$, with

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$|x_k| = \sup\{|x(t)|:\ 0 \le t \le k\}$, $k \ge 1$. By $|\cdot|$ we denote the Euclidean norm of the space R^n. As it was mentioned, our approach here is somewhat different than in [4, 5, 6].

It is well known (see, for instance [2, 3]) that the above topology/convergence means uniform convergence on any bounded interval $[0,T] \subseteq R_+$.

An initial condition of the form

$$x(0) = x^0 \in R^n, \quad \dot{x}(0) = v^0 \in R^n, \tag{2}$$

must be associated with (1), if we expect to get a unique solution to the Cauchy problem (1), (2).

Unlike our preceding papers, [8, 9, 12, 13], we will be interested here in finding global solutions to the problem (1), (2), i.e. belonging to the space $C(R_+, R^n)$.

The space $C(R_+, R^n)$ contains as subspaces (closed or not) many usual spaces appearing in the theory of differential or integral equations. An example is the space $BC(R_+, R^n)$ consisting of all bounded continuous maps from R_+ into R^n with the norm

$$\|x\| = \sup\{|x(t)|:\ t \in R_+\}, \tag{3}$$

$BC(R_+, R^n)$ is a Banach space. Other function spaces are encountered in the literature, such as the space $C_\ell(R_+, R^n)$, consisting of all maps in $BC(R_+, R^n)$ such that $\lim x(t)$, as $t \to \infty$, exists in R^n. The space $C_\ell(R_+, R^n)$ is a closed subspace of $BC(R_+, R^n)$, but $BC(R_+, R^n)$ is not closed in $C(R_+, R^n)$.

The main concern of this paper is finding adequate conditions on the data, more precisely on the operators L and V, such that the existence of solution to the problem (1), (2) is guaranteed (on the semi-axis R_+).

If we succeed to prove the existence of a solution to (1), (2), then we can draw some conclusions about its asymptotic behavior if we can show that it belongs to one of the function spaces we have mentioned above (like $BC(R_+, R^n)$, $C_\ell(R_+, R^n)$, or other function spaces).

2 An Auxiliary Result

We shall briefly discuss in this section a connection between causal operators on function spaces and classical/integral operators of Volterra type. For a more detailed discussion and further references, we send the reader to [1; Ch. 2 and Ch. 4], with more references.

The result we need is simply expressed by the formula

$$\int_0^t (Vx)(s)\,ds = \int_0^t K(t,s)x(s)\,ds, \quad t \in R_+, \tag{4}$$

where V stands for a linear causal operator on the space $C(R_+, R^n)$, and $K(t,s)$ denotes a matrix kernel of type $n \times n$.

More precisely, the result described by (4), states that for each V, there exists a measurable kernel $K(t,s)$, rendering the service described by (4). The formula (4) holds true for every $x \in C(R_+, R^n)$, and even in the most general case, $x \in L_{loc}(R_+, R^n)$. We notice that $K(t,s)$ must not be continuous. In order to assure the inclusion

$$\int_0^t K(t,s)x(s)\,ds \in C(R_+, R^n), \tag{5}$$

for each $x \in C(R_+, R^n)$, it suffices to deal with a locally integrable $K(t,s)$, $(t,s) \in \Delta = \{(t,s)\colon\ 0 \le s \le t\}$ satisfying also the condition

$$\lim_{h\to 0}\left(\int_0^t |K(t+h,s)-K(t,s)|\,ds + \int_t^{t+h} |K(t+h,s)|\,ds\right) = 0 \tag{6}$$

for each $t \in R_+$.

Let us notice that condition (6) on $K(t,s)$ is implied by (4), and the fact that V is a linear operator acting on $C(R_+, R^n)$. Hence, it takes continuous maps into continuous ones.

The kernel $K(t,s)$ from (4) automatically verifies other properties, if it does satisfy extra conditions. For instance, one frequently encountered property is

$$\sup\left\{\int_0^t |K(t,s)|\,ds\colon\quad t \in R_+\right\} < \infty, \tag{7}$$

is implied by the requirement that the subspace $BC(R_+, R^n)$ must be left invariant by the operator V.

In general, the connection between the operator V, and the properties of the kernel $K(t,s)$, is not always easy to be established. The problem of clarifying such connections is of great significance for applications.

For instance, an open problem is to establish the conditions on V, such that the associated kernel $K(t,s)$ admits a resolvent kernel $\widetilde{K}(t,s)$. See [3] for some discussion in this regard.

We shall formulate conditions for V, by means of the relationship (4). In other words, by imposing the adequate conditions on the associated kernel $K(t,s)$. In special situations, when V is chosen in a classical form, the connection may appear more transparent.

3 Equation (1) and Its Equivalent Forms

Let us return to equation (1) and make a few remarks that will help simplifying the coming considerations.

First, it is obvious that an additive constant to the operator L, i.e. a constant n-vector, does not change the equation (1). Hence, without loss of generality, we can assume

$$(Lx)(0) = \theta \in R^n, \tag{8}$$

for any x in the space $C(R_+, R^n)$. This property imposed on the causal operator L has been called by L. Neustadt [15], the “fixed initial value” property, and its significance in dealing with existence problems for (1) has been illustrated.

In case of classical Volterra operator

$$(Vx)(t) = f(t) + \int_0^t K(t,s,x(s))\,ds,$$

one obtains $(Vx)(0) = f(0) = \text{const}$, for any $x \in C(R_+, R^n)$, or in another underlying space.

As mentioned above we can substitute θ in (8) by any constant $c \in R^n$ without changing the equation.

By integrating both sides of (1) from 0 to $t > 0$ we obtain the functional differential equation

$$\dot{x}(t) - (Lx)(t) = v^0 + \int_0^t (Vx)(s)\,ds, \tag{9}$$

if we take into account (2) and (8).

The following equation is related to the equation (9),

$$\dot{x}(t) - (Lx)(t) = f(t), \quad t \in R_+. \tag{10}$$

As we know from [3, 7], the Cauchy problem for (10) with $x(0) = x^0 \in R^n$ can be represented in case of linear and continuous operator L on $C(R_+, R^n)$, by an integral formula, involving the Cauchy operator associated to L. It is sort of variation of parameters formula (Lagrange), and it looks

$$x(t) = X(t,0)\,x^0 + \int_0^t X(t,s)\,f(s)\,ds, \quad t \in R_+, \tag{11}$$

for any $f \in C(R_+, R^n)$. The Cauchy function (or kernel) $X(t,s)$ is defined by the formula

$$X(t,s) = I + \int_s^t \widetilde{K}_0(t,u)\,du, \tag{12}$$

for each $(t,s) \in \Delta$,

$$\Delta = \{(t,s)\colon \quad 0 \le s \le t\}, \tag{13}$$

where $\widetilde{K}_0(t,s)$ is the resolvent kernel corresponding to $K_0(t,s)$, $(t,s) \in \Delta$, from the representation

$$\int_0^t (Lx)(s)\,ds = \int_0^t K_0(t,s)\,x(s)\,ds, \quad t \in R_+. \tag{14}$$

In (14), $K_0(t,s)$ is measurable only, but the existence of $\widetilde{K}_0(t,s)$ is assured, for instance, if we accept the condition

$$K_0(t,s) \in L^\infty_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{15}$$

Let us apply formula (11) to the equation (9). We are led to the functional equation

$$x(t) = X(t,0)\,x^0 + \int_0^t X(t,s)\,v^0 ds + \int_0^t X(t,s)\int_0^s (Vx)(u)\,du\,ds. \tag{16}$$

Since we already assumed L to be linear, there results that (16), which is an equivalent equation to the problem (1), (2), is also linear when V is linear. Otherwise, it is a nonlinear functional equation for $x(t)$.

In case V is a linear operator acting on $C(R_+, R^n)$, we can use the representation (4), and (16) leads to another equivalent form of the problem (1), (2):

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0\,ds + \int_0^t X(t,s)\int_0^s K(s,u)x(u)\,du\,ds.$$

Interchanging the order of integration in the double integral, we obtain the functional differential equation

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0\,ds + \int_0^t \left(\int_u^t X(t,s)K(s,u)\,ds\right) x(u)\,du,$$

which can be rewritten as

$$x(t) = f(t) + \int_0^t K_1(t,u)x(u)\,du, \quad t \in R_+, \tag{17}$$

with

$$f(t) = X(t,0)\,x^0 + \int_0^t X(t,s)v^0\,ds, \quad t \in R_+, \tag{18}$$

and

$$K_1(t,u) = \int_u^t X(t,s)K(s,u)\,du, \quad 0 \le u \le t. \tag{19}$$

The equation (17) will be discussed in detail in the next section, and the existence result will be applied to the problem (1), (2).

Another case will be considered, when the operator V is not necessarily linear, but it is Lipschitz continuous.

4 The Linear Case: Equation (17)

We have to examine the function (18) and the kernel $K_1(t,u)$ to see if we can construct a solution in $C(R_+, R^n)$.

First, the function $f(t)$ from (18) is the solution of the functional differential equation $\dot{x}(t) - (Lx)(t) = v^0$, with the initial condition $x(0) = x^0$. Hence, $f(t)$ is a continuously differentiable function on R_+, with values in R^n.

Second, the kernel $K_1(t,u)$ given by (19) is a locally bounded function on Δ. More precisely, we can infer

$$K_1(t,s) \in L^\infty_{loc}(\Delta,\, \mathcal{L}(R^n, R^n)). \tag{20}$$

Indeed, we shall admit that $K(t,s)$ belongs to L^∞_{loc}, as mentioned above. Furthermore, formula (12) shows us that $X(t,s)$ is also locally bounded on Δ. This fact is a consequence of property that states: any kernel $K(t,s)$, which is locally bounded on Δ, admits a resolvent $\widetilde{K}(t,s)$, which is also locally bounded on Δ.

Therefore, the integral equation (17), whose kernel $K_1(t,s)$ satisfies (20), admits a resolvent kernel $\widetilde{K}_1(t,s)$, locally bounded on Δ, while its unique solution is represented by the resolvent formula

$$x(t) = f(t) + \int_0^t \widetilde{K}_1(t,s)\,f(s)\,ds, \quad t \in R_+, \tag{21}$$

for any function $f \in L^\infty_{loc}(R_+, R^n)$. In particular, (21) holds true when $f(t)$ is given by the formula (18).

In summarizing the discussion carried out above, we can state the following existence result for the problem (1), (2).

Theorem 4.1 *Consider the neutral functional differential equation (1), with the initial conditions (2). Assume the following conditions are satisfied:*

(i) *the operators L and V are linear, continuous and causal, acting on the space $C(R_+, R^n)$;*

(ii) *the kernels $K(t,s)$ and $K_0(t,s)$, occuring in the representations (4) and (14) are locally bounded on Δ, defined by (13).*

Then, there exists a unique solution $x(t)$, $t \in R_+$, of the problem (1), (2), for arbitrary initial data $x^0, v^0 \in R^n$. This solution is continuously differentiable on R_+.

The proof is immediate if we rely on the discussion preceding the statement of Theorem 1, the equivalence of (1), (2) with the equation (17) being the key ingredient.

Remark 4.1 The condition (6) is not the only condition that can be derived from the fact that V is acting on $C(R_+, R^n)$.

Indeed, from (6) we read

$$\int_0^t K(t,s)x(s)\,ds \in AC_{loc}(R_+, R^n), \tag{22}$$

for every $x \in C(R_+, R^n)$. The space AC_{loc} in (22) is a subspace of $C(R_+, R^n)$, and the inclusion

$$\int_0^t K(t,s)x(s)\,ds \in C(R_+, R^n), \quad t \in R_+, \tag{23}$$

tells us less than (22). Nevertheless, we prefer to use (23) instead of (22) for simplicity. For example, (23) implies

$$\lim_{h \to 0} \int_0^t |K(t+h,s) - K(t,s)| ds = 0 \quad t \in R_+. \tag{24}$$

Remark 4.2 Considering the resolvent formula (21) for the solution of equation (17), we can obtain more information about the solution of (1), (2), making extra assumptions on the kernel $\widetilde{K}_1(t,s)$. This kernel is determined, as shown by (19), by the properties of the operators L and V.

For instance, if we assume that $\widetilde{K}_1(t,s)$ satisfies the condition

$$\int_0^t |\widetilde{K}_1(t,s)|\,ds \le M < \infty, \quad t \in R_+, \tag{25}$$

and also

$$|X(t,0)| + \int_0^t |X(t,s)|\,ds \le N < \infty, \quad t \in R_+, \tag{26}$$

then the solution of the problem (1), (2) will verify the inclusion

$$x(t) \in L^\infty(R_+, R^n), \quad x^0, v^0 \in R^n. \tag{27}$$

The Proof follows immediately from the formulas (18) and (21).

Further properties of the solution can be obtained by imposing various types of estimates on the kernels $K_1(t,s)$ or $\widetilde{K}_1(t,s)$, as well as on $f(t)$.

The main problem is to establish the connection between the properties of the operators L and V, and the kernels occurring in the representations (4) and (14). This will be discussed in forthcoming papers.

5 A Nonlinear Case: Equation (16)

We shall rewrite equation (16) in the form

$$x(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx)(u)\,du\,ds, \quad t \in R_+, \tag{28}$$

where $f(t)$ is given by (18). Equation (28), with $f(t)$ defined by (18), is equivalent to our problem (1), (2). This is a functional integral equation and we shall treat it by the classical method of iteration/successive approximations. This approach will lead to an existence and uniqueness result in the space $C(R_+, R^n)$. Of course, the operator V is assumed to be acting on this space.

In order to simplify somewhat the procedure, we shall adopt a hypothesis which is part of the assumption (25) above. This hypothesis concerns only the linear operator L, which fully determines the Cauchy kernel $X(t,s)$, $0 \le s \le t$.

Namely, we assume in this section that

$$\int_0^t |X(t,s)|\,ds \le M < \infty, \quad (t,s) \in \Delta, \tag{29}$$

and we shall limit our consideration in regard to equation (28), only to those operators L for which (29) is satisfied.

Concerning the operator V, acting on the same space $C(R_+, R^n)$, we shall assume it verifies the Lipschitz type condition

$$|(Vx)(t) - (Vy)(t)| \le \lambda(t)\,|x(t) - y(t)|, \quad t \in R_+, \tag{30}$$

for any $x, y \in C(R_+, R^n)$. We also assume that $\lambda(t)$ is a nonnegative nondecreasing map from R_+ into itself.

In order to prove the existence and uniqueness of a solution to (28), we construct the sequence of successive approximations $\{x_k(t)\colon\ k \ge 0\}$, by letting $x_0(t) = f(t)$, and

$$x_{k+1}(t) = f(t) + \int_0^t X(t,s) \int_0^s (Vx_k(u))\,du\,ds, \tag{31}$$

for $k \ge 1,\ t \in R_+$.

We shall prove now that the sequence of successive approximations converges in $C(R_+, R^n)$. This means that the sequence converges uniformly on each bounded interval $[0,T] \subseteq R_+$. The limit of this sequence

$$x(t) = \lim_{k \to \infty} x_k(t), \quad t \in R_+, \tag{32}$$

will constitute the solution of our problem. As usual, if we subtract side by side the relationship (31) and the one corresponding to k instead of $k+1$, we find the following recurrent relation, valid for $k \ge 1$ and $t \in R_+$:

$$x_{k+1} - x_k(t) = \int_0^t X(t,s) \int_0^s [(Vx_k)(u) - (Vx_{k-1})(u)]\,du\,ds. \tag{33}$$

Taking into account (29) and (30), we obtain from (33) the following recurrent inequality:

$$|x_{k+1}(t) - x_k(t)| \le M \sup_{0 \le s \le t} \int_0^s \lambda(u)|x_k(u) - x_{k-1}(u)|\,du.$$

The above inequality leads immediately to

$$|x_{k+1}(t) - x_k(t)| \leq M \int_0^t \lambda(s) \sup_{0\leq u\leq s} |x_k(u) - x_{k-1}(u)|\, du. \tag{34}$$

Let us denote

$$y_k(t) = \sup_{0\leq s\leq t} |x_k(s) - x_{k-1}(s)|, \tag{35}$$

and rewrite (34) in the form

$$y_k(t) \leq M \int_0^t \lambda(s) y_k(s)\, ds, \quad k \geq 1. \tag{36}$$

We had to keep in mind the fact that $\sup\{|x_k(t) - x_{k-1}(t)|\colon\ t \in [0,T]\}$ is nondecreasing in T.

Now by induction, the recurrent inequality (36) leads to, if one assumes $y_1(t) \leq A$ on the interval $[0,T]$, $T > 0$ arbitrary,

$$y_{k+1}(t) \leq A\, \frac{M^k}{k!} \left(\int_0^t \lambda(u)\, du \right)^k, \quad t \in [0,T]. \tag{37}$$

The inequality (37) obviously implies the uniform convergence of the sequence of successive approximations, on any finite interval of R_+. Therefore, we have

$$\lim_{k\to\infty} x_k(t) = x(t) \in C(R_+, R^n). \tag{38}$$

The function $x(t)$ defined by (38) is a solution of equation (17), and this equation is equivalent to the problem (1), (2). While (38) shows that $x(t)$ is a continuous solution, it has actually better regularity properties, as stipulated in section 3.

Let us now formulate the main result of this section, related to our basic problem (1), (2).

Theorem 5.1 *Consider the initial value problem (1), (2), or equivalently the functional integral equation (17), under the following assumptions:*

(i) *The operator L is a linear continuous operator on the space $C(R_+, R^n)$.*

(ii) *The operator V is also acting on the space $C(R_+, R^n)$, and verifies the Lipschitz condition (30), with $\lambda(t)$ nondecreasing on R_+.*

Then, there exists a unique solution $x(t) \in C(R_+, R^n)$, of (1), (2), or (17), and it is continuously differentiable on R_+, as well as $\dot{x}(t) - (Lx)(t)$.

Remark 5.1 The uniqueness is proven in the same way we have shown the convergence of the successive approximations, and using estimates like (37).

Remark 5.2 The function $f(t)$ given by (18) is continuously differentiable. It would suffice to be just continuous.

Once we know the solution of our problem does exist, we can think of obtaining further properties, of asymptotic nature.

Namely, we shall drop the assumption (25) on the resolvent kernel $\widetilde{K}_1(t,s)$, and impose other conditions that can be verified more directly, on the operator V. These conditions are

$$(V\theta)(t) \equiv 0, \quad \lambda(t) \in L^1(R_+, R), \tag{39}$$

and they will help us to recognize the property of boundedness for the solution $x(t) \in BC(R_+, R^n)$.

Indeed, we see that the conditions of Theorem 5.1 are verified if we accept (39) and the Lipschitz condition with $\lambda(t)$ instead of the Lipschitz constant. But if we define $\tilde{\lambda}(t) = \sup\{\lambda(s) : \ 0 \le s \le t\}$, we find a function providing as $\lambda(t)$ does. Obviously, (39) implies $|(Vx)(t)| \le \lambda(t)|x(t)| \le \tilde{\lambda}(t)|x(t)|$. Hence, on behalf of Theorem 5.1 we have assured the existence and uniqueness of the solution.

We shall prove that the solution is actually in $BC(R_+, R^n)$, if we make the extra assumption (26) on $X(t,s)$.

From (26), (28) and (29), we obtain the integral inequality

$$|x(t)| \le N + M \int_0^t |(Vx)(s)|\, ds, \quad t \in R_+, \tag{40}$$

$x(t)$ being the solution in $C(R_+, R^n)$ for our problem. The inequality (40) and Lipschitz condition imply

$$|x(t)| \le N + M \int_0^t \lambda(s)\, |x(s)|\, ds, \quad t \in R_+. \tag{41}$$

The inequality (41) is of Gronwall type, and yields

$$|x(t)| \le M \exp\left(M \int_0^\infty \lambda(s)\, ds \right), \quad t \in R_+,$$

which shows that $x(t) \in BC(R_+, R^n)$, as it follows from second condition (39).

6 Some Final Remarks

As seen above, the existence of the resolvent kernel is very helpful. Besides the case $K_0(t,s) \in L^\infty_{loc}(\Delta, \mathcal{L}(R^n, R^n))$, the resolvent kernel does exist in other cases. For instance, when $K_0(t,s) \in L^2_{loc}(\Delta, \mathcal{L}(R^n, R^n))$. A parallel investigation of problem (1), (2) could be conducted in this case. For the general framework see [16].

Several problems of asymptotic behavior of solutions can be treated, with extra assumptions, in the framework used in this paper. For instance, looking for solutions in the space $C_\ell(R_+, R^n)$.

We shall close these remarks with a dynamical interpretation of equation (28), which is equivalent to our problem (1), (2), when the function $f(t)$ is chosen in the form (18).

Equation (17) describes the working of a feedback dynamical system, in which the linear plant is described by the equation

$$x(t) = f(t) + \int_0^t X(t,u) c(u)\, du, \tag{42}$$

where $c(u)$ represents the input, $c(u) \in E(R_+, R^n)$, while the feedback action is given by

$$c(t) = \int_0^t (Vx)(s)\, ds, \quad t \in R_+. \tag{43}$$

The information about the asymptotic behavior of the solution of (28) can be directly related to the motion of the dynamical system. The mathematical and engineering literatures are providing numerous applications of such systems. Some examples and further discussion, with ample references to literature, can be found in [1, 2, 10, 11, 14].

References

[1] Azbelev, N.V., Maksimov, V.P. and Rakhmatulina, L.F. *Introduction to the Theory of Linear Functional Differential Equations*. Atlanta, World Federation Publishers, 1996.

[2] Corduneanu, C. *Integral Equations and Stability of Feedback Systems*. New York, Academic Press, 1973.

[3] Corduneanu, C. *Integral Equations and Applications*. Cambridge, Cambridge University Press, 1991.

[4] Corduneanu, C. Existence of solutions for neutral functional differential equations with causal operators. *J. Diff Eqns* **168** (2000) 93–101.

[5] Corduneanu, C. Abstract Volterra equations: a survey. *Mathematical and Computer Modeling* **32** (2000) 1503–1528.

[6] Corduneanu, C. Second order functional equations of neutral type. *Dynamic Systems and Applications* **14** (2005) 83–89.

[7] Corduneanu, C. *Functional Equations with Causal Operators*. London and New York, Taylor and Francis, 2002.

[8] Corduneanu, C. and Mahdavi, M. On neutral functional differential equations with causal operators. *Systems Science and Its Applications*. Tianjin People's Publishing House, Tianjin, 1998, 43–48.

[9] Corduneanu, C. and Mahdavi, M. A class of second order functional differential equations of neutral type. *Mathematical Reports* **5**(55) (2003) 293–299.

[10] Hale, J.K. *Functional Differential Equations*. New York, Springer-Verlag, 1977.

[11] Lakshmikantham, V. and Rao, M.R.M. *Theory of Integro-Differential Equations*. Switzerland, Gordon and Breach, 1995.

[12] Mahdavi, M. Linear functional differential equations with abstract Volterra operators. *Differential and Integral Eqns* **8**(6) (1995) 1517–1523.

[13] Mahdavi, M. and Yizeng, L. Linear and quasilinear equations with abstract Volterra operators. In: *Volterra Equations and Applications* (C. Corduneanu and I.W. Sandberg, eds.). Gordon and Breach, Reading, U.K., 2000, 325–330.

[14] Miller, R. K. *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park, CA, 1971.

[15] Neustadt, L. *Optimization: A Theory of Necessary Conditions*. Princeton University Press, Princeton, 1976.

[16] Tricomi, F.G. *Integral Equations*. Interscience Publishers, New York, 1957.