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Estimations of Solutions Convergence of Hybrid Systems Consisting of Linear Equations with Delay

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Abstract: The logic-dynamical hybrid system given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay is investigated in the paper. The estimations of disturbances of such system are obtained. We consider the cases of stable and unstable subsystems. Besides the estimations of solutions of hybrid system given by a set of scalar subsystems are obtained.

Keywords: *Hybrid system; differential-difference equation; Lyapunov-Krasovsky functional; stable system*

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1 Introduction

Nowadays the disturbances in hybrid systems dynamic is an actual research problem [2,9]. Since in different branches such as medicine, ecology, construction of control systems, the state at a given moment in time essentially depends on the previous history, more adequate instrument for researching the dynamic of separate subsystems is formed by equations with delay [4-6].

Let the logic-dynamical system be given by a set of subsystems which are linear differential-difference equations with constant coefficients and constant delay

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau), \quad i = \overline{1,n}, \quad x(t) \in R^n, \quad t \geq 0. \tag{1}$$

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Each of these subsystems describes the dynamics on a fixed finite time interval $t_{i-1} \leq t < t_i$, $i = \overline{1, N}$, $t_0 = 0$. Subsystems can be stable or unstable. We suppose, the initial disturbance is in δ-vicinity of the origin. It is required to estimate the size of the deviation of solutions $x(t)$ of the logic-dynamical system (1) from the origin at the final moment $t = t_N$. We consider finite time intervals, and at switching times coordinates have no discontinuity, i.e.

$$\lim_{s \to +0} x(t_i - s) = \lim_{s \to +0} x(t_i + s), \quad i = \overline{1,\, N-1}, \tag{2}$$

and on separate time intervals the subsystems are systems of linear differential-difference equations such that, by virtue of a continuity, all solutions which start from δ-vicinity do not leave $\varepsilon(\delta)$-vicinity. On the contrary, for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that $|x(t_N)| < \varepsilon$, if $\|x(0)\|_\tau < \delta(\varepsilon)$. In the paper the mentioned values are calculated. Special attention is given to the case of unstable subsystems. Here and further the following vector and matrix norms are used

$$|A| = \left\{\lambda_{\max}(A^T A\right\}^{1/2},$$

$$|x(t)| = \left\{\sum_{i=1}^{n} x_i^2(t)\right\}^{1/2},$$

$$\|x(t)\|_\tau = \max_{-\tau \leq s \leq 0} \left\{|x(s+t)|\right\},$$

$$\|x(t)\|_{\tau,\beta} = \left\{\int_{-\tau}^{0} e^{\beta s} \left|x(t+s)\right|^2 ds\right\}^{1/2},$$

$\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ are the largest and smallest eigenvalues of the corresponding symmetric, positive definite matrices.

For the derivation of estimations the method of Lyapunov-Krasovsky functionals [7–9] is used.

Research of such type of logic-dynamic systems has been carried out earlier. In [10] the logic-dynamical system consisting of linear differential equations subsystems was examined. The method of quadratic Lyapunov functions was used. The Lyapunov's functions were built as non-autonomous quadratic forms $V(x,t) = x^T H(t) x$, $H(t) = e^{-tA^T} e^{-tA}$ by using a first integral. This kind of Lyapunov function allows to derive the most exact estimations of solutions, as level surfaces $V_i(x,t) = \alpha_i$, $i = \overline{1,\, N-1}$ of Lyapunov functions $V_i(x,t)$, $i = \overline{1,\, N-1}$, completely consisting of integral curves. However, the construction of such functions is connected with the presence of a matrix exponential e^{tA}, i.e. with the presence of a fundamental matrix of solutions. That is a strong condition.

In [11] it has been proposed to use autonomous Lyapunov functions with symmetric, positive definite matrices H_i, $i = \overline{1,\, N-1}$ which are calculated using a solution of the matrix Lyapunov equations

$$A_i^T H_i + H_i A_i = C_i$$

for $i = \overline{1,\, N-1}$. However this requires the asymptotic stability of matrices A_i, $i = \overline{1,\, N-1}$. Finally, in [12] estimations of disturbances of logic-dynamical system (1) without the requirement of asymptotic stability of matrices A_i, $i = \overline{1,\, N-1}$ has been obtained.

2 Estimations of solutions of stable subsystems

We'll first obtain some auxiliary results. We investigate the behavior of the solution $x(t)$ of a linear stationary subsystem with delay

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{3}$$

determined on an interval $t_0 \le t \le t_1$. For obtaining an estimation of solutions we use a functional of the form

$$V[x(t),t] = e^{\gamma t}\left\{ x^T(t)Hx(t) + \int_{-\tau}^{0} e^{\beta s} x^T(t+s)Gx(t+s)ds \right\}. \tag{4}$$

Let's denote

$$\begin{gathered}
\varphi_{11}(H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}, \quad \varphi_{12}(G,H) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)}, \\
\varphi_{21}(G,H) = \frac{\lambda_{\max}(H)}{\lambda_{\min}(G)}, \quad \varphi_{22}(G) = \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)} \quad , \\
S[G,H] = \begin{bmatrix} -A^T H - HA - G & -HB \\ -B^T H & G \end{bmatrix}.
\end{gathered} \tag{5}$$

The following statement holds.

Theorem 2.1 *Let there exist positive definite matrices G and H for which the matrix $S[G,H]$ is also positive definite. Then the system (3) is asymptotic stable and for its solutions $x(t)$ it follows the top exponential estimations of convergence hold:*

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_{\tau,\beta} \right] \exp\left\{ -\frac{1}{2}\varsigma t \right\}, \quad t \ge 0, \tag{6}$$

and

$$\|x(t)\|_{\tau,\beta} \le \left[\sqrt{\varphi_{21}\,(G,H)}\,|x(0)| + \sqrt{\varphi_{22}(G)}\,\|x(0)\|_{\tau,\beta} \right] \exp\left\{ -\frac{1}{2}\varsigma t \right\}, \quad t \ge 0 \tag{7}$$

for

$$\varsigma(\beta,\gamma) = \min\left\{ \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)},\ \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} \right] \right\}. \tag{8}$$

The value $\beta \ge 0$ can be arbitrary for

$$\lambda_{\min}(S[G,H]) \ge \lambda_{\max}(G).$$

And

$$\beta \le \frac{1}{\tau}\ln\left\{ \frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}(S[G,H])} \right\},$$

if

$$\lambda_{\min}(S[G,H]) < \lambda_{\max}(H).$$

The value γ satisfies a condition $\gamma \le \beta$.

Proof For the proof we use the Lyapunov-Krasovsky functional of the form (4) with positive definite matrices G and H. It satisfies the following bilateral estimations:

$$\begin{aligned} e^{\gamma t}\left\{\lambda_{\min}(H)|x(t)|^2+\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} &\le V[x(t),t] \\ &\le e^{\gamma t}\left\{\lambda_{\max}(H)|x(t)|^2+\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} \end{aligned} \tag{9}$$

We find an estimation for its derivative in force of system (3). We make a substitution $t+s=\xi$. Then the functional transforms to

$$V[x(t),t]=e^{\gamma t}\left\{x^T(t)Hx(t)+\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)\,d\xi\right\}. \tag{10}$$

We calculate a full derivative of the transformed functional (10) along solutions $x(t)$ of system (3). We obtain

$$\begin{aligned} \frac{d}{dt}V[x(t),t] = \gamma e^{\gamma t}&\left\{x^T(t)Hx(t)+\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\} \\ &+e^{\gamma t}\big\{\left[Ax(t)+Bx(t-\tau)\right]^T Hx(t)+x^T(t)H\left[Ax(t)+Bx(t-\tau)\right] \\ &+x^T(t)Gx(t)-e^{-\beta\tau}x^T(t-\tau)Gx(t-\tau)\big\} \\ &-e^{\gamma t}\left\{\beta\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\}. \end{aligned}$$

We transform the obtained expression as follows:

$$\begin{aligned} \frac{d}{dt}V[x(t),t] = -e^{\gamma t}&\left\{(\beta-\gamma)\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)d\xi\right\} \\ &-e^{\gamma t}\left(x^T(t),x^T(t-\tau)\right)\begin{bmatrix}-A^TH-HA-G & -HB\\ -B^TH & G\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix} \\ &+\gamma e^{\gamma t}x^T(t)Hx(t)+e^{\gamma t}\big(1-e^{-\beta\tau}\big)x^T(t-\tau)Gx(t-\tau). \end{aligned} \tag{11}$$

We suppose, as follows from the conditions of Theorem 1, there are positive definite matrices G and H for which the matrix $S[G,H]$ is also positive definite and $\beta\ge\gamma\ge 0$. Then we obtain

$$\begin{aligned} \frac{d}{dt}V[x(t),t] \le -e^{\gamma t}&\lambda_{\min}\big(S[G,H]\big)\big(|x(t)|^2+|x(t-\tau)|^2\big) \\ &+e^{\gamma t}\gamma\lambda_{\max}(H)|x(t)|^2+e^{\gamma t}\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\left|x(t-\tau)\right|^2 \\ &-e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}. \end{aligned}$$

Let's transform the obtained expression as follows

$$\begin{aligned} \frac{d}{dt}V[x(t),t] \le -e^{\gamma t}&\{\lambda_{\min}(S[G,H])-\gamma\lambda_{\max}(H)\}|x(t)|^2 \\ &-e^{\gamma t}\left\{\lambda_{\min}(S[G,H])-\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\right\}\left|x(t-\tau)\right|^2 \\ &-e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}. \end{aligned} \tag{12}$$

If the parameters of system and functional are

$$\lambda_{\min}(S[G,H]) \geq \lambda_{\max}(G)$$

then from inequality (12) it follows, that

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\left\{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right\}|x(t)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta} \tag{13}$$

for any $\beta \geq 0$. If

$$\lambda_{\min}(S[G,H]) < \lambda_{\max}(G),$$

then inequality (13) will be used for

$$0 \leq \beta < \frac{1}{\tau}\ln\left[\frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}(S[G,H])}\right].$$

We transform the right part of the inequality of quadratic forms (9) as

$$-e^{\gamma t}\lambda_{\max}(H)|x(t)|^2 - e^{\gamma t}\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta} \leq -V[x(t),t]\,. \tag{14}$$

Let's consider two cases.

1. Let's transform the inequality (14) as

$$-e^{\gamma t}|x(t)|^2 \leq -\frac{1}{\lambda_{\max}(H)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\left\|x(t)\right\|^2_{\tau,\beta}$$

and we substitute it in the first part of the inequalities (13). We obtain

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}V\left[x(t,t)\right]$$
$$- e^{\gamma t}\left\{(\beta-\gamma)\lambda_{\min}(G) - \left[\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\right\}\left\|x(t)\right\|^2_{\tau,\beta}\,.$$

If the parameters are

$$(\beta-\gamma)\lambda_{\min}(G) \geq \left[\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\,, \tag{15}$$

then

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}V[x(t),t].$$

Solving the obtained differential inequality, we get

$$V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \frac{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H)}{\lambda_{\max}(H)}, \quad t \geq 0. \tag{16}$$

From here

$$\zeta = \alpha + \gamma = \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)}\,.$$

2. We transform inequality (14) to the following form

$$-e^{\gamma t}\left\|x(t)\right\|^2_{\tau,\beta} \leq -\frac{1}{\lambda_{\max}(G)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}|x(t)|^2$$

and again we substitute it in the second part of the inequalities (13). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V[x(t),t] - e^{\gamma t}\left\{\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}\right\}|x(t)|^2.$$

And if parameters are such that

$$\lambda_{\min}(S[G,H]) - \gamma\lambda_{\max}(H) - (\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} > 0, \tag{17}$$

then

$$\frac{d}{dt}V\left[x(t)\right] \le -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V\left[x(t)\right].$$

Having integrated the obtained expression, we get

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = (\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \quad t \ge 0. \tag{18}$$

We get

$$\zeta = \alpha+\gamma = \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right].$$

For obtaining the required result we return to bilateral estimations of Lyapunov–Krasovsky functional (9). Using expressions (16), (18), we write down

$$e^{\gamma t}\left\{\lambda_{\min}(H)|x(t)|^2 + \lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\right\} \le V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t} \le e^{-\alpha t}\left\{\lambda_{\max}(H)\left|x(0)\right|^2 + \lambda_{\max}(G)\left\|x(0)\right\|^2_{\tau,\beta}\right\}.$$

It is possible to obtain two estimations. First, we get

$$|x(t)|^2 \le \left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(H)}\left|x(0)\right|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(H)}\left\|x(0)\right\|^2_{\tau,\beta}\right]e^{-(\alpha+\gamma)t}.$$

And, using denotations $\varphi_{11}(H)$, $\varphi_{12}(G,H)$, we obtain

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\left|x(0)\right| + \sqrt{\varphi_{12}(G,H)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t \ge 0.$$

Further it is possible to write down

$$\|x(t)\|^2_{\tau,\beta} \le \left[\frac{\lambda_{\max}(H)}{\lambda_{\min}(G)}\left|x(0)\right|^2 + \frac{\lambda_{\max}(G)}{\lambda_{\min}(G)}\left\|x(0)\right\|^2_{\tau,\beta}\right]e^{-(\alpha+\gamma)t}.$$

And, using designations $\varphi_{21}(G,H), \varphi_{22}(G)$, we obtain an inequality

$$\|x(t)\|_{\tau,\beta} \le \left[\sqrt{\varphi_{21}\left(G,H\right)}\left|x(0)\right| + \sqrt{\varphi_{22}(G)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \ge 0.$$

As follows from consideration of both cases we have

$$\varsigma = \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} \quad \text{for} \quad \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] \ge \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} \tag{19}$$

$$\varsigma = \frac{\beta\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] \quad \text{for} \quad \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right] < \frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}H)}. \tag{20}$$

Uniting these expressions, we obtain the statement of Theorem 2.1. □

3 Estimations of solutions of unstable subsystems

We consider a case where it is not possible to find matrices G and H for which the matrix $S[G,H]$ is positive definite. Let's denote

$$S[G,H,\gamma] = \begin{bmatrix} -A^T H - HA - \gamma H - G & -HB \\ -B^T H & G \end{bmatrix}. \tag{21}$$

Obviously, due to the choice of a scalar value $\gamma < 0$ the matrix $S[G,H,\gamma]$ can be made positive definite.

Lemma 3.1 *Let the matrices G, H be positive definite and let the following inequality hold*

$$\gamma < \frac{\lambda_{\min}\left[-A^T H - HA - G - HBG^{-1}B^T H\right]}{\lambda_{\max}(H)}. \tag{22}$$

Then the matrix $S[G,H,\gamma]$ is also positive definite.

Proof We introduce a vector $z^T(t,\tau) = \left(x^T(t), x^T(t-\tau)\right)$. The condition of positive definiteness of matrix $S[G,H,\gamma]$ is equivalent to positiveness of the minimal eigenvalue

$$\lambda_{\min}\left[S\left(G,H\right)\right] = \min_{|z|=1}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} > 0,$$

or to the condition

$$\min_{x(t-\tau)}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} > 0$$

at an arbitrary $x(t) \in R^n$. In braces the quadratic form is written down

$$\begin{aligned} z^T(t,\tau)S[G,H,\gamma]z(t,\tau) &= x^T(t)\left[-A^T H - HA - \gamma H - G\right]x(t) \\ &- x^T(t)HBx(t-\tau) - x^T(t-\tau)B^T Hx(t) + x^T(t-\tau)Gx(t-\tau). \end{aligned}$$

The necessary and sufficient condition for a minimum on a variable $x(t-\tau)$ is equality to zero of a partial derivative on $x(t-\tau)$ and positive definiteness of a matrix G, i.e.

$$\frac{\partial}{\partial x(t-\tau)}\left\{z^T(t,\tau)S[G,H,\gamma]z(t,\tau)\right\} = 0.$$

Calculating the derivative, we get

$$-B^T Hx(t) + Gx(t-\tau) = 0.$$

As the matrix G is positive definite, it is non special. From this it follows that $x(t-\tau) = G^{-1}B^T Hx(t)$. We calculate the value of the quadratic form in the obtained point $x(t-\tau)$

$$z^T(t,\tau)S[G,H,\gamma]z\left(t,\tau\right) = x^T(t)\left[-A^T H - HA - \gamma H - G - HBG^{-1}B^T H\right]x(t).$$

From this we obtain that the matrix $S[G,H,\gamma]$ is positive definite, if there are positive definite matrices G and

$$Q[G,H,\gamma] = -A^T H - HA - \gamma H - G - HBG^{-1}B^T H.$$

This expression is used for

$$\lambda_{\min}\left(Q[G,H,\gamma]\right) > \lambda_{\min}\left[-A^T H - HA - G - HBG^{-1}B^T H\right] - \gamma\lambda_{\max}(H) > 0.$$

From this we obtain inequality (22), i.e. the statement of the Lemma. □

Using the proved Lemma, we obtain the following statement.

Theorem 3.1 *Let there not be any positive definite matrices G, H for which the matrix $S[G,H]$ is also positive definite. If the value γ is chosen according to an inequality (22) and $\beta \geq \gamma$ then for the solutions $x(t)$ of system (3) there are truly top exponential estimations of convergence (6), (7)*

$$|x(t)| \leq \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_{\tau,\beta}\right] \exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0,$$

$$\|x(t)\|_{\tau,\beta} \leq \left[\sqrt{\varphi_{21}(G,H)}\,|x(0)| + \sqrt{\varphi_{22}(G)}\,\|x(0)\|_{\tau,\beta}\right] \exp\left\{-\frac{1}{2}\varsigma t\right\}, \quad t \geq 0,$$

and

$$\varsigma(\beta,\gamma) = \min\left\{\frac{\lambda_{\min}(S[G,H])}{\lambda_{\max}(H)} + \gamma,\ \beta\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)} + \gamma\left[1 - \frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right]\right\}. \qquad (23)$$

The value β can be arbitrary if

$$\lambda_{\min}\left(S[G,H,\gamma]\right) \geq \lambda_{\max}(G)$$

and

$$\beta \leq \frac{1}{\tau}\ln\left\{\frac{\lambda_{\max}(G)}{\lambda_{\max}(G) - \lambda_{\min}\left(S[G,H,\gamma]\right)}\right\}$$

if

$$\lambda_{\min}\left(S[G,H,\gamma]\right) < \lambda_{\max}(H).$$

Proof For the proof of the statements of Theorem 3.1 again we use a Lyapunov–Krasovsky functional of the form (4) with positive definite matrices G and H. We write the full derivative of the functional (10) along solutions $x(t)$ of system (3) as

$$\frac{d}{dt}V[x(t),t] = -e^{\gamma t}\left\{(\beta-\gamma)\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^T(\xi)Gx(\xi)\,d\xi\right\}$$
$$- e^{\gamma t}(x^T(t), x^T(t-\tau))\begin{bmatrix} -A^TH - HA - \gamma H - G & -HB \\ -B^TH & G\end{bmatrix}\begin{pmatrix} x(t) \\ x(t-\tau)\end{pmatrix}$$
$$+ e^{\gamma t}(1-e^{-\beta\tau})x^T(t-\tau)Gx(t-\tau).$$

Let the matrix $S[G,H]$ described in (4), be nonpositive definite. Then, as follows from the Lemma, if γ satisfies conditions (22), then the matrix $S[G,H,\gamma]$ will be positive definite and the following inequality holds

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)\left(|x(t)|^2 + |x(t-\tau)|^2\right)$$
$$+ e^{\gamma t}\left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\,|x(t-\tau)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\,\|x(t)\|^2_{\tau,\beta}.$$

Let's transform the obtained expression as follows

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)|x(t)|^2$$
$$- e^{\gamma t}\left\{\lambda_{\min}\left(S[G,H,\gamma]\right) - \left(1-e^{-\beta\tau}\right)\lambda_{\max}(G)\right\}|x(t-\tau)|^2 \qquad (24)$$
$$- e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\,\|x(t)\|^2_{\tau,\beta}.$$

If the parameters of system and functional are such that

$$\lambda_{\min}\left(S[G,H,\gamma]\right) \geq \lambda_{\max}(G),$$

then

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S[G,H,\gamma]\right)|x(t)|^2 - e^{\gamma t}(\beta-\gamma)\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta} \tag{25}$$

for arbitrary $\beta \geq 0$. If

$$\lambda_{\min}\left(S[G,H,\gamma]\right) < \lambda_{\max}(G),$$

then inequality (25) is used for

$$0 \leq \beta < \frac{1}{\tau}\ln\left[\frac{\lambda_{\max}(G)}{\lambda_{\max}(G)-\lambda_{\min}\left(S[G,H,\gamma]\right)}\right].$$

We transform the right part of inequality of quadratic forms (9) to the form of expression (14)

$$-e^{\gamma t}\lambda_{\max}(H)|x(t)|^2 - e^{\gamma t}\lambda_{\max}(G)\left\|x(t)\right\|^2_{\tau,\beta} \leq -V[x(t),t]$$

and we consider two cases.

1. Let's transform the right part of the inequality (14) as

$$-e^{\gamma t}|x(t)|^2 \leq -\frac{1}{\lambda_{\max}(H)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\left\|x(t)\right\|^2_{\tau,\beta}$$

and we substitute it in the first part of inequalities (25). We get

$$\begin{aligned}\frac{d}{dt}V[x(t),t] \leq & -\frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}V\left[x(t,t)\right] \\ & - e^{\gamma t}\left\{(\beta-\gamma)\lambda_{\min}(G) - \left[\lambda_{\min}\left(S[G,H],\gamma\right)\right]\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}\right\}\left\|x(t)\right\|^2_{\tau,\beta}.\end{aligned} \tag{26}$$

If the parameters are such that

$$(\beta-\gamma)\lambda_{\min}(G) \geq \lambda_{\min}\left(S[G,H,\gamma]\right)\frac{\lambda_{\max}(G)}{\lambda_{\max}(H)}, \tag{27}$$

then

$$\frac{d}{dt}V[x(t),t] \leq -\frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}V[x(t),t].$$

Solving the obtained differential inequality, we get

$$V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \frac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}, \quad t \geq 0. \tag{28}$$

2. Further we transform inequality (14) as follows:

$$-e^{\gamma t}\left\|x(t)\right\|^2_{\tau,\beta} \leq -\frac{1}{\lambda_{\max}(G)}V[x(t),t] + e^{\gamma t}\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}|x(t)|^2,$$

and we also substitute it in the second part of inequality (27). We get

$$\frac{d}{dt}V[x(t),t] \leq -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V[x(t),t] \\ - e^{\gamma t}\left\{\lambda_{\min}\left(S[G,H,\gamma]\right)-(\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)}\right\}|x(t)|^2.$$

And if parameters are

$$\lambda_{\min}\left(S[G,H,\gamma]\right)-(\beta-\gamma)\lambda_{\min}(G)\frac{\lambda_{\max}(H)}{\lambda_{\max}(G)} > 0$$

then

$$\frac{d}{dt}V\left[x(t)\right] \leq -(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}V\left[x(t)\right].$$

Having integrated it, we obtain

$$V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha=(\beta-\gamma)\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}, \quad t\geq 0. \tag{29}$$

Let's return to bilateral estimations of Lyapunov–Krasovsky functional (9). Using expressions (28), (29), we obtain

$$e^{\gamma t}\big\{\lambda_{\min}(H)|x(t)|^2+\lambda_{\min}(G)\left\|x(t)\right\|^2_{\tau,\beta}\big\} \leq V[x(t),t] \leq V\left[x(0),0\right]e^{-\alpha t} \\ \leq e^{-\alpha t}\left\{\lambda_{\max}(H)\left|x(0)\right|^2+\kappa_{\max}(G)\left\|x(0)\right\|^2_{\tau,\beta}\right\}.$$

From this we obtain

$$|x(t)| \leq \left[\sqrt{\varphi_{11}(H)}\left|x(0)\right|+\sqrt{\varphi_{12}(G,H)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t\geq 0,$$

$$\left\|x(t)\right\|_{\tau,\beta} \leq \left[\sqrt{\varphi_{21}\left(G,H\right)}\left|x(0)\right|+\sqrt{\varphi_{22}(G)}\left\|x(0)\right\|_{\tau,\beta}\right]\exp\left\{-\frac{1}{2}(\alpha+\gamma)t\right\}, \quad t\geq 0.$$

From the consideration of both cases we get the following expressions

$$\alpha+\gamma=\begin{cases}\dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}+\gamma, & \text{for } \beta\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}+\gamma\left[1-\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right]\geq \dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}(H)}, \\ \dfrac{\beta\lambda_{\min}(G)}{\lambda_{\max}(G)}, & \text{for } \beta\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}+\gamma\left[1-\dfrac{\lambda_{\min}(G)}{\lambda_{\max}(G)}\right]< \dfrac{\lambda_{\min}\left(S[G,H,\gamma]\right)}{\lambda_{\max}H)}.\end{cases}$$

Uniting these expressions, we obtain the statement of Theorem 3.1. □

Remark 3.1 As for the value $\|x(t)\|^2_{\tau,\beta}$ the top estimations hold

$$\left\|x(t)\right\|^2_{\tau,\beta}=\int\limits_{-\tau}^{0}e^{\beta s}\left|x(t+s)\right|ds \leq \max_{-\tau\leq s\leq 0}\left\{\left|x\left(t+s\right)\right|^2\right\}\int\limits_{-\tau}^{0}e^{\beta s}ds \\ \leq \frac{1}{\beta}\left(1-e^{-\beta\tau}\right)\left\|x(t)\right\|^2_{\tau} \leq \tau\left\|x(t)\right\|_{\tau}$$

where

$$\|x(t)\|_\tau = \max_{-\tau \le s \le 0} \{|x(t+s)|\},$$

then it is possible to transform the inequality (6) to the following

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)}\,|x(0)| + \sqrt{\varphi_{12}(G,H)}\,\|x(0)\|_\tau\right] e^{-\frac{1}{2}\varsigma t}, \quad t \ge 0,$$

or, even,

$$|x(t)| \le \left[\sqrt{\varphi_{11}(H)} + \sqrt{\varphi_{12}(G,H)}\right] \|x(0)\|_\tau \, e^{-\frac{1}{2}\varsigma t}, \quad t \ge 0. \tag{30}$$

Remark 3.2 As estimations of majorant type, they contain two free parameters β and γ, and in the second theorem γ can be negative. If put to the task of finding an "optimum estimation" for a given class of functionals it is possible to calculate the parameters β and γ precisely.

4 Estimations of solutions of scalar subsystems

Let's consider the scalar linear differential equation with constant delay

$$\dot{x}(t) = -ax(t) + bx(t-\tau), \quad a > 0, \quad 0 \le t \le t_1, \quad \tau > 0. \tag{31}$$

For the equation (31) the Lyapunov–Krasovsky functional (10) looks like

$$V[x(t),t] = e^{\gamma t}\left\{hx^2(t) + g\int_{-\tau}^{0} e^{\beta s} x^2(t+s)\,ds\right\}, \tag{32}$$

where $h > 0$, $g > 0$ are positive constants. We obtain estimations of the divergence of disturbances on a finite time interval. As $h > 0$, $g > 0$ are scalar values then

$$\lambda_{\min}(H) = \lambda_{\max}(H) = h, \quad \lambda_{\min}(G) = \lambda_{\max}(G) = g.$$

For the full derivative of functional (32) along solutions of the equation (31) the equality holds

$$\begin{aligned}\frac{d}{dt}V[x(t),t] &= \gamma e^{\gamma t}\left\{hx^2(t) + g\int_{t-\tau}^{t} e^{-\beta(t-\xi)} x^2(\xi)\,d\xi\right\}\\ &\quad + e^{\gamma t}\left\{2hx(t)\left[-ax(t) + bx(t-\tau)\right] + gx^2(t) - ge^{-\beta\tau}x^2(t-\tau)\right\}\\ &\quad - e^{\gamma t}\left\{\beta g\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\}.\end{aligned}$$

Let's transform it similarly to the form of (11)

$$\begin{aligned}\frac{d}{dt}V[x(t),t] &= -e^{\gamma t}\left\{(\beta-\gamma)g\int_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\}\\ &\quad - e^{\gamma t}\left(x(t), x(t-\tau)\right)\begin{bmatrix}2ah - g & -hb\\ -hb & g\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix}\\ &\quad + e^{\gamma t}\gamma hx^2(t) + e^{\gamma t}\left(1 - e^{-\beta\tau}\right)gx^2(t-\tau).\end{aligned} \tag{33}$$

4.1 Derivation of estimations of disturbances in the case of stable equation

Let's find $h > 0$, $g > 0$ from the condition of "maximal" positive definiteness of the matrix

$$S[g,h] = \begin{bmatrix} 2ah - g & -hb \\ -hb & g \end{bmatrix}.$$

If the parameters of equation (31) and the Lyapunov–Krasovsky functional (32) are

$$g\,(2ah - g) - h^2 b^2 > 0,$$

as follows from Silvester criterion, the matrix $S[g,h]$ is positive definite. As $h > 0$, $g > 0$, then, taking into account uniformity, we denote $h = 1$ and we transform the inequality to

$$g\,(2a - g) - b^2 > 0.$$

Function $F(g) = g(2a - g) - b^2$ with respect to the variable g represents a parabola with the branches directed downwards. And it reaches the extreme value at $g = a$. Thus "maximal positive definiteness" of matrixes $S[g,h]$ is reached at $g = a$. And the Lyapunov – Krasovsky functional (32) is chosen as

$$V[x(t),t] = e^{\gamma t}\left\{ x^2(t) + a\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)} x^2(\xi)d\xi \right\}. \tag{34}$$

In this case a matrix $S[g,h]$ looks like

$$S[g,h] = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix}. \tag{35}$$

Let's transform the expression for a full derivative (33) in view of $h = 1$, $g = a$ to the form similar to (12)

$$\begin{aligned} \frac{d}{dt}V[x(t),t] \le & -e^{\gamma t}\left\{\lambda_{\min}\left(S[g,h]\right) - \gamma\right\}|x(t)|^2 \\ & - e^{\gamma t}\left\{\lambda_{\min}\left(S[g,h]\right) - \left(1 - e^{-\beta\tau}\right)a\right\}|x(t-\tau)|^2 \\ & - e^{\gamma t}(\beta - \gamma)a\left\|x(t)\right\|^2_{\tau,\beta} \end{aligned}$$

If

$$\lambda_{\min}\left(S[g,h]\right) = a - |b|, \quad \lambda_{\min}\left(S[g,h]\right) - \left(1 - e^{-\beta\tau}\right)a = e^{-\beta\tau}a - |b|,$$

then

$$\beta < \frac{1}{\tau}\ln\frac{a}{|b|}. \tag{36}$$

Then for a full derivative the inequality such as (13) becomes

$$\frac{d}{dt}V[x(t),t] \le -e^{\gamma t}\left\{a - |b| - \gamma\right\}|x(t)|^2 - e^{\gamma t}(\beta - \gamma)a\left\|x(t)\right\|^2_{\tau,\beta}. \tag{37}$$

It is easy to see that for the functional (33) the following inequality holds:

$$-e^{\gamma t}|x(t)|^2 - e^{\gamma t}a\left\|x(t)\right\|^2_{\tau,\beta} \le -V[x(t),t]. \tag{38}$$

a) We transform (38) to

$$-e^{\gamma t}|x(t)|^2 \le -V[x(t),t] + e^{\gamma t} a \, \|x(t)\|^2_{\tau,\beta} \, . \tag{39}$$

Also we substitute it in the first part of (37). We obtain

$$\frac{d}{dt}V[x(t),t] \le -\left(a-|b|-\gamma\right) V[x(t),t] - e^{\gamma t}\left[(\beta-\gamma)a - \left(a-|b|-\gamma\right)a\right] \|x(t)\|^2_{\tau,\beta} \, .$$

And, if for the parameters $\beta > a-|b|$ holds then

$$\frac{d}{dt}V[x(t),t] \le -(a-|b|-\gamma)V[x(t),t].$$

And from this

$$V[x(t),t] \le V\left[x(0),0\right] e^{-(a-|b|-\gamma)t}, \quad t \ge 0. \tag{40}$$

b) We transform (38) to

$$e^{\gamma t}\, \|x(t)\|^2_{\tau,\beta} \le -\frac{1}{a}V[x(t),t] + e^{\gamma t}\frac{1}{a}|x(t)|^2. \tag{41}$$

Also we substitute it in the second part of (37). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t] + \left(\beta - a + |b|\right) \|x(t)\|^2_{\tau,\beta} \, .$$

And, if $\beta \le a-|b|$, then

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t].$$

We get

$$V[x(t),t] \le V\left[x(0),0\right] e^{-(\beta-\gamma)t}, \quad t \ge 0. \tag{42}$$

Uniting inequalities (40), (41), we obtain

$$V[x(t),t] \le V\left[x(0),0\right] e^{-\alpha t}, \quad t \ge 0 \tag{43}$$

if

$$\alpha = \begin{cases} a-|b|-\gamma & \text{for} \;\; \beta > a-|b|, \\ \beta-\gamma & \text{for} \;\; \beta \le a-|b|. \end{cases}$$

Let's transform the inequality (43) as

$$e^{\gamma t}|x(t)|^2 + e^{\gamma t} a \, \|x(t)\|^2_{\tau,\beta} \le \left[|x(0)|^2 + a \, \|x(0)\|^2_{\tau,\beta}\right] e^{-\alpha t}, \quad t \ge 0.$$

We get

$$|x(t)| \le \sqrt{|x(0)|^2 + a\, \|x(0)\|^2_{\tau,\beta}}\, e^{-\frac{1}{2}(\alpha+\gamma)t},$$

$$\|x(0)\|^{\le}_{\tau,\beta} \sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|^2_{\tau,\beta}}\, e^{-\frac{1}{2}(\alpha+\gamma)t}, \quad t \ge 0.$$

Let's denote

$$\varsigma = \min t\{a-|b|,\, \beta\}.$$

As the value β is chosen according to (36), finally the following most exact estimation of convergence is obtained.

Proposition 4.1 *Let the condition $a > |b|$ be satisfied. Then the equation (31) is asymptotically stable and for its solutions the exponential estimation of convergence is valid*

$$|x(t)| \leq \sqrt{|x(0)|^2 + a\,\|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}\varsigma t}, \quad \|x(0)\|^{\leq}_{\tau,\beta} \sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}\varsigma t}, \quad t \geq 0,$$

for

$$\varsigma = \min\left\{a - |b|,\ \frac{1}{\tau}\ln\frac{a}{|b|}\right\}.$$

4.2 Derivation of estimations of disturbances in the case of unstable equation

Let's transform the expression for a full functional (34) derivative to

$$\begin{aligned} \frac{d}{dt}V[x(t),t] = &-e^{\gamma t}\left\{(\beta-\gamma)g\int\limits_{t-\tau}^{t} e^{-\beta(t-\xi)}x^2(\xi)d\xi\right\} \\ &- e^{\gamma t}\left(x(t), x(t-\tau)\right)\begin{bmatrix}2a-g-\gamma h & -hb\\ -hb & g\end{bmatrix}\begin{pmatrix}x(t)\\ x(t-\tau)\end{pmatrix} \\ &+ e^{\gamma t}\left(1-e^{-\beta\tau}\right)gx^2(t-\tau). \end{aligned} \tag{44}$$

Similarly to the first case, we denote $h = 1$, $g = a$. Then

$$S\left[g,h,\gamma\right] = \begin{bmatrix}a-\gamma & -b\\ -b & a\end{bmatrix}, \quad \lambda_{\min}\left(S\left[g,h,\gamma\right]\right) = a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2}. \tag{45}$$

Let's suppose, that $a < |b|$, i.e. the equation is unstable. Then if

$$\gamma < \frac{a^2 - b^2}{a}, \tag{46}$$

the matrix $S\left[g, h.\gamma\right]$ is positive definite, i.e. $\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) > 0$ and expression for a full functional (34) derivative can be written down as

$$\begin{aligned} \frac{d}{dt}V[x(t),t] \leq &-e^{\gamma t}\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)|x(t)|^2 \\ &- e^{\gamma t}\left\{\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - \left(1-e^{-\beta\tau}\right)a\right\}|x(t-\tau)|^2 - e^{\gamma t}(\beta-\gamma)a\,\|x(t)\|^2_{\tau,\beta}. \end{aligned}$$

As the value

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - a = -\frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} < 0$$

is always negative, then if

$$\beta < \frac{1}{\tau}\ln\frac{a}{\frac{1}{2}\gamma + \sqrt{b^2 + \frac{1}{4}\gamma^2}} \tag{47}$$

it yields

$$\frac{d}{dt}V[x(t),t] \leq -e^{\gamma t}\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)|x(t)|^2 - e^{\gamma t}(\beta-\gamma)a\,\|x(t)\|^2_{\tau,\beta}. \tag{48}$$

1) We substitute inequality (39) in the first part of (48). We obtain

$$\frac{d}{dt}V[x(t),t] \le -\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)V[x(t),t] + e^{\gamma t}\left\{a\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) - (\beta-\gamma)a\right\}\left\|x(t)\right\|^2_{\tau,\beta}.$$

And, if inequality

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) < \beta - \gamma \tag{49}$$

holds, then

$$\frac{d}{dt}V[x(t),t] \le -\lambda_{\min}\left(S\left[g,h,\gamma\right]\right)V[x(t),t]. \tag{50}$$

From this, we have

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2}, \quad t \ge 0. \tag{51}$$

2) We substitute an inequality (41) in the second part of (48). We obtain

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t] + e^{\gamma t}\left\{-\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) + (\beta-\lambda)\right\}\left|x(t)\right|^2$$

and, if

$$\lambda_{\min}\left(S\left[g,h,\gamma\right]\right) \ge \beta - \gamma, \tag{52}$$

then

$$\frac{d}{dt}V[x(t),t] \le -(\beta-\gamma)V[x(t),t]. \tag{53}$$

We get

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad \alpha = \beta - \gamma, \quad t \ge 0. \tag{54}$$

Uniting expressions (51), (54) connected by conditions (49), (52) and having substituted instead of $\lambda_{\min}(S[g,h,\gamma])$ its value, we obtain

$$V[x(t),t] \le V\left[x(0),0\right]e^{-\alpha t}, \quad t \ge 0,$$

if

$$\alpha = \begin{cases} a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} & \text{for } \; a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} < \beta - \gamma, \\ \beta - \gamma, & \text{for } \; a - \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} \ge \beta - \gamma. \end{cases}$$

Let's denote $\alpha + \gamma = \varsigma$, and we obtain

$$\varsigma(\beta,\gamma) = \begin{cases} a + \frac{1}{2}\gamma - \sqrt{b^2 + \frac{1}{4}\gamma^2} & \text{for } \; a - \sqrt{b^2 + \frac{1}{4}\gamma^2} < \beta; \\ \beta & \text{for } \; a - \sqrt{b^2 + \frac{1}{4}\gamma^2} \ge \beta. \end{cases}$$

As the values β and γ satisfy the expressions

$$\beta < \frac{1}{2}\ln\frac{a}{\frac{1}{2}\gamma + \sqrt{b^2 + \frac{1}{4}\gamma^2}}, \quad \gamma < \frac{a^2 - b^2}{a},$$

the following result holds.

Proposition 4.2 *Let the condition $a < |b|$ be satisfied. Then the equation (31) is unstable and for its solutions the following exponential estimation holds*

$$|x(t)| \le \sqrt{|x(0)|^2 + a\,\|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}\varsigma t}, \quad \|x(0)\|^{\le}_{\tau,\beta}\sqrt{\frac{1}{a}\,|x(0)|^2 + \|x(0)\|^2_{\tau,\beta}}\,e^{-\frac{1}{2}\varsigma t}, \quad t \ge 0,$$

for

$$\varsigma = \frac{a^2 - b^2}{a}.$$

5 Estimations of solutions of hybrid systems

In the previous sections majorant estimations of solutions of stable and unstable subsystems were separately obtained. Now we shall consider whole hybrid system (1). On each of intervals $t_{i-1} \le t < t_i$, $i = \overline{1,N}$ let's select Lyapunov–Krasovsky functional of the form (4) with positive definite matrices H_i, G_i, $i = \overline{1,N}$. If there are positive definite matrices H_i, G_i, $i \in I$, such that matrices

$$S_i\left[G_i, H_i\right] = \begin{bmatrix} -A_i^T H_i - H_i A_i - G_i & -H_i B_i \\ -B_i^T H_i & G_i \end{bmatrix}, \quad i \in I$$

are positive definite, then we designate

$$N_i = \left[\sqrt{\varphi_{11}\left(H_i\right)} + \sqrt{\varphi_{12}\left(G_i, H_i\right)}\right] \exp\left\{\varsigma_i\left(\beta_i, \gamma_i\right)\tau\right\},$$

where the value $\beta_i > 0$ can be arbitrary at

$$\lambda_{\min}\left(S\left[G_i, H_i\right]\right) \ge \lambda_{\max}(G_i)$$

and

$$\beta_i \le \frac{1}{\tau}\ln\left\{\frac{\lambda_{\max}(G_i)}{\lambda_{\max}(G_i) - \lambda_{\min}\left(S\left[G_i, H_i\right]\right)}\right\},$$

if $\lambda_{\min}\left(S\left[G_i, H_i\right]\right) < \lambda_{\max}\left(H_i\right)$. The value γ satisfies the condition $\gamma \le \beta$. If such matrices H_i, G_i , $j \in J$ do not exist, then we assume

$$\gamma_j < \frac{\lambda_{\min}\left[-A_j^T H_j - H_j A_j - G_j - H_j B_j G_j^{-1} B_j^T H_j\right]}{\lambda_{\max}(H_j)},$$

and we denote

$$S\left[G_j, H_j, \gamma_j\right] = \begin{bmatrix} -A_j^T H_j - H_j A_j - \gamma_j H_j - G_j & -H_j B_j \\ -B_j^T H_j & G_j \end{bmatrix},$$

$$N_j = \left[\sqrt{\varphi_{11}\left(H_j\right)} + \sqrt{\varphi_{12}\left(G_j, H_j\right)}\right] \exp\left\{\varsigma_j\left(\beta_j, \gamma_j\right)\right\},$$

for

$$\varsigma_j\left(\beta_j, \gamma_j\right) = \min\left\{\frac{\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right)}{\lambda_{\max}(H_j)} + \gamma_j,\ \beta_j \frac{\lambda_{\min}(G_j)}{\lambda_{\max}(G_j)} + \gamma_j\left[1 - \frac{\lambda_{\min}(G_j)}{\lambda_{\max}(G_j)}\right]\right\}.$$

The value β_j can be arbitrary at

$$\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right) \ge \lambda_{\max}(G_j)$$

and

$$\beta_j \le \frac{1}{\tau} \ln \left\{ \frac{\lambda_{\max}(G_j)}{\lambda_{\max}(G_j) - \lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right)} \right\},$$

if

$$\lambda_{\min}\left(S\left[G_j, H_j, \gamma_j\right]\right) < \lambda_{\max}\left(H_j\right).$$

Theorem 5.1 *Let the initial state of the logic-dynamical hybrid system (1) satisfy the condition $\|x(0)\|_\tau < \delta$. Then at $t = t_N$ the following inequality holds*

$$\|x(t_N)\| \le \prod_{i=1}^{N} N_i \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \varsigma_i \left(t_i - t_{i-1}\right) \right\}.$$

Proof Let's consider the first time interval $t_0 \le t \le t_1$, $t_0 = 0$. If there are positive definite matrices G_1, H_1, for which the matrix $S[G_1, H_1]$ is also positive definite, then as follows from expression (30) of Remark 1, the following inequality holds:

$$\|x(t_1)\| \le \left[\sqrt{\varphi_1(H_1)} + \varphi\left(G_1, H_1\right)\right] \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - \tau)}.$$

If there are no such matrices, for arbitrary positive definite matrices G_1, H_1, there exists γ_1, for which the matrix $S[G_1, H_1, \gamma_1]$ is also positive definite. Again using expression (30) of Remark 1, we get

$$\|x(t_1)\| \le \left[\sqrt{\varphi_1(H_1)} + \varphi\left(G_1, H_1\right)\right] \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - t_0)}.$$

And for the moment $t = t_1$

$$\|x(t_1)\|_\tau \le N_1 \|x(t_0)\|_\tau \, e^{-\frac{1}{2}\varsigma_1 (t_1 - t_0)}$$

holds. Let us consider the next interval $t_1 \le t \le t_2$. As for the second interval a similar estimate

$$\|x(t_2)\|_\tau \le N_2 \|x(t_1)\|_\tau \, e^{-\frac{1}{2}\varsigma_2 (t_2 - t_1)}$$

holds we obtain

$$\|x(t_2)\|_\tau \le N_1 N_2 \|x(t_0)\|_\tau \exp\left\{ -\frac{1}{2} \left[\varsigma_1 \left(t_1 - t_0\right) + \varsigma_2 \left(t_2 - t_1\right)\right] \right\}.$$

Continuing the process further, for the moment $t = t_N$ we get

$$\|x(t_N)\| \le \prod_{i=1}^{N} N_i \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} \varsigma_i \left(t_i - t_{i-1}\right) \right\},$$

which was required to prove. □

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