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# Nonlinear Dynamics of a Two-Degrees of Freedom Hamiltonian System: Bifurcations and Integration ⋆

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**Abstract:** In this paper we treat the motion induced by a starting pulse on a system of two-degrees of freedom $s$, $\theta$. Decoupling the motion equations, we obtain the $s$-nonlinear ordinary differential equation

$$\ddot{s} = c^2 \frac{s}{(d^2 + s^2)^2} - \lambda^2 s,$$

where $(c, d, \lambda) > 0$, and the dots mean time derivatives. A bifurcation analysis has revealed the onset of periodic motions for $\lambda \neq 0$ (presence of elastic forces inside the system), whilst for $\lambda = 0$ nonperiodic motions will appear. Almost all the cases (five for $\lambda \neq 0$, three for $\lambda = 0$) have been integrated by obtaining $t = t(s)$ by means of the Jacobi elliptic functions.
The other (angle) coordinate $\theta$ has been in any case brought to the quadratures by knowing $s$.

**Keywords:** *Nonlinear differential equations; Hamiltonian systems; bifurcations; elliptic functions.*

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## 1 Introduction

The cases in which the ordinary differential equations (ode) can be integrated in closed form, or even reduced to the quadratures, are quite limited.

Since much time we are involved in the integrable systems with and without friction, contributing with closed form integrations of ode by means of higher transcendental functions [3, 4, 5]. In this frame we treated single degree of freedom systems.

This article tackles a system[1] whose planar, frictionless motion depends upon *two* Lagrangian coordinates (the displacement $s$ and the angle $\theta$). The couple of nonlinear ode with the coordinates tied, has been de-coupled and integrated. A bifurcation analysis has been carried out on the basis of the values of a certain parameter $\lambda$ which is controlling the elastic force inside the system: its description follows. An homogeneous straight pipe of mass $M$ and length $\ell$ can rotate on a horizontal plane, around its middle fixed point O, without friction.

A punctual body P of mass $m$ can flow frictionless inside it, acted by a spring which is at rest only when P≡O. Consequently the deformation of the spring entering its elastic potential, will coincide with the particle's coordinate $s$.

The movement is induced by a starting instantaneous pulse ($s_0$, $\dot{s}_0$, $\theta_0$, $\dot{\theta}_0$): the absence of any propelling force, drag, friction is assumed then it will persist indefinitely. Let the angle $\theta$ be the pipe axis inclination, and $s =$ OP be the instantaneous distance

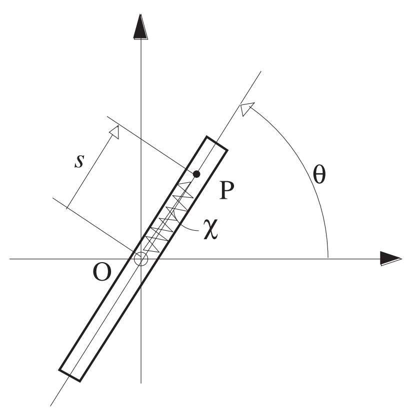


**Figure 1.1:** The elastic pendulum: a system's geometrical sketch.

of P from the pivot O. For the system $\mathcal{L}$-function is

$$\mathcal{L} = -\chi\frac{s^2}{2} + \frac{1}{2}J\dot{\theta}^2 + \frac{m}{2}\dot{s}^2 + \frac{m}{2}s^2\dot{\theta}^2,$$

being $J = \dfrac{1}{12}M\ell^2$ the pipe moment of inertia, and $\chi > 0$ a measure of the spring elastic stiffness.

[1] The problem has been introduced -and only sketched- on pages 279–280 of [6], where the Lagrange equations (1) and (2) are obtained, and the second order ode (6) integrated a first time. But a mistake occurred ($\dot{s}$ instead of $\dot{s}^2$). The second integration has not been carried out there, nor the motion any way analyzed, nor qualitatively discussed.

The above system could be termed as *elastic pendulum*, namely a non-circular pendulum obtained from the classic one, by replacing its unextensible, weightless rod between the body P and the suspension O, with the deformable constraint of a linear elastic spring OP. Of course the weight force on P is perpendicular to the motion's plane shown in Figure 1.1. The Lagrange equation for $s$ gives

$$\begin{aligned} & m\,\ddot{s} + \chi\,s - m\,s\,\dot{\theta}^2 = 0, \\ & s(0) = s_0, \quad \dot{s}(0) = \dot{s}_0, \end{aligned} \tag{1}$$

while, for $\mathcal{L}$ not depending upon $\theta$, we get the other one:

$$\begin{aligned} & \frac{d}{dt}\left(J\,\dot{\theta} + m\,s^2\,\dot{\theta}\right) = 0, \\ & \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0. \end{aligned} \tag{2}$$

By (2), putting $\ell = 2\sqrt{3}b$ and then $J = Mb^2$, we have immediately a first integral:

$$\left(Mb^2 + m\,s^2\right)\dot{\theta} = \dot{\theta}_0\left(Mb^2 + m\,s_0^2\right) \equiv c_1 \tag{3}$$

for $c_1$ being a positive constant depending on both the system characteristic and the initial conditions.

## 2 Bifurcation Analysis in the Presence of Elastic Force

Starting from (3), we have

$$\theta(t) = \theta_0 + \frac{c_1}{m}\int_0^t \frac{d\tau}{\gamma^2 + s^2(\tau)} \tag{4}$$

with

$$\gamma^2 = \frac{M}{m}b^2 > 0, \tag{5}$$

then $\theta$ is known if we succeed in evaluating $s(t)$. Moreover, if (3) is replaced in (1), we can get rid of $\ddot{\theta}$, obtaining

$$\ddot{s} = c^2\frac{s}{\left(d^2 + s^2\right)^2} - \lambda^2 s = f(s), \tag{6}$$

where all the constants below are positive, i.e.

$$c = \frac{c_1}{m},\ d = \frac{b}{m},\ \lambda^2 = \frac{\chi}{m}. \tag{7}$$

Of course the meaning of the parameter $\lambda$ is the presence, or the absence, $\lambda = 0$, of the elastic force inside the system. This will induce the system to bifurcate.

Following the Weierstraß method [8], we write the relevant time equation as

$$t = \pm\int_{s_0}^{s}\frac{du}{\sqrt{\Phi(u)}},$$

with

$$\Phi(s) = 2\int_{s_0}^{s} f(u)\,du + \dot{s}_0^2,$$

the sign has to be taken, according to the sign of the initial speed $\dot{s}_0^2$, or, if $\dot{s}_0^2 = 0$ according to the $f(s_0)$ sign, as it is well known, see e.g. [7] page 114 or [1] pages 287–292.

The $\Phi = 0$ roots' existence and kind, marks completely the motion, deciding its periodic or aperiodic nature. Obviously the reality condition $\Phi(s) \geq 0$ must be met: it is always satisfied in a neighborhood of $s_0^2$. We have

$$\Phi(s) = h^2 - \lambda^2 s^2 - \frac{c^2}{d^2 + s^2}, \tag{8}$$

where

$$h^2 = h^2(c, d, \lambda; s_0, \dot{s}_0) = \lambda^2 s_0^2 + \frac{c^2}{d^2 + s_0^2} + \dot{s}_0^2 > 0. \tag{9}$$

Therefore the motion reality condition stems from the positivity of the $4^{\text{th}}$ degree polynomial

$$p(s) = -\lambda^2 s^4 + (h^2 - d^2\lambda^2)s^2 + d^2 h^2 - c^2.$$

Such a problem is an elementary, but quite tedious exercise of Calculus. First notice that in any case, by construction, we have $p(s_0) = \dot{s}_0^2(d^2 + s_0^2) > 0$. The discussion is pivoted on the number of roots of its first derivative

$$p'(s) = 2s\left(h^2 - d^2\lambda^2 - 2s^2\lambda^2\right).$$

Therefore all the treatment is centered on the motion (6) which takes place along a rotating straight-line of variable inclination (4) during the time.

**Degeneracy**

If $\dot{\theta}_0 = 0$, then $c_1 = 0$, and (6) would return the elementary harmonic movement with period $2\pi/\lambda$. Then the presence of a nonzero initial pulse of angular speed $\dot{\theta}_0 \neq 0$ is essential: from now on our analysis will deal with non-degenerate cases only.

### 2.1 First case: $p'(s)$ has three real roots ($h^2 > d^2\lambda^2$)

Suppose first that

$$h^2 - d^2\lambda^2 > 0. \tag{10}$$

If (10) holds, the first derivative of $p(s)$ has three real roots, say

$$\hat{s} = 0, \quad \hat{s}_+ = \frac{\sqrt{h^2 - d^2\lambda^2}}{\lambda\sqrt{2}}, \quad \hat{s}_- = -\hat{s}_+.$$

Moreover (10) implies that $\hat{s} = 0$ is for $p(s)$ a relative minimum given by

$$p(0) = h^2 d^2 - c^2, \tag{11}$$

and at $\hat{s}_+$ and $\hat{s}_-$ $p(s)$ has two relative maxima whose common value is

$$p(\hat{s}_-) = p(\hat{s}_+) = \frac{\left(h^2 + d^2\,\lambda^2\right)^2}{4\lambda^2} - c^2.$$

Three following sub-cases are possible.

First, $p(s)$ has two real zeros if the condition

$$h^2 d^2 > c^2 \tag{12}$$

holds. This means that, as far as it concerns the $s$ coordinate, the particle oscillates periodically along the pipe between the symmetrical extremes

$$s_{\pm} = \pm \frac{1}{\lambda\sqrt{2}} \sqrt{h^2 - d^2\,\lambda^2 + \sqrt{\left(h^2 + d^2\,\lambda^2\right)^2 - 4\,c^2\,\lambda^2}}. \tag{13}$$

Second, if, vice-versa, we have

$$h^2 d^2 < c^2 \tag{14}$$

and if we have also

$$\frac{\left(h^2 + d^2\lambda^2\right)^2}{4\lambda^2} > c^2, \tag{15}$$

we are in the presence of four real zeros of $p(s)$ placed symmetrically on the real axis, i.e.: $-s_2 < -s_1 < 0 < s_1 < s_2$; the motion will be periodic between the two positive or the two negative roots, according to the sign of $s_0$. We have

$$s_2 = \frac{1}{\lambda\sqrt{2}} \sqrt{h^2 - d^2\lambda^2 + \sqrt{\left(h^2 + d^2\lambda^2\right)^2 - 4c^2\lambda^2}}, \tag{16}$$

$$s_1 = \frac{1}{\lambda\sqrt{2}} \sqrt{h^2 - d^2\lambda^2 - \sqrt{\left(h^2 + d^2\lambda^2\right)^2 - 4c^2\lambda^2}}. \tag{17}$$

The range of coordinates $(-s_1,\, s_1)$ is forbidden to the motion for the reality condition $p(s) > 0$ being not met.

Notice that the situation

$$h^2 d^2 < c^2, \quad \frac{\left(h^2 + d^2\lambda^2\right)^2}{4\lambda^2} < c^2$$

would be against the reality of the motion, as implying all the roots of $p(s)$ to be complex, the negativity of $p(s)$, and so forth.

Finally, third, if

$$h^2 d^2 = c^2 \tag{18}$$

we are faced with a double root at the origin for $p(s)$, whose form is

$$p(s) = s^2(h^2 - d^2\lambda^2 - \lambda^2 s^2).$$

Then it will be $p(s) \geq 0$ for

$$-\frac{1}{\lambda}\sqrt{h^2 - d^2\lambda^2} \leq s \leq \frac{1}{\lambda}\sqrt{h^2 - d^2\lambda^2},$$

but the double zero at the origin implies an asymptotic motion, not a periodic one. The motion will take place for positive or negative values of $s$ according to the sign of $s_0$. If $s_0 = 0$, the sign of $\dot{s}_0$ will determine the region of motion. Finally, if $s_0 = \dot{s}_0 = 0$, there will be no motion at all.

### 2.2 Second case: $p'(s)$ with one real root ($h^2 < d^2\lambda^2$)

Suppose that

$$h^2 - d^2\lambda^2 < 0. \tag{19}$$

Now $p(s)$ has only one stationary point in $\hat{s} = 0$ which is a maximum. The relative extremum is once more given by (11), and the motion reality is ensured again by (12).

The particle's movement is periodic between the roots singled out by (13).

### 2.3 Third case: $p'(s)$ has a real triple root ($h^2 = d^2\lambda^2$)

We now have

$$h^2 - d^2\lambda^2 = 0. \tag{20}$$

This means that $p(s) = -\lambda^2 s^4 + \lambda^2 d^4 - c^2,\ p'(s) = -4\lambda^2 s^3$. In order to meet the reality condition we must require that

$$d^2 > \frac{c}{\lambda}. \tag{21}$$

So the motion is periodic between the two real symmetric roots of $p(s) = 0$.

## 3 Integration: $\lambda \neq 0$

The time equation

$$t = \pm \int_{s_0}^{s} \frac{du}{\sqrt{h^2 - \lambda^2 u^2 - c^2(d^2+u^2)^{-1}}} \tag{22}$$

will be solved by transforming the integral (22) in a form studied in [2], involving the I and III kind canonical elliptic integrals. For the purpose, let we pass from $u$ to the new variable $\zeta$ defined by $u = \sqrt{\zeta^2 - d^2}$. Discarding the problem of the sign, we can take both $s_0$ and $s$ positive without loss of generality. Then (22) will be transformed into

$$t = \frac{1}{2} \int_{d^2+s_0^2}^{d^2+s^2} \sqrt{\frac{\zeta}{[-\lambda^2\zeta^2 + (h^2 + d^2\lambda^2)\zeta + c^2](\zeta - d^2)}}\, d\zeta. \tag{23}$$

The discriminant $\Delta$ of the second degree polynomial

$$q(\zeta) = -\lambda^2\zeta^2 + (h^2 + d^2\lambda^2)\zeta + c^2, \tag{24}$$

appearing in (23) is

$$\Delta = (h^2 - 2c\lambda + d^2\lambda^2)(h^2 + 2c\lambda + d^2\lambda^2), \tag{25}$$

and its positivity depends on the sign of the first factor $h^2 - 2c\lambda + d^2\lambda^2$. If we consider it as a function of $\lambda$, its discriminant is

$$\Delta_1 = c^2 - d^2 h^2. \tag{26}$$

Now we have to go back to the discussion about the Weierstraß function $\Phi$ introduced in (8).

1. Assume (10) and (12) hold.
2. Assume (10), (14) and (15) hold.
3. Assume (10) and (18) hold.
4. Assume (19) and (12) hold.
5. Assume (20) and (21) hold.

**Case 1**

The quantity $\Delta_1$ introduced in (26) is negative and so (25) is positive. This implies that $q(\zeta)$, see (24), has two real roots $\zeta_1 < \zeta_2$, and recalling that $\zeta_1 \leq d^2 + s_0^2 \leq d^2 + s^2 \leq \zeta_2$, we infer $s_0^2 \leq \zeta - d^2 \leq s^2$. We have now to locate the position of $d^2$ with respect to $\zeta_1$ and $\zeta_2$; this can be done because for (12) we have $q(d^2) = h^2 d^2 - c^2 > 0$ and this means that $\zeta_1 < d^2 < \zeta_2$. Now if we write (23) as

$$t = \frac{1}{2} \int_{d^2+s_0^2}^{d^2+s^2} \sqrt{\frac{\zeta}{(\zeta_2 - \zeta)(\zeta - d^2)(\zeta - \zeta_1)}}\, d\zeta, \tag{27}$$

we can use first the integrals 256.13 page 122, and then 339.01 page 203, of [2] to evaluate (27). In fact, first we write (27) as

$$t = \frac{1}{2} \left\{ \int_{d^2}^{d^2+s^2} R(\zeta)\, d\zeta - \int_{d^2}^{d^2+s_0^2} R(\zeta)\, d\zeta \right\},$$

where

$$R(\zeta) = \sqrt{\frac{\zeta}{(\zeta_2 - \zeta)(\zeta - d^2)(\zeta - \zeta_1)}}.$$

In such a way the time is expressed by

$$t(s) = A(d^2 + s^2) - A(d^2 + s_0^2), \tag{28}$$

where

$$A(y) = \frac{1}{d\sqrt{(\zeta_2 - \zeta_1)}} \left[ \zeta_1 F(\varphi_1(y), k_1) + (d^2 - \zeta_1) \Pi(\varphi_1(y), \alpha_1^2, k_1) \right]$$

and

$$\varphi_1(y) = \arcsin \sqrt{\frac{(\zeta_2 - \zeta_1)(y - d^2)}{(\zeta_2 - d^2)(y - \zeta_1)}}$$

is the amplitude of the elliptic integrals of I and III kind $F(\varphi_1, k_1)$ and $\Pi(\varphi_1, \alpha_1^2, k_1)$ of modulus $k_1$ and parameter $\alpha_1^2$:

$$k_1^2 = \frac{(\zeta_2 - d^2)\zeta_1}{(\zeta_2 - \zeta_1)d^2}, \quad \alpha_1^2 = \frac{\zeta_2 - d^2}{\zeta_2 - \zeta_1}.$$

Of course the oscillation period $T$ will be given by

$$\frac{T}{2} = A(d^2 + s_+^2) - A(d^2 + s_-^2).$$

**Case 2**

In such a case the inequality (15) ensures that the discriminant of $q(\zeta)$ is positive, in fact from (15) we infer that

$$h^2 + d^2\lambda^2 > 2c\lambda$$

and by this we get that the first factor of (25) is positive:

$$h^2 - 2c\lambda + d^2\lambda^2 > 2c\lambda - 2c\lambda = 0.$$

As before, we have to single out the location of $d^2$ with respect to the roots $\zeta_1$ and $\zeta_2$ of the polynomial $q(\zeta)$ introduced in (24). Taking into account that the condition (14) holds, we find out $q(d^2) = h^2 d^2 - c^2 < 0$ and then $d^2 \notin [\zeta_1, \zeta_2]$. To establish if $d^2$ lies on the left or on the right of $[\zeta_1, \zeta_2]$, we evaluate the half-sum $\Sigma$ of $\zeta_1$ and $\zeta_2$ and, by (10), we find

$$\Sigma - d^2 = \frac{h^2 - d^2\lambda^2}{2\lambda^2} > 0.$$

Therefore the inequality holds:

$$d^2 < \zeta_1 < \zeta_2.$$

Henceforth the integral (27) is again evaluated by means of the formulae 256.13 page 122, and 339.01 page 203, of [2], but now the lower extreme of integration is $\zeta_1$:

$$t(s) = \frac{1}{2}\left\{\int_{\zeta_1}^{d^2+s^2} R(\zeta)\,d\zeta - \int_{\zeta_1}^{d^2+s_0^2} R(\zeta)\,d\zeta\right\}.$$

The time is then expressed by

$$t(s) = B(d^2+s^2) - B(d^2+s_0^2), \tag{29}$$

where

$$B(y) = \frac{1}{\sqrt{\zeta_1(\zeta_2 - d^2)}}[d^2 F(\varphi_2(y), k_2) + (\zeta_1 - d^2)\Pi(\varphi_2(y), \alpha_2^2, k_2)],$$

with

$$k_2^2 = \frac{(\zeta_2 - \zeta_1)d^2}{(\zeta_2 - d^2)\zeta_1}, \quad \alpha_2^2 = \frac{\zeta_2 - \zeta_1}{\zeta_2 - d^2}$$

and

$$\varphi_2(y) = \arcsin\sqrt{\frac{(\zeta_2 - d^2)(y - \zeta_1)}{(\zeta_2 - \zeta_1)(y - d^2)}}.$$

Of course the oscillation period $T$ will be

$$\frac{T}{2} = B(d^2+s_2^2) - B(d^2+s_1^2).$$

**Case 3**

In this occurrence (asymptotic motion), the integration of (22) does not require elliptic integrals any longer, but elementary functions only. First, notice that solving with respect to $h$ in (18), the condition (10) becomes:

$$c^2 - d^4\lambda^2 > 0. \tag{30}$$

Therefore by (18), (22) gives

$$t(s) = \frac{1}{\lambda}\int_{s_0}^{s} \frac{1}{u}\sqrt{\frac{d^2+u^2}{\Lambda^2 - u^2}}\,du, \tag{31}$$

where, for (30)

$$\Lambda^2 = \frac{c^2 - d^4\lambda^2}{d^2\lambda^2} > 0.$$

The integration of (31) is elementary: $t(s) = \frac{1}{\lambda}[C(s) - C(s_0)]$, where

$$C(s) = \arctan\sqrt{\frac{d^2+s^2}{\Lambda^2 - s^2}} - \frac{d}{\Lambda}\operatorname{arctanh}\sqrt{\frac{\Lambda^2(d^2+s^2)}{d^2(\Lambda^2 - s^2)}}. \tag{32}$$

**Case 4 and Case 5**

In these last two situations the analytical treatment is the same as in the case 1, because we find $q(d^2) > 0$. This means that $\zeta_1 < d^2 < \zeta_2$ allowing us to repeat the integration seen in the case 1.

## 4 Bifurcation Analysis: $\lambda = 0$

The case of the absence of the elastic force, is a *free* motion of a $m$-particle pulsed by some speed on a rotating straight line. Putting $\lambda = 0$ in (6), we obtain

$$\ddot{s} = c^2 \frac{s}{(d^2 + s^2)^2} \tag{33}$$

with $c = \frac{c_1}{m}, \quad d = \frac{b}{m}$. Neither (27), nor the conclusions expressed by formulae (28) and (29) involving the real roots $\zeta_1$ and $\zeta_2$ of $q(\zeta)$, (24), can be used for this occurrence. It is now necessary to go back to the Weierstraß method in order to write the $\lambda = 0$ time equation[2]

$$t = \pm \int_{s_0}^{s} \frac{du}{\sqrt{\Phi^*(u)}} = \pm \int_{s_0}^{s} \sqrt{\frac{d^2 + u^2}{((h^*)^2 d^2 - c^2) + (h^*)^2 u^2}}\, du,$$

where

$$h^2(c, d, \lambda; s_0, \dot{s}_0)\big|_{\lambda=0} = (h^*)^2 = \frac{c^2}{d^2 + s_0^2} + \dot{s}_0^2 > 0$$

and with the usual cautions about the sign's choice. We can see three different situations

(a) $(h^*)^2 d^2 > c^2$, which implies $\Phi^*(s) > 0$ for any $s$ (aperiodic motion for any allowable $s$);

(b) $(h^*)^2 d^2 < c^2$, which implies $\Phi^*(s) > 0$ for $s^2 > c^2 (h^*)^{-2} - d^2$ and $\Phi^*(s) = 0$ for $s^2 = c^2 (h^*)^{-2} - d^2$ (simple root), (aperiodic motion with forbidden region);

(c) $(h^*)^2 d^2 = c^2$, which implies $\Phi^*(s) > 0$ for any $s > 0$ and $\Phi^*(0) = 0$, double root (asymptotic motion towards the origin).

The reader should be aware that the physical sense is fully met by the analytical discussion just done: in fact the elastic force disappearance is the physical cause leaving any periodicity from the straight-linear motions.

## 5 Integration: $\lambda = 0$

After the former discussion, we perform the relevant integration.

[2] We mark by a star $(*)$ the quantities $\Phi$ and $h$ in the case $\lambda = 0$. On the contrary, the same symbols $s_0$ and $\dot{s}_0$ have been kept for meaning the initial conditions also in the $\lambda = 0$ motion. If a $\lambda \neq 0$ motion previously took place, the last computed values by (6), will provide the initial conditions input for (33).

**Case (a)**

Let us write the numerator of $\Phi^*$ as

$$(h^*)^2\left(d^2 - \frac{c^2}{(h^*)^2}\right) + (h^*)^2 s^2 = (h^*)^2(\Gamma^2 + s^2),$$

where hypothesis (a) ensures that $0 < \Gamma^2 = d^2 - \frac{c^2}{(h^*)^2} < d^2$. Discarding the sign (i.e. we can take $\dot{s}_0 > 0$ with no loss of generality) we find

$$t(s) = \frac{1}{h^*}\left\{\int_0^s \sqrt{\frac{d^2+u^2}{\Gamma^2+u^2}}\,du - \int_0^{s_0} \sqrt{\frac{d^2+u^2}{\Gamma^2+u^2}}\,du\right\}. \tag{34}$$

Both integrals at the right hand side of (34) are once more evaluated in [2]: first we use integral 221.03 page 61, and then integral 321.02 page 198, obtaining

$$t = \frac{1}{h^*}[A_0(s) - A_0(s_0)], \tag{35}$$

where

$$\begin{aligned} A_0(y) = d\,[&F(\varphi_3(y), k_3) - E(\varphi_3(y), k_3) \\ &+ \operatorname{dn}(F(\varphi_3(y), k_3), k_3)\operatorname{tn}(F(\varphi_3(y), k_3), k_3)] \end{aligned} \tag{36}$$

is a function depending on $y$ through the amplitude $\varphi_3(y)$

$$\varphi_3(y) = \arctan\frac{y}{\Gamma}, \quad k_3^2 = \frac{d^2-\Gamma^2}{d^2} = \frac{c^2}{(h^*)^2 d^2},$$

and where $u = F(\varphi_3, k_3)$ and $E(\varphi_3, k_3)$ are the Legendre elliptic integrals of I and II kind with modulus $k_3$ and amplitude $\varphi_3$; $\operatorname{dn} u$, $\operatorname{tn} u$ are two Jacobian elliptic functions of argument $u$ and modulus $k_3$.

**Case (b)**

Once again let the numerator of $\Phi^*$ be written as $(h^*)^2(s^2 - \Theta^2)$, where

$$\Theta^2 = \frac{c^2}{(h^*)^2} - d^2 > 0. \tag{37}$$

In such a way, the relevant time equation, defined for $0 < \Theta \le s_0 \le s$, taking the square root's positive determination and minding (37), becomes

$$t = \frac{1}{h^*}\left\{\int_\Theta^s \sqrt{\frac{u^2+d^2}{u^2-\Theta^2}}\,du - \int_\Theta^{s_0} \sqrt{\frac{u^2+d^2}{u^2-\Theta^2}}\,du\right\}. \tag{38}$$

To evaluate the integrals in (38), we refer for the last time in this paper, to [2], integrals 211.03 page 82 and 321.02 page 198. We find

$$t = \frac{1}{h^*}[B_0(s) - B_0(s_0)], \tag{39}$$

where

$$\begin{aligned} B_0(y) = \frac{c}{(h^*)^2}\,[&F(\varphi_4(y), k_4) - E(\varphi_4(y), k_4) \\ &+ \operatorname{dn}(F(\varphi_4(y), k_4), k_4)\operatorname{tn}(F(\varphi_4(y), k_4), k_4)]\,, \end{aligned} \tag{40}$$

is a function of $y$ through the amplitude $\varphi_4(y)$:

$$\varphi_4(y) = \arccos \frac{\sqrt{c^2 - (h^*)^2 d^2}}{h^* y}, \quad k_4^2 = \frac{d^2 (h^*)^2}{c^2}. \tag{41}$$

As usually, $u = F(\varphi_4, k_4)$ and $E(\varphi_4, k_4)$ denote the Legendre elliptic integrals of I and II kind with modulus $k_4$ and amplitude $\varphi_4$. Furthermore $\operatorname{dn} u$, $\operatorname{tn} u$ are two Jacobi elliptic functions of argument $u$ and modulus $k_4$. Finally, notice that in (41) and in (40) we used the identity

$$\sqrt{d^2 + \Theta^2} = \sqrt{\frac{c^2}{(h^*)^2}} = \frac{c}{h^*}.$$

**Case (c)**

In such asymptotic sub-case, the time equation, for positive initial speed and for the spatial coordinate $s$, is

$$t(s) = \frac{1}{h^*} \int_{s_0}^{s} \frac{\sqrt{u^2 + d^2}}{u}\, du. \tag{42}$$

The integral in (42) is elementary:

$$t(s) = \frac{1}{h^*} [C_0(s) - C_0(s_0)], \tag{43}$$

where

$$C_0(y) = \sqrt{y^2 + d^2} - d \ln \frac{2\left(d + \sqrt{y^2 + d^2}\right)}{d^2 y}.$$

## 6 Conclusions

We summarize five points, without degeneracy, $c \neq 0$, i.e. with $\dot{\theta}_0 \neq 0$.

### (i) $s$-motions under elastic force

The $s$-motions we examined in the presence of the elastic force ($\lambda \neq 0$) are five, as grasped by the table

| Conditions | $p(s)$ behavior | $s$-motion | case |
|---|---|---|---|
| (10) & (12) | 2 real roots | symmetric oscillation | 1 |
| (10) & (14) | 4 real roots | asymmetric oscillation | 2 |
| (10) & (18) | double root $s = 0$ | asymptotic behavior | 3 |
| (19) & (12) | 2 real roots | symmetric oscillation | 4 |
| (20) & (21) | 2 real roots | symmetric oscillation | 5 |

which is self-explanatory.

Almost all of the $s$-motions with $\lambda \neq 0$ are oscillatory, except the case 3, which is asymptotic, and whose time law is depending upon elementary functions.

In the cases 1, 2, 4 and 5, time is linked to the coordinate $s$ by means of the I and III kind elliptic integrals, whose upper bound is algebraically tied to $s$. Each oscillatory motion, according to its initial conditions, can have a double nature: either symmetric or not symmetric, namely centered or not around the origin O of the reference.

### (ii) $s$-motions with no forces

An alternative situation is that of the spring cut-off ($\lambda = 0$): further nonlinear (but in no way oscillatory) $s$-motions have been so found (whose nature is decided by $(h^*)^2 d^2 \gtreqless c^2$) and ruled by different elliptic functions.

### (iii) The angle $\theta$

The time equation concerning the angle $\theta$ is given by (4), a formula which needs to know $s$ as a function of the time, and then the 5+3 analytical solutions linking $t$ to $s$. Even if for each case we gave the relevant plots of $s$ versus the time, it should be clear that nobody can invert *formally* the relevant functions[3] $t = t(s)$; and then the $\theta$-integral (4) requiring $s = s(t)$ cannot be in any way evaluated in closed form.

However our explicit formulae for $t = t(s)$ allow an easy tabulation of $s = s(t)$, and therefore one might implement some numerical integration algorithm for getting $\theta$ as a (tabular) function of the time.

The $\theta$ time-behavior will be always growing: rotations cannot in fact extinguish ever, because neither friction nor drag are consuming the initial pulse.

### (iv) Trajectory

The planar *trajectory* of P, see Figure 1.1, might be obtained in a polar reference, 0 being the pole, assuming $\theta$ as anomaly, and the absolute value of $s$ as radius. For the purpose, one should try to eliminate the time between $s = s(t)$ and $\theta = \theta(t)$. No hope this could be accomplished in closed form.

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[3] The functions we mean are expressed by the formulae: (28), (29), (32), (35), (39), (43), and those strictly tied to them.