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Existence and Exponential Stability of Almost Periodic Solutions for a Class of Neural Networks with Variable Delays [1]

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Abstract: In this paper, some sufficient conditions for the existence and exponential stability of almost periodic solutions for Cohen–Grossberg neural networks with variable delays are obtained by applying Banach fixed point theory and differential inequality techniques. Some previous results are improved and extended. Moreover, an example is given to illustrate that our results are feasible.

Keywords: *Cohen–Grossberg neural networks; almost periodic solutions; exponential stability.*

Mathematics Subject Classification (2000): 03C50, 34C27, 34D23, 34K50, 92B20.

1 Introduction

Recently, the behavior of dynamical systems has been widely investigated [1, 2, 3, 4]. Cohen–Grossberg neural networks, which were first proposed by Cohen and Grossberg in [5] are typical dynamical systems and have received increasing interesting due to their promising potential applications in many fields such as optimization, associative memory, pattern recognition, signal and image processing. The stability of Cohen–Grossberg neural network with or without delays has been widely studied by many researchers [6, 7, 8, 9]. Moreover, many sufficient conditions on the stability of equilibrium point for Cohen–Grossberg neural networks with constant coefficients have been available [10, 11, 12].

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As is well known, the investigation on the neural dynamical systems not only involves a discussion of stability, but also involves many other dynamical behavior such as periodic oscillatory behavior, almost periodic oscillatory properties, chaos and so on. There exist some results on the existence of periodic solutions of Cohen–Grossberg neural networks with variable coefficients [13, 14, 15, 16]. In practice, almost periodic oscillatory is more accordant. Some authors have researched almost periodic solutions for neural networks, and obtained several interesting results [17, 18, 19, 20]. However, To the best of our knowledge, few authors discuss almost periodic solutions for Cohen–Grossberg neural networks with variable coefficients [21].

In this paper, our objective is to study further Cohen–Grossberg neural networks with variable delays. By applying Banach fixed point theory, differential inequality techniques, we get some sufficient conditions ensuring the existence and exponential stability of almost periodic solutions for Cohen–Grossberg neural networks with variable delays. These conditions obtained are easy to check and in practice. Moreover, in this paper, the assumptions of boundedness, monotonicity, and differentiability for the activation functions are not available.

The rest of the paper is organized as follows. In Section 2, some notations, definitions and model description are given. The existence and uniqueness of almost periodic solutions is established in Section 3. In Section 4, we derive some sufficient conditions on exponential stability of almost periodic solutions. Finally, an example is given to demonstrate the validity of our results in Section 5.

2 Model Description and Preliminaries

Consider the Cohen–Grossberg neural networks with variable delays as follows:

$$\dot{x}_i(t) = -a_i(x_i(t))\Big[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}(t) f_j(x_j(t-\tau_j(t))) + I_i(t)\Big], \quad (1)$$

where $t \geq 0$, $i = 1, 2, \ldots, n$; n is the number of neurons, $x_i(t)$ is the state of neuron i at the time t; $a_i(x_i(t))$ and $b_i(x_i(t))$ represent an amplification function and an appropriately behaved function at the time t, respectively; $f_j(x_j)$ is the activation function of the j-th unit; $c_{ij}(t)$ and $d_{ij}(t)$ denote the neural connection at the time t; $I_i(t)$ is the external inputs at the time t, $\tau_j(t) > 0$ is transmission delay.

The initial conditions of system (1) are of the form $x_i(t) = \varphi_i(t)$, $t \in [-\tau, 0]$, $\tau = \max_{1\leq i\leq n} \tau_j(t)$, $\varphi_i \in C$ ($C \triangleq C[[-\tau, 0], R^n]$), and φ_i is assumed to be bounded and continuous on $[-\tau, 0]$.

Definition 2.1 [22, 23] Let $x(t)\colon\ R \to R^n$ be continuous in t. $x(t)$ is said to be *almost periodic* on R if, for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$ such that, for any interval with length $l(\varepsilon)$, there is a number $\delta = \delta(\varepsilon)$ in this interval such that $|x(t+\delta) - x(t)| < \delta l$, for any $t \in R$.

Throughout this paper, we assume that $c_{ij}(t)$, $d_{ij}(t)$, $I_i(t)$, $\varphi_i(t)$ are continuous almost periodic functions. For an arbitrary continuous function $f(t)\colon\ R \to R$, we define

$$\overline{f} = \sup_{t\in R} |f(t)|, \quad \underline{f} = \inf_{t\in R} |f(t)|.$$

We list some assumptions which will be used in this paper as follows:

(H1) $a_i(t)$ is continuous and $0 < \underline{a}_i \le a_i(t) \le \overline{a}_i$ for all $t \in R$, $i = 1, 2, \dots, n$.

(H2) There are positive constants k_i such that $\dot{b}_i(\cdot) \ge k_i$, $\dot{b}_i(\cdot)$ denotes the derivative of $b_i(\cdot)$, and $b_i(0) = 0$, $i = 1, 2, \dots, n$.

(H3) There are constants $\alpha_j > 0$ such that $|f_j(x) - f_j(y)| \le \alpha_j |x - y|$ for any x, $y \in R$, and $f_j(0) = 0$, $j = 1, \dots, n$.

Definition 2.2 The almost periodic solutions $x^*(t)$ of system (1) is said to be *global exponentially stable*, if there exist constants $\varepsilon > 0$ and $M \ge 1$ such that

$$|x_i(t) - x^*| \le M\|\varphi - \varphi^*\| e^{-\varepsilon t}, \quad t > 0, \quad i = 1, 2, \dots, n,$$

where φ^* is the initial value of x^*, $\|\varphi - \varphi^*\| = \sup\limits_{-\infty \le s \le 0} \max\limits_{1 \le i \le n} |\varphi_i(s) - \varphi_i^*(s)|$.

Definition 2.3 [21] Let $y \in R^n$ and $P(t, y)$ be a $n \times n$ continuous matrix defined on $R \times R^n$. For any continuous function $v(t)\colon\ R \to R^n$, the following system

$$\dot{y}(t) = P(t, v(t))y(t)$$

is said to be an *exponential dichotomy* on R if there exist constants $k, l > 0$, projection S and the fundamental matrix $Y_v(t)$ satisfying

$$\begin{aligned} \|Y_v(t) S Y_v^{-1}(s)\| &\le k e^{-l(t-s)} \quad \text{for} \quad t \ge s, \\ \|Y_v(t)(I - S) Y_v^{-1}(s)\| &\le k e^{-l(t-s)} \quad \text{for} \quad t \le s. \end{aligned}$$

Lemma 2.1 [21] *If the linear system $\dot{y}(t) = P(t, v(t))y(t)$ has an exponential dichotomy, then almost periodic system*

$$\dot{y}(t) = P(t, v(t))y(t) + g(t, v(t))$$

has a unique almost periodic solution $y(t)$ which can be expressed as follows:

$$y(t) = \int_{-\infty}^{t} Y_v(t) S Y_v^{-1}(s) g(s, v(s))\, ds - \int_{t}^{\infty} Y_v(t)(I - S) Y_v^{-1}(s) g(s, v(s))\, ds.$$

Lemma 2.2 [22, 23] *Assume that $e_i(t)$ is an almost periodic function and*

$$\lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} e_i(s)\, ds > 0, \quad i = 1, 2, \dots, n.$$

Then the linear system $\dot{y}(t) = e(t)y(t)$ admits an exponential dichotomy, where $e(t) = \mathrm{diag}\{e_i(t)\}$.

Definition 2.4 [24, 25] A real $n \times n$ matrix $W = (w_{ij})_{n \times n}$ is said to be an *M-matrix* if $w_{ij} \le 0$, $i, j = 1, 2, \dots, n$, $i \ne j$, and $W^{-1} \ge 0$, where W^{-1} denotes the inverse of W.

Lemma 2.3 [24, 25] *Let $W = (w_{ij})_{n \times n}$ with $w_{ij} \le 0$, $i, j = 1, 2, \dots, n$, $i \ne j$. Then the following statements are equivalent:*

(1) *W is an M-matrix;*

(2) *there exists a vector* $\eta = (\eta_1, \eta_2, \ldots, \eta_n) > (0, 0, \ldots, 0)$ *such that* $\eta W > 0$;

(3) *there exists a vector* $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > (0, 0, \ldots, 0)^{\mathrm{T}}$ *such that* $W\xi > 0$.

Lemma 2.4 [24, 25] *Let* $A \geq 0$ *be an* $n\times n$ *matrix and* $\rho(A) < 1$, *the* $(E_n - A)^{-1} \geq 0$, *where* $\rho(A)$ *denotes the spectral radius of* A.

From (H1), the antiderivative of $\frac{1}{a_i(x_i)}$ exists. We choose an antiderivative $g_i(x_i)$ of $\frac{1}{a_i(x_i)}$ that satisfies $g_i(0) = 0$. Obviously, $\dot{g}_i(x_i) = \frac{1}{a_i(x_i)}$. By $a_i(x_i) > 0$, we obtain that $g_i(x_i)$ is increasing with respect to x_i, and the inverse function $g_i^{-1}(x_i)$ of $g_i(x_i)$ is existential, continuous, and differentiable. So, $\dot{g_i}^{-1}(x_i) = a_i(x_i)$, where $\dot{g_i}^{-1}(x_i)$ is the derivative of $g_i^{-1}(x_i)$ with respect to x_i, and composition function $b_i(g_i^{-1}(z))$ is differentiable. Denote $u_i(t) = g_i(x_i(t))$. It is easy to see that $\dot{u}_i(t) = \dot{g}_i(x_i)\dot{x}_i(t) = \frac{\dot{x}_i(t)}{a_i(x_i)(t)}$ and $x_i(t) = g_i^{-1}(u_i)$. Substituting these equalities into system (1) gives that

$$\begin{aligned} \dot{u}_i(t) &= -b_i(g_i^{-1}(u_i(t))) + \sum_{j=1}^{n} c_{ij}(t) f_j(g_j^{-1}(u_j(t))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(t) f_j(g_j^{-1}(u_j(t - \tau_j(t)))) - I_i(t), \quad t \geq 0 \\ u_i(t) &= g_i(\varphi_i(t)) \triangleq \phi_i(t), \quad -\tau \leq t \leq 0. \end{aligned} \tag{2}$$

Considering $b_i(g_i^{-1}(u_i(t))) = \dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i} \cdot u_i(t)$, system (2) can be written as the following system:

$$\begin{aligned} \dot{u}_i(t) &= -e_i(u_i(t))u_i(t) + \sum_{j=1}^{n} c_{ij}(t) f_j(g_j^{-1}(u_j(t))) \\ &\quad + \sum_{j=1}^{n} d_{ij}(t) f_j(g_j^{-1}(u_j(t - \tau_j(t)))) - I_i(t), \quad t \geq 0, \\ u_i(t) &= \phi_i(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{3}$$

where $e_i(u_i)(t) \triangleq \dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i}$, $\dot{b}_i(g_i^{-1}(u_i(t)))|_{z=\varepsilon_i}$ denotes the derivative of $b_i(g_i^{-1}(z))$ at point $z = \varepsilon_i$, $z \in R$, ε_i is between 0 and $u_i(t)$.

Let $e_i(u_i)(t)$ be an almost periodic function, the system (1) has a unique almost periodic solution which is globally exponentially stable if and only if system (3) has a unique almost periodic solution which is globally exponentially stable.

It is easy to see that $|g_i^{-1}(u) - g_i^{-1}(v)| = |\dot{g}_i^{-1}(\mu)(u - v)| = |a_i(\mu)||u - v| \leq \overline{a}_i|u - v|$, where μ is between u and v.

For convenience, we introduce some notations. We will use $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$ to denote a column vector, in which the symbol $(^{\mathrm{T}})$ denotes the transpose of a vector. For matrix $A = (a_{ij})_{n\times n}$, A^{T} denotes the transpose of A, and E_n denotes the identity matrix of size n. A matrix or vector $A \geq 0$ means that all entries of A are greater than or equal to zero. $A > 0$ can be defined similarly. For matrices or vectors A and B, $A \geq B$ (rep. $A > B$) means that $A - B \geq 0$ (rep. $A - B > 0$).

3 Existence and Uniqueness of Almost Periodic Solutions

In this section, we shall discuss the existence and uniqueness of the almost periodic solution of system (3).

Theorem 3.1 *Suppose that (H1)–(H3) are satisfied, and* $\rho(\underline{A}^{-1}(\overline{C}+\overline{D}))<1$*, where* $\overline{C}=(\overline{c}_{ij}\alpha_j\overline{a}_j)_{n\times n}$, $\overline{D}=(\overline{d}_{ij}\alpha_j\overline{a}_j)_{n\times n}$, $\underline{A}=\mathrm{diag}(k_1\underline{a}_1,k_2\underline{a}_2,\dots,k_n\underline{a}_n)$*. Then, there exists exactly one almost periodic solution of system (3).*

Proof Set the vector $\hat{u}(t)=(\hat{u}_1(t),\ \hat{u}_2(t),\dots,\ \hat{u}_n(t))^{\mathrm{T}}$, for $\forall\, x\in R^n$, we define the norm: $\|\hat{u}(t)\|=\max_{1\le i\le n}|\hat{u}_i(t)|$. Let $\Lambda=\{\hat{u}(t)=\mathrm{col}\{\hat{u}_i(t)\mid \hat{u}(t)\colon\ R\to R^n$, is continuous almost periodic function$\}$. For any $\hat{u}\in\Lambda$, we define its induced model as follows:

$$\|\hat{u}\|=\sup_{t\in R}\|\hat{u}(t)\|=\sup_{t\in R}\max_{1\le i\le n}|\hat{u}_i(t)|.$$

Obviously, $(\Lambda,\|\cdot\|)$ is a Banach space. For any $\{\hat{u}_i(t)\}\in\Lambda$, consider the following system:

$$\begin{aligned}\dot{u}_i(t)&=-e_i(\hat{u}_i(t))u_i(t)+\sum_{j=1}^{n}c_{ij}(t)f_j(g_j^{-1}(\hat{u}_j(t)))\\&\quad+\sum_{j=1}^{n}d_{ij}(t)f_j(g_j^{-1}(\hat{u}_j(t-\tau_j(t))))-I_i(t),\end{aligned}\tag{4}$$

where $i=1,2,\dots,n$. From H(1) and H(2), we get $e_i(u_i(t))\ge k_i\underline{a}_i>0$ and

$$\lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T}e_i(u_i(s))\,ds\ge\lim_{T\to+\infty}k_i\underline{a}_i>0.$$

Similar to the analysis of [21], we know that following system:

$$\dot{U}(t)=Q(\hat{u}(t))U(t)$$

has an exponential dichotomy on R, where

$$Q(\hat{u})(t))=\mathrm{diag}(e_1(\hat{u}_1(t)),\ e_2(\hat{u}_2(t)),\dots,\ e_n(\hat{u}_n(t))).$$

Thus by Lemma 2.1 and Lemma 2.2, system (4) has a unique almost periodic solution $u_{\hat{u}}(t)$ which can be expressed as follows:

$$\begin{aligned}u_{\hat{u}}(t)=\mathrm{col}\bigg\{\int_{-\infty}^{t}e^{-\int_s^t e_i(\hat{u}(\sigma))d\sigma}\bigg[\sum_{j=1}^{n}c_{ij}(s)f_j(g_j^{-1}(\hat{u}_j(s)))\\+\sum_{j=1}^{n}d_{ij}(s)f_j\big(g_j^{-1}(\hat{u}_j(s-\tau_{ij}(s)))\big)-I_i(s)\bigg]ds\bigg\}.\end{aligned}\tag{5}$$

Now define a mapping $T\colon\ \Lambda\to\Lambda$ by setting

$$T_{\hat{x}}(t)=x_{\hat{x}}(t),\quad\forall\,\hat{x}\in\Lambda.$$

Next, we prove that T is a contraction mapping. For any $\forall\,\hat{x},x^*\in\Lambda$, from (H3) we have

$|T(\hat{u}(t))-T(u^*(t))|$

$$
\begin{aligned}
&= \Bigg(\Bigg|\int_{-\infty}^{t} e^{-\int_s^t e_1(\hat u(\sigma))d\sigma}\Bigg[\sum_{j=1}^{n} c_{1j}(s)\big(f_j(g_j^{-1}(\hat u_j(s))) - f_j(g_j^{-1}(u_j^*(s)))\big)\\
&+\sum_{j=1}^{n} d_{1j}(s)\big(f_j(g_j^{-1}(\hat u_j(s-\tau_{1j}(s)))) - f_j(g_j^{-1}(u_j^*(s-\tau_{1j}(s))))\big)\Bigg]ds\Bigg|,\ \ldots,\\
&\Bigg|\int_{-\infty}^{t} e^{-\int_s^t e_n(\hat u(\sigma))d\sigma}\Bigg[\sum_{j=1}^{n} c_{nj}(s)\big(f_j(g_j^{-1}(\hat u_j(s)) - f_j g_j^{-1}(u_j^*(s))))\big)\\
&+\sum_{j=1}^{n} d_{nj}(s)\big(f_j(g_j^{-1}(\hat u_j(s-\tau_{nj}(s)))) - f_j(g_j^{-1}(u_j^*(s-\tau_{nj}(s))))\big)\Bigg]ds\Bigg|\Bigg)^{\mathrm{T}}\\
&\le \Bigg(\int_{-\infty}^{t} e^{-k_1\underline{a}_1(t-s)}\Bigg[\sum_{j=1}^{n}\overline{c}_{1j}\alpha_1\overline{a}_1|\hat u_j(s)-u_j^*(s)| \qquad (6)\\
&+\sum_{j=1}^{n}\overline{d}_{1j}\alpha_1\overline{a}_1|\hat u_j(s-\tau_{1j}(s)) - x_j^*(s-\tau_{1j}(s))|\Bigg]ds,\ \ldots,\\
&\int_{-\infty}^{t} e^{-k_n\underline{a}_n(t-s)}\Bigg[\sum_{j=1}^{n}\overline{c}_{nj}\alpha_n\overline{a}_n|\hat u_j(s)-u_j^*(s)|\\
&+\sum_{j=1}^{n}\overline{d}_{nj}\alpha_n\overline{a}_n|\hat u_j(s-\tau_{nj}(s)) - u_j^*(s-\tau_{nj}(s))|\Bigg]ds\Bigg)^{\mathrm{T}}\\
&\le \Bigg(\sum_{j=1}^{n}(k_1\underline{a}_1)^{-1}(\overline{c}_{1j}+\overline{d}_{1j})\alpha_1\overline{a}_1\sup_{t\in R}|\hat u_j(t)-u_j^*(t)|,\ \ldots,\\
&\sum_{j=1}^{n}(k_n\underline{a}_n)^{-1}(\overline{c}_{nj}+\overline{d}_{nj})\alpha_n\overline{a}_n\sup_{t\in R}|\hat u_j(t)-u_j^*(t)|\Bigg)^{\mathrm{T}},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\Big(\sup_{t\in R}|(T(\hat u(t)) - T(u^*(t)))_1|,\ \ldots,\ \sup_{t\in R}|(T(\hat u(t)) - T(u^*(t)))_n|\Big)^{\mathrm{T}}\\
&\qquad\le \Bigg(\sum_{j=1}^{n}(k_1\underline{a}_1)^{-1}(\overline{c}_{1j}+\overline{d}_{1j})\alpha_1\overline{a}_1\sup_{t\in R}|\hat u_j(t)-u_j^*(t)|,\ \ldots,\\
&\qquad\sum_{j=1}^{n}(k_n\underline{a}_n)^{-1}(\overline{c}_{nj}+\overline{d}_{nj})\alpha_n\overline{a}_n\sup_{t\in R}|\hat u_j(t)-u_j^*(t)|\Bigg)^{\mathrm{T}} \qquad (7)\\
&\qquad\le F\Big(\sup_{t\in R}|\hat u_1(t)-u_1^*(t)|,\ \ldots,\ \sup_{t\in R}|\hat u_n(t)-u_n^*(t)|\Big)^{\mathrm{T}}
\end{aligned}
$$

where $F = \underline{A}^{-1}(\overline{C}+\overline{D})$. Let m be a positive integer. Then, from (7), we get

$$
\begin{aligned}
&\Big(\sup_{t\in R}|(T^m(\hat u(t)) - T^m(u^*(t)))_1|,\ \ldots,\ \sup_{t\in R}|(T^m(\hat u(t)) - T^m(u^*(t)))_n|\Big)^{\mathrm{T}}\\
&= \Big(\sup_{t\in R}|(T(T^{m-1}(\hat u(t)) - T^m(u^*(t))))_1|,\ldots,\sup_{t\in R}|(T(T^{m-1}(\hat u(t)) - T^m(u^*(t))))_n|\Big)^{\mathrm{T}}
\end{aligned}
$$

$$\le F\Big(\sup_{t\in R}|(T^{m-1}(\hat{u}(t))-T^{m-1}(u^*(t)))_1|,\dots,\sup_{t\in R}|(T^{m-1}(\hat{u}(t))-T^{m-1}(u^*(t)))_n|\Big)^{\mathrm{T}}$$
$$\le F^m\Big(\sup_{t\in R}|(T(\hat{u}(t))-T(u^*(t)))_1|,\dots,\sup_{t\in R}|(T(\hat{u}(t))-T(u^*(t)))_n|\Big)^{\mathrm{T}}$$
$$\le F^m\Big(\sup_{t\in R}|\hat{u}_1(t)-u_1^*(t)|,\dots,\sup_{t\in R}|\hat{u}_n(t)-u_n^*(t)|\Big)^{\mathrm{T}}. \quad (8)$$

Since $\rho(F)<1$, we obtain $\lim_{n\to+\infty}F^m=0$, which implies that there exists a positive integer N and a positive integer $\beta<1$ such that

$$F^N=(\underline{A}^{-1}(\overline{C}+\overline{D}))^N=(h_{ij})_{n\times n}, \quad \text{and} \quad \sum_{j=1}^{n}h_{ij}\le\beta, \quad i=1,2,\dots,n. \quad (9)$$

In view of (8) and (9), we have

$$\begin{aligned}|(T^N(\hat{u}(t))-T^N(u^*(t)))_i| &\le \sup_{t\in R}|(T^N(\hat{u}(t))-T^N(u^*(t)))_i|\\ &\le \sum_{j=1}^{n}h_{ij}\sup_{t\in R}|\hat{u}_j(t)-u_j^*(t)|\\ &\le \Big(\sup_{t\in R}\max_{1\le i\le n}|\hat{u}_j(t)-u_j^*(t)|\Big)\sum_{j=1}^{n}h_{ij}\le\beta\|\hat{u}(t)-u^*(t)\|,\end{aligned}$$

for all $t\in R$, $i=1,2,\dots,n$. It follows that

$$\|T^N(\hat{u}(t))-T^N(u^*(t))\|=\sup_{t\in R}\max_{1\le i\le n}|(T^N(\hat{u}(t))-T^N(u^*(t)))_i|\le\beta\|\hat{u}(t)-u^*(t)\|.$$

This implies that the mapping $T^N\colon\ \Lambda\to\Lambda$ is a contraction mapping.

By Banach fixed point theorem, there exists a unique fixed point $u^*\in\Lambda^*$ such that $Tu^*=u^*$. From (4) and (5), we know that u^* satisfies system (3), therefore, it is the unique almost periodic solution of system (3). We complete the proof. □

4 Exponential Stability of Almost Periodic Solutions

In this section, we shall discuss the global exponential stability of the almost periodic solution of system (3).

Theorem 4.1 *Suppose that (H1)–(H3) are satisfied, and the condition in Theorem 3.1 holds, then there exists exactly one almost periodic solution of system (3) which is exponentially stable, i.e. all other solutions of system (3) converge to this almost periodic solution exponentially.*

Proof By Theorem 3.1, we have known that system (3) has a unique almost periodic solution, then we shall prove the exponential stability of almost periodic solution.

Let $u(t)=(u_1(t),u_2(t),\dots,u_n(t))^{\mathrm{T}}$ be an arbitrary solution and $u^*(t)=(u_1^*(t),u_2^*(t),\dots,u_n^*(t))^{\mathrm{T}}$ be an almost periodic solution of system (3) with initial values $\phi(t)=(\phi_1(t),\phi_2(t),\dots,\phi_n(t))^{\mathrm{T}}$ and $\phi^*(t)=(\phi_1^*(t),\phi_2^*(t),\dots,\phi_n^*(t))^{\mathrm{T}}$, respectively. Set

$$y_i(t)=u_i(t)-(u_i^*(t), \quad F_j(y_j(t))=f_j(y_j(t)+(u_j^*(t))-f_j(u_j^*(t)),$$

where $i, j = 1, 2, \ldots, n$. It is easy to see that system (3) can be reduced to the following system:

$$\dot{y_i}(t) = -e_i(u_i(t))y_i(t) + \sum_{j=1}^{n} c_{ij}(t)F_j(y_j(t)) + \sum_{j=1}^{n} d_{ij}(t)F_j(y_j(t-\tau_{ij}(t))). \tag{10}$$

Since $\rho(F) = \rho(\underline{A}^{-1}(\overline{C}+\overline{D})) < 1$, it follows from Lemma 2.4 that $E_n - \underline{A}^{-1}(\overline{C}+\overline{D})$ is an M-matrix. In view of Lemma 2.3, there exists a constant vector $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\xi}_n)^{\mathrm{T}} > (0, 0, \ldots, 0)^{\mathrm{T}}$ such that

$$(E_n - \underline{A}^{-1}(\overline{C}+\overline{D}))\overline{\xi} > (0, 0, \ldots, 0)^{\mathrm{T}}.$$

That is,

$$-k_i\underline{a}_i\overline{\xi}_i + \sum_{j=1}^{n} \overline{\xi}_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n.$$

Therefore, we can choose a constant $d > 1$ such that

$$\xi = d\overline{\xi} > \sup_{\tau \le t \le 0} |y_i(t)|, \quad i = 1, 2, \ldots, n,$$

and

$$-k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i = \left[-k_i\underline{a}_i\overline{\xi}_i + \sum_{j=1}^{n} \overline{\xi}_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i \right] d < 0,$$

where $i = 1, 2, \ldots, n$. Set

$$M_i(\varepsilon) = \varepsilon\xi_i - k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\varepsilon\tau})\alpha_i\overline{a}_i, \quad i = 1, 2, \ldots, n.$$

Clearly, $M_i(\varepsilon)$, $i = 1, 2, \ldots, n$, are continuous functions on $[0, \omega_0]$. Since

$$M_i(0) = -k_i\underline{a}_i\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n,$$

we can choose a positive constant $\omega \in [0, \omega_0]$ such that

$$M_i(\omega) = (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n} \xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i < 0, \quad i = 1, 2, \ldots, n. \tag{11}$$

We consider the Lyapunov functional

$$V_i(t) = |y_i(t)|e^{\omega t}, \quad i = 1, 2, \ldots, n. \tag{12}$$

Obviously, for any $y_i(t) \neq 0$, $V_i(t) > 0$. Calculating the upper right derivative of $V_i(t)$ along the solution $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^{\mathrm{T}}$ of system (10) with the initial value $\overline{\phi} = \phi - \phi^*$, we have

$$\begin{aligned} D^+(V_i(t)) &\le -k_i\underline{a}_i|y_i(t)|e^{\omega t} + \sum_{j=1}^{n} \overline{c}_{ij}|y_i(t)|e^{\omega t} + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t-\tau_{ij}(t))|e^{\omega t} + \omega|y_i(t)|e^{\omega t} \\ &= \left[(\omega - k_i\underline{a}_i)|y_i(t)| + \sum_{j=1}^{n} \overline{c}_{ij}|y_i(t)|\alpha_i\overline{a}_i + \sum_{j=1}^{n} \overline{d}_{ij}|y_i(t-\tau_{ij}(t))|\alpha_i\overline{a}_i \right] e^{\omega t} \end{aligned} \tag{13}$$

where $i = 1, 2, \dots, n$. We claim that

$$V_i(t) = |y_i(t)|e^{\omega t} < \xi_i, \quad \text{for all} \quad t > 0, \quad i = 1, 2, \dots, n. \tag{14}$$

Contrarily, there must exist $i \in \{i = 1, 2, \dots, n\}$ and $t_i > 0$ such that

$$V_i(t_i) = \xi_i \quad \text{and} \quad V_j(t) < \xi_j, \quad \forall\, t \in (-\infty, t_i), \quad j = 1, 2, \dots, n, \tag{15}$$

which implies that

$$V_i(t_i) - \xi_i = 0 \quad \text{and} \quad V_j(t) - \xi_j < 0, \quad \forall\, t \in (-\infty, t_i), \quad j = 1, 2, \dots, n. \tag{16}$$

Together with (13) and (16), we obtain

$$\begin{aligned}
0 &\le D^+(V_i(t_i) - \xi_i) = D^+V_i(t_i) \\
&\le \left[(\omega - k_i\underline{a}_i)|y_i(t)| + \sum_{j=1}^{n}\overline{c}_{ij}|y_i(t)|\alpha_i\overline{a}_i + \sum_{j=1}^{n}\overline{d}_{ij}|y_i(t-\tau_{ij}(t))|\alpha_i\overline{a}_i\right]e^{\omega t} \\
&= (\omega - k_i\underline{a}_i)\xi_i + \alpha_i\overline{a}_i\left(\sum_{j=1}^{n}\overline{c}_{ij}|y_i(t_i)|e^{\omega t_i} + \sum_{j=1}^{n}\overline{d}_{ij}|y_i(t_i-\tau_{ij}(t_i))|e^{\omega(t_i-\tau_{ij}(t_i))}e^{\omega\tau_{ij}(t_i)}\right) \\
&\le (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n}\xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i.
\end{aligned} \tag{17}$$

Thus

$$0 \le (\omega - k_i\underline{a}_i)\xi_i + \sum_{j=1}^{n}\xi_j(\overline{c}_{ij} + \overline{d}_{ij}e^{\omega\tau})\alpha_i\overline{a}_i$$

which contradicts (11). Hence, (14) holds. It follows that

$$|y_i(t)| < \max_{1\le i\le n}\{\xi_i\}e^{-\omega t}. \tag{18}$$

Letting $\|\overline{\phi}\| = \|\phi - \phi^*\| > 0$, it follows from (18) that we can choose a constant $M > 1$ such that

$$|x_i(t) - x_i^*(t)| = |y_i(t)| \le \max_{1\le i\le n}\{\xi_i\}e^{-\omega t} \le M\|\phi - \phi^*\|e^{-\omega t}, \tag{19}$$

where $i = 1, 2, \dots, n$, $t > 0$. Thus, the almost periodic solution of system (3) is globally exponentially stable.

We complete the proof. □

Corollary 4.1 Suppose that (H1)–(H3) are satisfied, and $E_n - \underline{A}^{-1}(\overline{C} + \overline{D})$ is an M-matrix, then there exists exactly an almost periodic solution of system (3) which is exponentially stable, i.e. all other solutions of system (3) converge to this almost periodic solution exponentially.

Proof Notice that $E_n - \underline{A}^{-1}(\overline{C} + \overline{D})$ is an M-matrix, it follows that there exists a vector $\eta = (\eta_1, \eta_2, \dots, \eta_n)^{\mathrm{T}} > (0, 0, \dots, 0)^{\mathrm{T}}$ such that

$$(E_n - \underline{A}^{-1}(\overline{C} + \overline{D}))\eta > (0, 0, \dots, 0)^{\mathrm{T}}.$$

That is,

$$-k_i\underline{a}_i\eta + \sum_{j=1}^{n}(\overline{c}_{ij} + \overline{d}_{ij})\alpha_i\overline{a}_i\eta < 0, \quad i = 1, 2, \ldots, n.$$

Therefore, Corollary 4.1 follows immediately from Theorem 4.1. □

Remark 4.1 In Theorem 4.1 and Corollary 4.1, we do not need the assumptions on boundedness, monotonicity, and differentiability for the activation functions. Clearly, the proposed results are different from those in [5, 6, 14] and the references cited therein. Therefore, our results are new and they complement previously known results.

5 An Example

In this section, we give an example to illustrate that our results are feasible.

Example 5.1 Consider the following system with continuously distributed delays:

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{2} c_{ij}(t)f_j(x_j(t)) - \sum_{j=1}^{2} d_{ij}(t)f_j(x_j(t-\tau_j(t))) + I_i(t)\right], \tag{20}$$

where $i = 1, 2$. Let $f_j(x) = \frac{1}{2}(|x+1| - |x-1|)$, we have $\alpha_j = 1$ $(j = 1, 2)$.

Taking

$$(a_1(x_1(t)), a_2(x_2(t)))^{\mathrm{T}} = \left(2 - \frac{1}{10\pi}\arctan x_1(t), 2 + \frac{1}{10\pi}\arctan x_2(t)\right)^{\mathrm{T}},$$

$$(b_1(x_1(t)), b_2(x_2(t)))^{\mathrm{T}} = (x_1, x_2)^{\mathrm{T}}, \quad I_1(t) = \frac{9}{5}\sin t, \quad I_2(t) = \frac{9}{5}\cos t,$$

thus we obtain $\underline{a}_1 = \underline{a}_2 = 1$, $\overline{a}_1 = \overline{a}_2 = 3$, $\underline{b}_1 = \underline{b}_2 = \overline{b}_1 = \overline{b}_2 = 1$, $\overline{I}_1 = \overline{I}_1 = \frac{9}{5}$, $k_1 = k_2 = 1$. Let

$$\begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{13}\sin t & \frac{1}{13}\sin 2t \\ \frac{1}{13}\sin 3t & \frac{1}{13}\sin 4t \end{pmatrix},$$

$$\begin{pmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{13}\cos t & \frac{1}{13}\cos 2t \\ \frac{1}{13}\cos 3t & \frac{1}{13}\cos 4t \end{pmatrix}.$$

Noting that $\overline{c}_{11} = \overline{c}_{12} = \overline{c}_{21} = \overline{c}_{22} = \overline{d}_{11} = \overline{d}_{12} = \overline{d}_{21} = \overline{d}_{22} = \dfrac{1}{13}$, we get

$$\underline{A}^{-1}(\overline{C} + \overline{D}) = \begin{pmatrix} \frac{6}{13} & \frac{6}{13} \\ \frac{6}{13} & \frac{6}{13} \end{pmatrix}.$$

So, we have

$$\rho(\underline{A}^{-1}(\overline{C} + \overline{D})) = \frac{12}{13} < 1.$$

Thus, it follows from Theorem 3.1 and Theorem 4.1 that system (20) has exactly a unique almost periodic solution, which is globally exponentially stable.

Remark 5.1 System (20) is a simple form of Cohen-Grossberg neural networks with variable delays. In this system, $L_1^a = L_2^a = \frac{1}{30}$, $L_1^{ab} = L_2^{ab} = 1$. If we apply Corollary 4.1 in [15, 16], and choose $\eta = (\eta_1, \eta_2) = (1, 1)$, we obtain $\delta = \frac{26}{5}$, $\rho(K) = \frac{1800}{1781} > 1$, this doesn't satisfy the conditions in Corollary 4.1 in [15, 16]. So, the results in [15, 16] cannot be applicable to this system. This implies that our results are essentially new.

Remark 5.2 Since $f_1(x) = f_2(x) = \frac{1}{2}(|x+1| - |x-1|)$, we can easily verify that the assumptions of boundedness, monotonicity, and differentiability for the activation functions are not available. So, the proposed results in [5, 6, 14] and the references cited therein can not be applicable to system (20).

6 Conclusion

In this paper, the existence and exponential stability of almost periodic solutions for Cohen-Grossberg neural networks with variable delays are considered. Some new sufficient conditions are obtained by applying Banach fixed point theory and differential inequality techniques. Some previous results are improved and extended.

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