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Approximate Constraint-Following of Mechanical Systems under Uncertainty

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Abstract: We consider a mechanical system, which is required to obey a set of constraints. The system may contain uncertainty, which is possibly fast time-varying. We propose a robust control scheme that is motivated by the Nature's strategy. The control also takes into account the uncertainty for guaranteeing approximate constraint following.

Keywords: *Mechanical system; constraint; motion control; robust control.*

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1 Introduction

For a mechanical system to be confined to a set of constraints, constraint forces are needed. Out of many possible forms for such forces, in Lagrangean mechanics, it is postulated that the constraint forces should be governed by the Lagrange's form of d'Alembert's principle. In a sense this is what Joseph-Louis Lagrange asserted the *Nature* would do ([1]).

In the past, the majority of the efforts in constrained mechanical systems can be divided into two categories: the *passive constraint problem* and the *servo constraint problem*. In the passive constraint problem, the main focus is to investigate what the *Nature* will do in order to assure that the constraints are (strictly) obeyed. These include, for example, the *Maggi equation* [2,3], the *Boltzmann and Hamel equation* [3, 4], the *Gibbs*

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and Appell equation [4–6], and the *Udwadia and Kalaba equation* [7]. A more complete list can be found in [1].

In the servo constraint problem, on the other hand, the main focus is to find what the *engineer* should do, so that the constraints are followed. A survey on the use of geometric and algebraic approaches can be found in [8–12]. Most of the emphasis is on precise model-based control design. Prominent contributions which deal with uncertainty can be found in, e.g., [13–16] and their bibliographies.

In these two categories, the main differences are twofold. First, the *Nature* (i.e., the environment or the structure) is assumed to possess all information, be it the system or the surrounding. Therefore there is no uncertainty. The engineer, however, is always limited to a rather confined domain of knowledge. As a result, uncertainty tends to be inevitable in many applications.

Second, the *Nature* is only contended with a strict performance. That is, the constraints are to be precisely followed. The engineer, taking a more pragmatic viewpoint, can be settled with approximate constraint following.

This paper falls into the second category. The features of the current approach are fourfold. First, no state transformation is needed. One can choose any coordinate system to represent the system and design the control. Second, the uncertainty considered can be (possibly fast) time-varying. Third, no Lagrange multiplier is needed for control formulation; hence no force feedback. Fourth, no initial condition restrictions are imposed. The starting configuration of the mechanical system can be far away from the desired constraint.

2 Mechanical System Subject to Constraints

Consider the following mechanical system:

$$M(q(t),\sigma(t),t)\ddot{q}(t)+C(q(t),\dot{q}(t),\sigma(t),t)\dot{q}(t)+g(q(t),\sigma(t),t)=\tau(t). \tag{2.1}$$

Here $t\in R$ is the independent variable, $q\in R^n$ is the coordinate, $\dot{q}\in R^n$ is the velocity, $\ddot{q}\in R^n$ is the acceleration, $\sigma\in\Sigma\subset R^p$ is the uncertain parameter, and $\tau\in R^n$ is the control input. Furthermore, $M(q,\sigma,t)$ is the inertia matrix, $C(q,\dot{q},\sigma,t)\dot{q}$ is the Coriolis/centrifugal force, and $g(q,\sigma(t),t)$ is the gravitational force. The matrices/vector $M(q,\sigma,t)$, $C(q,\dot{q},\sigma,t)$, and $g(q,\sigma,t)$ are of appropriate dimensions. We assume that the functions $M(\cdot)$, $C(\cdot)$, and $g(\cdot)$ are continuous (this can be generalized to be Lebesgue measurable in t). In addition, the bounding set Σ is prescribed and compact.

Remark 2.1 The coordinate q can be selected based on the specifics of the problem and does not need to be the *generalized coordinate* [17].

The following constraints are proposed:

$$\sum_{i=1}^{n}A_{li}(q,t)\dot{q}_i=c_l(q,t),\quad l=1,\dots,m, \tag{2.2}$$

where $\dot{q}_i$ is the i-th component of $\dot{q}$, $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1, $m\leq n$. They are the *first order* form of the constraints. The constraints may not be integrable; and may be nonholonomic in general. The constraints can be put in matrix form

$$A(q,t)\dot{q}=c(q,t), \tag{2.3}$$

where $A = [A_{li}]_{m\times n},\ c = [c_1, c_2, \ldots, c_m]^{\mathrm{T}}$.

We now convert the first order form into *second order* form. Differentiating the constraint equations (2.2) with respect to t yields

$$\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i + \sum_{i=1}^{n}A_{li}(q,t)\ddot{q}_i = \frac{d}{dt}c_l(q,t), \tag{2.4}$$

where

$$\frac{d}{dt}\,A_{li}(q,t) = \sum_{k=1}^{n}\frac{\partial A_{li}(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial A_{li}(q,t)}{\partial t},$$
$$\frac{d}{dt}\,c_l(q,t) = \sum_{k=1}^{n}\frac{\partial c_l(q,t)}{\partial q_k}\dot{q}_k + \frac{\partial c_l(q,t)}{\partial t}.$$

Equation (2.4), the second order form of the constraints can be rewritten as

$$\sum_{i=1}^{n}A_{li}(q,t)\ddot{q}_i = -\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i + \frac{d}{dt}c_l(q,t) = b_l(q,\dot{q},t), \quad l = 1,\ldots,m, \tag{2.5}$$

or in matrix form

$$A(q,t)\ddot{q} = b(q,\dot{q},t), \tag{2.6}$$

where $b = [b_1, b_2, \ldots, b_m]^{\mathrm{T}}$.

Remark 2.2 For a given configuration, the *possible* velocity $\dot{q}$ is governed by (2.3) while the *possible* acceleration $\ddot{q}$ is governed by (2.6) ([17]). Advantages of considering second order constraints are discussed in, e.g., [18,19].

Remark 2.3 The constraint (2.6) is in fact a very general form. It includes typical constraints as illustrated in, e.g., [1,17]. It also includes a number of standard control objectives such as stability, trajectory tracking, and optimality. All one needs is to prescribe a desirable system dynamics and then convert it to the second order form.

Remark 2.4 Besides the "practical" constraint a dynamic system needs to meet, which are thoroughly discussed in, e.g., [1], there is also the "numerical" constraint. By this, we mean that it is possible that a system, under a prescribed constraint, while in numerical simulations, tends to have a numerical drift of constraints and integrals. Therefore it is desirable to impose an additional "numerical" constraint on the system to make sure there is no numerical drift. The standard technique such as in [20] can be applied. The "numerical" constraint can be combined with the "practical" constraint to form (2.6).

3 The Passive Constraint Problem

There are two categories of problems associated with constraints. In the *passive constraint problem*, the environment (or the structure) is to supply the constraint force in order for the system to comply with the constraint. In the *active (or servo) constraint problem*, the control input supplies the required force. We discuss the first in this section.

The *virtual displacement* [17] δq is such that $A(q,t)\delta q = 0$. Let $Q^c \in R^n$ denote the constraint force. The *Lagrange's form of d'Alembert's principle*, which in turn prescribes the constraint to be *ideal*, is such that the first order constraint virtual work vanishes:

$$\delta' W^{(c)} = Q^{c\mathrm{T}} \delta q = 0, \tag{3.1}$$

where $\delta' W^{(c)}$ is the (first order) constraint virtual work.

Assumption 1 For each $(q,t) \in R^n \times R$, $\sigma \in \Sigma$, $M(q,\sigma,t) > 0$.

Remark 3.1 The assumption on the positive definiteness of the inertia matrix will be vital in later development. In the past, it was often believed that this was always true, and therefore a fact rather than an assumption. However, there are counter examples, as listed in [21], when q is not selected to be the generalized coordinate (as the current case).

Definition 3.1 For given A and b, the constraint (2.6) is called *consistent* if there exists at least one solution $\ddot{q}$.

Assumption 2 The constraint (2.6) is consistent.

Theorem 3.1 ([7, p. 233]) *Consider the system (2.1) and the constraint (2.6). Subject to Assumptions 3.1 and 3.2, the constraint force*

$$\begin{aligned} Q^c =& M^{1/2}(q,\sigma,t)(A(q,t)M^{-1/2}(q,\sigma,t))^+ \\ &\times [b(q,\dot{q},t) + A(q,t)M^{-1}(q,\sigma,t)(C(q,\dot{q},\sigma,t)\dot{q} + g(q,\sigma,t))]. \end{aligned} \tag{3.2}$$

obeys the Lagrange's form of d'Alembert's principle (3.1) and renders the system to meet the constraint. Here "+" stands for the Moore-Penrose generalized inverse ([22, p. 337]).

Sketch of Proof: By the choice of (3.2), it can be shown that, with $\tau = Q^c$ in (2.1),

$$A\ddot{q} - b = A[M^{-1}(-C\dot{q} - g) + M^{-1}Q^c] - b = 0. \tag{3.3}$$

Furthermore, we have $Q^c \in \mathcal{R}(A^{\mathrm{T}})$ (note that $\delta q \in \mathcal{N}(A)$ and $\mathcal{R}(A^{\mathrm{T}}) \perp \mathcal{N}(A)$). □

Remark 3.2 The Lagrange's form of d'Alembert's principle renders the constraint force (3.2) to be the one with *minimum norm*, out of all possible alternative forces which can also meet (2.6) [7].

Remark 3.3 Theorem 1 suggests the strategy the *Nature* will undertake to meet the constraint. The constraint force is *model-based*. That is, it is based on the exact model information. Based on the theorem, one could apply the control input $\tau = Q^c$ to drive the system to meet (2.6), if the uncertainty *was* known. A more realistic consideration that the uncertainty is unknown is investigated in the next section.

4 Robust Servo Control Design

We now take the uncertainty into account while designing the control τ. Decompose the M, C, and g as follows:

$$\begin{aligned} M(q,\sigma,t) &= \bar{M}(q,t) + \Delta M(q,\sigma,t), \\ C(q,\dot{q},\sigma,t) &= \bar{C}(q,\dot{q},t) + \Delta C(q,\dot{q},\sigma,t), \\ g(q,\sigma,t) &= \bar{g}(q,t) + \Delta g(q,\sigma,t). \end{aligned} \tag{4.1}$$

Here $\bar{M}$, $\bar{C}$, and $\bar{g}$ denote the "nominal" portions with $\bar{M} > 0$ (this is always feasible since it is the designer's discretion), while ΔM, ΔC and Δg are the uncertain portions. The functions $\bar{M}(\cdot)$, $\Delta M(\cdot)$, $\bar{C}(\cdot)$, $\Delta C(\cdot)$, $\bar{g}(\cdot)$, and $\Delta g(\cdot)$ are all continuous. Let $D(q,t) := \bar{M}^{-1}(q,t)$, $\Delta D(q,\sigma,t) := M^{-1}(q,\sigma,t) - \bar{M}^{-1}(q,t)$, $E(q,\sigma,t) := \bar{M}(q,t)M^{-1}(q,\sigma,t) - I$ (hence $\Delta D(q,\sigma,t) = D(q,t)E(q,\sigma,t)$).

Assumption 3 For each $(q,t) \in R^n \times R$, $A(q,t)$ is of full rank.

Assumption 4 There exists $\hat{\rho}_E(\cdot)$: $R^n \times R \to (-1,\infty)$ such that for all $(q,t) \in R^n \times R$,

$$\frac{1}{2}\min_{\sigma\in\Sigma} \lambda_m \left(E(q,\sigma,t) + E^{\mathrm{T}}(q,\sigma,t)\right) \geq \hat{\rho}_E(q,t). \tag{4.2}$$

Remark 4.1 Suppose there is no uncertainty in M: $\bar{M} = M$, then $E = 0$ and hence one can choose $\hat{\rho}_E = 0$ to meet the assumption. By continuity, there is a (unidirectional) threshold for the allowable uncertainty in E. We note that a standard assumption in this area (see, e.g., [23]) that $\max_{\sigma\in\Sigma} \|E(q,\sigma,t)\| < 1$ for all $(q,t) \in R^n \times R$ is more restrictive than the current setting.

Assumption 5 For given $P \in R^{n\times n}$, $P > 0$, let

$$\Psi(q,t) := PA(q,t)D(q,t)D(q,t)A^{\mathrm{T}}(q,t)P.$$

There exists a scalar constant $\underline{\lambda} > 0$ such that

$$\inf_{(q,t)\in R^n\times R} \lambda_m\left(\Psi(q,t)\right) \geq \underline{\lambda}. \tag{4.3}$$

Remark 4.2 Under Assumptions 3.1, 3.2, and 4.1, the matrix $\Psi(q,t)$ is always positive definite. Thus all this assumption adds is that $\lambda_m(\Psi)$ is positively bounded from below.

Remark 4.3 Let $\beta(q,\dot{q},t) := A(q,t)\dot{q} - c(q,t)$. We consider the approximate constraint following problem. That is, it is possible that $\beta \neq 0$ (hence $A\ddot{q} \neq b$). This may be due to modelling uncertainty (and hence (3.2) can not be implemented by the designer; while it can be by the *Nature*). In addition, the system may not start with the constraint manifold in the beginning (i.e., $\beta \neq 0$ as $t = t_0$).

Consider the following control design:

$$\tau(t) = p_1(q(t),\dot{q}(t),t) + \hat{p}_2(q(t),\dot{q}(t),t) + \hat{p}_3(q(t),\dot{q}(t),t), \tag{4.4}$$

with

$$\hat{p}_2(q, \dot{q}, t) = -\kappa \bar{M}^{-1}(q, t) A^{\mathrm{T}}(q, t) P(A(q, t)\dot{q} - c(q, t)), \tag{5}$$
$$\hat{p}_3(q, \dot{q}, t) = -\hat{\gamma}(q, \dot{q}, t)\hat{\mu}(q, \dot{q}, t)\hat{\rho}(q, \dot{q}, t), \tag{6}$$

where $\epsilon, \kappa \in R,\ \epsilon, \kappa > 0$,

$$\hat{\gamma}(q, \dot{q}, t) = \begin{cases} \dfrac{(1+\hat{\rho}(q,t))^{-1}}{\|\hat{\mu}(q,\dot{q},t)\|}, & \text{if} \quad \|\hat{\mu}(q,\dot{q},t)\| > \epsilon, \\ \dfrac{(1+\hat{\rho}(q,t))^{-1}}{\epsilon}, & \text{if} \quad \|\hat{\mu}(q,\dot{q},t)\| \le \epsilon, \end{cases} \tag{7}$$

$$\hat{\mu}(q, \dot{q}, t) := \bar{M}^{-1}(q, t) A^{\mathrm{T}}(q, t) P(A(q, t)\dot{q} - c(q, t))\hat{\rho}(q, \dot{q}, t). \tag{8}$$

The function $\hat{\rho}(\cdot) : R^n \times R^n \times R \to R_+$ is chosen such that for all $(q, \dot{q}, t) \in R^n \times R^n \times R$,

$$\begin{aligned} \hat{\rho}(q, \dot{q}, t) \ge \max_{\sigma\in\Sigma} \| &PA(q,t)\Delta D(q,\sigma,t)(-C(q,\dot{q},t)\dot{q} - g(q,t) + p_1(q,\dot{q},t) + \hat{p}_2(q,\dot{q},t)) \\ &- PA(q,t)D(q,t)(\Delta C(q,\dot{q},\sigma,t)\dot{q} + \Delta g(q,\sigma,t))\|. \end{aligned} \tag{4.9}$$

Theorem 4.1 *Subject to Assumptions 3.1, 3.2, and 4.1-4.3, consider the system (2.1). The control (4.4) renders β uniformly bounded (that is, for any $r > 0$, there is a $d(r) < \infty$ such that if $\|\beta(q(t_0), \dot{q}(t_0), t_0)\| \le r$, then $\|\beta(q(t), \dot{q}(t), t)\| \le d(r)$ for all $t \ge t_0$) and uniformly ultimately bounded (that is, for any $r > 0$ and $\underline{d} > 0$ with $\|\beta(q(t_0), \dot{q}(t_0), t_0)\| \le r$, $\|\beta(q(t), \dot{q}(t), t)\| \le \bar{d}$ for any $\bar{d} > \underline{d}$ and all $t \ge t_0 + T(\bar{d}, r)$, where $T(\bar{d}, r) < \infty$). Furthermore, $\bar{d} \to 0$ as $\epsilon \to 0$.*

Proof Let $V(\beta) = \beta^{\mathrm{T}} P \beta$. For any given $\sigma(\cdot)$, the derivative of V along a trajectory is evaluated as (for simplicity, arguments of functions are sometimes omitted when no confusions are likely to arise):

$$\dot{V} = 2\beta^{\mathrm{T}} P(A\ddot{q} - b) = 2\beta^{\mathrm{T}} P \left\{ A\left[M^{-1}(-C\dot{q} - g) + M^{-1}(p_1 + \hat{p}_2 + \hat{p}_3) \right] - b \right\}. \tag{4.10}$$

After decomposing M^{-1}, C, and g, we have

$$\begin{aligned} &A[M^{-1}(- C\dot{q} - g) + M^{-1}(p_1 + \hat{p}_2 + \hat{p}_3)] - b \\ &\quad = A[(D + \Delta D)(-\bar{C}\dot{q} - \bar{g} - \Delta C\dot{q} - \Delta g) + (D + \Delta D)(p_1 + \hat{p}_2 + \hat{p}_3)] - b \\ &\quad = A[D(-\bar{C}\dot{q} - \bar{g} + p_1 + \hat{p}_2) + D(-\Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2) \\ &\qquad + (D + \Delta D)\hat{p}_3] - b. \end{aligned}$$

First, we recall that

$$A[D(-\bar{C}\dot{q} - \bar{g}) + Dp_1] - b = 0. \tag{4.11}$$

Next, by (4.9),

$$\begin{aligned} &\beta^{\mathrm{T}} PA[D(- \Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2)] \\ &\qquad \le 2\|\beta\|\|PA[D(-\Delta C\dot{q} - \Delta g) + \Delta D(-C\dot{q} - g + p_1 + \hat{p}_2)]\| \le 2\|\beta\|\hat{\rho}. \end{aligned} \tag{4.12}$$

Based on (4.5),

$$2\beta^{\mathrm{T}} PAD\hat{p}_2 = 2\beta^{\mathrm{T}} PAD(-\kappa \bar{M}^{-1} A^{\mathrm{T}} P(A\dot{q} - c)) = -2\kappa\eta^{\mathrm{T}}\eta = -2\kappa\|\eta\|^2, \tag{4.13}$$

where $\eta = \bar{M}^{-1}A^{\mathrm{T}}P\beta$. By $\Delta D = DE$, (4.6), and recalling that $\bar{M}^{-1} = D$,

$$\begin{aligned} 2\beta^{\mathrm{T}}PA(D+\Delta D)\hat{p}_3 &= 2\beta^{\mathrm{T}}PA(D+DE)(-\hat{\gamma}\hat{\mu}\hat{\rho}) \\ &= 2(DA^{\mathrm{T}}P\beta\hat{\rho})^{\mathrm{T}}(I+E)(-\hat{\gamma}\hat{\mu}) = 2\hat{\mu}^{\mathrm{T}}(I+E)(-\hat{\gamma}\hat{\mu}) \\ &= -2\hat{\gamma}\hat{\mu}^{\mathrm{T}}\hat{\mu} - 2\hat{\gamma}\hat{\mu}^{\mathrm{T}}E\hat{\mu} \le -2\hat{\gamma}\|\hat{\mu}\|^2 - \hat{\gamma}\lambda_m(E+E^{\mathrm{T}})\|\hat{\mu}\|^2 \\ &\le -2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2. \end{aligned} \tag{4.14}$$

As $\|\hat{\mu}\| > \epsilon$, by (4.7),

$$-2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{(1+\hat{\rho}_E)^{-1}}{\|\hat{\mu}\|}\,(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\|\hat{\mu}\|. \tag{4.15}$$

As $\|\hat{\mu}\| \le \epsilon$, then by (4.7),

$$-2\hat{\gamma}(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{(1+\hat{\rho}_E)^{-1}}{\epsilon}\,(1+\hat{\rho}_E)\|\hat{\mu}\|^2 = -2\,\frac{\|\hat{\mu}\|^2}{\epsilon}\,. \tag{4.16}$$

With (4.11)–(4.15), we have for $\|\hat{\mu}\| > \epsilon$,

$$\dot{V} \le -2\kappa\|\eta\|^2 - 2\|\hat{\mu}\| + 2\|\beta\|\hat{\rho} = -2\kappa\|\eta\|^2 - 2\|\hat{\mu}\| + 2\|\hat{\mu}\| = -2\kappa\|\eta\|^2.$$

As $\|\hat{\mu}\| \le \epsilon$,

$$\dot{V} \le -2\kappa\|\eta\|^2 - 2\,\frac{\|\hat{\mu}\|^2}{\epsilon} + 2\|\hat{\mu}\| \le -2\kappa\|\eta\|^2 + \frac{\epsilon}{2}\,.$$

Finally we conclude that

$$\dot{V} \le -2\kappa\|\eta\|^2 + \frac{\epsilon}{2}\,.$$

By Rayleigh's principle ([21]) and Assumption 4.3,

$$\|\eta\|^2 = \eta^{\mathrm{T}}\eta = \beta^{\mathrm{T}}PADDA^{\mathrm{T}}P\beta \ge \lambda_m(PADDA^{\mathrm{T}}P)\|\beta\|^2 \ge \underline{\lambda}\|\beta\|^2.$$

Therefore,

$$\dot{V} \le -2\kappa\underline{\lambda}\|\beta\|^2 + \frac{\epsilon}{2}\,.$$

Upon invoking arguments as in [23], we conclude uniform boundedness with

$$d(r) = \begin{cases} \sqrt{\dfrac{\lambda_M(P)}{\lambda_m(P)}}\,R, & \text{if} \quad r \le R, \\ \sqrt{\dfrac{\lambda_M(P)}{\lambda_m(P)}}\,r, & \text{if} \quad r > R, \end{cases}$$

$$R = \sqrt{\frac{\epsilon}{4\kappa\underline{\lambda}}}\,.$$

Uniform ultimate boundedness also follows with

$$\underline{d} = \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}}\,R,$$

$$T(\bar{d}, r) = \begin{cases} 0, & \text{if} \quad r \le \bar{d}\sqrt{\dfrac{\lambda_m(P)}{\lambda_M(P)}}\,, \\ \dfrac{\lambda_M(P)r^2 - (\lambda_m^2(P)/\lambda_M(P))\bar{d}^2}{2\kappa\underline{\lambda}\bar{d}^2(\lambda_m(P)/\lambda_M(P)) - (\epsilon/2)}\,, & \text{otherwise.} \end{cases} \tag{4.17}$$

Based on (4.17), the size of the uniform ultimate boundedness region $\bar{d} \to 0$ as $\epsilon \to 0$. □

Remark 4.4 With the uncertainty in presence and no restrictions on the initial condition, it is only reasonable to expect approximate constraint following, which is shown in the ultimate boundedness of β. In the special case when there is no uncertainty, i.e., $\Delta D \equiv 0$, $\Delta C \equiv 0$, and $\Delta g \equiv 0$, one may choose $\rho = 0$ and hence $\hat{p}_3 = 0$. This means $\tau = p_1 + \hat{p}_2$ and $\dot{V} \leq -2\kappa\underline{\lambda}\|\beta\|^2$. One therefore expects $\beta \to 0$ as $t \to \infty$. If, in addition, we choose $\hat{p}_2 = 0$, then $\dot{V} = 0$. This means if $\beta = 0$ initially (i.e., the constraint is met initially), then $\beta = 0$ for all $t \geq t_0$. This special case falls into Theorem 3.1, the perfect constraint following case.

5 Conclusions

We consider a mechanical system subject to a class of (possibly nonholonomic) constraints. The system contains uncertainty. The control design objective is to render the system to follow the constraint sufficiently close, even in the presence of uncertainty. Two robust control designs are proposed. They are motivated by a previous design which is based on the Lagrange's form of D'Alembert's principle; hence the Nature's action. The controls assure the uniform boundedness and uniform ultimate boundedness of the tracking error (denoted by β).

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