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Generalized Monotone Iterative Technique for Functional Differential Equations with Retardation and Anticipation

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Abstract: The method of monotone iterative technique together with coupled lower and upper solutions is employed to prove the existence of coupled extremal solutions when the forcing function is the sum of an increasing and decreasing functions. This is referred to as generalized monotone method. This will include the usual monotone method results as special cases. Further using uniqueness condition uniqueness results for functional differential equations involving retardation and anticipation are also established.

Keywords: *Generalized monotone method, equations with retardation and anticipation.*

Mathematics Subject Classification (2000): 34C12, 34K05, 34K99.

1 Introduction

Qualitative and quantitative study of the functional differential equations with retardation and anticipation has very useful applications. Such dynamic systems occur in chaotic epidemic model and financial models, specifically stock exchange models. A typical model that arises is of the form

$$\begin{aligned} x'(t) &= F(t, x(t), y(t+\tau)) - ax(t), \\ y'(t) &= G(x(t-\tau) - by(t). \end{aligned} \tag{1}$$

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See [3, 4, 5] for more details. Some numerical,computational and simulation methods are suggested for such equations in [3, 4, 5]. One formulation of such models [6] can be considered as

$$\begin{aligned} &x'(t) = f(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ &x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T. \end{aligned} \tag{2}$$

See [6] for other possible formulation. In [6] the authors developed existence theory for the general functional differential equations which involved with both retardation and anticipation, indicating other possible formulations. They achieved this by suitably applying results of [1]. Recently, in [7] the authors developed usual monotone iterative method by assuming the forcing function to be nondecreasing in the unknown function and its retardation term and non-increasing nature in the anticipation term. In this paper we develop the generalized monotone method as in [9, 11] for the functional differential equation with retardation and anticipation when the forcing function is the sum of a nondecreasing and non-increasing function in all its components. This yields the results of the usual monotone method [8, 10] as special cases. Using the method of coupled upper and lower solutions we develop sequences which converge to coupled minimal and maximal solutions. Further, using uniqueness condition we can prove that the nonlinear functional differential equations with retardation and anticipation problem has a unique solution. For more details on the monotone method and delay differential equations equations see [2] and the references therein.

2 Main Results

The usual monotone method developed in literature proves the existence of extremal solutions of

$$\begin{aligned} &x'(t) = f(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ &x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T, \end{aligned} \tag{3}$$

when $f(t, x, \phi, \psi)$ is either nondecreasing in x, ϕ, ψ or could be made nondecreasing by adding appropriate linear terms. This is precisely the onesided Lipschitz condition in x, ϕ, ψ. In this paper we develop monotone method for the following functional differential equation with retardation and anticipation, given by

$$\begin{aligned} x'(t) = f(t, x(t), x_t, x^t) + g(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T, \end{aligned} \tag{4}$$

where $\mathcal{C}_1 = C([-h_1, 0], R)$, $\mathcal{C}_2 = C([0, h_2], R)$, $\phi_0 \in \mathcal{C}_1$, $\psi_0 \in \mathcal{C}_2$ and $f, g \in C(I \times R \times \mathcal{C}_1 \times \mathcal{C}_2, R)$, $h_1, h_2 > 0$. Here and in what follows, the symbols $x_t = x_t(s) = x(t+s)$, $-h_1 \leq s \leq 0$, $x^t = x^t(\sigma) = x(t+\sigma)$, $0 \leq \sigma \leq h_2$, representing retardation and anticipation, respectively. We plan to employ the generalized monotone iterative technique for proving the existence of unique solution for (4) utilizing coupled lower and upper solutions for (4) if of two different types. Through this paper we assume that f is nondecreasing in all its components or could be made nondecreasing by adding appropriate linear functions whereas g is non-increasing in all its components. Before we proceed further, we need to list the following known results relative to linear functional differential inequalities in a suitable form [10].

Lemma 2.1 *Assume that*

(i) $p \in C([t_0 - h_1, T + h_2], R)$, *p is continuously differentiable on* $I = [t_0, T]$ *and*

$$p'(t) \le -Mp(t) - N \int_{-h_1}^{0} p_t(s)\, ds, \quad t \in I;$$

(ii) $p_{t_0}(s) \le 0$, $-h_1 \le s \le 0$, $p \in C^1([t_0 - h_1, t_0], R)$, $p'(s) \le \dfrac{\lambda}{T + h_1}$ *where* $\min_{[t_0 - h_1, t_0]} p(s) = -\lambda$, $\lambda \ge 0$ *and* $[M + Nh_1]\,(T + h_1) \le 1$.

Then $p(t) \le 0$ *on* $t_0 \le t \le T$.

This lemma is the suitable part of Lemma 2.1 in [10].

Lemma 2.2 *Suppose that* $p \in C([t_0 - h_1, T + h_2], R)$, $p'(s)$ *exists and is continuous on* I *and*

$$p'(t) \le -Lp(t) + N_1 \int_{-h_1}^{0} p_t(s)\, ds + N_2 \int_{0}^{h_2} p^t(\sigma)\, d\sigma, \quad t \in I,$$

where $L, N_1, N_2 > 0$ *satisfying* $N_1 h_1 + N_2 h_2 < L$. *Then* $p_{t_0} \le 0$, $p^T \le 0$ *implies* $p(t) \le 0$ *on* I.

Proof If the conclusion is false, there exists a $t_1 \in (t_0, T)$ and an $\epsilon > 0$ such that $p(t_1) = \epsilon$, $p(t) \le \epsilon$ on I. It then follows that

$$0 = p'(t_0) \le -L\epsilon + N_1 \epsilon h_1 + N_2 \epsilon h_2 < 0,$$

by assumptions proving $p(t) \le 0$ on I. □

Let us list the following assumptions relative to (4) for convenience.

We call α_0, β_0 as type I or of type II coupled lower and upper solutions of (4) respectively if (i) or (ii) below are satisfied.

(i) $\alpha_0, \beta_0 \in C^1(I, R)$ satisfies

$$\alpha_0'(t) \le f(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t) + g(t, \beta_0(t), \beta_{0t}, \beta_0^t), \quad \alpha_{0t_0} = \phi_1, \quad \alpha_0^T = \psi_1,$$
$$\beta_0'(t) \ge f(t, \beta_0(t), \beta_{0t}, \beta_0^t) + g(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t), \quad \beta_{0t_0} = \phi_2, \quad \beta_0^T = \psi_2,$$

such that $\phi_1 \le \phi_0 \le \phi_2$, $\psi_1 \le \psi_0 \le \psi_2$, $\alpha_0(t) \le \beta_0(t)$ on I and $\phi_1, \phi_2 \in \mathcal{C}_1$, $\psi_1, \psi_2 \in \mathcal{C}_2$.

(ii) $\alpha_0, \beta_0 \in C^1(I, R)$ satisfies

$$\alpha_0'(t) \le f(t, \beta_0(t), \beta_{0t}, \beta_0^t) + g(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t), \quad \alpha_{0t_0} = \phi_1, \quad \alpha_0^T = \psi_1,$$
$$\beta_0'(t) \ge f(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t) + g(t, \beta_0(t), \beta_{0t}, \beta_0^t), \quad \beta_{0t_0} = \phi_2, \quad \beta_0^T = \psi_2,$$

such that $\phi_1 \le \phi_0 \le \phi_2$, $\psi_1 \le \psi_0 \le \psi_2$, $\alpha_0(t) \le \beta_0(t)$ on I and $\phi_1, \phi_2 \in \mathcal{C}_1$, $\psi_1, \psi_2 \in \mathcal{C}_2$.

(iii)

$$f(t,x,\phi,\psi) = F(t,x,\phi,\psi) - M_1 x - N_1 \int_{-h_1}^{0} \phi(s)\, ds,$$

where $f(t,x,\phi,\psi)$ is nondecreasing in (x,ϕ,ψ) for each t,

$$g(t,x,\phi,\psi) = G(t,x,\phi,\psi) - M_2 x - N_2 \int_{-h_1}^{0} \phi(s)\, ds,$$

where $G(t,x,\phi,\psi)$ is non-increasing in (x,ϕ,ψ) for each t, whenever $\alpha_0(t) \le x \le \beta_0(t)$, $\alpha_{0t} \le \phi \le \beta_{0t}$, $\xi \in \mathcal{C}_2$ such that $M_1, N_1, M_2, N_2 \ge 0$. Also $M_1 + M_2 > 0$ and $N_1 + N_2 > 0$.

(iv) $\alpha_{0t_0} - \phi_0$, $\phi_0 - \beta_{0t_0}$ satisfying the assumptions (ii) of Lemma 2.1.

The type I and II of coupled lower and upper solutions assumed in (i)and (ii) in the assumption are utilized in [8, 9, 11] fruitfully. We are now in a position to state and prove our main result relative to coupled lower and upper solutions of type I.

Theorem 2.1 *Suppose that assumptions (i) to (iv) except (ii) are satisfied. Then there exist monotone sequences* $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ *such that* $\alpha_n(t) \to \rho(t)$, $\beta_n(t) \to r(t)$ *uniformly as* $n \to \infty$ *on* $[t_0 - h_1,\ T + h_2]$ *and that* (ρ, r) *are coupled minimal and maximal solutions of (4). That is* $\rho(t)$, $r(t)$ *satisfies*

$$\rho' = f(t,\rho,\rho_t,\rho^t) + G(t,r,r_t,r^t) - M_2\rho(t) - N_2 \int_{-h_1}^{0} \rho_t(s)\, ds, \tag{5}$$

$$r' = f(t,r,r_t,r^t) + G(t,\rho,\rho_t,\rho^t) - M_2 r(t) - N_2 \int_{-h_1}^{0} r_t(s)\, ds. \tag{6}$$

If, in addition,

(v) $f(t,x,\phi_1,\psi_1) - f(t,y,\phi_2,\psi_2)$

$$\le -L_1(x-y) + N_{11} \int_{-h_1}^{0} (\phi_1 - \phi_2)(s)\, ds + N_{12} \int_{0}^{h_2} (\psi_1 - \psi_2)(\sigma)\, d\sigma$$

and

$G(t,y,\phi_2,\psi_2) - G(t,x,\phi_1,\psi_1)$

$$\le -L_2(x-y) + N_{21} \int_{-h_1}^{0} (\phi_1 - \phi_2)(s) ds + N_{22} \int_{0}^{h_2} (\psi_1 - \psi_2)(\sigma)\, d\sigma$$

where $L_1, L_2, N_{11}, N_{12}, N_{21}, N_{22} \ge 0$, *such that* $(L_1 + L_2) > 0$, $(N_{11} + N_{21}) > 0$ $(N_{12} + N_{22}) > 0$ *for* $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0t} \le \phi_2 \le \phi_1 \le \beta_{0t}$, $\alpha_0^T \le \psi_1 \le \psi_2 \le \beta_0^T$ *and* $(N_{11} + N_{21} - N_2)h_1 + (N_{21} + N_{22})h_2 < L_1 + L_2 + M_2$, *holds, then* $\rho(t) = r(t) = x(t)$ *is the unique solution of (4) on* I.

Proof Consider the following linear problem for each $n = 1, 2, 3, \ldots$

$$\begin{aligned} \alpha_{n+1}' = {} & F(t,\alpha_n,\alpha_{nt},\alpha_n^t) - M_1\alpha_{n+1}(t) - N_1 \int_{-h_1}^{0} (\alpha_{(n+1),t}(s)\, ds \\ & + G(t,\beta_n,\beta_{nt},\beta_n^t) - M_2\alpha_{n+1}(t) - N_2 \int_{-h_1}^{0} \alpha_{(n+1),t}(s)\, ds \end{aligned} \tag{7}$$

$$\begin{aligned}\beta'_{n+1} = {} & F(t,\beta_n,\beta_{nt},\beta_n^t) - M_1\beta_{n+1}(t) - N_1\int_{-h_1}^{0}\beta_{(n+1),t}(s)\,ds \\ & + G(t,\alpha_n,\alpha_{nt},\alpha_n^t) - M_2\beta_{n+1}(t) - N_2\int_{-h_1}^{0}\beta_{(n+1),t}(s)\,ds\end{aligned} \tag{8}$$

with $\alpha_{(n+1)t_0} = \phi_0,\ \beta_{(n+1)t_0} = \phi_0$ and $\alpha_{n+1}^T,\ \beta_{n+1}^T$ are chosen such that

$$\alpha_0^T \le \alpha_n^T \le \alpha_{n+1}^T \le \psi_0 \le \beta_{n+1}^T \le \beta_n^T \le \beta_0^T \tag{9}$$

and α_n^T, β_n^T converge uniformly to ψ_0 on $[0,h_2]$ (see Remark 2.1).

Clearly each linear problem has a unique solution on $[t_0-h_1, T+h_2]$. We wish to show that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_2 \le \beta_1 \le \beta_0 \quad \text{on} \quad I. \tag{10}$$

We claim first that $\alpha_0 \le \alpha_1$ on I. Since $\alpha_0(t) \le \beta_0(t)$ on I and using (iv) we get

$$\begin{aligned}\alpha'_0(t) &\le f(t,\alpha_0(t),\alpha_{0t},\alpha_0^t) + g(t,\beta_0(t),\beta_{0t},\beta_0^t) \\ &\le F(t,\alpha_0(t),\alpha_{0t},\alpha_0^t) - M_1\alpha_0(t) - N_1\int_{-h_1}^{0}\alpha_{0t}(s)\,ds \\ &\quad + G(t,\beta_0(t),\beta_{0t},\beta_0^t) - M_2\alpha_0(t) - N_2\int_{-h_1}^{0}\alpha_{0t}(s)\,ds.\end{aligned}$$

Now set $p = \alpha_0 - \alpha_1$ so that it follows from (7), (9) and condition (i),

$$\begin{aligned}p' = \alpha'_0 - \alpha'_1 &\le F(t,\alpha_0,\alpha_{0t},\alpha_0^t) - F(t,\alpha_0,\alpha_{0t},\alpha_0^t) + G(t,\beta_0,\beta_{0t},\beta_0^t) - G(t,\beta_0,\beta_{0t},\beta_0^t) \\ &\quad + (M_1+M_2)(\alpha_1-\alpha_0) + (N_1+N_2)\int_{-h_1}^{0}(\alpha_{1t}-\alpha_{0t})(s)\,ds \\ &\le -(M_1+M_2)p - (N_1+N_2)\int_{-h_1}^{0}p_t(s)\,ds, \quad t\in I\end{aligned}$$

and

$$p_{t_0} = \alpha_{0t_0} - \alpha_{1t_0} \le 0.$$

By Lemma 2.1, in view of assumption (iii), this implies $\alpha_0 \le \alpha_1$ on I. Similarly, we can show that $\beta_1 \le \beta_0$ on I.

Next we prove that $\alpha_1 \le \beta_1$ on I. Setting $p = \alpha_1 - \beta_1$, we obtain in view of (7), (8), for $t \in I$,

$$\begin{aligned}p' = \alpha'_1 - \beta'_1 &= F(t,\alpha_0,\alpha_{0t},\alpha_0^t) - F(t,\beta_0,\beta_{0t},\beta_0^t) + G(t,\beta_0,\beta_{0t},\beta_0^t) - G(t,\alpha_0,\alpha_{0t},\alpha_0^t) \\ &\quad - (M_1+M_2)(\alpha_1-\beta_1) - (N_1+N_2)\int_{-h_1}^{0}(\alpha_{1t}-\beta_{1t})(s)\,ds.\end{aligned}$$

Since $F(t,x,\phi,\psi)$ and $G(t,x,\phi,\psi)$ is nondecreasing and non-increasing in (x,ϕ,ψ) respectively for each t, we get

$$p' = \alpha'_1 - \beta'_1 \le -(M_1+M_2)(p(t)) - (N_1+N_2)\int_{-h_1}^{0}p_t(s)\,ds$$

and from Lemma 2.1 it follows that $p(t) \leq 0$ which proves that $\alpha_1(t) \leq \beta_1(t)$ on I. As a result, it follows that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 \quad \text{on} \quad I. \tag{11}$$

Now suppose that for some $k > 1$, we have

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \quad \text{on} \quad I. \tag{12}$$

We shall show that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad \text{on} \quad I. \tag{13}$$

To do this, let $p = \alpha_k - \alpha_{k+1}$ so that $p_{t_0} = 0$ and

$$\begin{aligned} p' = \alpha_k' - \alpha_{k+1}' &= F(t, \alpha_{(k-1)}, \alpha_{(k-1)t}, \alpha_{(k-1)}^t) - M_1(\alpha_k) - N_1 \int_{-h_1}^0 \alpha_{kt}(s)\, ds \\ &+ G(t, \beta_(k-1), \beta_{(k-1)t}, \beta_(^t k-1)) - M_2\alpha_k(t) - N_2 \int_{-h_1}^0 \alpha_{(k),t}(s)\, ds \\ &- (F(t, \alpha_k, \alpha_{kt}, \alpha_k^t) - M_1\alpha_{k+1}(t) - N_1 \int_{-h_1}^0 \alpha_{(k+1),t}(s) ds) \\ &- (G(t, \beta_n, \beta_{nt}, \beta_k^t) - M_2\alpha_{k+1}(t) - N_2 \int_{-h_1}^0 \alpha_{(k+1),t}(s) ds). \end{aligned}$$

Substituting for F in terms of f and using the monotone nature of f, G and simplifying we get

$$p' \leq (M_1 + M_2)(\alpha_k - \alpha_{(k+1)} + (N_1 + N_2) \int_{-h}^0 (\alpha_{kt} - \alpha_{k+1t})(s)\, ds \quad \text{and} \quad p_{t_0} = 0.$$

This implies by Lemma 2.1 that $\alpha_k \leq \alpha_{k+1}$ on I. Similarly, we can show that $\beta_{k+1} \leq \beta_k$ on I. To prove $\alpha_{k+1} \leq \beta_{k+1}$ on I, consider $p = \alpha_{k+1} - \beta_{k+1}$ so that $p_{t_0} = 0$ and arguing as before, one can show that

$$p' \leq -(M_1 + M_2)p - (N_1 + N_2) \int_{-h_1}^0 p_t(s)\, ds,$$

and $p_{t_0} = 0$, which yields $\alpha_{k+1} \leq \beta_{k+1}$, on I. Thus we have (13) and therefore by induction, we see that (10)is valid on I. This together with (9) follows that (10) is also true on $[t_0, T + h_2]$.

Since the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are bounded by (10), employing the standard arguments [8, 9] namely Ascoli-Arzela and Dini theorems, one can conclude that $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly on $[t_0, T]$, that is, $\alpha_n \to \rho$, $\beta_n \to r$ uniformly on $[t_0, T]$.

Also, it is easy to show that (ρ, r) satisfy (5) and (5) respectively with $\rho \leq r$ on I and $\rho^T = r^T$. To show that (ρ, r) are coupled minimal and maximal solutions of (4), let $x(t)$ be any solution of (4) with $x_{t_0} = \phi_0$, $x^T = \psi_0$ such that $\alpha_0 \leq x \leq \beta_0$ on I. Then it is enough to show that $\rho \leq x \leq r$ since by definition of (ρ, r) we already have $\rho^T = x^T = r^T$. Setting $p = \alpha_1 - x$ so that $p_{t_0} = 0$ and

$$\begin{aligned} p' = \alpha_1' - x' &= F(t, \alpha_0, \alpha_{0t}, \alpha_0^t) - M_1\alpha_1 - N_1 \int_{-h_1}^0 \alpha_{1t}(s)\, ds - f(t, x, x_t, x^t) \\ &+ G(t, \beta_0, \beta_{0t}, \beta_0^t) - M_2\alpha_1 - N_2 \int_{-h_1}^0 \alpha_{1t}(s)\, ds - g(t, x, x_t, x^t). \end{aligned}$$

Since $\alpha_0 \le x(t) \le \beta_0$, substituting for (F, g) in terms of (f, G) from (iii) and using the nondecreasing and non-increasing nature of f, G the above equation simplifies to the

$$p' \le -((M_1 + M_2)(\alpha_1 - x) - (N_1 + N_2)\int_{-h_1}^{0} (\alpha_{1t} - x_t)(s)\, ds.$$

Thus we get from Lemma 2.1, $\alpha_1 \le x$ on I. Similarly, $x \le \beta_1$ on I. By proceeding similarly and by induction, it is easy to show that $\alpha_{n+1} \le x \le \beta_{n+1}$ on I for all n. Hence (ρ, r) are coupled minimal and maximal solutions of (4).

If, in addition, condition (v) holds, since $\rho \le r$, we let $p = r - \rho$ and find using $\rho^t \le r^t$ and (v),

$$p' = r' - \rho' = f(t, r, r_t, r^t) - f(t, \rho, \rho_t, \rho^t) + G(t, \rho, \rho_t, \rho^t) - M_2 r - N_2 \int_{h_1}^{0} r_t(s)\, ds$$
$$- \left\{ G(t, r, r_t, r^t) - M_2 \rho - N_2 \int_{h_1}^{0} \rho_t(s) ds \right\}.$$

Using (v) this simplifies to

$$p' \le -L_1(r - \rho) + N_{11}\int_{-h_1}^{0} (r_t - \rho_t)(s) ds + N_{12}\int_{0}^{h_2} (r^t - \rho^t(\sigma) d\sigma$$
$$+ (-L_2 - M_2)(r - \rho) + (N_{21} - N_2)\int_{-h_1}^{0} (r_t - \rho_t)(s) ds + N_{22}\int_{0}^{h_2} (r^t - \rho^t(\sigma)\, d\sigma$$
$$\le -(L_1 + L_2 + M_2)(p) + (N_{11} + N_{21} - N_2)\int_{-h_1}^{0} (p_t)(s) ds + (N_{12} + N_{22})\int_{0}^{h_2} p^t(\sigma)\, d\sigma$$

and

$$p_{t_0} = 0, \quad p^T = 0.$$

This implies by Lemma 2.2, $p(t) \le 0$ on I, which means $\rho = r$ on I. This proves that $x = \rho = r$ is the unique solution of (4) with $x_{t_0} = \phi_0$, and $x^T = \psi_0$. The proof is therefore complete. □

Here we recall the remark of [7] for completion of our Theorem 2.1.

Remark 2.1 A simple choice of (9) would be to take for α_n^T, β_n^T, suitable translates of ψ_0 such that $\alpha_n^T = \psi_0 - \epsilon_n$, $\beta_n^T = \psi_0 + \eta_n$, with $\alpha_n(T) = \psi_0(T) - \epsilon_n$, $\beta_n(T) = \psi_0(T) + \eta_n$, for each n, where ϵ_n, $\eta_n > 0$ are monotone sequences tending to zero as $n \to \infty$. To make life simpler still, one can assume that $\alpha_0^T = \beta_0^T = \psi_0$. Note also that given any ϕ_0 with $\alpha_{0t_0} \le \phi_0 \le \beta_{0t_0}$, ψ_0 need to satisfy the inequality $\alpha_1(T) \le \psi_0(T) \le \beta_1(T)$ so that the choice (9) is possible.

Remark 2.2 If $g \equiv 0$ then the results of [7] related to (3) can be obtained as a special case of Theorem 2.1.

Recall that the method of lower and upper solutions provides existence results in the closed set generated by lower and upper solutions. In general coupled upper and lower solutions of type II can be easily constructed. See [9, 11] for details. Generalized monotone method using coupled upper and lower solutions of type II will be discussed elsewhere.

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