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Constrained Linear Quadratic Regulator: Continuous-Time Case

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Abstract: This paper deals with the linear quadratic regulator with constraints on the state and the input vectors. Such an optimization problem has a wide applications in industry like chemical and manufacturing industries. Our goal in this paper consists of developing an efficient numerical algorithm to solve such problem. Our technique relays on an iterative approach that uses the solution of the standard linear quadratic regulator as an initial guess for the optimal solution and then iteratively, the solution is improved by designing a controller that compensates for the violation of the constraints at each iteration. A numerical example is given to show the effectiveness of this algorithm.

Keywords: *Linear systems; linear quadratic regulator; constrained input; constrained state.*

Mathematics Subject Classification (2000): 49N10, 49N35.

1 Introduction

The linear quadratic regulator (LQR) is one of the most studied control problem in the literature. It will require many pages to cite all the works that were reported in the literature on the subject. In fact there are many variants. If we restrict ourselves to the case of LQR with constrained states and inputs, this variant consists of designing a state feedback controller that drives the state from a nonzero initial condition to zero by respecting simultaneously the constraints on the state and the control vectors.

This control problem has many applications in industry. In fact to motivate our study, let us consider a deterministic manufacturing system that produces n-items that

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can be stocked in a storage with finite size for each part and delivered to the market according to a given demand. Therefore, the inventory control problem can be stated as a constrained linear quadratic regulator problem.

This type of problem has been tackled by many authors among them we quote the works [5, 3, 4, 6, 7, 11, 9, 10, 2, 13]. In these references efficient algorithms have been developed to solve numerically the optimality conditions for the linear quadratic regulator with constraints on the states and/or the inputs. Both versions (continuous-time and discrete-time) have been tackled. Pytlak [8] presents many numerical methods for nonlinear optimal control problems with state constraints.

Our goal in this paper consists of solving the linear quadratic regulator with constrained states and inputs. To determine the control law, we develop a numerical method that uses the standard linear regular as an initial guess solution and iteratively, we improve the control law using the error at each iteration. Our idea in this paper, consists of considering the linear regulator problem as an initial guess. Based on this solution another optimization problem is formulated in which the state constraints are relaxed while the control constraints are maintained. Then, an iterative procedure is developed to solve the problem at hand while satisfying systems constraints.

The rest of this paper is organized as follows. In Section 2, the constrained linear quadratic regulator problem is stated and some results are recalled to facilitate the understanding of results. Section 3 contains the main of the paper and presents the steps of our algorithm. Section 4 provides a numerical example to show the effectiveness of the developed algorithm.

2 Problem Statement

Let us consider the class of continuous-time linear systems with the following dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ and $u(t) \in R^m$ represent respectively the state and the control of the system at time t, the matrices A and B are assumed to be known and constant, and x_0 is the initial condition.

The standard formulation of the linear quadratic regulator consists of minimizing the following cost function

$$J = \frac{1}{2}\int_0^T \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] dt, \tag{2}$$

where $Q \in R^{n\times n}$ and $R \in R^{m\times m}$ are two given matrices such that $Q \geq 0$ and $R > 0$ and $T > 0$ is a given finite time.

Under the assumption that the linear system is stabilizable and detectable it can be shown that the solution of this optimization problem is given by (see [1])

$$u^{\star}(t) = Kx(t), \tag{3}$$

where $K = -R^{-1}B^{\mathrm{T}}P(t)$ with $P(t)$ is the solution of the following Riccati equation

$$-\dot{P} = A^{\mathrm{T}}P(t) + P(t)A - PBR^{-1}B^{\mathrm{T}}P + Q. \tag{4}$$

As it was pointed out in the introduction, more often practical systems have constraints either on the state or the input or on both of them. Therefore, the previous formulation doesn't represent the real case and the constraints either on the state or the control or on both should be included in the previous formulation. The corresponding formulation is referred to as constrained linear quadratic regulator. For more details on this formulation either for the continuous-time or the discrete-time version, we refer the reader to [3, 4, 6, 7] and the references therein. This formulation is given by:

$$Pc\colon \begin{cases} \min J = \dfrac{1}{2}\displaystyle\int_0^T \left[x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\right] dt, \quad \text{subject to:} \\ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ \underline{x} \le x(t) \le \bar{x}, \quad \underline{u} \le u(t) \le \bar{u}, \end{cases} \tag{5}$$

where $\underline{x}$, $\bar{x}$, $\underline{u}$ and $\bar{u}$ are known vectors and the other parameters keep the same definitions as before.

This optimization problem does not have an analytical solution as it is the case for the previous one and the only way to solve it is to proceed numerically. In the literature, we can find some numerical methods that solve such problem. For more details on this subject, we refer the reader to [5, 3, 4, 6, 7, 8] and the references therein. Our goal in this paper is to solve this problem and to propose a numerical algorithm that solves efficiently the optimization problem Pc. The next section will provide such algorithm and in Section 4, a numerical example is provided to show the effectiveness of this algorithm.

3 Main Results

To solve the optimization problem Pc some attempts have been proposed in the literature for more details on this topics we refer the reader to [3, 4, 6, 7] and the references therein. Here we will propose a new way that solves the problem Pc iteratively starting from an initial solution that we can get from the unconstrained optimization problem. Then, subsequently by correcting the error between desired trajectory and the one at iteration k, we can design a controller that compensates for this error which will be added to the one at iteration k. At the end of the algorithm we end up with the desired control and the trajectory that satisfy all the system constraints.

Let us denote by $\widehat{x}(t)$ and $\widehat{u}(t)$ the optimal trajectory and the optimal control for the unconstrained linear quadratic control problem. The link between the optimal control and the optimal trajectory is given by:

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + B\widehat{u}(t), \quad \widehat{x}(0) = x_0, \tag{6}$$

where $\widehat{u}(t) = -R^{-1}B^{\mathrm{T}}P(t)\widehat{x}(t)$ with $P(t)$ is the solution of the Riccati equation (4).

It is obvious that this solution is not the optimal one and some corrections are needed to be done to make it closer to the optimal solution. For this purpose by denoting by $x^\star(t)$ and $u^\star(t)$ respectively the optimal trajectory and the optimal control of the constrained linear quadratic regulator, we have:

$$\begin{aligned} \dot{x}^\star(t) &= Ax^\star(t) + Bu^\star(t), \quad x^\star(0) = x_0, \\ u^\star(t) &= \widehat{u}(t) + \Delta u^\star(t), \end{aligned}$$

with $\Delta u^\star(t)$ is a control law that we have to determine that will correct the trajectory of the system and then reduces the error.

Notice that $x(t)$ and $u(t)$ are linked to the optimal solution of the standard linear quadratic regular by the following expressions:

$$x(t) = \widehat{x}(t) + e(t), \quad u(t) = \widehat{u}(t) + \Delta u(t).$$

Using now these expressions, the cost function and the previous constraints become respectively:

$$\begin{aligned}\min J = \frac{1}{2}\int_0^T &\left[\widehat{x}^{\mathrm{T}}(t)Q\widehat{x}(t) + \widehat{u}^{\mathrm{T}}(t)R\widehat{u}(t)\right] dt \\ &+ \int_0^T \left[\widehat{x}^{\mathrm{T}}(t)Qe(t) + \frac{1}{2}\, e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t)\right] dt\end{aligned}$$

subject to:

$$\dot{\widehat{x}}(t) + \dot{e}(t) = A\widehat{x}(t) + Ae(t) + B\widehat{u}(t) + B\Delta u(t)$$

and

$$\underline{x} \le \widehat{x}(t) + e(t) \le \bar{x}, \quad \underline{u} \le \widehat{u}(t) + \Delta u(t) \le \bar{u}.$$

Assume that we are now at the first iteration, i.e.: $k = 1$, in which $x^k(t) = \hat{x}(t)$ and $u^k(t) = \hat{u}(t)$ are known. From the constraints on the states and the knowledge of $x^k(t)$, we can determine precisely the maximum and minimum values as well as the corresponding time instant as which $x^k(t)$ trajectories violate these constraints. Let us now denote by t^k_{ij}, $j = 1, \ldots, p_p$ (where p_i is a finite positive integer) the corresponding instants at which the maximum or the minimum violations occur and by $e_i^{\star k}(t^k_{ij})$ the value of the i-th component of the maximum and the minimum error at time t^k_{ij} that we should compensate. This imposes the following constraints which have to be satisfied in our optimization problem:

$$e_i^k(t^k_{ij}) = e_i^{\star k}(t^k_{ij}), \quad j = 1, \ldots, p_i, \quad i = 1, \ldots, n.$$

Therefore, our original problem can be transformed to the following one that has only inequality constraints on the input (for simplicity, the iteration number k will be dropped while deriving the necessary conditions for optimality)

$$\min \Delta J = \int_0^T \left[\widehat{x}^{\mathrm{T}}(t)Qe(t) + \frac{1}{2}e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t)\right] dt$$

subject to

$$\begin{aligned}\dot{e}(t) &= Ae(t) + B\Delta u(t), \quad e(0) = 0, \\ e_i(t_{ij}) &= e_i^{\star}(t_{ij}), \quad j = 1, \ldots, p_i, \quad i = 1, \ldots, n, \\ \underline{\Delta u(t)} &\le \Delta u(t) \le \overline{\Delta u(t)},\end{aligned}$$

with $\underline{\Delta u(t)} = \underline{u} - \widehat{u}(t)$, $\overline{\Delta u(t)} = \bar{u} - \widehat{u}(t)$.

To solve this problem, let us write the corresponding Hamiltonian:

$$\begin{aligned}H(e, \Delta u, t) = \widehat{x}^{\mathrm{T}}(t)Qe(t) &+ \frac{1}{2}\, e^{\mathrm{T}}(t)Qe(t) + \widehat{u}^{\mathrm{T}}(t)R\Delta u(t) + \frac{1}{2}\Delta u^{\mathrm{T}}(t)R\Delta u(t) \\ &+ \lambda^{\mathrm{T}}\left[Ae + B\Delta u\right] + \sum_{i=1}^{n}\sum_{j=1}^{p} \pi_{ij}\left[e_i(t) - e_i^{\star}(t)\right]\delta(t - t_{ij}),\end{aligned}$$

where λ is the costate vector, π_{ij} is the Lagrange multiplier and $\delta(t)$ is the Dirac delta function defined as follows:

$$\delta(t - t_{ij}) = \begin{cases} 1, & \text{if } t = t_{ij}, \\ 0, & \text{elsewhere.} \end{cases}$$

Based on optimization theory, the necessary conditions of optimality give

$$\frac{\partial H}{\partial \Delta u} = 0$$

which implies

$$R\widehat{u} + R\Delta u + B^{\mathrm{T}}\lambda = 0$$

that gives in turn

$$\Delta u = -\widehat{u} - R^{-1}B^{\mathrm{T}}\lambda.$$

The feasible control law, $\Delta u + \widehat{u}$ that minimizes the Hamiltonian while satisfying the previous constraints on control is given by [12]:

$$\Delta u + \widehat{u} = \begin{cases} \underline{u}, & \text{if } -R^{-1}B^{\mathrm{T}}\lambda < \underline{u}\ , \\ -R^{-1}B^{\mathrm{T}}\lambda, & \text{if } \ \underline{u} \le \Delta u(t) \le \overline{u}, \\ \overline{u}, & \text{if } -R^{-1}B^{\mathrm{T}}\lambda > \overline{u}\ . \end{cases} \tag{7}$$

The second necessary optimality condition for our problem is

$$\frac{\partial H}{\partial \lambda} = \dot{e} = Ae + B\Delta u \tag{8}$$

with $e(0) = 0$.

The third necessary optimality condition for our problem is

$$\frac{\partial H}{\partial e} = -\dot{\lambda},$$

which implies that

$$\dot{\lambda} = -Q\widehat{x} - Qe - A^{\mathrm{T}}\lambda - \pi(t) \tag{9}$$

with $\lambda(T) = 0, \quad \pi(t) = [\pi_1(t)\delta(t - t_{1j}), \dots, \pi_p(t)\delta(t - t_{p_{nj}})]$.

The last necessary optimality condition gives:

$$\frac{\partial H}{\partial \pi_{ij}} = e_i(t_{ij}) - e_i^{\star}(t_{ij}), \quad i = 1, \dots, n, \quad j = 1, \dots, p_i.$$

The error $e_i(t_{ij}) - e_i^{\star}(t_{ij})$ at iteration m is used to update π_{ij} employing the following expression:

$$\pi_{ij}^{m+1} = \pi_{ij}^{m} + \alpha \left[e_i^{m}(t_{ij}) - e_i^{\star}(t_{ij}) \right], \tag{10}$$

where α can be chosen following the well know optimization techniques.

To solve our optimization we need to determine $\lambda(t)$ that comes from (9) that its self depends on $\pi(t)$ that we should estimate, and $e(t)$ that be can obtained from (8) that in turn depends on $\Delta u(t)$ that we should determine once we know $\lambda(t)$. All the equations are coupled and one way of obtaining a solution to this problem is numerically solve the problem. Once the optimization problem is solved, we update the trajectories, $x^k(t)$, $u^k(t)$ by $e^k(t)$ and $\Delta u^k(t)$ to get the new trajectories $x^{k+1}(t) = x^k(t) + e^k(t)$ and $u^{k+1}(t) = u^k(t) + \Delta u^k(t)$ and then repeat the whole process till the $\sup_{ij} e_i^{\star}(t_{ij}^k)$ is less than a specified given value. The steps of our algorithm are summarized by:

Algorithm 3.1

1. Initialization: Choose $\bar{\epsilon}_x > 0,\ \bar{\epsilon}_\lambda > 0,\ \bar{\epsilon}_\pi > 0,$ and let $k = 1,\ l = 1,$ and $m = 1$ (the numbers of iterations for $x(t)$, $\lambda(t)$ and $\pi(t)$), and solve the standard LQR to get $\widehat{x} = x^k$ and $\widehat{u} = u^k$.

2. Identify the values of $e_i^\star(t_{ij}^k)$ and the corresponding instants t_{ij}^k at the iteration k for each trajectory.

3. Guess $\pi_{ij}^{km}(t)$.

4. Guess $\lambda^{klm}(t)$.

5. Compute Δu^{klm} using (7), and solve (8) to determine $e^{klm}(t)$.

6. Solve (9) to get the trajectory $\lambda^{k(l+1)m}$ at the iteration k and m.

7. Compute the error on λ as follows

$$\varepsilon_\lambda = \sqrt{\int_0^T \|\lambda^{klm}(t) - \lambda^{k(l+1)m}(t)\|^2\, dt}.$$

   Test: If $\varepsilon_\lambda > \bar{\epsilon}_\lambda$, use the computed $\lambda(t)$ at this iteration as a guess for $\lambda(t)$, put $l = l + 1$ and go to Step 5, otherwise continue.

8. Update $\pi_{ij}^{kl}(t)$ using for example

$$\pi_{ij}^{kl(m+1)} = \pi_{ij}^{klm} + \alpha\left(e_i^{klm}(t_{ij}^k) - e_i^\star(t_{ij}^k)\right),$$

   where α can be chosen following one of the well known optimization techniques; and compute the error as:

$$\varepsilon_\pi = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{p_i} \|e_i^{klm}(t_{ij}^k) - e_i^\star(t_{ij}^k)\|^2}.$$

   Test: If $\varepsilon_\pi > \bar{\epsilon}_\pi$, put $l = 1$ and $m = m + 1$ and solve (9) to get new trajectory for λ and go to Step 5, otherwise continue.

9. Calculate the new trajectory $x(t)$ and $u(t)$ at the iteration $k+1$ using the following:

$$x^{k+1} = x^k + e^k, \quad u^{k+1} = u^k + \Delta u^k.$$

10. Identify the values of $e_i^\star(t_{ij})$ and the corresponding instants t_{ij} at the iteration $k + 1$ for each trajectory and compute the error using the following

$$\varepsilon_x = \sup_{ij}\{e_i^\star(t_{ij})\} \ \text{ for the } e_i^\star(t_{ij}) \text{ computed at this step.}$$

11. Test: If $\varepsilon_x > \bar{\epsilon}_x$, increase k by 1, put $l = 1$, $m = 1$ and go to Step 3, else record the trajectories and the controls and stop.

In the next section a numerical example with lower bounds on the states and the control is provided to show the validness of our approach. Our algorithm has been programmed using Fortran language on Pentium PC. The computation time is very acceptable and for the one we are presenting is less than one second.

4 Numerical Example

To show the effectiveness of our algorithm, let us consider a linear system with the following data:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2.36 & -13.6 & -12.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.79 & 2.68 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \end{bmatrix}.$$

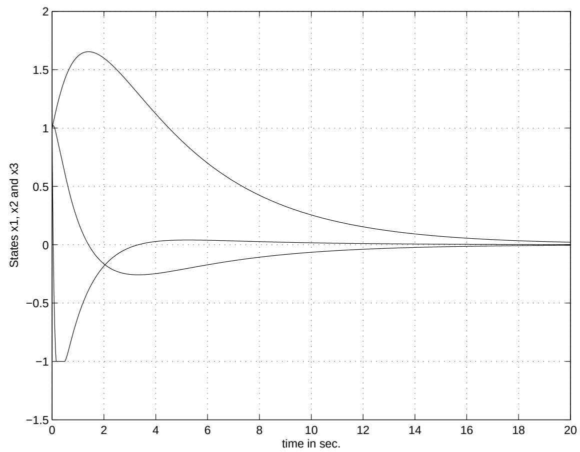

Figure 4.1: Behavior of the state variables x_1, x_2 and x_3.

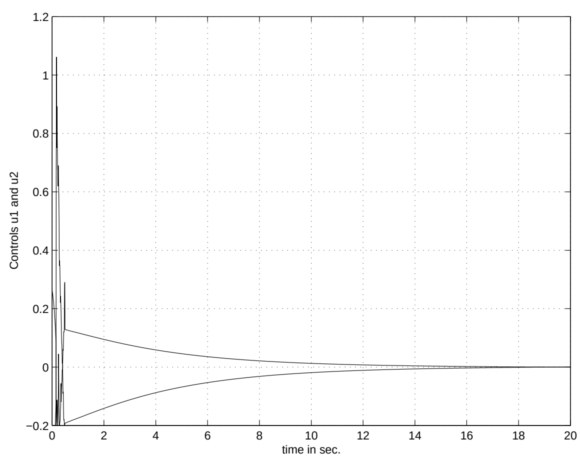

Figure 4.2: Behavior of the control variables u_1 and u_2.

As it can be seen in Figures 1 and 2, the obtained suboptimal states and control trajectories satisfy all the required system constraints.

5 Conclusion

This paper dealt with the constrained linear quadratic regulator for the class of linear continuous-time. The constraints are on both the control and the state vectors. A procedure is developed in which the original problem is converted to another one which has only constraints on control. By solving this new problem iteratively, it is possible to get the solution of the original one. The illustrative example shows the applicability of the developed technique.

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