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Compact Explicit MPC Law with Guarantees of Feasibility for Reference Tracking

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Abstract: The present paper deals with the constrained model predictive control for linear time invariant systems. Even if these techniques reached a considerable maturity in the last decade, the feasibility problems remain a sensitive point at least for applications which involve tracking of challenging reference signals, most often in conjunction with restrictive physical limitations. The main goal here is the adaptation (enlargement) of the set of feasible trajectories. Two strategies are discussed: the tuning of the predictive control parameters and the reference governor schemes. Specifically on the former direction it will be shown that a compact piecewise affine "feedback control law" with guarantees of feasibility can be constructed. This compactness is given by mixing the explicit formulations of the predictive law and those of the reference adjustment mechanism.

Keywords: *Optimal control; predictive control; multiparametric optimization.*

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1 Introduction

Model Predictive Control (MPC) has imposed itself as a flexible optimization based technique with versatile constraints handling capabilities due to its time-domain formulation (see the up-to-date monographies [15], [6], [19], [9]). In the same time, the optimization fundament imposes the feasibility as a crucial demand as long as it represents the main ingredient for the stability of the entire closed loop [16].

For the regulation problem, the necessary and sufficient conditions of MPC feasibility are based on pseudo-infinite prediction horizons or, similarly, on terminal constraints [11] designed in concordance with *positive invariant sets* principles [4]. The reference

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tracking problem [5] is somehow more laborious but it can take advantage of the prediction capabilities of MPC and thus deliver excellent control performances. However, the infeasibility threatening becomes severe in this case [22], mainly when the reference contradicts with the imposed constraints or when the a priori information regarding the set-point is limited. The modification of the trajectory to be followed may induce the reduction of the feasible domain and even lead to infeasibility.

In order to deal with this phenomenon several strategies can be followed and a possible taxonomy can differentiate between: infeasibility avoidance and feasibility recovery. The first category contains the predictive laws which deal with the feasibility in terms of a robustness issue. This elegant approach implies that the exogenous reference signal can be described either statistically, either by deterministic models and by consequence included in the control law synthesis. The main criticisms are linked to the numerical difficulties in solving the new optimizations (for example the computational aspects involving min-max problems) and often to the conservative performances due to the worst case combination of set-points which have to be considered.

The feasibility recovery strategies follow a different philosophy. The idea is to design a control law which optimizes the performances in relation with an usual reference signal and to treat the eventual infeasible situation either through the enhancement of the feasible domain either by increasing the prediction horizon, either by the on-line adjustment of the trajectory to be followed (using a so-called "reference governor"). In the first case, the feasible domain is enlarged but remains limited, another disadvantage being the relative augmentation of the set of decision variables and related constraints. The "reference governor" method (see [3, 7] and the citations therein) replaces the set-points with the best admissible reference found as the solution of an optimization problem apart MPC.

The current paper revisits the main concepts related to the feasibility of MPC and extends their definitions to the reference tracking case. The structure of the feasible domain and the limitations on the reference signal are further considered together with their links to the MPC parameters. As infeasibility avoidance technique, this paper focuses on the reference governor schemes, the main goal being to obtain a control law with guarantees of feasibility. Due to the fact that both ingredients (MPC and reference governors) are represented by multiparametric optimization problems (mpOP) [2, 12], their explicit formulation is obtained and the result gathered in a compact form. The main contribution will be the construction of this piecewise affine control law, feasible over any initial set. The conservativeness and the compromise between the memory needs and the performance of the evaluation mechanism are some of the addressed issues.

In the following, Section 2 reminds the MPC problem, the explicit formulation and states some definitions related with the feasibility. Section 3 deals with the reference tracking problems, analyzing the feasibility limitations. In Section 4, the infeasibility avoidance mechanism is integrated in the predictive control scheme resulting in the compact MPC with guarantees of feasibility. Finally, Section 5 presents some study cases and section VI the conclusions.

2 Model Predictive Control

2.1 Constrained model predictive control

MPC implies the idea of minimizing a cost index based on the predicted plant evolution. For the regulation to origin, consider the discrete time LTI system in a state-space description:

$$\Sigma_P : \begin{cases} x_{t+1} = Ax_t + Bu_t, t \geq 0, x_0 = x^0 \in X_0, \\ y_t = Hx_t, \\ Cx_t + Du_t \leq \gamma, \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $y_t \in \mathbb{R}^r$ are the state, input and output vectors and X_0 is the set of initial conditions. It is assumed throughout that the pair (A, B) is stabilizable. The inequality constraints, given by $\gamma \in \mathbb{R}^q$, $D \in \mathbb{R}^{q\times m}$, $C \in \mathbb{R}^{q\times n}$, describe a polyhedral region including the origin. At each sampling time, the current state, $x = x_t$ (assumed available), is used to find the optimal open-loop control sequence $\mathbf{k}_u^* = [u_{t|t}^T, \ldots, u_{t+N-1|t}^T]^T \in \mathbb{R}^{N\times m}$:

$$\begin{gathered} \mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} x_{t+N|t}^T P x_{t+N|t} + \sum_{k=0}^{N-1} \left\{ x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t} \right\}, \\ \begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t},\ k \geq 0, \\ Cx_{t+k|t} + Du_{t+k|t} \leq \gamma,\ 0 \leq k \leq N-1, \\ x_N \in X_N, \end{cases} \end{gathered} \tag{2}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ are weighting matrices and the pair $(Q^{1/2}, A)$ is detectable. P is characterizing the terminal cost while X_N is the associated terminal set. The prediction horizon - N, together with the matrices P, Q and R are the knobs of this construction based on optimization.

The first part of the resulting optimal open-loop control sequence (2) - $u_{t|t}^*$, is effectively applied and the whole procedure is restarted following the "receding horizon principle" which provides the MPC law with all the advantages of a closed-loop control law.

2.2 Multiparametric optimization. Explicit solutions

The optimization problem (2) is tractable as it has mN decision variables and qN constraints. It can be rewritten after simple matrix manipulations as:

$$\begin{aligned} &\arg\min_{\mathbf{k}_u} \mathbf{k}_u^T H \mathbf{k}_u + \mathbf{k}_u^T F x + x^T G x, \\ &\text{subject to}: \quad A_{in}\mathbf{k}_u \leq b_{in} + B_{in} x. \end{aligned} \tag{3}$$

This is known in the literature as the multiparametric quadratic problem (mpQP) and its solution is represented by a piecewise linear and continuous function [2, 20, 17]:

$$\mathbf{k}_u^*(x) = K_i * x + \kappa_i, \text{ for } x \in D_i\,, \tag{4}$$

where D_i are convex polyhedral regions in $\mathbb{R}^n$. MPC uses only the first component of this optimal solution:

$$u^{MPC}(x) = K_i^{MPC} * x + \kappa_i^{MPC}, \text{ with}\, i\, \text{such that}\, x \in D_i\,, \tag{5}$$

and $K_i^{MPC}, \kappa_i^{MPC}$ the first components of K_i, κ_i.

Lately, efficient algorithms are available [13] to develop these explicit solutions and thus the constrained predictive control policy can dispose of an additional design and analysis tool.

2.3 Feasibility within constrained MPC

Definition 2.1 The feasible set for the MPC law (2), is the set of all states for which a control sequence $\mathbf{k}_u^*(x)$ exists:

$$X_f = \bigcup_i D_i \,, \tag{6}$$

with D_i as in (4). X_f is a convex polyhedron and corresponds to the projection of the parameterized polyhedron [18] formed by the constraints in (3), onto the state space.

Definition 2.2 A set $X \in \mathbb{R}^n$ is positively invariant with respect to the dynamic $x_{t+1} = Ax_t$ if and only if $\forall x_t \in X \Rightarrow x_{t+1} \in X$.

In the literature, one can find exhaustive results regarding the feasibility of MPC laws [11]. In the following a classification of the infeasibility for MPC based on the relation with the set of initial conditions, X_0 is introduced:

- $\mathbf{X_0} \setminus \mathbf{X_f} \neq \emptyset$.

  The MPC law is infeasible w.r.t. X_0, because it exists at least one initial condition $x^0 \in X_0$ such that $u^{MPC}(x^0)$ is not well defined.

- $\mathbf{X_0} = \mathbf{X_f}$

  The MPC law is feasible w.r.t. X_0 if and only if X_f is positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$.

- $\mathbf{X_0} \subset \mathbf{X_f}$

  The MPC law is feasible w.r.t. X_0 if and only if it exists a set $\Omega \subset \mathbb{R}^n$, positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$, which satisfies $X_0 \subset \Omega \subset X_f$.

This classification does underline the role of the initial conditions and also the threatening represented by the inappropriate tuning of the MPC which may lead to self-generated infeasibility [21]. However, the set X_f depends on the MPC parameters and thus constructive methods do exist for designing feasible laws for the regulation problem (1-2).

Definition 2.3 [8] The maximal admissible set for (1) under a constant feedback law $u = Kx$ is represented by:

$$O_\infty = \left\{ x \in \mathbb{R}^n | (C(A+BK)^k + DK(A+BK)^{k-1})x \leq \gamma, \forall k \geq 0 \right\}. \tag{7}$$

Definition 2.4 A set $X \subset \mathbb{R}^n$ is called control invariant for the system $x_{t+1} = Ax_t + Bg(x_t)$ if there exists a function $g(x_t)$ such that X is positive invariant with respect to this dynamic. The maximal control invariant set - C_∞, is the control invariant set containing all other control invariant sets.

Remark 2.1 X is control invariant only if $X \subset C_\infty$.

Given these definitions, a design sketch for a feasible MPC law could be:

- Compute C_∞.
- If $X_0 \setminus C_\infty \neq \emptyset$ stop ; there is no MPC control law feasible for all $x \in X_0$.

- Choose a stabilizing control law $u = Kx$ and compute the maximal admissible set O_∞. Fix $X_N = O_\infty$.
- Find $N < \infty$ for the problem (2) such that $X_0 \subset X_f$.

Remark 2.2 This discussion was dedicated to the feasibility analysis as it represents the main issue of the current paper. It must be mentioned that the feasibility of the predictive control law is not guaranteeing the stability but is one of its major ingredients (see [16]). If all the signals of the system are directly or indirectly bounded, then a BIBO stability is achieved once the infeasibility is avoided.

3 MPC Feasibility For Tracking Systems

A case which is often studied in the literature [5], [14] is the one of "constant reference tracking". The necessary and sufficient condition in the unconstrained case for such reference signals is resumed by the following assumption.

Assumption 1: The pencil matrix:

$$A_c = \begin{bmatrix} A - I & B \\ H & 0 \end{bmatrix} \tag{8}$$

is invertible.

Given the discrete-time dynamical system (1) with the associated mixed state-inputs constraints it is worth to recall that the set of admissible constant references $\mathcal{Y}_r$ which the system will be able to track will be given by:

$$\mathcal{Y}_r = \left\{ y_r | \, [CH^T(H^TH)^{-1} - D(H(A-I)^{-1}B)^{-1}]y_r \le \gamma \right\}. \tag{9}$$

Remark 3.1 If the polyhedron described by:

$$PC = \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix} \middle| Cx_t + Du_t \le \gamma \right\} \tag{10}$$

is bounded, then the admissible constant references $\mathcal{Y}_r$ will be also bounded.

In the following it is considered the reference tracking in the general case by relaxing this "piece-wise constant" assumption for the reference signal. However, the results obtained for this family of set-point can be found as particular cases of reference management.

3.1 Classification of general reference tracking problems

The problem of regulating the system state to origin using optimal control sequences over receding horizons can be extended to the reference tracking problems. A classification of these tracking problems [1] might be:

1) The model-following problem: The reference signal is the output of a known linear model:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t, z_0 \in Z_0, \\ r_t = H_z z_t \end{cases} \tag{11}$$

with $r_t \in \mathbb{R}^p$, A_z stable, the pair (A_z, H_z) observable and $Z_0 \in \mathbb{R}^{n_z}$ the set of initial conditions. The model predictive control is supposed to find the optimal control sequence

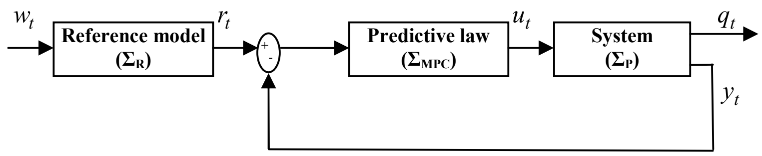

Figure 3.1: Classical MPC tracking scheme (q_t-the tracking quality).

for the system (1) such that the output y tracks the incoming reference r. By recasting this problem on the form:

$$\begin{aligned}
&\tilde{x}_{t+1} = \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_z & 0 \\ 0 & A \end{bmatrix}}_{\tilde{A}} \tilde{x}_t + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\tilde{B}} u_t, \\
&\tilde{C}\tilde{x}_t + Du_t \leq \tilde{\gamma}, \\
&\tilde{C} = \begin{bmatrix} 0 & C \end{bmatrix}, \ \tilde{D} = D, \ \tilde{\gamma} = \gamma
\end{aligned} \tag{12}$$

and putting $\tilde{Q} = \begin{bmatrix} H_z & -H \end{bmatrix} Q \begin{bmatrix} H_z & -H \end{bmatrix}^T$, the classical MPC design techniques for the problem (1-2) can be applied to assure the existence of a feasible law for all initial conditions $\tilde{x}_0 \in \tilde{X}_0 = \left\{ \begin{bmatrix} z_0 \\ x_0 \end{bmatrix} \middle| z_0 \in Z_0; x_0 \in X_0 \right\}$.

2) The tracking problem: The desired trajectory is the output of a known linear model (Figure 3.1) excited by an exogenous signal, partially known:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t + B_z w_t, \ z_0 \in Z_0, \\ r_t = H_z z_t \end{cases} \tag{13}$$

with $Z_0 \in \mathbb{R}^{n_z}$ and considering that the same assumptions as for the previous case are satisfied. Note that w_k can be the output of a high-order, time-varying system.

All the information on the exogenous signal w_t should be incorporated in the reference model such that the MPC law could improve the prediction accuracy. For example if the signal w_t is supposed to be known in advance over a horizon $N_w - \left\{ w_{t|t}, \ldots, w_{t+N_w-1|t} \right\}$, then the reference model to be used for the MPC design is:

$$A_z \leftarrow \left[\begin{array}{c|cccc} A_z & B_z & 0 & \cdots & 0 \\ \hline & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ & \vdots & \vdots & \ddots & 1 \\ & 0 & 0 & \cdots & 0 \end{array} \right], \quad B_z \leftarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$H_z \leftarrow \begin{bmatrix} H_z \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \quad z_t \leftarrow \begin{bmatrix} z_t \\ w_t \\ w_{t+1} \\ \vdots \\ w_{t+N_w-1} \end{bmatrix}, \quad w_t \leftarrow w_{t+N_w}.$$

With this reference description, one can construct the augmented model as in (12) and proceed to the MPC synthesis based on the minimization of a cost function weighting the tracking error and the control effort. Following developments will focus on the influence of the exogenous signal on the feasibility of the overall control scheme.

3.2 Infeasibility within reference tracking

Even if the MPC design is reduced to the classical case, the feasibility requirements must be fulfilled with respect to the set of initial conditions in the augmented space $\tilde{X}_0$ and not only for X_0. In this respect, the terminal set must be based on the model (12) - $\tilde{x}_N \in \tilde{X}_N$ and not only $x_N \in X_N$.

The MPC law will be characterized by a polyhedral set of feasible points in the augmented state space as in (6):

$$\tilde{X}_f = \{\tilde{x} | \Lambda \tilde{x} \leq \lambda\} = \left\{ \begin{bmatrix} z^T & x^T \end{bmatrix}^T | \Lambda_z z + \Lambda_x x \leq \lambda \right\}. \tag{14}$$

The evolution of the reference model is independent of the chosen control action at time t:

$$z_{t+1} = A_z z_t + B_z w_t$$

and thus at time $t+1$ the infeasibility phenomenon entails that $x_{t+1} \notin X_f(z_{t+1}) = \{x | \Lambda_x x \leq \lambda - \Lambda_z z_{t+1}\}$.

But in the same time the MPC law is designed such that in the absence of exogenous signal ($w_t = 0$), the feasibility is preserved (a necessary condition for the constraint fulfilment over the prediction horizon). This means that $x_{t+1} \in X_f(\hat{z}_{t+1}) = \{x | \Lambda_x x \leq \lambda - \Lambda_z \hat{z}_{t+1}\}$ with $\hat{z}_{t+1} = A_z z_t$ (absence of exogenous signal). Based on this observation the infeasibility of the MPC tracking scheme at time t can be classified as:

- $\mathbf{B_z w_t = 0}$

  The MPC law is feasible because $\hat{z}_{t+1} = z_{t+1}$.

- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) \neq \emptyset}$ (Figure 3.2 left)

  If $x_{t+1} \in D$ the MPC law is feasible.

  If $x_{t+1} \notin D$ the MPC law is infeasible due to the incompatibility between the current state and the exogenous signal entering the reference model.

- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) = \emptyset}$ (Figure 3.2 right)

  Infeasibility due to a jump of the reference model state which overwhelms the closed loop tracking capabilities.

Remark 3.2 The only degree of freedom one can dispose within MPC to diminish the infeasibility risk is to enlarge the set $\tilde{X}_f$. This can be done by augmenting the prediction horizon. Unfortunately this manoeuvre is increasing the complexity of the resulting control law. Another limitation towards this augmentation is the fact that $\tilde{X}_f$ can never go beyond $\tilde{C}_\infty$.

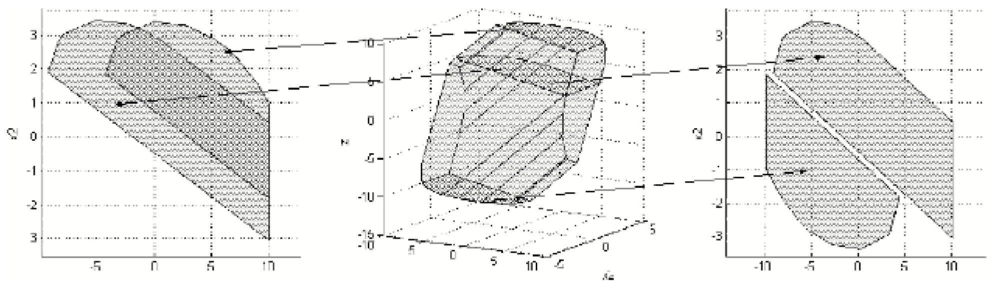

Figure 3.2: Example of feasible set $\tilde{X}_f$ (center). $\mathbf{D} = \mathbf{X_f(z_{t+1})} \cap \mathbf{X_f(\hat{z}_{t+1})} \neq \emptyset$ (left). $\mathbf{D} = \mathbf{X_f(z_{t+1})} \cap \mathbf{X_f(\hat{z}_{t+1})} = \emptyset$ (right).

3.3 Feasible bounds expressed as multiparametric optimization solutions

In the previous discussion the feasible set $\tilde{X}_f$ has been considered as the extended version of a parameterized polyhedron $X_f(z_t)$ with parameters given by the state of the reference model. If the goal is to describe the family of feasible references, then a dual approach has to be considered, with the characterization of the infeasibility by the fact that $z_{t+1} \notin Z_f(x_{t+1}) = \{ z | \Lambda_z z \leq \lambda - \Lambda_x x_{t+1} \}$. Using this set, in the SISO case, the feasibility can be characterized based on the limits of the feasible reference signal r_t or of the feasible exogenous signal w_t as in Table 3.1.

$r_t^{\min}(x_t) \leq r_t \leq r_t^{\max}(x_t)$	$w_t^{\min}(x_t, z_t) \leq w_t \leq w_t^{\max}(x_t, z_t)$
$r_t^{\min}(x_t) = \min\limits_z H_z z$ s.t. $\Lambda_z z \leq \lambda - \Lambda_x x_t$	$w_t^{\min} = \min\limits_w w$ s.t. $\Lambda_z B_z w \leq \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$
$r_t^{\max}(x_t) = \max\limits_z H_z z$ s.t. $\Lambda_z z \leq \lambda - \Lambda_x x_t$	$w_t^{\max} = \max\limits_w w$ s.t. $\Lambda_z B_z w \leq \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$

Table 3.1: Feasibility conditions.

The feasible bounds $r_t^{\min}, r_t^{\max}, w_t^{\min}, w_t^{\max}$ are solutions of linear multiparametric optimization problems. Their expression provides a hint about the way the feasibility can be recovered by the adjustment of the reference to the corresponding feasible limitation once this is violated.

4 Feasibility recovery

The idea resumed in Table 3.1 is that at each sampling time, the feasibility of the MPC scheme depends on whether the reference signal is contained in a safe region (X_r):

$$r_t \in X_r(x_t, z_t) \Leftrightarrow H_z z_t \in X_r(x_t, z_t). \tag{15}$$

As long as the exogenous signal is given, and the state of the system is supposed to be known, the only degree of freedom available to force the feasibility of the control law is the adjustment of the reference, to follow the best feasible approximation of the reference

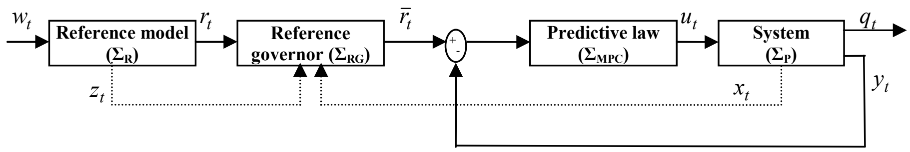

Figure 4.1: Reference governor scheme for MPC feasibility.

signal. This technique has already a wide experience, being known in the literature as the reference governor scheme.

4.1 Reference governor

A reference governor (RG) is based on the idea that for the reference tracking scheme in (Figure 3.1) a MPC law can be designed such that the closed loop performance, feasibility and stability requirements are fulfilled for the model-following problem (in the absence of the exogenous excitation). Once the MPC law description is available, the associate feasible set $\tilde{X}_f$ is also available. Based on this assumption, the goal of the RG is to replace the reference r_t by:

$$\bar{r}_t = \bar{r}_t(x_t, r_t) \tag{16}$$

such that $\bar{r}_t$ is the best approximation of r_t and the MPC law is well defined. Due to the fact that r_t is the output of the reference model, one can rewrite (16) as $\bar{r}_t(z_t, x_t)$. The best approximation must be judged with respect to a cost index. According to this principle the adjusted reference might be:

$$\begin{aligned} &\bar{r}_t = H_z \bar{z}_t^*, \\ &\bar{z}_t^*(z_t, x_t) = \arg\min_{\bar{z}_t}(H_z\bar{z}_t - H_z z_t)^T S (H_z \bar{z}_t - H_z z_t), \\ &\qquad \text{such that} \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} \in \tilde{X}_f, \end{aligned} \tag{17}$$

where the matrix S weights the deviation of the references. This mathematical formulation of the RG underlines the fact that the information needed for the optimization (17) is restricted to the current measurements (Figure 4.1).

4.2 Compact MPC law with guarantees of feasibility

An important detail regarding the optimization in (17) is the dependence of the argument $\bar{z}_t^*$ on the set of parameters $\{z_t, x_t\}$. The set of constraints depends exclusively on the vector x_t while the dependence on z_t comes exclusively from the cost index. If there is no other restriction added to (17), then the reference model state can be any value $z_t \in \mathbb{R}^{n_z}$. This aspect is decisive from the feasibility point of view because indirectly, all the constraints on z_t represent limitations on the reference or on the exogenous signal. As a first result, by applying the MPC law $u^{MPC}(\bar{z}_t, x_t)$ instead of $u^{MPC}(\tilde{z}_t, x_t)$, one can obtain a predictive law with guarantees of feasibility at each sampling time.

Despite this advantage the cascaded implementation of the reference governor with the MPC law is quite demanding on-line. A first step is to observe as in [12] that the RG

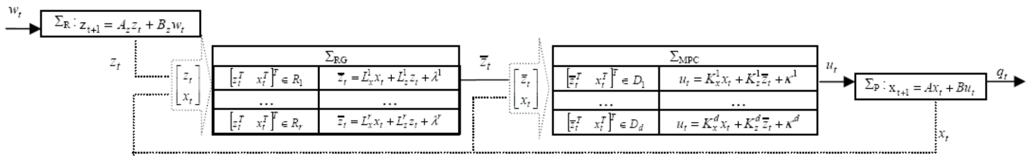

Figure 4.2: Explicit description of reference governor and MPC law.

implies in fact a multiparametric quadratic problem (17) for which the explicit solution does exist as in (4).

$$\bar{z}_t(z_t, x_t) = L_i * \begin{bmatrix} z_t \\ x_t \end{bmatrix} + l_i, \text{ for } \begin{bmatrix} z_t \\ x_t \end{bmatrix} \in R_i\,, \tag{18}$$

which can be further written componentwise:

$$\bar{z}_t(z_t, x_t) = L_{i_x} * x_t + L_{i_z} * z_t + l_i. \tag{19}$$

The union $R = \bigcup_i R_i$ represents the domain where the guarantees of feasibility are accomplished. The number of regions R_i is depending on the freedom allowed for the family of references. For diminishing the complexity of the explicit formulation (18), only a part of the cutting R can be retained, the one which is critical for the MPC feasibility.

Regarding the explicit implementation of the MPC law, as already mentioned (5), it is expressed as a piecewise linear continuous function:

$$u(\bar{z}_t, x_t) = K_i * \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} + \kappa_i, \quad \text{for } \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} \in D_i\,, \tag{20}$$

or written componentwise:

$$u_t(\bar{z}_t, x_t) = K_{i_x} * x_t + K_{i_z} * \bar{z}_t + \kappa_i. \tag{21}$$

Using these explicit formulations of the RG and MPC blocks, one can conclude that the real time implementation of the predictive scheme with guarantees of feasibility comes to a successive look-up table positioning for the evaluation of the control law at each sampling time (Figure 4.2).

Given these two piecewise linear (PWA) functions, and the fact that their evaluations depend on the depth of the binary search tree for each look-up table, the natural question is whether, the two functions can be compacted in a single control law including both the MPC and the RG mechanism (MPC-RG) and being "everywhere" feasible.

Proposition 4.1 *Let two piecewise linear and continuous functions:*

$$\begin{aligned} &f : R \to D,\ R = \bigcup_{i=1}^{r} R_i, \\ &\qquad f(x) = A_{f_i} x + b_{f_i}\ \forall x \in R_i, \\ &g : D \to F,\ \ D = \bigcup_{j=1}^{d} D_j, \\ &\qquad g(x) = A_{g_j} x + b_{g_j}\ \forall x \in D_j, \end{aligned} \tag{22}$$

then a composed piecewise affine and continuous function exists such that:

$$\begin{aligned} h: R \to F, \quad R = \bigcup_{k=1}^{n_h} DR_k, \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x)),\ \forall x \in DR_k, \end{aligned} \tag{23}$$

with D, R, F, D_i, R_j, DR_k convex sets.

Proof For the existence of the cutting $R = \bigcup_{i=1}^{n_h} DR_i$ it is sufficient to construct for each $R_i, i = 1, .., r$ the subsets:

$$DR_{ij} = \{x|\, x \in R_i \text{ and } f(x) \in D_j\}, j = 1, .., d. \tag{24}$$

From hypothesis $f(R_i) \subset D$ and by retaining only the nonempty subsets of this construction on can obtain:

$$R_i = \bigcup_{j}^{DR_{ij} \neq \emptyset} DR_{ij}. \tag{25}$$

Now, by associating for each DR_{ij} :

$$A_{h_{ij}} \leftarrow A_{g_j} A_{f_i}, \quad b_{h_{ij}} \leftarrow A_{g_j} b_{f_i} + b_{g_j} \tag{26}$$

a finite n_h is obtained such that:

$$\begin{aligned} h: R \to F, \quad R = \bigcup_{k=1}^{n_h} DR_k \equiv \bigcup_{i,j}^{DR_{ij} \neq \emptyset} DR_{ij}, \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x))\ \forall x \in DR_k. \ \square \end{aligned} \tag{27}$$

Remark 4.1 For the resulting function $h(.)$, the number of subsets in the definition domain will satisfy $h_h \leq d * r$ and due to the fact that the evaluation mechanism for $f(.), g(.), h(.)$ is logarithmic in the number of partitions [23], it follows that the complexity of evaluation for $h(.)$ is inferior to the sequential evaluation of $f(.)$ and $g(.)$.

The previous result assures the existence of a compact law (MPC-RG), which inherits the qualities of the RG mechanism and the MPC performances (Figure 4.3). It can be noticed that the intermediary adjusted reference $\bar{r}_k$ and the associated governed reference model state $\bar{z}_k$ needs no evaluation, all this process being nested in the MPC-RG law (Figure 4.4).

Certainly, the on-line evaluation of the control action is optimized by this formulation but one may wonder about the price to be paid. As it was already observed in the literature related to the explicit formulations, the on-line computational gains are obtained by increasing the memory needs. Is the case also for the MPC-RG law which needs to store a much more complex look-up table than the original MPC law. On the contrary, the RG explicit formulation stores the entire explicit solution (19) and not only the first component as it is the case for the MPC. This fact burdens in some extent the complexity of the sequential scheme. The compact MPC-RG law does not suffer from this point of view and thus memory storage disadvantages are mitigated.

The following algorithm resumes the MPC-RG design.

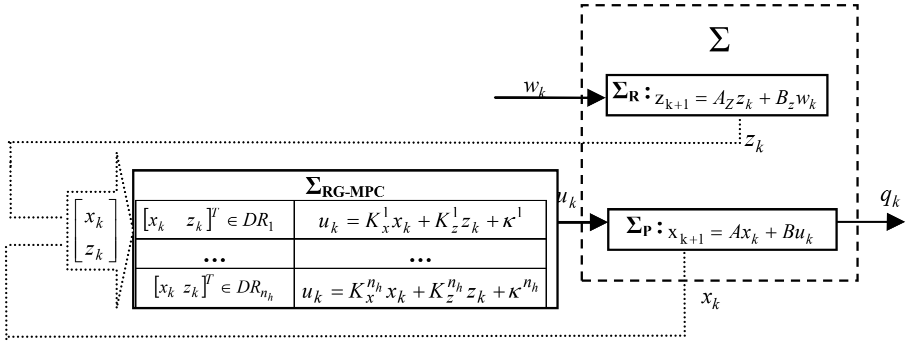

Figure 4.3: MPC with guarantees of feasibility.

Algorithm:

1. Find the explicit MPC for the model-following problem.

$$u = K_x^i x + K_z^i \bar{z} + k^i, \text{ for } \begin{bmatrix} x^T & \bar{z}^T \end{bmatrix}^T \in D_i, i = 1, .., d.$$

2. Determine the MPC feasible set $\tilde{X}_f = \bigcup_{i=1}^{d} D_i$.
3. Construct the explicit form of the RG:

$$\bar{z} = L_x^i x + L_z^i z + \lambda^i, \text{for } \begin{bmatrix} x^T & z^T \end{bmatrix}^T \in R_i, i = 1, .., r.$$

4. Build the compact MPC-RG law:
   For all $i = 1, .., r$
      For all $j = 1, .., d$
         Compute $DR_{ij} = \{x | \, x \in R_i \; et \; f(x) \in D_j\}$
         If DR_{ij} is nondegenerate store it together with:

$$u = \underbrace{(K_x^i + K_z^i L_x^j)}_{K_x^{ij}} x + \underbrace{K_z^i L_z^j}_{K_z^{ij}} z + \underbrace{(k^i + K_z^i \lambda^j)}_{k^{ij}}$$

      end
   end □

Proposition 4.2 *The compact MPC-RG law enjoys the following properties:*

i) *MPC-RG is a piecewise linear and continuous function of the extended state* $\tilde{x}_t = \begin{bmatrix} z_t^T & x_t^T \end{bmatrix}^T$.

ii) *If* $\tilde{x}_t \in \tilde{X}_f$ *then the MPC and MPC-RG are equivalent.*

iii) $y_k \to \hat{r}$ *if* $r_k \to r$ *where* $\hat{r}$ *is the best feasible approximation of* r *with respect to the criterium in (17).*

iv) *If the original MPC law was designed for zero steady error for constant references* r*, the MPC-RG law has a finite settling time to the best feasible approximation* $\hat{r}$.

Proof The property *i)* is a consequence of the fact that the composition of two piecewise linear and continuous functions inherits the same properties.

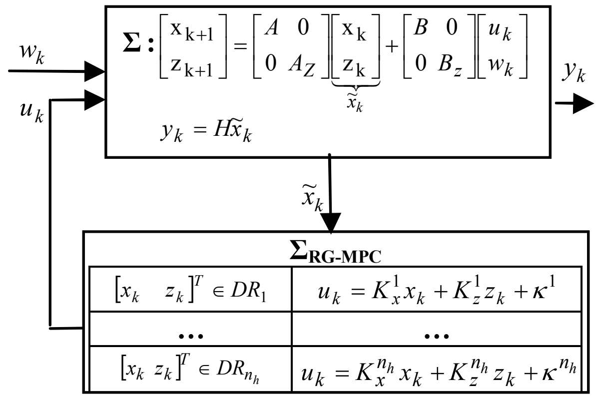

Figure 4.4: Compact MPC law with guarantees of feasibility.

For *ii)* it is sufficient to see that if the condition $\tilde{x}_t \in \tilde{X}_f$ is verified then (17) becomes an unconstrained optimization problem and thus $\bar{z}_t = z_t$ implying the equivalence of the MPC law with the MPC-RG version.

In order to demonstrate *iii*) the continuity of the RG must be taken into account to generate a sequence of references leading to a steady set-point $H_z\tilde{z}_k \to \hat{r}$. Further due to the stabilization properties of the MPC law this position will be regulated $y_k \to \hat{r}$. If the obtained steady output $\hat{r}$ is not corresponding to the best approximation then the optimality of the RG is denied leading to a contradiction.

The point *iv*) is a special case of the former problem. $\square$

5 Example

Consider the discrete version of the double integrator:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k. \end{aligned} \tag{28}$$

For the exemplification of the feasibility analysis, an autonomous reference model is considered at the beginning:

$$\begin{aligned} z_{k+1} &= 0.99 z_k, \\ r_k &= z_k. \end{aligned} \tag{29}$$

The system has to obey a set of constraints:

$$\begin{aligned} &-1 \leq u_k \leq 1, \\ &x_k \in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | -10 \leq x_1, x_2 \leq 10 \right\}. \end{aligned} \tag{30}$$

One can construct based on (28-30) the extended model and develop the MPC law for the regulation case. The first step is to find the optimal control law in the unconstraint case for a infinite cost index as in (2) with $Q = 1, R = 10$:

$$u_k = \begin{bmatrix} 0.23335 & 0.67756 \end{bmatrix} x_k - 0.22664 z_k$$

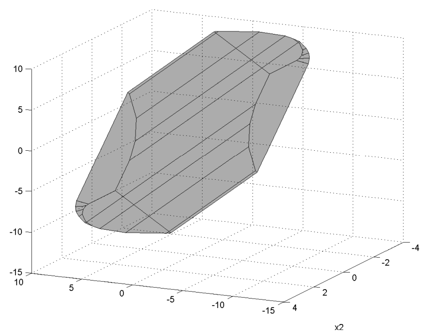

Figure 5.1: The maximal admissible set O_∞.

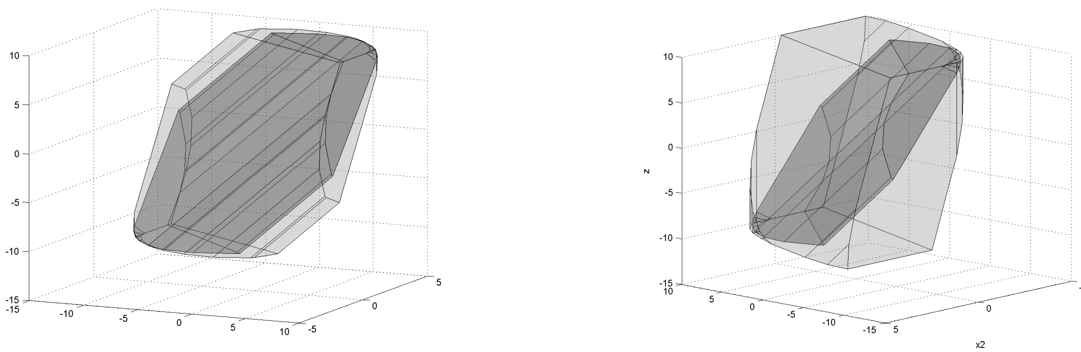

Figure 5.2: a) $\tilde{X}_f$ for $N = 1$, b) $\tilde{X}_f$ for $N = 4$.

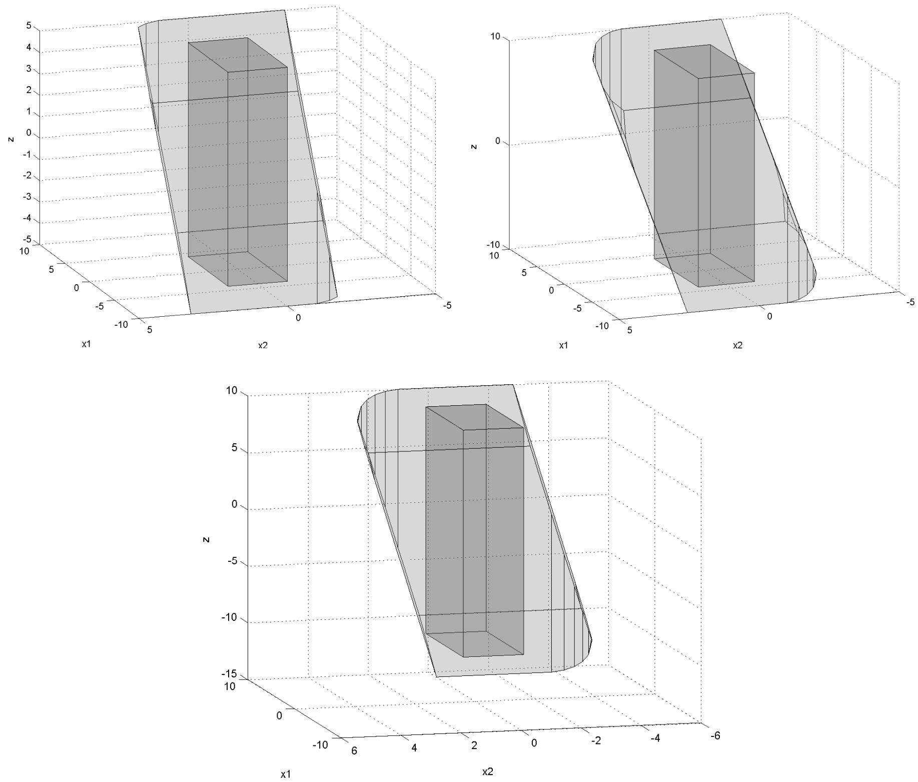

Figure 5.3: Intersection of $\tilde{X}_f$ with the set of initial conditions. a) $N = 4$ and $-5 \leq z \leq 5$ b) $N = 4$ and $-10 \leq z \leq 10$ c) $N = 5$ and $-10 \leq z \leq 10$.

and further find the maximal admissible set O_∞(Figure 5.1).

The predictive law can be synthesized using it as a terminal invariant set. As mentioned in Section 3, the prediction horizon has a decisive influence on the shape of the feasible set for the MPC law. In Figure 5.2, it is presented the feasible set $\tilde{X}_f$ for the MPC laws with prediction horizon $N = 1$ (Figure 5.2a) and $N = 4$ (Figure 5.2b).

The feasibility guarantee has to be given with respect to a set of initial conditions. Choosing:

$$\begin{aligned} &z_0 \in Z_0 = \{ z \in \mathbb{R} | -5 \leq z \leq 5 \}, \\ &x_0 \in X_0 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| -4 \leq x_1 \leq 4; -1 \leq x_2 \leq 1 \right\} \end{aligned}$$

a MPC law with $N = 4$ is guaranteed to be feasible (Figure 5.3a). If the set of initial conditions for the reference model is changed to:

$$z_0 \in Z_0 = \{ z \in \mathbb{R} | -10 \leq z \leq 10 \},$$

the infeasibility is no longer guaranteed (Figure 5.3b). By augmenting the prediction horizon to $N = 5$, the feasibility is retrieved (Figure 5.3c).

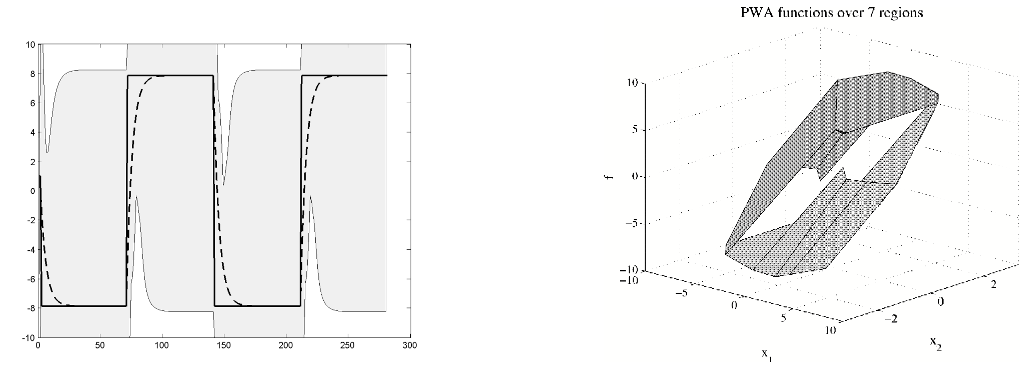

Figure 5.4: a) Time-domain simulation $w_k^{min} \leq w_k \leq w_k^{max}$ (dotted line — r_k); b) The explicit description of the reference limitations.

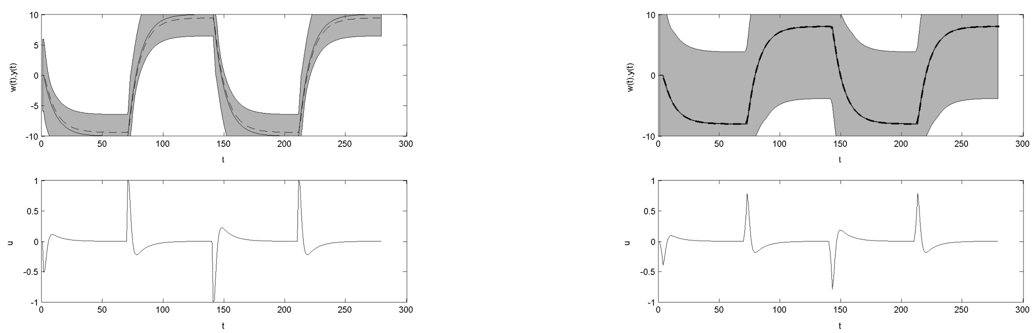

Figure 5.5: Feasible points along a given trajectory: a) $N = 1$, b) $N = 4$.

For the trajectory tracking problem, the following reference model is considered:

$$\begin{aligned} z_{k+1} &= 0.85 z_k + 0.15 w_k, \\ r_k &= z_k. \end{aligned} \tag{31}$$

The MPC law can be synthesized with respect to the set of constraints:

$$\begin{aligned} &-1 \leq u_k \leq 1, \\ &x_k \in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | -10 \leq x_1, x_2 \leq 10 \right\}, \\ &-10 \leq r_k \leq 10. \end{aligned} \tag{32}$$

The exogenous signal will affect the evolution and using the formulations in Table 3.1, one can obtain a time domain simulation of the feasible limitations of w_k (Figure 5.4a) or the explicit solution for $r_k^{min}(x_k)$, $r_k^{max}(x_k)$ (Figure 5.4b).

The prediction horizon plays a decisive role in the feasibility limitations as it can be seen in Figure 5.5.

If the feasible set available is not satisfying the feasibility demands for the tracking problem, then an avoiding redundancy mechanism can be synthesized in terms of a reference governor (RG). In order to give a slight idea about the complexity of the explicit

N	NZ_{MPC}	NZ_{RG}	Combinations to be explored	NZ_{MPC-RG}
1	5	37	185	79
2	23	154	3542	597
3	99	627	62073	3979
4	421	2373	999033	25114

Table 5.1: Explicit formulations.

	ST_{RG+MPC}			ST_{MPC-RG}		
N	Nodes	Depth	Memory(bytes)	Nodes	Depth	Memory(bytes)
1	176+6	8+4	27616+1136	228	8	25402
2	1479+71	11+6	173928 + 8408	1756	12	212963

Table 5.2: Optimal search tree complexity.

solution, one can find in Table 5.1 the number of zones NZ_* for prediction horizons going from 1 to 4.

The column NZ_{MPC} contains the number of domains for the predictive control explicit solution. Their union will form $\tilde{X}_f$ which is further the base for the RG scheme. The column NZ_{RG} presents the number of zones exclusively for the reference avoidance scheme. If these two blocks function independently, then the number of combinations to be explored are represented by the product $NZ_{MPC} * NZ_{RG}$ which is reported in the fourth column of the table. It can be observed that for large prediction horizon this indicator becomes very large and the natural question is whether all of them are representing valid combinations of the extended (reference model + system) state. The answer is given by the compact MPC law with guarantees of feasibility, constructed by the composition of the MPC and RG descriptions. The fifth column in Table 5.1 contains the number of zones NZ_{MPC-RG} for its explicit formulation.

One conclusion appears: the compact $MPC-RG$ scheme is avoiding the exploration of useless combinations but in the same time its number of zones is larger than the number of zones needed for the sequential implementation:

$$NZ_{MPC} + NZ_{RG} \leq NZ_{MPC-RG} \leq NZ_{MPC} * NZ_{RG}.$$

From the point of view of the on-line evaluation, the second inequality is relevant. From the point of view of the memory needed to store the explicit solution, the first inequality mainly who reflects the comparison between the sequential implementation or the compact implementation.

In order to get a better insight on the on-line evaluation vs. memory used compromise, the search tree for the explicit solutions can be constructed and their complexity compared (Table 5.2).

Unfortunately, due to the huge number of zones to be explored, only the $N = 1$ and $N = 2$ are obtainable in reasonable time. However, the results are insightful, and for example for $N = 1$ one can observe that the depth of the search tree for the compact formulation is equivalent with the depth of the RG. This means that in the worst case the RG evaluation is equivalent with MPC-RG evaluation which on its turn is far less expensive that the sequential evaluation of MPC and RG (composed depth of the search

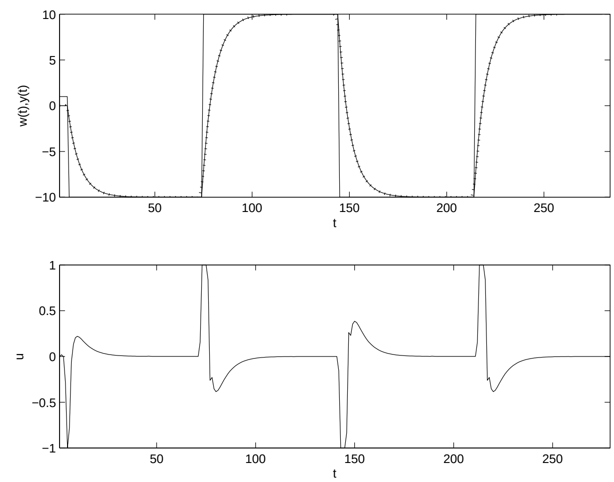

Figure 5.6: Simulation of the evolution under the MPC-RG law. In the upper part, the exogenous signal w, the effective reference signal r and the system output y (dotted). In the lower part the input signal.

tree 12). The price to be paid as already mentioned is transparent in the memory needs. But due to the fact that the RG scheme needs the storage of the complete explicit solution, for $N = 1$ this disadvantage is mitigated and it can be noticed that the sequential scheme is much memory involved that the compact scheme. For $N = 2$ the evaluation mechanism has to deal with a $12 vs. 17$ depth search tree. The memory used for the compact scheme is still comparable with the sequential case. For larger prediction horizons the differences from the memory point of view become more evident. As a conclusion the choice between the compact or the sequential scheme are dependent on the aspiration towards a small on-line evaluation time on one hand and the memory available on the other hand.

x_t^1	x_t^1	z_t	w_t	w_{t+1}	w_{t+2}	w_{t+3}
-9.843	0.078	-10	-10	10	10	10
-8.76	0.578	-10	10	10	10	10
-7.18	1.07	-7.14	10	10	10	10
9.842	-0.078	10	10	-10	-10	-10
8.763	-0.578	10	-10	-10	-10	-10
7.18	-1.07	7.14	-10	-10	-10	-10

Table 5.3: Infeasible combinations for Figure 5.6.

The fact that the infeasibility avoidance scheme is effective can be illustrated by a time domain simulation as the one in Figure 5.6.

The reference is adjusted for the combinations corresponding to the jumps on the reference like the ones enumerated in Table 5.3.

The compact MPC scheme with guarantees of feasibility can prove to be versatile even for extreme reference signals which are outside the operating zone of the constrained system. For example, in Figure 5.7, the reference is bringing the MPC law to infeasibility at almost every sample time. The RG can adjust it to the best admissible value proving the good convergence properties of the scheme.

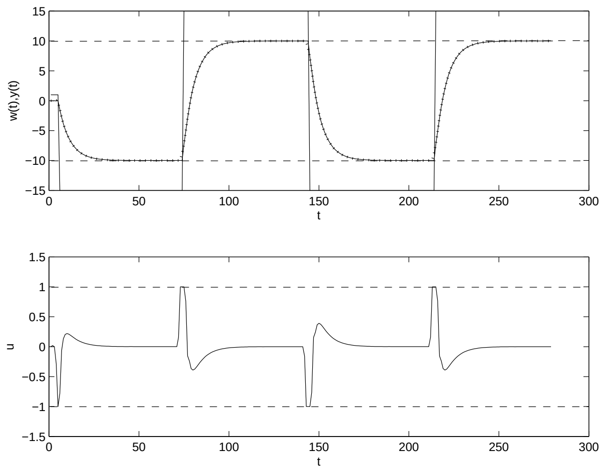

Figure 5.7: Simulation of the evolution under the MPC-RG law. In the upper part, the exogenous signal w, the effective reference signal r and the system output y (dotted). In the lower part the input signal.

6 Conclusions

The feasibility problem within the model predictive control framework was treated with a special attention to the tracking problems. Based on the feasibility results existing for the regulation case, the limitations of the feasible trajectories are established and the infeasible behavior classified.

In order to avoid the infeasibility, a reference adjustment mechanism based on the idea of a "reference governor" was used. Based on the fact that both the MPC and the RG are in fact formulated as multiparametric optimization problems, their explicit formulation in terms of piecewise affine functions was proposed.

The independent implementation of the RG in conjunction MPC law was shown to be not optimal from the point of view of the evaluation mechanism. A compact predictive law with guarantees of feasibility was constructed, optimal from this point of view. Its application might be considered if the memory demands are not overwhelming.

References

[1] Anderson, B. and Moore, J. *Linear Optimal Control*. Prentice Hall, 1971.

[2] Bemporad, A., Morari, M., Dua, V. and Pistikopoulos, E. The explicit linear quadratic regulator for constrained systems. *Automatica* **38** (2002) 3–20.

[3] Bemporad, A. and Mosca, E. Fulfiling hard constraints in uncertain linear systems by reference managing. *Automatica* **34** (1996) 451–461.

[4] Blanchini, F. Set invariance in control. *Automatica* **35** (1999) 1747–1467.

[5] Blanchini, F. and Miani, S. Set based constant reference tracking for Continuous-time Constrained Systems. *Nonlinear Dynamics and Systems Theory* **1**(2) (2001) 121–131.

[6] Camacho, E.F. and Bordons, C. *Model Predictive Control*. Springer, London, 2003.

[7] Casavola, A. Mosca, E. and Papini, M. Control under Constraints: An application of the command governor approach to inverted pendulum. *IEEE Trans. Control System Technology* **12** (2004) 193–204.

[8] Gilbert, E.G. and Tan, K.T. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Trans. Automat. Control* **36** (1991) 1008–1020.

[9] Goodwin, G.C., Seron, M.M. and De Dona, J.A. *Constrained Control and Estimation*. Springer-Verlag, London, 2004.

[10] Kerrigan, E. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control.* PhD thesis, University of Cambridge, 2000.

[11] Kerrigan, E.C. and Maciejowski, J.M. Invariant Sets for Constrained Nonlinear Discrete-time Systems with Application to Feasibility in Model Predictive Control. In *39th IEEE Conference on Decision and Control*, Sydney, 2000.

[12] Kogiso, K. and Hirata, K. A reference governor in a piecewise state affine function: The implementation and validation. In *42nd IEEE Conference on Decision and Control*, Maui, 2003.

[13] Kvasnica, M., Grieder, P. Baotic, M. and Morari, M. *Multi-parametric Toolbox (MPT)*. Available online at: http://control.ee.ethz.ch/ mpt/, 2004.

[14] Limon, D., Alvarado, I., Alamo, T. and Camacho, E.F. MPC for tracking of piece-wise constant references for constrained linear systems. *Proc. 16th IFAC World Congress*, Prague, 2005.

[15] Maciejowski, J.M. *Predictive Control with constraints*. Prentice Hall, Harlow, 2002.

[16] Mayne, D.Q., Rawlings, J.B., Rao, C.V.and Scokaert, P.O.M. Constrained model predictive control: Stability and optimality. *Automatica* **36** (2000) 789–814.

[17] Olaru, S. and Dumur, D. A Parameterized Polyhedra Approach for Explicit Constrained Predictive Control. In *43rd IEEE Conference on Decision and Control*, Bahamas, 2004.

[18] Olaru, S. and Dumur, D. Avoiding Constraints Redundancy in Predictive Control Optimization Routines. *IEEE Trans. Automat. Control* **50** (2005) 1459-1466.

[19] Rossiter, J.A. *Model-Based Predictive Control*. CRC Press, Boca Raton, 2003.

[20] Seron, M.M., Goodwin, G.C. and De Dona, J.A. Characterisation of Receding Horizon Control for Constrained Linear Systems. *Asian Journal of Control* **5** (2002) 271–286.

[21] Scockaert, P.O.M. and Clarke, D. Stability and feasibility in constrained predictive control. In: *Advances in model-based predictive control*. Oxford University Press, 1994a, 217-230.

[22] Scockaert, P.O.M. and Clarke, D. Stabilising properties of constrained predictive control. *IEE Proceedings Control Theory Applications*, **141** (1994b) 295-304.

[23] Tondel, P., Johansen, T.A. and Bemporad, A. Evaluation of piecewise affine control via binary search tree. *Automatica* **39** (2003) 945–950.