Nonlinear Dynamics and Systems Theory, **8** (4) (2008) 317–328

Method of Lines to Hyperbolic Integro-Differential Equations in $\mathbb{R}^n$

D. Bahuguna [1*], J. Dabas [1] and R.K. Shukla [2]

[1] *Department of Mathematics*
Indian Institute of Technology, Kanpur - 208 016, India.
[2] *Invertis Institute of Engineering Technology*
Bareilly, India

Received: October 18, 2007; Revised: September 18, 2008

Abstract: In this work we consider a hyperbolic integro-differential equation in $\mathbb{R}^n$. We reformulate it into an evolution equation in a suitable Hilbert space and establish the existence and uniqueness of a strong solution using the method of lines and the theory of semigroups of contractions in a Hilbert space.

Keywords: *Wave equation; semigroups; contractions; method of lines; strong solution.*

Mathematics Subject Classification (2000): 34G20, 35L90,35L15, 47D03.

1 Introduction

In this paper we are concerned with the following perturbed wave equation in $\mathbb{R}^n$,

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(x,t) - \Delta w(x,t) = f(x,t) + \int_0^t k(t-s)\Delta w(x,s)\, ds, \ (x,t) \in \mathbb{R}^n \times (0,T], \\ w(x,0) = f_1(x), \quad \frac{\partial w}{\partial t}(x,0) = f_2(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where Δ denotes the n-dimensional Laplacian, the unknown real valued function w is to be defined on $\mathbb{R}^n \times [0,T]$, $0 < T < \infty$, k is a real valued function defined on $[0,T]$, the real valued function f is defined on $\mathbb{R}^n \times [0,T]$, the real valued functions f_i are defined on $\mathbb{R}^n$, $i = 1, 2$.

The problem (1) with $k \equiv 0$ has been extensively treated by many authors, see, for instance, Yosida [21, 22] and Pazy [18]. Our aim is to reformulate (1) as a first order

[*] Corresponding author: dhiren@iitk.ac.in

evolution integro-differential equation in a suitable product Hilbert space and apply the theory of semigroups together with the method of time discretization in time to establish the existence and uniqueness of solutions.

One may formulate (1) as a second order evolution equation in a Hilbert space. Goldstein [13] has used semigroup of bounded linear operators to second order evolution equations. But we use the ideas of Pazy [18] to put (1) in a product Hilbert space taken as a Cartesian product of two Hilbert spaces. It turns out that the operator associated with the differentiation is an infinitesimal generator of a group of contractions in the chosen product Hilbert space. We incorporate the properties of such operators with the method of lines to establish the existence of a strong solution.

As a model for the foregoing equation we consider equations of the form:

$$u_t(x,t) = (k(x)u_x(x,t))_x + \int_0^t G(t,s)(\sigma(u_x(x,s)))_x ds + h(x,t). \tag{2}$$

Such equations have physical application; for example, they arise in problems concerned with heat flow in materials with memory. Linear versions of equation (2) are treated in [16] and [17]; the nonlinear versions are treated in [15] and similar equations are also treated in [14]. If we replace $u_t(x,t)$ by $u_{tt}(x,t)$ we obtain an equation arising in the theory of viscoelasticity [8]. Our results for the evolution integro-differential equation may also be applied to the heat conduction problem for a material with memory (cf. Liu and Ezzinbi [12]),

$$u_t(x,t) = u_{xx}(x,t) + \int_0^t k(t-s)u_{xx}(x,s)\, ds + f(x,t), \quad t>0, \quad x \in (a,b), \tag{3}$$

$$u(x,0) = u_0(x). \tag{4}$$

The above problem can be teated as a particular case of our study even in the case when $(a,b) = \mathbb{R}$. whereas in [12] it is a bounded interval only. In [7] the authors study the following functional integro-differential equation in the product Hilbert space $\mathcal{H} := H_0^1(0,1) \times L^2(0,1)$,

$$\begin{cases} \frac{du}{dt} - Au = \int_0^t k(t,s)Au(s)ds + F(t,u_t), \\ u_0 = \phi, \end{cases} \tag{5}$$

where $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is shown to be the infinitesimal generator of a contraction semigroup in $\mathcal{H}$ and the nonlinear function $F : [0,T] \times \mathcal{C}_0 \to \mathcal{H}$. Here the space $\mathcal{C}_t := C([-T,t];\mathcal{H})$, $t \in [0,T]$, is the Banach space of all continuous functions from $[-T,t]$ into $\mathcal{H}$ endowed with the supremum norm. By the application of the method of lines the existence and uniqueness of a strong solution is proved.

Our aim is to apply Rothe's method to establish the existence and uniqueness of a strong solution which in turn will guarantee the well-posedness of (1). The method of lines is a powerful tool for proving the existence and uniqueness of solutions to evolution equations. This method is oriented towards the numerical approximations. For instance, we refer to Rektorys [19] for a rich illustration of the method applied to various interesting physical problems. For the application of the method of lines to nonlinear differential and Volterra integro-differential equations (VIDEs) in which bounded, though nonlinear, operators appear inside the integrals, see Kacur [9, 10], Rektorys [19], Bahuguna and Raghavendra [4]. Recently, Bahuguna and Dabas [6] have considered a nonlocal problem arising in the population dynamics using Rothe's Method. In the present study we extend the application of the method of lines to a class of nonlinear VIDEs in which differential operators occur inside the integrals and hence are unbounded. Motivation for considering such problems arises from the theory of wave propagation under the influence of damping, see Bahuguna [2], and Bahuguna and Shukla [3] and references cited therein.

2 Preliminaries

Here we briefly describe the spaces required in subsequent analysis. For details we refer to Pazy [18]. Let $\Omega \subset \mathbb{R}^n$ be a domain with sufficiently smooth boundary $\partial\Omega$ and $\overline{\Omega}$ be the closure of Ω in $\mathbb{R}^n$. Let $m \in \mathbb{N}\cup\{0\}$ and let $C^m(\Omega)$ ($C^m(\overline{\Omega})$) be the set of all m-times continuously differentiable real valued functions on Ω ($\overline{\Omega}$). Let $C_0^m(\Omega)$ be the subset of all functions in $C^m(\Omega)$ having compact support in Ω. For $1 \le p < \infty$, we define a norm in $C^m(\Omega)$ by

$$\|u\|_{m,p}^p = \int_\Omega \sum_{|\alpha|\le m} |D^\alpha u(x)|^p \, dx, \quad u \in C^m(\Omega),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}\cup\{0\}$, $i = 1, 2, \dots, n$, is a multi-index with $|\alpha| = \sum_{i=1}^n \alpha_i$ and D^α is the partial differential operator

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots x_n^{\alpha_n}}.$$

Let $C^{m,p}(\Omega)$ be the subset of all $u \in C^m(\Omega)$ such that $\|u\|_{m,p} < \infty$. For $p = 2$ we also define the inner product

$$(u, v)_m = \int_\Omega \sum_{|\alpha|\le m} (D^\alpha u)(D^\alpha v) \, dx.$$

The Banach spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are defined as the completion of $C^{m,p}(\Omega)$ and $C_0^{m,p}(\Omega)$ with respect to the norm $\|.\|_{m,p}$, respectively. For $p = 2$, we denote the Hilbert spaces $W^{m,2}(\Omega)$ and $W_0^{m,2}(\Omega)$ as $H^m(\Omega)$ and $H_0^m(\Omega)$, respectively. For $\Omega = \mathbb{R}^n$, we write $H^m = H^m(\mathbb{R}^n)$. If given function w defined on $\mathbb{R}^n \times [0, T]$ into $\mathbb{R}$ such that for each $t \in [0, T]$, $w(., t) \in H^0$, then we may identify w from $[0, T]$ into H^0 by $w(t)(x) = w(x, t)$, $x \in \mathbb{R}^n$. In addition, if $\frac{\partial w}{\partial t}(., t) \in H^0$ for each $t \in [0, T]$ then $\frac{dw}{dt}$ is also defined as a function from $[0, T]$ into H^0. We now consider the product Hilbert space $H = H^1 \times H^0$ which is the completion of $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\begin{aligned} |||u|||^2 &= |||(u_1, u_2)|||^2 = \left(\int_{\mathbb{R}^n} (|u_1(x)|^2 + |\nabla u_1(x)|^2 + |u_2(x)|^2) \, dx \right), \\ & (u_1, u_2) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n). \end{aligned} \tag{6}$$

The Hilbert space H^m may also be characterized as the space of all functions $f \in$ such that the Fourier transform $\hat{f}$ has the property that $(1 + |\xi|^2)^{k/2} \hat{f}(\xi)$ is in H^0 as a function of $\xi \in \mathbb{R}^n$.

In order to write (1) as an evolution equation in H we take $u = (u_1, u_2)$ where $u_1 = w$ and $u_2 = \frac{\partial w}{\partial t}$. We define the operator $A : D(A) \subset H \to H$ as $D(A) = H^2 \times H^1$ with $Au = A(u_1, u_2) = (u_2, \Delta u_1) - 2(u_1, u_2)$.

Thus, the problem (1) is equivalently represented by the following evolution equation in H as

$$\frac{du(t)}{dt} = Au(t) + F(t, u(t)) + \int_0^t k(t - s) Bu(s) \, ds, \; t \in (0, T], \; u(0) = u_0, \tag{7}$$

where $D(B) = H^2 \times H^0$ with $Bu = B[u_1, u_2] = [0, \Delta u_1]$ for $u \in D(B)$ and $u_0 = (f_1, f_2)$, under the assumption that $(f_1, f_2) \in H^1 \times H^0$ and $F : [0,T] \times H \to H$ given by $F(t,u) = [0, 2u + f(t)]$, $f : [0,T] \to H^0$, $f(t)(x) = f(x,t)$.

3 Existence and Uniqueness of Solutions

In this section we continue to use the notations and notions introduced in the earlier sections and consider the well-posedness of (7).

We observe some of the properties of the operators A and B and assume Lipschitz continuity of the kernel k.

(P1) The operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 of group $T(t)$ in H. (Cf. Pazy [18] Theorem 7.4.5, pp. 222–223.)

(P2) $D(A) \subset D(B)$ and $Bu = Au + Pu$ where $P : H^1 \times H^1 \to H$ is a bounded linear operator given by $Pu = P[u_1, u_2] = [-u_2, 0] + 2[u_1, u_2]$ with $|||Bu||| \le C(|||Au||| + |||u|||)$ for $u \in D(A)$ and $|||Pu||| \le C|||u|||$ for $u \in D(P)$.

(P3) The function $F : [0,T] \times H \to H$ satisfies

$$|||F(t,u) - F(s,v)||| \le L_F[|t-s| + |||u-v|||],$$

for $t, s \in [0,T]$ and $u, v \in H$. The function $k : [0,T] \to \mathbb{R}$ is Lipschiz continuous with $k(0) = 0$.

By $C^{0,1}([0,T]; H)$ we denote the Banach space of all Lipschitz continuous functions from $[0,T]$ into H endowed with the norm

$$\|u\|_{C^{0,1}} = \max_{0 \le t \le T} |||u(t)||| + \sup \left\{ \frac{|||u(t) - u(s)|||}{|t-s|} : t, s \in [0,T], t \ne s \right\}.$$

We have the following main result.

Theorem 3.1 *If (P1), (P2) and (P3) are satisfied then for each $(f_1, f_2) \in H^2 \times H^1$, there exists a $u \in C^{0,1}([0,T]; H)$ with $u(0) = u_0$ satisfying (7) a.e. on $[0,T]$. Furthermore, if $k \in C^1[0,T]$, then the solution u is unique.*

4 Basic Lemmas

We shall prove Theorem 3.1 with the help of several lemmas. We devide the interval $[0,T]$ into subintervals $[t^l_{j-1}, t^l_j]$, $t^l_j = j.h$, $h = T/l$, $j = 0, 1, 2, \ldots, l$. We set $u^l_0 = u_0$ for all $l \in \mathbb{N}$. For $j = 1, 2, \ldots, l$, we define by u^l_j the unique solutions of each of the equations

$$\frac{u - u^l_{j-1}}{h} - Au = F^l_j + h \sum_{i=0}^{j-1} k^l_{ji}\, Bu^l_i, \tag{8}$$

where

$$F^l_j = F(t^l_j, u^l_{j-1}), \tag{9}$$
$$k^l_{ji} = k(t^l_j - t^l_i). \tag{10}$$

The existence of unique u^l_j satisfying (8) is ensured by Corollary 7.4.4 in Pazy [18] which is stated here as follows.

Lemma 4.1 *Let* $A_0 : (D(A_0) \subset H \to H$ *with* $D(A_0) = D(A)$ *and* $A_0[u_1, u_2] = [u_2, \Delta u_1]$ *Then for any* $v \in H$, *and* $0 < |\lambda| < 1/2$, *the equation*

$$u - \lambda A_0 u = v$$

has a unique solution $u \in D(A_0)$ *satisfying the estimate*

$$|||u||| \leq (1 - 2|\lambda|)^{-1} |||v|||.$$

Furthermore A_0 *is the infinitesimal generator of a group* $\{S(t) : t \in \mathbb{R}\}$ *of bounded linear operators in* H *with*

$$\|S(t)\|_{B(H)} \leq e^{2|t|},$$

where $B(H)$ *denotes the Banach space of bounded linear operators in* H *with the norm* $\|.\|_{B(H)}$.

Remark 4.1 From Lemma 4.1 it follows that A is the infinitesimal generator of the contraction group $\{T(t) : t \in \mathbb{R}\}$ of bounded linear operators in H where $T(t) = e^{-2|t|} S(t)$. Therefore by Lumer-Phillips Theorem (cf. Pazy [18]) that A is m-dissipative, i.e.,

$$(Au, u) \leq 0, \quad u \in D(A)$$

and $R(I - \lambda A) = H$ for all $\lambda > 0$.

In order to ensure unique solution $u_j^l \in D(A)$ of (8) with the help of Lemma 4.1 we rewrite it as

$$u - hAu = u_{j-1}^l + hF_j^l + h^2 \sum_{i=0}^{j-1} k_{ji}^l B u_i^l.$$

Now, the existence of a unique u_j^l satisfying (8) follows from the m-dissipativity of A.

Now, we show that $\delta u_j^l = (u_j^l - u_{j-1}^l)/h$ lie in a ball of fixed radius independent of the discretization parameters j, h and l. For convenience, we shall denote by C a generic constant, i.e., KC, e^{KC}, etc., will be replaced by C where K is a positive constant independent of j, h and l.

We shall use later the following lemma due to Sloan and Thomee [20].

Lemma 4.2 *Let* $\{w_l\}$ *be a sequence of nonnegative real numbers satisfying*

$$w_l \leq \alpha_l + \sum_{i=0}^{l-1} \beta_i w_i, \quad l > 0,$$

where $\{\alpha_l\}$ *is a nondecreasing sequence of nonnegative real numbers and* $\beta_l \geq 0$. *Then*

$$w_l \leq \alpha_l \exp\{\sum_{i=0}^{l-1} \beta_i\}, \quad l > 0.$$

Furthermore, we also require the following lemma for later use.

Lemma 4.3 *Let $C > 0$, $h > 0$ and let $\{\alpha_j\}_{j=1}^l$ be a sequence of nonnegative real numbers satisfying*

$$\alpha_j \leq (1+Ch)\alpha_{j-1} + Ch^2 \sum_{i=1}^{j-1} \alpha_i + Ch, \quad 2 \leq j \leq l. \tag{11}$$

Then

$$\alpha_j \leq (1+Ch)^j [\alpha_1 + jCh^2 \sum_{i=1}^{j-1} \alpha_i + jCh], \quad 2 \leq j \leq l.$$

Proof From (11)

$$\begin{aligned} \alpha_{j-1} &\leq (1+Ch)\alpha_{j-2} + Ch^2 \sum_{p=1}^{j-2} \alpha_p + Ch \\ &\leq (1+Ch)\alpha_{j-2} + Ch^2 \sum_{p=1}^{j-1} \alpha_p + Ch. \end{aligned} \tag{12}$$

Putting in (11)

$$\alpha_j \leq (1+Ch)^2 \alpha_{j-2} + Ch^2[1 + (1+Ch)] \sum_{p=1}^{j-1} \alpha_p + Ch[1 + (1+Ch)]. \tag{13}$$

By repeating the above process we get

$$\begin{aligned} \alpha_j &\leq (1+Ch)^{(j-1)}\alpha_1 + Ch^2[1 + (1+Ch) + \cdots + (1+Ch)^{(j-1)}] \sum_{p=1}^{j-1} \alpha_p \\ &\quad + Ch[1 + (1+Ch) + \cdots + (1+Ch)^{(j-1)}] \\ &\leq (1+Ch)^j [\alpha_1 + jCh^2 \sum_{p=1}^{j-1} \alpha_p + jCh]. \end{aligned} \tag{14}$$

This completes the proof of the lemma. □

Lemma 4.4 *There exists a constant C independent of j, h and l such that*

$$|||\delta u_j^l||| \leq C.$$

Proof From (8) for $j = 1$ we get

$$\delta u_1^l - hA\delta u_1^l = Au_0 + F_1^l + hk_{10}^l Bu_0.$$

Lemma 4.1 implies that

$$|||\delta u_1^l||| \leq |||Au_0 + F_1^l + hk_{10}^l Bu_0||| \leq C.$$

Hence $|||Au_1^l||| \leq C$. Let $2 \leq j \leq l$. Subtracting (8) for $j-1$ from (8) for j, we get

$$\delta u_j^l - hA\delta u_j^l = \delta u_{j-1}^l + F_j^l - F_{j-1}^l + hk_{jj-1}^l Bu_{j-1}^l + \sum_{i=0}^{j-2} [k_{ji}^l - k_{j-1i}^l] Bu_i^l.$$

Applying Lemma 4.1 again, we get

$$\begin{aligned}
|||\delta u_j^l||| &\leq |||\delta u_{j-1}^l||| + |||F_j^l - F_{j-1}^l||| + h|k_{j(j-1)}^l||||Bu_{j-1}^l||| \\
&\quad + h\sum_{i=0}^{j-2} |k_{ji}^l - k_{(j-1)i}^l||||Bu_i^l||| \\
&\leq (1+Ch)|||\delta u_{j-1}^l||| + Ch^2 \sum_{i=0}^{j-2} |||\delta u_i^l||| + Ch^2 \sum_{i=0}^{j-1} |||Au_i^l||| + Ch \\
&\leq (1+Ch) \max_{1\leq p\leq j-1} |||\delta u_p^l||| + Ch^2 \sum_{i=0}^{j-1} |||Au_i^l||| + Ch \qquad (15)
\end{aligned}$$

From (8), for $2 \leq i \leq j$, we have

$$\begin{aligned}
|||Au_i^l||| &\leq |||\delta u_i^l||| + |||F_i^l||| + Ch\sum_{p=1}^{i-1} |||Bu_p^l||| \\
&\leq C(1 + \max_{1\leq p\leq i} |||\delta u_p^l|||) + Ch + Ch\sum_{p=1}^{i-1} |||Au_p^l|||. \qquad (16)
\end{aligned}$$

Applying Lemma 4.2 in (16), we get

$$|||Au_i^l||| \leq Ce^{CT}(1 + \max_{1\leq p\leq i} |||\delta u_p^l|||). \qquad (17)$$

Using (17) in (15), we have

$$\begin{aligned}
\max_{1\leq p\leq j} |||\delta u_p^l||| &\leq (1+Ch) \max_{1\leq p\leq j-1} |||\delta u_p^l||| \\
&\quad + Ch^2 \sum_{p=1}^{j-1} \max_{1\leq p\leq i} |||\delta u_p^l||| + Ch. \qquad (18)
\end{aligned}$$

To use Lemma 4.3 in (18), we take $\alpha_j = \max_{1\leq p\leq j} |||\delta u_p^l|||$ and the fact that $(1+Ch)^j \leq e^{CT}$, $2 \leq j \leq l$ and $\alpha_1 \leq C$ to get the estimate

$$\max_{1\leq p\leq j} |||\delta u_p^l||| \leq C + Ch\sum_{p=1}^{j-1} \max_{1\leq p\leq j-1} |||\delta u_p^l|||. \qquad (19)$$

Again we apply Lemma 4.2 to get the required estimate. This completes the proof of the lemma. □

Now, using the discrete points u_j^l, we introduce the following sequences of functions defined from $[0,T]$ into H.

Definition 4.1 We define the Rothe sequence $\{U^l\} \subset C([0,T];H)$ given by

$$U^l(t) = u_{j-1}^l + (t - t_{j-1}^l)\delta u_j^l, \quad t \in [t_{j-1}^l, t_j^l],\ j = 1,2,\ldots,l.$$

Definition 4.2 We define the sequence $\{X^l\}$ of step function from $[0,T]$ into H given by

$$X^l(0) = u_0^l, \quad X^l(t) = u_{j-1}^l, \quad t \in (t_{j-1}^l, t_j^l],\ j = 1,2,\ldots,n,\ l.$$

Remark 4.2 Each of the functions $\{U^l\}$ is Lipschitz continuous with uniform Lipschitz constant, i.e.,

$$|||U^l(t) - U^l(s)||| \leq C|t-s|, \quad t, s \in [0,T].$$

Furthermore,

$$|||U^l(t) - X^l(t)||| \leq \frac{C}{l}.$$

From (8), for $t \in (0,T]$, we may write

$$\frac{d^-}{dt}U^l(t) - AX^l(t) = f^l(t) + \int_0^t K^l(s)ds \tag{20}$$

where $f^l(t) = F(t_j^l, X^l(t))$ for $t \in [t_{j-1}^l, t_j^l]$ and

$$K^l(0) = 0, \quad K^l(t) = h\sum_{i=0}^{j-1} k_{ji}^l B u_i^l, \quad t \in (t_{j-1}^l, t_j^l]. \tag{21}$$

In the next lemma we prove the local uniform convergence of the Rothe sequence.

Lemma 4.5 *There exist a subsequence $\{U^{l_k}\}$ of $\{U^l\}$ and a function $u : [0,T] \to D(A)$ such that $U^{l_k} \to u$ in $C([0,T]; H)$, and $AU^{l_k}(t) \rightharpoonup Au(t)$ uniformly in H as $k \to \infty$ where $\rightharpoonup$ denotes the weak convergence in H. Furthermore, $Au(t)$ is weakly continuous on $[0,T]$.*

Proof Since $\{U^l(t)\}$ and $\{AX^l(t)\}$ are uniformly bounded in the Hilbert space H, there exist weakly convergent subsequences $\{U^{l_k}(t)\}$ and $\{AX^{l_k}(t)\}$ (we take the same indices without loss of generality otherwise we first take the subsequence $\{U^{l_k}(t)\}$ of $\{U^l(t)\}$ and then take the subsequence $\{U^{l_{k_l}}(t)\}$ and $\{AX^{l_{k_l}}(t)\}$ of $\{U^{l_k}(t)\}$ and $\{AX^{l_k}(t)\}$, respectively). Thus, there exist functions $u, w : [0,T] \to H$ such that $U^{l_k}(t) \rightharpoonup u(t)$ and $AX^{l_k}(t) \rightharpoonup w(t)$ as $k \to \infty$. Also, we have $X^{l_k}(t) \rightharpoonup u(t)$ as $k \to \infty$. Let χ_Q be the characteristic function of a set $Q \subset \mathbb{R}^n$ and let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r > 0$, be the open ball in $\mathbb{R}^n$ of radius r centered at the origin.

Let

$$\begin{aligned} X_r^l(t) &= (\chi_{B_r} X_1^l(t), \chi_{B_r} X_2^l(t)), \quad l = 1, 2, \ldots, \\ u_r(t) &= (\chi_{B_r} u_1(t), \chi_{B_r} u_2(t)), \\ w_r(t) &= (\chi_{B_r} w_1(t), \chi_{B_r} w_2(t)). \end{aligned}$$

Clearly, $\{X_r^{l_k}(t)\}$ and $\{AX_r^{l_k}(t)\}$ are uniformly bounded and $X_r^{l_k}(t) \rightharpoonup u_r(t)$ and $AX_r^{l_k}(t) \rightharpoonup w_r(t)$ as $k \to \infty$. Since $\chi_{B_r} X_1^{l_k}(t) \in H^2(B_{r+\epsilon}) \cap H_0^1(B_{r+\epsilon})$, by the equivalence of the norms $\|u\|_{H^2(B_{r+\epsilon}) \cap H_0^1(B_{r+\epsilon})}$ and $\|\Delta u\|_{L^2(B_{r+\epsilon})}$ in $H^2(B_{r+\epsilon}) \cap H_0^1(B_{r+\epsilon})$ (cf. inequality (6.3.9) on page 214 in Atkinson and Han [1]), it follows that $(\partial_{x_i} \chi_{B_r} X_1^l(t), \chi_{B_r} X_2^l(t)) \in D^1(\mathbb{R}^n) \times D^1(\mathbb{R}^n)$, for $i = 1, 2, \ldots, n$, where ∂_{x_i} is the distributional partial derivative with respect to the variable x_i and $D^1(\mathbb{R}^n)$ is the space introduced by Lieb and Loss [11] in the sense that a function $f \in D^1(\mathbb{R}^n)$ if it is in $L_{loc}^1(\mathbb{R}^n)$, its distributional derivative $\partial_{x_i} f$ is in $L^2(\mathbb{R}^n)$, for $i = 1, 2, \ldots, n$, and f vanishes at infinity. Hence we may apply Theorem 8.6 of Lieb and Loss [11] to conclude that $X_r^{l_k}(t) \to u_r(t)$ as $k \to \infty$ and hence $U_r^{l_k}(t) \to u_r(t)$ as $k \to \infty$ in H.

Now, For $\epsilon > 0$, there exist $r_\epsilon > 0$ such that

$$|||U^{l_k}(t) - u(t)|||^2 < |||U_r^{l_k}(t) - u_r(t)|||^2 + \frac{\epsilon}{2}.$$

Now we choose k_0 sufficiently large such that

$$|||U_r^{l_k}(t) - u_r(t)||||| < \frac{\epsilon}{2}, \quad k \geq k_0.$$

Hence

$$|||U^{l_k}(t) - u_r(t)||| < \epsilon, \quad k \geq k_0.$$

Thus $U^{l_k}(t) \to u(t)$ as $k \to \infty$ in H. Since U^{l_k} is Lipschitz continuous with uniform Lipschitz constant, it follows that $\{U^{l_k}\}$ is equi-continuous in $C([0,T]; H)$ and $\{U^{l_k}(t)\}$ is relatively compact in H. Hence by Asoli-Arzela theorem, $U^{l_k} \to u$ as $k \to \infty$ in $C([0,T]; H)$. From the properties of the operator A we have $u(t) \in D(A)$ and $Au(t) = w(t)$. To show the weak continuity of $Au(t)$ in t, let $\{t_k\} \subset [0,T]$ such that $t_k \to t$ as $k \to \infty$, $t \in [0,T]$. Then $u(t_k) \to u(t)$ and since $|||Au(t_k)||| \leq C$, there exists a subsequence $\{Au(t_{k_p})\} \subset \{Au(t_k)\}$ such that $Au(t_{k_p}) \rightharpoonup z(t)$ as $p \to \infty$. Since $u(t_{k_p}) \to u(t)$ and $Au(t_{k_p}) \rightharpoonup z(t)$ as $p \to \infty$, it follows as above that $u(t) \in D(A)$ and $Au(t) = z(t)$. Hence $Au(t)$ is weakly continuous. This completes the proof of the lemma. □

Remark 4.3 Since $Bx = Ax + Px$ for $x \in D(A)$ and $P : H^1 \times H^0 \to H$ is a bounded linear operator, $Bu(t)$ is weakly continuous on $[0,T]$.

Lemma 4.6 *$Au(t)$ and $Bu(t)$ are Bochner integrable on $[0,T]$.*

For the proof of this lemma we refer to Bahuguna and Raghavendra [4]

Lemma 4.7 *Let $\{K^l(t)\}$ be the sequence of functions defined by (21) and*

$$K(\phi)(t) = \int_0^t k(t-s)\phi(s)ds,$$

where $\phi : [0,T] \to H$ is Bochner integrable. We have

$$K^{l_k}(t) \rightharpoonup K(Bu)(t),$$

uniformly on $[0,T]$ as $k \to \infty$.

Proof We first show that $K^{l_k}(t) - K(X^{l_k})(t) \to 0$ uniformly on $[0,T]$ as $p \to \infty$. For $t \in (t_{j-1}^{l_k}, t_j^{l_k}]$, we have

$$\begin{aligned} K^{l_k}(t) - K(X^{l_k})(t) &= h\sum_{i=0}^{j-1} k_{ji}^{l_k} Bu_i^{l_k} - \int_0^t k(t-s)BX^{l_k}(s)\, ds \\ &= \sum_{i=0}^{j-2} \int_{t_i^{l_k}}^{t_{i+1}^{l_k}} [k_{ji}^{l_k} - k(t-s)\, ds] Bu_{i+1}^{l_k} \\ &\quad -[\int_{t_{j-1}^{l_k}}^{t} k(t-s)\, ds] Bu_j^{l_k}. \end{aligned}$$

Since $|||Bu_j^{l_k}||| \leq C$, we have

$$\begin{aligned} |||K^{l_k}(t) - K(X^{l_k})(t)||| &\leq C \sum_{i=0}^{j-2} \int_{t_i^{l_k}}^{t_{i+1}^{l_k}} |k_{ji}^{l_k} - k(t-s)| ds \\ &\leq C \int_{t_{j-1}^{l_k}}^{t} |k(t-s)| ds \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

Now we show that $K(X^{l_k})(t) \rightharpoonup \int_0^t k(t-s)Bu(s)\, ds$ uniformly as $k \to \infty$. For any $v \in H$, We note that $(Bu(t), v)$ is continuous hence we may write

$$\left(\int_0^t k(t-s)Bu(s)\, ds, v\right) = \int_0^t k(t-s)(Bu(s), v)\, ds.$$

Now, for any $v \in H$,

$$\begin{aligned} (K(X^{l_k})(t), v) &= \left(\int_0^t k(t-s)BX^{l_k}(s) ds, v\right) \\ &= \sum_{i=0}^{j-2} \int_{t_i^{l_k}}^{t_{i+1}^{l_k}} k(t-s)(BX^{l_k}(s), v) ds \\ & \int_{t_{j-1}^{l_k}}^{t} k(t-s)(BX^{l_k}(s), v) ds \to \int_0^t k(t-s)(Bu(s), v)\, ds, \end{aligned}$$

as $k \to \infty$. This completes the proof of the lemma. □

5 Proof of Theorem 3.1

In this section we prove Theorem 3.1 with the help of lemmas of the previous section.

Proof Let $v \in H$ be any element. For $t \in (0, T]$, we have

$$(U^{l_k}(t), v) - (Au(t), v) = (u_0, v) + \int_0^t (K^{l_k}(s) + f^{l_k}(s), v)\, ds.$$

Passing to the limit as $p \to \infty$ using bounded convergence theorem and Lemmas 4.5 and 4.7, we have

$$(u(t), v) - (Au(t), v) = (u_0, v) + \int_0^t (K(u)(s) + F(s, u(s)), v)\, ds.$$

Using the continuity of the integrands on the right hand side, we have $(u(t), v)$ is continuously differentiable and

$$\frac{d}{dt}(u(t), v) - (Au(t), v) = (F(t, u(t)) + \int_0^t (k(t-s)Bu(s)\, ds, v). \tag{22}$$

Since $u(t)$ is differentiable a.e. on $[0, T]$, we may take $\frac{d}{dt}$ inside the inner product for a.e. $t \in [0, T]$, hence

$$\frac{du(t)}{dt} - Au(t) = F(t, u(t)) + \int_0^t k(t-s)Bu(s)\, ds, \quad \text{a.e. } t \in [0, T],\ u(0) = u_0. \tag{23}$$

Now we prove the uniqueness under the assumption that $k \in C^1[0,T]$. Let u_1 and u_2 be two solutions of (7) and let $u = u_1 - u_2$. Then

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s)) \\ &\quad + \int_0^s k(s-\tau)Au(\tau)\, d\tau]\, ds \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^s k(s-\tau)T(t-s)Au(\tau))\, d\tau \right) ds \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_\tau^t k(s-\tau)T(t-s)Au(\tau))\, ds \right) d\tau \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta-\tau)T(\eta)Au(\tau))\, d\eta \right) d\tau. \end{aligned} \tag{24}$$

Since $u(\tau) \in D(A)$ for $\tau \in [0,T]$, we have $T(\eta)Au(\tau) = \frac{\partial}{\partial \eta}(T(\eta)u(\tau))$ (cf. Theorem 1.2.4 in Pazy). Thus, we have

$$\begin{aligned} u(t) &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k(t-\eta-\tau)\frac{\partial}{\partial \eta}(T(\eta)u(\tau))\, d\eta \right) d\tau \\ &= \int_0^t T(t-s)[F(s,u_1(s)) - F(s,u_2(s))]ds \\ &\quad + k(0)\int_0^t T(t-\tau)u(\tau)d\tau - \int_0^t k(t-\tau)u(\tau)d\tau \\ &\quad + \int_0^t \left(\int_0^{t-\tau} k'(t-\eta-\tau)T(\eta)u(\tau)d\eta \right) d\tau. \end{aligned} \tag{25}$$

Now taking the norm and using the fact that $\|T(t)\| \le 1$, we have

$$\max_{0 \le r \le t} |||u(r)||| \le C \int_0^t \max_{0 \le r \le s} |||u(r)|||\, ds.$$

Gronwall's inequality implies that $u(t) \equiv 0$. This completes the proof of the theorem. □

References

[1] Atkinson, K. and Han, W. *Theoretical Numerical Analysis*. Springer-Verlag, New York, 2001.

[2] Bahuguna, D. Rothe's method to strongly damped wave equations. *Acta. Applicandae Mathematicae.* **38** (1995) 185–196.

[3] Bahuguna, D. and Shukla, R., Approximations of solutions to second order semilinear integrodifferential equations.*Numer. Funct. Anal. Optim.* **24**(3-4) (2003) 365–390.

[4] Bahuguna, D. and Raghavendra, V. Rothe's method to parabolic integro-differntial equations via abstract integro-differential equations. *Appl. Anal.* **33**(3-4) (1989) 153–167.

[5] Bahuguna, D. and Shukla, R.K. Partial functional differential equations and applications to population dynamics. *Nonlinear Dyn. Syst. Theory* **5** (2005) 345–356.

[6] Bahuguna, D. and Dabas, J. Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition. *Nonlinear Dyn. Syst. Theory* **8** (2008) 7–19.

[7] Bahuguna, D. and Dabas, J. Existence and uniqueness of a solution to a partial integro-differential equation by the method of lines. *E. J. Qualitative Theory of Diff. Equ.* **4** (2008) 1–12.

[8] Dafermos, C. An abstract Volterra equation with applications to linear viscoelasticity. *J. diff. Eqns* **7** (1970) 554–559.

[9] Kacur, J. Application of Rothe's method to perturbed linear hyperbolic equations and variantional inequalities, *Czechoslovak Mathematical Journal.* **34**(109) (1984) 92–106.

[10] Kacur, J. Method of Rothe in evolution equations, *Lecture Notes in Mathematics Springer, Berlin.* **1192** (1985) 23–34.

[11] Lieb, E.H. and Loss, M. *Analysis.* New Delhi, Narosa Publishing House, 1997.

[12] Liu, J.H. and Ezzinbi, K. Non-autonomous integrodifferential equations with nonlocal conditions. *J. Integral Equations* **15** (2003) 79–93.

[13] Goldstein, J.A. Semi-groups and second order differential equations. *J. Func. Anal.* **4** (1969) 50–70.

[14] Grabmuller, H. Singular Petrubation Techniques Applied to Zntegrodifferential Equations. *Research Notes in Mathematics*. London, Pitman, 1978.

[15] Herrmann, R. and Nachlinger, R. An uniqueness and wave propogation in nonlinear heat conductors with memory. *J. math. Anal. Appl.* **50** (1975) 530–547.

[16] Nachlinger, R. and Wheels, L. A uniqueness theorem for rigid heat conductors with memory, *Q. appl. Math.* **3** (1973) 267–273.

[17] Nunziato, J. On heat conduction in materials with memory. *Q. appl. Math.* **29** (1971) 187–204.

[18] Pazy, A. *Semi-groups of Linear Operators and Applications to Partial Differential Equations.* Berlin, Springer Verlag, 1983.

[19] Rektorys, K. *The Method of Discretization in Time and Partial Differential Equations.* Dordrecht-Boston-London, Reidel, 1982.

[20] Sloan, I.H. and Thomee, V. Time discretization of an integro-differential equation of parabolic type. *SIAM J. Numer. Anal.* **23** (1986) 1052–1061.

[21] Yosida, K. On the group embeded in the material complete ring. *Japan J. Math.* **13** (1936) 61–80.

[22] Yosida, K. *Functional Analysis (6-th edition)*. Berlin, Springer Verlag, 1980.