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A Study on Stabilization of Nonholonomic Systems Via a Hybrid Control Method

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Abstract: In this paper, we consider a hybrid control strategy for stabilization of nonholonomic systems. In particular, we deal with a typical nonholonomic system, namely a two-wheeled vehicle. We first rewrite the system in a chained form, and then transform it into a nonholonomic integrator (NHI) system. Finally, we apply and modify the hybrid control method for the NHI system, so that the entire system is exponentially stable. We provide a simulation example to demonstrate the effectiveness of the transformation and the control, and give some analysis together with an example for the case where there are constraints on control inputs. We also extend the discussion to the case of four-wheeled vehicles.

Keywords: *Nonholonomic system; two(four)-wheeled vehicle; hybrid control; chained form; nonholonomic integrator; exponential stability/stabilization; switching strategy.*

Mathematics Subject Classification (2000): 93B50, 93B51, 93C10, 93C95, 93D15, 93D20.

1 Introduction

It is known that many mechanical systems are subject to nonholonomic velocity constraints (for example, wheeled mobile robots [2], tractor-trailer (or car-trailor) systems [3], free-floating space [4], etc.), and these constraints can be modelled as symmetrically affine systems [5,6]. Since such nonholonomic systems do not satisfy the so-called Brockett's stabilizability condition [7], they can not be asymptotically stabilized to their equilibrium points by any continuously differentiable, time invariant, state feedback control laws [7,8]. For this reason, there have been a large quantity of works on the stabilization

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problem of nonholonomic systems in the last two decades, including the efforts of finding continuous, time varying control laws [9,10], discontinuous ones [8,11,12] and middle strategies (discontinuous and time varying) [13,14].

In this paper, we consider a hybrid control strategy for this challenging problem. In particular, we deal with a typical nonholonomic system, namely a two-wheeled vehicle. The significant difference from the abovementioned existing work is that we seek the possibility of achieving exponential stabilization for such a nonholonomic system. For this purpose, we first rewrite the model of the two-wheeled vehicle in a chained form, and then transform it into a nonholonomic integrator (NHI) system [7]. Finally, we apply and modify the hybrid control method, which was originally proposed in [1], for the NHI system. We demonstrate by a simulation example the effectiveness of the transformation and the control. After that, we discuss the case where there are constraints on the control inputs and propose using the idea of bounded functions in the hybrid control method. We also extend our discussion to the case of four-wheeled vehicles. By choosing an alternative control input, we can reduce the stabilization of four-wheeled vehicles to the same control problem as for two-wheeled vehicles, and thus can apply the same approach.

The remainder of this paper is organized as follows. In Section 2, we describe the system of a two-wheeled vehicle and then transform it into an NHI system. In Section 3, we present the hybrid control strategy and the simulation result, and give two important remarks, which concern the switching time interval and the method of dealing with singularities. Section 4 considers the case where there exist constraints on the control inputs, and Section 5 discusses the extension to the case of four-wheeled vehicles. Finally, we give some concluding remarks in Section 6.

2 System Description and Transformation

We deal with a two-wheeled vehicle as depicted in Figure 2.1, which is known as a typical nonholonomic system. Let (x, y) denote the position of the vehicle, let θ be the angle with respect to the x-axis and let $\bar{v}_1$ be the velocity of the vehicle in its body direction. If we view $\bar{v}_1 = u_1$, $\dot{\theta} = u_2$ as control inputs, we obtain the vehicle's system described by

$$\left\{ \begin{array}{rcl} \dot{x} &=& u_1 \cos\theta, \\ \dot{y} &=& u_1 \sin\theta, \\ \dot{\theta} &=& u_2. \end{array} \right. \tag{1}$$

Note that this is a three-dimensional symmetrically affine system with two control inputs.

In this paper, we propose transforming the system (1) into a chained form, and then transforming the chained form into an NHI system. More precisely, we first let $\mu_1 = u_1 \cos\theta$, $\mu_2 = u_2$ to rewrite (1) as

$$\left\{ \begin{array}{rcl} \dot{x} &=& \mu_1, \\ \dot{y} &=& \mu_1 \tan\theta, \\ \dot{\theta} &=& \mu_2. \end{array} \right. \tag{2}$$

In (2), we let $z_1 = x$, $z_2 = \tan\theta$, $z_3 = y$, $v_1 = \mu_1$, $v_2 = (\sec^2\theta)\mu_2$ to obtain

$$\left\{ \begin{array}{rcl} \dot{z}_1 &=& v_1, \\ \dot{z}_2 &=& v_2, \\ \dot{z}_3 &=& z_2 v_1, \end{array} \right. \tag{3}$$

which is a chained form.

Next, we apply the idea in [6] to transform the chained form (3) into an NHI system. More precisely, if we define the new variables

$$x_1 = z_1\,, \quad x_2 = z_2\,, \quad x_3 = -2z_3 + z_1 z_2\,, \tag{4}$$

then the NHI system

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{x}_2 = v_2, \\ \dot{x}_3 = x_1 v_2 - x_2 v_1, \end{cases} \tag{5}$$

is obtained from (3) easily.

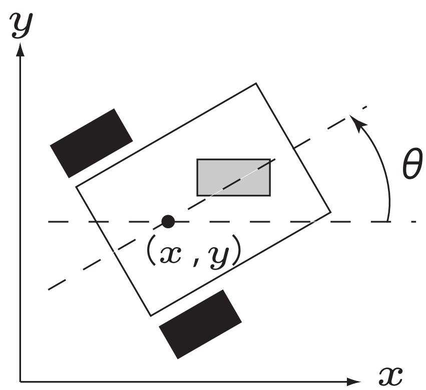

Figure 2.1: A two-wheeled vehicle.

It is not difficult to obtain the relation between (x, y, θ) in (1) and (x_1, x_2, x_3) in (5) as

$$x = x_1\,, \quad y = \frac{-x_3 + x_1 x_2}{2}\,, \quad \theta = \tan^{-1} x_2 \tag{6}$$

and the relation between (u_1, u_2) in (1) and (v_1, v_2) in (5) as

$$u_1 = \frac{v_1}{\cos\theta}\,, \qquad u_2 = v_2 \cos^2\theta\,. \tag{7}$$

These relations imply that if we can design a controller $v = [v_1 \;\; v_2]^T$ to make the NHI system (5) asymptotically/exponentially stable, then the controller u computed by (7) stabilizes the original nonholonomic system (1) asymptotically/exponentially.

3 Hybrid Control and Simulation

Since the control problem has been reduced to stabilizing the NHI system (5), we propose applying the hybrid control method in [1]. Define the functions

$$\pi_1(w) = 0.5(1 - e^{-\sqrt{w}}), \quad \pi_2(w) = 1.7\pi_1(w), \quad \pi_3(w) = 2.5\pi_1(w), \quad \pi_4(w) = 4\pi_1(w), \tag{8}$$

and the overlapping regions

$$\begin{aligned} R_1 &= \left\{ x \in R^3 : 0 \leq x_1^2 + x_2^2 \leq \pi_2(x_3^2) \right\}, \\ R_2 &= \left\{ x \in R^3 : \pi_1(x_3^2) \leq x_1^2 + x_2^2 \leq \pi_4(x_3^2) \right\}, \\ R_3 &= \left\{ x \in R^3 : \pi_3(x_3^2) \leq x_1^2 + x_2^2 \right\}, \\ R_4 &= \{0\}. \end{aligned} \tag{9}$$

Then, we define the control strategy

$$v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T = g_\sigma(x), \tag{10}$$

where σ is a piecewise constant switching signal, which is continuous from the right at every point and takes value on a finite set $J = \{1, 2, 3, 4\}$, and

$$\begin{aligned} & g_1(x) \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g_2(x) \triangleq \begin{bmatrix} x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \\ & g_3(x) \triangleq \begin{bmatrix} -x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \quad g_4(x) \triangleq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

The switching signal σ is defined recursively by

$$\begin{aligned} \sigma &= \phi(x, \sigma^-), \\ \phi(x, j) &= \begin{cases} j, & \text{if } x \in R_j \\ \max\{i \in J : x \in R_i\}, & \text{if } x \notin R_j. \end{cases} \end{aligned} \tag{12}$$

Therefore, the above control strategy is a hybrid control which is composed of four continuous-time controllers and a state-dependent switching law. It has been shown in [1] that exponential stability is obtained for the NHI system (5) by using the above hybrid control method. Therefore, as explained before, the original nonholonomic system (1) is also exponentially stabilized by the hybrid controller defined by (7) and (10)-(12).

The simulation results are described in Figures 3.1 – 3.5, where the initial state is $x_1 = 0.25$, $x_2 = 0.15$, $x_3 = 1.0$. Figure 3.1 describes how the switching signal changes with $w_1 = x_3^2$, $w_2 = x_1^2 + x_2^2$. Figure 3.2 and Figure 3.3 respectively show that both the NHI system (5) and the original system (1) are exponentially stable. Figure 3.4 and Figure 3.5 depict the switchings in control inputs.

In the end of this section, we give two important remarks concerning the discussion in this section.

First, as also pointed out in [1], the time interval between consecutive switchings in the switching law is bounded away from zero, not only on any finite time interval but also as time goes to infinity. Therefore, chattering phenomena will not happen. Here, we give more precise description, though similar to that appeared in [1], so that the readers can follow the design precept.

Let $\bar{t}$ denote any time instant at which σ switches from Mode 2 (Controller 2) to Mode 3 (Controller 3). Then one must have

$$w_2(\bar{t}) = \pi_4(w_1(\bar{t})). \tag{13}$$

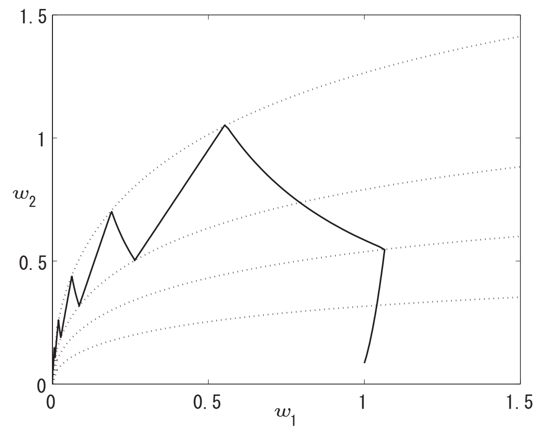

Figure 3.1: Switchings.

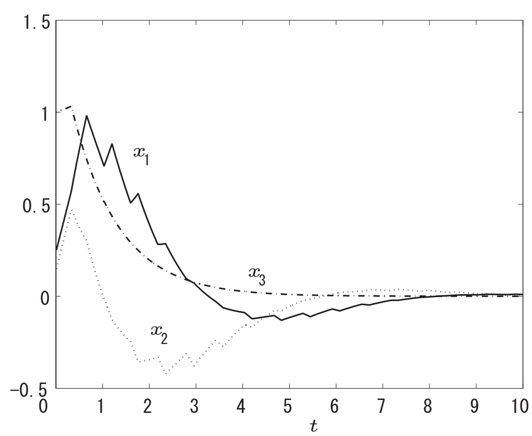

Figure 3.2: The states of the NHI (5).

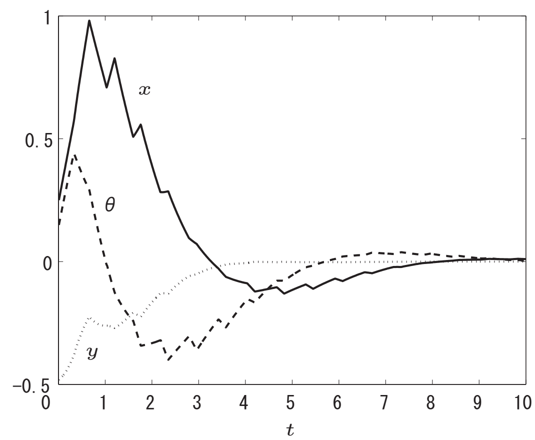

Figure 3.3: The states of the the vehicle (1).

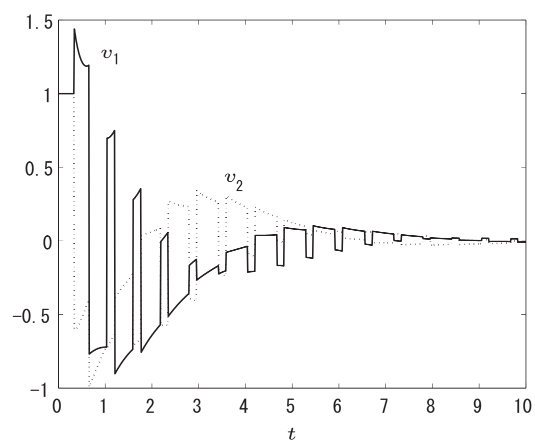

Figure 3.4: The control inputs of the NHI (5).

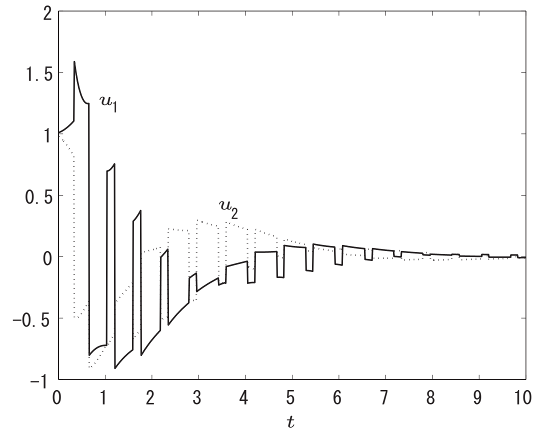

Figure 3.5: The control inputs of the vehicle (1).

Suppose that σ switches back to Mode 2 after some time interval Δt, which implies that

$$w_2(\bar{t} + \Delta t) = \pi_3(w_1(\bar{t} + \Delta t))\,. \tag{14}$$

Since $\dot{w_2} = -2w_2$ on the interval $[\bar{t}, \bar{t} + \Delta t)$, we obtain

$$w_2(\bar{t} + \Delta t) = w_2(\bar{t})e^{-2\Delta t}\,, \tag{15}$$

and thus

$$\Delta t = \frac{1}{2}\log\frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t} + \Delta t))}\,. \tag{16}$$

Noting the fact that w_1 is decreasing for all $t \geq \bar{t}$ and π_3 is monotone nondecreasing, which leads to $\pi_3(w_1(\bar{t} + \Delta t)) \leq \pi_3(w_1(\bar{t}))$, we conclude that

$$\Delta t \geq \frac{1}{2}\log\frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t}))}\,. \tag{17}$$

Therefore, we can adjust the switching time interval by choosing the ratio between π_4 and π_3 (we used the ratio $\frac{4}{2.5} = 1.6$ in (8)). For example, if we desire $\Delta t \geq 2$, then we choose $\pi_4(w_1) = \exp(4)\pi_3(w_1)$. In this case, the switchings are done as described in Figure 3.6.

Next, we give a remark on the system transformation from the original system (1) to the NHI system (5). In (2) or (7), it is easy to understand that we can't obtain the original control input u in singular points such as $\theta = \pm\frac{\pi}{2}$ (which means that the vehicle is located towards the vertical direction). To say it in other words, the consideration for the NHI system (5) does not cover the case of $\theta = \pm\frac{\pi}{2}$. To overcome this difficulty, we can use an alternative transformation which doesn't result in singular points. That is, we let $\mu_1 = u_2$, $\mu_2 = u_1$ to rewrite (1) as

$$\dot{x} = \mu_2\cos\theta, \qquad \dot{y} = \mu_2\sin\theta, \qquad \dot{\theta} = \mu_1. \tag{18}$$

Then, we let $z_1 = \theta$, $z_2 = -x\cos\theta - y\sin\theta$, $z_3 = -x\sin\theta + y\cos\theta$, $v_1 = \mu_1$, $v_2 = (x\sin\theta - y\cos\theta)\mu_1 - \mu_2$ in (18) to obtain

$$\dot{z}_1 = v_1, \qquad \dot{z}_2 = v_2, \qquad \dot{z}_3 = z_2 v_1, \tag{19}$$

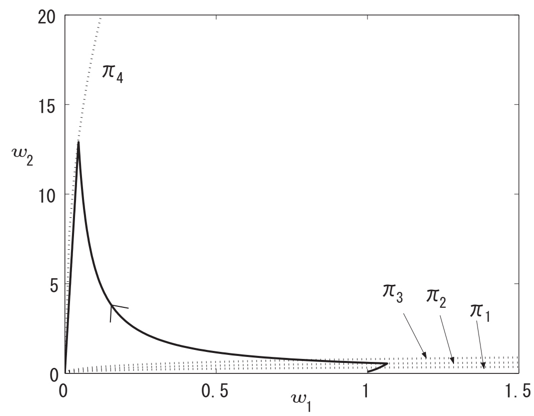

Figure 3.6: Switchings when $\pi_4(w_1) = \exp(4)\pi_3(w_1)$.

which is also a chained form. The remaining discussion is the same as before.

4 Constrained Control Input

In this section, we give some analysis and simulation in the case where there exist constraints on the control inputs. Ref. [15] considered the asymptotic stabilization problem for nonholonomic mobile robots under constraints on control inputs, but it is found that the convergence rate is very slow there (only asymptotic stability is guaranteed there). Here, we suggest using the bounded function proposed in [15] for the hybrid controller (11).

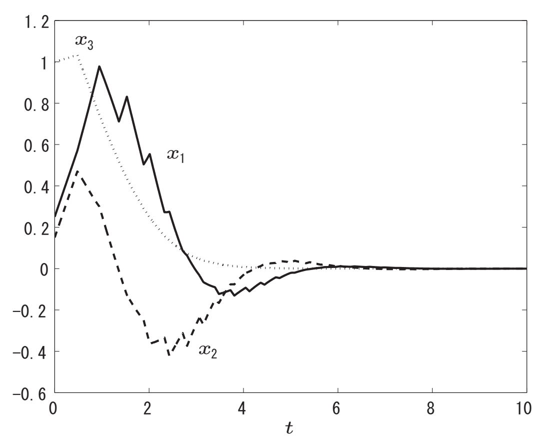

Figure 4.1: The states in the saturated case ($x_1(0) = 0.25$, $x_2(0) = 0.15$, $x_3(0) = 1.0$).

Suppose that due to physical environment and/or actuator capability limitation, we need imposing certain constraints on the control inputs. For simplicity, we consider here

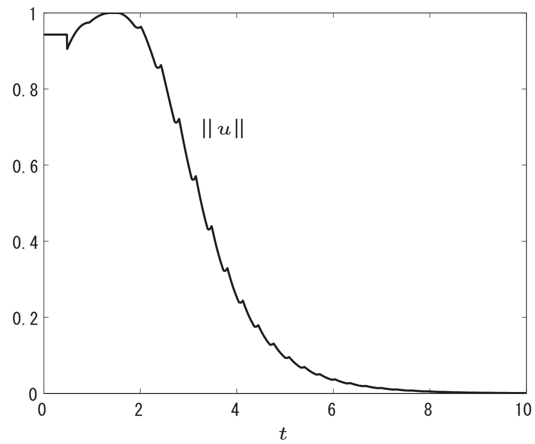

Figure 4.2: The norm of the control input in the saturated case ($x_1(0) = 0.25$, $x_2(0) = 0.15$, $x_3(0) = 1.0$).

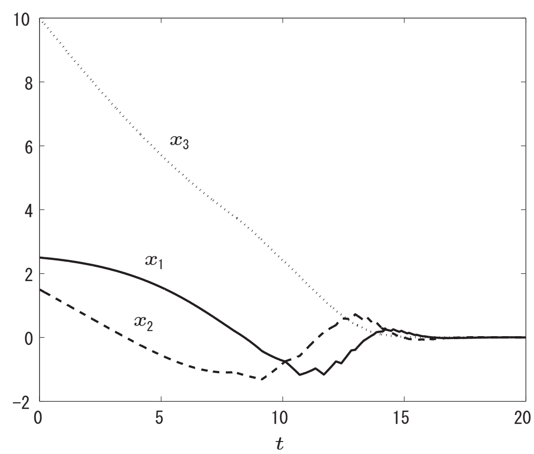

Figure 4.3: The states in the saturated case ($x_1(0) = 2.5$, $x_2(0) = 1.5$, $x_3(0) = 10$).

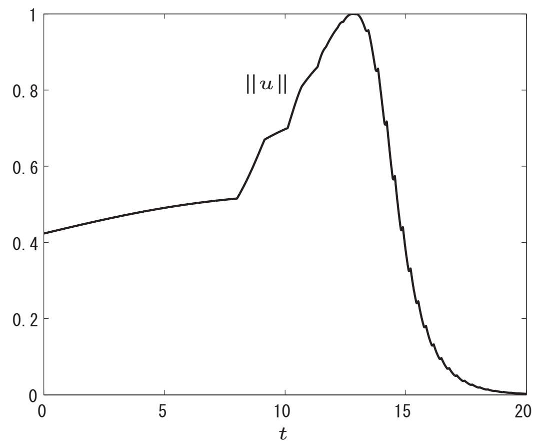

Figure 4.4: The norm of the control input in the saturated case ($x_1(0) = 2.5$, $x_2(0) = 1.5$, $x_3(0) = 10$).

the case where the constraints can be imposed directly on the NHI system (5) as

$$\left\| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\| \leq r\,, \tag{20}$$

where r is a positive scalar indicating the constraint bound. Then, utilizing the idea of bounded function in the control input vector (11), we propose the new controller candidates

$$\bar{g}_i = \frac{2r}{1 + g_i^T g_i} g_i\,, \quad i = 1, 2, 3, 4 \tag{21}$$

instead of (11). Note that the constraints are satisfied with the above controllers since

$$\bar{g}_i^T \bar{g}_i = \frac{4r^2}{(1 + g_i^T g_i)^2} g_i^T g_i \leq r^2\,. \tag{22}$$

Now, we consider the same system with constrained controller under $r = 1$. The simulation result with the same initial state ($x_1 = 0.25$, $x_2 = 0.15$, $x_3 = 1.0$) is shown in Figure 4.1 and Figure 4.2. Figure 4.1 tells that the system is also exponentially stabilized, and Figure 4.2 tells that the constraint on the control inputs is satisfied.

Since the controller switchings depend on the initial state significantly, we increase the initial state to $x_1 = 2.5$, $x_2 = 1.5$, $x_3 = 10$. Then, the simulation result is shown in Figure 4.3 and Figure 4.4. We see that we have also obtained desired exponential stability under the constrained control inputs.

5 Extension to Four-Wheeled Vehicles

In this section, we extend our consideration to the case of four-wheeled vehicles, which are depicted in Figure 5.1.

We let (x, y) and (x_f, y_f) be the coordinates of the middle point of the rear tire axle and that of the front tire, respectively, and let L be the length from (x, y) to (x_f, y_f).

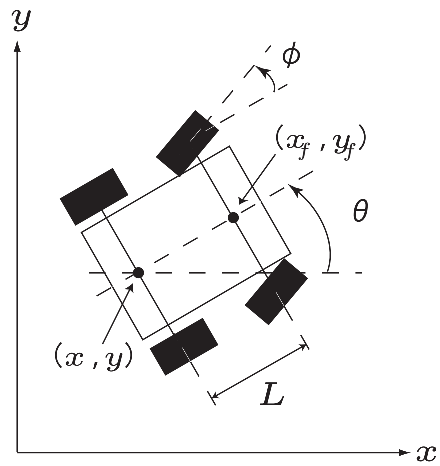

Figure 5.1: A four-wheeled vehicle.

Define θ and $\bar{v}_1$ as in the two-wheeled vehicle, and let ϕ be the angle with respect to its body direction. If we view $\bar{v}_1 = u_1$, $\dot{\phi} = u_2$ as control inputs, we obtain the vehicle's system described by

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = u_1 \dfrac{\tan\phi}{L}, \\ \dot{\phi} = u_2. \end{cases} \tag{23}$$

Note that this is a four-dimensional symmetrically affine system with two inputs. We can transform this system into four-dimensional chained form. However, we find that it is hard to apply the aforementioned control method since the obtained chained form can not be transformed further into the NHI system. For this reason, we propose choosing the control inputs as $\bar{v}_1 = u_1$, $\bar{v}_1 \tan\phi = u_2$, and rewrite the vehicle's system as

$$\begin{cases} \dot{x} = u_1 \cos\theta, \\ \dot{y} = u_1 \sin\theta, \\ \dot{\theta} = \dfrac{u_2}{L}. \end{cases} \tag{24}$$

Since (24) is a three-dimensional symmetrically affine system with two inputs, we can use the same approach as in Section II to transform this system into an NHI system, and then apply the aforementioned hybrid control strategy for the system. Note that the relation between (x, y, θ) in (24) and (x_1, x_2, x_3) in (5) is the same as in the case of two-wheeled vehicles, and the relation between (u_1, u_2) in (24) and (v_1, v_2) in (5) is

$$u_1 = \frac{v_1}{\cos\theta}, \qquad u_2 = v_2 L \cos^2\theta\,. \tag{25}$$

6 Concluding Remarks

We have considered a hybrid control strategy for stabilization of a class of nonholonomic systems, namely two(four)-wheeled vehicle systems. We first rewrite the system in a chained form, and then transform it into a nonholonomic integrator (NHI) system. Finally, we have applied the hybrid control method proposed in [1] for the obtained NHI system. The key point is that the transformations are returnable and the switching time interval can be adjusted easily. We have shown that it is possible to extend the results to the case involving constrained control inputs.

Future research includes the hybrid control for extended NHI forms (for example, those with even dimension) and for robust performance of nonholonomic systems.

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