Nonlinear Dynamics and Systems Theory, 8(3) (2008) 237–254

Approximation of Solutions to a Class of Second Order History-valued Delay Differential Equations

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Received: July 12, 2007; Revised: June 3, 2008

Abstract: In this paper we shall study the approximations of solutions to a class of second order history-valued delay differential equations in a separable Hilbert space. Using a pair of associated nonlinear integral equations and projection operators we consider a pair of approximate nonlinear integral equations. We first show the existence and uniqueness of solutions to this pair of approximate integral equations and then establish the convergence of the sequences of the approximate solutions to the solution and the pair of associated integral equations, respectively. Also, we consider the Faedo–Galerkin approximations of the solution and prove some convergence results. Finally, we give an example.

Keywords: *Second order history-valued delay differential equations; analytic semigroup; Banach fixed point theorem; Faedo-Galerkin approximation.*

Mathematics Subject Classification (2000): 34K30, 35R10, 47D06.

1 Introduction

We consider the following second order history-valued abstract delay differential equation in a separable Hilbert space $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$:

$$
\begin{aligned}
u''(t) + Av(t) = f(t, u(t), v(t), u(t-\tau), v(t-\tau)), \quad t \in (0, T], \\
u(t) = h(t), \quad v(t) = g(t), \quad t \in [-\tau, 0],
\end{aligned}
\tag{1}
$$

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where A is a closed linear operator defined on a dense subset of H and $v(t) = u'(t)$ for all $t \in [-\tau, T]$. We assume that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\colon\ t \geq 0\}$ in H and the nonlinear map f is defined from $[0,T] \times H^4$ into H satisfying certain conditions to be specified later.

Regarding the earlier works on existence, uniqueness, regularity and stability of various types of solutions to evolutions equations, delay differential equations and neutral functional differential equations under different conditions, we refer to Bahuguna and Muslim [1, 2, 3], Bahuguna *et al* [4], Wei *et al* [5], Balachandran and Chandrasekaran [6], Lin and Liu [7], Alaoui [8], Adimy [9], Hernandez and Henriquez [10, 11], Blasio and Sinestrari [12], Jeong [13], Rhandi [14] and the references cited in these papers.

The related results for the approximation of solutions to the first order evolution equations with and without delay can be found in Bahuguna and Muslim [1, 2], Henriquez [15] and Muslim [16].

Initial studies concerning existence, uniqueness and finite-time blow-up of solutions for the following equation

$$\begin{aligned} u'(t) + Au(t) &= g(u(t)), \quad t \geq 0,\\ u(0) &= \phi, \end{aligned}$$

have been considered by Segal [17], Murakami [18] and Heinz and Von Wahl [19]. Bazley [20, 21] has considered the following semilinear wave equation

$$\begin{aligned} u''(t) + Au(t) &= g(u(t)), \quad t \geq 0,\\ u(0) &= \phi, \quad u'(0) = \psi, \end{aligned} \tag{2}$$

and has established the uniform convergence of approximations of solutions to (2) using the results of Heinz and von Wahl [19]. Goethel [22] has proved the convergence of approximations of solutions to (2) but assumed g to be defined on the whole of H. Based on the ideas of Bazley [20, 21], Miletta [23] has proved the existence and convergence of approximate solutions to (2).

The authors Bahuguna and Muslim [2] have considered the following first order retarded integro-differential equation

$$\begin{aligned} u'(t) + Au(t) &= Bu(t) + Cu(t-\tau) + \int_{-\tau}^{0} a(\theta) Lu(t+\theta)\, d\theta,\ \ 0 < t \leq T < \infty,\ \ \tau > 0,\\ u(t) &= h(t), \quad t \in [-\tau, 0] \end{aligned} \tag{3}$$

in a separable Hilbert space and studied the approximation of solution of the above problem under the conditions when $-A$ is the infinitesimal generator of an analytic semigroup, $B,\ C$ and L are nonlinear continuous operators suitably defined on H.

In [23], Miletta has established the convergence of Faedo-Galerkin approximation of the solution to

$$u'(t) + Au(t) = M(u(t)), \quad u(0) = \phi,$$

in a separable Hilbert space where A satisfies the same condition as in this paper and M is a nonlinear map defined on $D(A^\alpha)$, for some $\alpha,\ 0 < \alpha < 1,$ which satisfies a Lipschitz condition in a ball in $D(A^\alpha)$.

Despite the widespread use of the Faedo-Galerkin method (in many applications it is referred to as the method of harmonic balance), the convergence behaviour in many cases is not known. Bazely [20, 21] has proved the uniform convergence of the approximation solution of the nonlinear wave equation

$$u''(t) + Au(t) + M(u(t)) = 0, \quad u(0) = \phi, \quad u'(0) = \psi,$$

on any closed subinterval $[0, T]$ of the existence of the solution.

2 Preliminaries

We note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c > 0$ large enough, $-(A + cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence without loss of generality we suppose that

$$\|e^{-tA}\| \le M \quad \text{for} \quad t \ge 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \le \alpha \le 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ being dense in X.

In view of the facts mentioned above we have the following Lemma for an analytic semigroup $\{e^{-tA},\ t \ge 0\}$ (cf. Pazy [24], pp. 195–196).

Lemma 2.1 *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA},\ t \ge 0\}$ with $\|e^{-tA}\| \le M$, for $t \ge 0$ and $0 \in \rho(-A)$. Then we have the following*

(i) *$D(A^\alpha)$ for $0 \le \alpha \le 1$ is a Banach space endowed with the norm $\|\cdot\|_\alpha$,*

(ii) *For $0 < \beta \le \alpha$, the embedding $H_\alpha \hookrightarrow H_\beta$ is continuous,*

(iii) *A^α commutes with e^{-tA} and there exists a constant $C_\alpha > 0$ depending on α such that*

$$\|A^\alpha e^{-tA}\| \le C_\alpha t^{-\alpha}, \quad t > 0,$$

(iv) *There exists a constant C such that*

$$\|A^{-\alpha}\| \le C, \quad \textit{for} \quad 0 \le \alpha \le 1.$$

We assume that the linear operator A satisfies the following assumption.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_2 \ldots$$

and a corresponding complete orthonormal system of eigenfunctions $\{\phi_i\}$, i.e.,

$$A\phi_i = \lambda_i \phi_i \quad \text{and} \quad \langle \phi_i, \phi_j \rangle = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied then $-A$ generates an analytic semigroup $\{e^{-tA}: \ t \geq 0\}$ in H. Further assume that the maps h, g and f satisfy the following hypotheses.

(H2) The maps $h, g \in \mathcal{C}_0^1$ are locally Hölder continuous on $[-\tau, 0]$.

We define the two new functions $\tilde{h}$ and $\tilde{g}$ given by

$$\tilde{h}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0), & t \in [0, T] \end{cases} \tag{4}$$

and

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ g(0), & t \in [0, T]. \end{cases} \tag{5}$$

(H3) The nonlinear map f is defined from $[0, T] \times D(A) \times D(A^\alpha) \times D(A) \times D(A^\alpha)$ into H and there exists a nondecreasing function L_f from $[0, \infty)$ into $[0, \infty)$ depending on some $r_1 > 0$ such that

$$\begin{aligned} \|f(t, u_1, &v_1, w_1, z_1) - f(s, u_2, v_2, w_2, z_2)\| \\ &\leq L_f(r_1)\{|t-s|^\theta + \|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha + \|w_1 - w_2\|_1 + \|z_1 - z_2\|_\alpha\}, \end{aligned}$$

for all $t, s \in [0, T]$, $\theta \in (0, 1]$, and (u_1, v_1), (u_2, v_2), (w_1, z_1),$(w_2, z_2) \in B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t)))$ where $B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t))) = \{(x_1, y_1) \in D(A) \times D(A^\alpha): \|x_1 - \tilde{h}(t)\|_1 + \|y_1 - \tilde{g}(t)\|_\alpha \leq r_1\}$.

3 Approximate Integral Equations

The existence of solutions to equation (1) is closely associated with the following pair of integral equations

$$u(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - & \\ \int_0^t (e^{-(t-s)A} - I)A^{-1} f(s, u(s), v(s), u(s-\tau), v(s-\tau))\, ds, & t \in [0, T], \end{cases} \tag{6}$$

$$v(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA}g(0) + & \\ \int_0^t e^{-(t-s)A} f(s, u(s), v(s), u(s-\tau), v(s-\tau))\, ds, & t \in [0, T]. \end{cases} \tag{7}$$

By a solution (u, v) to equations (6)–(7) on $[-\tau, T]$, we mean a pair of functions $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ for some $0 < \alpha < 1$ satisfying (6)–(7), where $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ is the Banach space $C([-\tau, T], D(A) \times D(A^\alpha))$ of all continuous functions from $[-\tau, T]$ into $D(A) \times D(A^\alpha)$ endowed with the norm

$$\|(u, v)\|_{C_{T,1} \times C_{T,\alpha}} = \|u\|_{T,1} + \|v\|_{T,\alpha},$$

where

$$\|u\|_{T,1} = \sup_{-\tau \leq t \leq T} \|Au(t)\| = \sup_{-\tau \leq t \leq T} \|u(t)\|_1$$

and

$$\|v\|_{T,\alpha} = \sup_{-\tau \le t \le T} \|A^\alpha v(t)\| = \sup_{-\tau \le t \le T} \|v(t)\|_\alpha.$$

Let $0 < T_0 < \infty$ be an arbitrary fixed real number and

$$L(R) = (1+R)2F_R(T_0), \tag{8}$$

where

$$F_R(T_0) = 2L_f(R)[T_0^\theta + R + \|\tilde{h}\|_{T,1} + \|\tilde{g}\|_{T,\alpha}] + \|f_n(0,0,0,0,0)\|. \tag{9}$$

Let $0 < T \le T_0$ be such that

$$\sup_{0\le t\le T} \left\{ \|(e^{-tA} - I)g(0)\| + \|(e^{-tA} - I)A^\alpha g(0)\| \right\} < \frac{R}{3}$$

and

$$T < \min\left\{ T_0,\ \frac{R}{3}\big[(M+1)L(R)\big]^{-1},\ \left[\frac{R}{3}(1-\alpha)[L(R)C_\alpha]^{-1}\right]^{\frac{1}{1-\alpha}} \right\}.$$

Let H_n denote the finite dimensional subspace of H spanned by $\{\phi_0, \phi_1, \dots, \phi_n\}$ and for each $n = 0,1,2,\dots,$ $P^n :\ H \to H_n$ be the corresponding projection operators. For each n we define $f_n :\ [0,T_0] \times D(A) \times D(A^\alpha) \times D(A) \times D(A^\alpha) \to H$ such that $f_n(t,u,v,w,z) = f(t, P^n u, P^n v, P^n w, P^n z)$, where $(u,v),(w,z) \in D(A) \times D(A^\alpha)$ and $t \in [0,T_0]$.

Let $W_R = B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g}))$, where

$$B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g})) = \{(y_1,y_2) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha :\ \|y_1 - \tilde{h}\|_{T,1} + \|y_2 - \tilde{g}\|_{T,\alpha} \le R\}.$$

Define a map S_n on W_R such that $S_n(u,v) = (\hat{u}, \hat{v})$ with

$$\hat{u}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - & \\ \int_0^t (e^{-(t-s)A} - I)A^{-1} f_n(s, u(s), v(s), u(s-\tau), v(s-\tau))ds, & t \in [0,T], \end{cases} \tag{10}$$

$$\hat{v}(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA}g(0) + \int_0^t e^{-(t-s)A} f_n(s, u(s), v(s), u(s-\tau), v(s-\tau))ds, & t \in [0,T]. \end{cases} \tag{11}$$

Theorem 3.1 *If all the assumptions (H1)–(H3) are satisfied then there exists a unique* $(u_n, v_n) \in W_R$ *such that* $S_n(u_n, v_n) = (u_n, v_n)$ *for each* $n = 0,1,2,\dots$.

Proof We claim that $S_n :\ W_R \to W_R$. For this we need to show that the map $t \mapsto (S_n(u,v))(t)$ is continuous from $[-\tau, T]$ into $D(A) \times D(A^\alpha)$ with respect to the norm $\|\cdot\|_1 + \|\cdot\|_\alpha$. For $t \in [-\tau, 0]$ we have

$$\|\hat{u}(t_2) - \hat{u}(t_1)\|_1 + \|\hat{v}(t_2) - \hat{v}(t_1)\|_\alpha = \|h(t_2) - h(t_1)\|_1 + \|g(t_2) - g(t_1)\|_\alpha. \tag{12}$$

For $t_1, t_2 \in (0,T]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
[\hat{u}(t_2)-\hat{u}(t_1)]+[\hat{v}(t_2)-\hat{v}(t_1)] &= [(e^{-t_2A}-e^{-t_1A})(-A)^{-1}g(0)]+[(e^{-t_2A}-e^{-t_1A})g(0)]\\
&+\int_{t_1}^{t_2}[e^{-(t_2-s)A}-I](-A)^{-1}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds\\
&+\int_0^{t_1}\left[e^{-(t_2-s)A}-e^{-(t_1-s)A}\right]\\
&\times(-A)^{-1}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds\\
&+\int_{t_1}^{t_2}e^{-(t_2-s)A}f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds\\
&+\int_0^{t_1}[(e^{-(t_2-s)A}-e^{-(t_1-s)A})]f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))ds.
\end{aligned}
$$

Hence from the above equation we get

$$
\begin{aligned}
\|\hat{u}(t_2)-\hat{u}(t_1)\|_1+\|\hat{v}(t_2)-\hat{v}(t_1)\|_\alpha &\le \|(e^{-t_2A}-e^{-t_1A})g(0)\|+\|(e^{-t_2A}-e^{-t_1A})g(0)\|_\alpha\\
&+\int_{t_1}^{t_2}\|e^{-(t_2-s)A}-I\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\
&+\int_0^{t_1}\|e^{-(t_2-s)A}-e^{-(t_1-s)A}\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\
&+\int_{t_1}^{t_2}\|A^\alpha e^{-(t_2-s)A}\|\,\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\
&+\int_0^{t_1}\|A^\alpha(e^{-(t_2-s)A}-e^{-(t_1-s)A})\|\\
&\times\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds.
\end{aligned}
$$

We calculate the above inequality as follows

$$
\begin{aligned}
&\int_{t_1}^{t_2}\|e^{-(t_2-s)A}-I\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\
&\qquad\le (M+1)L(R)(t_2-t_1)
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&\int_{t_1}^{t_2}\|e^{-(t_2-s)A}A^\alpha\|\,\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\
&\qquad\le L(R)C_\alpha\int_{t_1}^{t_2}(t_2-s)^{-\alpha}ds = L(R)C_\alpha\frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}\,.
\end{aligned}
\tag{14}
$$

Part (d) of Theorem 2.6.13 in Pazy [24] implies that for $0<\vartheta\le 1$ and $x\in D(A^\vartheta)$, we have

$$\|(e^{-tA}-I)x\|\le C'_\vartheta t^\vartheta\|x\|_\vartheta. \tag{15}$$

If $0<\vartheta<1$ and $0<\alpha+\vartheta<1$, then $A^\alpha y\in D(A^\vartheta)$ for any $y\in D(A^{\alpha+\vartheta})$. Therefore, for $t\in[0,T]$ and $s\in(0,T]$, we have

$$
\begin{aligned}
\|(e^{-tA}-I)A^\alpha e^{-sA}x\| &\le C'_\vartheta t^\vartheta\|A^\alpha e^{-sA}x\|_\vartheta = C'_\vartheta t^\vartheta\|A^{\alpha+\vartheta}e^{-sA}x\|\\
&\le C'_\vartheta C_{\alpha+\vartheta}t^\vartheta s^{-(\alpha+\vartheta)}\|x\|.
\end{aligned}
\tag{16}
$$

Hence from (16) we get

$$\begin{aligned}\|(e^{-(t_2-s)A}-e^{-(t_1-s)A})A^{\alpha}\| &= \|(e^{-(t_2-t_1)A}-I)A^{\alpha}e^{-(t_1-s)A}\|\\ &\le C'_{\vartheta}C_{\alpha+\vartheta}(t_2-t_1)^{\vartheta}(t_1-s)^{-(\alpha+\vartheta)}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{t_1}\|(e^{-(t_2-s)A}-e^{-(t_1-s)A})A^{\alpha}\|\,\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds&\\ \le C'_{\vartheta}C_{\alpha+\vartheta}L(R)(t_2-t_1)^{\vartheta}\int_0^{t_1}(t_1-s)^{-(\alpha+\vartheta)}ds&\\ \le C'_{\vartheta}C_{\alpha+\vartheta}L(R)\,\frac{T_0^{1-(\alpha+\vartheta)}}{1-(\alpha+\vartheta)}\,(t_2-t_1)^{\vartheta}.&\end{aligned} \tag{17}$$

Also, from (16), we have

$$\begin{aligned}\|(e^{-tA}-I)e^{-sA}x\| &\le C'_{\vartheta}t^{\vartheta}\|e^{-sA}x\|_{\vartheta} = C'_{\vartheta}t^{\vartheta}\|A^{\vartheta}e^{-sA}x\|\\ &\le C'_{\vartheta}C_{\vartheta}t^{\vartheta}s^{-\vartheta}\|x\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|e^{-(t_2-s)A}-e^{-(t_1-s)A}\| &= \|(e^{-(t_2-t_1)A}-I)e^{-(t_1-s)A}\|\\ &\le C'_{\vartheta}C_{\vartheta}(t_2-t_1)^{\vartheta}(t_1-s)^{-\vartheta}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{t_1}\|e^{-(t_2-s)A}-e^{-(t_1-s)}\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds&\\ \le C'_{\vartheta}C_{\vartheta}L(R)(t_2-t_1)^{\vartheta}\int_0^{t_1}(t_1-s)^{-\vartheta}ds&\\ \le C'_{\vartheta}C_{\vartheta}L(R)\,\frac{T_0^{1-\vartheta}}{1-\vartheta}\,(t_2-t_1)^{\vartheta}.&\end{aligned} \tag{18}$$

From inequalities (13), (14), (17) and (18), it follows that $S_n(u,v)(t)$ is continuous from $[-\tau,T]$ into $D(A)\times D(A^{\alpha})$ with respect to the norm $\|\cdot\|_1+\|\cdot\|_{\alpha}$. Next we want to show that $S_n(u,v)\in W_R$ i.e., $(\hat u,\hat v)\in W_R$. Now if $t\in[-\tau,0]$ then we have

$$\|\hat u(t)-\tilde h(t)\|_1+\|\hat v(t)-\tilde g(t)\|_{\alpha}=0.$$

Now, if $t\in(0,T]$, then we have

$$\begin{aligned}\|\hat u(t)-\tilde h(t)\|_1+\|\hat v(t)-\tilde g(t)\|_{\alpha} &\le \|(e^{-tA}-I)g(0)\|+\|(e^{-tA}-I)A^{\alpha}g(0)\|\\ &+\int_0^t\|e^{-(t-s)A}-I\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\ &+\int_0^t\|e^{-(t-s)A}A^{\alpha}\|\|f(s,P^nu(s),P^nv(s),P^nu(s-\tau),P^nv(s-\tau))\|ds\\ &\le \frac{R}{3}+(M+1)L(R))(T_0)T+C_{\alpha}L(R)(T_0)\int_0^t(t-s)^{-\alpha}ds\\ &\le \frac{R}{3}+(M+1)L(R)T+C_{\alpha}L(R)\,\frac{T^{1-\alpha}}{1-\alpha}\le R.\end{aligned}$$

Taking the supremum over $[-\tau, T]$, we get

$$\|\hat{u} - \tilde{h}\|_{T,1} + \|\hat{v} - \tilde{x}_1\|_{T,\alpha} \leq R,$$

which implies that $S_n(u,v) \in W_R$. Hence, S_n maps W_R into W_R. Now to complete the proof of this theorem it only remains to show that S_n is a strict contraction mapping on W_R.

If $t \in [-\tau, 0]$, and $(u_1, v_1), (u_2, v_2) \in W_R$, then we have

$$\begin{aligned}
&\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 + \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\
&\qquad \leq \int_0^t \|e^{-(t-s)A} - I\| \, \|f(s, P^n u_1(s), P^n v_1(s), P^n u_1(s-\tau), P^n v_1(s-\tau)) \\
&\qquad\quad - f(s, P^n u_2(s), P^n v_2(s), P^n u_2(s-\tau), P^n v_2(s-\tau))\| ds \\
&\qquad + \int_0^t \|e^{-(t-s)A} A^\alpha\| \, \|f(s, P^n u_1(s), P^n v_1(s), P^n u_1(s-\tau), P^n v_1(s-\tau)) \\
&\qquad\quad - f(s, P^n u_2(s), P^n v_2(s), P^n u_2(s-\tau), P^n v_2(s-\tau))\| ds.
\end{aligned}$$

From assumption (H3), we get

$$\begin{aligned}
\|f(t, &P^n u_1(t), P^n v_1(t), P^n u_1(t-\tau), P^n v_1(t-\tau)) \\
&- f(t, P^n u_2(t), P^n v_2(t), P^n u_2(t-\tau), P^n v_2(t-\tau))\| \\
&\leq F_R(T_0)[\|u_1(s) - u_2(s)\|_1 + \|v_1(s) - v_2(s)\|_\alpha \\
&\quad + \|u_1(s-\tau) - u_2(s-\tau)\|_1 + \|v_1(s-\tau) - v_2(s-\tau)\|_\alpha] \\
&\leq \frac{2RF_R(T_0)}{R}(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha)}).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f(t, &P^n u_1(t), P^n v_1(t), P^n u_1(t-\tau), P^n v_1(t-\tau)) \\
&- f(t, P^n u_2(t), P^n v_2(t), P^n u_2(t-\tau), P^n v_2(t-\tau))\| \\
&\leq 2F_R(T_0)(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}) \\
&\leq \frac{L(R)}{R}(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 + \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\
&\qquad \leq [(M+1)2F_R(T_0)T + 2C_\alpha F_R(T_0)] \int_0^t (t-s)^{-\alpha} ds \, (\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T\alpha}) \\
&\qquad \leq \frac{1}{R}\left[(M+1)L(R)T + C_\alpha L(R)\,\frac{T^{1-\alpha}}{1-\alpha}\right](\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}) \\
&\qquad \leq \frac{2}{3}\,(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).
\end{aligned}$$

Taking the supremum over $[-\tau, T]$, we get

$$\|\hat{u}_1 - \hat{u}_2\|_{T,1} + \|\hat{v}_1 - \hat{v}_2\|_{T,\alpha} \leq \frac{2}{3}\,(\|u_1 - u_2\|_{T,1} + \|v_1 - v_2\|_{T,\alpha}).$$

Thus S_n is a strict contraction mapping on W_R. Hence, there exists a unique pair $(u_n, v_n) \in W_R$ such that

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0) - (e^{-tA} - I)A^{-1}g(0) - \\ \int_0^t (e^{-(t-s)A} - I)A^{-1} f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))ds, & t \in [0, T], \end{cases} \tag{19}$$

and

$$v_n(t) = \begin{cases} g(t), & t \in [-\tau, 0], \\ e^{-tA}g(0) + \int_0^t e^{-(t-s)A} f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))ds, & t \in [0, T]. \end{cases} \tag{20}$$

The equations (19)–(20) are known as a pair of approximate solutions related to the given problem (1). □

Corollary 3.1 Let all the assumptions (H1)–(H3) hold. If $(h(t), g(t)) \in D(A) \times D(A)$ for all $t \in [-\tau, 0]$ then $(u_n(t), v_n(t)) \in D(A) \times D(A^\vartheta)$ for all $t \in [-\tau, T]$, where $0 \le \vartheta < 1$.

Proof From Theorem 3.1, we have the existence of a unique pair $(u_n, v_n) \in B_R(\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha, (\tilde{h}, \tilde{g}))$ satisfying (19)–(20). By Theorem (1.2.4) in Pazy [24], we have for $x \in H$, $\int_0^t e^{-tA}xds \in D(A)$ and if $x \in D(A)$ then $e^{-tA}x \in D(A)$. Thus the result follows from these facts and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$. □

Corollary 3.2 If all the conditions (H1)–(H3) hold then for $g(0) \in D(A)$ there exists a constant V_0 independent of n such that

$$\|v_n(t)\|_\vartheta \le V_0, \quad \text{where} \quad 0 \le \vartheta < 1, \quad -\tau \le t \le T.$$

Proof If $t \in [-\tau, 0]$, then from equation (20), we get the following

$$\|v_n(t)\|_\vartheta \le \|A^\vartheta g(0)\|.$$

If $t \in (0, T]$, then we have

$$\begin{aligned} \|v_n(t)\|_\vartheta &\le \|e^{-tA}A^\vartheta g(0)\| \\ &\quad + \int_o^t \|e^{-(t-s)A}A^\vartheta\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\| ds \\ &\le M\|g(0)\|_\vartheta + C_\vartheta L(R)\frac{T^{1-\vartheta}}{1-\vartheta} \le V_0'. \end{aligned}$$

This completes the proof of the Corollary. □

Corollary 3.3 If all the conditions (H1)–(H3) are hold then for $h(0) \in D(A)$ there exist a constant V_1 independent of n such that

$$\|u_n(t)\|_1 \le V_1, \quad \text{for all} \quad -\tau \le t \le T.$$

Proof If $t \in [-\tau, 0]$, then from equation (19) $\|v_n(t)\|_1 \le \|Ag(0)\|$.

If $t \in (0, T]$, then we have

$$\begin{aligned}\|u_n(t)\|_1 &\le \|h(0)\|_1 + \|(e^{-tA} - I)g(0)\| \\ &\quad + \int_o^t \|(e^{-(t-s)A} - I)\| \|f(s, P^n u(s), P^n v(s), P^n u(s-\tau), P^n v(s-\tau))\|\, ds \\ &\le \|h(0)\|_1 + (M+1)\|g(0)\| + (M+1)L(R)T \le V_1'.\end{aligned}$$

This completes the proof of the Theorem. □

4 Convergence of Approximate Solutions

In this section we will establish the convergence of the solution $(u_n, v_n) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ of approximate integral equations to a unique solution (u, v) of equation (1).

For proving the convergence, we need the following stronger assumption on the nonlinear map f than (H3).

(H3′) The nonlinear map f is defined from $[0, T] \times D(A) \times D(A^\alpha) \times D(A) \times D(A^\alpha)$ into $D(A^\beta)$ for $0 < \alpha < \beta < 1$ and there exists a nondecreasing function $\tilde{L}_f$ from $[0, \infty)$ into $[0, \infty)$ depending on some $r_1 > 0$ such that

$$\begin{aligned}&\|f(t, u_1, v_1, w_1, z_1) - f(s, u_2, v_2, w_2, z_2)\|_\beta \\ &\qquad \le \tilde{L}_f(r_1)\{|t-s|^\theta + \|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha + \|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha\}\end{aligned}$$

for all $t, s \in [0, T]$, $\theta \in (0, 1]$ and $(u_1, v_1), (u_2, v_2), (w_1, z_1), (w_2, z_2) \in B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t)))$, where $B_{r_1}(D(A) \times D(A^\alpha), (\tilde{h}(t), \tilde{g}(t))) = \{(x_1, y_1) \in D(A) \times D(A^\alpha) : \|x_1 - \tilde{h}(t)\|_1 + \|y_1 - \tilde{g}(t)\|_\alpha \le r_1\}$.

We can easily observe that the conditions (H3′) is stronger than (H3) because the same condition is satisfied in $D(A^\beta)$ rather than in H. Now, we are in a position to state a theorem.

Theorem 4.1 *Let (H1), (H2) and (H3′) be satisfied and* $(h(0), g(0)) \in D(A) \times D(A)$. *Then,*

$$\lim_{m \to \infty} \sup_{\{n \ge m,\ -\tau \le t \le T\}} \{\|u_n - u_m\|_{T,1} + \|v_n - v_m\|_{T,\alpha}\} = 0,$$

where u_n and v_n are given by (19) and (20) respectively.

Proof For $n \ge m$, we have

$$\begin{aligned}&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau), v_n(t-\tau)) - f_m(t, u_m(t), v_m(t), u_m(t-\tau), v_m(t-\tau))\| \\ &\le \|f_n(t, u_n(t), v_n(t), u_n(t-\tau), v_n(t-\tau)) - f_n(t, u_m(t), v_m(t), u_m(t-\tau), v_m(t-\tau))\| \\ &+ \|f_n(t, u_m(t), v_m(t), u_m(t-\tau), v_m(t-\tau)) - f_m(t, u_m(t), v_m(t), u_m(t-\tau), v_m(t-\tau))\| \\ &\le L_f(R)[\|P^n u_n(t) - P^n u_m(t)\|_1 + \|P^n v_n(t) - P^n v_m(t)\|_\alpha \\ &\quad + \|P^n u_n(t-\tau) - P^n u_m(t-\tau)\|_1 + \|P^n v_n(t-\tau) - P^n v_m(t-\tau)\|_\alpha \\ &\quad + \|(P^n - P^m) u_m(t)\|_1 + \|(P^n - P^m) v_m(t)\|_\alpha \\ &\quad + \|(P^n - P^m) u_m(t-\tau)\|_1 + \|(P^n - P^m) v_m(t-\tau)\|_\alpha].\end{aligned}$$

Also, we can see that

$$\begin{aligned}\|(P^n-P^m)v_m(t)\|_\alpha &= \|A^\alpha(P^n-P^m)v_m(t)\| = \|A^{\alpha-\vartheta}(P^n-P^m)A^\vartheta v_m(t)\| \\ &\le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|(P^n-P^m)A^\vartheta v_m(t)\| \le \frac{\|A^\vartheta v_m(t)\|}{\lambda_m^{\vartheta-\alpha}}\end{aligned}$$

and

$$\begin{aligned}\|(P^n-P^m)v_m(t-\tau)\|_\alpha &= \|A^\alpha(P^n-P^m)v_m(t-\tau)\| = \|A^{\alpha-\vartheta}(P^n-P^m)A^\vartheta v_m(t-\tau)\| \\ &\le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|(P^n-P^m)A^\vartheta v_m(t-\tau)\| \le \frac{\|A^\vartheta v_m(t-\tau)\|}{\lambda_m^{\vartheta-\alpha}}.\end{aligned}$$

For convenience, we denote

$$\xi_{m,n}(t) = \|u_n(t)-u_m(t)\|_1 + \|v_n(t)-v_m(t)\|_\alpha$$

and

$$\xi_{m,n}(t-\tau) = \|u_n(t-\tau)-u_m(t-\tau)\|_1 + \|v_n(t-\tau)-v_m(t-\tau)\|_\alpha.$$

Thus, we have

$$\begin{aligned}&\|f_n(t,u_n(t),v_n(t),u_n(t-\tau)),v_n(t-\tau)) - f_m(t,u_m(t),v_m(t),u_m(t-\tau),v_m(t-\tau))\| \\ &\quad\le L_f(R)[\xi_{m,n}(t)+\xi_{m,n}(t-\tau)+\|(P^n-P^m)u_m(t)\|_1 \\ &\qquad + \frac{\|v_m(t)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}} + \|(P^n-P^m)u_m(t-\tau)\|_1 + \frac{\|v_m(t-\tau)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}] \\ &\quad\le 2L_f(R)\left[\{\|u_n-u_m\|_{t,1}+\|v_n-v_m\|_{t,\alpha}\} + \|(P^n-P^m)u_m\|_{t,1} + \frac{\|v_m\|_{t,\vartheta}}{\lambda_m^{\vartheta-\alpha}}\right].\end{aligned} \tag{21}$$

Now, from the pair of integral equations (19)–(20), for any $0 < t_0' < t < T_0$, we have

$$\begin{aligned}&\|u_n(t)-u_m(t)\|_1 + \|v_n(t)-v_m(t)\|_\alpha \\ &\quad\le \left\{\int_0^{t_0'}\|e^{-(t_0'-s)A}-I\| + \int_{t_0'}^t \|e^{-(t-s)A}-I\|\right\} \\ &\quad\times\big[\|f_n(s,u_n(s),v_n(s),u_n(s-\tau),v_n(s-\tau)) \\ &\qquad - f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\big]ds \\ &\quad+\left\{\int_0^{t_0'}\|e^{-(t_0'-s)A}A^\alpha\| + \int_{t_0'}^t \|e^{-(t-s)A}A^\alpha\|\right\} \\ &\quad\times\big[\|f_n(s,u_n(s),v_n(s),u_n(s-\tau),v_n(s-\tau)) \\ &\qquad - f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\big]ds.\end{aligned} \tag{22}$$

By using the estimate of the inequality (21) in the inequality (22), we get

$$\begin{aligned}
&\|u_n(t)-u_m(t)\|_1+\|v_n(t)-v_m(t)\|_\alpha\\
&\quad\le A_1t_0'+L(R)\int_{t_0'}^t\left((M+1)+\frac{C_\alpha}{(t-s)^\alpha}\right)ds\left(\|(P^n-P^m)u_m\|_{T,1}+\frac{V_0}{\lambda_m^{\vartheta-\alpha}}\right)\\
&\qquad+L(R)\int_{t_0'}^t\left((M+1)+\frac{C_\alpha}{(t-s)^\alpha}\right)\{\|u_n-u_m\|_{s,1}+\|v_n-v_m\|_{s,\alpha}\}ds\\
&\quad\le A_1t_0'+C(R,T)B_{mn}+N_1\int_{t_0'}^t\frac{1}{(t-s)^\alpha}\{\|u_n-u_m\|_{s,1}+\|v_n-v_m\|_{s,\alpha}\}ds,
\end{aligned}\tag{23}$$

where

$$\begin{gathered}
B_{mn}=B_{mn}^1+B_{mn}^2,\quad B_{mn}^1=\|(P^n-P^m)u_m\|_{T,1},\quad B_{mn}^2=\frac{V_0}{\lambda_m^{\vartheta-\alpha}},\\
C(R,T)=L(R)\left((M+1)T+\frac{C_\alpha T^{1-\alpha}}{1-\alpha}\right),\\
N_1=L(R)(T^\alpha+1)\max\{(M+1),C_\alpha\}
\end{gathered}$$

and

$$\begin{aligned}
A_1=\{(M+1)+C_\alpha(t_0-t_0')^{-\alpha}\}2L_f(R)\Big[&\{\|u_n-u_m\|_{t_0',1}+\|v_n-v_m\|_{t_0',\alpha}\}\\
&+\|(P^n-P^m)u_m\|_{t_0',1}+\frac{V_0}{\lambda_m^{\vartheta-\alpha}}\Big]t_0'.
\end{aligned}\tag{24}$$

Now we replace t by $t+\theta$ in the inequality (23), where $\theta\in[t_0'-t,0]$, we get

$$\begin{aligned}
\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha&\le A_1t_0'+C(R,T)B_{mn}\\
+N_1\int_{t_0'}^{t+\theta}(t+\theta-s)^{-\alpha}\{\|u_n-u_m\|_{s,1}+\|v_n-v_m\|_{s,\alpha}\}\,ds.&
\end{aligned}\tag{25}$$

We put $s-\theta=\gamma$ in inequality (25) and get

$$\begin{aligned}
&\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\\
&\quad\le A_1t_0'+C(R,T)B_{mn}+N_1\int_{t_0'-\theta}^t(t-\gamma)^{-\alpha}\{\|u_n-u_m\|_{\gamma,1}+\|v_n-v_m\|_{\gamma,\alpha}\}d\gamma\\
&\quad\le A_1t_0'+C(R,T)B_{mn}+N_1\int_{t_0'}^t(t-\gamma)^{-\alpha}\{\|u_n-u_m\|_{\gamma,1}+\|v_n-v_m\|_{\gamma,\alpha}\}d\gamma.
\end{aligned}\tag{26}$$

Thus

$$\begin{aligned}
&\sup_{t_0'-t\le\theta\le0}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}\\
&\le A_1t_0'+C(R,T)B_{mn}+N_1\int_0^t(t-\gamma)^{-\alpha}\{\|u_n-u_m\|_{\gamma,1}+\|v_n-v_m\|_{\gamma,\alpha}\}d\gamma.
\end{aligned}\tag{27}$$

We have

$$\begin{aligned}\sup_{-\tau-t\leq\theta\leq 0}&\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}\\ &\leq \sup_{0\leq\theta+t\leq t_0'}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}\\ &+\sup_{t_0'-t\leq\theta\leq 0}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}.\end{aligned} \tag{28}$$

By using the inequalities (26) and (27) in the inequality (28), we get

$$\begin{aligned}&\sup_{-\tau\leq t+\theta\leq t}\{\|u_n(t+\theta)-u_m(t+\theta)\|_1+\|v_n(t+\theta)-v_m(t+\theta)\|_\alpha\}\\ &\leq 2A_1t_0'+C(R,T)B_{mn}+N_1\int_0^t(t-\gamma)^{-\alpha}\{\|u_n-u_m\|_{\gamma,1}+\|v_n-v_m\|_{\gamma,\alpha}\}d\gamma.\end{aligned} \tag{29}$$

Hence, from Gronwall's Lemma and taking the limit as $m\to\infty$ on both sides, we get the required result, since $B_{mn}\to 0$ as $m\to\infty$ provided $\|(P^n-P^m)u_m\|_{T,1}\to 0$ as $m\to\infty$ for $-\tau\leq t\leq T$. Since $B_{mn}^2\to 0$ as $m\to\infty$, hence to prove that $B_{mn}\to 0$, we only need to prove that for $-\tau\leq t\leq T$, $\|(P^n-P^m)u_m(t)\|_1\to 0$ as $m\to\infty$. We can easily check that for every $x\in H$ and $\eta<0$

$$\|A^\eta(P^n-P^m)x\|\leq\lambda_m^\eta\|(P^n-P^m)x\|\leq\lambda_m^\eta\|x\|. \tag{30}$$

From the equation (19), for any $t\in[-\tau,0]$ we have

$$\|A(P^n-P^m)u_m(t)\|=\|(P^n-P^m)Ah(0)\|. \tag{31}$$

For $t\in(0,T]$, we have

$$\begin{aligned}&\|A(P^n-P^m)u_m(t)\|\leq\|(P^n-P^m)Ah(0)\|+(M+1)\|(P^n-P^m)g(0)\|\\ &+(M+1)\int_0^t\|(P^n-P^m)f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\,ds.\end{aligned} \tag{32}$$

Since $A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(t-\tau))\in H$, hence from inequality (30), we have

$$\begin{aligned}&\|(P^n-P^m)f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\\ &\qquad\leq\|A^{-\beta}(P^n-P^m)A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\|\|\\ &\qquad\leq\frac{1}{\lambda_m^\beta}\|A^\beta f_m(s,u_m(s),v_m(s),u_m(s-\tau),v_m(s-\tau))\|\\ &\qquad\leq\frac{1}{\lambda_m^\beta}\tilde F_R(T_0),\end{aligned} \tag{33}$$

where

$$\tilde F_R(T_0)=2\tilde L_f(R)[T_0^\theta+R+\|\tilde h\|_{T,1}+\|\tilde g\|_{T,\alpha}]+\|f_n(0,0,0,0,0)\|. \tag{34}$$

Using the inequality (33) in the inequality (32), we get

$$\|(P^n-P^m)u_m(t)\|_1\leq\|(P^n-P^m)Ax_0\|+(M+1)\{\|(P^n-P^m)x_1\|+\frac{1}{\lambda_m^\beta}T(\tilde F_R(T_0), \tag{35}$$

which tend to zero as $m\to\infty$ for $0\leq t\leq T$. Hence from (32) and (35) we get the required result. This completes the proof of the theorem. □

Theorem 4.2 *If (H1)–(H2) and (H3′) are satisfied and* $(h(0), g(0)) \in D(A) \times D(A)$ *then there exists a pair of functions* $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ *such that* $(u_n, v_n) \to (u, v)$ *as* $n \to \infty$ *in* $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ *and* (u, v) *satisfies (6)–(7) on* $[-\tau, T]$.

Proof Theorem 4.1 implies that there exists $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ such that (u_n, v_n) converges to (u, v) in $\mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$. Since $(u_n, v_n) \in W_R$ for each n, (u, v) is also in W_R. Further, we have

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le \|f(t, P^n u_n(t), P^n v_n(t), P^n u_n(t-\tau)), P^n v_n(t-\tau)) \\
&\qquad - f(t, P^n u(t), P^n v(t), P^n u(t-\tau), P^n v(t-\tau))\| \\
&\qquad + \|f(t, P^n u(t), P^n v(t), P^n u(t-\tau), P^n v(t-\tau)) \\
&\qquad - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\|.
\end{aligned}$$

Hence from the above inequality, we have

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le L_f(R)[\|P^n u_n(t) - P^n u(t)\|_1 + \|P^n v_n(t) - P^n v(t)\|_\alpha \\
&\qquad + \|P^n u_n(t-\tau) - P^n u(t-\tau)\|_1 + \|P^n v_n(t-\tau) - P^n v(t-\tau)\|_\alpha \\
&\qquad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha \\
&\qquad + \|(P^n - I)u(t-\tau)\|_1 + \|(P^n - I)v(t-\tau)\|_\alpha] \\
&\quad \le L_f(R)[\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha \\
&\qquad + \|u_n(t-\tau) - u(t-\tau)\|_1 + \|v_n(t-\tau) - v(t-\tau)\|_\alpha \\
&\qquad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha \\
&\qquad + \|(P^n - I)u(t-\tau)\|_1 + \|(P^n - I)v(t-\tau)\|_\alpha].
\end{aligned}$$

Thus finally we get

$$\begin{aligned}
&\|f_n(t, u_n(t), v_n(t), u_n(t-\tau)), v_n(t-\tau)) - f(t, u(t), v(t), u(t-\tau), v(t-\tau))\| \\
&\quad \le 2L_f(R)[\|u_n - u\|_{T,1} + \|v_n - v\|_{T,\alpha}|(P^n - I)u\|_{T,1} + \|(P^n - I)v\|_{T,\alpha}].
\end{aligned} \tag{36}$$

Hence, by using the inequality (36) and the bounded convergence theorem we can see easily that the pair of functions (u, v) must be given by equations (6)–(7). □

5 Faedo-Galerkin Approximations

From the previous sections we know that for any $-\tau \le T < \infty$ we have a unique pair $(u, v) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ satisfying the integral equations (6)–(7).

Also we have a unique pair $(u_n, v_n) \in \mathcal{C}_T^1 \times \mathcal{C}_T^\alpha$ which is the solution of the approximate integral equations (19)–(20).

If we project the equations (19)–(20) onto H_n, we get the Faedo-Galerkin approximation $(\hat{u}_n(t), \hat{v}_n(t)) = (P^n u_n(t), P^n v_n(t))$ satisfying

$$\hat{u}_n(t) = \begin{cases} P^n h(t), & t \in [-\tau, 0], \\ P^n h(0) - (e^{-tA} - I)A^{-1}P^n g(0) - \int_0^t (e^{-(t-s)A} - I)A^{-1} \times & \\ P^n f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))\, ds, & t \in [0, T], \end{cases} \tag{37}$$

$$\hat{v}_n(t) = \begin{cases} P^n g(t), & t \in [-\tau, 0], \\ e^{-tA} P^n g(0) + & \\ \int_0^t e^{-(t-s)A} P^n f_n(s, u_n(s), v_n(s), u_n(s-\tau), v_n(s-\tau))\, ds, & t \in [0, T]. \end{cases} \tag{38}$$

The solution (u, v) of (6)–(7) and $(\hat{u}_n, \hat{v}_n)$ of (37)–(38), have the representations

$$\begin{aligned} u(t) &= \sum_{i=0}^{\infty} \alpha_i(t)\phi_i, \quad \alpha_i(t) = \langle u(t), \phi_i \rangle, \quad i = 0, 1, \dots, \\ v(t) &= \sum_{i=0}^{\infty} \beta_i(t)\phi_i, \quad \beta_i(t) = \langle v(t), \phi_i \rangle, \quad i = 0, 1, \dots, \end{aligned} \tag{39}$$

and

$$\begin{aligned} \hat{u}_n(t) &= \sum_{i=0}^{n} \alpha_i^n(t)\phi_i, \quad \alpha_i^n(t) = \langle \hat{u}_n(t), \phi_i \rangle, \quad i = 0, 1, \dots, n, \\ \hat{v}_n(t) &= \sum_{i=0}^{n} \beta_i^n(t)\phi_i, \quad \beta_i^n(t) = \langle \hat{v}_n(t), \phi_i \rangle, \quad i = 0, 1, \dots, n. \end{aligned} \tag{40}$$

Now, we shall show the convergence of (α_i^n, β_i^n) to (α_i, β_i). It can be easily checked that

$$A[u(t) - \hat{u}(t)] = \sum_{i=0}^{\infty} \lambda_i(\alpha_i(t) - \alpha_i^n(t))\phi_i$$

and

$$A^{\alpha}[v(t) - \hat{v}(t)] = \sum_{i=0}^{\infty} \lambda_i^{\alpha}(\beta_i(t) - \beta_i^n(t))\phi_i.$$

Thus, we have

$$\|A[u(t) - \hat{u}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^2 |\alpha_i(t) - \alpha_i^n(t)|^2$$

and

$$\|A^{\alpha}[v(t) - \hat{v}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} |\beta_i(t) - \beta_i^n(t)|^2.$$

Hence, we have the following convergence theorem.

Theorem 5.1 *Let (H1), (H2) and (H3') be satisfied and* $(h(0), g(0)) \in D(A) \times D(A)$. *Then,*

$$\lim_{n\to\infty} \sup_{-\tau \leq t \leq T} \left\{ \sum_{i=0}^{n} \lambda_i^2 |\alpha_i(t) - \alpha_i^n(t)|^2 + \sum_{i=0}^{n} \lambda_i^{2\alpha} |\beta_i(t) - \beta_i^n(t)|^2 \right\} = 0.$$

The assertion of Theorem 5.1 follows from the facts mentioned above and from the following proposition.

Theorem 5.2 *Let (H1), (H2) and (H3') be satisfied and let T be any number such that $0 < T < \infty$, and $(h(0), g(0)) \in D(A) \times D(A)$. Then,*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ -\tau\leq t\leq T\}} \{\|A[\hat{u}_n(t) - \hat{u}_m(t)]\| + \|A^\alpha[\hat{v}_n(t) - \hat{v}_m(t)]\|\} = 0.$$

Proof For $n \geq m$, we have

$$\begin{aligned}
&\|A(\hat{u}_n(t) - \hat{u}_m(t))\| + \|A^\alpha(\hat{v}_n(t) - \hat{v}_m(t))\| \\
&\quad = \|A(P^n u_n(t) - P^m u_m(t))\| + \|A^\alpha(P^n v_n(t) - P^m v_m(t))\| \\
&\quad \leq \|AP^n(u_n(t) - u_m(t))\| + \|A(P^n - P^m)u_m(t)\| \\
&\quad\quad + \|A^\alpha P^n(v_n(t) - v_m(t))\| + \|A^\alpha(P^n - P^m)v_m(t)\| \\
&\quad \leq \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_1 + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\beta v_m\|.
\end{aligned}$$

Hence, the result follows directly from Theorem 4.1. □

6 Example

Let $H = L^2((0,1);\mathbb{R})$. Consider the following partial delay differential equations

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2}(t,x) - \frac{\partial^2 w}{\partial x^2}(t,x) \\
&\quad = F\Big(t, x, \frac{\partial^2 w}{\partial x^2}(t,x), \frac{\partial^2 w}{\partial x \partial t}(t,x), \frac{\partial^2 w}{\partial x^2}(t-\tau,x), \frac{\partial^2 w}{\partial x \partial t}(t-\tau,x)\Big), \\
&\qquad x \in (0,1), \quad t > 0, \\
&w(\xi,x) = h_1(\xi,x), \quad \frac{\partial w}{\partial t}(\xi,x) = g_1(\xi,x) \quad \text{for all} \quad \xi \in [-\tau,0], \quad x \in (0,1) \\
&\text{and} \quad w(t,0) = w(t,1) = 0, \quad t \in [0,T], \quad 0 < T < \infty,
\end{aligned} \tag{41}$$

where F is a sufficiently smooth nonlinear function, h_1 and g_1 are given locally Hölder continuous functions on $[-\tau, 0]$.

We define an operator A as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = H_0^1(0,1). \tag{42}$$

Here clearly the operator A is self-adjoint with the compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm

$$\|x\|_{1/2} = \|A^{1/2}x\|, \quad x \in D(A^{1/2}),$$

and we denote this space by $H_{1/2}$.

The equation (41) can be reformulated as the following abstract equation in H:

$$\begin{aligned}
&\frac{d^2 u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) = f\left(t, u(t), \frac{du}{dt}(t), u(t-\tau), \frac{du}{dt}(t-\tau)\right), \quad t > 0, \\
&u(t) = h(t), \quad u'(t) = g(t) \quad \text{for all} \quad t \in [-\tau, 0],
\end{aligned} \tag{43}$$

where $u(t)(x) = w(t,x)$, $h(t)(x) = h_1(t,x)$, $g(t)(x) = g_1(t,x)$, the linear operator A is given by equation (42) and the function f is defined from $[0,T] \times D(A) \times D(A^{1/2}) \times D(A) \times D(A^{1/2})$ into H such that

$$\begin{aligned} f\left(t,\, u(t),\, \frac{du}{dt}(t),\, u(t-\tau),\, \frac{du}{dt}(t-\tau)\right)(x) \\ = F\left(t,\, x,\, \frac{\partial^2 w}{\partial x^2}(t,x),\, \frac{\partial^2 w}{\partial x \partial t}(t,x),\, \frac{\partial^2 w}{\partial x^2}(t-\tau,x),\, \frac{\partial^2 w}{\partial x \partial t}(t-\tau,x)\right). \end{aligned}$$

It can be verified that the assumptions of Theorem 3.1 for (43) are satisfied and hence the existence of a unique solution of (43) is guaranteed which in turn ensures the existence of a unique solution to (41).

Acknowledgements

The authors wish to thank the referees for their comments and valuable suggestions. The second author would like to thank the National Board of Higher Mathematics for providing the financial support to carry out this work under its research project No. NBHM/40/5/2006-R&D-II/6517.

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