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Observer Design for a Class of Nonlinear Systems with Non-Full Relative Degree

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Abstract: The paper proposes a method for observer design for a class of nonlinear systems. We decompose the system using a weaker concept than the relative degree. We provide sufficient conditions for global asymptotic stability of the error dynamics. The observer design is carried out by means of a change of coordinates combined with a high gain technique. In particular, our approach results in an observer gain vector field which is extraordinary easy to compute.

Keywords: *Nonlinear system; coordinate change; observer design; Moore-Penrose inverse.*

Mathematics Subject Classification (2000): 93B17, 93B29, 93B50, 93C10.

1 Introduction

We consider the problem of observer design for nonlinear single-input single-output systems. A particularly interesting class of design methods use differential geometric concepts. These design methods are based on various normal forms. In [19, 3], the observer canonical form consisting of a linear output map and linear dynamics driven by a nonlinear output injection is used. The resulting observer has exactly linear error dynamics, i.e., nonlinearities are compensated exactly. The approaches suggested in [13, 9, 10, 5] rely on the observability canonical form, which has significantly weaker existence conditions than the observer canonical form. In the observability canonical form, the observer is designed by a high-gain technique with a constant observer gain, i.e., the nonlinearities are not compensated but dominated by a linear part. For an implementation of the

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observer in the original coordinates one gets a Luenberger-like observer with a possibly nonlinear gain vector field.

In the last decade, new approaches have been developed for nonlinear systems that are not uniformly observable. Several approaches use Kalman-like decompositions, see [1, 17, 18, 26]. For example, the observer design method suggested in [17] uses the Byrnes-Isidori normal form [6, 7] in almost the same way as the observability canonical form is used in [13, 9, 10]. Similarly, the partial observer canonical form used in [18] generalizes the design method given in [19].

For the observability canonical form, the change of coordinates is explicitly given in terms of iterated Lie derivatives of the system's output map. In contrast to that, the transformation into the Byrnes-Isidori normal form is not unique. Although this non-uniqueness offers some degrees of freedom that may be utilized by an experienced control engineer, it makes a symbolic implementation by means of computer algebra systems less straight forward.

In this work we use a weaker concept than the well-known relative degree [10]. The observer design uses a normal form similar to the Byrnes-Isidori normal form. Our work is strongly related to [17], but in contrast to [17] we exploit possible degrees of freedom in the change of coordinates to obtain an explicit expression of the observer gain. Similar as in [16], our approach may even be applicable for systems with ill-defined relative degree. This paper extends preliminary results presented in [26].

The paper is structured as follows. In Section 2 we suggest a decomposition of the system. Section 3 presents an observer and conditions for global asymptotic convergence of the error dynamics. The main contribution of the paper is presented in Section 4, where we suggest a new approach to compute the observer gain. The design method is demonstrated on an example in Section 5.

2 Preliminaries

Consider a nonlinear single-input single-output system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1}$$

with smooth maps $f, g : \Omega \to \mathbb{R}^n$ and $h : \Omega \to \mathbb{R}$ defined on an open and connected subset $\Omega \subseteq \mathbb{R}^n$. We assume that Ω is positively invariant under the dynamics of (1). The notation used in this paper is common in context of differential-geometric control theory (see [14, 22]). In particular, the Lie derivative of h along f is given by $L_f h(x) = \langle dh(x), f(x)\rangle$, where $dh = h'$ denotes the gradient of h and $\langle\cdot,\cdot\rangle$ is the inner product. Iterated Lie derivatives are defined by $L_f^{k+1}h(x) = L_f(L_f^k h(x))$ with $L_f^0 h(x) = h(x)$. The Lie bracket of two vector fields f and g is given by $[f, g](x) = g'(x)f(x) - f'(x)g(x)$. The Euclidean norm of a vector x is noted by $\|x\|$.

The decomposition of system (1) is based on the following assumption [10]:

A1 System (1) has an *observation relative degree* $r < n$ in Ω, i.e., $L_g L_f^k h(x) = 0\ \forall x \in \Omega$ and for $k = 0, \dots, r-2$, and $\exists x \in \Omega$ with $L_g L_f^{r-1} h(x) \neq 0$. Moreover, the covector fields dh, $dL_f h$, $\dots$, $dL_f^{r-1} h$ are linearly independent in Ω.

The concept of an observation relative degree is weaker than the well-known relative degree. In particular, assumption A1 may hold for systems with ill-defined relative degree. Clearly, if system (1) has an uniform relative degree r it also has the observation relative degree r. In this case, the covector fields occurring in A1 are linearly independent [14, Lemma 4.1.1].

Assumption A1 guarantees that for each $x_0 \in \Omega$ there exists a neighborhood $\mathcal{U} \subseteq \Omega$ and smooth functions $\phi_{r+1}, \dots, \phi_n : \mathcal{U} \to \mathbb{R}$ such that the map $(z, \eta) = \Phi(x)$ defined by

$$\begin{aligned} z_i &= L_f^{i-1} h(x) \quad \text{for } i = 1, \dots, r; \\ \eta_j &= \phi_{j+r}(x) \quad \text{for } j = 1, \dots, n-r \end{aligned} \tag{2}$$

with $z = (z_1, \dots, z_r)^T$ and $\eta = (\eta_1, \dots, \eta_{n-r})^T$ is a local diffeomorphism. This diffeomorphism transforms system (1) into

$$\dot{z} = Az + b(\alpha(z,\eta) + \beta(z,\eta)u), \quad y = c^T z, \tag{3a}$$

$$\dot{\eta} = q(z,\eta) + p(z,\eta)u, \tag{3b}$$

with possibly nonlinear maps

$$\begin{aligned} \alpha(z,\eta) &= L_f^r h(\Phi^{-1}(z,\eta)), & \beta(z,\eta) &= L_g L_f^{r-1} h(\Phi^{-1}(z,\eta)), \\ q_i(z,\eta) &= L_f \phi_{r+i}(\Phi^{-1}(z,\eta)), & p_i(z,\eta) &= L_g \phi_{r+i}(\Phi^{-1}(z,\eta)) \end{aligned}$$

for $i = 1, \dots, n-r$. The triple (A, b, c) is in Brunovsky form

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{r \times r}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^r, \quad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^r. \tag{4}$$

The functions $\phi_{r+1}, \dots, \phi_n$ are not uniquely determined. If system (1) has a well-defined relative degree, these functions can be chosen such that

$$\forall x \in \mathcal{U}\colon \quad L_g \phi_i(x) = 0 \quad \text{for} \quad i = r+1, \dots, n, \tag{5}$$

where in the normal form (3b) we have $p_i \equiv 0$ for $i = 1, \dots, n-r$. In this special case, equation (3) becomes the Byrnes-Isidori normal form [7], because the second subsystem (3b) of (3) does not explicitly depend on the input u. In general, the choice of the functions ϕ_i in (5) is rather difficult (see [15]), and only in particular cases (e.g. textbook examples) the choice is easy.

3 Observer Setup

We propose an observer for system (1) based on the form (3). The first subsystem (3a) is observable since $z_1 = y$, $z_2 = \dot{y}, \dots, z_r = y^{(r-1)}$. For this subsystem we design a high-gain observer [13, 9, 10]. Similar as in [17] we suggest an observer of the structure

$$\dot{\hat{z}} = A\hat{z} + b(\alpha(\hat{z},\hat{\eta}) + \beta(\hat{z},\hat{\eta})u) + k(y - c^T \hat{z}), \tag{6a}$$

$$\dot{\hat{\eta}} = q(\hat{z},\hat{\eta}) + p(\hat{z},\hat{\eta})u \tag{6b}$$

with the constant gain vector $k \in \mathbb{R}^r$. In the original coordinates the observer (6) has the form

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + l(\hat{x})(y - h(\hat{x})) \tag{7}$$

with the gain

$$l(\hat{x}) = (\Phi'(\hat{x}))^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix}. \tag{8}$$

Because in (6) the observer gain interacts only with the first subsystem (6a), we augment in (8) the gain vector k by an $(n-r)$-dimensional zero vector.

The convergence analysis for system (1) and observer (7) is carried out in the normal form (3) and (6), respectively. The observation errors $\tilde{z} = z - \hat{z}$ and $\tilde{\eta} = \eta - \hat{\eta}$ are governed by the error dynamics

$$\dot{\tilde{z}} = (A - kc^T)\tilde{z} + b(\alpha(z,\eta) + \beta(z,\eta)u - \alpha(\hat{z},\hat{\eta}) - \beta(\hat{z},\hat{\eta})u), \tag{9a}$$

$$\dot{\tilde{\eta}} = q(z,\eta) + p(z,\eta)u - q(\hat{z},\hat{\eta}) - p(\hat{z},\hat{\eta})u. \tag{9b}$$

The dynamics of subsystem (9a) can be influenced by the gain vector $k = (k_1, \ldots, k_r)^T$. The linear part of this subsystem has the characteristic polynomial

$$\det(\lambda I - (A - kc^T)) = \lambda^r + k_1\lambda^{r-1} + \cdots + k_{r-1}\lambda + k_r. \tag{10}$$

We need the following assumptions:

A2 The map Φ given in (2) is defined on whole Ω and diffeomorphic onto its image $\Phi(\Omega)$, i.e., Φ is a global diffeomorphism.

A3 The maps α and β are globally Lipschitz, i.e., there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$|\alpha(z,\eta) + \beta(z,\eta)u - \alpha(\hat{z},\hat{\eta}) - \beta(\hat{z},\hat{\eta})u| \le \gamma_1\|z - \hat{z}\| + \gamma_2\|\eta - \hat{\eta}\| \tag{11}$$

for all $(z,\eta), (\hat{z},\hat{\eta}) \in \Phi(\Omega)$ and bounded u.

A4 There exist a positive definite matrix $P_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ and constants $\gamma_3, \gamma_4 > 0$ such that for $V_2(\tilde{\eta}) = \tilde{\eta}^T P_2 \tilde{\eta}$ we have

$$\frac{\partial V_2(\tilde{\eta})}{\partial \tilde{\eta}} (q(z,\eta) + p(z,\eta)u - q(\hat{z},\hat{\eta}) - p(\hat{z},\hat{\eta})u) \le \gamma_3\|z - \hat{z}\|^2 - \gamma_4\|\eta - \hat{\eta}\|^2 \tag{12}$$

for all $(z,\eta), (\hat{z},\hat{\eta}) \in \Phi(\Omega)$ and any bounded u.

From A1 we already concluded that Φ is a local diffeomorphism. Conditions for Φ to be a global diffeomorphism as required in A2 are presented in [28, 4, 25]. A3 is a standard assumption in high gain design [24]. Assumption A4 means that the subsystem (3b) possesses a global steady state solution property [1]. If one considers the full state z of the observable subsystem (3a) as an output, the function V_2 becomes a global exponential-decay output-to-state stable (OSS) Lyapunov function [27]. This is a cruicial difference to assumption H3 of [17], where a classical (not OSS) Lyapunov function is required for the second subsystem. Note that the property A4 depends on the coordinate transformation (2), especially on the choice of the functions ϕ_i for $i = r+1, \ldots, n$. Similar as H3 in [17], this property is difficult to check.

Theorem 3.1 *Consider system* (1) *with the observer* (7)*. Assume that the input u is bounded and conditions A1-A4 hold, where r denotes the observations relative degree. Then, there exist a vector $k \in \mathbb{R}^r$ such that* $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$ *for all* $x(0), \hat{x}(0) \in \Omega$.

The proof of Theorem 3.1 is shown in the appendix. In [9, 10], the vector k is chosen such that the roots $\lambda_1, \dots, \lambda_r$ of (10) are placed at $\lambda_i = -\theta^i$ for $i = 1, \dots, r$ and sufficiently large $\theta > 0$. The technique suggested in [13] corresponds to the multiple root $\lambda_i = -\theta$ for $i = 1, \dots, r$. A general discussion about the computation of the constant observer gain for high-gain design can be found in [24].

4 Observer Gain Based on the Moore-Penrose Inverse

Up to now, one has to compute the $n-r$ gradients $d\phi_{r+1}, \dots, d\phi_n$. To obtain the Byrnes-Isidori normal form, these gradients must additionally satisfy (5). In the following, we consider a special choice of the nonlinear observer gain vector field (8), which can be computed directly, i.e., without an explicit knowledge of these gradients.

The Jacobian matrix of the transformation (2) is split up into two parts, where the first r rows consisting of gradients of Lie derivatives form a reduced observability matrix

$$Q(x) = \begin{pmatrix} dh(x) \\ \vdots \\ dL_f^{r-1}h(x) \end{pmatrix}. \tag{13}$$

There remaining $n-r$ rows are collected in a matrix

$$R(x) = \begin{pmatrix} d\phi_{r+1}(x) \\ \vdots \\ d\phi_n(x) \end{pmatrix}. \tag{14}$$

Matrix Q results directly from system (1), whereas the matrix R is not uniquely determined. However, the observer gain in (7) depends also on R:

$$l(x) = (\Phi'(x))^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{Q(x)}{R(x)} \end{pmatrix}^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix}. \tag{15}$$

In the following we suggest an observer gain, in which the matrix R does not occur. In particular, if the rows of (13) and (14) are orthogonal to each other, that is

$$\forall\, x \in \Omega : \; R(x)Q^T(x) = 0, \tag{16}$$

the observer gain (15) becomes

$$l(x) = \begin{pmatrix} \frac{Q(x)}{R(x)} \end{pmatrix}^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} = \Big(Q^+(x) \,|\, R^+(x) \Big) \begin{pmatrix} k \\ 0 \end{pmatrix} = Q^+(x)\, k, \tag{17}$$

which depends explicitly only on the reduced observability matrix Q, where Q^+ denotes the Moore-Penrose inverse of Q, see [21, 23].

Clearly, the gain (17) is a special case of (15). The crucial question is whether the functions $\phi_{r+1}, \dots, \phi_n$ in (14) can be chosen such that condition (16) holds. To formulate the existence conditions, we consider the matrix $Q^+(x)$. This matrix is well-defined and smooth on Ω because of A1. The columns of Q^+ are vector fields:

$$Q^+(x) = (\tau_1(x), \dots, \tau_r(x))\,. \tag{18}$$

We need the following assumptions:

A5 The distribution

$$\Delta(x) = \mathrm{span}\{\tau_1(x), \ldots, \tau_r(x)\} \tag{19}$$

spanned by the columns of $Q^+(x)$ is involutive, i.e., for every two vector fields $\tau_1, \tau_2 \in \Delta$ there holds $[\tau_1, \tau_2] \in \Delta$.

A6 The vector fields $\tau_1, \ldots, \tau_r$ are complete.

Theorem 4.1 *Suppose system* (1) *fulfills A1, A5 and A6. Then, there exists a global diffeomorphism of the form* (2)*, for which the observer gain* (15) *becomes* (17)*.*

Proof The rows of Q are linearly independent due to A1. Therefore, the distribution Δ is regular with rank r. Then, the distribution Δ is integrable by the Theorem of Frobenius [14, p. 23], i.e., for any $x_0 \in \Omega$ there exist a neighbourhood $\mathcal{U}$ and smooth functions $\phi_{r+1}, \ldots, \phi_n$ such that

$$\forall x \in \mathcal{U}: \quad \langle d\phi_i(x), \tau_j(x)\rangle = 0 \tag{20}$$

for $j = 1, \ldots, r$ and $i = r+1, \ldots, n$. In addition the covector fields $d\phi_{r+1}, \ldots, d\phi_n$ are linearly independent. Equation (20) is equivalent to $R(x)Q^+(x) = 0$ for all $x \in \mathcal{U}$. From $Q^+ = Q^T(QQ^T)^{-1}$ we get (16) on $\mathcal{U}$. Therefore, the observer gain (15) becomes (17) on $\mathcal{U}$.

Now, we want to address global aspects. The proof of the Theorem of Frobenius is constructive, see [14]. In particular, the construction of the maps $\phi_{r+1}, \ldots, \phi_n$ is based on the flows of the vector fields $\tau_1, \ldots, \tau_r$. These vector fields are complete by A6. Moreover, we can always augment $\Delta(x)$ to $\mathbb{R}^n$ using a basis of complete vector fields $\tau_{r+1}, \ldots, \tau_n$. The map $\Psi(z, \eta) = \varphi_{z_1}^{\tau_1} \circ \cdots \circ \varphi_{z_r}^{\tau_r} \circ \varphi_{\eta_1}^{\tau_{r+1}} \circ \cdots \circ \varphi_{\eta_{n-r}}^{\tau_n}(x_0)$ with arbitrary $x_0 \in \Omega$, in which $\varphi_t^{\tau_i}$ denotes the flow of a vector field τ_i, is a global diffeomorphism onto Ω due to the completeness of the vector fields $\tau_1, \ldots, \tau_n$, see [25]. This implies that $\Phi := \Psi^{-1}$ is also a global diffeomorphism. The maps $\phi_{r+1}, \ldots, \phi_n$ are the last $n-r$ components of Ψ^{-1}. Therefore, these maps are defined on whole Ω, and (20) holds globally.

Instead of (5), which can be written as $\langle d\phi_i, g\rangle = 0$, the additional functions $\phi_{r+1}, \ldots, \phi_n$ now satisfy (20). However, to obtain the observer gain (17) we neither have to compute the functions ϕ_i nor the gradients $d\phi_i$ for $i = r+1, \ldots, n$. If we also have $g \in \Delta$, the second subsystem does not directly depend on u, i.e., in this case the observer design is carried out in the Byrnes-Isidori form as in [17], but without an explicit computation the zero dynamics. Condition A5 always holds for $r \in \{1, n\}$, where for $r = n$ we also have $Q(x) = \Phi'(x)$ and $Q^+(x) = (\Phi'(x))^{-1}$, by which we obtain the observer gain given in [13, 9, 10].

Note that one could in principle design an observer gain vector field like (17) with an arbitrary generalized inverse [2, 8] of the reduced observability matrix (13) to project the correction term $k(y - h(\hat{x}))$ to observable dynamics of the first subsystem. However, the crucial contribution of Theorem 4.1 is the insight, that the Moore-Penrose inverse used in (17) is part of a change of coordinates (2). Our observer is similar to that in [20] and [11], but derived by a different framework.

Combining Theorem 3.1 and 4.1 results in the following conclusion.

Corollary 4.1 *Consider system* (1) *with the observer* (7) *and the observer gain* (17)*. Assume that the input u is bounded and conditions A1 and A3–A6 hold, where r denotes the observation relative degree. Then, there exist a vector* $k \in \mathbb{R}^r$ *such that* $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$ *for all* $x(0)$, $\hat{x}(0) \in \Omega$*.*

5 Example

Consider the system

$$\dot{x} = \begin{pmatrix} x_1x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2+2x_3 \\ 1 \\ 0 \end{pmatrix} u, \quad y = x_4, \tag{21}$$

taken from [14, p. 146] on $\Omega = \mathbb{R}^4$. From $L_g h(x) \equiv 0$ and $L_g L_f h(x) = 2(1+x_3)$ we conclude that system (21) has the observation relative degree $r = 2$. System (21) also has relative degree $r = 2$ if $x_3 \neq -1$. The first two components of the transformation (2) are $\phi_1(x) = h(x) = x_4$ and $\phi_2(x) = L_f h(x) = x_2 + x_1^2$. First, we design the observer as in [17] based on the Byrnes-Isidori normal form. The components ϕ_3 and ϕ_4 must satisfy (5), that is

$$L_g\phi_i(x) = (2+2x_3)\frac{\partial\phi_i}{\partial x_2} + \frac{\partial\phi_i}{\partial x_3} = 0 \tag{22}$$

for $i = 3, 4$. Two independent choices are $\phi_3(x) = x_2 - 2x_3 - x_3^2$ and $\phi_4(x) = x_4$, from which we obtain the Jacobian matrix

$$\Phi'(x) = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 2x_1 & 1 & 0 & 0 \\ \hline 0 & 1 & -2-2x_3 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right). \tag{23}$$

This Jacobian is singular if $x_3 = -1$. The singularity occurs on the same set where the relative degree is not defined. As a consequence, the observer gain (15) given by

$$l(x) = \begin{pmatrix} 0 \\ k_2 \\ \dfrac{k_2}{2+2x_3} \\ k_1 \end{pmatrix} \tag{24}$$

has a pole for $x_3 = -1$, i.e., the gain (24) is not defined for all $x \in \Omega$.

Now, we consider the approach suggested in Sect. 4. We have

$$Q^+(x) = (\tau_1(x), \tau_2(x)) = \begin{pmatrix} 0 & \dfrac{2x_1}{4x_1^2+1} \\ 0 & \dfrac{1}{4x_1^2+1} \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{25}$$

where the vector fields τ_1 and τ_2 are the first and second column of Q^+, respectively. Condition A5 is fulfilled since $[\tau_1, \tau_2] \equiv 0$. Note that these vector fields are complete. The observer gain can be computed directly from (17). We get

$$l(x) = \begin{pmatrix} \dfrac{2k_2x_1}{4x_1^2+1} \\ \dfrac{k_2}{4x_1^2+1} \\ 0 \\ k_1 \end{pmatrix}. \tag{26}$$

In several simulation scenarios, the observers (7) with the gain vector fields (24) and (26) behave similarly. However, in contrast to (24) the new observer gain (26) is well-defined for all $x \in \Omega$.

6 Conclusion

We addressed the problem of observer design for the special class of nonlinear systems. Similar as in [1, 17, 18, 26], the approach is based on a decomposition of the system into an observable and a possibly unobservable subsystem. In contrast to previous work, this paper is dedicated to the actual computation of the observer gain vector field. We exploit degrees of freedom to get an observer gain, whose symbolic computation is straightforward. In particular, the observer gain is an immediate generalization of the gain vector used in [13, 9, 10].

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Appendix A. Proof of Theorem 3.1

The following lemma is a straight consequence of [10, Lemma 3.11]:

Lemma 1 *Given A, b, c defined in* (4) *and arbitrary constants $\nu, \rho > 0$. Then, there exist a vector $k \in \mathbb{R}^r$ and a positive definite matrix P such that*

$$(A - kc^T)^T P + P(A + kc^T) + \nu P b b^T P + \rho I < 0, \tag{27}$$

i.e., the matrix on the left hand side of (27) *is negative definite.*

Now, we prove Theorem 3.1.

We consider system (1) and observer (7) in the transformed coordinates, namely (3) and (6). We have to show that the equilibrium point $(\tilde{z}, \tilde{\eta}) = (0, 0)$ of the error dynamics (9) is asymptotically stable. We choose the candidate Lyapunov function $V(\tilde{z}, \tilde{\eta}) = \tilde{z}^T P \tilde{z} + \tilde{\eta}^T P_2 \tilde{\eta}$, where the positive definite matrix P will be specified later and the positive definite matrix P_2 is taken from A3. Then, we have

$$\begin{aligned}\dot{V}(\tilde{z},\tilde{\eta})\big|_{(9)} &= \tilde{z}^T \left[(A-kc^T)^T P + P(A+kc^T)\right] \tilde{z} + \gamma_3\|\tilde{z}\|^2 - \gamma_4\|\tilde{\eta}\|^2 \\ &\quad + 2\,\tilde{z}^T Pb\,[\alpha(z,\eta)+\beta(z,\eta)u-\alpha(\hat{z},\hat{\eta})-\beta(\hat{z},\hat{\eta})u]\end{aligned} \tag{28}$$

for all $(z,\eta),(\hat{z},\hat{\eta}) \in \Phi(\Omega)$ and bounded u. Using (11), the inequality

$$\begin{aligned}\alpha(z,\eta)+\beta(z,\eta)u-\alpha(\hat{z},\hat{\eta})-\beta(\hat{z},\hat{\eta})u &\leq 2\gamma_1|\tilde{z}^T Pb|\cdot\|\tilde{z}\| + 2\gamma_2|\tilde{z}^T Pb|\cdot\|\tilde{\eta}\| \\ &\leq \left(\gamma_1^2+\frac{\gamma_2^2}{\mu}\right)\tilde{z}Pbb^TP\tilde{z} + \tilde{z}^T\tilde{z} + \mu\,\tilde{\eta}^T\tilde{\eta}\end{aligned}$$

holds for arbitrary $\mu > 0$, see [10, 24]. We set $\nu = \gamma_1^2 + \gamma_2^2/\mu$, $\rho = \gamma_3 + 1$, where (28) becomes

$$\dot{V}(\tilde{z},\tilde{\eta})\big|_{(9)} \leq \tilde{z}^T \left[(A-kc^T)^T P + P(A+kc^T) + \nu Pbb^TP + \rho I\right] \tilde{z} - (\gamma_4-\mu)\,\tilde{\eta}^T\tilde{\eta}.$$

This quadratic form is negative definite if we choose $\mu \in (0,\gamma_4)$ and take P and k from Lemma 6.