Nonlinear Dynamics and Systems Theory, 8(3) (2008) 255–268

An Extension of Barbashin–Krasovskii–LaSalle Theorem to a Class of Nonautonomous Systems

Radu Balan *

Department of Mathematics and CSCAMM, University of Maryland, College Park, MD 20742

Received: July 07, 2007; Revised: April 6, 2008

Abstract: In this paper we give an extension of the Barbashin-Krasovskii-LaSalle theorem to a class of time-varying dynamical systems, namely the class of systems for which the restricted vector field to the zero-set of the time derivative of the Liapunov function is time invariant and this set includes some trajectories. Our goal is to improve the sufficient conditions for the case of uniform asymptotic stability of the equilibrium. We obtain an extension of a well-known result of linear zero-state detectability to nonlinear systems, as well as a robust stabilizability result of nonlinear affine control systems.

Keywords: *Invariance Principle; Liapunov functions; detectability; robust stabilizability.*

Mathematics Subject Classification (2000): 34D20, 93D20.

1 Introduction and Main Results

Let us consider the following time-varying dynamical system:

$$\dot{x} = f(t, x), \quad x \in D, \ t \in R, \tag{1}$$

where D is a domain in R^n containing the origin ($0 \in D \subset R^n$). About f we suppose the following:

1) $f(t, 0) = 0$, for any $t \in R$;

2) uniformly continuous in t, uniformly in $x \in D$, i.e. $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ such that $\forall t_1, t_2 \in R, |t_1 - t_2| < \delta_\varepsilon$ and $\forall x \in D$, $\| f(t_1, x) - f(t_2, x) \| < \varepsilon$;

3) uniformly local Lipschitz continuous in x for any $t \in R$, i.e. for any compact set $K \subset D$, there exists a positive constant $L_K > 0$ such that:

$$\|f(t, x) - f(t, y)\| \le L_K \|x - y\| \quad \text{for any } x, y \in K \text{ and } t \in R.$$

* Corresponding author: rvbalan@math.umd.edu

4) bounded in time, that means there exists a continuous function $M : D \to R$ such that:

$$\|f(t,x)\| \leq M(x) \quad \text{for any } t \in R.$$

With these hypotheses we know that for any $(t_0, x_0) \in R \times D$ there exists a unique solution of the Cauchy problem:

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \tag{2}$$

with the initial data (t_0, x_0). We denote by $x(t; t_0, x_0)$ this solution. One can define this solution for $t \in (t_0 - T, t_0 + T)$ where $T = \sup_{r>0, B_r(x_0) \subset D} \frac{r}{\|f\|_{B_r(x_0)}}$, the supremum is taken over all positive radius such that the ball centered around x_0, $B_r(x_0) = \{x \in R^n | \|x - x_0\| < r\}$, is completely included in D and $\|f\|_{B_r(x_0)} = \sup_{(t,x) \in R \times \bar{B}_r(x_0)} \|f(t,x)\|$ is a supremum norm of f with respect to $B_r(x_0)$ (where is no confusion we denote $B_r = B_r(0)$). The function $\gamma_{t,t_0}(x_0) = x(t; t_0, x_0)$ is well defined for some bounded open set S, $\gamma_{t,t_0} : S \to U \subset D$ (with U open and bounded) and it is Lipschitz continuous with a Lipschitz constant given by $L = exp(L_U |t - t_0|)$ (L_U being the Lipschitz constant associated to f, as above, on the compact set $\bar{U}$). All these results can be found in any textbook of differential equations (for instance see [6]).

Our concern regards the stability behaviour of the equilibrium point $\bar{x} = 0$. First we recall some definitions about stability (in Liapunov sense).

Definition 1.1 We say the equilibrium point $\bar{x} = 0$ for (1) is *uniformly stable*, if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $t_0 \in R$ and $x_0 \in R$ with $\|x_0\| < \delta_\varepsilon$ the solution $x(t; t_0, x_0)$ is defined for all $t \geq t_0$ and furthermore $\|x(t; t_0, x_0)\| < \varepsilon$, for every $t > t_0$.

Definition 1.2 We say that the equilibrium point $\bar{x} = 0$ for (1) is *uniformly asymptotic stable*, if it is uniformly stable and there exists a $\delta > 0$ such that for any $t_0 \in R$ and $x_0 \in D$ with $\|x_0\| < \delta$ the solution $x(t; t_0, x_0)$ is defined for every $t \geq t_0$ and $\lim_{t \to \infty} x(t; t_0, x_0) = 0$.

If in the definition of uniform stability we interchange "there exists $\delta_\varepsilon > 0$" with "for any $t_0 \in R$" (thus δ will depend on ε and t_0, $\delta_{\varepsilon, t_0}$) then the equilibrium is said (just) *stable*. If we proceed the same in the second definition we obtain that the equilibrium is *asymptotic stable*. For time-invariant systems there is no distinction between uniform stability and stability, or uniform asymptotic stability and asymptotic stability. In general case, the uniform (asymptotic) stability implies (asymptotic) stability, but the converse is not true (see for instance [7]).

We say that the dynamics (1) has a *positive invariant set* N if for any $t_0 \in R$ and $x_0 \in N$ the solution $x(t; t_0, x_0) \in N$ for all $t \geq t_0$ for which it is well-defined. Then it makes sense to consider the *dynamics restricted to* N, i.e. the function:

$$X : R^+ \times R \times N \to N, \quad X(\tau; t_0, x_0) = x(\tau + t_0; t_0, x_0),$$

where τ runs up to a maximal value depending on (t_0, x_0). Moreover, by considering the case of f from (1) we obtain that $X(\tau; t_0, 0) = 0$, for any $\tau > 0$, $t_0 \in R$. Therefore we may define the corresponding stability properties of the restricted dynamics as above, where we replace D by N.

The main result of this paper is given by the following theorem:

Theorem 1.1 *Consider the time-varying dynamical system* (1) *for which f has the properties* $1) - 4)$. *Suppose there exists a function $V : D \to R$ of class $\mathcal{C}^1$ such that:*

$H1)$ $V(x) \geq 0$ *for every* $x \in D$ *and* $V(0) = 0$*;*

$H2)$ *There exists a continuous function* $W : D \to R$ *such that*

$$\frac{dV}{dt}(t,x) = \nabla V(x) \cdot f(t,x) \leq W(x) \leq 0.$$

$H3)$ *Let* $E = \{x \in D | W(x) = 0\}$ *denote the zero-set (or kernel) of* W*; suppose that f restricted to E is time-invariant (i.e.* $f(t,x) = f(t_0,x)$*, for every* $t \in R$ *and* $x \in E$*). Let us denote by N the maximal positive invariant set in E, i.e. for any* $x_0 \in N$ *and* $t_0 \in R$, $x(t;t_0,x_0) \in N$*, for every* $t \in [t_0, t_0 + T_{x_0})$ *in the maximal interval of definition of the solution.*

Then the dynamics (1) *has at* $\bar{x} = 0$ *an uniformly asymptotic stable equilibrium point if and only if the dynamics restricted to N has an asymptotic stable equilibrium at* $\bar{x} = 0$.

Even if it has appeared in the literature in a more general setting (I refer to [23]), it is worth mentioning the form the invariance principle takes in this context:

Theorem 1.2 (Invariance principle) *Consider the time-varying dynamical system* (1) *for which f has the properties* $1) - 4)$. *Suppose there exists a function* $V : D \to R$ *of class* $\mathcal{C}^1$ *such that:*

$H1)$ *It is bounded below, i.e.* $V(x) \geq V_0$ *for any* $x \in D$ *for some* $V_0 \in R$*;*

$H2)$ *There exists a continuous function* $W : D \to R$ *such that*

$$\frac{dV}{dt}(t,x) = \nabla V(x) \cdot f(t,x) \leq W(x) \leq 0.$$

$H3)$ *Let* $E = \{x \in D | W(x) = 0\}$ *denote the zero-set (or kernel) of* W*; suppose that f resticted to E is time-invariant (i.e.* $f(t,x) = f(t_0,x)$*, for any* $t \in R$ *and* $x \in E$*). Let us denote by N the maximal positive invariant set included in E, i.e. for any* $x_0 \in N$ *and* $t_0 \in R$, $x(t;t_0,x_0) \in N$*, for any* $t \in [t_0, t_0 + T_{x_0})$ *in the maximal interval of definition of the solution.*

Then any bounded trajectory of (1) *tends to N, i.e. if* (t_0, x_0) *is the initial data for a bounded solution included in D then:*

$$\lim_{t \to \infty} d(x(t;t_0,x_0), N) = 0. \tag{3}$$

Remark 1.1 There are two directions in which Theorem 1.1 generalizes the well-known Barbashin–Krasovskii–LaSalle's theorem (see [15], [16] or [14]); firstly it requires V to be only nonnegative and not strictly positive, secondly it applies to the case of time-varying dynamical systems. Several extensions were presented in literature dealing with the stability result.

The first result that I am referring to is Lemma 5 from [5]. In that lemma only autonomous systems are considered and the restricted dynamics is required to be attractive in the sense that all trajectories should tend to the origin. I point out that only the requirement of attractivity is not enough; this can be seen in a trivial case, namely the 2 dimensional system given by Vinograd (conform [7]), for which the origin is an attractive equilibrium but not stable, and take $V \equiv 0$. I need to point out also that, for the purposes of their paper [5], their Lemma 5 can be replaced by Theorem 3.1 of this paper without affecting the other results from their paper.

A second result has appeared in [22] but not in a general and explicit form as here. In fact in [22] the author is concerned with the stability of the large-scale systems which are already decomposed in triangular form. Thus, this result solves the problem only in the case when we can perform the observability decomposition of the dynamics (1) with respect to the output $W(x)$. This case requires a supplementary condition, namely the codistribution span by $dW, dL_fW, \ldots, dL_f^nW$ to be of constant rank on D (see [12]). Among other requirements, this geometric condition implies also that N is a manifold, whereas we do not assume here this rather strong assumption.

I acknowledge the existence of a recently published paper that deals with a similar extension of the Liapunov theorem, yet only for autonomous systems ([10]). However, we were unaware of this result at the time we were working in this field (i.e. 1993–1995). More recently, in [11], the authors extended to time-varying systems these previous results. It is interesting to note, based on this last paper and historical references therein, the autonomous version of these results were first stated and proved by Boulgakov and Kalitine in [3]. Compared to [11], here we present a stabilizability result (Theorem 4.1) tailored specifically for affine nonlinear control systems.

Remark 1.2 Some other papers deal with extensions of the invariance principle for nonautonomous systems. In two special cases, when the system is either asymptotically autonomous (in [23]) or asymptotically almost periodic (in [19]), the bounded solution tends to the largest pseudo-invariant set in E. However they use the classical Liapunov theorems to obtain the uniform boundedness of the solutions. Thus they require the existence of a strictly positive definite function playing the rôle of Liapunov function, while here we require only nonnegativeness of the Liapunov-like function. In other approaches an additional auxiliary function is assumed and by means of extra conditions the time in E is controlled (see the results of Salvadori or Matrosov, e.g. in [20]). In a third approach an extra condition on $\dot{V}$ is considered without any additional condition on the vector field; such an approach is considered in [1].

Remark 1.3 The condition that the restricted dynamics to be uniformly asymptotically stable is necessary and sufficient. Thus it is a center-manifold-type result where a knowledge about a restricted dynamics to some invariant set implies the same property of the whole dynamics. We point out here that the set N does not need to be a manifold.

Remark 1.4 One could expect that simple stability of the restricted dynamics would imply uniform stability of the restricted dynamics. But this is not true as we can see from the following example:

Example 1.1 Consider the following autonomous planar system:

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = -y^3 \end{cases}, \quad (x, y) \in R^2, \tag{4}$$

The solution of the system is given by $(x, y) \to \left(x + ln(1 + y^2t), \frac{y}{\sqrt{1+y^2t}} \right)$. It is obvious that the equilibrium is not stable but if we take $V = y^2$ we have $\frac{dV}{dt} = -2y^4$ and on the set $E = N = \{(x, 0),\ x \in R\}$ the dynamics is trivial stable $\dot{x} = 0$.

The problem is not the nonisolation of the equilibrium, but the existence of some invariant sets in any neighborhood of the equilibrium;

Remark 1.5 Theorem 1.2 is the natural generalization of the invariance principle to the class of systems considered in this paper. The conclusion of this theorem applies only to bounded trajectories. Thus we have to know apriori which solutions are bounded. Since they are bounded we can extend them indefinitely in positive time. Thus it makes sense to take the limit $t \to \infty$ in (3). We mention that a more general invariance principle can be obtained even under weaker conditions than those from here (see [23]).

The organization of the paper is the following: in the next section we give the proof of these results. In the third section we consider the autonomous case and we present the systemic consequences related to the nonlinear Liapunov equation and a special type of zero-state detectability. In the fourth section we consider a nonlinear Riccati equation (or Hamilton-Jacobi equation) and we present a result of robust stabilizability by output feedback. The last section contains the conclusions and is followed by the bibliography.

2 Proof of the Main Results

We prove by contradiction the uniform stability of the equilibrium. For this, we construct a $\mathcal{C}^1$-convergent sequence of solutions that are going away from the origin and whose limit is a trajectory, thus contradicting the hypothesis.

For the uniform asymptotic stability, we prove first that the ω-limit set of bounded trajectories is included in N (implicitly proving the invariance principle — Theorem 1.2) and then we adapt a classical trick (used for instance in Theorem 34.2 from [7]) that the convergence of trajectories in ω-limit set will attract the convergence of the bounded trajectory itself. In both steps we use essentially the time-invariant property of f restricted to E. In proving the uniform stability we also obtain that the solution can be defined on the whole positive real set (can be completely extended in future).

Theorem 1.2 (the invariance principle) will follow simply from a lemma that we state during the proof of uniform attractivity.

First we need a lemma.

Lemma 2.1 *Let f be a vector field defined on a domain D and having the properties 1-4 as above. Let $(t_i)_i$ be a sequence of real numbers and $(w_i)_i$, $w_i : [a,b] \to D$ be a sequence of trajectories for the time-translated vector field f with t_i, i.e. $\dot{w}_i(t) = f(t+t_i, w_i(t))$.*

If the trajectories are uniformly bounded, i.e. there exists $M > 0$ such that $\|w_i\|_\infty < M$, for any i, then we can extract a subsequence, denoted also by $(w_i)_i$, uniformly convergent to a function w in $\mathcal{C}^1([a,b];D)$, i.e. $w_i \to w$ and $\dot{w}_i \to \dot{w}$ both uniformly in $\mathcal{C}^0([a,b];D)$.

Proof We apply the Ascoli-Arzelà lemma twice: first to extract a subsequence such that $(w_i)_i$ is uniformly convergent and second to extract further another subsequence such that $(\dot{w}_i)_i$ is uniformly convergent. Then we obtain that $\lim_i \frac{d}{dt} w_i = \frac{d}{dt} \lim_i w_i$.

1. We verify that $(w_i)_i$ are uniformly bounded and equicontinuous. The uniformly boundedness comes from $\|w_i\|_\infty < M$. The equicontinuity comes from the uniformly boundedness of the first derivative. Indeed, since $\|w_i\| \le M$, the closed ball $\bar{B}_M$ is compact and $f(t,\cdot)$ is continuous on $\bar{B}_M$, there exists a constant A such that $\|f(t,x)\| \le A$, for any $(t,x) \in R \times \bar{B}_M$. Then

$$\|\dot{w}_i(t)\| = \|f(t+t_i, w_i(t))\| \le A, \text{ for any } i \text{ and } t \in [a,b].$$

Thus $(w_i)_i$ is relatively compact and we can extract a subsequence, that we denote also by $(w_i)_i$, which is uniformly convergent to a function $w \in \mathcal{C}^0([a,b];D)$.

2. We prove that $(\dot{w}_i)_i$ is relatively compact. We have already proved the uniform boundedness $\|\dot{w}_i\|_\infty \leq A$. For the equicontinuity we use both the uniform continuity in t and uniform local Lipschitz continuity in x, of f. Let L_M be the uniform Lipschitz constant corresponding to the compact set $\bar{B}_M$. Then

$$\|\dot{w}_i(t_1) - \dot{w}_i(t_2)\| = \|f(t_i + t_1, w_i(t_1)) - f(t_i + t_2, w_i(t_2))\| \leq$$

$$\|f(t_i + t_1, w_i(t_1)) - f(t_i + t_2, w_i(t_1))\| + \|f(t_i + t_2, w_i(t_1)) - f(t_i + t_2, w_i(t_2))\|.$$

Let $\varepsilon > 0$ be arbitrarily. Then we choose δ_1 such that $\|f(s_1, x) - f(s_2, x)\| < \frac{\varepsilon}{2}$, for any $|s_1 - s_2| < \delta_1$ and $x \in \bar{B}_M$. On the other hand: $\|f(t_i + t_2, w_i(t_1)) - t(t_i + t_2, w_i(t_2))\| \leq L_M \|w_i(t_1) - w_i(t_2)\| \leq L_M A |t_1 - t_2|$. Then we choose $\delta = \min(\delta_1, \frac{\varepsilon}{2 L_M A})$. Then the left-hand side from the above inequality is also bounded by $\frac{\varepsilon}{2}$ for any t_1, t_2 with $|t_1 - t_2| < \delta$. Thus $\|\dot{w}_i(t_1) - \dot{w}_i(t_2)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for any i and $t_1, t_2 \in [a,b]$, $|t_1 - t_2| < \delta$.

We can now extract a second subsequence from $(w_i)_i$ such that $(\dot{w}_i)_i$ is also uniformly convergent and this ends the proof of lemma. □

Proof of uniform stability in Theorem 1.1.

Let us assume that the equilibrium is not uniformly stable. Then there exists $\varepsilon_0 > 0$ such that for any δ, $0 < \delta < \varepsilon_0$ there are x_0, t and $\Delta > 0$ such that $\|x_0\| < \delta$ and $\|x(t + \Delta; t, x_0)\| = \varepsilon_0$, $\|x(t + \tau; t, x_0)\| < \varepsilon_0$, for $0 \leq \tau < \Delta$. We choose ε_0 (eventually by shrinking it) such that $\bar{B}_{\varepsilon_0} \cap N$ is included in the attraction domain of the origin (for the restricted dynamics).

By choosing a sequence $(\delta_i)_i$ converging to zero we obtain sequences $(x_{0i})_i$, $(t_i)_i$ and $(\Delta_i)_i$ such that: $\|x_{0i}\| \to 0$ and $\|x(t_i + \Delta_i; t_i, x_{0i})\| = \varepsilon_0$.

Let $\delta < \varepsilon_0$ be such that for any $z_0 \in B_\delta \cap N$ we have $\|x(t; 0, z_0)\| < \frac{\varepsilon_0}{2}$ for any $t > 0$ (such a choice for δ is possible since the dynamics restricted to N is stable). Let i_0 be such that $\delta_i < \delta$, for $i > i_0$. We denote by $(u_i)_{i > i_0}$ the time moments such that $\|x(t_i + u_i; t_i, x_{0i})\| = \delta$ and $\|x(t; t_i, x_{0i})\| > \delta$ for $t > t_i + u_i$. Since the spheres $\bar{S}_{\varepsilon_0}$ and $\bar{S}_\delta$ are compact we can extract a subsequence (indexed also by i) such that both $x_i = x(t_i + \Delta_i; t_i, x_{0i})$ and $y_i = x(t_i + u_i; t_i, x_{0i})$ are convergent to x^*, respectively to y^*; $x_i \to x^*$, $y_i \to y^*$, $\|x^*\| = \varepsilon_0$, $\|y^*\| = \delta$. Since V is continuously nonincreasing on trajectories and $\lim_i V(x_{0i}) = 0$, we get $V(x^*) = V(y^*) = 0$. Therefore $x^*, y^* \in N$.

Suppose $\|f(x,t)\| \leq A$ on $\bar{B}_{\varepsilon_0}$, for some $A > 0$. Then one can easily prove that $\Delta_i - u_i \geq \frac{\varepsilon_0 - \delta}{A} = T_1$, for any $i > i_0$ (i.e. the flight time between two spheres of radius δ and ε_0 has a lower bound).

Define now the time-translated vector fields $f_i(t,x) = f(t + t_i + u_i, x)$ and denote by $w_i : [0, T_1] \to \bar{B}_{\varepsilon_0}$ the time-translated solutions $w_i(t) = x(t + t_i + u_i; t_i, x_{0i})$. Then: $\dot{w}_i(t) = f_i(t, w_i(t))$, $0 \leq t \leq T_1$. By applying Lemma 2.1 we get a subsequence uniformly convergent to a trajectory $w^1 : [0, T_1] \to \bar{B}_{\varepsilon_0} \cap N$, such that $w^1(0) = \lim_i w_i(0) = y^*$ and $\|w^1(t)\| > \delta$, for $0 < t \leq T_1$. If $\|w^1(T_1)\| < \varepsilon_0$ we obtain that $\Delta_i - u_i - T_1 > \frac{\varepsilon_0 - \|w^1(T_1)\|}{A}$, for some $i \geq i_1 > i_0$. Then, we denote $T_2 = T_1 + \frac{\varepsilon_0 - \|w^1(T_1)\|}{A}$ and we repeat the scheme. We obtain another sequence which is uniformly convergent to a trajectory $w^2 : [0, T_2] \to \bar{B}_{\varepsilon_0} \cap N$ such that $w^2(0) = y^*$, $\|w^2(t)\| > \delta$, $0 < t \leq T_2$ and $w^2(t) = w^1(t)$, for $0 \leq t \leq T_1$.

Thus we extend each trajectory $w^k : [0, T_k] \to \bar{B}_{\varepsilon_0} \cap N$ to a trajectory $w^{k+1} : [0, T_{k+1}] \to \bar{B}_{\varepsilon_0} \cap N$ such that $T_{k+1} \geq T_k$, $w^{k+1}(t) = w^k(t)$ for $0 \leq t \leq T_k$ and $\|w^{k+1}(t)\| > \delta$, for $0 < t \leq T_{k+1}$.

We end this sequence of extensions in two cases:

1) $\lim_k T_k = T^* < +\infty$ (the limit may be reached in a finite number of steps), in which case we have $\lim_k \|w^k(T_k)\| = \varepsilon_0$ and thus $\lim_k w^k(T_k) = x^*$; or:

2) $\lim_k T_k = +\infty$.

In the first case we obtain a trajectory $w^* : [0, T^*] \to \bar{B}_{\varepsilon_0} \cap N$ such that $w^*(0) = y^*$, $w^*(T^*) = x^*$ with $\|w^*(0)\| = \delta$ and $\|w^*(T^*)\| = \varepsilon_0$. But this is a contradiction with the choice of δ (and of stability of the restricted dynamics).

In the second case we obtain a trajectory $w^* : [0, \infty) \to \bar{B}_{\varepsilon_0} \cap N$ such that $\|w^*(0)\| = \delta < \varepsilon_0$ and $\|w^*(t)\| > \delta$ for $t > 0$. Thus $\lim_{t\to\infty} w^*(t) \neq 0$ contradicting the assumption that $\bar{B}_{\varepsilon_0} \cap N$ is included in the attraction domain of the origin. Now the proof is complete. $\square$.

For the proof of uniformly attractivity we recall a few definitions and results.

Definition 2.1 A point x^* is called *ω-limit point* for the trajectory $x(t; t_0, x_0)$ if there exists a sequence $(t_k)_k$ such that $\lim_{k\to\infty} t_k = \infty$, $x(t; t_0, x_0)$ is defined for all $t > t_0$ and $\lim_k x(t_k; t_0, x_0) = x^*$. The set of all ω-limit points is called the ω-limit set and is denoted by $\Omega(t_0, x_0)$. It characterizes the trajectory $x(t; t_0, x_0)$ and it depends on the initial data (t_0, x_0).

Theorem 2.1 (Birkoff's limit set theorem, see [4]) *A bounded trajectory approaches its ω-limit set, i.e.* $\lim_{t\to\infty} d(x(t; t_0, x_0), \Omega(t_0, x_0)) = 0$, *where* $d(p, S) = \inf_{x\in S} \|p - x\|$ *is the distance between the point p and the set S.*

There is also a very useful result about uniformly continuous functions.

Lemma 2.2 (Barbălat's lemma, see [2]) *If $g : [t_0, \infty) \to \infty$ is a uniformly continuous function such that the following limit exists and is finite,* $\lim_{t\to\infty} \int_{t_0}^{t} g(\tau) d\tau$, *then* $\lim_{t\to\infty} g(t) = 0$.

Proof of uniform attractivity in Theorem 1.1

We already know that $\bar{x} = 0$ is uniformly stable. What we have to prove is the uniform attractivity.

Let $\varepsilon_0 > 0$ be chosen with the following properties: for any t_0 and $x_0 \in D \cap \bar{B}_{\varepsilon_0}$ the positive trajectory $x(t; t_0, x_0)$ is bounded by ε_1 (i.e. $x(t; t_0, x_0) \in B_{\varepsilon_1}$); for any t_1 and $x_1 \in D \cap B_{\varepsilon_1}$ the trajectory $x(t; t_1, x_1)$, $t > t_1$, is bounded by some M; and for any $x_2 \in N \cap B_{\varepsilon_1}$ the trajectory $x(t; t_0, x_2)$ tends to the origin $\lim_{t\to\infty} x(t; t_0, x_2) = 0$. We are going to prove that $\lim_{t\to\infty} x(t; t_0, x_0) = 0$.

Let us consider the ω-limit set $\Omega(t_0, x_0)$. It is enough to prove that $\Omega(t_0, x_0) = \{0\}$, because of Birkoff's limit set theorem.

Let $x^* \in \Omega(t_0, x_0)$ and suppose $x^* \neq 0$. Let us denote by $x(t) = x(t; t_0, x_0)$ and $g(t) = \nabla V(x(t)) \cdot f(t, x(t))$. Since the solution is continuous and bounded, so is $g(t)$. On the other hand

$$V(x(t)) = V(x_0) + \int_{t_0}^{t} g(\tau) d\tau.$$

Since $\dot{x}(t) = f(t, x(t))$ and $x(t)$ is bounded we obtain that it is also uniformly continuous. Thus $g(t)$ is also uniformly continuous (recall we have assumed $f(\cdot, x)$ is uniformly continuous in t). Let $(t_k)_k$ be a sequence that renders x^* a ω-limit point. Then $\lim_k V(x(t_k)) = V(\lim_k x(t_k)) = V(x^*)$. Since $V(x(t))$ is a decreasing function bounded below, there exists the limit: $\lim_{t\to\infty} V(x(t)) = V(x^*)$. Now, applying Barbălat's lemma we obtain $\lim_{t\to\infty} g(t) = 0$ or $W(x^*) = 0$. Thus $\Omega(t_0, x_0) \subset E$, the kernel of W.

In this point we need a result about the behaviour of solutions starting at x^*. We mention that the following lemma is a consequence of Theorem 3 from [23]. But, since we are under stronger conditions, we have found a simpler proof that we are going to present here (our conditions are stronger because we need to obtain uniform stability and consequently boundedness of the solutions when Liapunov function is only positive semidefinite, which overall means a weaker condition).

Lemma 2.3 *The positive trajectory starting at x^* is included in E and thus the Ω-limit set is a positive invariant set included in N.*

Proof Let $\tau > 0$ be an arbitrary time interval. Let $(t_k)_k$ be the sequence that renders x^* a ω-limit point for the trajectory $x(t) = x(t; t_0, x_0)$. Then, if we denote by $x_k = x(t_k)$ we have $\lim_k x_k = x^*$. Consider the following sequence of functions: $w_k : [0, \tau] \to D,\ w_k(t) = x(t + t_k; t_k, x^*)$. We have chosen x_0, t_0 such that all these functions are bounded by M, i.e. $\|w_k\|_\infty < M$. We have $w_k(0) = x^*$ and $V(w_k(t)) \le V(x^*)$. Let us denote by $y_k^t = x(t+t_k)$, for any $0 \le t \le \tau$, and let L_M be the Lipschitz constant of f on the compact $\bar{B}_M$. Then: $\|y_k^t - w_k(t)\| \le e^{L_m t}\|x_k - x^*\|$ and, since $\lim_k x_k = x^*$ we get $\lim_k \|y_k^t - w_k(t)\| = 0$. On a hand, since $V(x^*) = \lim_{t\to\infty} V(x(t))$ and V is nonincreasing on trajectories we have $V(y_k^t) > V(x^*)$ and also $\lim_k V(y_k^t) = V(x^*) = \lim_k V(w_k(t))$. On the other hand, since $(w_k)_k$ are uniformly bounded we apply Lemma 2.1 and we obtain a subsequence uniformly convergent to a function $w \in \mathcal{C}^1([0, \tau]; D \cup \bar{B}_M)$. Obviously $V(w(t)) = V(x^*)$ for any $0 \le t \le \tau$. Thus $W(w(t)) = 0$ and $w(t) \in E$. On the other hand, since f is continuous in (t, x) we obtain that w is an integral curve of f, i.e. $\dot{w}(t) = f(t_*, w(t))$, for $0 \le t \le \tau$ and any t_*. In particular, for $t_* = t_k$ we get $w(t)$ is a solution of the same equation as $w_k(t)$ and $w(0) = w_k(0) = x^*$. By the uniqueness of the solution they must coincide. Then $x(t + t_k; t_k, x^*) \in E$ for $0 \le t \le \tau$. But τ was arbitrarily; thus $x(t; t_0, x^*) \in E$ for any t and then $x^* \in N$. □

Since the trajectory starting at x^* is included in N, it should converge to the origin (the equilibrium point). Let us denote by $\varepsilon = \frac{\|x^*\|}{2}$. From uniform stability there exists a $\delta > 0$ such that for any $\tilde{x} \in D$, $\|\tilde{x}\| < \delta$ implies $\|x(t_2; t_1, \tilde{x})\| < \varepsilon$, for any $t_2 > t_1$. Let Δt be a time interval such that $\|x(t; 0, x^*)\| < \frac{\delta}{2}$ for any $t > \Delta t$. We consider the compact set C, the $\frac{\delta}{2}$-neighborhood of the compact curve $\Gamma = \{x(t; 0, x^*) | 0 \le t \le \Delta t\}$:

$$C = \{x \in D | d(x, \Gamma) \le \frac{\delta}{2}\} = \bigcup_{t\in[0,\Delta t]} B_{\delta/2}(x(\bar{t}; 0, x^*))$$

which is the union of the closed balls centered at $x(t; 0, x^*)$ and of radius $\frac{\delta}{2}$. We set $\delta_1 = \frac{\delta}{2} exp(-L_C \Delta t)$ where L_C is the uniform Lipschitz constant of f on the compact set C. Since the solution is uniformly Lipschitz with respect to the initial point x_0 we have that for any $t_1 \in R$ and x_1 such that $\|x_1 - x^*\| < \delta_1$ we get: $\|x(t_1 + \Delta t; t_1, x_1) - x(\Delta t; 0, x^*)\| < \frac{\delta}{2}$ and then $\|x(t_1 + \Delta t; t_1, x_1)\| < \delta$. Furthermore, from the choice of δ we obtain that $\|x(t_1 + \tau; t_1, x_1)\| < \varepsilon$, for any $\tau > \Delta t$ or $\|x(t_1 + \tau; t_1, x_1) - x^*\| > \varepsilon$, for any $\tau > \Delta t$.

Now we pick a t_n such that $\|x(t_n;t_0,x_0) - x^*\| < \delta_1$. Then, from the previous discussion $\|x(t_n + \tau;t_0,x_0) - x^*\| > \varepsilon$, for any $\tau > \Delta t$ which contradicts the limit $\lim_k x(t_k;t_0,x_0) = x^*$. This contradiction comes from the hypothesis that $x^* \neq 0$. Thus $\Omega(t_0,x_0) = \{0\}$ and now the proof is complete. □

Proof of Theorem 1.2 (The invariance principle)
If $x(t;t_0,x_0)$ is a bounded trajectory then, from Birkoff's limit set theorem it approaches its ω-limit set. On the one hand we can use Barbălat's lemma and prove that W vanishes on ω-limit set of bounded trajectories. On the other hand, as we have proved in Lemma 2.3, the ω-limit set is invariant and included in N. Thus the bounded trajectory approaches the set N. □

3 The Autonomous Case: Consequences in Nonlinear Control Theory

Consider the following inputless nonlinear control system:

$$S\left\{\begin{array}{l} \dot{x} = f(x) \\ y = h(x) \end{array}\right. , \quad x \in D \subset R^n, \ y \in R^p, \tag{5}$$

such that $f(0) = 0$, $h(0) = 0$ and D a neighborhood of the origin. Suppose f is local Lipschitz continuous and h continuous on D. Then denote by $x(t,x_0)$ the flow generated by f on D (i.e. the solution of $\dot{x} = f(x)$, $x(0) = x_0$), by $E = ker\ h = \{x \in D | h(x) = 0\}$, the kernel of h and by N the maximal positive invariant set included in E, i.e. the set $N = \{\tilde{x} \in D | h(x(t,\tilde{x})) = 0$ for any $t \geq 0$ such that $x(t,\tilde{x})$ has sense$\}$.

We present two concepts of detectability for (5). The first one has been used by many authors (see for instance [13]).

Definition 3.1 The pair (h,f) is called *zero-state detectable* (or *z.s.d.*) if $\bar{x} = 0$ is an attractive point for the dynamics restricted to N, i.e. there exists an $\varepsilon_0 > 0$ such that for any $x_0 \in N$, $\|x_0\| < \varepsilon_0$, $\lim_{t\to\infty} x(t,x_0) = 0$.

Definition 3.2 The pair (h,f) is called *strong zero-state detectable* (or *strong z.s.d.*) if $\bar{x} = 0$ is an asymptotical stable equilibrium point for the dynamics restricted to N, i.e. it is zero-state detectable and for some ε_0 and for any $x_0 \in N$ with $\|x_0\| < \varepsilon_0$, $\lim_{t\to\infty} x(t,x_0) = 0$.

We see that strong z.s.d. implies z.s.d., but obviously the converse is not true.

In this framework, as a consequence of the main result we can state the following theorem.

Theorem 3.1 *For the inputless nonlinear control system* (5) *with f local Lipschitz continuous and h continuous, consider the following nonlinear Liapunov equation:*

$$\nabla V \cdot f + \|h\|^q = 0 \tag{6}$$

or the following nonlinear Liapunov inequality:

$$\nabla V \cdot f + \|h\|^q \leq 0 \tag{7}$$

for some $q > 0$. Suppose there exists a positive semidefinite solution of (6) *or* (7) *of class $\mathcal{C}^1$ defined on D such that $V(0) = 0$.*

Then the pair (h,f) is strong zero-state detectable if and only if $\bar{x} = 0$ is an asymptoticaly stable equilibrium for the dynamics (5).

Below we give an example.

Example 3.1 Consider the dynamics:

$$\begin{array}{l} \dot{x}_1 = -x_1^3 + \Psi(x_2) \\ \dot{x}_2 = -x_2^3 \end{array} \quad , \quad (x_1, x_2) \in R^2, \tag{8}$$

where $\Psi : R \to R$ is local Lipschitz continuous, $\Psi(0) = 0$ and there exist constants $a > 0$, $b \geq 1$ such that:

$$|\Psi(x)| \leq a|x|^b, \quad \forall x_2.$$

If we choose as output function $h(x) = x_2^2$ we see that the pair (h, f) is strong zero-state detectable; indeed, the set $E = \{x \in R^2 | h(x) = 0\} = \{(x_1, 0) | x_1 \in R\}$ and the dynamics restricted to E is $\dot{x}_1 = -x_1^3$ which is asymptotically stable.

Now, if we choose $V(x) = \frac{x_2^2}{2}$ we have $\dot{V} = -x_2^4$ and thus V is a solution of the Liapunov equation (6) with $q = 2$. Then, the equilibrium is asymptotically stable, as a consequence of the Theorem 3.1.

On the other hand we can explicitely solve for x_2: $x_2(t) = \frac{x_{20}}{\sqrt{2(1+x_{20}^2 t)}}$ and then we have: $|\Psi(x_2(t))| \leq C(1 + Bt)^{-1/2}$ for some $B, C > 0$ and any $t \geq 0$. Now the asymptotic stability follows as a consequence of Theorem 68.2 from [7] (stability under perturbation).

4 An Application to Robust Stabilizability

We present here, as an application, a robust stabilizability result for a nonlinear affine control system. In fact it is an absolute stability result about a particular situation. More general results about absolute stability for nonlinear affine control system will appear in a forthcoming paper. We base our approach on the existence of a positive semidefinite solution of some Hamilton-Jacobi equation or inequality. Discussions about solutions of this type of equation may be found in [21].

Consider the following single input–single output control system:

$$\left\{ \begin{array}{lcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array} \right. \quad , \quad x \in D \subset R^n, \ \ u, y \in R, \tag{9}$$

where f and g are local Lipschitz continuous vector fields on a domain D including the origin, h is a local Lipschitz real-valued function on D, and $f(0) = 0$, $h(0) = 0$. Consider also a local Lipschitz output feedback:

$$\varphi : R \to R, \quad \varphi(0) = 0. \tag{10}$$

We define now two classes of perturbations associated to this feedback. Let $a > 0$ be a positive real number. The first class contains time-invariant perturbations:

$$P_1 = \{p : R \to R \ , \ p \text{ is local Lipschitz}, \ p(0) = 0 \text{ and } |p(y)| < a|\varphi(y)| \ , \ \forall y \neq 0\}$$

while the second class is composed by time-varying perturbations:

$$P_2 = \{p : R \times R \to R \ , \ p(y,t) \text{ is local Lipschitz in } y \text{ for } t \text{ fixed and uniformly continuous in } t$$
$$\text{for any } y \text{ fixed} \ , \ p(0,t) \equiv 0 \text{ and there exists } \varepsilon > 0 \text{ such that } |p(y,t)| < (a - \varepsilon)|\varphi(y)| \ , \ \forall y \neq 0, t\}$$

Now we can define more precisely the concept of robust stability.

Definition 4.1 We say the feedback (10) *robustly stabilizes* the system (9) *with respect to the class* $P_1 \cup P_2$ if for any perturbation $p \in P_1 \cup P_2$ the closed-loop with the perturbed feedback $\varphi + p$ has an asymptotically stable equilibrium at the origin.

In other words, we require that the origin to be asymptotically stable for the dynamics:

$$\dot{x} = f(x) + g(x)(\varphi(h(x)) + p(h(x), t)) \tag{11}$$

for any $p \in P_1 \cup P_2$. Since the null function belongs to P_1, the feedback φ itself must stabilize the closed-loop too.

With these preparations we can state the result.

Theorem 4.1 *Consider the nonlinear affine control system* (9) *and the feedback* (10)*. Suppose the pair* (h, f) *is strong zero-state detectable and suppose the following Hamilton-Jacobi equation:*

$$\nabla V \cdot f + (\frac{1}{2}\nabla V \cdot g + \varphi \circ h)^2 - (1 - a^2)(\varphi \circ h)^2 = 0, \quad V(0) = 0 \tag{12}$$

or inequality:

$$\nabla V \cdot f + (\frac{1}{2}\nabla V \cdot g + \varphi \circ h)^2 - (1 - a^2)(\varphi \circ h)^2 \leq 0, \quad V(0) = 0 \tag{13}$$

has a positive semidefinite solution V *of class* $\mathcal{C}^1$ *on* D*.*

Then the feedback φ *robustly stabilizes the system* (9) *with respect to the class* $P_1 \cup P_2$*.*

Proof Let us consider a perturbation $p \in P_1 \cup P_2$. Then, the closed-loop dynamics is given by (11). We compute the time derivative of the solution V of (12) with respect to this dynamics:

$$\frac{dV}{dt} = \nabla V \cdot f(x) + \nabla V \cdot g(x)(\varphi(h(x)) + p(h(x), t)).$$

After a few algebraic manipulations we get:

$$\frac{dV}{dt} \leq -(\frac{1}{2}\nabla V \cdot g - p \circ h)^2 + (p \circ h)^2 - a^2(\varphi \circ h)^2.$$

Now, for $p \in P_1$, $\frac{dV}{dt}$ is time-independent and we may take for instance:

$$W(x) = (p(h(x)))^2 - a^2(\varphi(h(x)))^2 \leq 0.$$

For $p \in P_2$, $\frac{dV}{dt}$ is time-dependent and we define:

$$W(x) = -(2a\varepsilon - \varepsilon^2)(\varphi(h(x)))^2 \leq 0.$$

Either a case or the other, we obtain (recall the definitions of P_1 and P_2):

$$\frac{dV}{dt} \leq W(x) \leq 0.$$

The kernel-set of W is given by:

$$E = \{x \in D |\ W(x) = 0\} = \{x \in D |\ h(x) = 0\}.$$

We see that the closed-loop dynamics (11) restricted to E is simply given by $\dot{x} = f(x)$ and is time-independent. Moreover, since we have supposed (h, f) is strong zero-state detectable, it follows that the restricted dynamics to the maximal positive invariant set in E has an asymptotically stable equilibrium at the origin. Now, applying Theorem 1.1, the result follows. □

Let us consider now an example.

Example 4.1 Consider the following planar nonlinear control system:

$$\begin{cases} \dot{x}_1 = -x_1^3 + u, \\ \dot{x}_2 = -x_2^3, \\ y = x_2^3. \end{cases} \tag{14}$$

We are interested to find how robust the feedback $\varphi(y) = y$ is, i.e. how large we can choose a such that φ robustly stabilizes the system (14) with respect to the class $P_1 \cup P_2$.

The Hamilton-Jacobi equation (12) takes the form:

$$-x_1^3 \frac{\partial V}{\partial x_1} - x_2^3 \frac{\partial V}{\partial x_2} + (\frac{1}{2}\frac{\partial V}{\partial x_1} + x_2^3)^2 - (1 - a^2)x_2^6 = 0$$

or:

$$-x_1^3 \frac{\partial V}{\partial x_1} - x_2^3 \frac{\partial V}{\partial x_2} + \frac{1}{4}(\frac{\partial V}{\partial x_1})^2 + x_2 \frac{\partial V}{\partial x_1} + a^2 x_2^6 = 0.$$

A solution of this equation is:

$$V(x_1, x_2) = \frac{a^2}{4} x_2^4.$$

For any $a > 0$ it is positive semidefinite and the system (14) is strong zero-state detectable. Thus, as a consequence of Theorem 4.1, we can choose a arbitrary large such that φ robustly stabilizes the system (14) with respect to the class $P_1 \cup P_2$.

On the other hand, for any feedback Φ, local Lipschitz and:

$$|\Phi(y)| \leq a|y| \quad \text{for some } a > 0,$$

we have seen in the previous example that the closed-loop has an asymptotically stable equilibrium at the origin.

5 Conclusions

In this paper we study an extension of Barbashin-Krasovskii-LaSalle and Invariance Principle to a class of time-varying dynamical systems. We impose two type of conditions on the vector field: one is regularity (we require uniformly continuity with respect to t and uniformly local Lipschitz continuoity and boundedness with respect to x); the other condition requires the vector field to be time-invariant on the zero-set E of an auxiliary function. In this setting we find that the asymptotic behaviour of the dynamics restricted to the largest positive invariant set in E determines the asymptotic stability character of the full dynamics.

Then we study two applications in control theory. The first application concerns the notion of detectability. We give another definition for this notion, called strong zero-state detectability and we show how the existence of a positive semidefinite solution of the Liapunov equation or inequation is related to the asymptotic stability of the equilibrium.

We obtain a nonlinear equivalent of the linear well-known result: if the pair (C, A) is detectable and there exists a positive solution $P \geq 0$ of the Liapunov algebraic equation $A^T P + PA + C^T C = 0$, then the matrix A has all eigenvalues with negative real part.

The second application is on the problem of robust stabilizability. We give sufficient conditions such that a given feedback robustly stabilizes the closed-loop with respect to two sector classes of perturbations (time-invariant and time-varying). The condition is formulated in term of the existence of a positive solution of some Hamilton-Jacobi equation or inequality.

Interesting open questions are to find extensions of the results presented here to the class of switched linear systems (see [8] for an excellent starting point), and to the class of large scale systems (see [18] for a novel approach).

References

[1] Aeyels, D. Asymptotic stability of nonautonomous systems by Liapunov's direct method. *Systems & Control Letters* **25** (1995) 273–280.

[2] Barbălat, I. Systèmes d'èquations différentielles d'oscilations non linéaires. *Rev. de mathématiques pures et appliquées* **IV 2** (1959) 267–270.

[3] Boulgakov, N.G. and Kalitine, B. La généralisation de la deuxieme methode de Liapunov. I. Theory. *Izvestiya Akademii Nauk BSSR* **(3)** (1978) 32–36. [Russian]

[4] Birkoff, M.G.D. Quelques Théorèmes sur le Mouvement des Systemès Dynamiques. *Bull. Soc. Math. de France* **40** (1912) 305–323, reprinted in *Collected Mathematical Papers* **1** (AMS-1950) 654–672.

[5] Brocket, R.W. and Willems, J.L. Frequency Domain Stability Criteria - Part I. *IEEE Trans. on Autom. Contr.* **AC-10** (1965) 255–261.

[6] Corduneanu, C. *Principles of Differential and Integral Equations*. Chelsea Publishing Company, 2nd Edition, 1977.

[7] Hahn, W. *Stability of Motion*. Springer-Verlag, New York, 1967.

[8] Hespanha, J.P. Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invariance Principle. *IEEE Trans. on Autom. Contr.* **49** (2004) 470–482.

[9] Hirsch, M. W. and Smale, S. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, New York and London, 1974.

[10] Iggidr, A. Kalitine, B. and Outbib, R. Semi-Definite Lyapunov Functions Stability and Stabilization. *Mathematical Control Signals Systems* **9** (1996) 95–106.

[11] Iggidr, A. and Sallet, G. On the stability of nonautonomous systems. *Automatica* **39** (2003) 167–171.

[12] Isidori, A. *Nonlinear Control Systems*. Springer-Verlag, New York, 1989.

[13] Isidori, A. and Astolfi, A. Nonlinear H_∞-Control via Measurement Feedback. *Journal of Mathematical Systems, Estimation, and Control* **2** (1992) 31–44.

[14] Khalil, H.K. *Nonlinear Systems*. Macmillan Publishing Company, 1993.

[15] LaSalle, J.P. Some Extension of Liapunov's Second Method. *IRE Transactions on Circuit Theory* **CT-7** (1960) 520–527.

[16] LaSalle, J.P. Stability Theory for Ordinary Differential Equations. *J. of Diff. Eqns.* **4** (1968) 57–65.

[17] Lee, T.-C. and Liaw, D.-C. and Chen, B.-S. A General Invariance Principle for Nonlinear Time-varying Systems and Its Applications. *IEEE Trans. on Autom. Contr.* **AC-46** (2001) 1989–1993.

[18] Martynyuk, A.A. and Slyn'ko, V.I. Solution of the Problem of Constructing Liapunov Matrix Function for a Class of Large Scale Systems. *Nonlin. Dyn. Syst. Theory* **1**(2) (2001) 193–203.

[19] Miller, R.K. Asymptotic Behaviour of Solutions of Nonlinear Differential Equations. *Trans. of the AMS* **115** (1965) 400–416.

[20] Rouche, N. and Habets, P. and Laloy, M. *Stability Theory by Liapunov's Direct Method*. Springer Verlag, New York, 1977.

[21] van der Schaft, A.J. On a State Space Approach to Nonlinear H_∞ Control. *Systems & Control Letters* **16** (1991) 1–8.

[22] Vidyasagar, H. Decomposition Techniques for Large-Scale Systems with Nonadditive Interactions: Stability and Stabilizability. *IEEE Trans. on Autom. Contr.* **AC-25** (1980) 773–779.

[23] Yoshizawa, T. Asymptotics Behaviour of Solutions of a System of Differential Equations. *Contributions to Differential Equations* **1** (1963) 371–387.