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A Successive Approximation Algorithm to Optimal Feedback Control of Time-varying LPV State-delayed Systems

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Abstract: In this paper problem of finite-time optimal state feedback control for a class of time-varying linear parameter-varying (LPV) systems with a known delay in the state vector under quadratic cost functional is investigated via a successive approximation algorithm. The method of successive approximation algorithm results an iterative scheme, which successively improves any initial control law ultimately converging to the optimal state feedback control. On the other hand, by manipulating linear matrix inequalities imposed by Generalized-Hamiltonian-Jacobi-Bellman method and the polynomially parameter-dependent quadratic (PPDQ) functions, sufficient conditions with high precision are given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay.

Keywords: *Linear parameter-varying systems; time-delay; successive approximation algorithm; optimal control.*

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1 Introduction

The investigation of the optimal control is of importance in modern control theory. The theory and the application of optimal control for linear time invariant (LTI) systems have been developed perfectly. For the convenient implementation, many suboptimal control methods have risen which do not pursue the optimal control performance indexes. In the literature, some computational methods were stated to solve finite-time optimal control problem of the LTI systems, time varying systems, second order linear systems, singular perturbed systems, nonlinear systems with quadratic cost functions [15, 22, 28-30, 37].

Over the last three decades, considerable attention has been paid to robustness analysis and control of linear systems affected by structured real parameters. Linear parameter-varying (LPV) systems have gained a lot of interest as they provide a systematic means of computing gain-scheduled controllers, especially those related to vehicle and aerospace control [2-6, 9, 18, 21].

Generally speaking, a LPV system is a linear system in which the system matrices are fixed functions of a known parameter vector. A LPV system can be viewed as a nonlinear system that is linearized along a trajectory determined by the parameter vector. Hence, the parameter vector of an LPV system corresponds to the operating point of the non-linear system. In the LPV framework, it is assumed that the parameter vector is measurable for control process. In many industrial applications, like flight control and process control, the operating point can indeed be determined from measurement, making the LPV approach viable, see for example [7, 32, 36, 39]. Concerning unknown parameter vector, an adaptive method has been presented for robust stabilization with performance of LPV systems in [27].

For LPV systems, establishing stability via the use of constant Lyapunov functions is conservative. To investigate the stability of LPV systems one needs to resort the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability, see [10, 12-14, 16, 17, 19, 33, 43]. Bliman proposed the problem of robust stability for LPV systems with scalar parameters in [13]. Also, he developed some conditions for robust stability in terms of solvability of some linear matrix inequalities (LMIs) without conservatism. Moreover, the existence of a polynomially parameter-dependent quadratic (PPDQ) Lyapunov function for systems, which are robustly stable, is investigated in [14]. However, as for LPV systems, synthesis problems that are solved by classic control theory lead to difficult computations. People have studied optimal control of LPV systems for decades.

On the other hand, time delays are often present in engineering systems, which have been generally regarded as a main source on instability and poor performance [11, 31]. Therefore, the stabilization of LPV state-delayed systems is a field of intense research [38-41, 44]. Generally, a way to ensure stability robustness with respect to the uncertainty in the delays is to employ stability criteria valid for any nonnegative value of the delays that is delay-independent results. This assumption that no information on the value of the delay is known is often coarse in practice. Recently, systematic ways of the use of PPDQ functions in the state feedback control and output feedback control for LPV systems with time-delay in the state vector were proposed in [23-26]. It was shown that the PPDQ Lyapunov-Krasovskii functions make some sufficient conditions to investigate robust stability analysis of LPV systems in LMIs.

In this paper, we provide a systematic way to finite-time state feedback control problem for time-varying LPV systems with a constant delay in the state vector under quadratic cost functional via a successive approximation algorithm (SAA). This paper is essentially an extension of the SAA of the linear and nonlinear systems presented in [8] to the optimal control problem of the time-varying LPV state-delayed systems. The method of SAA results an iterative scheme, which successively improves any initial control law ultimately converging to the optimal state feedback control. On the other hand, by manipulating LMIs imposed by Generalized-Hamiltonian-Jacobi-Bellman (GHJB) method and the PPDQ functions, sufficient conditions with high precision are given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay.

The notations used throughout the paper are fairly standard. The matrices I_n , 0_n and $0_{n\times p}$ are the identity matrix, the $n \times n$ and $n \times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes} = 1$, $M^{p\otimes} := M^{(p-1)\otimes} \otimes M$. Let $\hat{J}_k, \tilde{J}_k \in \Re^{k\times(k+1)}$, and $v^{[k]}$ be defined by $\hat{J}_k := [I_k, \quad 0_{k\times 1}]$, $\tilde{J}_k := [0_{k\times 1}, \quad I_k]$ and $v^{[k]} = [1, v, \dots, v^{k-1}]^T$, respectively, which have essential roles for polynomial manipulations [11]. Finally given a signal $x(t)$,$\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; i.e., $\|x(t)\|_2^2 = \int_0^\infty x^T(t)x(t)\,dt$.

2 Problem Description

Consider in the following a class of time-varying LPV state-delayed system

$$\begin{cases} \dot{x}(t) = A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u(t), \\ x(t) = \phi(t), \quad t \in [-h, 0], \end{cases} \tag{1}$$

where the constant parameter h is time delay and $\phi(t)$ is the continuous vector valued initial function, also $x(t) \in \Re^n$ and $u(t) \in \Re^l$ are the state vector and the control input, respectively. Moreover, the parameter-dependent matrices $A(t;\rho)$, $A_d(t;\rho)$ and $B(t;\rho)$ are expressed as

$$[A(t;\rho)\ A_d(t;\rho)\ B(t;\rho)] = [A_0(t)\ A_{0d}(t)\ B_0(t)] + \sum_{i=1}^{m} \rho_i [A_i(t)\ A_{id}(t)\ B_i(t)],$$

where $A_0(t), \cdots, A_m(t), A_{0d}(t), \cdots, A_{md}(t)$ and $B_0(t), \cdots, B_m(t)$ are known constant matrices of appropriate dimensions. Furthermore, it is known that the vector $\rho = [\rho_1, \rho_2, \cdots, \rho_m] \in \Re^m$ is contained in a priori given set whereas the actual curve of the vector ρ is unknown but can be measured online for control process. In the sequel, we make the following definitions for the system (1).

Definition 2.1 A finite-time state feedback $u(t) = -K(t;\rho)x(t)$ for $t \in [0, T]$ with $K(t;\rho) \in \Re^{m\times n}$ is said to achieve global asymptotic stability of the system (1) if the closed-loop system

$$\dot{x}(t) = (A(t;\rho) - B(t;\rho)K(t;\rho))x(t) + A_d(t;\rho)x(t-h) \tag{2}$$

is globally asymptotic stable in the Lyapunov sense.

According to Definition 2.1, the main objective of the paper is to develop an iterative technique for finite-time optimal control problem of the time-varying LPV state-delayed system (1), which minimizes the following cost functional with respect to some $u^*(t;\rho)$:

$$J = \|x(T)\|_{Q_0}^2 + \int_0^T \|x(t)\|_Q^2 + \|u(t)\|_R^2\ dt. \tag{3}$$

Definition 2.2 A polynomially parameter-dependent quadratic (PPDQ) function is said to any quadratic function $x^T(t)S(\rho)x(t)$ such $S(\rho)$ is defined as

$$S(\rho) := (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T S_k (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n) \tag{4}$$

for a certain $S_k \in \Re^{(k^m n)\times(k^m n)}$. The integer $k-1$ is called the degree of the PPDQ function of $S(\rho)$.

3 Finite-Time Optimal Control Problem

Before deriving the main results, a preliminary Lemma is reviewed.

Lemma 3.1 *(Schur Complement lemma) Given constant matrices Ψ_1, Ψ_2 and Ψ_3 where $\Psi_1 = \Psi_1^T$ and $\Psi_2 = \Psi_2^T > 0$, then $\Psi_1 + \Psi_3^T \Psi_2^{-1} \Psi_3 < 0$ if and only if*

$$\begin{bmatrix} \Psi_1 & \Psi_3^T \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0 \quad \textit{or equivalently}, \quad \begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^T & \Psi_1 \end{bmatrix} < 0.$$

In the literature, extensions of the Lyapunov method to the Lyapunov-Krasovskii method have been proposed for time-delayed systems [11, 31]. To investigate the delay-independent asymptotically stability analysis of the closed-loop system (2), we define in the following a class of PPDQ Lyapunov-Krasovskii functions of the degree $k-1$

$$V(x(t);\rho) = x^T(t)P_\rho(t)x(t) + \int_{t-h}^{t} x^T(\sigma)Q_\rho(\sigma)x(\sigma)\,d\sigma, \tag{5}$$

where the positive-definite matrices $P_\rho(t) := P(t;\rho) \in \Re^{n\times n}$ and $Q_\rho(t) := Q(t;\rho) \in \Re^{n\times n}$ with the following forms

$$P_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k(t) (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n), \tag{6}$$

$$Q_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Q_k(t) (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n), \tag{7}$$

where the positive-definite matrices $\{P_k(t), Q_k(t)\} \in \Re^{(k^m n)\times(k^m n)}$ are to be determined.

Definition 3.1 Given an admissible control $u(t;\rho)$, which ensures the asymptotic stability of the closed-loop system (2). The function $V(x(t);\rho)$ in (5) satisfies the Generalized-Hamiltonian-Jacobi-Bellman (GHJB) inequality, written $GHJB(V,u) < 0$, if

$$\frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}(A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u(t)) + \|x(t)\|_Q^2 + \|u(t)\|_R^2 < 0, \tag{8}$$

where $V(T,x) = \|x(T)\|_{Q_0}^2$.

Remark 3.1 Generally, the Hamiltonian-Jacobi equation being nonlinear is very difficult to solve. Recently, a new approach for solving the Hamiltonian-Jacobi equation for a fairly large class of Hamiltonian systems has been studied in [1].

To improve the performance of an arbitrary control $u^{(0)}$ we minimize the following function

$$u^{(1)} = arg \min_{u\in A_J(D)} \{ \frac{\partial V}{\partial t} + \frac{\partial V^T}{\partial x}(A(t;\rho)x(t) + A_d(t;\rho)x(t-h) + B(t;\rho)u^{(0)})$$

$$+\|x(t)\|_Q^2+\left\|u^{(0)}\right\|_R^2\}=\frac{-1}{2}R^{-1}B^T(t;\rho)\frac{\partial V^{(0)}}{\partial x}$$
$$=-R^{-1}B^T(t;\rho)P_\rho^{(0)}(t)x(t), \qquad (9)$$

where $D := [0,T]\times\Omega$, and Ω is a compact set of $\Re^n$ containing a ball around the origin and $A_J(D)$ is the set of admissible controls. In the infinite-time case, the initial control law is required to be stabilizing for the SAA to converge. For the finite-time this is not the case; in particular we may chose $u^{(0)}=0$. However, $u^{(0)}$ provides a degree of freedom which maybe judiciously chosen to speed the convergence of the algorithm.

The cost of $u^{(1)}$ is given by the solution of the equation $GHJB(V^{(1)},u^{(1)})<0$. In [34], it has been shown that $V^{(1)}(t,x)\le V^{(0)}(t,x)$ for each $(t,x)\in D$ and the convergence does not get stuck in local minimum, i.e., if $V^{(i+1)}(t,x)=V^{(i)}(t,x)$ for a fixed i, then $V^i(t,x)=V^*(t,x)$. Based on this fact, we assume that a unique optimal control u^* exists and is an admissible control. Then the optimal cost is given by the solution to the GHJB inequality, i.e.,

$$\frac{\partial V^*}{\partial t}+\frac{\partial V^{*T}}{\partial x}(A(t;\rho)x(t)+A_d(t;\rho)x(t-h)+B(t;\rho)u^*(t))+\|x(t)\|_Q^2+\|u^*(t)\|_R^2<0. \qquad (10)$$

From the solution to the GHJB inequality (10) we obtain an optimal control law as

$$u^*(t)=\frac{-1}{2}R^{-1}B^T(t;\rho)\frac{\partial V^*}{\partial x}$$
$$:= -K(t;\rho)x(t),\quad t\in[0,T], \qquad (11)$$

where $K(t;\rho)=R^{-1}B^T(t;\rho)P_\rho^*(t)$ and the optimal cost is

$$J(x(0),u^*)=\phi^T(0)P_\rho^*(0)\phi(0).$$

Remark 3.2 For the finite-time version of the problem, there is generally a unique solution to GHJB (under appropriate conditions), which brings up the question of obtaining the solution relevant to the infinite-time problem as the limit of the unique solution of the finite-time one. This question is investigated in [42] for nonlinear systems affine in the control and the disturbance, and with a cost function quadratic in the control, where the control is not restricted to lie in a compact set. It establishes the existence of a well-defined limit, and also obtains a result on global asymptotic stability of closed-loop system under the H_∞ controller and the corresponding worst-case disturbance.

Noting to the expressions (5), (10) and (11), we find

$$x^T(t)(\dot P_\rho(t)+A^T(t;\rho)P_\rho(t)+P_\rho(t)A(t;\rho)-P_\rho(t)B(t;\rho)R^{-1}B^T(t;\rho)P_\rho(t)+Q_\rho(t)+Q)x(t)$$
$$-x^T(t-h)Q_\rho(t)x(t-h)+x^T(t)P_\rho(t)A_d(t;\rho)x(t-h)+(A_d(t;\rho)x(t-h))^TP_\rho(t)x(t)<0. \qquad (12)$$

Then, the aforementioned inequality is rewritten as

$$X^T(t)M_\rho(t)X(t)<0, \qquad (13)$$

where the new vector $X(t)=[x^T(t),x^T(t-h)]^T$ is an augmented state and the parameter-dependent matrix $M_\rho(t)$ is defined as

$$M_\rho(t)=\begin{bmatrix}\tilde\Sigma_{11} & P_\rho(t)A_{d\rho}(t)\\ * & -Q_\rho(t)\end{bmatrix}, \qquad (14)$$

where

$$\tilde{\Sigma}_{11} = \dot{P}_\rho(t) + A_\rho^T(t)P_\rho(t) + P_\rho(t)A_\rho(t) - P_\rho(t)B_\rho(t)R^{-1}B_\rho^T(t)P_\rho(t) + Q_\rho(t) + Q.$$

Remark 3.3 Stability of the time-varying LPV state-delayed system (1) can be provided by finding the positive-definite solutions $P_\rho(t)$ and $Q_\rho(t)$ to the associated parameter-dependent matrix inequality $M_\rho(t) < 0$.

In the iterative step, we assume a non-singular solution and that $V^{(i)}(x(t);\rho)$ has the form

$$V^{(i)}(x(t);\rho) = x^T(t)P_\rho^{(i)}(t)x(t) + \int_{t-h}^{t} x^T(\sigma)Q_\rho^{(i)}(\sigma)x(\sigma)\,d\sigma \tag{15}$$

and we let the new control be

$$u^{(i)}(t;\rho) := -K^{(i)}(t;\rho)x(t) = -R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t)x(t). \tag{16}$$

By substituting $V^{(i)}(x(t);\rho)$ and the control (16) into (10) and using Schur Complement Lemma, the following parameter-dependent LMI is easily obtained

$$\begin{bmatrix} \hat{\Sigma}_{11}^{(i)} & P_\rho^{(i-1)}(t)B_\rho(t) & P_\rho^{(i)}(t)A_{d\rho}(t) \\ * & -R & 0 \\ * & * & -Q_\rho^{(i)}(t) \end{bmatrix} < 0, \tag{17}$$

where

$$\hat{\Sigma}_{11}^{(i)} = \dot{P}_\rho^{(i)}(t) + P_\rho^{(i)}(t)(A_\rho(t) - B_\rho(t)R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t))$$
$$+(A_\rho(t) - B_\rho(t)R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t))^T P_\rho^{(i)}(t) + Q_\rho^{(i)}(t) + Q.$$

Remark 3.4 A general framework for relaxing parameter-dependent LMI problems into parameter-independent LMIs (conventional form) has been investigated in [5]. However, application of the PPDQ Lyapunov functions as a new tool for relaxing parameter dependency of the matrix inequalities will be stated in the next section.

4 Parameter-Dependent LMI Relaxations

In this section the PPDQ functions as the basis functions are used to relax parameter-dependent LMIs into conventional parameter-independent LMI problems by utilizing some positives-definite Lagrange multiplier matrices (see for instance [24,24]).

Lemma 4.1 *Let the degree of the PPDQ Lyapunov function $P_\rho^{(i)}(t)$ be $k-1$. The parameter-dependent matrix $P_\rho^{(i)}(t)B_\rho(t)$ can be represented as*

$$P_\rho^{(i)}(t)\,B_\rho(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T H_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_l), \tag{18}$$

where the matrix $H_k^{(i)}(t) \in \Re^{((k+1)^m n)\times((k+1)^m l)}$ which depends linearly on the matrix $P_k^{(i)}(t)$ is defined as

$$H_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes B_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t)). \tag{19}$$

Proof According to the structures of the parameter-dependent matrices $P_\rho^{(i)}(t)$ and $B_\rho(t)$, one has

$$P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k^{(i)}(t)(\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)(B_0(t) + \sum_{i=1}^{m} \rho_i B_i(t))$$

and using the property of $(\vartheta^{[k]} \otimes I_n)B_i(t) = (I_k \otimes B_i(t))(\vartheta^{[k]} \otimes I_l)$ one finds

$$P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k^{(i)}(t)(I_{k^m} \otimes (B_0(t) + \sum_{i=1}^{m} \rho_i B_i(t)))(\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l)$$

or

$$\begin{aligned} P_\rho^{(i)}(t)\,B_\rho(t) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_k^{(i)}(t)((I_{k^m} \otimes B_0(t))(\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l) \\ +(I_{k^m} \otimes B_1(t))(\rho_m^{[k]} \otimes \ldots \otimes \rho_1 \rho_1^{[k]} \otimes I_l) + \cdots \\ +(I_{k^m} \otimes B_m(t))(\rho_m \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l)), \end{aligned}$$

then by repeated use of the properties $\hat{J}_k \rho_i^{[k+1]} = \rho_i^{[k]}$ and $\tilde{J}_k \rho_i^{[k+1]} = \rho_i \rho_i^{[k]}$ the matrix $H_k^{(i)}(t)$ in (19) is obtained. □

According to Lemma 4.1 for the matrix $A_{d\rho}(t)$, we have:

$$P_\rho^{(i)}(t)\,A_{d\rho}(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n), \quad (20)$$

where the matrix $S_k^{(i)}(t)$ is expressed in the form

$$S_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes A_{0d}(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_{id}(t)). \quad (21)$$

Therefore, the PPDQ Lyapunov function of degree k for the positive-definite matrix $R_\rho^{(i)}(t) = A_\rho^T(t)P_\rho^{(i)}(t) + P_\rho^{(i)}(t)A_\rho(t)$ is written as

$$R_\rho^{(i)}(t) := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T R_k^{(i)}(t)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n) \quad (22)$$

and from Lemma 4.1 the matrix $R_k^{(i)}(t)$ in (22), which depends linearly on the matrix $P_k^{(i)}(t)$ is obtained as follows:

$$\begin{aligned} R_k^{(i)}(t) = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t)) \\ +(\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t))^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n). \end{aligned} \quad (23)$$

Similarly, the constant positive-definite matrices $R \in \Re^{l\times l}$, and $Q \in \Re^{n\times n}$ can be represented as

$$R = (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_l)^T (\hat{J}_k^{m\otimes} \otimes I_l)^T \bar{R}_k (\hat{J}_k^{m\otimes} \otimes I_l)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_l), \quad (24)$$

and

$$Q = (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T (\hat{J}_k^{m\otimes} \otimes I_n)^T \bar{Q}_k (\hat{J}_k^{m\otimes} \otimes I_n)(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n), \quad (25)$$

where the certain matrices $\bar{R}_k$ and $\bar{Q}_k$ are defined, respectively, as

$$\bar{R}_k = diag\,(R, \underbrace{0_l, \cdots, 0_l}_{(k^m - 1)\ elements}),$$

and

$$\bar{Q}_k = diag\,(Q, \underbrace{0_n, \cdots, 0_n}_{(k^m - 1)\ elements}).$$

We are now in the position to state our main result in the following Theorem.

Theorem 4.1 *For a given positive parameter k if there exist positive-definite matrices $P_k^{(i)}(t)$, $Q_k^{(i)}(t)$ and the set of positive definite Lagrange multipliers $\hat{Q}_{i,k}^{(1)}(t), \hat{Q}_{i,k}^{(2)}(t)$ and $\hat{Q}_{i,k}^{(3)}(t)$ for $i = 1, 2, \cdots, m$ to the following parameter-independent differential linear matrix inequality (DLMI),*

$$\begin{bmatrix} \Sigma_{11} & H_k^{(i-1)}(t) & S_k^{(i)}(t) \\ * & \Sigma_{22} & 0 \\ * & * & \Sigma_{33} \end{bmatrix} < 0, \quad (26)$$

where

$$\Sigma_{11} = (\hat{J}_k^{m\otimes} \otimes I_n)^T \dot{P}_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n) + \hat{R}_k^{(i)}(t)$$
$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T (Q_k^{(i)}(t) + \bar{Q}_k)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$
$$\Sigma_{22} = -(\hat{J}_k^{m\otimes} \otimes I_l)^T \bar{R}_k (\hat{J}_k^{m\otimes} \otimes I_l)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l}),$$

and

$$\Sigma_{33} = -(\hat{J}_k^{m\otimes} \otimes I_n)^T Q_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$

with

$$\hat{R}_k^{(i)}(t) = \{\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t) - (\hat{J}_k^{m\otimes} \otimes B_0(t)$$

$$+ \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t))R^{-1}H_k^{(i-1)^T}(t)\}^T P_k^{(i)}(t)(\hat{J}_k^{m\otimes} \otimes I_n)$$

$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)}(t)\{\hat{J}_k^{m\otimes} \otimes A_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i(t)$$

$$-(\hat{J}_k^{m\otimes} \otimes B_0(t) + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i(t))R^{-1}H_k^{(i-1)^T}(t)\},$$

then the parameter-dependent finite-time state feedback control

$$u^{(i)}(t;\rho) = -R^{-1}B_\rho^T(t)P_\rho^{(i-1)}(t)x(t), \quad t \in [0,T] \tag{27}$$

achieves global asymptotic stability for the time-varying LPV state-delayed system (1) with the quadratic cost function (3).

Proof By substituting the relations (18)-(25) into the parameter-dependent LMI (17), one parameter-dependent matrix inequality is obtained which includes left- and right-multiplication of the (26) by

$$\begin{bmatrix} \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n & 0 & 0 \\ * & \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_l & 0 \\ * & * & \rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n \end{bmatrix},$$

and its transpose. Then, it can be concluded that the LMI (26), which included the positive-definite Lagrange multipliers $\hat{Q}_{1,k}^{(1)}, \cdots, \hat{Q}_{m,k}^{(1)}, \hat{Q}_{1,k}^{(2)}, \cdots, \hat{Q}_{m,k}^{(2)}$ and $\hat{Q}_{1,k}^{(3)}, \cdots, \hat{Q}_{m,k}^{(3)}$, is a sufficient condition to fulfil the parameter-dependent matrix inequality (17) for any vector ρ contained in a priori given set. □

It is essential in this result that the matrices $P_k^{(i)}(t)$ and $Q_k^{(i)}(t)$ are calculated independently from the parameter vector ρ and thereafter $P_\rho^{(i)}(t)$, $Q_\rho^{(i)}(t)$ and the control law are found analytically by (6), (7) and (27), respectively.

Remark 4.1 The solution to the DLMI in (26) can be obtained by discretizing the time interval $[0,T]$ into equally spaced time instances $\{t_j, j = 1, \cdots, N, t_N = T, t_0 = 0\}$ [35], where

$$t_j - t_{j-1} := \kappa = N^{-1}T, \quad j = 1, \cdots, N.$$

The discretized DLMI problem thus becomes one of finding, at each $\kappa \in [1,N]$, $P_k^{(i)_{j-1}}(:= P_k^{(i)}(t_{j-1}))$ that satisfies

$$\begin{bmatrix} \hat{\Sigma}_{11} & H_k^{(i-1)_j} & S_k^{(i)_j} \\ * & \hat{\Sigma}_{22} & 0 \\ * & * & \hat{\Sigma}_{33} \end{bmatrix} < 0 \tag{28}$$

with $P_k^{(i)_N} = Q_0$ and

$$\hat{\Sigma}_{11} = (\hat{J}_k^{m\otimes} \otimes I_n)^T(-P_k^{(i)_{j-1}} + P_k^{(i)_j})(\hat{J}_k^{m\otimes} \otimes I_n) + \kappa \hat{R}_k^{(i)_j}(t)$$
$$+\kappa(\hat{J}_k^{m\otimes} \otimes I_n)^T(Q_k^{(i)_j}(t) + \bar{Q}_k)(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\kappa \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\kappa \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(1)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$
$$\hat{\Sigma}_{22} = -\kappa^{-1}(\hat{J}_k^{m\otimes} \otimes I_l)^T \bar{R}_k(\hat{J}_k^{m\otimes} \otimes I_l)$$
$$+\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}l})$$
$$-\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l})^T \hat{Q}_{i,k}^{(2)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}l}),$$
$$\hat{\Sigma}_{33} = -\kappa^{-1}(\hat{J}_k^{m\otimes} \otimes I_n)^T Q_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)}(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n})$$
$$-\kappa^{-1} \sum_{i=1}^{m}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n})^T \hat{Q}_{i,k}^{(3)}(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}),$$
$$H_k^{(i-1)_j} := H_k^{(i-1)}(t_j)$$
$$= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i-1)_j}(\hat{J}_k^{m\otimes} \otimes B_0^j + \sum_{l=1}^{m} \hat{J}_k^{(m-l)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(l-1)\otimes} \otimes B_i^j),$$

and

$$S_k^{(i)_j} := S_k^{(i)}(t_j)$$
$$= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes A_{0d}^j + \sum_{l=1}^{m} \hat{J}_k^{(m-l)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(l-1)\otimes} \otimes A_{ld}^{(j)}),$$

with

$$\hat{R}_k^{(i)_j} := \hat{R}_k^{(i)}(t_j)$$
$$= \{\hat{J}_k^{m\otimes} \otimes A_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i^j - (\hat{J}_k^{m\otimes} \otimes B_0^j$$
$$+ \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i^j)R^{-1} H_k^{(i-1)_j^T}\}^T P_k^{(i)_j}(\hat{J}_k^{m\otimes} \otimes I_n)$$
$$+(\hat{J}_k^{m\otimes} \otimes I_n)^T P_k^{(i)_j}\{\hat{J}_k^{m\otimes} \otimes A_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i^j$$
$$-(\hat{J}_k^{m\otimes} \otimes B_0^j + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_i^j)R^{-1} H_k^{(i-1)_j^T}\},$$

where $A_i^j := A_i(t_j)$, $A_{ld}^j := A_{ld}(t_j)$ and $B_i^j := B_i(t_j)$.

Remark 4.2 From the DLMI (26) in Theorem 4.1, it can be concluded that a unique positive-definite solution to (8) exists, then $V^{(0)}(x(t);\rho) \geq V^{(1)}(x(t);\rho) \geq \cdots \geq V^{*}(x(t);\rho)$ with equality holding if and only if $V^{(i)}(x(t);\rho) \equiv V^{*}(x(t);\rho)$. Furthermore $V^{(i)}(x(t);\rho) \rightarrow V^{*}(x(t);\rho)$ and $u^{(i)}(x(t);\rho) \rightarrow u^{*}(x(t);\rho)$ pointwise for all $x(t), \rho$ and $t \in [0,T]$.

Remark 4.3 It is observed that the discretized DLMI (28) is linear in $P_k^{(i)_j}$, $Q_k^{(i)_j}, \hat{Q}_{1,k}^{(1)}, \cdots, \hat{Q}_{m,k}^{(1)}, \hat{Q}_{1,k}^{(2)}, \cdots, \hat{Q}_{m,k}^{(2)}$ and $\hat{Q}_{1,k}^{(3)}, \cdots, \hat{Q}_{m,k}^{(3)}$ thus the standard LMI techniques, [20], can be exploited to find the positive-definite solutions. It is also seen from the above results that the choice of appropriate parameter $k-1$ as the degree of the PPDQ Lyapunov functions of the matrix $P_k^{(i)}(t)$ and $Q_k^{(i)}(t)$ play the role of freedom of design in the control law.

5 Conclusion

A successive approximation algorithm was used to generate the finite-time optimal feedback gains for a class of time-varying LPV state-delayed systems under quadratic cost functional. The method of SAA was developed, which successively improves any initial control law ultimately converging to the optimal state feedback control. By manipulating LMIs imposed by Generalized-Hamiltonian-Jacobi-Bellman method and the PPDQ functions, sufficient conditions with high precision were given to guarantee asymptotic stability of the time-varying LPV state-delayed systems independent of the time delay. In this paper, the results are presented on the delay-independent stability conditions case, and the extension of the results to delay-dependent stability conditions is a topic currently under study.

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